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APPROXIMATE SOLUTIONS FOR INDIFFERENCE PRICING UNDER GENERAL UTILITY FUNCTIONS

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Abstract. With the aid of Taylor-based approximations, this paper presents results for pricing insurance contracts by using indifference pricing under general utility functions. We discuss the connection between the resulting "theoretical" indifference prices and the pricing rule-of-thumb that practitioners use: Best Estimate plus a "Market Value Margin". Furthermore, we compare our approximations with known analytical results for exponential and power utility.

Keywords: *Indifference pricing, nontradable insurance risk, Taylor approximation, general utility*

Introduction

Due to the untradable insurance risk, pricing of life insurance contracts take place in an incomplete market setup. In such markets, there exist a series of equivalent martingale measures and generally no unique price can be achieve by arbitrage theory. An alternative to the arbitrage theory for pricing the contingent claims in the incomplete market is utility-based approach (c.f. Hodges and Neuberger (1989)). Henderson and Hobson (2004) provide an overview of utility indifference pricing. In the problem of pricing contingent claims in incomplete markets, this approach takes account of the fact that the investors's attitude towards those unhedgable risks. For instance, Hodges' seller's price is the price which leaves an economic agent indifferent between the optimal utility he obtains from selling a certain contingent claim and

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investing his money in a self-financing portfolio and that he obtains from investing his money in a self-financing portfolio. The solution to these optimal terminal wealth problems is well known (See e.g. Karatzas et al. (1987) and Cox and Huang (1989)) and it can be expressed as the inverse function of the first derivative of the utility. In most of the existing literature, the analysis is carried out under either constant absolute risk aversion (exponential utility) or constant relative risk aversion (power utility). For the exponential utility, explicit solution can be achieved for the utility maximization problem, whereas for the power utility, there is no closed-form solution because the resulting partial differential equation (PDE) is highly nonlinear and it can be solved numerically.

In the present paper, we start with an insurance company which issues a fairly popular type of life insurance contracts, unit-linked types of contracts. It can be a with-profit contract as introduced in Bacinello (2001) or a French participating contract in Briys and de Varenne (1994) as well as an equity-linked product etc. The payoffs of these unit-linked types of contracts are contingent not only on the untradable insurance risk but on the evolution of some tradable asset(s). In other words, in contrast to the previous literature[1] which consider payoffs like $g(y_T)$, i.e. the payoff depends on the untradable uncertainty y_T only, our payoff functions are generalized to $g(S_T, y_T)$, where S_T denotes the final value of the tradable asset S. Under *general utility functions*, we determine the Hodges's seller's price (premiums) for the issued liabilities. It is important to generalize the utility class because many unit-linked types of insurance contracts sold by insurance companies cannot be priced using exponential or power utility. However, there are usually no explicit closed-form solution in the indifference pricing theory when a general utility function comes into consideration. In other words, without approximation we have to solve the problem numerically, then it becomes much more difficult to interpret the results.

Therefore, in the present work, we are interested in exploring approximations of the indifference price for more general utility functions via Taylor-series approximation. To this end, we shall mention the papers of Henderson and Hobson (2002) and Henderson (2002) where they approximate the *power* indifference pricing with respect to the number of the contingent claims. The former paper deals with claims which

[1]C.f. for instance Musiela and Zariphopoulou (2001), Henderson and Hobson (2002) and Henderson (2002).

are units of the non-traded asset, and the latter considers more general European claims. In comparison with them, our analysis is carried out not only for power but more general utilities. Further, our approximation is not developed around the number of the contingent claims. Simple asymptotical results can be obtained when an approximation is done around the number (when the number approaches 0), i.e. the indifference price reduces to the expected value under the minimal martingale measure (c.f. Davis (2004) for the convergence result). However, for life insurance liabilities which deal with a large portfolio problem, this convergence result around the number becomes not highly relevant.

We are not the first that uses utility indifference in an insurance context. Møller (2003a, 2003b) determine fair premiums and optimal strategies under financial variance and standard deviation principles for some insurance contracts with financial risk. These principles can be derived via a utility indifference argument. Our analysis differs from his by considering more general utility functions and we are more interested in developing approximate solutions.

Although our model is set up to find an approximate solution to pricing insurance contracts, our results should be suitable for specific utility functions and regular non-traded contingent claims $g(y_T)$. Therefore, in order to verify our results and examine the goodness of the approximation, we apply our results to price contingent claims whose payoffs depend on the nontradable risk only, for both the exponential and power utility function. More specifically, we compare our results with some existing results e.g Musiela and Zariphopoulou (2001) and Henderson (2002). Our approximate prices coincide with the results obtained in both literature.

The remainder of the paper is structured as follows: In section 1, we derive the dual formulation of an indifference pricing problem and section 2 focuses on Taylor-series approximations to achieve approximations of the indifference price for general utility functions. Section 3 demonstrates the application of our pricing approach in exponential and power utility in order to examine our approximate results. In the subsequent Section 4, we investigate the impact of the unhedgeable risk on the optimal wealth and strategy. Section 5 concludes the paper.

1. Derivation of dual formulation

Using the dual formulation approach of Rogers (2001), we will derive the dual formulation of an indifference pricing problem. The indifference pricing problem can be formulated as follows as an incomplete markets optimal utility problem:

$$\begin{aligned} \max_{\theta} \quad & \mathbb{E}[U(X_T - g(S_T, y_T))] \\ s.t. \quad & dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_1(t) \\ & dy_t = a(t,\omega)dt + b(t,\omega)(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)). \end{aligned} \tag{1.1}$$

X_T denotes the wealth at time T, S denotes the traded asset (that follows Black-Scholes dynamics where μ is the drift rate and σ the volatility), y is the non-traded (insurance) process and both define an insurance claim $g(S_T, y_T)$. The variable θ denotes the optimal investment strategy that leads to the wealth X_T at time T.

The investor can only trade in the stock S, hence the wealth process only depends on the Brownian Motion W_1 and the wealth dynamics are given by

$$dX_t = (rX + \theta(\mu - r))dt + \theta\,\sigma\, dW_1(t), \tag{1.2}$$

with r denoting the deterministic interest rate. In Musiela and Zariphopoulou (2001) the problem described above is solved analytically via the route of an HJB problem for the special case of exponential utility and the insurance claim being a function $g(y_T)$ only. We are interested in solving the problem via the dual formulation, i.e. by introducing a Lagrange multiplier process Λ that forces the final wealth X_T to be a solution of (1.2). Along the lines of Rogers (2001), we can find the dual formulation as follows. Let us consider the positive process:

$$d\Lambda_t = \Lambda_t(\alpha(t,\omega) + \beta_1(t,\omega)dW_1(t) + \beta_2(t,\omega)dW_2(t)), \tag{1.3}$$

where α, β_1, β_2 will be determined later, and all these parameters are not necessarily deterministic. Using the derivation in Rogers (2001), we can express the dynamic optimization problem (1.1) as a static Lagrangian optimization problem:

$$\begin{aligned} L(\Lambda) \;=\; \max_{X,\theta} \mathbb{E}\Bigg[& U(X_T - g(S_T, y_T)) - \Lambda_T X_T + \Lambda_0 X_0 - \int_0^T \Lambda_T \Big((\alpha + r) X_t \\ & + \theta\sigma\beta_1(t,\omega) + \theta(\mu - r) \Big) dt \Bigg]. \end{aligned} \tag{1.4}$$

Note that in this static Lagrangian formulation the dynamics of y do not explicitly enter and also the Lagrange volatility parameter β_2 is not explicitly present. There are however still available in the "background" and will later serve to determine the optimal solution for the Lagrange function L. Let us begin with solving the "inner maximization" of the Lagrangian (1.4). With slight abuse of notation we can derive the following first order condition:

$$\frac{\partial L}{\partial X_T} = \mathbb{E}[U'(X_T - g(S_T, y_T)) - \Lambda_T] = 0 \quad \Rightarrow \; X_T^* = g(S_T, y_T) + I(\Lambda_T). \tag{1.5}$$

This is the "Cox-Huang" (c.f Cox and Huang (1989) and Karatzas et al. (1987)) condition for the optimal wealth X_T^* (including the non-hedgeable claim $g(S_T, y_T)$), where $I(.)$ denotes the inverse function of $U'(.)$.

Furthermore we find:

$$\frac{\partial L}{\partial X_t} = \mathbb{E}\left[-\int_0^T \Lambda_T(\alpha + r)dt\right] = 0 \quad \Rightarrow \; \alpha(t,\omega) = -r, \tag{1.6}$$

$$\frac{\partial L}{\partial \theta} = \mathbb{E}\left[-\int_0^T \Lambda_T(\sigma\,\beta_1(t,\omega) + (\mu - r))dt\right] = 0 \quad \Rightarrow \; \beta_1(t,\omega) = -\frac{\mu - r}{\sigma}. \tag{1.7}$$

These two results are also very nicely in line with the "Cox-Huang" since they imply that the Lagrange multiplier Λ is actually a pricing kernel (or deflator) that prices all assets driven by the Brownian Motion W_1. If fact, the prices obtained in this way are fully consistent with the arbitrage-free prices in the Black-Scholes economy. Note also that when $g(S_T, y_T)$ does not depend on y, then we have a complete market pricing problem which can be solved with the Cox-Huang formalism.

If we substitute the results found in (1.5)-(1.7) back into the Lagrangian (1.4), we now obtain the reduced Lagrangian L^*:

$$\begin{aligned} L^*(\Lambda) &= \mathbb{E}\left[U(I(\Lambda_T)) - \Lambda_T(I(\Lambda_T) + g(S_T, y_T)) + \Lambda_0 X_0\right] \\ &= \mathbb{E}\left[\tilde{U}(\Lambda_T) - \Lambda_T g(S_T, y_T) + \Lambda_0 X_0\right], \end{aligned} \tag{1.8}$$

where the function $\tilde{U}$ denotes the convex dual of the utility function $U(.)$. This is a well-known result and is also derived e.g. in Henderson and Hobson (2004).

As noted above, the Lagrange function Λ has not been fully specified yet. Hence, for each choice of Λ, the function $L^*(\Lambda)$ will give an upper bound for the maximization

problem (1.1). The tightest possible upper bound will be given by minimizing the function $L^*(\Lambda)$ for all Λ (and in particular by choosing the remaining parameter Λ_0 and process β_2).

The dual formulation of (1.1) therefore ought to be

$$\min_{\Lambda_0} \left[\tilde{U}(\Lambda_T) - \Lambda_T g(S_T, y_T) + \Lambda_0 X_0 \right]. \tag{1.9}$$

To begin with, from (1.3) we can explicitly represent the Lagrange multiplier Λ as

$$\begin{aligned} \Lambda_T &= \Lambda_0 \exp\{-rT\} M_1^{0,T} M_2^{0,T} \\ \text{with} \quad M_1^{t,T} &= \exp\left\{ -\int_t^T \frac{\mu - r}{\sigma} dW_1(s) - \frac{1}{2} \int_t^T \frac{(\mu - r)^2}{\sigma^2} ds \right\} \\ M_2^{t,T} &= \exp\left\{ \int_t^T \beta_2(s,\omega) dW_2(s) - \int_t^T \frac{1}{2} (\beta_2(s,\omega))^2 ds \right\}. \end{aligned} \tag{1.10}$$

M_1 and M_2 are change-of-measure exponential martingales which act on the Brownian Motions W_1 and W_2 respectively. Let us consider the minimization of L^* with respect to Λ_0. The first order condition is given by

$$\frac{dL^*}{d\Lambda_0} = \mathbb{E}[\tilde{U}'(\Lambda_T) e^{-rT} M_1^{0,T} M_2^{0,T} - e^{-rT} M_1^{0,T} M_2^{0,T} g(S_T, y_T) + X_0] = 0 \tag{1.11}$$

$$\Rightarrow X_0 = e^{-rT} \mathbb{E}^{**}[I(\Lambda_T) + g(S_T, y_T)]. \tag{1.12}$$

We have used the fact that $\tilde{U}'(\Lambda_T) = -I(\Lambda_T)$. In addition, $\mathbb{E}^{**}$ denotes the expectation with respect to the measure $\mathbb{P}^{**}$ which is induced by $M_1^{0,T} M_2^{0,T}$. This result indicates that Λ_0 is determined in order to make (1.12) binding.

Not surprisingly, we are also interested in the evolution of the optimal wealth at any $t \in (0, T)$. First, we reformulate Λ_T as a function of Λ_t, i.e. $\Lambda_T = \Lambda_t e^{-r(T-t)} M_1^{t,T} M_2^{t,T}$. According to the law of iterated expected value, (1.9) can be rewritten as

$$\begin{aligned} & \min_{\Lambda_0} \mathbb{E}\left[\mathbb{E}_t \left[\tilde{U}(\Lambda_T) - \Lambda_T g(S_T, y_T) + \Lambda_0 X_0 \right] \right] \\ = & \min_{\Lambda_0} \mathbb{E}\left[\mathbb{E}_t \left[\tilde{U}(\Lambda_T) - \Lambda_T g(S_T, y_T) + \Lambda_t X_t - \int_0^t (\Lambda_s dX_s + X_s d\Lambda_s + dX_s d\Lambda_s) \right] \right] \end{aligned}$$

Throughout the paper, we use $\mathbb{E}_t[x] := \mathbb{E}[x|\mathcal{F}_t]$ to denote the expected value conditional on the information structure $\mathcal{F}_t$ and $\mathbb{E}[x] := \mathbb{E}_0[x]$. Taking the first order

condition of L^* with respect to Λ_0 leads to

$$\frac{dL^*}{d\Lambda_0} = \frac{dL^*}{d\Lambda_t}\frac{d\Lambda_t}{d\Lambda_0} = \mathbb{E}\left[\mathbb{E}_t\left[\left((\tilde{U}'(\Lambda_T) - g(S_T, y_T))\frac{d\Lambda_T}{d\Lambda_t} + X_t\right)\frac{d\Lambda_t}{d\Lambda_0}\right]\right] = 0.$$

Since $d\Lambda_T/d\Lambda_t = e^{-r(T-t)}M_1^{t,T}M_2^{t,T}$, we finally obtain

$$X_t = e^{-r(T-t)}\mathbb{E}_t^{**}[I(\Lambda_T) + g(S_T, y_T)]. \tag{1.13}$$

The optimal wealth at time t is given by the conditional expected discounted of optimal final wealth X_T^* under the measure $\mathbb{P}^{**}$.

1.1. **Determining $\beta_2(t,\omega)$ by solving HJB in dual form.** In this subsection, we use HJB approach to solve the dual problem directly in order to obtain the optimal $\beta_2(.,\omega)$, i.e. we are dealing with the following optimization problem:

$$\begin{aligned}
\min_{\beta_2} \quad & \mathbb{E}\left[\tilde{U}(\Lambda_T) - \Lambda_T g(S_T, y_T)\right] + \Lambda_0 X_0 \\
s.t.\ d\Lambda_t &= \Lambda_t\left[-r\,dt - \frac{\mu - r}{\sigma}dW_1(t) + \beta_2(t,\omega)dW_2(t)\right] \\
dS_t &= \mu\, S_t\, dt + \sigma\, S_t\, dW_1(t) \\
dy_t &= a(t,\omega)dt + b(t,\omega)(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)).
\end{aligned} \tag{1.14}$$

We can now define the indirect dual utility

$$f(t,\Lambda,S,y) = \mathbb{E}\left[\tilde{U}(\Lambda_T) - \Lambda_T g(S_T, y_T)|t,\Lambda_t,S_t,y_t\right] := \mathbb{E}_t\left[\tilde{U}(\Lambda_T) - \Lambda_T g(S_T, y_T)\right]. \tag{1.15}$$

Please note that we have ignored the constant $\Lambda_0 X_0$ and will revisit this in Section 1.2 when we discuss the indifference price. The indirect dual utility $f(t,\Lambda,S,y)$ follows the PDE

$$\begin{aligned}
& f_t + (-r)\Lambda_t f_\Lambda + a(t,\omega)f_y + \mu\, S_t\, f_S + \left(\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 + \frac{1}{2}(\beta_2(t,\Lambda,S,y))^2\right)\Lambda_t^2 f_{\Lambda\Lambda} \\
& + \frac{1}{2}(b(t,\omega))^2 f_{yy} + \frac{1}{2}\sigma^2 S_t^2 f_{SS} + b(t,\omega)\left(\rho\,\frac{\mu - r}{\sigma} + \sqrt{1-\rho^2}\beta_2(t,\Lambda,S,y)\right)\Lambda_t f_{\Lambda\, y} \\
& + b(t,\omega)\,\rho\,\mu\, S_t\, f_{y\,S} - (\mu - r)S_t\,\Lambda_t\, f_{\Lambda\, S} = 0,
\end{aligned} \tag{1.16}$$

where $f_v := \frac{\partial f}{\partial v}$; $f_{uv} := \frac{\partial f^2}{\partial u\,\partial v}$. Based on the assumption that we will follow the optimal policy for $t < s \le T$, the optimal choice of $\beta_2(t,\omega)$ at time t is given by maximizing (1.16) over $\beta_2(t,\omega)$. This leads to:

$$\beta_2^*(t,\omega) = \beta_2^*(t,\Lambda_t,S,y) = -\frac{\sqrt{1-\rho^2}\, b(t,\omega)\, f_{\Lambda\, y}}{\Lambda_t f_{\Lambda\Lambda}}. \tag{1.17}$$

Substituting this optimal value back to (1.16) results in:

$$\begin{aligned} & f_t + (-r)\Lambda_t f_\Lambda + a(t,\omega) f_y + \mu\, S_t\, f_S + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \Lambda_t^2 f_{\Lambda\Lambda} + \frac{1}{2}(b(t,\omega))^2 f_{yy} \\ & + \frac{1}{2}\sigma^2 S_t^2 f_{SS} + b(t,\omega)\,\rho\,\frac{\mu - r}{\sigma}\Lambda_t f_{\Lambda\, y} + b(t,\omega)\,\rho\,\mu\, S_t\, f_{y\, S} - (\mu - r) S_t\, \Lambda_t\, f_{\Lambda\, S} \\ = \;& \frac{1}{2}\frac{(1-\rho^2)(b(t,\omega))^2 (f_{\Lambda y})^2}{f_{\Lambda\Lambda}} = \frac{1}{2} f_{\Lambda\Lambda}\Lambda_t^2 (\beta_2^*(t,\Lambda_t,S,y))^2. \end{aligned} \tag{1.18}$$

This is a nonlinear PDE which is difficult to solve. For the case of exponential utility, it can be solved with similar technique as Musiela and Zariphopoulou (2001).

Now one possibility to approximate this nonlinear PDE for f is as follows. As the first step, we neglect the righthand side of (1.18), which removes the nonlinear term. In other words, we set $\beta_2 = 0$ first and then Λ_T is reduced to $\Lambda_T^{(0)} = \Lambda_0^{(0)} \exp\{-rT\} M_1^{0,T} = \Lambda_t^{(0)} \exp\{-r(T-t)\} M_1^{t,T}$, which corresponds to the state price deflator under the minimal martingale measure. Let $f_t^{(0)}$ denote the solution to the linear PDE with the righthand side of (1.18) equals zero. Feynman-Kac formula which establishes a link between PDEs and stochastic processes tells hat the solution can be written as an expectation:

$$f^{(0)}(t,\Lambda,S,y) = \mathbb{E}_t\left[\tilde{U}(\Lambda_T^{(0)}) - \Lambda_T^{(0)} g(S_T, y_T)\right].$$

According to the expression in (1.17), we obtain:

$$\beta_2^{(0)}(t,\omega) = \beta_2^{(0)}(t,\Lambda_t^{(0)},S,y) = -\frac{\sqrt{1-\rho^2}\, b(t,\omega)\, f_{\Lambda\, y}^{(0)}}{\Lambda_t^{(0)} f_{\Lambda\Lambda}^{(0)}}. \tag{1.19}$$

$\beta_2^{(0)}$ is the approximate version of (1.17) and used later to determine the approximate indifference price. Plugging $f^{(0)}$ in the righthand side of (1.18) leads to

$$\begin{aligned} & f_t^{(1)} + (-r)\Lambda_t f_\Lambda^{(1)} + a(t,\omega) f_y^{(1)} + \mu\, S_t\, f_S^{(1)} + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 (\Lambda_t)^2 f_{\Lambda\Lambda}^{(1)} \\ & + \frac{1}{2}(b(t,\omega))^2 f_{yy}^{(1)} + \frac{1}{2}\sigma^2 S^2 f_{SS}^{(1)} + b(t,\omega)\,\rho\,\frac{\mu - r}{\sigma}\Lambda_t f_{\Lambda\, y}^{(1)} + b(t,\omega)\,\rho\,\mu\, S_t\, f_{y\, S}^{(1)} \\ & -(\mu - r) S_t\, \Lambda_t\, f_{\Lambda\, S}^{(1)} = \frac{1}{2} f_{\Lambda\Lambda}^{(0)} (\Lambda_t)^2 (\beta_2^{(0)}(t,\Lambda_t^{(0)},S,y))^2. \end{aligned} \tag{1.20}$$

This PDE is still solvable in closed form and an extended Feynman-Kac formula provides the solution to $f^{(1)}(t,\Lambda_t,S,y)$:

$$f^{(1)}(t,\Lambda_t,S,y) = f^{(0)}(t,\Lambda_t,S,y) + \mathbb{E}_t\left[\int_t^T \frac{1}{2} f^{(0)}_{\Lambda\Lambda}(\Lambda_s)^2 (\beta_2^{(0)}(s,\Lambda_s^{(0)},S,y))^2 d\,s\right]. \quad (1.21)$$

$f^{(1)}(t,\Lambda_t,S,y)$ is the approximation we propose. In principle, it is possible to continue with $f^{(2)}$, $f^{(3)}$ and so on, but we use $f^{(1)}$ only.

In the remainder of this section, let us have a close look at $\beta_2^*(t,\Lambda_t,S,y)$ and $\beta_2^{(0)}(t,\Lambda_t^{(0)},S,y)$ given in (1.17) and (1.19). Due to the relation $\partial\tilde{U}(\Lambda_T)/\partial\Lambda_T = -I(\Lambda_T)$ and the linearity of expectation $\partial\mathbb{E}[\,]/\partial\Lambda = \mathbb{E}\,[\partial/\partial\Lambda]$, we have

$$\begin{aligned} f_\Lambda &= \frac{\partial\mathbb{E}_t\left[\tilde{U}(\Lambda_T) - \Lambda_T g(S_T,y_T)\right]}{\partial\Lambda_t} = \mathbb{E}_t\left[\left(\tilde{U}'(\Lambda_T) - g(S_T,y_T)\right)\frac{\partial\Lambda_T}{\Lambda_t}\right] \\ &= \mathbb{E}_t\left[\left(-I(\Lambda_T) - g(S_T,y_T)\right)e^{-r(T-t)} M_1^{t,T} M_2^{t,T}\right] \\ &= -e^{-r(T-t)}\mathbb{E}_t^{**}[I(\Lambda_T) + g(S_T,y_T)], \end{aligned}$$

where the expectation $\mathbb{E}^{**}$ is taken under the probability measure $\mathbb{P}^{**}$ which corresponds to the probability measure induced by $M_1^{0,T}M_2^{0,T}$. Using similar derivations we find

$$\begin{aligned} f_{\Lambda y} &= -e^{-r(T-t)}\mathbb{E}_t^{**}\left[g_{y_T}(S_T,y_T)\frac{\partial y_T}{\partial y_t}\right] \\ f_{\Lambda\Lambda} &= -e^{-r(T-t)}\mathbb{E}_t^{**}\left[I'(\Lambda_T)\frac{\Lambda_T}{\Lambda_t}\right]. \end{aligned}$$

As a result, $\beta_2^*(t,\Lambda_t,S,y)$ given in (1.17) can be alternatively expressed as

$$\beta_2^*(t,\Lambda_t,S,y) = \frac{\mathbb{E}_t^{**}\left[\sqrt{1-\rho^2}\,b(t,\omega)\,g_{y_T}(S_T,y_T)\frac{\partial y_T}{\partial y_t}\right]}{-\mathbb{E}_t^{**}\left[I'(\Lambda_T)\Lambda_T\right]}. \quad (1.22)$$

Recall that $\beta_2^{(0)}(t,\Lambda_t^{(0)},S,y)$ depends on $\Lambda_t^{(0)}$ which is a function of W_1 and not related to W_2, hence, we obtain

$$\begin{aligned} f^{(0)}_{\Lambda y} &= -e^{-r(T-t)}\mathbb{E}_t^{*}\left[g_{y_T}(S_T,y_T)\frac{\partial y_T}{\partial y_t}\right] \\ f^{(0)}_{\Lambda\Lambda} &= -e^{-r(T-t)}\mathbb{E}_t^{*}\left[I'(\Lambda_T^{(0)})\frac{\Lambda_T^{(0)}}{\Lambda_t^{(0)}}\right]. \end{aligned}$$

Since $\Lambda_T^{(0)} = \Lambda_t^{(0)} M_1^{t,T}$, the expectation $\mathbb{E}^*$ is now taken under the probability measure $\mathbb{P}^*$ which corresponds to the probability measure induced by $M_1^{0,T}$. As a result, an

alternative expression for the approximate $\beta^{(0)}(t, \Lambda_t^{(0)}, S, y)$ is given by

$$\beta_2^{(0)}(t, \Lambda_t^{(0)}, S, y) = \frac{\mathbb{E}_t^* \left[\sqrt{1-\rho^2}\, b(t,\omega)\, g_{y_T}(S_T, y_T) \frac{\partial y_T}{\partial y_t} \right]}{-\mathbb{E}_t^* \left[I'(\Lambda_T^{(0)}) \Lambda_T^{(0)} \right]}. \tag{1.23}$$

Furthermore, the optimal terminal wealth expression in (1.5) can be reformulated into

$$X_T^* - g(S_T, y_T) = I(\Lambda_T) \quad \Rightarrow \Lambda_T = U'(X_T^* - g(S_T, y_T)). \tag{1.24}$$

As it holds $I'(\Lambda_T) = 1/U''(I(\Lambda_T))$, we obtain

$$-I'(\Lambda_T)\Lambda_T = -\frac{U'(X_T^* - g(S_T, y_T))}{U''(X_T^* - g(S_T, y_T))} = \frac{1}{R(X_T^* - g(S_T, y_T))} = \mathcal{T}(X_T^* - g(S_T, y_T)), \tag{1.25}$$

where $R(x) = -U''(x)/U'(x)$ stands for the Arrow-Pratt measure of absolute risk-aversion (see Arrow (1970) and Pratt (1964)) and is a measure of the absolute amount of wealth an individual is willing to expose to risk as a function of changes in wealth. $\mathcal{T}(x) := 1/R(x)$ is defined as the inverse of the absolute risk aversion and called risk tolerance. To sum up, the expression for $\beta_2^*(t, \Lambda_t, S, y)$ is rewritten as

$$\beta_2^*(t, \Lambda_t, S, y) = \frac{\mathbb{E}_t^{**} \left[\sqrt{1-\rho^2} b(t,\omega)\, g_{y_T}(S_T, y_T) \frac{\partial y_T}{\partial y_t} \right]}{\mathbb{E}_t^{**} \left[\mathcal{T}(X_T^* - g(S_T, y_T)) \right]}. \tag{1.26}$$

The higher the expected absolute risk aversion (or the lower the expected risk tolerance), the higher the optimal β_2^*.

Following the same reasonings, we can describe the approximate $\beta_2^{(0)}(t, \Lambda_t^{(0)}, S, y)$ as a function of tolerance too:

$$\beta_2^0(t, \Lambda_t^{(0)}, S, y) = \frac{\mathbb{E}_t^* \left[\sqrt{1-\rho^2} b(t,\omega)\, g_{y_T}(S_T, y_T) \frac{\partial y_T}{\partial y_t} \right]}{\mathbb{E}_t^* \left[\mathcal{T}(X_T^{(0)}) \right]}, \tag{1.27}$$

where $X_T^{(0)}$ is the optimal terminal wealth when the claim $g(S_T, y_T)$ is not available. Please note that we have $\mathcal{T}(X_T^{(0)})$ rather than $\mathcal{T}(X_T^* - g(S_T, y_T))$. This is due to the fact that in the denominator we are taking the expectation of $I'(\Lambda_T^{(0)})\Lambda_T^{(0)}$ instead of $I'(\Lambda_T)\Lambda_T$. Concerning $\Lambda_T^{(0)}$, it holds

$$X_T^{(0)} = I(\Lambda_T^{*0}) \quad \Rightarrow \Lambda_T^{(0)} = U'(X_T^{(0)}).$$

1.2. **(Approximate) indifference price.** By investing in some self-financing hedging strategies, the indifference pricing principle states the insurance company shall

be indifferent between the utility he obtains from not issuing the insurance liability $g(S_T, y_T)$ and that he obtains from issuing $g(S_T, y_T)$. From now on, we put the superscript ^{*0} on the parameters to denote the optimal values we obtain for the case without issuing the liability, and the superscript $^{*\pi}$ is used to denote the optimal values we obtain for the case with issuing the liability[2]. For instance, Λ_T^{*0} is the optimal Lagrangian level Λ_T when no liability is issued, whereas $\Lambda_T^{*\pi}$ is the optimal Λ_T when $g(S_T, y_T)$ is issued.

The derivation in Section 1.1 provides us an approximate optimal indirect utility (at time 0) when the insurance company issues the insurance liability $g(S_T, y_T)$:

$$U^{*\pi} = f^{(1)}(0, \Lambda_0^{*0}, S, y) + \Lambda_0^{*\pi}(X_0 + \pi_0), \tag{1.28}$$

where π_0 is the utility indifference price we are looking for. First, after substituting (1.23) to (1.21), $f^{(1)}(t, \Lambda_t, S, y)$ can be further calculated:

$$\begin{aligned}
& f^{(1)}(t, \Lambda_t^{*0}, S, y) \\
= \;& f^{(0)}(t, \Lambda_t^{*0}, S, y) + \mathbb{E}_t\left[\int_t^T \frac{1}{2} f_{\Lambda\Lambda}^{(0)}(\Lambda_s^{*0})^2 (\beta_2^{(0)}(s, \Lambda^{*0}, S, y))^2 d\, s\right] \\
= \;& f^{(0)}(t, \Lambda_t^{*0}, S, y) + \frac{1}{2}\mathbb{E}_t\left[\int_t^T e^{-r(T-s)}\Lambda_s^{*0} \frac{\left(\mathbb{E}_s^*\left[\sqrt{1-\rho^2}\, b(s,\omega)\, g_{y_T}(S_T, y_T)\frac{\partial y_T}{\partial y_s}\right]\right)^2}{-\mathbb{E}_s^*\left[I'(\Lambda_T^{*0})\Lambda_T^{*0}\right]} d\, s\right] \\
= \;& \mathbb{E}_t\left[\tilde{U}(\Lambda_T^{*0}) - \Lambda_T^{*0} g(S_T, y_T)\right] \\
& + \frac{1}{2} e^{-rT}\Lambda_0^{*0}\mathbb{E}_t^*\left[\int_t^T \frac{\left(\mathbb{E}_s^*\left[\sqrt{1-\rho^2}\, b(s,\omega)\, g_{y_T}(S_T, y_T)\frac{\partial y_T}{\partial y_s}\right]\right)^2}{-\mathbb{E}_s^*\left[I'(\Lambda_T^{*0})\Lambda_T^{*0}\right]} d\, s\right].
\end{aligned} \tag{1.29}$$

From step 2 to step 3 we use $f_{\Lambda\Lambda}^{(0)}(\Lambda_s^{*0})^2 = -e^{-r(T-s)}\Lambda_s^{*0}\mathbb{E}_s^*[I'(\Lambda^{*0})\Lambda_T^{*0}]$.
On the other side, the case without issuing insurance liabilities corresponds to a complete market setting, where the initial optimal indirect utility is given by

$$U^{*0} = \mathbb{E}[\tilde{U}(\Lambda_T^{*0})] + \Lambda_0^{*0} X_0. \tag{1.30}$$

[2]The parameters with the superscript $^{(0)}$ in Section 1.1 indeed coincide the ones with ^{*0} used in this section.

Proposition 1.1 (Approximate indifference price for $g(S_T, y_T)$ via HJB approach). *The approximate indifference price for $g(S_T, y_T)$ via HJB approach is given as follows*:

$$\pi_0 \approx e^{-rT}\mathbb{E}^*[g(S_T, y_T)] - \frac{e^{-rT}}{2}\mathbb{E}^*\left[\int_0^T \frac{\left(\mathbb{E}_t^*\left[\sqrt{1-\rho^2}\, b(t,\omega)\, g_{y_T}(S_T, y_T)\frac{\partial y_T}{\partial y_t}\right]\right)^2}{\mathbb{E}_t^*[I'(\Lambda_T^{*0})\Lambda_T^{*0}]}\, d\, t\right]. \tag{1.31}$$

Proof: Utility indifference indicates $U^{*0} = U^{*\pi}$, i.e.

$$\begin{aligned}\mathbb{E}[\tilde{U}(\Lambda_T^{*0})] + \Lambda_0^{*0} X_0 &= \mathbb{E}[\tilde{U}(\Lambda_T^{*0})] - e^{-rT}\,\Lambda_0^{*0}\,\mathbb{E}^*[g(S_T, y_T)] + \Lambda_0^{*\pi} X_0 + \Lambda_0^{*\pi}\pi_0 \\ &\quad + \frac{1}{2}e^{-rT}\Lambda_0^{*0}\mathbb{E}^*\left[\int_0^T \frac{\left(\mathbb{E}_t^*\left[\sqrt{1-\rho^2}\, b(t,\omega)\, g_{y_T}(S_T, y_T)\frac{\partial y_T}{\partial y_t}\right]\right)^2}{-\mathbb{E}_t^*\left[I'(\Lambda_T^{*0})\Lambda_T^{*0}\right]} d\, s\right].\end{aligned}$$

In principle, we can calculate $\Lambda_0^{*\pi}$ from the first order condition $\partial U^{*\pi}/\partial\Lambda_0^{*\pi} = 0$ and obtain an approximate optimal value for Λ_0^{*0} because it depends on the approximate value of $f^{(1)}$. Since we would end up an approximate value anyway, in this place, we assume $\Lambda_0^{*0} \approx \Lambda_0^{*\pi}$. This leads to the approximate indifference price for the insurance claim $g(S_T, y_T)$ via HJB approach:

$$\pi_0 \approx e^{-rT}\mathbb{E}^*[g(S_T, y_T)] - \frac{e^{-rT}}{2}\mathbb{E}^*\left[\int_0^T \frac{\left(\mathbb{E}_t^*\left[\sqrt{1-\rho^2}\, b(t,\omega)\, g_{y_T}(S_T, y_T)\frac{\partial y_T}{\partial y_t}\right]\right)^2}{\mathbb{E}_t^*[I'(\Lambda_T^{*0})\Lambda_T^{*0}]}\, d\, t\right].$$

□

The indifference price consists of two parts: the first part corresponds to the expected discounted payoff under the minimal martingale measure $\mathbb{P}^*$ and reflects the "best estimate" what practitioners call in terms of pricing life insurance liabilities. The second part has the same sign as the absolute risk aversion coefficient (or risk tolerance). Under risk aversion, we always obtain positive term. This can be described as "market value margin" which suggests the insurance company to charge an additional cash amount due to the unhedgeable risk. For a given numerator of β_2, the higher the expected absolute risk aversion (or the lower the risk tolerance), the higher the "market value margin", the higher the indifference price. The size of the market value margin depends on the interplay between the denominator and numerator of β_2.

An assumption made in the HJB-approach is that we are in a Markovian setting. In the next section, we start with the results obtained by the dual formulation and

develop Taylor-series approximations of the indifference price to value insurance contracts. A more general approach "martingale representation approach" is used to determine β_2^*.

2. Taylor-series approximations of the indifference price

Again, the case without issuing insurance liabilities leads to the following (dual) formulation of the optimal investment problem:

$$U^{*0} = \min_{\Lambda} \mathbb{E}[\tilde{U}(\Lambda_T) + \Lambda_0 X_0]. \tag{2.1}$$

Let us denote the optimal choice for Λ by $\Lambda_T^{*0} = \Lambda_0^{*0} e^{-rT} M_1^{0,T}$. Note that in the complete market case, we have that $M_2^{0,T} = 1$. The remaining parameter Λ_0^{*0} is a solution of the first-order condition

$$\mathbb{E}[e^{-rT} M_1^{0,T} I(\Lambda_T^{*0})] = X_0. \tag{2.2}$$

In the incomplete market case with an insurance liability $g(S_T, y_T)$, the indifference price π_0 is given by solving:

$$U^{*\pi} = \mathbb{E}[\tilde{U}(\Lambda_T^{*\pi}) - \Lambda_T^{*\pi} g(S_T, y_T) + \Lambda_0^{*\pi}(X_0 + \pi_0)] \tag{2.3}$$

where $\Lambda_T^{*\pi} = \Lambda_0^{*\pi}\, e^{-rT} M_1^{0,T} M_2^{0,T}$ denotes the optimal choice of Λ that minimizes the dual utility on the right-hand side of (2.3). Please note that (2.3) is an implicit equation in π_0 as $\Lambda_0^{*\pi}$ and $\Lambda_T^{*\pi}$ both depend on π_0. The first-order condition for $\Lambda_0^{*\pi}$ is given by:

$$\mathbb{E}\left[e^{-rT} M_1^{0,T} M_2^{0,T} (I(\Lambda_T^{*\pi}) + g(S_T, y_T))\right] \;=\; X_0 + \pi_0, \tag{2.4}$$

Let us also recall that the function $\tilde{U}$ is defined as $\tilde{U}(\Lambda) = U(I(\Lambda)) - \Lambda I(\Lambda)$. If we combine this definition with (2.1) and (2.3) we obtain

$$\begin{aligned} & \mathbb{E}\left[U(I(\Lambda_T^{*0})) - \Lambda_T^{*0} I(\Lambda_T^{*0}) + \Lambda_0^{*0} X_0\right] \\ = \;& \mathbb{E}\left[U(I(\Lambda_T^{*\pi})) - \Lambda_T^{*\pi}(I(\Lambda_T^{*\pi}) + g(S_T, y_T)) + \Lambda_0^{*\pi}(X_0 + \pi_0)\right]. \end{aligned} \tag{2.5}$$

According to (2.2) and (2.4), we obtain the simplified expression

$$\mathbb{E}\left[U(I(\Lambda_T^{*\pi}))\right] - \mathbb{E}\left[U(I(\Lambda_T^{*0}))\right] = 0. \tag{2.6}$$

This expression should not come as a surprise, as we have simply recovered the primal formulation of the indifference price by noting that $I(\Lambda_T)$ represents the optimal wealth X_T^* at time T.

2.1. **Result for Λ_0.** Up to now all our expressions have been exact. Let us now try to make some progress by investigating some Taylor-expansion of the expressions we have considered in $\Lambda_T^{*\pi}$ around Λ_T^{*0}.

Let us first start with (2.6). If we note that the derivative $\frac{\partial U(I(\Lambda))}{\partial \Lambda} = U'(I(\Lambda))I'(\Lambda) = \Lambda I'(\Lambda)$, then (2.6) combined with Taylor expansion leads to

$$\begin{aligned}
& \mathbb{E}\left[\Lambda_T^{*0}\, I'(\Lambda_T^{*0})(\Lambda_T^{*\pi} - \Lambda_T^{*0})\right] \approx 0 \\
\Rightarrow\quad & \mathbb{E}\left[\Lambda_T^{*0}\, I'(\Lambda_T^{*0})\, \Lambda_0^{*\pi} e^{-rT} M_1^{0,T}\, M_2^{0,T}\right] \approx \mathbb{E}\left[\Lambda_T^{*0}\, I'(\Lambda_T^{*0})\, \Lambda_0^{*0} e^{-rT} M_1^{0,T}\right] \\
\Rightarrow\quad & \Lambda_0^{*\pi}\mathbb{E}\left[\Lambda_T^{*0}\, I'(\Lambda_T^{*0})\, e^{-rT} M_1^{0,T}\right] \mathbb{E}[M_2^{0,T}] \approx \Lambda_0^{*0}\mathbb{E}\left[\Lambda_T^{*0}\, I'(\Lambda_T^{*0})\, e^{-rT} M_1^{0,T}\right] \\
\Rightarrow\quad & \Lambda_0^{*\pi}\mathbb{E}[M_2^{0,T}] \approx \Lambda_0^{*0} \\
\Rightarrow\quad & \Lambda_0^{*\pi} \approx \Lambda_0^{*0}. \qquad (2.7)
\end{aligned}$$

In the third line we bring the constants $\Lambda_0^{*\pi}$ and Λ_0^{*0} outside the expectation operator, and we use the fact that M_2 is independent from Λ_0^{*0}. In the fourth line we have divided out the common factor, and in the fifth line we have used the fact that M_2 is a martingale with expectation 1. This leads to the result $\Lambda_0^{*\pi} \approx \Lambda_0^{*0}$.

2.2. **Derivation of $\beta_2(t,\omega)$ via martingale representative theorem.** In an incomplete market, there exist uncertainties which cannot be hedged by trading in the market's financial instruments. However, fictitious risky assets which are perfectly correlated with the unhedgeable uncertainties can be created to complete the financial market. On the one hand, the optimal strategies (also for the fictitious assets) can be derived under the fictitious complete market. The resulting strategies are functions of "market prices of the uncertainties". On the other hand, due to the untradablility of the uncertainties, the hedging demand for these uncertainties shall be as small as possible or equal to zero. If we equate the optimal strategies for the fictitious assets to the low or zero demand for the untradable uncertainties, we are able to determine the the parameters determining the "market prices of the uncertainties". The idea of "completing" the incomplete financial market (by creating a fictitious risky asset) goes back to Karatzas et al. (1991) and is recently used by Keppo et al. (2007) who develop a computation scheme for the optimal strategy in a model setup with an unhedgeable endowment. Henceforth, we address this idea to determine β_2. In our model setup, we would like to find a market completion where the optimal hedging demand for W_2 (unhedgeable risk) equal to zero. More

precisely, we need to write down the martingale representation for $X_t^{*\pi}$ and let the coefficient of dW_2 equal to zero. Due to (1.13), in order to obtain the martingale representation for the optimal wealth, we just need to use Malliavin calculus and write down the generalized Clarke-Ocone formulae for the expressions $I(\Lambda_T^{*\pi})$ and $g(S_T, y_T)$.

According to the expression of $\Lambda_T^{*\pi}$

$$\begin{aligned}\Lambda_T^{*\pi} &= \Lambda_0^{*\pi} \exp\Bigg\{ -rT - \frac{\mu-r}{\sigma} W_1(T) - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T + \int_0^T \beta_2(s,\omega) dW_2(s) \\ &\qquad -\frac{1}{2}\int_0^T (\beta_2(s,\omega))^2 ds \Bigg\},\end{aligned}$$

we obtain the Mallivian expansion (under $\mathbb{P}^{**}$)

$$\begin{aligned}I(\Lambda_T^{*\pi}) &= \mathbb{E}^{**}[I(\Lambda_T^{*\pi})] + \int_0^T \mathbb{E}_t^{**}\left[-\frac{\mu-r}{\sigma} I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}\right] dW_1^{**}(t) + \int_0^T \Bigg(\mathbb{E}_t^{**}\left[\beta_2(t,\omega)\right. \\ &\qquad \left. I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}\right] + E_t^{**}\left[I(\Lambda_T^{*\pi}) \int_t^T D_t(\beta_2(u,\omega)) dW_2^{**}(u)\right]\Bigg) dW_2^{**}(t). \qquad (2.8)\end{aligned}$$

Moreover, the generalized Clarke-Ocone formula for $g(S_T, y_T)$ under $\mathbb{P}^{**}$ is given by

$$\begin{aligned}g(S_T, y_T) &= \mathbb{E}^{**}\left[g(S_T, y_T)\right] + \int_0^T \left(\mathbb{E}_t^{**}\left[g_{S_T}(S_T, y_T)\,\sigma S_T\right] + \mathbb{E}_t^{**}\left[g_{y_T}(S_T, y_T) b(t,\omega)\rho \frac{\partial y_T}{\partial y_t}\right]\right) \\ &\qquad dW_1^{**}(t) + \int_0^T \Bigg(\mathbb{E}_t^{**}\left[b(t,\omega)\sqrt{1-\rho^2} g_{y_T}(S_T, y_T)\frac{\partial y_T}{\partial y_t}\right] \\ &\qquad + \mathbb{E}_t^{**}\left[g(S_T, y_T)\int_t^T D_t(\beta_2(u,\omega)) dW_2^{**}(u)\right]\Bigg) dW_2^{**}(t). \qquad (2.9)\end{aligned}$$

Based on the martingale representations for $I(\Lambda_T^{*\pi})$ and $g(S_T, y_T)$ together with (1.13), we come to the following alternative expression for $X_t^{*\pi}$:

$$\begin{aligned}X_t^{*\pi} &= X_0 e^{rt} + \int_0^t \Bigg(\mathbb{E}_u^{**}\left[-\frac{\mu-r}{\sigma} I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}\right] + \Big(\mathbb{E}_u^{**}\left[g_{S_T}(S_T, y_T)\,\sigma S_T\right] \\ &\qquad + \mathbb{E}_u^{**}\left[g_{y_T}(S_T, y_T) b(u,\omega)\rho\frac{\partial y_T}{\partial y_u}\right]\Big)\Bigg) dW_1^{**}(u) + \int_0^t \Bigg(\mathbb{E}_u^{**}\left[\beta_2(u,\omega) I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}\right] \\ &\qquad + \mathbb{E}_u^{**}\left[b(u,\omega)\sqrt{1-\rho^2} g_{y_T}(S_T, y_T)\frac{\partial y_T}{\partial y_u}\right] + E_u^{**}\Big[\left(I(\Lambda_T^{*\pi}) + g(S_T, y_T)\right) \\ &\qquad \int_u^t D_u(\beta_2(s,\omega)) dW_2^{**}(s)\Big]\Bigg) dW_2^{**}(u). \qquad (2.10)\end{aligned}$$

In the optimal control problem, we will follow the optimal policy for $t < s \leq T$ not the past. Hence, the integral in the last line of (2.10) shall not play a role in determining β_2. Furthermore, since dW_2^{**} is an unhedgeable uncertainty, the hedge demand for W_2^{**} shall be equal to 0. This leads to the following $\beta_2(t,\omega)$:

$$\beta_2(t,\omega)\cdot\mathbb{E}_t^{**}[I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}] \;=\; -\mathbb{E}_t^{**}\left[b(t,\omega)\sqrt{1-\rho^2}g_{y_T}(S_T,y_T)\frac{\partial y_T}{\partial y_t}\right].$$

Or alternatively it holds

$$\beta_2^*(t,\omega) = -\frac{\mathbb{E}_t^{**}\left[b(t,\omega)\sqrt{1-\rho^2}g_{y_T}(S_T,y_T)\frac{\partial y_T}{\partial y_t}\right]}{\mathbb{E}_t^{**}[I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}]}. \tag{2.11}$$

It is noted that $\beta_2^*(t,\omega)$ value resulting from the martingale representation is more general than the optimal $\beta_2^*(t,\Lambda,s,y)$ achieved by HJB-approach given in (1.22) because for the latter case it is necessary to assume that we are in a Markovian setting.

The approximate representation of $\beta_2(t,\omega)$ is developed according to the Taylor-series expansion of $I(\Lambda_T^{*\pi})$, i.e.

$$I(\Lambda_T^{*\pi}) \approx I(\Lambda_T^{*0}) + I'(\Lambda_T^{*0})\Lambda_T^{*0}(M_2^{0,T}-1) \approx I(\Lambda_T^{*0}) + I'(\Lambda_T^{*0})\Lambda_T^{*0}\ln M_2^{0,T}. \tag{2.12}$$

Here we use approximate $M_2^{0,T}-1$ by $\ln M_2^{0,T}$. Since we are only interested in calculating the approximation solution for $\beta_2(t,\omega)$, we just need to write down the generalized Clarke-Ocone formula for $I'(\Lambda_T^{*0})\Lambda_T^{*0}\ln M_2^{0,T}$ and $g(S_T,y_T)$ which are related to $\beta_2(t,\omega)$. The martingale representation of $g(S_T,y_T)$ is already given in (2.9) and that of $I'(\Lambda_T^{*0})\Lambda_T^{*0}\ln M_2^{0,T}$ owns the following expression:

$$\begin{aligned}
& I'(\Lambda_T^{*0})\Lambda_T^{*0}\ln M_2^{0,T} \\
= \;& \mathbb{E}^{**}[I'\Lambda_T^{*0})\Lambda_T^{*0}\ln M_2^{0,T}] + \int_0^T\left(\mathbb{E}_t^{**}\left[(I'(\Lambda_T^{*0})\Lambda_T^{*0}\beta_2(t,\omega)\right]\right. \\
& \left.+\mathbb{E}_t^{**}\left[I'(\Lambda_T^{*0})\Lambda_T^{*0}\ln M_2^{0,T}\int_t^T D_t(\beta_2(u,\omega))dW_2^{**}(u)\right]\right)dW_2^{**}(t) \\
& +\int_0^T \mathbb{E}_t^{**}\left[\ln M_2^{0,T}(I''(\Lambda_T^{*0})\Lambda_T^{*0}+I'(\Lambda_T^{*0}))\Lambda_T^{*0}(-\frac{\mu-r}{\sigma})\right]dW_1^{**}(t).
\end{aligned} \tag{2.13}$$

Develop an approximation for $X_t^{*\pi}$ similarly as in (2.10) and let the hedge demand for dW_2^{**} equal 0, we obtain an approximate expression for β_2:

$$\beta_2(t,\omega) \approx \frac{\mathbb{E}_t^{*}\left[b(t,\omega)\sqrt{1-\rho^2}g_{y_T}(S_T,y_T)\frac{\partial y_T}{\partial y_t}\right]}{-\mathbb{E}_t^{*}[I'(\Lambda_T^{*0})\Lambda_T^{*0}]}. \tag{2.14}$$

Similarly, (2.14) is more general than (1.23) because the Makovian assumption is not needed here.

2.3. **Approximate indifference price.** We now want to turn our attention to finding an (approximate) expression for the price π_0. By rewriting (2.4) we obtain:

$$\begin{aligned}\pi_0 &= e^{-rT}\mathbb{E}\left[M_1^{0,T}M_2^{0,T}\left(I(\Lambda_T^{*\pi})+g(S_T,y_T)\right)\right]-X_0\\ &= e^{-rT}\mathbb{E}^{**}\left[I(\Lambda_T^{*\pi})+g(S_T,y_T)\right]-X_0\\ &= e^{-rT}\mathbb{E}^{**}\left[g(S_T,y_T)+\left(I(\Lambda_T^{*\pi})-I(\Lambda_T^{*0})\right)\right]=X_0^{*\pi}-X_0. \qquad (2.15)\end{aligned}$$

In the third line we have used (2.2) and the fact that since Λ_T^{*0} is W_1-measurable we have $e^{-rT}\mathbb{E}^{**}[I(\Lambda_T^{*0})] = e^{-rT}\mathbb{E}^{*}[I(\Lambda_T^{*0})] = X_0$. Up to this point the derivation has been exact.

From (2.15) we once again see that the indifference price π_0 consists of two components. The first component is the expected value under the martingale measure $\mathbb{P}^{**}$ of the insurance payoff $g(S_T, y_T)$. The second component is the wealth difference $I(\Lambda_T^{*\pi}) - I(\Lambda_T^{*0})$ between the "pre-insurance" and "post-insurance" portfolios. More specifically, the second component quantifies the compensation that is required for the *unhedgeable risk* due to writing the insurance claim $g(S_T, y_T)$. It is this second component which makes the indifference price operator a non-linear operator. When the unhedgeable part of the risk disappears, the price operator reduces to the familiar (linear) risk-neutral martingale pricing operator. More detailed about the compensations caused by the unhedgeable risk will be provided in Section 4.

Proposition 2.1 (Approximate indifference price for $g(S_T, y_T)$ via Taylor expansion). *The approximate indifference price for $g(S_T, y_T)$ via Taylor expansion is given as follows*:

$$\begin{aligned}\pi_0 &\approx e^{-rT}\left(\mathbb{E}^*[g(S_T,y_T)]-\frac{1}{2}\mathbb{E}^*[I'(\Lambda_T^{*0})\Lambda_T^{*0}]\mathbb{E}^*\left[\int_0^T(\beta_2^*(t,\omega))^2dt\right]\right)\\ &= e^{-rT}\left(\mathbb{E}^*[g(S_T,y_T)]-\frac{1}{2}\mathbb{E}^*[I'(\Lambda_T^{*0})\Lambda_T^{*0}]\mathrm{Var}^*[M_2^{0,T}]\right),\end{aligned}$$

where "Var*" *denotes a variance under the probability measure $\mathbb{P}^*$ induced by $M_1^{0,T}$.*

PROOF: We continue the calculation (2.15) with another Taylor-expansion

$$\begin{aligned}
& \pi_0 e^{rT} \\
\approx\ & \mathbb{E}^{**}\left[g(S_T, y_T) + I'(\Lambda_T^{*0})(\Lambda_T^{*\pi} - \Lambda_T^{*0})\right] \\
=\ & \mathbb{E}^{*}\left[g(S_T, y_T) M_2^{0,T}\right] + \mathbb{E}^{**}\left[I'(\Lambda_T^{*0})\Lambda_T^{*0}(M_2^{0,T} - 1)\right] \\
\approx\ & \mathbb{E}^{*}[g(S_T, y_T)(\ln M_2^{0,T} + 1)] + \mathbb{E}^{*}[I'(\Lambda_T^{*0})\Lambda_T^{*0}]\mathbb{E}^{**}[M_2^{0,T} - 1] \\
=\ & \mathbb{E}^{*}[g(S_T, y_T)] + \mathbb{E}^{*}[g(S_T, y_T) \cdot \ln M_2^{0,T}] + \mathbb{E}^{*}[I'(\Lambda_T^{*0})\Lambda_T^{*0}]\mathbb{E}^{**}[M_2^{0,T} - 1]. \quad (2.16)
\end{aligned}$$

From line 2 to 3 we use $\ln M_2^{0,T} \approx M_2^{0,T} - 1$. In order to calculate the second term in (2.16) we represent $\ln M_2^{0,T}$ and $g(S_T, y_T)$ in martingale forms via Mallivian calculus:

$$\begin{aligned}
\ln M_2^{0,T} \ =\ & \mathbb{E}^{**}[\ln M_2^{0,T}] + \int_0^T \left(\beta_2(t,\omega) + \mathbb{E}_t^{**}\left[\ln M_2^{0,T} \int_t^T D_t(\beta_2(u,\omega)) dW_2^{**}(u) \right] \right) \\
& dW_2^{**}(t),
\end{aligned}$$

and $g(S_T, y_T)$ is already given in (2.9). According to (2.14), we obtain

$$\mathbb{E}_t^{**}\left[b(t,\omega)\sqrt{1-\rho^2} g_{y_T}(S_T, y_T) \frac{\partial y_T}{\partial y_t} \right] \approx -\beta_2(t,\omega) E_t^{*}[I'(\Lambda_T^{*0})\Lambda_T^{*0}].$$

Hence, it holds:

$$\mathbb{E}^{*}[g(S_T, y_T) \cdot \ln M_2^{0,T}] = -\mathbb{E}^{*}[I'(\Lambda_T^{*0})\Lambda_T^{*0}]\mathbb{E}^{*}\left[\int_0^T (\beta_2(t,\omega))^2 dt \right].$$

Moreover, we approximate the last term of (2.16) as follows:

$$\begin{aligned}
\mathbb{E}^{**}[M_2^{0,T} - 1] \ \approx\ & \mathbb{E}^{**}[\ln M_2^{0,T}] \\
=\ & \mathbb{E}^{*}[M_2^{0,T} \ln M_2^{0,T}] \approx \mathbb{E}^{*}[(\ln M_2^{0,T} + 1) \ln M_2^{0,T}] \\
=\ & \mathbb{E}^{*}\left[\int_0^T (\beta_2(t,\omega))^2 dt \right] - \frac{1}{2}\mathbb{E}^{*}\left[\int_0^T (\beta_2(t,\omega))^2 dt \right] \\
=\ & \frac{1}{2}\mathbb{E}^{*}\left[\int_0^T (\beta_2(t,\omega))^2 dt \right].
\end{aligned}$$

Here we use the approximation $M_2^{0,T} - 1 \approx \ln M_2^{0,T}$ twice.

□

Via Taylor approximation, the indifference price π_0 can be decomposed into exactly two parts: $\mathbb{E}^{*}[g(S_T, y_T)]$ plus a correction term proportional to the variance of the martingale $M_2^{0,T}$. This approximate indifference price coincides with the one given in

Proposition 1.1 and suggests the practitioners to use "best estimate plus a correction term" to price insurance liabilities.

3. Illustrative examples

Although the present work is designed to use a more tractable approach to price life insurance contracts, the results of the paper shall be suitable for the other untradable contingent claims. Hence, in this section, we would examine our result and investigate the quality of the approximation by comparing it with Musiela and Zariphopoulou (2001) where the untradable contingent claim depends on y_T only and the agent owns an exponential utility. Further, Henderson and Hobson (2002) and Henderson (2002) where power utility is discussed are served as a comparison basis, too.

3.1. **Exponential utility.** For an exponential utility $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, it holds $U'(x) = e^{-\gamma x}$, $I(\Lambda) = -\frac{1}{\gamma}\ln(\Lambda)$ and $\tilde{U}(\Lambda) = \frac{1}{\gamma}(\Lambda\ln(\Lambda) - \Lambda)$, in addition, $I'(\Lambda) = -\frac{1}{\gamma*\Lambda}$. Further, for those contingent claims whose terminal payments depend on the evolution of y_T only, i.e., $g(S_T, y_T) = g(y_T)$, $\beta_2(t,\omega)$ value expressed in (2.11) is reduced to

$$\beta_2(t,\omega) \;=\; \gamma\mathbb{E}_t^{**}\left[b(t,\omega)\sqrt{1-\rho^2}\,g_{y_T}(y_T)\right].$$

For the specific specification of y in Musiela and Zariphopoulou (2001), it follows $\frac{\partial y_T}{\partial y_t} = 1$. According to Proposition 1.1 or 2.1, the approximate indifference price of $g(y_T)$ is determined by

$$\begin{aligned}\pi_0 \;&\approx\; e^{-rT}\left(\mathbb{E}^*[g(y_T)] + \frac{1}{2}\frac{1}{\gamma}\int_0^T \mathbb{E}^*[(\beta_2(t,\omega))^2]dt\right)\\ &=\; e^{-rT}\left(\mathbb{E}^*[g(y_T)] + \frac{1}{2}\gamma\int_0^T \mathbb{E}^*\left[\left(\mathbb{E}_t^{**}[\,(b(t,\omega))^2\,(1-\rho^2)\,g_{y_T}(y_T)]\right)^2\right]dt\right). \quad (3.1)\end{aligned}$$

On the other hand, the indifference price for this special utility is given by the exact Musiela and Zariphopoulou (2001) price formula:

$$\pi_0^{MZ} \;=\; e^{-rT}\,\frac{1}{\gamma(1-\rho^2)}\,\ln\mathbb{E}^*\left[e^{\gamma(1-\rho^2)g(y_T)}\right]$$

We can interpret the expectation on the right-hand side as the moment-generating function of $g(y_T)$ with parameter $\gamma(1-\rho^2)$. Hence, up to second order we can

approximate the price-formula as

$$\begin{aligned}\pi_0^{MZ} &\approx e^{-rT}\,\frac{1}{\gamma(1-\rho^2)}\,\ln\left(\exp\left\{\gamma(1-\rho^2)\mathbb{E}^*[g(y_T)]+\frac{1}{2}\gamma^2(1-\rho^2)^2\mathrm{Var}^*[g(y_T)]\right\}\right)\\ &\approx e^{-rT}\left(\mathbb{E}^*[g(y_T)]+\frac{1}{2}\,\gamma\,(1-\rho^2)\,\mathrm{Var}^*[g(y_T)]\right). \qquad (3.2)\end{aligned}$$

Further, if we decompose $g(y_T)$ by writing down the generalized Clarke-Ocone formula as follows:

$$\begin{aligned}g(y_T) &= \mathbb{E}^{**}\left[g(y_T)\right]+\int_0^T \mathbb{E}_t^{**}\left[g_{y_T}(y_T)b(t,\omega)\rho\right]dW_1^{**}(t)\\ &\quad+\int_0^T \mathbb{E}_t^{**}[b(t,\omega)\sqrt{1-\rho^2}g_{y_T}(y_T)]dW_2^{**}(t),\end{aligned}$$

we obtain

$$\mathrm{Var}^*[g(y_T)] = \int_0^T \mathbb{E}^*\left[\left(\mathbb{E}_t^{**}\left[(b(t,\omega))^2 g_{y_T}(y_T)\right]\right)^2\right]dt.$$

From this we can infer that the expressions (3.1) and (3.2) coincide.

3.2. **Power utility.** For a power utility $U(x)=\frac{x^{1-\eta}}{1-\eta}$, $\eta>0$ and $\eta\neq 1$, it holds $U'(x)=x^{-\eta}$, $I(\Lambda)=\Lambda^{-\frac{1}{\eta}}$ and $\tilde{U}(\Lambda)=\frac{\eta\Lambda^{1-\frac{1}{\eta}}}{1-\eta}$, in addition, $I'(\Lambda)=-\frac{1}{\eta}\Lambda^{-\frac{1}{\eta}-1}$. Furthermore, the following relation hold particularly for the power utility function:

$$\mathbb{E}_t^*[I'(\Lambda_T^{*0})\Lambda_T^{*0}] = -\frac{1}{\eta}\mathbb{E}_t^*\left[I(\Lambda_T^{*0})\right]=-\frac{1}{\eta}e^{r(T-t)}X_t^{*0}.$$

The prices given in Propositions 1.1 and 1.2 are the seller's indifference price, while in Henderson (2002), buyer's price is derived. In order to compare our results with Henderson (2002), we adapt our results to buyer's price, i.e. the approximate buyer's indifference price is given by

$$\begin{aligned}\pi_0 &\approx e^{-rT}\mathbb{E}^*[g(S_T,y_T)]+\frac{e^{-rT}}{2}\mathbb{E}^*\left[\int_0^T\frac{\left(\mathbb{E}_t^*\left[\sqrt{1-\rho^2}\,b(t,\omega)\,g_{y_T}(S_T,y_T)\frac{\partial y_T}{\partial y_t}\right]\right)^2}{\mathbb{E}_t^*[I'(\Lambda_T^{*0})\Lambda_T^{*0}]}\,d\,t\right]\\ &= e^{-rT}\mathbb{E}^*[g(S_T,y_T)]\\ &\quad-\frac{\eta}{2}\mathbb{E}^*\left[\int_0^T\frac{e^{-2rT+rt}\left(\mathbb{E}_t^*\left[\sqrt{1-\rho^2}\,b(t,\omega)\,g_{y_T}(S_T,y_T)\frac{\partial y_T}{\partial y_t}\right]\right)^2}{X_t^{*0}}\,d\,t\right]. \qquad (3.3)\end{aligned}$$

On the other side, as mentioned in the introduction, Henderson (2002) develops an approximation in framework of a power utility for the contingent claims of the form $kg(y_T)$, where k is a constant and can be interpreted as the number of the claims.

She obtains an approximate utility indifference price:

$$\begin{aligned}\pi_0^{He}(k) &= ke^{-rT}\mathbb{E}^*[g(y_T)] - \frac{k^2}{2}\frac{\eta}{X_0}b^2(1-\rho^2) \\ &\quad \mathbb{E}^*\left[\int_0^T e^{-rt}\frac{y_t^2 e^{-2r(T-t)}\left(\mathbb{E}_t^*[g_{y_T}(y_T)]\right)^2}{(X_t^{*0}/X_0)}\right]dt + o(k^2) \qquad (3.4)\end{aligned}$$

Since our result is more general and in order to compare it with Henderson (2002), we have to fit our parameters to their modeling setup. First, it holds $g(S_T, y_T) = k\, g(y_T)$. Second, the y-process in Henderson is assumed to follow a geometric Brownian motion, i.e. $a(t,\omega) = ay_t,\ b(t,\omega) = by_t$, where a, b is a constant. Combining these with (3.3), we obtain

$$\pi_0 \approx e^{-rT}\mathbb{E}^*[kg(y_T)] - \frac{\eta}{2}\mathbb{E}^*\left[\int_0^T \frac{e^{-2rT+rt}\left(\sqrt{1-\rho^2}\, b\, y_t \mathbb{E}_t^*\left[k\, g_{y_T}(y_T)\right]\right)^2}{X_t^{*0}}\, d\, t\right].$$

This coincides with the Henderson's (2002) approximate power utility indifferent price.

4. Impact of the unhedgeable risk on optimal wealth ("surpluses") and strategy

In order to gain some insights into the impact of the unhedgeable risk on the optimal wealth, we compare the optimal wealth derived for the case without and with the unhedgeable risk.

Abstracting from the unhedgeable risk, the optimal wealth is described as

$$X_T^{*0} = I(\Lambda_T^{*0}).$$

Whereas with the unhedgeable risk, we obtain the following relation:

$$X_T^{*\pi} - g(S_T, y_T) = I(\Lambda_T^{*\pi}).$$

Now we apply the Taylor expansion to $I(\Lambda_T^{*\pi})$ at Λ_T^{*0} and achieve the approximation as follows:

$$I(\Lambda_T^{*\pi}) \approx I(\Lambda_T^{*0}) + I'(\Lambda_T^{*0})\Lambda_T^{*0}(M_2^{0,T} - 1). \qquad (4.1)$$

Hence, the optimal wealth $X_T^{*\pi}$ can be described approximately by

$$X_T^{*\pi} \approx g(S_T, y_T) + I(\Lambda_T^{*0}) + I'(\Lambda_T^{*0})\Lambda_T^{*0}(M_2^{0,T} - 1).$$

Based on the approximate version of $X_T^{*\pi}$, we achieve an approximate value for $X_t^{*\pi}$ by using the relation (1.13):

$$X_t^{*\pi} \approx e^{-r(T-t)} \left(\mathbb{E}_t^{**}[g(S_T, y_T)] + \mathbb{E}_t^{**}[I(\Lambda_T^{*0})] + \mathbb{E}_t^{**}[I'(\Lambda_T^{*0})\Lambda_T^{*0}(M_2^{0,T} - 1)] \right). \quad (4.2)$$

Since Λ_T^{*0} depends on the W_1 only and $M_2^{0,T}$ on W_2 only, besides W_1 and W_2 are independent, we can rewrite (4.2) to

$$\begin{aligned} X_t^{*\pi} &\approx e^{-r(T-t)} \left(\mathbb{E}_t^{**}[g(S_T, y_T)] + \mathbb{E}_t^{*}[I(\Lambda_T^{*0})] + \mathbb{E}_t^{*}[I'(\Lambda_T^{*0})\Lambda_T^{*0}]\mathbb{E}_t^{**}[(M_2^{0,T} - 1)] \right) \\ &= X_t^{*0} + e^{-r(T-t)} \left(\mathbb{E}_t^{**}[g(S_T, y_T)] + \mathbb{E}_t^{*}[I'(\Lambda_T^{*0})\Lambda_T^{*0}]\mathbb{E}_t^{**}[(M_2^{0,T} - 1)] \right). \quad (4.3) \end{aligned}$$

With the introduction of the unhedgeable insurance risk, the optimal wealth $X_t^{*\pi}$ differs from that obtained in a complete market setting X_t^{*0} by the size

$$X_t^{*\pi} - X_t^{*0} = e^{-r(T-t)} \left(\mathbb{E}_t^{**}[g(S_T, y_T)] + \mathbb{E}_t^{*}[I'(\Lambda_T^{*0})\Lambda_T^{*0}]\mathbb{E}_t^{**}[(M_2^{0,T} - 1)] \right), \quad 0 \le t \le T. \quad (4.4)$$

Basically, the optimal wealth margin process $(X_t^{*\pi} - X_t^{*0})_{t\in[0,T]}$ hinges on the realization of $W_1^*(t)$ (or $W_1(t)$) as $\mathbb{E}_t^*[I'(\Lambda_T^{*0})\Lambda_T^{*0}]$ is generally a function of $W_1^*(t)$ (or $W_1(t)$). An exception here is the exponential utility in which this expected value becomes a constant:

$$\mathbb{E}^*[I'(\Lambda_T^{*0})\Lambda_T^{*0}] = -\frac{1}{\gamma}.$$

The optimal wealth margin is accordingly given by

$$X_t^{*\pi} - X_t^{*0} \approx e^{-r(T-t)} \left(\mathbb{E}^{**}[g(S_T, y_T)] - \frac{1}{\gamma}\mathbb{E}^{**}[(M_2^{0,T} - 1)] \right). \quad (4.5)$$

This indicates that using the exponential utility has a consequence that the optimal wealth margin is not a function of W_1 or the tradable risky asset S. Therefore, only investment in the riskless asset (or cash amount) is necessary to compensate the wealth loss caused by the unhedgeable risk.

When we take into consideration other general utility functions, the additional investment in he risky (tradable) asset cannot always be determined explicitly. However, we can conclude that the additional investments (either in the risky assets or in both the risky and riskless assets) depend on the risk attitude of the agent, because $X_t^{*\pi} - X_t^{*0}$ can be expressed alternatively as (c.f. (1.24)):

$$X_t^{*\pi} - X_t^{*0} = e^{-r(T-t)} \left(\mathbb{E}_t^{**}[g(S_T, y_T)] - \mathbb{E}_t^{*}[\mathcal{T}(X_T^{*0})]E_t^{**}[(M_2^{0,T} - 1)] \right), \quad 0 \le t \le T.$$

$\mathcal{T}(.)$ is again the risk tolerance. The lower the risk tolerance (or the higher the absolute risk aversion), the more investments are needed to compensate the optimal wealth margin.

Let us have a close look at the strategy henceforth. First we can read from the martingale representation of the optimal wealth in (2.10) that amount invested in the hedgeable risk W_1^{**} at time t is given by:

$$\mathbb{E}_t^{**}\left[-\frac{\mu-r}{\sigma}\, I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}\right] + \mathbb{E}_t^{**}\left[g_{S_T}(S_T,y_T)\,\sigma S_T\right] + \mathbb{E}_t^{**}\left[g_{y_T}(S_T,y_T)b(t,\omega)\rho\right]. \quad (4.6)$$

In contrast, in a complete market setup, the resulting amount invested in the hedgeable risk W_1^* $(= W_2^*)$ at time t is determined by

$$\mathbb{E}_t^{*}\left[-\frac{\mu-r}{\sigma}\, I'(\Lambda_T^{*0})\,\Lambda_T^{*0}\right].$$

This is the Merton's (1971) optimal portfolio derived for the original problem of optimal consumption and portfolio choice in continuous time. Due to

$$\begin{aligned}\mathbb{E}_t^{**}\left[I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}\right] &\approx E_t^{**}\left[I'(\Lambda_T^{*0})\,\Lambda_T^{*0} + \left(I''(\Lambda_T^{*0})\Lambda_T^{*0} + I'(\Lambda_T^{*0})\right)\right)(\Lambda_T^{*\pi} - \Lambda_T^{*0})\Big] \\ &= E_t^{*}\left[I'(\Lambda_T^{*0})\,\Lambda_T^{*0}\right] + E_t^{*}\left[I''(\Lambda_T^{*0})(\Lambda_T^{*0})^2 + I'(\Lambda_T^{*0}))\Lambda_T^{*0}\right]\mathbb{E}_t^{**}\left[M_2^{0,T} - 1\right],\end{aligned}$$

the difference between the "new" and the original Merton strategy can be approximated by

$$\begin{aligned}&\mathbb{E}_t^{**}\left[-\frac{\mu-r}{\sigma}\, I'(\Lambda_T^{*\pi})\,\Lambda_T^{*\pi}\right] - \mathbb{E}_t^{*}\left[-\frac{\mu-r}{\sigma}\, I'(\Lambda_T^{*0})\,\Lambda_T^{*0}\right] \\ \approx\ & \frac{\mu-r}{\sigma} E_t^{*}\left[I''(\Lambda_T^{*0})(\Lambda_T^{*0})^2 + I'(\Lambda_T^{*0}))\Lambda_T^{*0}\right]\mathbb{E}_t^{**}\left[M_2^{0,T} - 1\right] \\ =\ & \frac{\mu-r}{\sigma} E_t^{*}\left[I''(\Lambda_T^{*0})(\Lambda_T^{*0})^2 + I'(\Lambda_T^{*0}))\Lambda_T^{*0}\right]\underbrace{\left(\exp\left\{\int_t^T (\beta_2(u,\omega))^2 du\right\} - 1\right)}_{>0}.\end{aligned}$$

Further, it is noted that

$$\begin{aligned}I''(\Lambda_T^{*0})(\Lambda_T^{*0})^2 &= \Lambda_T^{*0} - \Lambda_T^{*0}\,\frac{U'(\Lambda_T^{*0})}{U''(\Lambda_T^{*0})}\,\frac{U'''(\Lambda_T^{*0})}{U''(\Lambda_T^{*0})} - \frac{U'(\Lambda_T^{*0})}{U''(\Lambda_T^{*0})} \\ I'(\Lambda_T^{*0}) &= \frac{U'(\Lambda_T^{*0})}{U''(\Lambda_T^{*0})}.\end{aligned}$$

Accordingly, the difference between the "new" and the original Merton strategy can be further expressed

$$\begin{aligned}
&\frac{\mu - r}{\sigma}\left(\mathbb{E}_t^{**}\left[\Lambda_T^{*0}\right] - \mathbb{E}_t^{**}\left[\Lambda_T^{*0}\,\frac{U'(\Lambda_T^{*0})}{U''(\Lambda_T^{*0})}\,\frac{U'''(\Lambda_T^{*0})}{U''(\Lambda_T^{*0})}\right]\right)\left(\exp\left\{\int_t^T(\beta_2(u,\omega))^2du\right\} - 1\right)\\
= \;& \frac{\mu - r}{\sigma}\mathbb{E}_t^{**}\left[\Lambda_T^{*0}(1 - C(\Lambda_T^{*0})\right]\left(\exp\left\{\int_t^T(\beta_2(u,\omega))^2du\right\} - 1\right)
\end{aligned}$$

where $R(.) = -\frac{U''(.)}{U'(.)}$ is the coefficient of absolute risk aversion and $P(.) = -\frac{U'''(.)}{U''(.)}$ is the coefficient of absolute prudence introduced by Kimball (1990). Risk aversion has interpretations for investors' investment activities in financial market, whereas prudence explains the saving decision of the investor. Kimball (1990) show that an individual with a larger coefficient of absolute risk aversion has a larger risk premium than the other individual at any given wealth level, whereas an individual with a larger coefficient of absolute prudence has a larger equivalent precautionary premium than the other at any given level of savings. Further, the notation $C(.)$ is defined by the ratio $P(.)/R(.)$ and represents the concept cautiousness. It measures the strength of an investor's motives to hedge the down-side risk of his investment using convex-payoff contracts. There exists the negative relation between the cautiousness and relative risk aversion, i.e. if investor A is more cautious than investor B, he would be less relative risk averse than investor B, and if A is more relative risk averse than B, A is less cautious than B (a detailed discussion on this topic can be found e.g. in Kimball (1990) and Huang (2000). Given an increasing and concave utility function, we can obtain an alternative expression for the cautiousness, i.e.

$$C(x) = -\frac{R'(x)}{(R(x))^2} + 1,$$

and an alternative description for the "new" and original strategy:

$$\begin{aligned}
&\frac{\mu - r}{\sigma}\left(\mathbb{E}_t^{**}\left[\Lambda_T^{*0}\right] - \mathbb{E}_t^{**}\left[\Lambda_T^{*0}\,\frac{U'(\Lambda_T^{*0})}{U''(\Lambda_T^{*0})}\,\frac{U'''(\Lambda_T^{*0})}{U''(\Lambda_T^{*0})}\right]\right)\left(\exp\left\{\int_t^T(\beta_2(u,\omega))^2du\right\} - 1\right)\\
= \;& \frac{\mu - r}{\sigma}\mathbb{E}_t^{**}\left[\Lambda_T^{*0}\frac{R'(\Lambda_T^{*0})}{(R(\Lambda_T^{*0}))^2}\right]\left(\exp\left\{\int_t^T(\beta_2(u,\omega))^2du\right\} - 1\right).
\end{aligned}$$

From this we can read that those utility functions with constant absolute risk aversion, i.e. $R'(x) = 0$, the "new" and original strategy can be approximately considered equal. Exponential utility belongs to this category. This result conforms to the conclusion of the optimal wealth margin we draw for the exponential utility, i.e. it is unnecessary to invest more in W_1 but just in cash amounts to compensate the wealth

loss. However, for decreasing absolute risk aversion type utilities which disclose an investor's utility most likely, as widely agreed in the literature, the "new strategy" is smaller than the original one because $R'(x) < 0$. This is due to the unhedgeable insurance risk component. The investor becomes exposed to the risk of (unexpectedly) underperforming with respect to his optimal wealth-target. Therefore, he becomes "more worried" about investing in the risky asset and consequently he acts as if he is more risk-averse, and hence holds less of the risky asset (i.e. a lower exposure to W_1).

5. Conclusion

Under general utility class, this paper develops an approximate pricing framework for life insurance liabilities using utility indifference. The resulting approximate indifference price has a nice connection to the pricing rule-of-thumb that practitioners use: best estimate plus a "Market Value Margin". The best estimate corresponds to the expected discounted value of the contingent claim, where the expectation is taken under the minimal Martingale measure. The "Market Value Margin" can be interpreted as a safety load which usually depends on the expected risk aversion coefficient (except the exponential utility case).

Although our main purpose is to develop tractable market-consistent tractable approximate prices for life insurance contracts, our results also apply to other untradable contingent claims and our approximate formulae lead to the same results obtained in the existing literature.

Furthermore, we investigate the question of how the untradable insurance risk affects the optimal wealth process and what risk management strategies the insurance company can use to compensate the wealth loss.

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