Network for Studies on Pensions, Aging and Retirement

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Discussion Paper 2008 - 008

February 15, 2008

ANALYTICAL APPROXIMATIONS FOR PRICES OF SWAP RATE DEPENDENT EMBEDDED OPTIONS IN INSURANCE PRODUCTS

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This version: February 15, 2008 [1]

Abstract

Life insurance products have profit sharing features in combination with guarantees. These so-called embedded options are often dependent on or approximated by forward swap rates. In practice, these kinds of options are mostly valued by Monte Carlo simulations. However, for risk management calculations and reporting processes, lots of valuations are needed. Therefore a more efficient method to value these options would be helpful. In this paper analytical approximations are derived for these kinds of options, based on an underlying multi-factor Gaussian interest rate model. The analytical approximation for options with direct payment is almost exact while the approximation for compounding options is also satisfactory. In addition, the proposed analytical approximation can be used as a control variate in Monte Carlo valuation of options for which no analytical approximation is available, such as similar options with management actions. Furthermore, it's also possible to construct analytical approximations when returns on additional assets (such as equities) are part of the profit sharing rate.

Keywords: embedded options, insurance products, analytical approximation, Gaussian interest rate model, Fair Value of Liabilities, IFRS 4 Phase 2, Solvency 2, Monte Carlo simulation, Control Variate technique

1. Introduction

In recent years there has been an increasing amount of attention of the insurance industry for market valuation of insurance liabilities. Important drivers of this development are IFRS 4 Phase 2 and Solvency 2, that both are expected to be implemented around 2011. IFRS 4 Phase 2 will define an accounting model for insurance contracts. In the document "Preliminary Views on

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Insurance Contracts" (May 2007, discussion paper) the IASB states that "..the Board's preliminary view is that the inputs used to develop estimates of cash flows should, as far as possible, be consistent with observed market prices..". At this moment, most insurers are reporting their liabilities on a book value basis, where the economic assumptions are often not directly linked to the financial market.

Solvency 2 will lead to a change in the regulatory required solvency capital for insurers. At this moment this capital requirement is a fixed percentage of the mathematical reserve or the risk capital. Under Solvency 2 these solvency requirements will be risk-based, and market values of assets and liabilities will be the basis for these calculations.

An important part of the market valuation of liabilities is the valuation of embedded options. Embedded options are options that have been sold to the policyholders and are often the more complex features in insurance products. An embedded option that is very common in insurance products in Europe, is a profit sharing rule based on a (moving average) fixed income rate, in combination with a minimum guarantee. This fixed income rate is either from an external source or could be the book value return on a fixed income portfolio. For example, in the Netherlands the profit sharing is often based on the so-called u-yield, which is more or less an average return of several treasury rates. In other parts of Europe, the book value return on the fixed income portfolio is often the basis for the profit sharing. In practice the exact rates are difficult to determine and to project forward, and implied volatilities from the market are not available. Therefore often the euro swap rate is used as a proxy. So what remains is the valuation of an option on a moving or weighted average of forward and historic swap rates.

Most insurers use Monte Carlo simulations for the valuation of their embedded options. The advantage of this is that many kinds of options can be valued with it (also the more complex ones) and that it gives one uniform simulation framework that is applicable for various embedded options. However, an important disadvantage is the computational time it requires. Embedded option calculations are required for Fair Value reporting, Market Consistent Embedded Value, Asset Liability Management, product development and pricing, Economic Capital calculations and Mergers & Acquisitions. For most of these purposes several calculations are required. For the calculation of Economic Capital for example 20.000 or more simulations are used and in each of these scenario's the market value of liabilities (and thus the value of embedded options) has to be calculated. Also for other purposes, often sensitivities and analysis of changes are necessary. If an insurer then also exists of several business units or legal entities, the total computational time can be significant. Therefore, analytical solutions for the valuation of embedded options would be very helpful.

In this paper analytical approximations are derived for the above mentioned swap rate dependent embedded options. The underlying interest rate model is a multi-factor Gaussian model. This model is very appealing because of its analytical tractability. Also, the model implicitly accounts for the volatility skew to some extent, what is important for these kind of options because those are in most cases not at-the-money. Because of this the model is often used in practice (in most cases the 1-factor or 2-factor hull-white variant). Analytical approximations are derived for the case of direct payment of profit sharing, as well as for the case of compounding profit sharing. In case of (very) complex options with management actions, the analytical approximation for the direct payment case can be used as a control variate in combination with Monte Carlo simulation, reducing the computational time to a great extent.

It could well be that an insurance company has other kinds of embedded options for which no analytical approximations are available. These embedded options probably have to be valued using Monte Carlo simulation. Since the multi-factor Gaussian models are often used in practice, the analytical approximation for the swap rate dependent options can in that case be used in conjunction with the simulation model that may be required for the valuation of other embedded options. This results in a consistent underlying interest rate model for the valuation of embedded options, despite the fact that perhaps some of the options are valued with Monte Carlo simulations and others with analytical formulas.

The basis for the analytical approximation is the result of Schrager and Pelsser (2006), who have developed an approximation for swaption prices for affine term structure models (of which the multi-factor Gaussian models are a subset). They determine the dynamics of the swap rate under the relevant swap measure and these dynamics are approximated by replacing some low-variance martingales by their time zero values. This technique is already used extensively in the context of Libor Market Models and given the results of Schrager and Pelsser, it also proves to work well in an affine setting. By use of the Change of Numeraire techniques developed by Geman et al (1995)[2], the result of Schrager and Pelsser can be used to derive analytical approximations for swap rate dependent options.

Most of the existing literature on valuation of embedded options in insurance products focuses on Unit Linked products, With-Profits products or Guaranteed Annuity Options, see for example Schrager and Pelsser (2004), Nielsen and Sandmann (2002), Sheldon and Smith (2004), Castellani et al (2007), Prieul et al (2001), Wilkie et al (2003) and Grosen and Jorgensen (2001). However, to our knowledge there has been little focus on profit sharing based on (moving average) fixed income rates, despite this being one of the most common types of profit sharing in Europe. Our contribution to the existing literature is that we provide analytical approximations for these kinds of profit sharing. Analytical approximations for direct payment of profit sharing and for compounding profit sharing are given, while a combination with returns on other assets (such as equities) is also possible. In addition, the proposed analytical approximation can be used as a control variate in Monte Carlo valuation of options for which no analytical approximation is available, such as similar options with management actions. This potentially reduces the number of simulations required to a great extent.

The remainder of the paper is organized as follows. First, in section 2 the characteristics of the swap rate dependent embedded options are described. In section 3 the underlying Gaussian interest model is given. In section 4 the Schrager-Pelsser result for swaptions is repeated and this is applied to the direct payment case in section 5. In section 6 possibilities are given for more complex embedded options. Then numerical examples are worked out in section 7 and conclusions are given in section 8.

[2] For more information about this subject, see for example Pelsser (2004) or Brigo and Mercurio (2006).

2. Swap rate dependent embedded options

Traditional non-linked life insurance products often guarantee a certain insured amount. Common practice was (and often still is) to calculate the price of this insurance by discounting the expected cash flows with a relatively low interest rate, called the technical interest rate. Often this is combined with profit sharing, where some reference return is paid out to the policyholder if this exceeds the technical interest rate, possibly under subtraction of a margin. There exist various types of profit sharing, such as:

- Profit sharing based on an external reference index
- Profit sharing based on the (book or market value) return on the underlying investment portfolio
- Profit sharing based on the performance and profits of the insurance company
- Profit sharing of the so-called with-profits products, where regular and terminal bonuses are given though the life of the product, based on the return of the underlying investment portfolios. The terms of these policies often contain management actions that allow the insurance companies to reduce the risks of these products.

In most cases where the profit sharing rate depends on a certain fixed income rate, the exact profit sharing rate is either very complex or not fully known, or implied volatilities from the market are not available. In practice, these kinds of options are often valued using an (average) forward swap rate as an approximation for the profit sharing rate. The profit sharing payoff *PS(t)* in year *t* is in that case:

(2.1) $$PS(t) = L(t)\,Max\{c\,(R(t) - K(t)), 0\}$$

where *L(t)* is the profit sharing basis, c is the percentage that is distributed to the policyholder and *K(t)* is the strike of the option. The strike equals the sum of the technical interest rate *TR(t)* and a margin. In most cases, either the margin or the c is used for the benefits of the insurer. *R(t)* is the profit sharing rate and is a (weighted) average of historic and forward swap rates.

The profit sharing as described in (2.1) is a call option on a rate *R(t)* and has to be valued using option valuation techniques. The profit sharing is either paid directly or is being compounded and paid at the end of the contract.

Example 1 – book value return on underlying portfolio
One of the most common forms of profit sharing across the European life insurance business is the one where the profit sharing rate is based on the book value return of the underlying fixed income portfolio[3]. To be able to value this option, assumptions have to be made about the reinvestment strategy. One way to tackle this problem is to assume a certain average turnover rate δ and a reinvestment strategy favoring *m*-year maturity assets. This means that the book value return of the portfolio can be modeled as follows:

(2.2) $$R(t) = (1-\delta)\,R(t-1) + \delta\, y_{t,t+m}(t)$$

[3] This is common practice in for example France, Germany, Italy, Czech, Switzerland and Norway.

where $y_{t,t+m}(t)$ is the m-year swap rate at time t. The book value return on time t can also be expressed in terms of the current book value return $R(0)$, leading to an exponentially weighted moving average:

$$(2.3) \quad R(t) = (1-\delta)^t \, R(0) + \sum_{i=0}^{t} y_{i,i+m}(i)\,(1-\delta)^{t-i}\,\delta$$

being a weighted combination of forward swap rates and the current book value return.

Another approach that is often used is approximating the book value return by a moving average of swap rates:

$$(2.4) \quad R(t) = \frac{1}{n} \sum_{i=t-n+1}^{t} y_{i,i+m}(i)$$

□

where n (= $1/\delta$) is the number of fixings of the moving average.

Example 2 – "u-rate" profit sharing in the Netherlands
In the Netherlands the most common form of profit sharing is based on a moving average of the so-called u-rate. The u-rate is the 3-months average of u-rate-parts, where the subsequent u-rate-parts are weighted averages of an effective return on a basket of government bonds. This leads to a complicated expression, and no implied volatilities are available for government bonds. Therefore, it is common practice in the Netherlands to approximate the u-rate or the u-yield parts by a swap-rate[4]. That means that the profit sharing rate is approximated by a moving average of swap rates, as in (2.4). □

In the following sections analytical approximations are developed for prices of the above mentioned embedded options and of other possible forms of profit sharing where the profit sharing rate depends on or is approximated by forward swap rates. Note that the developed formulas are approximating swap rate dependent embedded options. When considering the results or using the formulas one always has to be aware of the fact that the first error is introduced when the swap rate is being used as a proxy for the profit sharing rate.

3. The underlying interest rate model

The underlying interest rate model for the valuation is the class of multi-factor Gaussian models. Special cases of this class of models are the 1-factor and 2-factor Hull-White model, which are often used in practice. These models are very appealing because of their analytical tractability.

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[4] Historical data that shows that u-rate parts have behaved similarly as swap rates in the past, is available upon request.

In the swaption market, the observed implied Black volatility is varying for different strike levels, leading to the so-called volatility skew. This volatility skew exists because the market apparently does not believe in lognormally distributed swap rates. Instead, the volatility skew seems to indicate a distribution that is closer to the normal distribution[5]. Therefore, the Gaussian models implicitly account for the volatility skew to a certain extent. This is also an appealing property of these models in the context of embedded options in insurance products, since these options are in most cases not at-the-money.

It is very well possible that insurance companies are going to use Monte Carlo simulations as well as analytical formulas for the pricing of their embedded options. This could be the case for example when the insurer also wrote embedded options that are too complex to value analytically. When using both techniques, it is important that the underlying stochastic interest rate model is consistent, so that the pricing of the various embedded options is consistent. Since the Gaussian models are often used in practice for more complex options, the analytical approximation developed in this paper can be used in conjunction with the Monte Carlo simulation model that may be required for the valuation of other embedded options.

The Gaussian interest rate models are a special case of the affine term structure models introduced by Duffie and Kan (1996). The *m*-factor Gaussian model describes the stochastic process for the instantaneous short rate as follows[6]:

$$(3.1) \quad r(t) = \mathbf{1}'Y(t) + \alpha(t)$$

$$(3.2) \quad dY(t) = -AY(t)dt + \sum dW^Q(t)$$

where $W^Q(t)$ is a m-dimensional Brownian motion under the risk-neutral measure and A and Σ are m x m matrices. A is a diagonal matrix.

The function $\alpha(t)$ is chosen is such a way that the fit with the initial term structure is perfect. The covariance matrix of the Y-variables is equal to $\Sigma\Sigma$'.

The analytical tractability of this model makes it possible to obtain bond prices analytically, from which swap and zero rates can be derived. The price at time t of a zero bond maturing at time T is given by:

$$(3.3) \quad D(t,T) = \exp\left(C(t,T) - \sum_{i=1}^{m} B^{(i)}(t,T)Y^{(i)}(t) \right)$$

where $B^{(i)}(t,T) = 1 / A_{(ii)}\left(1 - \exp(-A_{(ii)}(T-t))\right)$

[5] See Levin (2004) for a discussion on this issue.
[6] See Brigo & Mercurio (2006) for an extensive explanation of and pricing formulas for the 2-factor Gaussian model.

The expression for $C(t,T)$ is given in for example Brigo and Mercurio (2006) for the 1-factor and 2-factor case. However, since it is not used for the remaining part of the paper, we don't repeat it here.

The analytical tractability of the model and the implicit accounting for the skew make the Gaussian models relatively easy to implement, while there are also more possibilities for analytical approximations (or solutions) for embedded options.

4. The Schrager-Pelsser result for swaptions

Schrager and Pelsser (2006) have developed an approximation for swaption prices for affine term structure models. In this section their main result for the Gaussian models is repeated.

The swap rate $y_{n,N}$ is the par swap rate at which a person would like to enter into a swap contract with a value of 0, starting at time T_n (first payout at time T_{n+1}) and lasting until T_N. The swap rate at time t is given by:

$$(4.1) \qquad y_{n,N}(t) = \frac{D(t,T_n) - D(t,T_N)}{\sum_{k=n+1}^{N} \Delta_{k-1}^{Y} D(t,T_k)} = \frac{D(t,T_n) - D(t,T_N)}{P_{n+1,N}(t)}$$

where Δ_{k-1}^{Y} is the market convention for the calculation of the daycount fraction for the swap payment at T_k. When using $P_{n+1,N}(t)$ as a numéraire, all $P_{n+1,N}(t)$ rebased values must be martingales under the measure $Q^{n+1,N}$, associated with this numeraire. That means that $y_{n,N}$ is a martingale under this so-called swap measure, which is introduced by Jamshidian (1998). When applying Ito's Lemma to the model defined in (3.1) and (3.2) the following dynamics for the swap rate $y_{n,N}(t)$ under the swap measure result:

$$(4.2) \qquad dy_{n,N}(t) = \frac{\partial y_{n,N}(t)}{\partial Y(t)} \Sigma \, dW^{n+1,N}(t)$$

Where $dW^{n+1,N}$ is a m-dimensional Brownian motion under the swap measure $Q^{n+1,N}$ corresponding to the numéraire $P_{n+1,N}(t)$. Schrager and Pelsser (2006) determine the partial derivatives in (4.2), which are stochastic, and approximate these by replacing low-variance martingales by their time zero values. This technique is already used extensively in the context of Libor Market Models[7] and given the results of Schrager and Pelsser, it also proves to work well in an affine setting. This approximation makes the swap rate volatility deterministic and thus leads to a normally distributed forward swap rate. The approach described leads to an analytical approximation for the integrated variance of $y_{n,N}$ (associated with a T_n x T_N swaption) over the interval $[0,T_n]$ (for the proof, see appendix A):

[7] See Brigo and Mercurio (2006) and Gatarek (2003).

(4.3) $$\sigma^2_{n,N} \approx \sum_{i=1}^{m}\sum_{j=1}^{m} \hat{\Sigma}_{(ij)}\, \tilde{C}^{(i)}_{n,N} \tilde{C}^{(j)}_{n,N} \left[\frac{e^{\left[A_{(ii)}+A_{(jj)}\right]T_n} - 1}{A_{(ii)} + A_{(jj)}} \right]$$

where $\hat{\Sigma}_{(ij)}$ is the element (i,j) of $\Sigma\Sigma'$ and

(4.4) $$\tilde{C}^{(i)}_{n,N} = \frac{1}{A_{(ii)}} \left[e^{-A_{(ii)}T_n} D^P(0,T_n) - e^{-A_{(ii)}T_N} D^P(0,T_N) - y_{n,N}(0) \sum_{k=n+1}^{N} \Delta^Y_{k-1} e^{-A_{(ii)}T_k} D^P(0,T_k) \right]$$

where $D^P(t,T_n) = D(t,T_n) / P_{n+1,N}(t)$, the bond price normalized by the numéraire.

The result is an easy to implement analytical approach to calibrate Gaussian models to the swaption market. A nice by-product of the approach (as opposed to other approaches for approximating swaption prices) is that the dynamics of the swap rates are approximated. These approximate dynamics can be used for approximating prices of other swap-rate dependent options.

5. Analytical approximation – direct payment

Assume that the profit sharing rate at time T_i is a weighted average of τ-year maturity swap rates with weights w_k and the averaging period is from time T_{i-s} to time T_i :

(5.1) $$R(T_i) = \sum_{k=T_{i-s}}^{T_i} w_k\, y_{k,k+\tau}(k)$$

where $\Sigma w_k = 1$.

In case of direct payment of profit sharing, the embedded option is in fact a strip of options that mature at time T_i (i=1,2,…) and lead to a direct payment of $R(T_i)$ on these dates. Since the individual $y_{k,k+\tau}(k)$'s are approximately normally distributed (see section 4), $R(T_i)$ is also approximately normally distributed. So to value the option the expectation and the variance of $R(T_i)$ have to be approximated under the T_i-forward measure and feed into a Gaussian option formula for each time T_i. For determining the variance of $R(T_i)$ the covariance's of the $y_{k,k+\tau}(k)$'s with the $y_{l,l+\tau}(k)$'s have to be specified.

5.1 Determining the expectation of $R(T_i)$

The above means that each individual option has to be priced in the T_i-forward measure. To come to the expectations of *R(Ti)* under the right measure the following steps are necessary:

a) For each (forward) swap rate $y_{n,N}$ a change of measure has to be done from the swap measure $Q^{n+1,N}$ to the T_n-forward measure Q^{Tn}.

b) If the payoff of the option on the average of the swap rates is at time T_i, for each of the individual swap rates observed at time (T_{i-s}), a change of measure has to be done from the (T_{i-s})-forward measure to the T_i-forward measure.

The corrections mentioned above can be interpreted as convexity corrections (a) and timing corrections (b). The formulas for these corrections are given in (5.2) and (5.3), of which the proofs are given in appendix B. Note that due to the changes of measure it's not guaranteed that the quality of the approximation will remain. Therefore, this will be tested in section 7.

The convexity correction $CC_{n,N}(T_n)$ for time $T_n > 0$ for the swap rate $y_{n,N}$ is:

$$(5.2) \quad CC_{n,N}(T_n) \approx \sum_{i=1}^{m}\sum_{j=1}^{m} \hat{\Sigma}_{(ij)}\, \tilde{C}_{n,N}^{(i)} \tilde{G}_{n,N}^{(j)} \left[\frac{e^{[A_{(ii)}+A_{(jj)}]T_n} - 1}{A_{(ii)} + A_{(jj)}} \right]$$

$$where \;\; \tilde{G}_{n,N}^{(j)} = \frac{1}{A_{(jj)}} \left[e^{-A_{(jj)}T_n} - \sum_{k=n+1}^{N} \Delta_{k-1}^{Y}\, e^{-A_{(jj)}T_k}\, D^P(0,T_k) \right]$$

The timing correction $TC_{n,N}(T_n,T_{n+u})$ representing a change of measure from time $T_n > 0$ to T_{n+u} is:

$$(5.3) \quad TC_{n,N}(T_n,T_{n+u}) \approx \sum_{i=1}^{m}\sum_{j=1}^{m} \hat{\Sigma}_{(ij)}\, \tilde{C}_{n,N}^{(i)} \tilde{H}_{T_n,T_{n+u}}^{(j)} \left[\frac{e^{[A_{(ii)}+A_{(jj)}]T_n} - 1}{A_{(ii)} + A_{(jj)}} \right]$$

$$where \;\; \tilde{H}_{T_n,T_{n+u}}^{(j)} = \frac{1}{A_{(jj)}} \left[e^{-A_{(jj)}(T_{n+u})} - e^{-A_{(jj)}T_n} \right]$$

For $T_n < 0$, the convexity corrections and the timing corrections are 0. Note that in the derivation of (5.2) also stochastic terms are replaced by their time zero values, leading to a deterministic convexity correction.

The expectation $\mu_{R(T_i)}$ of $R(T_i)$ becomes:

$$(5.4) \quad \mu_{R(T_i)} \approx \sum_{k=T_{i-s}}^{T_i} w_k \left[y_{k,k+\tau}(0) + CC_{k,k+\tau}(k) + TC_{k,k+\tau}(k,T_i) \right]$$

The convexity correction is positive and the timing correction is negative, so they are partly offsetting each other. The formulas (5.2) and (5.3) have the same structure as in case of the swaptions in section 4, so the implementation is not much more complicated than that.

5.2 Determining the variance of $R(T_i)$

Given that the drift term is deterministic, the change of measure has no impact on the volatility, so expression (4.3) can be used to determine the variance of $R(T_i)$. The variance $\sigma_{R(T_i)}^2$ of $R(T_i)$ is:

(5.5) $$\sigma^2_{R(T_i)} = \sum_{k=T_{i-s}}^{T_i} \sum_{l=T_{i-s}}^{T_i} w_k w_l \, Cov[y_{k,k+\tau}(k), y_{l,l+\tau}(l)]$$

where *Cov(.)* is the covariance between the swap rates. From stochastic calculus we know:

(5.6) $$Cov\left[\int_0^t f(u)dW_u , \int_0^s g(u)dW_u \right] = \int_0^s f(u)g(u)du$$

Using this and expression (4.3) the covariance between swap rates is

(5.7) $$\begin{aligned} Cov[y_{k,k+\tau}(k), y_{l,l+\tau}(l)] &\approx \int_0^{k\wedge l} e^{As'} \, diag(\tilde{C}_{k,k+\tau}) \Sigma \Sigma' diag(\tilde{C}_{l,l+\tau}) e^{As} \, ds \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} \hat{\Sigma}_{(ij)} \, \tilde{C}^{(i)}_{k,k+\tau} \tilde{C}^{(j)}_{l,l+\tau} \left[\frac{e^{\left[A_{(ii)} + A_{(jj)}\right](k\wedge l)} - 1}{A_{(ii)} + A_{(jj)}} \right] \end{aligned}$$

where $k \wedge l$ = min(k,l).

5.3 Pricing formulas

The total value of the embedded option is the sum of the values of the strip of options that mature at time T_i (i=1,2,…). The profit sharing specified in (2.1) is in fact a call option on the normally distributed rate $R(T_i)$ with expectation (5.4) and variance (5.5) under the T_i-forward measure.

Let $\varphi_{\mu,\sigma}(\cdot)$ be the density of a Gaussian random variable with mean μ and standard deviation σ, $\Phi_{\mu,\sigma}$ the corresponding distribution function and $\Phi = \Phi_{0,1}$.

The value at time 0 of the profit sharing payoff $PS(T_i)$ at time *Ti* is[8]:

(5.8) $$\begin{aligned} V[PS(T_i)] &= D(0,T_i)\, L(T_i)\, c\, E^{T_i}\left[Max\{R(T_i) - K(T_i), 0\}\right] \\ &= D(0,T_i)\, L(T_i)\, c \int_{K(T_i)}^{\infty} (x - K(T_i))\, \varphi_{\mu_{R(Ti)},\sigma_{R(Ti)}}(x)\, dx \\ &= D(0,T_i)\, L(T_i)\, c \left[(\mu_{R(T_i)} - K(T_i))\, \Phi\left(\frac{\mu_{R(T_i)} - K(T_i)}{\sigma_{R(T_i)}} \right) + \sigma_{R(T_i)}\, \varphi\left(\frac{K(T_i) - \mu_{R(T_i)}}{\sigma_{R(T_i)}} \right) \right] \end{aligned}$$

The total value of the profit sharing at time 0 is then:

(5.9) $$V\left[PS\right] = \sum_i V\left[PS(T_i)\right]$$

[8] These results can be derived in a similar fashion in case of a put-option on rate $R(T_i)$.

When the profit sharing payoff at a time > 0 is dependent on observations at a time < 0, a slight adjustment has to be done. In that case the expectation to be valued is:

$$
\begin{aligned}
V\left[PS(T_i)\right] &= D(0,T_i)\,L(T_i)\,E^{T_i}\left[Max\{R(T_i) - K(t),0\}\right] \\
(5.10)\qquad &= D(0,T_i)\,L(T_i)\,E^{T_i}\left[Max\{R(T_i)_{t>0} + R(T_i)_{t\le 0} - K(t),0\}\right] \\
&= D(0,T_i)\,L(T_i)\,E^{T_i}\left[Max\{R(T_i)_{t>0} - K^*(t),0\}\right]
\end{aligned}
$$

$$
where\ R(T_i)_{t\le 0} = \sum_{k=T_{i-s}}^{T_j=0} w_k\, y_{k,k+\tau}(k),\ R(T_i)_{t>0} = \sum_{k=T_j}^{T_i} w_k\, y_{k,k+\tau}(k)\ and\ K^*(t) = K(t) - R(T_i)_{t\le 0}
$$

So these profit sharing options can be priced with a relatively simple and relatively easy to implement Gaussian option formula.

6. Valuation for more complex profit sharing rules

In section 5 an analytical approximation is derived for the case of direct payment of the profit sharing payoff specified in (2.1). However, in practice other variants of this profit sharing exist, such as:

1) Compounding variant of the profit sharing in (2.1)
2) Profit sharing including the return on an additional asset
3) (Compounding) profit sharing with additional management actions or other complex features

For 1) and 2), an analytical approximation can be derived in line with the approximation developed in section 5. For 3), either volatility scaling or Monte Carlo simulation will be necessary. In case of Monte Carlo simulation, the approximation in (5.8) can be used as a control variate, potentially reducing the amount of simulations necessary to a great extent.

6.1 Compounding profit sharing

It is also common that profit sharing is not paid directly, but is compounded and paid out at the end of the contract term. Valuation of this option with Monte Carlo simulation often takes a significant amount of time. The reason for this is the dependency of the profit sharing rates with the future cash flows, resulting in the need to use the original liability cash flow model in a stochastic way. An analytical approximation would significantly (even more than in the direct payment case) reduce computational time, since these formulas can be used as input for the liability cash flow model without the need to run these stochastically.

Let the maturity of the product be T_n and total payoff $L(T_n)$ be of the form:

$$
(6.1)\qquad L(T_n) = L(0)\prod_{i=0}^{n} s(T_i)\left[1 + TR(T_i) + Max\{c\,(R(T_i) - K(T_i)),0\}\right]
$$

where the definition of the variables is as in (2.1) and $s(T_i)$ is the probability that the policyholder stays in the portfolio.

The distribution of the right term of (6.1) is unknown so there is no analytical expression for this payoff. However, if we assume that the $R(T_i)$'s are independent (which is obviously a crude assumption in this case), the expectation of $L(T_n)$ under the T_n-forward measure is:

$$\text{(6.2)} \quad \begin{aligned} E^{T_n}\left[L(T_n)\right] &= E^{T_n}\left[L(0)\prod_{i=0}^{n} s(T_i)\left[1 + TR(t) + Max\{c\,(R(T_i) - K(T_i)),0\}\right]\right] \\ &= L(0)\prod_{i=0}^{n} s(T_i)\left[1 + TR(T_i) + E^{T_n}\left(Max\{c\,(R(T_i) - K(T_i)),0\}\right)\right] \end{aligned}$$

where the latter expectations can be calculated with (5.8), excluding the term $D(0,T_i)\ L(T_i)$.
Note that this expectation has to be determined under the T_n-forward measure by making a timing correction to time T_n using formula (5.3).

The value of the compounding profit sharing option would then be:

$$\text{(6.3)} \quad \begin{aligned} &V\left[PS\right] = D(0,T_n)\left[E^{T_n}\left[L(T_n)\right] - K\right] \\ &where\ K = \prod_{i=1}^{n}\left[1 + TR(T_i)\right] \end{aligned}$$

Despite the crude assumption on independence, the analytical approximation could still work well. When the expected $R(T_i)$'s are low, the impact of the compounding effect is relatively low, resulting in a relatively good approximation of the time value of the option. When the expected $R(T_i)$'s are high, the impact of the compounding effect is relatively high and the quality of the approximation will be less (in terms of time value). However, in this case the total value of the option will also be high and the impact of approximation errors in the time value on the total value will be less. This reasoning is being tested in section 7.

Instead of using this analytical approximation, it is also possible to use Monte Carlo simulation with the analytical approximation of (5.8) as a control variate, reducing the amount of simulations needed significantly. This technique is further described in paragraph 6.3.

6.2 Profit sharing including the return on an additional asset

In some cases the underlying investment portfolio also contains additional non fixed income assets. This means that the profit sharing rate is a combination of a (weighted) moving average of swaprates and the return on additional assets. The profit sharing rate could then be expressed as:

$$\text{(6.4)} \quad R^*(T_i) = \sum_{k=T_{i-s}}^{T_i} w_k^{FI}\, y_{k,k+\tau}(k) + \sum_j w_j^S\, r_{S_j}$$

where w_j^S is the weight in additional asset S_j, r_{S_j} is the return on that asset and
$\sum w_k^{FI} + \sum w_l^S = 1$.

Assume that the additional asset class S_j follows a standard geometric Brownian motion under the risk neutral measure Q:

(6.5) $$dS_j(t) = S_j(t)\left[r(t)\,dt + \sigma_{S_j} dW_{S_j}^Q(t)\right]$$

In this case there is an analytical expression for the distribution of r_{Sj} and the covariance's with $y_{k,k+\tau}$, under normally distributed stochastic interest rates in a T-forward measure. The analytical expression for the distribution of r_{Sj} is worked out in Brigo and Mercurio (2006) for the 1-factor model and the result is similar for multi-factor models. The covariance's with $y_{k,k+\tau}$ can be determined using (5.6) and the formulas in Brigo and Mercurio (2006).

In practice, often r_{Sj} is a book value return. The specification of this book value return can be complex and possibly differs for every insurance company. Often, Monte Carlo simulations are necessary. However, an alternative is the approach described above, where the volatility parameters σ_{Sj} can be calibrated to results of Monte Carlo simulation or derived from historical patterns of book value returns relative to total returns.

6.3 Additional management actions or other complex features

In some cases the insurer has added management actions or other complexities to the profit sharing rules, mainly to lower the risk exposure for the insurer. In most cases, it's not possible to properly value these options analytically. Other possibilities would then be:

a) Use a volatility scaling factor that is calibrated to results obtained with Monte Carlo simulation and use this as input for the analytical approximation in (5.8) and (6.3).
b) Value the option with Monte Carlo simulation, using the analytical approximation in (5.8) as a control variate.

Both possibilities are described below.

a) Volatility scaling factor

When the impact of the management actions or complexities is expected to be low or in cases where it is sufficient to use an approximation, one could use a volatility scaling factor $f(T_i)$, such that:

(6.6) $$\sigma_{R(T_i)}^{Adj} = \left[1 + f(T_i)\right]\sigma_{R(T_i)}$$

The factor $f(T_i)$ can be calibrated for each time T_i to output from Monte Carlo simulation.
This approach can be useful when lots of valuations are needed, for example for Economic Capital or Asset Liability Management calculations.

b) Control Variate technique

When the impact of the management actions or complexities is significant and exact valuation is necessary, Monte Carlo simulation can be used in conjunction with the control variate technique. For a thorough description of the control variate technique, see for example Glasserman (2004). When using the control variate technique, the value of the profit sharing is:

$$(6.7)\quad V[PS] = V[PS]^{sim} - b\left(X^{sim} - E[X]\right)$$

where $V[PS]^{sim}$ is the simulated value of the profit sharing option, X^{sim} is the simulated value of another asset and *E[X]* is the expected value of *X*, which is assumed to be known. When choosing the proper control variate, the standard error of the Monte Carlo estimate can be reduced significantly. This means that significantly less simulations are needed to come to an estimate with the same quality as an ordinary Monte Carlo estimation.

The coefficient b that minimizes the standard error of the Monte Carlo estimation is given by:

$$(6.8)\quad b = \frac{Cov(PS, X)}{Var(X)}$$

The control variate technique is most effective when the correlation between *PS* and *X* is high. Since the management actions or complexities are added to a profit sharing as in (2.1), the correlation between this profit sharing and the direct payment variant of (2.1) is probably very high. Therefore, using the direct payment option of section 5 as a control variate would significantly reduce the number of simulations necessary. This can be implemented by adding the approximate dynamics (A.4) to the simulations to determine X^{sim} and using (5.8) to determine *E[X]*.

An example of the benefits of this technique is the following. In section 7 the quality of the approximation (5.8) is assessed. For testing this quality, the option values coming from (5.8) were in first instance compared with the result of 1.000.000 Monte Carlo simulations. The result from the simulations is seen as the "true" value, since the standard error of the estimation is sufficiently low for this number of simulations. Now when we use the same option (valued under the approximate dynamics) as a control variate and (5.8) as its expected value, only 1.000 simulations are needed to come to the same standard error. Of course in this case the correlation between the option to be valued and the control variate is almost maximal, but one could imagine that in case of more complex options the reduction of the number of simulations needed would still be substantial.

7. Numerical examples

In this section the results of the approximation formulas will be shown for 2 example products and compared with the "true" values resulting from Monte Carlo simulation. When considering the results one has to be aware of the fact that before using the approximation already "errors" are introduced in the valuation, for example in the calibration of the interest rate model to market prices and by using the swap rate as a proxy for the profit sharing rate.

7.1 Example 1: 10-year average of 7-year swap rate, direct payment

This example is a specification of (2.1) and (2.4) with direct payment. This specification is for example commonly applied in pricing the u-rate profit sharing in the Netherlands, where the 7-year swap rate is often used as a proxy for the u-rate. Also, as in (2.4) it can be interpreted as a proxy for profit sharing based on the book value return on a underlying fixed income portfolio with an assumed turnover rate of 10% and a reinvestment strategy favoring 7-year maturity assets (on average). The underlying interest rate model used is a 2-factor Gaussian interest rate model.

The data used for the profit sharing basis and the technical interest rates are based on an example portfolio of a long term pension insurance portfolio, with cash flows up to 50 years ahead. This data is given in appendix C, along with the yield curve, implied volatility matrix and the specific parameter setting of the 2-factor Gaussian interest rate model. A margin of 0.5% is applied and c is assumed to be 1.

The analytical approximation described in section 5 is tested with Monte Carlo simulation, where 5000 (antithetic) simulations are used in combination with the control variate technique described in paragraph 6.3[9]. The results are given in table 1, where the total value of the option is given for both approaches and for different yield curve, volatility, mean reversion and strike sensitivities.

Table 1: comparison analytical / Monte Carlo approach, example 1

Total option value	*Analytical*	*Monte Carlo*	*error*	*%*
Base scenario	103.32	103.19	0.12	0.12%
Interest rates: + 1.5%	207.41	207.20	0.21	0.10%
Interest rates: - 1.5%	37.10	36.88	0.22	0.59%
Volatilities: + 0.15%	131.36	130.96	0.40	0.31%
Volatilities: - 0.15%	76.08	75.99	0.08	0.11%
Mean reversion: + 1.5%	93.16	93.03	0.13	0.14%
Mean reversion: -1.5%	116.26	116.00	0.26	0.22%
Strike: +1%	35.82	36.00	-0.18	-0.49%
Strike: -1%	238.18	237.73	0.46	0.19%

The table shows that the quality of the analytical approximation is excellent for all calculated scenarios. Note that the error as a percentage of the total value of the insurance liabilities would be around 0.01% in most cases. The analytical approximation is potentially more exact than most Monte Carlo methods used in practice, since the number of simulations used is often less than 1.000.000 and this specific control variate technique is not used yet.

7.2 Example 2: 10-year average of 7-year swap rate, compounding option

In this example the value of compounded profit sharing is calculated for a savings product with maturity 20. The compounding profit sharing is of form (6.1), where again the 10-year average of 7-year swap rates is used as the profit sharing rate. The assumed technical interest rate is 3.5%, $s(T_i)$ is assumed to be 1 and a margin of 0.5% is applied. The fund value at the start of the projection is 1.000.

[9] Note, as described in paragraph 6.3, that 1.000 simulations in combination with the described control variate technique leads to a similar standard error as 1.000.000 simulation without the control variate technique.

The analytical approximation described in paragraph 6.1 is tested with Monte Carlo simulation, where 100.000 (antithetic) simulations are used. The results are given in table 2, where again the total value of the option is given for both approaches and for different yield curve, volatility, mean reversion and strike sensitivities.

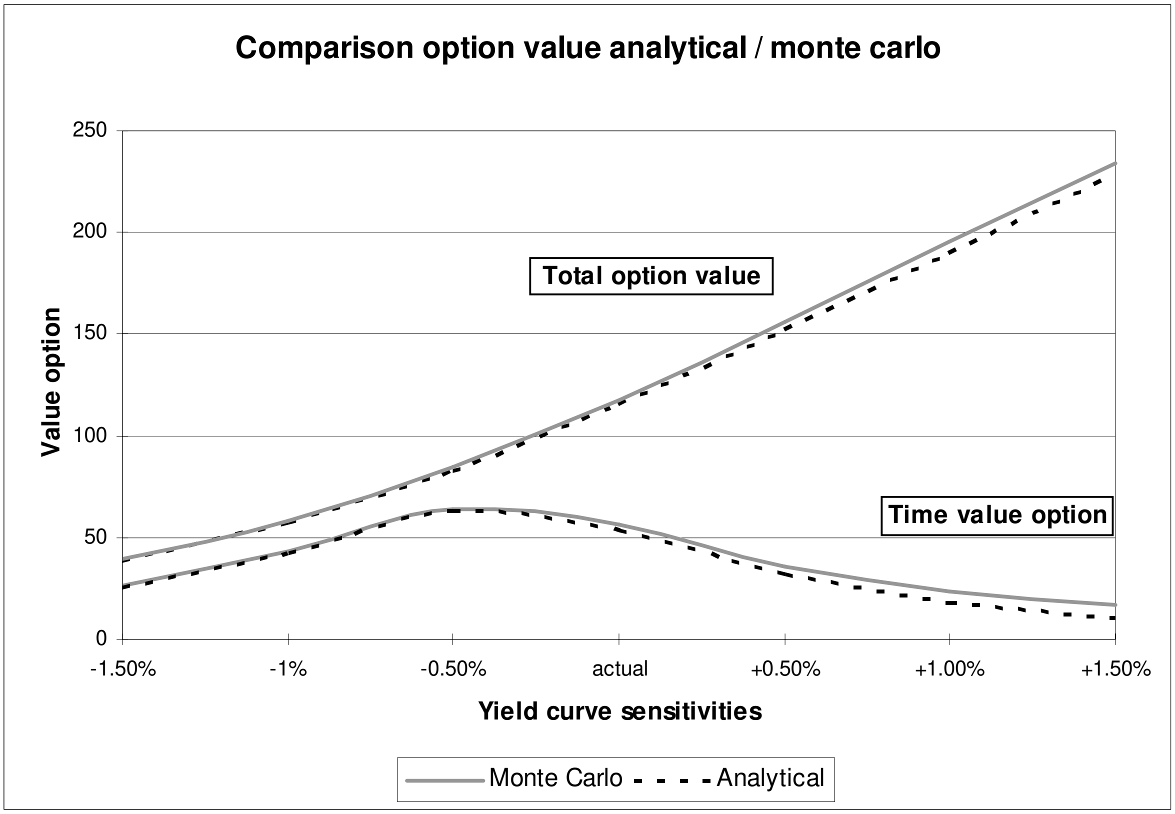
The table shows that the quality of the analytical approximation is reasonable for all calculated scenarios. Note that the error as a percentage of the initial fund value is less than 0.5% in most cases. The assumption of independent profit sharing rates over time introduces an extra error. However, considering the "errors" made earlier in the process (calibration of interest rate model, approximation with swap rate) and the quality of the assumptions usually made for non-economic parameters (mortality, lapses), the error could still be considered as being acceptable.

Table 2: comparison analytical / Monte Carlo approach, example 2

Total option value	***Analytical***	***Monte Carlo***	***error***	***%***
Base scenario	115.37	117.47	-2.10	-1.78%
Interest rates: + 1.5%	228.08	234.12	-6.05	-2.58%
Interest rates: - 1.5%	38.95	39.57	-0.62	-1.56%
Volatilities: + 0.15%	136.24	140.19	-3.96	-2.82%
Volatilities: - 0.15%	94.83	96.19	-1.36	-1.41%
Mean reversion: + 1.5%	109.01	110.83	-1.82	-1.65%
Mean reversion: -1.5%	122.81	125.80	-2.99	-2.38%
Strike: +1%	32.03	32.21	-0.17	-0.54%
Strike: -1%	276.14	282.00	-5.86	-2.08%

As mentioned in paragraph 6.1 the quality of the approximation (in terms of time value of the option) is less when the impact of the compounding is relatively high. However, since the total value of the option is higher in this case, the error will still be reasonable in terms of the total value of the option (as shown in the table above). This effect is also shown in figure 1, where the results of the analytical and the Monte Carlo approach are given for different yield curve sensitivities.

Figure 1: comparison analytical / Monte Carlo approach, example 2



8. Conclusions

In this paper analytical approximations are derived for prices of swap rate dependent embedded options in insurance products. In practice these options are often valued using Monte Carlo simulations. However, for risk management calculations and reporting processes, lots of valuations are needed and therefore a more efficient method to value these options would be helpful. The basis for the approximations is the result of Schrager and Pelsser (2006), who derived an approximate distribution for the forward swap rates under the relevant swap measure. After some changes of measure, this result is used to derive analytical approximations for swap rate dependent embedded options, given an underlying multifactor Gaussian interest rate model.

The analytical approximation for options with direct payment is almost exact while the approximation for compounding options is also satisfactory. For similar options with additional management actions that significantly impact the option value, no analytical approximation is possible. However, using the analytical approximation for an option with direct payment as a control variate, the number of Monte Carlo simulations can be reduced significantly for these kinds of options. Furthermore, it's also possible to construct analytical approximations when returns on additional assets (such as equities) are part of the profit sharing rate.

Appendix A: proof of (4.3)

Each element of the vector of derivatives of (4.2) can be written as:

(A.1)
$$\frac{\partial y_{n,N}(t)}{\partial Y^{(i)}(t)} = -B^{(i)}(t,T_n)\,D^P(t,T_n) + B^{(i)}(t,T_N)\,D^P(t,T_N) + y_{n,N}(t)\sum_{k=n+1}^{N}\Delta_{k-1}^{Y}\,B^{(i)}(t,T_k)\,D^P(t,T_k)$$

where $D^P(t,T_n) = D(t,T_n) / P_{n+1,N}(t)$, the bond price normalized by the numéraire.

Note that since bond prices in this model are stochastic, the volatility of the swap rate is stochastic as well. The approximation of Schrager and Pelsser consists of replacing the stochastic terms $D^P(t,T_i)$ by their time zero values $D^P(0,T_i)$. This results in:

(A.2)
$$\frac{\partial y_{n,N}(t)}{\partial Y^{(i)}(t)} \approx -B^{(i)}(t,T_n)\,D^P(0,T_n) + B^{(i)}(t,T_N)\,D^P(0,T_N) + y_{n,N}(0)\sum_{k=n+1}^{N}\Delta_{k-1}^{Y}\,B^{(i)}(t,T_k)\,D^P(0,T_k) = \overline{\overline{\frac{\partial y_{n,N}(t)}{\partial Y^{(i)}(t)}}}$$

This approximation makes the swap rate volatility deterministic and thus leads to a normally distributed forward swap rate. Furthermore, we can rewrite

(A.3)
$$B^{(i)}(t,T) = \frac{1}{A_{(ii)}} - \frac{e^{-A_{(ii)}T}}{A_{(ii)}}\,e^{A_{(ii)}t}$$

Using this, (A.2) can be split in a time dependent part and a constant part:

(A.4)
$$\overline{\overline{\frac{\partial y_{n,N}(t)}{\partial Y^{(i)}(t)}}} = \frac{1}{A_{(ii)}}\,e^{A_{(ii)}t}\left[e^{-A_{(ii)}T_n}\,D^P(0,T_n) - e^{-A_{(ii)}T_N}\,D^P(0,T_N) - y_{n,N}(0)\sum_{k=n+1}^{N}\Delta_{k-1}^{Y}\,e^{-A_{(ii)}T_k}\,D^P(0,T_k)\right] = e^{A_{(ii)}t}\,\tilde{C}_{n,N}^{(i)}$$

So in the approximate model, the swap rate at time T_n is given by:

(A.5)
$$\int_0^{T_n} dy_{n,N}(s) = \int_0^{T_n} \frac{\partial y_{n,N}(t)}{\partial Y(t)} \Sigma\, dW^{n+1,N}(t) \approx \int_0^{T_n} \overline{\overline{\frac{\partial y_{n,N}(t)}{\partial Y(t)}}} \Sigma\, dW^{n+1,N}(t)$$
$$= \int_0^{T_n} e^{As'}\, diag(\tilde{C}_{n,N})\, \Sigma\, dW^{n+1,N}(t)$$

$$where\ e^{As} = \begin{bmatrix} e^{A_{(11)}s} \\ \vdots \\ e^{A_{(mm)}s} \end{bmatrix} and\ diag(\tilde{C}_{n,N}) = \begin{bmatrix} \tilde{C}^{(1)}_{n,N} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \tilde{C}^{(m)}_{n,N} \end{bmatrix}$$

By using Ito's Isometry, this leads to an analytical expression for the integrated variance of $y_{n,N}$ (associated with a T_n x T_N swaption) over the interval $[0,T_n]$:

(A.6)
$$\sigma^2_{n,N} \approx \int_0^{T_n} e^{As'}\, diag(\tilde{C}_{n,N}) \Sigma \Sigma' diag(\tilde{C}_{n,N}) e^{As}\, ds$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \hat{\Sigma}_{(ij)}\, \tilde{C}^{(i)}_{n,N} \tilde{C}^{(j)}_{n,N} \left[\frac{e^{\left[A_{(ii)} + A_{(jj)}\right]T_n} - 1}{A_{(ii)} + A_{(jj)}} \right]$$

Appendix B: proofs of (5.2) and (5.3)

Proof of (5.2)

A change of measure has to be done from the swap measure $Q^{n+1,N}$ to the T_n-forward measure Q^{Tn}. In this case the Radon-Nikodym derivative is:

(B.1)
$$\frac{dQ^{T_n}}{dQ^{n+1,N}} = \rho(t) = \frac{D(t,T_n) / D(0,T_n)}{\sum_{k=n+1}^{N} \Delta^Y_{k-1} D(t,T_k) \Big/ \sum_{k=n+1}^{N} \Delta^Y_{k-1} D(0,T_k)}$$

Then using Ito's Lemma leads to:

(B.2)
$$d\rho(t) = \kappa(t)\, \rho(t)\, dW^{T_n}$$

where $\kappa(t)$ is an 1 x m vector with for each element $\kappa^{(i)}(t)$:

(B.4)
$$\kappa^{(i)}(t) = -B^{(i)}(t,T_n) + \sum_{k=n+1}^{N} \Delta^Y_{k-1} B^{(i)}(t,T_k) D^P(t,T_k)$$

Now like in appendix A replacing the stochastic terms $D^P(t,T_i)$ by their time zero values $D^P(0,T_i)$ and using (A.3) results in:

(B.5) $$\kappa^{(i)}(t) \approx \frac{1}{A_{(ii)}} e^{A_{(ii)}t} \left[e^{-A_{(ii)}T_n} + \sum_{k=n+1}^{N} \Delta_{k-1}^{Y} e^{-A_{(ii)}T_k} D^P(0,T_k) \right] = e^{A_{(ii)}t} \widetilde{G}_{n,N}^{(i)}$$

Using (A.4) and integrating $dy_{n,N}$ leads to the following formula for the convexity correction $CC_{n,N}(T_n)$ for time $T_n > 0$ for the swap rate $y_{n,N}$:

(B.6) $$\begin{aligned} CC_{n,N}(T_n) &\approx \int_0^{T_n} e^{As'} \, diag(\widetilde{C}_{n,N}) \Sigma \Sigma' diag(\widetilde{G}_{n,N}) e^{As} \, ds \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} \hat{\Sigma}_{(ij)} \widetilde{C}_{n,N}^{(i)} \widetilde{G}_{n,N}^{(j)} \left[\frac{e^{[A_{(ii)} + A_{(jj)}]T_n} - 1}{A_{(ii)} + A_{(jj)}} \right] \end{aligned}$$

$$\text{where } \widetilde{G}_{n,N}^{(j)} = \frac{1}{A_{(jj)}} \left[e^{-A_{(jj)}T_n} - \sum_{k=n+1}^{N} \Delta_{k-1}^{Y} e^{-A_{(jj)}T_k} D^P(0,T_k) \right]$$

Proof of (5.3)

In this case the Radon-Nikodym derivative is:

(B.7) $$\begin{aligned} \frac{dQ^{T_{n+u}}}{dQ^{T_n}} &= \rho(t) = \frac{D(t,T_{n+u}) / D(0,T_{n+u})}{D(t,T_n) / D(0,T_n)} \\ &= \frac{D(0,T_n)}{D(0,T_{n+u})} \exp\left[A(t,T_{n+u}) - A(t,T_n) - \left(\sum_{i=1}^{m} B^{(i)}(t,T_{n+u}) - \sum_{i=1}^{m} B^{(i)}(t,T_n) \right) Y^{(i)}(t) \right] \end{aligned}$$

Then using the same procedure as above:

(B.8) $$\kappa^{(i)}(t) = B^{(i)}(t,T_n) - B^{(i)}(t,T_{n+u}) = \frac{1}{A_{(ii)}} e^{A_{(ii)}t} \left[e^{-A_{(ii)}T_{n+u}} - e^{-A_{(ii)}T_n} \right] = e^{A_{(ii)}t} \widetilde{H}_{n,n+u}^{(i)}$$

Using (A.4) and integrating $dy_{n,N}$ leads the following formula for the timing correction *TCn,N*(T_n,T_{n+u}) representing a change of measure from time $T_n > 0$ to T_{n+u}:

(B.9) $$\begin{aligned} TC_{n,N}(T_n,T_{n+u}) &\approx \int_0^{T_n} e^{As'} \, diag(\widetilde{C}_{n,N}) \Sigma \Sigma' diag(\widetilde{H}_{T_n,T_{n+u}}) e^{As} \, ds \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} \hat{\Sigma}_{(ij)} \widetilde{C}_{n,N}^{(i)} \widetilde{H}_{T_n,T_{n+u}}^{(j)} \left[\frac{e^{[A_{(ii)} + A_{(jj)}]T_n} - 1}{A_{(ii)} + A_{(jj)}} \right] \end{aligned}$$

$$\text{where } \widetilde{H}_{T_n,T_{n+u}}^{(j)} = \frac{1}{A_{(jj)}} \left[e^{-A_{(jj)}(T_{n+u})} - e^{-A_{(jj)}T_n} \right]$$

Appendix C: input example 1

In this appendix the data and assumptions are given that are used for example 1. The data used for the profit sharing basis *L(t)* and the technical interest rates *TR(t)* are based on an example portfolio of a long term pension insurance portfolio and are given in table 3.

Table 3: used data for profit sharing basis and technical interest rate

Time	TR(t)	L(t)	Time	TR(t)	L(t)
0	3.8%	1,000	25	3.6%	655
1	3.7%	1,043	26	3.6%	625
2	3.7%	1,066	27	3.5%	594
3	3.7%	1,060	28	3.5%	563
4	3.7%	1,054	29	3.5%	532
5	3.7%	1,046	30	3.5%	501
6	3.7%	1,038	31	3.5%	470
7	3.7%	1,028	32	3.5%	440
8	3.7%	1,016	33	3.5%	410
9	3.7%	1,004	34	3.5%	381
10	3.7%	991	35	3.5%	353
11	3.7%	976	36	3.5%	326
12	3.7%	961	37	3.5%	300
13	3.7%	944	38	3.5%	275
14	3.6%	926	39	3.5%	251
15	3.6%	907	40	3.4%	228
16	3.6%	887	41	3.4%	206
17	3.6%	865	42	3.4%	186
18	3.6%	842	43	3.4%	167
19	3.6%	819	44	3.4%	149
20	3.6%	794	45	3.4%	132
21	3.6%	768	46	3.4%	116
22	3.6%	741	47	3.4%	102
23	3.6%	713	48	3.4%	89
24	3.6%	684	49	3.4%	77

The swap curve used is from ultimo 2006 and the parameters of the 2 factor Gaussian interest rate model are calibrated to the swaption implied volatility surface at the same date. This information is given in table 4 (where σ and a belong to factor 1 and η and b to factor 2).

Table 4: swap curve, implied volatility surface and parameters 2F Gaussian model

Swaption ATMF Volatility Surface											
Expiry/Tenor	**1Y**	**2Y**	**3Y**	**4Y**	**5Y**	**7Y**	**10Y**	**15Y**	**20Y**	**25Y**	**30Y**
1Y	14.3%	14.0%	14.4%	14.5%	14.5%	14.3%	14.0%	13.5%	13.0%	13.0%	12.9%
2Y	14.6%	14.9%	15.0%	14.9%	14.8%	14.6%	14.2%	13.6%	13.2%	13.0%	12.9%
3Y	15.1%	15.0%	15.1%	15.0%	14.8%	14.5%	14.1%	13.6%	13.2%	13.1%	12.9%
4Y	15.1%	15.0%	15.0%	14.8%	14.6%	14.2%	13.9%	13.4%	13.1%	13.0%	12.8%
5Y	14.8%	14.8%	14.7%	14.5%	14.3%	14.0%	13.6%	13.2%	12.9%	12.8%	12.6%
7Y	14.0%	14.0%	14.0%	13.9%	13.6%	13.4%	13.1%	12.9%	12.4%	12.4%	12.2%
10Y	13.0%	13.1%	13.1%	13.0%	12.8%	12.7%	12.5%	12.1%	11.8%	11.6%	11.4%
15Y	12.0%	12.0%	12.0%	12.0%	12.0%	12.0%	12.0%	11.6%	11.2%	11.0%	10.9%
20Y	11.6%	11.6%	11.6%	11.7%	11.8%	11.8%	11.8%	11.2%	10.8%	10.5%	10.5%
30Y	11.1%	11.1%	11.2%	11.3%	11.3%	11.3%	11.3%	10.7%	10.6%	10.7%	10.7%

Time	**Swap Rate**
1	4.08%
2	4.14%
3	4.12%
4	4.12%
5	4.13%
6	4.14%
7	4.15%
8	4.16%
9	4.18%
10	4.20%
15	4.28%
20	4.31%
30	4.29%
40	4.25%
50	4.20%

Parameters	
σ	0.51%
a	2.75%
η	0.28%
b	2.75%
rho	0.497

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