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## **TESTING FOR MEAN-COHERENT REGULAR RISK SPANNING**

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# Testing for Mean-Coherent Regular Risk Spanning

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## Abstract

Coherent risk measures have received considerable attention in the recent literature. Coherent *regular* risk measures form an important subclass: they are empirically identifiable, and, when combined with mean return, they are consistent with second order stochastic dominance. As a consequence, these risk measures are natural candidates in a mean-risk trade-off portfolio choice. In this paper we develop a mean-coherent regular risk spanning test and related performance measure. The test and the performance measure can be implemented by means of a simple semi-parametric instrumental variable regression, where instruments have a direct link with the stochastic discount factor. We illustrate applications of the spanning test and the performance measure for several coherent regular risk measures, including the well known expected shortfall.

*Keywords:* portfolio choice, coherent risk, spanning test

*JEL Classification:* G11, D81.

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# 1 Introduction

Introduced by Artzner et al. (1999), coherent risk measures received considerable attention in the recent literature. Indeed, coherent risk measures satisfy a set of properties desirable from the perspective of risk management, motivated by regulatory concerns. With additional requirements considered by Kusuoka (2001), making a risk measure among other things empirically identifiable, we get the class of so-called coherent *regular* risk (CRR) measures, see Bassett et al. (2004). A particular CRR measure is expected shortfall, which has become especially popular in theoretical and empirical applications due to its computational tractability.<sup>1</sup> In parallel with these developments in the risk measure theory, there is also an increasing understanding that risk measures alternative to the industry-standard variance can (and maybe should be) used in asset allocation decisions. Indeed, the variance as risk measure treats overperformance equally as underperformance. Starting with Markowitz (1953), who suggested the use of the semi-variance instead of the variance, many alternative risk measures, treating underperformance differently from overperformance, have been proposed, see, for example, Pedersen and Satchell (1998). In particular, also CRR measures found their way to the optimal portfolio choice theory by means of expected shortfall. Rockafellar and Uryasev (2000) suggest an efficient numerical method to solve an in-sample analog of the mean-expected shortfall portfolio optimization problem. Bertsimas et al. (2004) elaborate on the method further. Bassett et al. (2004) show that the mean-expected shortfall optimization problem can be seen as a constrained quantile regression, for which very efficient numerical methods have been developed.<sup>2</sup> They also suggest a point mass approximation for a general CRR measure and show that the mean-CRR optimal portfolio problem with such an approximation can be solved by quantile regression algorithms.

Portfolio choice based on expected utility might be considered as a bench-

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<sup>1</sup>See, for example, Acerbi and Tasche (2002), Tasche (2002), and Bertsimas et al. (2004) for theoretical properties of expected shortfall; and Bassett et al. (2004), Kerkhof and Melenberg (2004), and Polbennikov and Melenberg (2005) for practical applications.

<sup>2</sup>See Barrodale and Roberts (1974), Koenker and D'Orey (1987), and Portnoy and Koenker (1997).

mark to evaluate the choice of risk measure. For instance, the variance as risk measure in a mean-variance portfolio choice corresponds to expected utility with a quadratic utility index or when asset returns jointly follow an elliptically symmetric distribution. But, otherwise, a mean-variance optimal portfolio is not consistent with second order stochastic dominance. On the other hand, CRR measures, when combined with expected return, turn out to be consistent with second order stochastic dominance. Indeed, De Giorgi (2005) introduces portfolio choice based upon a reward-risk tradeoff, isotonic with respect to second order stochastic dominance. This latter isotonicity requirement means that for the reward one should take the mean return, while risk measures based upon particular Choquet integrals qualify as appropriate risk measures. Expected shortfall and, more generally, CRR measures are such Choquet integral based risk measures. As a consequence, mean-CRR optimal portfolios are consistent with second order stochastic dominance. For the special case of the mean-expected shortfall trade-off this has already been demonstrated by, for example, Ogryczak and Ruszczyński (2002).

As noticed by Bassett et al. (2004), an alternative justification for mean-CRR efficient portfolios can be given from the point of view of an investor who maximizes a Choquet expected utility with a linear utility index and a convex distortion of the original probability. This framework is an alternative to the expected utility paradigm developed by Ramsey (1931), von Neumann and Morgenstern (1944), and Savage (1954), see Schmeidler (1989), Yaari (1987), and Quiggin (1982). While in the classical expected utility theory the utility index bears the entire burden of representing the decision maker's attitude towards risk, Choquet expected utility theory introduces the possibility that preferences may require a distortion of the original probability assessments. The cumulative prospect theory, as developed by Tversky and Kahneman (1992) and Wakker and Tversky (1993), is also closely aligned with the Choquet approach.

Mean-CRR efficient portfolios lead to mean-CRR efficient frontiers. For example, Tasche (1999) calculates expected shortfall based risk contributions and discusses a mean-expected shortfall based capital asset pricing theory (CAPM). See also Kalkbrenner (2005) for an axiomatic approach to capital allocation.

Then a natural question to ask is whether analogs of statistical methods, well known in the mean-variance portfolio analysis,<sup>3</sup> can be developed in the mean-CRR case. In this paper we develop a simple mean-CRR spanning test, which can be used to check whether the mean-CRR frontier of a set of assets spans the frontier of a larger set of assets. We show that, analogous to the mean variance spanning test developed by Huberman and Kandel (1987), the mean-CRR spanning test can be performed as a significance test for the intercept coefficient in a simple linear regression model. The difference, however, is that in case of the mean-CRR spanning a semi-parametric instrumental variable (IV) estimation technique should be applied. The instrumental variable has a direct link to a corresponding stochastic discount factor. We illustrate applications of this spanning test for several CRR measures, including expected shortfall and the point mass CRR approximation, suggested by Bassett et al. (2004), and compare the results to their mean-variance analogs. Though quite different in approach, our analysis is similar in spirit to the analysis of Gourieroux and Monfort (2005), who analyze statistical properties of efficient portfolios in a constrained parametric expected utility optimization setup. De Giorgi and Post (2004) use a general equilibrium framework to develop a mean-CRR CAPM assuming that *all* agents have mean-CRR preferences, while we only assume this for the investor under consideration. Moreover, De Giorgi and Post (2004) consider atomic probability distributions of asset returns, while we focus on non-atomic return distributions.

The remainder of this paper is structured as follows. Section 2 briefly describes coherent regular risk (CRR) measures. In section 3 we introduce the mean-CRR problem and derive the risk contributions of a CRR measure. Spanning tests and their limit distributions are presented in section 4. Section 5 discusses the relation between the instrumental variable and a corresponding stochastic discount factor and gives a related performance measure applying Chen and Knez (1996). Empirical applications of the mean-CRR spanning test are given in section 6. Section 7 concludes.

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<sup>3</sup>See the survey by DeRoos and Nijman (2001).

## 2 Coherent regular risk (CRR) measures

Artzner et al. (1999) follow the axiomatic approach to define a risk measure coherent from a regulator's point of view. They relate a risk measure to the regulatory capital requirement and deduce four axioms which should be satisfied by a "rational" risk measure. We discuss these axioms below. Let  $\mathcal{X} = L_\infty(\Omega, \mathcal{F}, P)$  be a set of (essentially) bounded real valued random variables.<sup>4</sup>

**Definition 1** *A mapping  $\rho : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$  is called a coherent risk measure if it satisfies the following conditions for all real valued random variables  $X, Y \in \mathcal{X}$ :*

- *Monotonicity: if  $X \leq Y$ , then  $\rho(X) \geq \rho(Y)$ .*
- *Translation Invariance: if  $m \in \mathbb{R}$ , then  $\rho(X + m) = \rho(X) - m$ .*
- *Positive Homogeneity: if  $\lambda \geq 0$ , then  $\rho(\lambda X) = \lambda\rho(X)$ .*
- *Subadditivity:  $\rho(X + Y) \leq \rho(X) + \rho(Y)$ .*

These axioms are natural requirements for any risk measure that reflects a capital requirement for a given risk. The monotonicity property, which, for example, is not satisfied by the variance and other risk measures based on second moments, means that the downside risk of a position is reduced if the payoff profile is increased. Translation invariance is motivated by the interpretation of the risk measure  $\rho(X)$  as a capital requirement, i.e.,  $\rho(X)$  is the amount of the capital which should be added to the position to make  $X$  acceptable from the point of view of the regulator. Thus, if the amount  $m$  is added to the position, the capital requirement is reduced by the same amount. Positive homogeneity says that riskiness of a financial position grows in a linear way as the size of the position increases. This assumption is not always realistic as the position size can directly influence risk, for example, a position can be large enough that the time required to liquidate

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<sup>4</sup> $\Omega$  is the set of states,  $\mathcal{F}$  denotes the  $\sigma$ -algebra, and  $P$  is the probability measure. Delbaen (2000) extends the definition of coherent risk measure to the general probability space  $L_0(\Omega, \mathcal{F}, P)$  of all equivalence classes of real valued random variables.

it depends on its size. Withdrawing the positive homogeneity axiom leads to a family of convex risk measures, see Föllmer and Schied (2002).<sup>5</sup> The subadditivity property, which is not satisfied by the widely implemented value-at-risk, allows one to decentralize the task of managing the risk arising from a collection of different positions: If separate risk limits are given to different desks, then the risk of the aggregate position is bounded by the sum of the individual risk limits. The subadditivity is also closely related to the concept of risk diversification in a portfolio of risky positions.

Kusuoka (2001) investigates another two conditions for coherent risk measures

- *Law Invariance:* if  $P[X \leq t] = P[Y \leq t] \forall t$ , then  $\rho(X) = \rho(Y)$ .
- *Comonotonic Additivity:* if  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are measurable and non-decreasing, then  $\rho(f \circ X + g \circ X) = \rho(f \circ X) + \rho(g \circ X)$ .

The intuition of the two axioms is simple: the Law of Invariance means that financial positions with the same probability distribution should have the same risk. This property allows identification from an empirical point of view. The second condition of Comonotonic Additivity refines slightly the subadditivity property: subadditivity becomes additivity when two positions are comonotone. In fact, Comonotonic Additivity strengthens the concept of "perfect dependence" between two random variables. Indeed, if two random variables are monotonic transformations of the same third random variable, the risk of their combination should be equal to the sum of their separate risks. A risk measure that satisfies these two axioms will be referred to as a *regular* risk measure, see Bassett et al. (2004). Coherent regular risk measures will be abbreviated as CRR measures.

A risk measure that has received considerable attention is expected shortfall,<sup>6</sup> defined as

$$s_\alpha(X) = -\alpha^{-1} \int_0^\alpha F^{-1}(t) dt, \quad (1)$$

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<sup>5</sup>However, see De Giorgi (2005) on homogenization of risk measures.

<sup>6</sup>Here we use the terminology of Acerbi and Tasche (2002). In fact, variants of this risk measure have been suggested under a variety of names, including conditional value-at-risk (CVaR) by Rockafellar and Uryasev (2000) and tail conditional expectation by Artzner et al. (1999).

where  $F$  stands for the cumulative distribution function of the random variable  $X$ . An important characterization result, modifications of which are obtained by Kusuoka (2001) and Tasche (2002), is

**Theorem 1** *A risk measure  $\rho : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$  defined on  $\mathcal{X} = L_\infty(\Omega, \mathcal{F}, P)$ , with  $P$  non-atomic, is a CRR measure if and only if it has a representation*

$$\rho(X) = \int_0^1 s_\alpha(X) d\phi(\alpha), \quad (2)$$

where  $\phi$  is a probability measure defined on the interval  $[0, 1]$ .

As it is mentioned by Tasche (2002), the class of spectral measures of risk introduced by Acerbi (2002) is equivalent to the class of CRR measures. Notice also, that a coherent regular risk (CRR) measure corresponds to a Choquet expectation over  $F^{-1}(t)$  with a concave distortion probability function.<sup>7</sup> Indeed, a Choquet expectation over  $F^{-1}(t)$  with a distortion probability  $\nu$  is

$$\rho(X) = - \int_0^1 F^{-1}(t) d\nu(t).$$

If we substitute expression (1) for expected shortfall into equation (2) the relation between the distortion probability  $\nu$  and the probability measure  $\phi$  in (2) can be found

$$\nu'(t) = \int_t^1 \alpha^{-1} d\phi(\alpha). \quad (3)$$

We call the function  $\nu'(t)$  a Choquet distortion probability density function (pdf). Since  $\phi$  is a probability measure it follows that  $\nu(t)$  has to be a concave function. Hence the probability distortion  $\nu$  acts to increase the likelihood of the least favorable outcomes, and to depress the likelihood of the most favorable ones. This is the reason why, for example, Bassett et al. (2004) call a CRR measure a pessimistic risk measure. Through the Choquet representation, CRR measures can be related to the family of non-additive, or dual, or rank-dependent uncertainty choice theory formulations of Schmeidler (1989), Yaari (1987), and Quiggin (1982).

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<sup>7</sup>This corresponds to a convex distortion in case the risk measure is defined as Choquet expectation over  $X$ , instead of  $F^{-1}(t)$ , see Bassett et al. (2004).

A nice way to approximate a CRR measure by a weighted sum of Dirac's point mass functions<sup>8</sup> was suggested by Bassett et al. (2004). The point mass function  $\delta_\tau(\alpha)$  is defined through the integral  $\int_{-\infty}^x \delta_\tau(\alpha) d\alpha = I(x \geq \tau)$ . Let  $\phi(\alpha) = \sum_{k=1}^m \phi_k \delta_{\tau_k}(\alpha)$ , with  $\phi_k \geq 0$ ,  $\sum \phi_k = 1$ , then the CRR measure in (2) can be rewritten as

$$\rho(X) = \sum_{k=1}^m \phi_k S_{\tau_k}(X). \quad (4)$$

Clearly, expected shortfall is a particular case of this approximation. We use this approximation in our empirical applications of the mean-CRR spanning test in section 6.

### 3 Mean-CRR portfolios and risk contributions

In this section we first use the CRR-measures to formulate optimal portfolio choice problems, and then we generalize the risk contribution results for the case of expected shortfall obtained by Bertsimas et al. (2004) and Tasche (1999) to general CRR measures.

Consider a portfolio of  $p$  assets whose random returns are described by the random vector  $R = (R_1, \dots, R_p)'$  having a joint density with the finite mean vector  $\mu = E[R]$ . For simplicity, assume that the joint distribution of  $R$  is continuous. Let  $\theta = (\theta_1, \dots, \theta_p)'$  be portfolio weights, so that the total random return on the portfolio is  $Z = R'\theta$  with distribution function  $F_z$ . This allows us to view a CRR measure of a portfolio as a function of portfolio weights  $\rho(\theta) = \rho(R'\theta)$ . An optimization problem for a mean-CRR efficient portfolio can now be formulated in full analogy with the mean-variance case

$$\min_{\theta \in \mathbb{R}^p} \rho(R'\theta) \text{ s.t. } \mu'\theta = m, \iota'\theta = 1 \quad (5)$$

where  $m$  is the required expected portfolio return and  $\iota$  is a  $p \times 1$  vector of ones.

The fact that a CRR measure can be written as a Choquet expectation over  $F^{-1}(t)$  with a concave distortion function  $\nu$  (or, equivalently, as a

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<sup>8</sup>Notice, that such an approximation also corresponds to a piecewise linear approximation of the concave probability distortion function  $\nu$  in the Choquet expectation.

Choquet expectation over  $Z$  with a convex distortion function), means that the optimization problem (5) is isotonic with second order degree stochastic dominance, see, for instance, De Giorgi (2005). In combination with the empirical identifiability (due to the law invariance condition), makes optimal mean-CRR portfolio choice attractive, both from a theoretical and an empirical point of view. Moreover, as explained in Bassett et al. (2004), a CRR measure can be approximated by a finite sum of expected shortfalls. A sample analog of a mean-CRR problem with this finite sum approximation can be reformulated as a linear program and efficiently solved, see Portnoy and Koenker (1997), Rockafellar and Uryasev (2000), and Polbennikov and Melenberg (2005), making mean-CRR optimal portfolio choice also practically feasible. In summary, a CRR measure is a natural choice for a risk measure in case of a portfolio choice based on a mean-risk trade-off. In the companion paper, Polbennikov and Melenberg (2005), we also derive the asymptotic distribution of the mean-CRR portfolio weights  $\theta$  and consider special cases of a point mass approximation of a CRR measure and expected shortfall.

In the remainder of this section, we consider the risk contribution results obtained by Bertsimas et al. (2004) and Tasche (1999) for the case of expected shortfall and generalize them to a general CRR measure. This result, being interesting by itself,<sup>9</sup> is needed for the mean-CRR spanning test, which is to follow. An alternative expression for the risk contribution of a CRR measure is obtained by Tasche (2002).

**Proposition 1** *If the distribution of the returns  $R$  has a continuous density, then the CRR contributions of assets in  $R$  are given by the gradient vector*

$$\nabla_{\theta}\rho(\theta) = -E \left[ R \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) \right]. \quad (6)$$

**Proof.** See Appendix A. ■

The second proposition gives the expression for the Hessian of a CRR measure. This result is a generalization of the expression given in Bertsimas et al. (2004) for expected shortfall.

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<sup>9</sup>One can interpret risk contributions as an amount of required capital for a particular asset in the portfolio.

**Proposition 2** *If the distribution of the returns  $R$  has a continuous density, then the Hessian of a CRR measure is given by the matrix*

$$\nabla_{\theta}^2 \rho(\theta) = E \left[ \frac{\phi'(F_z(Z)) f_z(Z)}{F(Z)} \text{Cov}(R|Z) \right], \quad (7)$$

where  $f_z$  is the probability density function of the portfolio return  $Z$ .

**Proof.** See Appendix B. ■

Note that (7) implies the convexity of a CRR measure  $\rho(\theta)$  because the conditional covariance matrix  $\text{Cov}(R|Z)$  is positive semi-definite and the other terms are positive. This means that the mean-CRR portfolio problem (5) is well defined.

## 4 Mean-CRR spanning test

In this section we present the mean-CRR-spanning test. First, Tasche (1999) shows that an analog of the two fund separation theorem holds for a  $\tau$ -homogeneous risk measure satisfying certain regularity conditions. In particular, the differentiability of the risk measure with respect to the asset weights at the optimal point is required, see the discussion in Tasche (1999). A risk measure  $\rho(X)$  is called  $\tau$ -homogeneous if for any  $t > 0$  it satisfies  $\rho(tX) = t^\tau \rho(X)$ . The CRR measure is a homogeneous risk measure of degree one. Any  $\tau$ -homogeneous risk-efficient portfolio can be represented as a linear combination of the risk-free asset (assumed to be present) and a *risk-market portfolio*. The risk-market portfolio  $Z = R'\theta^*$  can be characterized by the maximal Sharpe-risk ratio, so that the following relation holds:

$$\mu - \iota r_f = \frac{\mu_z - r_f}{\tau \rho((R - \mu)'\theta^*)} \nabla_{\theta} \rho((R - \mu)'\theta^*),$$

where  $r_f$  is the risk-free rate,  $\mu$  is the vector of the expected returns,  $\mu_z$  is the expected return of the risk-market portfolio  $Z$ , and  $\iota$  is a vector of ones. Notice, that this relation for the risk-efficient portfolio includes the risk contribution vector  $\nabla_{\theta} \rho((R - \mu)'\theta^*)$ . Using equation (6) we obtain the following expression for risk contributions entering the characterization of the risk-market portfolio.

$$\nabla_{\theta} \rho((R - \mu)'\theta^*) = -E \left[ (R - \mu) \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) \right] = -\text{Cov}(R, \nu'(F(Z))),$$

where  $\nu'(F_z(s)) = \int_{F_z(s)}^1 \alpha^{-1} d\phi(\alpha)$  is the Choquet distortion probability density function. Thus, the characterization of an efficient portfolio for a CRR measure (2) becomes

$$\mu - r_f = \frac{\text{Cov}(R, \nu'(F_z(Z)))}{\text{Cov}(Z, \nu'(F_z(Z)))} (\mu_z - r_f). \quad (8)$$

This expression says that the expected excess return on any asset in a CRR market portfolio is proportional to the expected excess return of the CRR market portfolio with the coefficient proportional to the covariance between the asset return and the distorted cumulative distribution function of the risk-market portfolio  $Z$ . This characterization can be used for a spanning test. For expositional simplicity we derive the spanning test for a single asset, potentially to be included in the portfolio under consideration. The extension to the multiple asset case is straightforward.

Let  $Y$  be a random return of an asset for which we want to perform a spanning test. Denote by  $\mu_y$  its expected return. Under the spanning hypothesis this asset is redundant for the portfolio, i.e., its weight in the portfolio is zero. This means that under the spanning hypothesis the CRR-market portfolio  $Z$  does not change. Clearly, the characterization (8) should hold. It is straightforward to see that the relation (8) can be reformulated in terms of the semi-parametric instrumental variable (IV) regression

$$Y_i^e = \alpha + \beta Z_i^e + \epsilon_i, \quad (9)$$

$$E[\epsilon_i] = 0, \quad (10)$$

$$E[V_i \epsilon_i] = 0, \quad (11)$$

where  $Y_i^e = Y_i - r_f$ ,  $Z_i^e = Z_i - r_f$ , and  $V_i = \nu'(F_z(Z_i))$  is the semi-parametric instrument, which depends on the distribution  $F_z$  of the optimal portfolio return  $Z$ . The restriction imposed by the spanning hypothesis on the regression (9) is

$$\alpha = 0,$$

$$\beta \text{Cov}(Z, V) - \text{Cov}(Y, V) = 0.$$

Thus, the mean-CRR spanning test is a test on significance of the intercept parameter  $\alpha$  in the semi-parametric IV regression (9). Denote by  $W_i =$

$(1, V_i)'$  the two instruments of (9), and by  $X_i = (1, Z_i^e)'$  the regressors. Then, the IV estimator is given by

$$\hat{\gamma} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \left( \frac{1}{n} \sum_{i=1}^n \widehat{W}_i X_i' \right)^{-1} \frac{1}{n} \sum_{i=1}^n \widehat{W}_i Y_i^e. \quad (12)$$

where  $\widehat{W}$  stands for a non-parametric estimation of the instrumental variable  $W$ , which depends on the Choquet distortion pdf  $\nu'(F_z(Z))$ . The estimation of this functional is straightforward

$$\nu'(F_n(s)) = \int_{F_n(s)}^1 \alpha^{-1} d\phi(\alpha),$$

where  $F_n(s)$  is a consistent estimator of  $F_z$ .<sup>10</sup> Notice, that the methods developed by Newey (1994) to derive the asymptotic variance of a semi-parametric estimator are fully applicable to our semi-parametric IV case. The asymptotic distribution of the parameters can be determined (under appropriate regularity conditions) by

$$\sqrt{n}(\hat{\gamma} - \gamma) = \left( \frac{1}{n} \sum_{i=1}^n \widehat{W}_i X_i' \right)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{W}_i \epsilon_i + o_p(1), \quad (13)$$

We consider two cases. First, we ignore the estimation inaccuracy in the CRR market portfolio weights. This corresponds to the case where we assume a certain traded portfolio to be the CRR market portfolio, for example, the S&P 500 index. Then we consider the case when the estimation inaccuracy in the CRR market portfolio weights is taken into account. This corresponds, for instance, to the case where we want to test whether some chosen portfolio, likely based on estimated mean returns and probably some optimal criterion, is indeed optimal from the point of view of mean-CRR efficiency.

## 4.1 Spanning for a given CRR efficient portfolio

Suppose that the returns of the CRR market portfolio are observable, i.e, we do not need to take into account estimation inaccuracy in the CRR portfolio

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<sup>10</sup>For example, the usual empirical distribution function  $F_n(s) = n^{-1} \sum_{i=1}^n I(Z_i \leq s)$  can be used.

weights. Applying the Law of Large Numbers and the Central Limit Theorem to (13), we obtain

$$\frac{1}{n} \sum_{i=1}^n \widehat{W}_i X_i' \rightarrow_p E [W_i X_i'] = G, \quad (14)$$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \widehat{W}_i \epsilon_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(Z_i, \epsilon_i) + o_p(1) \rightarrow_d N(0, E[\psi\psi']), \quad (15)$$

where  $\psi = (\psi_1(Z, \epsilon), \psi_2(Z, \epsilon))'$  is a  $2 \times 1$  vector with the components  $\psi_1$  and  $\psi_2$  being the influence functions of the functionals<sup>11</sup>

$$\begin{aligned} \phi_1(F) &= \int \epsilon dF(Z, \epsilon), \\ \phi_2(F) &= \int \epsilon \nu'(F_z(Z)) dF(Z, \epsilon). \end{aligned}$$

The influence function of the first functional  $\phi_1(F)$  is obvious. The influence function of the functional  $\phi_2(F)$  is derived in the Appendix. The results are

$$\psi_1(Z, \epsilon) = \epsilon, \quad (16)$$

$$\psi_2(Z, \epsilon) = \chi(Z, \epsilon) - E[\chi(Z, \epsilon)], \quad (17)$$

where

$$\chi(Z, \epsilon) = \int_{F_z(Z)}^1 (\epsilon - E[\epsilon | Z = F_z^{-1}(\alpha)]) \alpha^{-1} d\phi(\alpha). \quad (18)$$

Finally, the asymptotic result for the semi-parametric IV estimator in (13) is

$$\sqrt{n}(\widehat{\gamma} - \gamma) \rightarrow_d N(0, G^{-1} E[\psi\psi'] G'^{-1}),$$

with the components of the influence function  $\psi$  given in equations (16), (17), and (18). The asymptotic distribution of the intercept  $\alpha$  is

$$\sqrt{n}(\widehat{\alpha} - \alpha) \rightarrow_d N(0, [G^{-1} E[\psi\psi'] G'^{-1}]_{11}), \quad (19)$$

where the sub-index 11 stands for the (1,1)-component of the asymptotic covariance matrix of the semi-parametric IV estimator.

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<sup>11</sup> $W\epsilon = (\epsilon, V\epsilon)'$ .

The mean-CRR spanning test is equivalent to the significance test of the intercept coefficient. Notice, that this result is close in spirit to the mean-variance spanning test developed by Huberman and Kandel (1987). They propose to test the mean-variance spanning by means of a significance test on the intercept coefficient in an OLS regression similar to (9), but with a mean-variance market portfolio excess return  $Z^e$  instead of the CRR one.

#### 4.1.1 Example: Expected Shortfall

A particular CRR measure which has recently received a lot of attention is expected shortfall  $s_\tau(X)$ , defined in (1). It is well known that a sample analog of a mean-expected shortfall portfolio problem can be reformulated as a linear program and solved efficiently, see Bertsimas et al. (2004) and Bassett et al. (2004). Our results immediately yield the mean-expected shortfall spanning test. We start with the instrumental variable  $V$ , which is used to estimate regression (9):

$$V = \Gamma_F(Z) = \tau^{-1} I(F_z(Z) \leq \tau).$$

The function  $\chi(Z, \epsilon)$  in (18) becomes

$$\chi(Z, \epsilon) = \tau^{-1} (\epsilon - E[\epsilon | Z = F_z^{-1}(\tau)]) I(F_z(Z) \leq \tau).$$

The result for the mean-expected shortfall spanning test is immediately obtained by means of equation (19) with

$$G = E \begin{bmatrix} 1 & Z^e \\ V & Z^e V \end{bmatrix}, \quad (20)$$

and

$$E[\psi\psi'] = \begin{bmatrix} \text{var}(\epsilon) & \text{cov}(\epsilon, \chi) \\ \text{cov}(\epsilon, \chi) & \text{var}(\chi) \end{bmatrix}. \quad (21)$$

An interesting observation is that in case of expected shortfall the components  $\text{var}(\chi)$  and  $\text{cov}(\epsilon, \chi)$  are mainly determined by the usual IV part  $\tau^{-1}\epsilon I(F_z(Z) \leq \tau)$  of the function  $\chi$ . This is because the non-parametric adjustment is effectively constant. The shift which appears at the  $\tau$  quantile brings a negligible correction to the covariance matrix  $E[\psi\psi']$ . This means that, when performing a usual IV inference without taking into account the non-parametric adjustment, one only makes a very small error.

### 4.1.2 Example: CRR point mass approximation

As suggested by Bassett et al. (2004), one can approximate a CRR measure (2) by taking a point mass probability distribution on the interval  $[0, 1]$ . In this case the exogenous probability  $\phi(\alpha)$  in the definition (2) becomes

$$\phi(\alpha) = \sum_{k=1}^m \phi_k I(\alpha \geq \tau_k),$$

where the weights  $\phi_k$  sum up to one. A point mass approximation (PMA) of a CRR measure becomes a weighted sum of expected shortfalls

$$\rho(Z) = \sum_{k=1}^m \phi_k s_{\tau_k}(Z).$$

As shown in Polbennikov and Melenberg (2005), a sample analog of a the mean-PMA CRR portfolio problem can be reformulated as a linear program and efficiently solved with existing numerical algorithms. The spanning test results of this section are applicable for the mean-PMA CRR spanning as well. The instrumental variable  $V$  of regression (9) becomes

$$V = \Gamma_F(Z) = \sum_{k=1}^m \phi_k \tau_k^{-1} I(F_z(Z) \leq \tau_k).$$

The function  $\chi(Z, \epsilon)$  in expression (17) for the influence function of the functional  $\phi_2(F) = E[\epsilon V]$  becomes

$$\chi(Z, \epsilon) = \sum_{k=1}^m \phi_k \tau_k^{-1} I(F_z(Z) \leq \tau_k) (\epsilon - E[\epsilon | Z = F_z^{-1}(\tau_k)]).$$

The spanning test, equivalent to the significance test of the intercept in the IV regression (9), is performed by means of equation (19) with expressions for  $G$  and  $E[\psi\psi']$  given in (20) and (21), respectively.

## 4.2 Estimation inaccuracy in market portfolio weights

The Mean-CRR spanning test (19) obtained in subsection (4.1) ignores the potential estimation inaccuracy in the weights of the CRR market portfolio  $Z$ . This is reasonable if one wants to test a CRR version of the Capital

Asset Pricing Model (CAPM) with a given market index as a CRR market portfolio. Alternatively, one could form *a priori* beliefs about the portfolio weights, so that they are not considered as having estimation inaccuracy. In this section we discuss an adjustment required to the limit distribution (19) of the intercept coefficient  $\alpha$  of the IV regression (9) in the case one also wants to take into account the error resulting from the estimation of the market portfolio weights. Our setup is quite general, as we consider an investor who wants to test his/her portfolio for CRR optimality, but whose portfolio is determined by solving some (arbitrary) optimization problem.

In principle, an alternative approach to test for mean-CRR spanning would be a straightforward significance test for the weight of the new asset in the market efficient portfolio. However, to implement this test one needs to re-derive the whole CRR market portfolio with the new asset included. This approach is similar in spirit to the mean-variance spanning test of Britten-Jones (1999). In this paper, however, we would like to separate the estimation of the market portfolio and the test for mean-CRR spanning for new candidate assets. The advantage is that one does not need to re-derive the market portfolio weights every time a new spanning test needs to be performed. All we need are asset returns and weights of the "old" market portfolio, which need to be derived only once.

Suppose, that the limit distribution of the market efficient portfolio weights  $\hat{\theta}$  resulting from the solution of an optimization problem<sup>12</sup> is characterized by an influence function  $\xi(R^e, Z)$ , i.e.,

$$\begin{aligned}\sqrt{n}(\hat{\theta} - \theta) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi(R_i^e, Z_i) + o_p(1), \\ E\xi &= 0, \quad E\xi\xi' < \infty,\end{aligned}$$

where  $R^e$  is a vector of asset returns in excess of the risk free rate  $r_f$ . The result (15) has to be adjusted in a straightforward way to take into account the estimation inaccuracy in the portfolio weights

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n W_i \epsilon_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi(Z_i, \epsilon_i) + \nabla_{\theta} E \begin{bmatrix} \epsilon \\ \epsilon \nu'(F_z(Z)) \end{bmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi(R_i^e, Z_i) + o_p(1).$$

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<sup>12</sup>In the Appendix D we consider the case of the mean-CRR portfolio weights.

It is straightforward to show that

$$M \equiv \nabla_{\theta} E \begin{bmatrix} \epsilon \\ \epsilon \nu'(F_z(Z)) \end{bmatrix} = -\beta E \begin{bmatrix} R^{e'} \\ R^{e'} \nu'(F_z(Z)) \end{bmatrix}.$$

Given the expressions for the components of the vector  $\psi(Z, \epsilon)$  provided in (16), (17), and (18) we obtain

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n W_i \epsilon_i = \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta(R_i^e, Z_i, \epsilon_i) + o_p(1) \equiv \begin{bmatrix} I_2 & M \end{bmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{bmatrix} \psi(Z_i, \epsilon_i) \\ \xi(R_i^e, Z_i) \end{bmatrix} + o_p(1),$$

with limit distribution

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n W_i \epsilon_i \rightarrow_d N(0, E[\zeta \zeta']).$$

Finally, the spanning test result (19) becomes

$$\sqrt{n}(\hat{\alpha} - \alpha) \rightarrow_d N(0, [G^{-1} E[\zeta \zeta'] G'^{-1}]_{11}). \quad (22)$$

The last step that remains is to find the influence function  $\xi(R^e, Z)$  of the estimated market portfolio weights  $\hat{\theta}$ . We report the relevant formulas for a mean-CRR market portfolio in the Appendix D, referring for the derivation details to the paper by Polbennikov and Melenberg (2005). The considered cases are mean-CRR, with as special cases mean-expected shortfall, and mean-PMA CRR.

## 5 Stochastic discount factor, instrumental variables, and performance measurement

In this section we demonstrate that, if considered as a pricing model, system (9) implies a linear relation between a stochastic discount factor (that can be used to price the assets) and the instrumental variable  $V$ . From the perspective of the mean-CRR portfolio this can be interpreted as a model of general equilibrium where the portfolio choices are based on the mean-CRR optimization. In this case the instrumental variable  $V$  is given by the Choquet distortion probability density function  $\nu'(F_z(Z))$ . Alternatively,

there could be an investor who makes his/her portfolio choice according to mean-CRR optimization. In this case, the assets in the portfolio should satisfy

$$\begin{aligned} R^e &= \beta Z^e + \epsilon, \\ E[\epsilon] &= 0, \quad E[V\epsilon] = 0, \end{aligned}$$

and the stochastic discount factor should be an affine function of the instrumental variable  $V$ , which can then be interpreted as the single risk factor. Notice, however, that this single risk factor is not a return on a portfolio. This means that we cannot construct a simple test of a zero intercept in a linear regression equation of the excess return  $R^e$  on the (non-existing) excess return " $V^e$ ". Instead, our spanning test, based on a linear regression but with an instrumental variable, allows one to perform a zero intercept test.

The general statement regarding the stochastic discount factor and the instrumental variable  $V$  is as follows.

**Proposition 3** *Suppose that the asset excess returns satisfy*

$$R^e = \beta Z^e + \epsilon, \tag{23}$$

$$E[\epsilon] = 0, \tag{24}$$

$$E[V\epsilon] = 0, \tag{25}$$

where  $Z^e$  is a global market factor, and  $V$  is the global market instrumental variable. Then

$$m = \frac{1}{r_f \text{Cov}(Z^e, V)} (E[Z^e V] - E[Z^e]E[V]) \tag{26}$$

is a valid stochastic discount factor.

**Proof.** We need to show that for any (relevant) asset return  $R$ , the pricing equation  $E[mR] = 1$  is satisfied. Notice that from the stated version of the modified CAPM model it follows that  $R = r_f + \beta Z^e + \epsilon$ . Then, substituting the expression (26) for the stochastic discount factor  $m$ , we obtain

$$\begin{aligned} E[mR] &= \frac{r_f}{r_f \text{Cov}(Z^e, V)} (E[Z^e V] - E[Z^e]E[V]) + \\ &\quad \frac{\beta}{r_f \text{Cov}(Z^e, V)} (E[Z^e V]E[Z^e] - E[Z^e]E[VZ^e]) = 1 \end{aligned}$$

■

The mean-CRR portfolio model (5) implies a specific choice of the instrumental variable  $V$  in (26), namely

$$V = \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha).$$

As we have shown, the stochastic discount factor  $m$  should be an affine function of this instrument. This means that the proposed spanning test (19) can also be viewed as a test for the validity of a model for the stochastic discount factor in (26).

Similar results are obtained by De Giorgi and Post (2004), who show in a general equilibrium framework with only mean-CRR investors and atomic probability distributions of asset returns, that the stochastic discount factor characterizing the equilibrium is a function of the market portfolio return with the exact shape of the stochastic discount factor determined by the probability distortion function.

Given the stochastic discount factor in (26) valid for returns satisfying (23)-(25), we can introduce a performance measure, following Chen and Knez (1996), for returns not yet marketed according to this stochastic discount factor. This performance measure is defined as  $kE[m(R - R^{ref})]$  with  $R$  a non-marketed return,  $R^{ref}$  an already marketed return, satisfying conditions (23)-(25), and  $k$  some constant. Straightforward calculations show that, in case one chooses  $k = r_f$ , the performance measure equals the intercept  $\alpha$  of the IV regression (9)-(11). This yields an alternative interpretation for the spanning test, comparable to Jensen's  $\alpha$  and its relationship with mean-variance spanning tests.

## 6 Empirical examples

### 6.1 Testing the world capital index for market efficiency

In this subsection we consider an application of the mean-CRR spanning test to capital market indexes of different countries. In particular, we test

the Morgan Stanley World Capital Index for mean-expected shortfall and mean-point mass approximated (PMA) CRR market efficiency with respect to inclusion of individual country indexes. This exercise is similar in spirit to Cumby and Glen (1990), who test the world index for mean-variance efficiency using the mean-variance spanning test. The data is available from Thomson Datastream. In our analysis we use the Morgan Stanley World Capital Market Index, individual country indexes denominated in local currencies, and currency exchange rates. The countries in the data set are divided into four groups based on geography and development level: American developing economies, Asian developing economies, European developing economies, and OECD countries. In the category of American developing countries we consider Argentina (ARG), Brazil (BRA), Chile (CHIL), Peru (PER), Mexico (MEX), and Venezuela (VEN). The group of Asian developing economies includes China (CHI), India (INDIA), Indonesia (INDO), Malaysia (MAL), Pakistan (PAK), Philippines (PHIL), Sri-Lanka (SRIL) and Thailand (THAIL). The Czech Republic (CZE), Hungary (HUN), Poland (POL), Romania (ROM), Russia (RUS), and Turkey (TURK) are the European developing economies. Finally, Australia (AU), Canada (CAN), The Euro zone (EU), Japan (JAP), South Korea (KOR), the United Kingdom (UK) and the United States (US) constitute the OECD group. As we want to exclude the effects of the Asian and Russian crisis (August 1998) on the world capital markets, we consider the time period from January 3, 1999 to May 12, 2005. We use daily US dollar index returns for our analysis. The US one month interbank rate is taken as a risk-free interest rate.

Table 1 shows descriptive sample statistics of the country index returns. The empirical return distributions are typically skewed and fat tailed.

### **TABLE 1 HERE**

Table 2 shows the result of the world index (WRLD) efficiency tests. The table reports annualized regression intercept coefficients, which can be interpreted as performance measures (see discussion in section 5), and significance levels of the mean-variance (MV), mean-expected shortfall (ShF), and the mean-PMA CRR (PMA) market efficiency tests with respect to inclusion of individual country indexes. The expected shortfall probability threshold

is chosen to be 5%, while probability thresholds for PMA CRR are taken at the levels of 5%, 10%, 15%, 20%, and 25% with equal weights of 20%. Significance levels of joint spanning tests for inclusion of country groups as a whole are reported in the table as well.

## TABLE 2 HERE

We see that the market efficiency tests with different risk measures (variance, expected shortfall, and PMA CRR) lead to similar conclusions in terms of both performance measures and significance levels. In most cases the market efficiency of the WRLD index cannot be rejected at the usual significance levels. A strong rejection of the efficiency hypothesis is observed for Mexico, Romania, Russia, and Canada (5% significance level). The capital indexes of these countries significantly outperform the world market index at the 5% significance level. Indeed, Russian and Romanian markets have shown a significant growth over the past decade. The spanning hypothesis is also rejected for Pakistan at the 10% significance level.

The performance  $\alpha$ 's reported in the table for the mean-variance efficiency tests are, in fact, Jensen's  $\alpha$ 's. The rest of the performance measures (for mean-CRR efficiency) have a similar interpretation. The results of Table 2 indicate that Romanian and Russian capital market indexes outperform the world market index approximately by 36% and 68%, respectively. Moreover, this outperformance is statistically significant at the 1% significance level.

The fact that the mean-CRR spanning tests perform at similar significance levels with the mean-variance spanning tests is encouraging. It shows that the mean-CRR spanning tests for country indexes work reasonably well. Moreover, for a moderate levels of skewness and kurtosis in the index return distributions the different risk measures are statistically equivalent and can be used interchangeably. This is in line with findings in Polbennikov and Melnberg (2005) who perform a systematic comparison of the mean-variance and the mean-CRR approaches in portfolio management. De Giorgi and Post (2004), who test mean-variance and mean-CRR efficiency of the US market index for the beta and momentum portfolios, obtain similar results: the mean-variance and mean-CRR performance measures as well as the test

statistics are comparable.<sup>13</sup>

## 6.2 Testing for mean-CRR spanning in portfolios of credit instruments

Our second example concerns portfolios of credit instruments. In particular, we consider collateralized debt obligations (CDO) as elementary entries of the portfolio. This example is chosen for two reasons. First, since CDO return distributions are not symmetric the mean-variance and mean-CRR market efficient portfolios are likely to be different. As a result, the outcomes of the spanning tests might be different as well. Second, CDO tranches are becoming very popular financial instruments among investors, for example, hedge funds, insurance companies, etc. The past several years have seen an increasingly growing market for CDO tranches. This means that the problem of finding an optimal portfolio of CDOs is relevant for practical applications. The mean-variance approach might not be good idea in this case due to significantly asymmetric returns.

A collateralized debt obligation (CDO) is a structure of fixed income securities whose cash flows are linked to the incidence of default in a pool of debt instruments. These debts may include loans, emerging market corporate or sovereign debt, and subordinate debt from structured transactions. The fundamental idea behind a CDO is that one can take a pool of defaultable bonds or loans and issue securities whose cash flows are backed by the payments due on the loans or bonds. Using a rule for prioritizing the cash flow payments to the issued securities, it is possible to redistribute the credit risk of the pool of assets to create securities with a variety of risk profiles. In our example we consider the simplest case of investing in securities linked to the total pool of the underlying debt, while receiving a fixed interest payment in exchange.

In the industry the analysis of CDOs is usually exclusively based on theoretical models. This is due to the fact that historical data on defaults, and especially joint defaults, is very sparse. Another reason is that the speci-

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<sup>13</sup>The test by De Giorgi and Post (2004), however, seems to ignore the effect of the non-parametric estimation of the pricing kernel.

fication of the full joint default probabilities is too complex: for example, for a CDO with 50 obligors there are  $2^{50}$  joint default events. CDO models differ in their complexity: while some of them admit analytical solutions for loss distribution functions, others require Monte-Carlo simulation techniques. However, as soon as one wants to construct an optimal mean-risk portfolio from several CDOs, no closed form solution is usually available. Therefore, a Monte-Carlo simulation is the only alternative. In our example, we use a simple one factor large homogeneous portfolio model to construct the return distributions of the CDOs.<sup>14</sup> Here we briefly outline the model.

The model assumes that a portfolio of loans consists of a large number of credits with the same default probability  $p$ . In addition, it is assumed that the default of a firm (obligor) is triggered when the normally distributed value of its assets  $V_n(T)$  falls below a certain level  $K$ . Without loss of generality we can standardize the developments of the firm values such that  $V_n(T) \sim N(0, 1)$ . In this case the default barrier level is the same for all obligors and equals  $K = \Phi^{-1}(p)$ . To introduce a default correlation structure it is assumed that the firm values are driven by a factor model

$$V_n(T) = \sqrt{\varrho}Y + \sqrt{1 - \varrho}\epsilon_n,$$

where  $Y$  is the systematic factor for all obligors in the pool of credits, and  $\epsilon_n$  is the idiosyncratic risk of a firm. The higher the correlation coefficient  $\varrho$ , the higher the probability of a joint default in the pool. Notice that, conditional on the factor  $Y$ , defaults are independent. The individual default probability conditional on the realization  $y$  of the systematic factor  $Y$  is

$$p(y) = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\varrho}y}{\sqrt{1 - \varrho}}\right).$$

Conditional on the realization  $y$  of  $Y$ , the individual defaults happen independently from each other. Therefore, in a very large portfolio, as we assume to be the case, the law of large numbers ensures that the fraction of obligors that actually defaults is almost surely equal to the individual default probability.

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<sup>14</sup>We use a simplified form of the firm's value model due to Vasicek (1997). Similar approach is used in Belkin et al. (1998) and Finger (1999).

For purposes of our analysis we simulate returns of three CDOs using the described one factor model. The steps that we take are as follows:

- We simulate 10,000 realizations of three factors  $(y_{1i}, y_{2i}, y_{3i})$  from the three-variate standard normal distribution with the identity correlation matrix.<sup>15</sup>
- From the simulated factors we generate fractions of obligors that actually default in the pool  $j = \{1, 2, 3\}$  using the formula

$$x_{ji} = \Phi \left( \frac{\Phi^{-1}(p_j) - \sqrt{\varrho_j} y_{ji}}{\sqrt{1 - \varrho_j}} \right)$$

with individual default probabilities  $p_j, j = \{1, 2, 3\}$  of 2.5%, 5%, and 7.5%; and default correlations  $\varrho_j, j = \{1, 2, 3\}$  of 0.15, 0.1, and 0.05.

- Finally, for each CDO  $j$  we obtain the returns  $R_{ji}$

$$R_{ji} = (1 + r_j)(1 - x_{ji}) - 1,$$

where  $r_j$  is the risk premium for holding pool  $j$  of defaultable obligors. We choose these risk premiums to be 4%, 10%, and 12%, correspondingly.

Even though the parameter choice in our simulation may seem ad-hoc, there are two reasons which make it plausible for a realistic situation. First, depending on the credit rating and the investment horizon, individual default probabilities can vary in a wide range from 0.00% (for one year default probability of an Aaa rated company) to almost 50% (for ten years default probability of a B rated company), according to Moody's, see Table 3 with historical cumulative default probabilities for the period 1970-2004. The default probabilities that we choose fall in this range. Second, it is possible to redistribute the credit risk of the pool of assets to create securities with a variety of risk profiles, which makes many possible combinations of parameters justified.

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<sup>15</sup>In principle, it is possible to make returns on the 3 CDOs dependent by introducing positive or negative correlations among the factors.

### TABLE 3 HERE

Table 4 shows descriptive statistics of the simulated returns of the three CDOs. The distributions of the returns are substantially skewed and fat tailed. The CDO with the smallest default correlation among obligors is the closest to the normal distribution.

### TABLE 4 HERE

From the simulated credit pool returns we construct three market portfolios:<sup>16</sup> mean-variance (MV), mean-expected shortfall (ShF) and mean-PMA CRR (PMA). In addition, we consider returns of CDO1 hypothesizing its market efficiency. The probability threshold for expected shortfall is chosen to be 5%. The probability thresholds for PMA CRR measure are chosen to be 5%, 10%, 15%, 20%, and 25% with equal weights of 20%. For these four portfolios (CDO1, MV, ShF, and PMA) we perform mean-variance, mean-expected shortfall, and mean-CRR PMA spanning tests with respect to inclusion of CDO2 and CDO3. Table 5 reports significance levels of these tests.

### TABLE 5 HERE

The results indicate a statistical difference between mean-variance and mean-CRR market portfolios. For the mean-CRR market portfolios (Mkt. ShF and Mkt. PMA), the mean-variance spanning tests result in strong rejection. At the same time, for the mean-variance market portfolio (Mkt. MV) mean-CRR spanning tests result in rejection as well. The difference between the mean-expected shortfall market portfolio (Mkt. ShF) and the mean-PMA CRR market portfolio (Mkt. PMA) with respect to the inclusion of CDO2 and CDO3 turns out to be significant as well.

In this exercise the mean-variance and the mean-CRR spanning tests do not produce similar results any more. The reason is the asymmetrically distributed returns. Skewness of the returns makes the variance a bad risk measure from the point of view of a CRR investor. Therefore, the mean-variance optimal portfolio is not recognized as a mean-CRR efficient one by

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<sup>16</sup>We assume a zero risk-free rate.

the mean-CRR spanning test. This exercise demonstrates the applicability of the mean-CRR spanning test to portfolios of credit instruments or other portfolios with comparable characteristics. It shows that the correct choice of the risk measure becomes increasingly important for assets with asymmetric returns.

## 7 Conclusion

In this paper we consider coherent regular risk (CRR) measures as an alternative to the conventional variance in a mean-risk optimal portfolio problem. Following trends in the recent literature on expected shortfall we derive useful properties of CRR measures. In particular, expressions for risk contributions and the Hessian of a CRR measure are obtained.

Our main contribution is the regression-based test for mean-CRR spanning. We show that this test can be performed in the spirit of Huberman and Kandel (1987) as a significance test of the intercept coefficient in a semi-parametric instrumental variable regression. The instrument in this regression is a functional, depending on a certain choice of the CRR measure.

We derive the limit distribution of the regression intercept coefficient to test for mean-CRR spanning. The resulting asymptotic covariance matrix is the variance of the usual IV estimator with an adjustment for the non-parametric part. In case of mean-expected shortfall or mean-PMA CRR portfolios this adjustment is likely to be negligible so that the non-parametric part can be ignored. Further, we illustrate how the estimation error in the mean-CRR portfolio weights can be incorporated in the spanning test.

The instrumental variable in the semi-parametric IV regression can be used to construct a stochastic discount factor of a CRR version of the CAPM. In particular, we show that this stochastic discount factor is an affine function of this instrumental variable. This allows for an alternative interpretation of our spanning test in terms of a performance measure similar in spirit to the way the performance measure Jensen's  $\alpha$  is related to the mean-variance spanning test.

Finally, as an empirical application, we use the mean-CRR spanning test to test for CRR efficiency of the world capital market index. In particular,

we test for mean-expected shortfall and mean-PMA CRR efficiency with respect to the inclusion of individual country indexes. We find that the mean-CRR and mean-variance spanning tests produce similar significance levels. In addition we consider spanning tests for simulated returns of simplistic CDOs. We show that due to asymmetry of the return distributions mean-variance and mean-CRR spanning tests produce statistically different results.

## A Proof of proposition 1

First, notice that expected shortfall of the portfolio return  $Z$  can be expressed as

$$s_\alpha(Z) = -\alpha E [ZI(F_z(Z) \leq \alpha)],$$

where  $I(A)$  is the usual indicator function. This means that a CRR measure of the portfolio  $Z$  is

$$\begin{aligned} \rho(\theta) &= -\int_0^1 \alpha^{-1} E [ZI(F_z(Z) \leq \alpha)] d\phi(\alpha) \\ &= -E \left[ Z \int_0^1 \alpha^{-1} I(F_z(Z) \leq \alpha) d\phi(\alpha) \right] \\ &= -E \left[ Z \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) \right]. \end{aligned}$$

The distribution function  $F_z(\cdot)$  is continuously differentiable with respect to portfolio weights  $\theta$  since the distribution of the returns  $R$  has a continuous density. Therefore, we can calculate the risk contributions of a CRR measure in a straightforward way. Notice, that portfolio  $Z = R'\theta$  and its distribution function  $F_z$  depend on the portfolio weights  $\theta$ . Then, applying the chain rule to the expression for a CRR measure  $\rho(\theta)$ , we obtain

$$\begin{aligned} \nabla_\theta \rho(\theta) &= -\nabla_\theta E \left[ Z \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) \right] \\ &= -E \left[ R \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) - Z \frac{\phi'(F_z(Z))}{F_z(Z)} (f_z(Z)R + \nabla_\theta F_z(s)|_{s=Z}) \right]. \end{aligned} \tag{27}$$

To finish the derivation we need to calculate the gradient  $\nabla_\theta F_z(s)$ . It suffices to derive only component  $j$  of this vector, the rest being analogous. Denote

by  $\theta_j$  the portfolio weight of asset  $R_j$  and by  $\theta_{-j}$  the vector of portfolio weights of the rest of the assets, which we denote by  $R_{-j}$ . Further, let  $Z_{-j} = R'_{-j}\theta_{-j}$  be the portfolio of assets  $R$  excluding asset  $j$ . Denote by  $F_{z_{-j}|R_j}$  and  $f_{z_{-j}|R_j}$  the conditional probability and density functions of return  $Z_{-j}$  conditional on return  $R_j$ . Then we can express the cumulative probability function  $F_z$  of portfolio  $Z$  through the expectation of the conditional probability  $F_{z_{-j}|R_j}$

$$\begin{aligned} F_z(s) = E[I(R'\theta \leq s)] &= E[E[I(R'_{-j}\theta_{-j} \leq s - R_j\theta_j) | R_j]] \\ &= E[F_{z_{-j}|R_j}(s - R_j\theta_j)]. \end{aligned}$$

Now the calculation of the derivative of  $F_z(s)$  with respect to weight  $\theta_j$  is straightforward

$$\begin{aligned} \frac{\partial F_z(s)}{\partial \theta_j} &= -E[f_{z_{-j}|R_j}(s - R_j\theta_j)R_j] = -E[f_{z|R_j}(s)R_j] \\ &= -\int_{-\infty}^{+\infty} \frac{f_{z,R_j}(s, R_j)}{f_{R_j}(R_j)} R_j dF_{R_j}(R_j) = -f_z(s)E[R_j|Z = s], \end{aligned}$$

where  $f_{z|R_j}$  is the conditional density function of the portfolio return  $Z$  conditional on return  $R_j$  of the asset  $j$ , and  $f_{z,R_j}$  is their joint probability density function. Stacking the components into one vector yields

$$\nabla_{\theta} F_z(s) = -f_z(s)E[R|Z = s],$$

Then, substituting this expression into equation (27), we obtain the result for the CRR risk contributions

$$\begin{aligned} \nabla_{\theta} \rho(\theta) &= -E\left[R \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) - Z \frac{\phi'(F_z(Z))}{F_z(Z)} (f_z(Z)R - f_z(Z)E[R|Z])\right] \\ &= -E\left[R \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha)\right], \end{aligned}$$

which concludes the proof.

## B Proof of proposition 2

The proof is straightforward

$$\begin{aligned}
\nabla_{\theta}^2 \rho(\theta) &= -\nabla_{\theta} E \left[ R \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) \right] \\
&= E \left[ R \frac{\phi'(F_z(Z))}{F_z(Z)} (f_z(Z)R' + \nabla_{\theta} F_z(s)|_{s=Z}) \right] \\
&= E \left[ \frac{\phi'(F_z(Z))f_z(Z)}{F_z(Z)} (RR' - RE[R'|Z]) \right] \\
&= E \left[ \frac{\phi'(F_z(Z))f_z(Z)}{F_z(Z)} \text{Cov}(R|Z) \right].
\end{aligned}$$

## C Influence function of semi-parametric IV regressors

As discussed in subsection 4.1, the derivation of the limit distribution of parameters in semi-parametric IV regression (13) requires the derivation of the influence function of the functional

$$\phi_2(F) = \int \epsilon \nu'(F(Z)) dF(z, \epsilon). \quad (28)$$

We refer to van der Vaart (1998) and Newey (1994) for the methodology and appropriate regularity conditions. The idea is that the functional delta method applies to a functional  $\phi_2(\cdot) : D_F \rightarrow \mathbb{R}$  satisfying Hadamard differentiability, so that for any square root consistent estimator  $F_n$  of the function  $F$

$$\begin{aligned}
\sqrt{n}(\phi_2(F_n) - \phi_2(F)) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_2(z_i, \epsilon_i) + o_p(1), \\
\psi_2(z, \epsilon) &= \frac{d}{dt} [\phi_2((1-t)F + t\delta_x)]_{t=0}.
\end{aligned}$$

The function  $\psi_2(F)$  is known as the *influence function* of the functional  $\phi_2$ . We rewrite the functional (28) as an expectation:

$$\phi_2(F) = E[\epsilon \nu'(F(Z))],$$

where the random variable  $\epsilon$  stands for the error term of the linear model (9),  $Z$  is the random variable corresponding to the return on the CRR market portfolio, and  $F$  is the cdf of  $Z$ .

Denote by  $g(Z)$  the projection of  $\epsilon$  on  $Z$ , i.e.,  $g(Z) = E[\epsilon|Z]$ . Introduce a "misspecified" joint distribution function  $F_\theta(z, \epsilon)$  along the path  $\theta$ , such that  $F_0$  is the true distribution function. Then we can calculate the influence function from the pathwise derivative of the given functional, using the pathwise derivative (see Newey (1994)):

$$\frac{dE_\theta [g_\theta(Z) \nu'(F_\theta)]}{d\theta} = \frac{\partial E_\theta [g(Z) \nu'(F)]}{\partial \theta} + \frac{\partial E [g_\theta(Z) \nu'(F)]}{\partial \theta} + \frac{\partial E [g(Z) \nu'(F_\theta)]}{\partial \theta},$$

where we denote  $F_\theta$  as the "misspecified" marginal distribution function of  $Z$  corresponding to the "misspecification" of the joint distribution function  $F_\theta(z, \epsilon)$ ,  $g_\theta(Z)$  as the "misspecified" conditional expectation of  $\epsilon$  given  $Z$ ,  $E_\theta[\cdot]$  as the expectation under the "misspecified" distribution  $F_\theta(z, \epsilon)$ . Formally:

$$\begin{aligned} F_\theta(z) &= \int_{-\infty}^z \int_{-\infty}^{\infty} dF_\theta(s, t), \\ g_\theta(z) &= \int_{-\infty}^{\infty} t dF_\theta(t|Z=z), \\ E_\theta[\cdot] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdot dF_\theta(s, t). \end{aligned}$$

From the expression for the pathwise derivative we can see that the influence function of the functional (28) can be represented as a superposition of three influence functions of the misspecified functionals:

$$\psi_2(z, \epsilon) = \psi_A(z, \epsilon) + \psi_B(z, \epsilon) + \psi_C(z, \epsilon),$$

with

$$\left[ \frac{dE_\theta [g_\theta(Z) \nu'(F_\theta)]}{d\theta} \right]_{\theta=0} = E \left[ \psi_2(Z, \epsilon) \left( \frac{\partial \ln dF_\theta}{\partial \theta} \right)_{\theta=0} \right].$$

Further, we calculate the separate pathwise derivative and find the influence function of the functional (28). The first part of the influence function is easy to find:

$$\frac{\partial E_\theta [g(Z) \nu'(F)]}{\partial \theta} = E \left[ g(Z) \nu'(F) \frac{\partial \ln dF_\theta}{\partial \theta} \right],$$

so that the first part of the influence function is

$$\psi_A(z, \epsilon) = g(z) \nu'(F(z)) \quad (29)$$

For the second part, we find, using the chain rule and the definition of the projection  $g_\theta$ :

$$\begin{aligned} \frac{\partial E [g_\theta(Z) \nu'(F)]}{\partial \theta} &= \frac{\partial E_\theta [g_\theta(Z) \nu'(F)]}{\partial \theta} - \frac{\partial E_\theta [g(Z) \nu'(F)]}{\partial \theta} \\ &= \frac{\partial E_\theta [(\epsilon - g(Z)) \nu'(F)]}{\partial \theta} \\ &= E \left[ (\epsilon - g(Z)) \nu'(F) \frac{\partial \ln dF_\theta}{\partial \theta} \right], \end{aligned}$$

so that the second part of the influence function is

$$\psi_B(z, \epsilon) = (\epsilon - g(z)) \nu'(F(z)). \quad (30)$$

To calculate the last part of the influence function we directly apply the definition of the influence function to the functional  $E[g(Z) \nu'(F_\theta)]$ :

$$\begin{aligned} \psi_C(z, \epsilon) &= \frac{d}{dt} \left[ \int g(s) \nu'((1-t)F + t\delta_z) dF \right]_{t=0} = \quad (31) \\ &= \int g(s) (\delta_z - F) d\nu'(F). \end{aligned}$$

The influence function of the functional (28) is the superposition of the three calculated influence functions (29), (30) and (31):

$$\begin{aligned} \psi(z, \epsilon) &= \psi_A(z, \epsilon) + \psi_B(z, \epsilon) + \psi_C(z, \epsilon) \\ &= \chi(z, \epsilon) - E[\chi(Z, \epsilon)], \end{aligned} \quad (32)$$

$$\chi(z, \epsilon) = \epsilon \nu'(F(z)) + \int_z^\infty g(s) d\nu'(F(s)). \quad (33)$$

Substituting the expression for Choquet distortion pdf  $\nu'(t)$  from (3) into (33) we obtain the final result

$$\chi(z, \epsilon) = \int_{F(z)}^1 (\epsilon - E[\epsilon|Z = F^{-1}(\alpha)]) \alpha^{-1} d\phi(\alpha).$$

## D Influence function of CRR efficient portfolio weights

Results on the asymptotic distribution of mean-CRR efficient portfolio weights are obtained in companion paper by Polbennikov and Melenberg (2005). Here we briefly restate the results without the derivation details.

Let  $R^e$  be a vector of the asset excess returns  $(R_1 - r_f, \dots, R_p - r_f)$ , and  $Z = r_f + R^e \theta$  be a portfolio of these assets. The mean-CRR portfolio problem can be formulated as

$$\min_{\theta \in \mathbb{R}^p} E \left[ -Z \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) \right] \text{ s.t. } E[Z] = m,$$

where  $m$  is the expected return on the efficient portfolio. From an econometric perspective this problem is a standard constrained extremum estimation problem, so that the limit distribution of resulting portfolio weights can be found in the usual way, see Gouriéroux and Monfort (2005) and Polbennikov and Melenberg (2005). The asymptotic distribution results can be equivalently expressed through the estimator influence function. Here we report the final results. The influence function of the mean-CRR optimal portfolio weights is

$$\xi(R^e, Z) = H^{-1} \begin{bmatrix} bCC'H^{-1} - I_p \\ -bC \end{bmatrix}' \begin{bmatrix} \psi_{\nabla f} - \lambda \psi_{\nabla g} \\ \psi_g \end{bmatrix},$$

where we use notations similar with Polbennikov and Melenberg (2005). The vector  $C$  stands for the gradient of the constraint function with respect to portfolio weights  $C = E[R^e]$ . The scalar  $\lambda$  is the Lagrange multiplier

$$\lambda = -E \left[ t' R^e \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha) \right] (t' E[R^e])^{-1},$$

where  $\iota$  stands for a  $(p \times 1)$  vector of ones. The matrix  $H$  is the Hessian of the objective function with respect to portfolio weights evaluated at the optimum

$$H = E \left[ \frac{\phi'(F_z(Z))f_z(Z)}{F_z(Z)} \text{Cov}(R^e|Z) \right].$$

The functions  $\psi_{\nabla f}$  and  $\psi_{\nabla g}$  are the influence functions of the objective and constraint function gradient functionals, respectively. The expressions for them are given by

$$\begin{aligned} \psi_{\nabla f} &= \chi_{\nabla f} - E[\chi_{\nabla f}], \\ \chi_{\nabla f} &= - \int_{F_z(Z)}^1 (R^e - E[R^e|Z = F^{-1}(\alpha)]) \alpha^{-1} d\phi(\alpha), \\ \psi_{\nabla g} &= R^e - E[R^e]. \end{aligned}$$

The function  $\psi_g$  is the influence function of the constraint functional,  $\psi_g = Z - m$ . Finally, the scalar  $b$  is a notation

$$b = (C'H^{-1}C)^{-1}.$$

The asymptotic distribution of the mean-CRR optimal portfolio weights is

$$\sqrt{n}(\hat{\theta} - \theta) \rightarrow_d N(0, E[\xi\xi']).$$

## D.1 Expected shortfall

The asymptotic result for the mean-expected shortfall optimal weights is a special case of the mean-CRR weights considered above with  $\phi(\alpha) = I(\alpha \geq \tau)$ . Substituting this expression into the corresponding formulas yields

$$\begin{aligned} \lambda &= -\tau^{-1} E[\iota' R^e I(F_z(Z) \leq \tau)] (\iota' E[R^e])^{-1}, \\ H &= \frac{f(F_z^{-1}(\tau))}{\tau} \text{Cov}(R^e|Z = F_z^{-1}(\tau)), \\ \chi_f &= -\tau^{-1} I(F_z(Z) \leq \tau) (R^e - E[R^e|Z = F_z^{-1}(\tau)]). \end{aligned}$$

The expression for the influence function of the mean-expected shortfall portfolio weights follows immediately.

## D.2 Point mass approximation (PMA) of a CRR

The point mass approximation of a CRR measure suggested by Bassett et al. (2004) takes

$$\phi(\alpha) = \sum_{k=1}^m \phi_k I(\alpha \geq \tau_k).$$

This is also a special case of a CRR measure. Therefore, the derived asymptotic results for a mean-CRR portfolio weights still apply. We have

$$\begin{aligned} \lambda &= - \sum_{k=1}^m \phi_k \tau_k^{-1} E[l'R^e I(F_z(Z) \leq \tau_k)] (l'E[R^e])^{-1}, \\ H &= E[\nabla_{\theta}^2 f] = \sum_{k=1}^m \phi_k \tau_k^{-1} f(F_z^{-1}(\tau_k)) \text{Cov}(R^e | Z = F_z^{-1}(\tau_k)), \\ \chi_f &= - \sum_{k=1}^m \phi_k \tau_k^{-1} I(F_z(Z) \leq \tau_k) (R^e - E[R^e | Z = F_z^{-1}(\tau_k)]). \end{aligned}$$

The expression for the influence function of the mean-PMA CRR portfolio weights follows immediately.

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Table 1: Annualized descriptive statistics of country capital index returns.

<b>Descriptive Statistics of Country Index Returns</b>					
Indexes	Mean	Median	Kurtosis	Skewness	Volatility
America					
WRLD	0.6%	9.1%	5.1	0.10	15.2%
ARG	1.2%	0.0%	32.9	-1.43	33.2%
BRA	15.9%	7.2%	7.4	0.13	30.8%
CHIL	11.3%	4.9%	4.5	-0.18	15.2%
PER	9.8%	6.6%	16.4	-0.23	14.3%
MEX	15.5%	19.8%	6.2	0.13	22.7%
VEN	-3.1%	-7.3%	75.0	-2.98	32.4%
Asia					
CHI	1.7%	0.0%	9.6	1.05	22.0%
INDIA	18.5%	24.7%	7.0	-0.47	27.1%
INDO	10.5%	1.2%	8.8	0.09	35.9%
MAL	13.5%	0.0%	26.5	1.97	19.7%
PAK	22.5%	18.6%	7.2	0.10	28.4%
PHIL	-2.4%	-8.1%	51.5	3.43	22.2%
SHRIL	10.6%	0.0%	51.1	0.99	23.2%
THAIL	11.3%	1.4%	6.6	0.46	29.5%
Europe					
CZE	14.5%	29.7%	7.2	-0.02	24.0%
HUN	12.9%	11.5%	5.0	0.16	26.0%
POL	23.5%	4.1%	8.8	-0.20	32.5%
ROM	37.7%	30.5%	6.3	-0.09	34.7%
RUS	70.2%	0.0%	22.9	1.78	53.5%
TURK	-2.5%	5.9%	4.2	0.02	19.9%
OECD					
AU	10.6%	13.5%	5.9	-0.34	16.6%
CAN	12.3%	18.3%	6.2	-0.41	17.3%
EU	5.4%	2.6%	4.4	-0.06	21.4%
JAP	4.8%	0.0%	4.4	-0.10	22.6%
KOR	19.4%	11.5%	5.0	-0.11	36.5%
UK	2.5%	5.1%	4.7	-0.11	18.0%
US	1.5%	0.0%	5.1	0.18	19.5%

Table 2: Efficiency tests of the Morgan Stanley world capital index (WRLD). The table reports performance measures  $\alpha$ 's and p-values of the mean-variance (MV), mean-expected shortfall (ShF) and mean-PMA CRR (PMA) spanning tests. Probability threshold for expected shortfall is 5%. Probability thresholds for PMA CRR are 5%, 10%, 15%, 20%, and 25% with equal weights of 20%.

Indexes	Efficiency of WRLD index					
	MV		ShF		PMA	
	Perf.	$\alpha$	p-value	Perf.	$\alpha$	p-value
America						
ARG	-0.8%	0.948	-1.0%	0.937	-0.8%	0.953
BRA	14.8%	0.182	15.2%	0.170	15.1%	0.172
CHIL	9.1%	0.095	9.3%	0.088	9.3%	0.089
PER	6.7%	0.231	6.7%	0.231	6.7%	0.231
<b>MEX</b>	<b>14.6%</b>	<b>0.042</b>	<b>14.6%</b>	<b>0.040</b>	<b>14.7%</b>	<b>0.040</b>
VEN	-6.1%	0.633	-6.1%	0.634	-6.0%	0.639
Joint	*	0.291	*	0.275	*	0.279
Asia						
CHI	-1.6%	0.851	-1.7%	0.847	-1.7%	0.842
INDIA	15.8%	0.133	16.3%	0.125	16.2%	0.126
INDO	7.7%	0.581	8.3%	0.556	8.2%	0.561
MAL	10.4%	0.175	10.4%	0.175	10.5%	0.172
<b>PAK</b>	<b>19.2%</b>	<b>0.085</b>	<b>18.9%</b>	<b>0.091</b>	<b>19.3%</b>	<b>0.083</b>
PHIL	-5.5%	0.529	-5.4%	0.533	-5.2%	0.551
SHRIL	7.3%	0.422	7.6%	0.406	7.3%	0.422
THAIL	8.8%	0.438	9.1%	0.427	9.1%	0.425
Joint	*	0.396	*	0.389	*	0.387
Europe						
CZE	12.3%	0.177	12.8%	0.162	12.6%	0.166
HUN	10.7%	0.277	10.8%	0.273	10.8%	0.272
POL	20.3%	0.112	20.4%	0.110	20.4%	0.111
<b>ROM</b>	<b>36.0%</b>	<b>0.006</b>	<b>36.0%</b>	<b>0.006</b>	<b>36.2%</b>	<b>0.006</b>
<b>RUS</b>	<b>67.9%</b>	<b>0.001</b>	<b>67.4%</b>	<b>0.001</b>	<b>67.7%</b>	<b>0.001</b>
TURK	-2.8%	0.497	-2.9%	0.490	-2.8%	0.493
Joint	*	<b>0.004</b>	*	<b>0.004</b>	*	<b>0.003</b>
OECD						
AU	7.9%	0.217	8.0%	0.210	8.0%	0.211
<b>CAN</b>	<b>11.0%</b>	<b>0.033</b>	<b>11.1%</b>	<b>0.031</b>	<b>11.1%</b>	<b>0.031</b>
EU	3.8%	0.613	4.1%	0.588	3.9%	0.600
JAP	2.5%	0.776	2.6%	0.766	2.6%	0.764
KOR	17.6%	0.206	17.9%	0.200	18.0%	0.198
UK	1.2%	0.832	1.5%	0.788	1.3%	0.818
US	1.3%	0.697	1.1%	0.747	1.2%	0.719
Joint	*	0.452	*	0.443	*	0.437

Table 3: Moody's cumulative default probabilities by letter rating from 1-10 years, 1970-2004. For more details see Default and Recovery Rates of Corporate Bond Issuers, 1920-2004, Special Comment, Moody's Investors Service, Global Credit Research, January 2005.

	<b>Cumulative Default Probability (%) to Year, 1970-2004</b>					
<b>Rating</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>6</b>	<b>8</b>	<b>10</b>
Aaa	0.00	0.00	0.04	0.21	0.41	0.63
Aa	0.00	0.00	0.12	0.29	0.47	0.61
A	0.02	0.08	0.36	0.67	1.04	1.48
Baa	0.19	0.54	1.55	2.59	3.65	4.89
Ba	1.22	3.34	8.27	12.98	16.64	20.11
B	5.81	12.93	25.33	35.1	42.89	48.64

Table 4: Descriptive statistics of the simulated CDO returns. Sample correlation matrix is given at the bottom of the table. Returns are simulated from the one-factor large homogeneous portfolio model.

	<b>Simulation Parameters</b>		
Def. Prob.:	2.5%	5%	7.5%
Def. Corr.:	0.15	0.1	0.05
Risk Prem.:	4%	10%	12%
	<b>Sample Return Statistics</b>		
Min.:	-27.00%	-30.80%	-16.80%
1st Qu.:	0.64%	2.78%	1.58%
Median:	2.25%	5.47%	4.20%
Mean:	1.40%	4.52%	3.60%
3rd Qu.:	3.16%	7.27%	6.29%
Max.:	3.99%	9.83%	10.80%
Std. Dev.	2.72%	3.86%	3.68%
Skew.	-2.63	-1.75	-1.03
Kurtos.	13.85	8.35	4.62
CDO1	1.00	0.01	0.00
CDO2	0.01	1.00	0.01
CDO3	0.00	0.01	1.00

Table 5: Spanning tests for simulated credit portfolio returns. The table reports p-values of mean-variance (MV Span), mean-expected shortfall (ShF Span), and mean-PMA CRR (PMA Span), spanning tests for assets CDO2 and CDO3. Four market efficient portfolios are considered: CDO1; mean-variance market portfolio (Mkt. MV); mean-expected shortfall market portfolio (Mkt. ShF); and mean-PMA CRR market portfolio (Mkt. PMA). The probability threshold for expected shortfall is 5%. The probability thresholds for PMA CRR are 5%, 10%, 15%, 20%, and 25% with equal weights of 20%.

Returns		Efficient Portfolios			
		CDO1	Mkt. MV	Mkt. ShF	Mkt. PMA
MV Span	CDO2	0.000	1.000	0.000	0.003
	CDO3	0.000	1.000	0.000	0.000
	All	0.000	1.000	0.000	0.000
ShF Span	CDO2	0.000	0.000	0.995	0.000
	CDO3	0.000	0.000	0.887	0.053
	All	0.000	0.000	0.989	0.000
PMA Span	CDO2	0.000	0.001	0.001	0.871
	CDO3	0.000	0.002	0.004	0.943
	All	0.000	0.001	0.000	0.916