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Abstract

Although most of the theory development concerning risk measures has concentrated on convex or even coherent risk measures, nonconvex risk measures are used in practice, the prime example of course being Value-at-Risk. The purpose of this paper is to investigate the relations between various notions of time consistency for nonconvex risk measures. We focus on three notions in particular which, as we show, all satisfy a certain compatibility property that other notions of consistency do not always have. One of these notions is the strong consistency (also called dynamic consistency) that has received most attention in the literature. Despite the fact that the other two notions are weaker, we show that they are still strong enough to make consistent updates unique if they exist at all. We give a number of sufficient and necessary conditions that an aggregate risk measure has to satisfy for consistent updating to be possible.

Keywords: convex risk measures; acceptability measures; weak time consistency.

MSC2000: 91B30, 91B28.

1 Introduction

Risk measures are used for various purposes, including regulation, margin setting, and asset pricing; see for instance [1, 3]. In most if not all of these applications, evaluations at more than one point in time are of interest. It is then natural to ask the question in what sense evaluations at different time points are consistent with each other. Time consistency of convex and coherent risk measures has been investigated extensively in recent years; see for instance [10, 20, 22, 6, 5, 2]. In this paper we investigate notions of time consistency without assuming the convexity axiom. Interest in nonconvex risk measures is motivated

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in part by practice; the risk measure most commonly used in industry, Value-at-Risk, is in fact nonconvex. From a mathematical point of view, it is of interest to observe that it is possible to develop the theory to some extent without the convexity axiom. The notion of convexity especially in its disaggregate (conditional) version is a strong property, and various distinctions are lost when this property is assumed. By not assuming convexity, we can see the roles of weaker properties that are discussed below such as the local property, the complementarity property, and the property of closedness under isolation. In particular it turns out that the local property is already enough to ensure several properties of interest, and we also highlight the special role of the complementarity property which is even weaker than the local property. A consequence of not assuming convexity that we do have to accept is that a representation theory in terms of test measures and associated threshold functions is unavailable. Consequently we characterize important properties, such as existence of consistent updates, in other ways, for instance in terms of properties of the acceptance set.

In the literature on time consistency of risk measures, attention has been paid in particular to the notion of *dynamic consistency*, which is analogous to the tower property of conditional expectations. It has been already been argued by several authors, however, that dynamic consistency is rather a strong requirement to impose. Klöppel and Schweizer [15, Section 7.2] show that a simple coherent risk measure based on one-sided moments cannot be updated in a dynamically consistent way. Tutsch [24, p. 100, 113] considers dynamic consistency unsuitable for updating. Kupper and Schachermayer [16] prove, under mild technical conditions, that imposing both dynamic consistency and distribution invariance reduces the class of dynamic risk measures to the one-parameter family of entropic risk measures. Whether dynamic consistency should be taken as a normative postulate is debatable, according to Schied [23, Rem. 3.5]. From an application perspective, it may not be easy, given specific purposes of evaluation, to construct risk measures that behave well with respect to a range of criteria, including ease of interpretation, creation of proper incentives, and consistency across different levels of aggregation. When it is not possible to meet the strongest requirements that might be imposed, work may be facilitated by the formulation of less stringent demands, for instance based on weaker notions of consistency.

In the literature on statistical decision theory, the concept of dynamic consistency has been discussed extensively as well. Within the context of multiple-stage decision problems, an agent is considered to act in a dynamically consistent way if decisions taken at a particular node are the ones that were planned for that situation as part of a strategy devised at an earlier stage. Machina [17] argues that preferences that are not of the expected utility type are incompatible with the combination of dynamic consistency and an axiom called *consequentialism* [12], which expresses that decisions depend only on events that may still occur in a given situation, and in particular do not depend on contingencies that were considered possible at an earlier stage but that have not materialized. Consequentialism may be compared to the *regularity* property that is used in the literature on risk measures (cf. (2.8) below). Faced with the incompatibility of the three properties of dynamic consistency, consequentialism, and non-EU preferences, Machina suggests that the axiom of consequentialism might be discarded. Hanany and Klibanoff [13] prove a number of nonexistence results for updating rules of non-expected utility under various forms of dynamic consistency, and they propose a specific non-consequentialist approach in order to maintain a certain form of dynamic consistency without having to assume preferences based on expected utility.

In the context of finance and insurance, dropping the idea of consequentialism (taken in the sense of the regularity property) may not be appropriate. On the other hand, it is perhaps not necessary to impose dynamic consistency in a context such as regulation, where the "measure" that is computed represents a *reserve* that is normally not used, rather than a *price* that will certainly need to be paid when the transaction under consideration is accepted. Several weaker notions of time consistency were proposed and discussed by Tutsch [24], Penner [19], and Roorda and Schumacher [21]. These notions will return in our discussion below.

The main results of this paper may be summarized as follows. We focus on three notions of time consistency, one of which is the standard notion of strong consistency; the other two are weaker notions which we call conditional consistency and sequential consistency, following terminology in [21]. We prove that each of these notions enjoys a *compatibility property*, which makes it possible to define consistency of sequences or even continuous-time families of risk measures in a natural way. We show that, under each of the three notions be it in some cases under additional sensitivity conditions, there is *at most one* consistent update of a given aggregate risk measure to a risk measure formulated at a finer level of aggregation. We provide an *explicit recipe* for obtaining this consistent update. The construction of this update, which we call the *refinement update*, is based on the application of the notion of conditional capital requirement as defined by Detlefsen and Scandolo [6] to an operation on acceptance sets as proposed by Tutsch [24]. The existence of consistent updates of a given risk measure can therefore decided by verifying whether the refinement update (which always exists) is consistent. We also provide some *alternative characterizations* of the existence of consistent updates. In terms of the *relations* between the three notions of consistency, we prove that under some conditions strong consistency implies sequential consistency, and sequential consistency implies conditional consistency.

In this paper we consider the evaluation of *payoffs* (random variables) rather than of *payoff streams* (random processes) as for instance in [5] and [14]. The consideration of payoff streams leads to more complicated notation, and the limitation to single payments may not incur a large loss of generality if it is assumed that there is some point in the future at which a final account settlement takes place; admittedly, this may not always be a natural assumption. Transfer to such a final settlement time would in particular be feasible when it is assumed that loans may be taken out against payments that will be received later, and that payments received at one stage may be safely carried over to a later stage. Of course it can be argued, as in [14], that agents may still prefer to receive payments now rather than later (even when market rates of interest are applied), due for instance to credit issues or to the nonexistence in reality of absolutely safe transfer. On the other hand, if one feels that credit events should be taken into account in constructing a risk measure in a given situation, then such events can be and perhaps should be modeled explicitly. As in most of the literature on risk measures, we shall limit ourselves to bounded random variables; methods for extending results from this case to the unbounded case are provided in [4].

The literature on risk measures that has developed following the work of Artzner et al. [2] is marked by variations in sign conventions and terminology. The term "monetary utility function" that has been used in a number of recent papers is a little long-winded as noted by Jobert and Rogers [14], but the alternative term "valuation" used by them may be too suggestive of applications in pricing rather than for instance in regulation. Since we want to avoid such possible suggestions, we add one letter and use "evaluation" instead, following terminology of Peng [18]. The sign conventions that we use are the same as for instance in [5]: the outcomes of random variables are interpreted as gains, and positive values of evaluation functionals correspond to acceptable positions.

The paper is organized as follows. Preliminaries with mostly well known material are presented in Section 2. The refinement update is defined in Section 3. Then, in Sections 4, 5, and 6, we discuss conditional consistency, sequential consistency, and strong consistency. Conclusions follow in Section 7.

2 Basic definitions and properties

In this section we list some basic definitions and properties and fix notation. Most of the material is standard and the basic properties are well known (see for instance [6, 5, 9]). The notion of complementarity does not seem to have received much attention in other papers, however; also we prove some auxiliary results that may not have appeared as such before.

2.1 Standing assumptions and notation

Throughout the paper we use a probability space $(\Omega, \mathcal{F}, P)$. The terms "measurable" and "almost surely" without further specification mean $\mathcal{F}$-measurable and P-almost surely, respectively. The complement of an event $F \in \mathcal{F}$ is denoted by F^c. We write $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$. Elements of L^∞ will be referred to as random variables but also as "payoffs" or "positions". All equalities and inequalities applied to random variables are understood to hold almost surely; also, convergence is almost sure convergence unless indicated otherwise.

Given a nonempty set $\mathcal{S} \subset L^\infty$, $\operatorname{ess\,sup} \mathcal{S}$ is defined as the least element in the a.s.-equivalence classes of measurable functions from Ω to $\mathbb{R} \cup \{\infty\}$ that dominate all elements of $\mathcal{S}$ in the almost sure sense (see for instance [9]); $\operatorname{ess\,inf} \mathcal{S}$ is defined similarly. We use $\inf X$ and $\sup X$ to refer to the essential infimum and the essential supremum, respectively, of an element X of L^∞. We also use inf and sup in the usual sense to refer to the infimum and supremum of a collection of real numbers; this should not lead to confusion.

We will use $\mathcal{F}_t$ as a generic notation for a sub-σ-algebra, which allows us to denote associated objects by subscript or superscript t. In particular, the set $L^\infty(\Omega, \mathcal{F}_t, P)$ of essentially bounded $\mathcal{F}_t$-measurable functions will be denoted by L_t^∞. Other subscripts such as s and u will be used as well in situations where several sub-σ-algebras play a role, with obvious notational conventions.

Given a random variable $X \in L^\infty$, the variable $\|X\|_t \in L_t^\infty$ defined by $\|X\|_t = \operatorname{ess\,inf}\{m \in L_t^\infty \,|\, m \geq |X|\}$ is referred to as the $\mathcal{F}_t$*-conditional norm* of X. The notation $\|X\|$ (without subscript) refers to the usual L^∞-norm of X. Since $\mathcal{F}_t$ is a sub-σ-algebra of $\mathcal{F}$, we have $L_t^\infty \subset L^\infty$ and $\|X\|_t \leq \|X\|$ for all $X \in L^\infty$. The trivial sub-σ-algebra $\{\emptyset, \Omega\}$ is denoted by $\mathcal{F}_0$.

While the usual interpretation of a sub-σ-algebra $\mathcal{F}_t$ is that of representing information available at time t, there is nothing in the developments below that prevents other possible interpretations, for instance information available to a particular agent, as mentioned in [6]. The sub-σ-algebra $\mathcal{F}_t$ may also represent information which is available in principle but which may be used or not used at the discretion of a regulatory authority, or it may represent a certain aggregation level within an organization.

A subset $\mathcal{S}$ of L^∞ will be called *real-convex* if $\lambda X + (1-\lambda)Y \in \mathcal{S}$ for all $X, Y \in \mathcal{S}$ and $\lambda \in \mathbb{R}$ such that $0 \leq \lambda \leq 1$. This is of course the usual notion of convexity, but we want to have a term that emphasizes the difference with the notion of $\mathcal{F}_t$-convexity (see (2.16) below).

To avoid proliferation of quantifiers, we sometimes replace a propositional expression of the form "$\forall Z \in \mathcal{Z} : p(Z)$" by the statement "$p(Z)$ $(Z \in \mathcal{Z})$", especially when the set membership relation $Z \in \mathcal{Z}$ is already suggested by notational conventions. We use the customary shorthand of writing $\{p(Z)\}$ instead of $\{\omega \in \Omega \,|\, p(Z(\omega))\}$ when Z is a random variable and $p(z)$ is a statement operating on reals such as $z \geq 0$.

2.2 Conditional evaluations

The following are order-related properties of functions $\phi : L^\infty \to L^\infty$:

- *normalization*:

$$\phi(0) = 0 \tag{2.1}$$

- *monotonicity*:

$$X \le Y \Rightarrow \phi(X) \le \phi(Y) \quad (X, Y \in L^\infty) \tag{2.2}$$

- *sensitivity*:

$$X \le 0,\ \phi(X) = 0 \ \Rightarrow\ X = 0 \quad (X \in L^\infty) \tag{2.3}$$

- *strong sensitivity*:

$$X \le Y,\ \phi(X) = \phi(Y) \ \Rightarrow\ X = Y \quad (X, Y \in L^\infty) \tag{2.4}$$

- *continuity from above*:

$$X_n \searrow X \Rightarrow \phi(X_n) \searrow \phi(X) \quad (X_n \in L^\infty, n = 1, 2, \ldots;\ X \in L^\infty). \tag{2.5}$$

We will need the following result. The argument in the proof is similar to the reasoning in the proof of [9, Thm. 4.31].

LEMMA 2.1 *Let $\phi : L^\infty \to L^\infty$ be a normalized monotonic mapping that is continuous from above, and let $X \in L^\infty$. If there exists a bounded sequence $(X_n)_{n\ge1}$ such that $X_n \to X$ and $\phi(X_n) \ge 0$ for all n, then $\phi(X) \ge 0$.*

PROOF Define $Y_n = \operatorname{ess\,sup}_{m\ge n} X_m$; then $Y_n \searrow X$ so that $\phi(X) = \lim_{n\to\infty} \phi(Y_n)$. By monotonicity and normalization, we have $\phi(Y_n) \ge \phi(X_n) \ge 0$ for all n, so that $\lim_{n\to\infty} \phi(Y_n) \ge 0$ and the stated result follows. □

Next we list some aggregation-related properties. Let a sub-σ-algebra $\mathcal{F}_t$ be given. A mapping $\phi_t : L^\infty \to L^\infty_t$ will be called an *$\mathcal{F}_t$-conditional evaluation* if it is normalized and monotonic in the sense of (2.1) and (2.2), and the following property holds:

- *$\mathcal{F}_t$-translation invariance*:[1]

$$\phi_t(X + C) = \phi_t(X) + C \quad (X \in L^\infty,\ C \in L^\infty_t). \tag{2.6}$$

The following property is satisfied by any $\mathcal{F}_t$-conditional evaluation [6, Prop. 1,2], [5, Prop. 3.3]:

- *$\mathcal{F}_t$-local property*:

$$\phi_t(1_F X + 1_{F^c} Y) = 1_F \phi_t(X) + 1_{F^c} \phi_t(Y) \quad (F \in \mathcal{F}_t;\ X, Y \in L^\infty). \tag{2.7}$$

[1] "Conditional additive homogeneity" would be an alternative term for this property.

Under the normalization assumption, the local property is equivalent to the following property [6, Prop. 1]:

- $\mathcal{F}_t$*-regularity*:
$$\phi_t(1_F X) = 1_F \phi_t(X) \quad (F \in \mathcal{F}_t;\ X \in L^\infty). \tag{2.8}$$

A mapping $\phi_t : L^\infty \to L_t^\infty$ is said to be a *concave* $\mathcal{F}_t$-conditional evaluation if in addition to the properties (2.1), (2.2), and (2.6), it satisfies the following property:

- $\mathcal{F}_t$*-concavity*:
$$\phi_t(\Lambda X + (1-\Lambda)Y) \geq \Lambda\phi_t(X) + (1-\Lambda)\phi_t(Y) \quad (X, Y \in L^\infty;\ \Lambda \in L_t^\infty,\ 0 \leq \Lambda \leq 1). \tag{2.9}$$

A concave $\mathcal{F}_t$-conditional evaluation is called *coherent* if it satisfies the following property:

- $\mathcal{F}_t$*-positive homogeneity*:
$$\phi_t(\Lambda X) = \Lambda\phi_t(X) \quad (X \in L^\infty;\ \Lambda \in L_t^\infty,\ \Lambda \geq 0). \tag{2.10}$$

We prove a few lemmas relating to sensitivity and strong sensitivity.

Lemma 2.2 *Let ϕ be a strongly sensitive and monotonic mapping from L^∞ to L^∞, and let $\mathcal{F}_t$ be a sub-σ-algebra. If $X, Y \in L_t^\infty$ are such that $\phi(1_F X) \geq \phi(1_F Y)$ for all $F \in \mathcal{F}_t$, then $X \geq Y$.*

Proof Let the assumptions of the lemma hold. If $X \not\geq Y$, then there exist $F \in \mathcal{F}_t$ with $P(F) > 0$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ such that $1_F X \leq 1_F(Y - \varepsilon)$. We have

$$\phi(1_F Y) \leq \phi(1_F X) \leq \phi(1_F(Y - \varepsilon)) \leq \phi(1_F Y)$$

so that all inequalities above are in fact equalities. It then follows from the strong sensitivity of ϕ that $1_F = 0$, i. e. $P(F) = 0$, and we have a contradiction. □

Corollary 2.3 *Let ϕ be a strongly sensitive and monotonic mapping from L^∞ to L^∞, and let $\mathcal{F}_t$ be a sub-σ-algebra. If $X, Y \in L_t^\infty$ are such that $\phi(1_F X) = \phi(1_F Y)$ for all $F \in \mathcal{F}_t$, then $X = Y$.*

Lemma 2.4 *Let ϕ be a normalized, monotonic, and sensitive mapping from L^∞ to L^∞, and let $\mathcal{F}_t$ be a sub-σ-algebra. If $X \in L_t^\infty$ is such that $\phi(1_F X) \geq 0$ for all $F \in \mathcal{F}_t$, then $X \geq 0$.*

Proof Apply the argument in the proof of Lemma 2.2 with $Y = 0$, noting that $\phi(Y) = \phi(0) = 0$ and replacing strong sensitivity with sensitivity. □

2.3 Acceptance sets

The *acceptance set* of a normalized monotonic mapping $\phi : L^\infty \to L^\infty$ is defined by

$$\mathcal{A}(\phi) = \{X \in L^\infty \,|\, \phi(X) \geq 0\}.$$

The acceptance set of an $\mathcal{F}_t$-conditional evaluation satisfies a number of properties which we express here for a general set $\mathcal{S} \subset L^\infty$:

- *acceptance of zero*:

$$0 \in \mathcal{S} \tag{2.11}$$

- *solidness*:

$$X \in \mathcal{S},\ Y \geq X \Rightarrow Y \in \mathcal{S} \quad (Y \in L^\infty) \tag{2.12}$$

- $\mathcal{F}_t$*-nonnegativity*:[2]

$$X \in L_t^\infty \cap \mathcal{S} \Rightarrow X \geq 0. \tag{2.13}$$

Below we shall refer to these three properties as the “basic conditions”. The acceptance set of an $\mathcal{F}_t$-conditional evaluation also always has the following properties:

- $\mathcal{F}_t$*-local property*:

$$X, Y \in \mathcal{S} \;\Rightarrow\; 1_F X + 1_{F^c} Y \in \mathcal{S} \quad (F \in \mathcal{F}_t) \tag{2.14}$$

- $\mathcal{F}_t$*-closedness*:

$$X_n \in \mathcal{S}\ (n = 1, 2, \dots),\ \|X_n - X\|_t \to 0 \;\Rightarrow\; X \in \mathcal{S} \quad (X \in L^\infty). \tag{2.15}$$

The $\mathcal{F}_t$-closedness property follows from the inequality $|\phi_t(X) - \phi_t(Y)| \leq \|X - Y\|_t$ [5, Prop. 3.3]. If ϕ_t is concave, then its acceptance set satisfies

- $\mathcal{F}_t$*-convexity*:

$$\Lambda X + (1 - \Lambda) Y \in \mathcal{S} \quad (X,\, Y \in \mathcal{S};\ \Lambda \in L_t^\infty,\ 0 \leq \Lambda \leq 1). \tag{2.16}$$

Additional properties of subsets of L^∞ that we shall use are

- *negative cone exclusion*:

$$X \in \mathcal{S},\ X \leq 0 \Rightarrow X = 0 \tag{2.17}$$

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[2]The term “normalization” is sometimes used for properties (2.11) and (2.13) together. This phrase may be too simple however since it does not indicate that the defined notion depends on $\mathcal{F}_t$.

- *closedness under $\mathcal{F}_t$-isolation*:[3]

$$X \in \mathcal{S} \Rightarrow 1_F X \in \mathcal{S} \quad (X \in L^\infty,\ F \in \mathcal{F}_t). \tag{2.18}$$

- *$\mathcal{F}_t$-complementarity*:

$$1_F X \in \mathcal{S},\ 1_{F^c} X \in \mathcal{S} \Rightarrow X \in \mathcal{S} \quad (X \in L^\infty,\ F \in \mathcal{F}_t). \tag{2.19}$$

Clearly, a normalized monotonic mapping is sensitive if and only if its acceptance set satisfies the negative cone exclusion property. Both closedness under $\mathcal{F}_t$-isolation and the $\mathcal{F}_t$-complementarity property express forms of compliance of the acceptance set $\mathcal{S}$ with the aggregation level $\mathcal{F}_t$. Closedness under $\mathcal{F}_t$-isolation may be interpreted as expressing that the acceptance set $\mathcal{S}$ does not allow compensation of adverse outcomes at the aggregation level associated to $\mathcal{F}_t$. The $\mathcal{F}_t$-complementarity property is perhaps more easily interpreted in the following equivalent formulation:

$$1_F X = 0,\ 1_{F^c} Y = 0 \Rightarrow X + Y \in \mathcal{S} \quad (X, Y \in \mathcal{S};\ F \in \mathcal{F}_t). \tag{2.20}$$

Looked at this way, the notion of $\mathcal{F}_t$-complementarity refers to a situation in which only one of two contracts will pay off (for instance a call and a put option written on the same underlying, with the same strike and the same time of maturity), and states that when both contracts are acceptable, and on the basis of information at time t it is possible to determine which one will pay off, then their combination is acceptable as well. The property of complementarity with respect to the full σ-algebra $\mathcal{F}$ will be referred to simply as "complementarity". This property may be expressed by the following condition:

$$X, Y \in \mathcal{S},\ P\big(\{X = 0\} \cup \{Y = 0\}\big) = 1 \ \Rightarrow\ X + Y \in \mathcal{S}. \tag{2.21}$$

Example 2.5 Given a probability measure $Q \ll P$ and a measurable function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that $E^Q(u(X)) \in [-\infty, \infty)$ is well-defined for all $X \in L^\infty$, one may define a subset $\mathcal{S} \subset L^\infty$ by

$$\mathcal{S} = \{X \in L^\infty \mid E^Q(u(X)) \geq u(0)\}. \tag{2.22}$$

Such sets are closely related to the shortfall risk measures discussed by Föllmer and Schied [8]. Usually some additional conditions are imposed on the function u, for instance requiring that it should be increasing and concave. The complementarity property however is always satisfied for sets of the form (2.22), as shown in the proposition below.

Proposition 2.6 *The set $\mathcal{S}$ defined by (2.22) satisfies the complementarity property.*

[3] Another term that might be used here is "$\mathcal{F}_t$-localization", but the term "localization" is perhaps already overused.

Proof We verify the criterion (2.19) with $\mathcal{F}_t = \mathcal{F}$. For any $F \in \mathcal{F}$ and $X \in L^\infty$, we have $u(1_F X) = 1_F u(X) + 1_{F^c} u(0)$. Therefore, if $X \in L^\infty$ and $F \in \mathcal{F}$ are such that both $1_F X \in \mathcal{S}$ and $1_{F^c} X \in \mathcal{S}$, then

$$\begin{aligned} E^Q(u(X)) &= E^Q(1_F u(X)) + E^Q(1_{F^c} u(X)) \\ &= E^Q(u(1_F X)) - Q(F^c)u(0) + E^Q(u(1_{F^c} X)) - Q(F)u(0) \\ &\geq u(0) \end{aligned}$$

so that $X \in \mathcal{S}$. □

The following proposition shows that the complementarity property and the property of closedness under isolation are, so to say, constituent parts of the local property.

Proposition 2.7 *Let $\mathcal{F}_t$ be a sub-σ-algebra. A set $\mathcal{S} \subset L^\infty$ that satisfies $0 \in \mathcal{S}$ has the $\mathcal{F}_t$-local property if and only if it has both the $\mathcal{F}_t$-complementarity property and the property of closedness under $\mathcal{F}_t$-isolation.*

Proof First, assume that $\mathcal{S}$ has the local property. For any $X \in \mathcal{S}$ and $F \in \mathcal{F}_t$, we have $1_F X = 1_F X + 1_{F^c} 0 \in \mathcal{S}$, so that $\mathcal{S}$ is closed under $\mathcal{F}_t$-isolation. To prove the complementarity property, let $X \in L^\infty$ and $F \in \mathcal{F}_t$ be such that $1_F X \in \mathcal{S}$ and $1_{F^c} X \in \mathcal{S}$. Writing $X = 1_F(1_F X) + 1_{F^c}(1_{F^c} X)$, we see that the local property implies that $X \in \mathcal{S}$.

Conversely, assume now that $\mathcal{S}$ is closed under $\mathcal{F}_t$-isolation and has the $\mathcal{F}_t$-complementarity property. Take $X, Y \in \mathcal{S}$, and $F \in \mathcal{F}_t$, and write $Z = 1_F X + 1_{F^c} Y \in \mathcal{S}$. We need to prove that $Z \in \mathcal{S}$. Note that $1_F Z = 1_F X \in \mathcal{S}$ and $1_{F^c} Z = 1_{F^c} Y \in \mathcal{S}$ by the closedness under $\mathcal{F}_t$-isolation of $\mathcal{S}$. By the $\mathcal{F}_t$-complementarity, this suffices to show that indeed $Z \in \mathcal{S}$. □

The following two examples show that the properties of closedness under isolation and the complementarity property do not imply each other.

Example 2.8 Let Q be a measure that is absolutely continuous with respect to P, let α be a number strictly between 0 and 1, and define $\mathcal{S} = \{X \in L^\infty \mid Q(X < 0) < \alpha\}$. Such a VaR-type acceptance set is closed under $\mathcal{F}_t$-isolation for any sub-σ-algebra $\mathcal{F}_t$, but in general does not have the $\mathcal{F}_t$-complementarity property and so in general also does not have the $\mathcal{F}_t$-local property.

Example 2.9 Let Ω consist of two points, say $\Omega = \{\omega_1, \omega_2\}$; let P be the uniform measure and let $\mathcal{F}_1$ be the power set of Ω. The space L^∞ can in this case be identified with $\mathbb{R}^2$. Consider the set $\mathcal{S} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \geq 0\}$. This set is obviously solid and contains 0. We have $(x_1, x_2) \in \mathcal{S}$ for all x_1 and x_2 such that $(x_1, 0) \in \mathcal{S}$ and $(0, x_2) \in \mathcal{S}$, so that the $\mathcal{F}_1$-complementarity property is satisfied. However, the set $\mathcal{S}$ is not closed under $\mathcal{F}_1$-isolation; for instance $(-1, 1) \in \mathcal{S}$ but $1_{\omega_1}(-1, 1) = (-1, 0) \notin \mathcal{S}$.

In the second example, note that the set $\mathcal{S}$ is $\mathcal{F}_0$-convex, where $\mathcal{F}_0$ is the trivial sub-σ-algebra, but not $\mathcal{F}_1$-convex.

2.4 Conditional capital requirements

Given an arbitrary set $\mathcal{S} \subset L^\infty$, one may, following [6], define a mapping from L^∞ to L_t^∞ by

$$\phi_{\mathcal{S}}^t(X) = \operatorname{ess\,sup}\{Y \in L_t^\infty \mid X - Y \in \mathcal{S}\}. \tag{2.23}$$

The mapping defined in this way is called the *conditional capital requirement* induced by $\mathcal{F}_t$ and $\mathcal{S}$. The following proposition gives conditions under which this mapping is a conditional evaluation. The proposition as given here is somewhat stronger than [6, Prop. 3], since an assumption of $\mathcal{F}_t$-convexity is replaced by an assumption of real-convexity. The generalization is possible on the basis of arguments in [5]. For the convenience of the reader we summarize the proof.

PROPOSITION 2.10 *Let $\mathcal{S}$ be a subset of L^∞ that satisfies the basic properties (2.11) acceptance of zero), (2.12) (solidness), and (2.13) (conditional nonnegativity) with respect to a given sub-σ-algebra $\mathcal{F}_t$. Then the conditional capital requirement $\phi_{\mathcal{S}}^t$ defined by (2.23) is a conditional evaluation with respect to $\mathcal{F}_t$. If in addition $\mathcal{S}$ is real-convex, then $\phi_{\mathcal{S}}^t$ is a concave $\mathcal{F}_t$-conditional evaluation.*

PROOF Take $X \in L^\infty$, and let $Y \in L_t^\infty$ be such that $X - Y \in \mathcal{S}$. It follows from the solidness of $\mathcal{S}$ that then also $\|X\|_t - Y \in \mathcal{S}$. Since $\|X\|_t - Y \in L_t^\infty$, the $\mathcal{F}_t$-nonnegativity of $\mathcal{S}$ implies that $Y \leq \|X\|_t$. This shows that the essential supremum in (2.23) is finite-valued (actually $\phi_{\mathcal{S}}^t(X) \leq \|X\|_t$) so that indeed $\phi_{\mathcal{S}}^t(X) \in L_t^\infty$ for every $X \in L^\infty$.

The $\mathcal{F}_t$-nonnegativity of $\mathcal{S}$ and the assumption that $0 \in \mathcal{S}$ together imply that $\operatorname{ess\,inf} L_t^\infty \cap \mathcal{S} = 0$ so that $\phi_{\mathcal{S}}^t(0) = 0$ as required. The monotonicity property (2.2) of $\phi_{\mathcal{S}}^t$ is immediate from the solidness of $\mathcal{S}$. The conditional translation property (2.6) of $\phi_{\mathcal{S}}^t$ follows, in fact without any assumptions on the set $\mathcal{S}$, from the corresponding property of the essential supremum.

Finally we consider the conditional concavity. It follows as in the proof of Prop. 3 in [6] that the real-convexity of $\mathcal{S}$ implies that the mapping $\phi_{\mathcal{S}}^t$ is $\mathcal{F}_0$-concave, i.e., $\phi_{\mathcal{S}}^t(\lambda X + (1 - \lambda)Y \geq \lambda\phi_{\mathcal{S}}^t(X) + (1-\lambda)\phi_{\mathcal{S}}^t(Y)$ for $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$ and $X, Y \in L^\infty$. It was shown in [5, Prop. 3.3] that monotonicity and conditional translation invariance of a conditional evaluation ϕ_t together imply the conditional local property as well as the inequality $\phi_t(X) - \phi_t(Y) \leq \|X - Y\|_t$ in L_t^∞, and in the same paper it is shown that the latter two properties together with $\mathcal{F}_0$-concavity imply $\mathcal{F}_t$-concavity [5, Rem. 3.4]. $\square$

Every conditional evaluation is the conditional capital requirement of some set $\mathcal{S} \subset L^\infty$; in particular,

$$\phi_t = \phi_{\mathcal{A}(\phi_t)} \tag{2.24}$$

(cf. [6, §2.3], [5, Prop. 3.9]). We will also need the following statement concerning the acceptability set of the conditional capital requirement of a given set (cf. [5, Prop. 3.10]).

PROPOSITION 2.11 *Let $\mathcal{S} \subset L^\infty$ satisfy the basic conditions of Prop. 2.10, so that $\phi^t_{\mathcal{S}}$ is a conditional evaluation. Then $\mathcal{A}(\phi^t_{\mathcal{S}})$ is the smallest subset of L^∞ that contains $\mathcal{S}$, is $\mathcal{F}_t$-closed, and satisfies the $\mathcal{F}_t$-local property.*

In particular it follows from this that $\mathcal{A}(\phi^t_{\mathcal{S}}) = \mathcal{S}$ when $\mathcal{S}$ is $\mathcal{F}_t$-closed and has the $\mathcal{F}_t$-local property. So the five conditions (2.11–2.15) are not only necessary but also sufficient for a set $\mathcal{S} \subset L^\infty$ to be the acceptance set of an $\mathcal{F}_t$-conditional evaluation.

3 The refinement update

3.1 Refinement of acceptance sets

An acceptance set that is designed for a given level of aggregation typically allows for a certain amount of compensation between good and bad outcomes at finer levels of aggregation. If one now wants to construct a related acceptance set at a finer level than the given one, it is natural to look for the least restrictive set that complies with the given set and disallows compensation at the finer level. This idea is expressed in the following definition.

DEFINITION 3.1 Given a set $\mathcal{S} \subset L^\infty$ such that $0 \in \mathcal{S}$, the *$\mathcal{F}_t$-refinement* of $\mathcal{S}$ is

$$\mathcal{S}^t = \{X \in L^\infty \mid 1_F X \in \mathcal{S} \text{ for all } F \in \mathcal{F}_t\}. \tag{3.1}$$

It is easy to see that indeed $\mathcal{S}^t$ is the largest subset of $\mathcal{S}$ that is closed under $\mathcal{F}_t$-isolation. The above definition was proposed by Tutsch [24, p. 88] in the situation in which the set $\mathcal{S}$ is the acceptance set of a conditional evaluation ϕ_s, and she refers to the above set as the *acceptance set of ϕ_s with respect to $\mathcal{F}_t$*. Tutsch notes [24, Bem. 3.1.10] that the refinement $\mathcal{S}^t$ need not be $\mathcal{F}_t$-conditionally convex. Even when we strengthen the assumption by requiring that ϕ_s is concave and weaken the conclusion to the $\mathcal{F}_t$-complementarity property, the implication still does not hold in general. This is demonstrated in the following example.

EXAMPLE 3.2 Consider a sample space consisting of four elements, say $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Let $\mathcal{F}$ be the power set of Ω, and let P be the uniform measure. The space L^∞ may in this case be identified with $\mathbb{R}^4$. Let the set $\mathcal{S}$ be defined by

$$\mathcal{S} = \{x \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 \geq 0,\ x_1 + 2x_2 + 2x_3 + x_4 + 1 \geq 0\}.$$

It is easily verified that the set $\mathcal{S}$ is convex and satisfies the basic properties (2.11–2.13) with respect to the trivial σ-algebra $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Consider the sub-$\sigma$-algebra $\mathcal{F}_t$ generated by $F = \{\omega_1, \omega_2\}$. The corresponding refinement of $\mathcal{S}$ is given by

$$\mathcal{S}^t = \{x \in \mathbb{R}^4 \mid x_1 + x_2 \geq 0,\ x_3 + x_4 \geq 0,\ x_1 + 2x_2 + 1 \geq 0,\ 2x_3 + x_4 + 1 \geq 0,\\ x_1 + 2x_2 + 2x_3 + x_4 + 1 \geq 0\}.$$

Consider now the positions $X = (1, -1, 0, 0)$ and $Y = (0, 0, -1, 1)$. We have $X \in \mathcal{S}^t$ and $Y \in \mathcal{S}^t$, but $1_F X + 1_{F^c} Y = (1, -1, -1, 1) \notin \mathcal{S}^t$, so that the set $\mathcal{S}^t$ does not have the $\mathcal{F}_t$-complementarity property.

The following three propositions state conditions on a given set $\mathcal{S}$ which guarantee that its $\mathcal{F}_t$-refinement $\mathcal{S}^t$ will have certain desirable properties. The first proposition shows that the conditions of Prop. 2.10 hold for $\mathcal{S}^t$ under conditions on $\mathcal{S}$ which do not involve any $\mathcal{F}_t$-related properties. This is a basic result because it allows us to define a refinement of conditional evaluations by means of the construction of that proposition.

Proposition 3.3 *Let $\mathcal{S}$ be a subset of L^∞ that satisfies the three properties (2.11) (acceptance of zero), (2.12) (solidness), and (2.17) (negative cone exclusion). Let $\mathcal{F}_t$ be a sub-σ-algebra. Then the $\mathcal{F}_t$-refinement $\mathcal{S}^t$ of $\mathcal{S}$ is solid, contains the zero function, and satisfies the conditional nonnegativity property (2.13) with respect to $\mathcal{F}_t$.*

Proof The refinement $\mathcal{S}^t$ trivially inherits the properties of solidness and acceptance of zero. It remains to show the conditional nonnegativity property. Suppose there exists $X \in L^\infty_t \cap \mathcal{S}^t$ such that $X \not\geq 0$. Then there exist $\varepsilon > 0$ and $F \in \mathcal{F}_t$ with $P(F) > 0$ such that $1_F X \leq -\varepsilon 1_F$. Since $X \in \mathcal{S}^t$ and $F \in \mathcal{F}_t$, we have $1_F X \in \mathcal{S}$. By the solidness of $\mathcal{S}$ it then follows that $-\varepsilon 1_F \in \mathcal{S}$, which is incompatible with the negative cone exclusion property. □

The second proposition in this series gives sufficient conditions for $\mathcal{F}_t$-closedness of the refinement $\mathcal{S}^t$. Here we assume that the set $\mathcal{S}$ is obtained as the acceptance set of a normalized monotonic mapping, as will be the case in the applications below. Again no $\mathcal{F}_t$-related properties of $\mathcal{S}$ are assumed.

Proposition 3.4 *Suppose that $\mathcal{S} = \mathcal{A}(\phi)$ where $\phi : L^\infty \to L^\infty$ is a normalized monotonic mapping, and let $\mathcal{F}_t$ be a sub-σ-algebra. If ϕ is continuous from above, then the $\mathcal{F}_t$-refinement $\mathcal{S}^t$ is $\mathcal{F}_t$-closed.*

Proof Take $X \in L^\infty$, and let $(X_n)_{n\geq 1}$ be a sequence of payoffs $X_n \in \mathcal{S}^t$ such that $\|X_n - X\|_t \to 0$. Note that we then also have $X_n \to X$. Take $F \in \mathcal{F}_t$; we want to show that $1_F X \in \mathcal{S}$. By Egorov's theorem, we can find for any $m \in \mathbb{N}$ a set $G_m \in \mathcal{F}_t$ with $P(G_m) > 1 - \frac{1}{m}$ such that the convergence of $\|X_n - X\|_t$ to 0 is uniform on G_m. In particular it follows, for fixed

m, that $(1_{G_m}\|X_n - X\|_t)_{n\geq 1}$ is a bounded sequence, which implies that $(1_{G_m}X_n)_{n\geq 1}$ is a bounded sequence as well. From $X_n \to X$ it follows that $1_{G_m\cap F}X_n \to 1_{G_m\cap F}X$. Moreover, since $G_m \in \mathcal{F}_t$ and $X_n \in (\mathcal{A}(\phi))^t$, we have $\phi(1_{G_m\cap F}X_n) \geq 0$ for all n. By Lemma 2.1, it follows that $\phi(1_{G_m\cap F}X) \geq 0$. Now, the sequence $(1_{G_m\cap F}X)_{m\geq 1}$ is a bounded sequence that converges to $1_F X$ and that satisfies $\phi(1_{G_m\cap F}X) \geq 0$ for all m. Using Lemma 2.1 again, we conclude that $\phi(1_F X) \geq 0$. Since $F \in \mathcal{F}_t$ was arbitrary, it follows that $X \in \mathcal{S}^t$. □

Finally, we present sufficient conditions for $\mathcal{S}^t$ to have the $\mathcal{F}_t$-local property. Here we do use an $\mathcal{F}_t$-related assumption on $\mathcal{S}$. This assumption is not satisfied by the set $\mathcal{S}$ that was used in Example 3.2.

Proposition 3.5 *If $\mathcal{S} \subset L^\infty$ has the $\mathcal{F}_t$-complementarity property, then its $\mathcal{F}_t$-refinement $\mathcal{S}^t$ has the $\mathcal{F}_t$-local property.*

Proof Take X and Y in $\mathcal{S}^t$, and take $G \in \mathcal{F}_t$. We need to show that $1_G X + 1_{G^c}Y \in \mathcal{S}^t$, which means by definition that $1_F(1_G X + 1_{G^c}Y) \in \mathcal{S}$ for all $F \in \mathcal{F}_t$. Take $F \in \mathcal{F}_t$. Because $X \in \mathcal{S}^t$ and $F \cap G \in \mathcal{F}_t$, we have $1_{F\cap G}X \in \mathcal{S}$, and likewise, $1_{F\cap G^c}Y \in \mathcal{S}$. Moreover, we clearly have

$$1_{G^c}1_{F\cap G}X = 0, \quad 1_G 1_{F\cap G^c}Y = 0.$$

It follows from the $\mathcal{F}_t$-complementarity property that $1_F(1_G X+1_{G^c}Y) = 1_{F\cap G}X+1_{F\cap G^c}Y \in \mathcal{S}$ as required. □

We continue with some comments concerning the relation between refinement and the complementarity property. If $\mathcal{S}$ has the complementarity property with respect to the full σ-algebra $\mathcal{F}$ (we have called this the complementarity property without further specification, cf. (2.21)), then it follows from Prop. 3.5 that its refinement with respect to any given sub-σ-algebra $\mathcal{F}_t$ has the $\mathcal{F}_t$-local property. As shown in the following proposition, the complementary property is in fact necessary for this generic feature to hold.

Proposition 3.6 *Let $\mathcal{S} \subset L^\infty$ be such that $0 \in \mathcal{S}$, and assume that, for any sub-σ-algebra $\mathcal{F}_t$ of $\mathcal{F}$, the $\mathcal{F}_t$-refinement $\mathcal{S}^t$ satisfies the $\mathcal{F}_t$-local property. Then $\mathcal{S}$ has the complementarity property.*

Proof Let $X, Y \in \mathcal{S}$ be such that $P\big(\{X = 0\} \cup \{Y = 0\}\big) = 1$. Define $F = \{X \neq 0\} \in \mathcal{F}$, and take $\mathcal{F}_t = \{\emptyset, F, F^c, \Omega\}$. We have $1_F X = X \in \mathcal{S}$ and $1_{F^c}X = 0 \in \mathcal{S}$, so that $X \in \mathcal{S}^t$, and likewise $Y \in \mathcal{S}^t$. By assumption the set $\mathcal{S}^t$ has the $\mathcal{F}_t$-local property, so that $1_F X + 1_{F^c}Y \in \mathcal{S}^t$. Since $1_F X = X$, $1_{F^c}Y = Y$, and $\mathcal{S}^t \subset \mathcal{S}$, this implies $X + Y \in \mathcal{S}$. □

The following proposition shows that the $\mathcal{F}_t$-refinement of a set $\mathcal{S} \subset L^\infty$ that has the complementarity property not only satisfies the $\mathcal{F}_t$-complementarity property, as follows from Prop. 3.5, but even satisfies the full complementarity property.

Proposition 3.7 *Let $\mathcal{F}_t$ be a sub-σ-algebra, and let $\mathcal{S} \subset L^\infty$. If $\mathcal{S} \subset L^\infty$ has the complementarity property, then the same holds for its $\mathcal{F}_t$-refinement $\mathcal{S}^t$.*

Proof Take two complementary payoffs $X, Y \in \mathcal{S}^t$, i.e. $\{X = 0\} \cup \{Y = 0\} = \Omega$, and take $F \in \mathcal{F}_t$. By definition of the refinement $\mathcal{S}^t$, we have $1_F X \in \mathcal{S}$ and $1_F Y \in \mathcal{S}$. Moreover, the payoffs $1_F X$ and $1_F Y$ are complementary. It follows that $1_F X + 1_F Y \in \mathcal{S}$. Therefore, we have $1_F(X + Y) \in \mathcal{S}$ for all $F \in \mathcal{F}_t$, which proves that $X + Y \in \mathcal{S}^t$. □

3.2 Refinement of conditional evaluations

We now use the refinement operation, which acts on subsets of L^∞, to define an update operation which acts on conditional evaluations. Let an $\mathcal{F}_s$-conditional evaluation ϕ_s and a sub-σ-algebra $\mathcal{F}_t \supset \mathcal{F}_s$ be given. It follows from Prop. 3.3 that, if ϕ_s is sensitive, then its acceptance set $\mathcal{A}(\phi_s)$ satisfies the nonnegative cone exclusion property and the other properties needed to ensure, by Prop. 2.10, that a conditional capital requirement can be defined on the basis of the $\mathcal{F}_t$-refinement of $\mathcal{A}(\phi_s)$. We define the update as follows.

Definition 3.8 Let a sensitive $\mathcal{F}_s$-conditional evaluation ϕ_s be given, and let $\mathcal{F}_t$ be a sub-σ-algebra such that $\mathcal{F}_t \supset \mathcal{F}_s$. The $\mathcal{F}_t$*-refinement update* of ϕ_s is the $\mathcal{F}_t$-conditional evaluation ϕ_s^t that may be described in terms of the notation introduced in (2.23) by $\phi_s^t = \phi_{(\mathcal{A}(\phi_s))^t}$, or more directly by

$$\phi_s^t(X) = \operatorname{ess\,sup}\{Y \in L_t^\infty \mid \phi_s(1_F(X - Y)) \geq 0 \text{ for all } F \in \mathcal{F}_t\}. \tag{3.2}$$

It should be noted that the acceptance set of a non-coherent conditional evaluation ϕ_s^t is not necessarily equal to the $\mathcal{F}_t$-refinement $(\mathcal{A}(\phi_s))^t$ of the acceptance set of ϕ_s; in general it may be larger (cf. Prop. 2.11 and Example 3.2). Consequently, some positions may be admitted by ϕ_s^t which are not admitted according to the $\mathcal{F}_t$-refinement of the acceptance set of ϕ_s. This issue will be discussed extensively in the next section.

By definition, we have $\phi_s^t(X) \geq 0$ if $\phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_t$. Under strong sensitivity, we can also prove an implication with reversed inequalities.

Proposition 3.9 *Let ϕ_s be a strongly sensitive $\mathcal{F}_s$-conditional evaluation, and let $\mathcal{F}_t \supset \mathcal{F}_s$. For all $X \in L^\infty$, the following implication holds:*

$$\forall F \in \mathcal{F}_t : \phi_s(1_F X) \leq 0 \;\;\Rightarrow\;\; \phi_s^t(X) \leq 0. \tag{3.3}$$

Proof Suppose that the assumptions of the proposition and the left hand side of (3.3) are satisfied. By the definition (3.2), to show that $\phi_s^t(X) \leq 0$ we have to prove that $Y \leq 0$ for any $Y \in L_t^\infty$ such that

$$\phi_s(1_F(X - Y)) \geq 0 \quad (F \in \mathcal{F}_t). \tag{3.4}$$

Therefore, take $Y \in L_t^\infty$ and suppose that (3.4) holds. Define $F = \{Y \geq 0\} \in \mathcal{F}_t$; we then have $1_F Y \geq 0$. By (3.4), the monotonicity of ϕ_s, and the left hand side of (3.3), we have

$$0 \leq \phi_s(1_F(X-Y)) \leq \phi_s(1_F X) \leq 0.$$

It follows that all inequalities in the above are actually equalities, and the strong sensitivity of ϕ_s implies that $1_F Y = 1_{Y \geq 0} Y = 0$, or in other words, $Y \leq 0$. □

4 Conditional consistency

4.1 Definition and uniqueness

Conditional consistency is the weakest form of time consistency that we consider in this paper.

Definition 4.1 Let two sub-σ-algebras $\mathcal{F}_s$ and $\mathcal{F}_t$ be given with $\mathcal{F}_s \subset \mathcal{F}_t$, and let ϕ_s and ϕ_t be conditional evaluations with respect to $\mathcal{F}_s$ and $\mathcal{F}_t$, respectively. We say that ϕ_s and ϕ_t are *conditionally consistent*, or that ϕ_t is a *conditionally consistent $\mathcal{F}_t$-update* of ϕ_s, if the following condition holds:

$$\phi_t(X) \geq 0 \;\Leftrightarrow\; \forall F \in \mathcal{F}_t : \phi_s(1_F X) \geq 0 \quad (X \in L^\infty). \tag{4.1}$$

If we write $\mathcal{A}_s = \mathcal{A}(\phi_s)$ and $\mathcal{A}_t = \mathcal{A}(\phi_t)$, and we use the notation for refinements of acceptance sets as introduced in (3.1), then the requirement for conditional consistency can simply be written as

$$\mathcal{A}_t = \mathcal{A}_s^t. \tag{4.2}$$

While conditional consistency is a relatively weak notion compared to the notions of consistency that are considered below, it is strong enough to ensure that updates that are consistent in this sense must be unique, if they exist at all. This is shown in the proposition below. More specifically, the proposition shows that the only possible conditionally consistent $\mathcal{F}_t$-update of a given $\mathcal{F}_s$-conditional evaluationis the refinement update that was defined in the previous section. The refinement update is always defined but it may not be conditionally consistent; if it is not, then no conditionally consistent update exists. The proof follows readily from the close relation between conditional consistency and the definition of the refinement update.

Theorem 4.2 *Let two sub-σ-algebras $\mathcal{F}_s$ and $\mathcal{F}_t$ be given with $\mathcal{F}_s \subset \mathcal{F}_t$. Let ϕ_s be an $\mathcal{F}_s$-conditional evaluation, and let ϕ_s^t denote its $\mathcal{F}_t$-refinement update. Suppose that a conditionally consistent $\mathcal{F}_t$-update ϕ_t of ϕ_s exists; then $\phi_t = \phi_s^t$.*

PROOF Let ϕ_t be a conditionally consistent $\mathcal{F}_t$-update of ϕ_s. By the relation (2.24), we have $\phi_t = \phi_{\mathcal{A}(\phi_t)} = \phi_{(\mathcal{A}(\phi_s))^t} = \phi_s^t$. □

In the *coherent* case, the refinement update is (under a sensitivity assumption) always conditionally consistent. This is shown in the example below. The fact that coherent risk measures can be updated in a conditionally consistent way by conditioning of the representing test measures was shown in [21, Thm. 7.1] in the context of finite probability spaces.

EXAMPLE 4.3 Let $\mathcal{Q}$ be a collection of probability measures that are all absolutely continuous with respect to the reference measure P. Let $\mathcal{F}_s$ be a sub-σ-algebra, and define an $\mathcal{F}_s$-conditional evaluation by

$$\phi_s(X) = \operatorname{ess\,inf}_{Q\in\mathcal{Q}} E_s^Q X \tag{4.3}$$

where $E_s^Q X$ denotes $E^Q[X \mid \mathcal{F}_s]$. Assume that ϕ_s is sensitive; a sufficient but not necessary condition for this is that the collection $\mathcal{Q}$ contains at least one measure that is equivalent to P. Let $\mathcal{F}_t$ be a sub-σ-algebra such that $\mathcal{F}_t \supset \mathcal{F}_s$. We can then show that the $\mathcal{F}_t$-conditional evaluation defined by $\phi_t(X) = \operatorname{ess\,inf}_{Q\in\mathcal{Q}} E_t^Q X$ is a conditionally consistent update of ϕ_s. For this, we need to show that $\phi_t(X) \geq 0$ if and only if $\phi_s(1_F X) \geq 0$. First, let $X \in L^\infty$ be such that $\phi_t(X) \geq 0$. Take $F \in \mathcal{F}_t$ and $Q \in \mathcal{Q}$. It follows from $\phi_t(X) \geq 0$ that $E_t^Q X \geq 0$ and hence $1_F E_t^Q X \geq 0$, so that $E_s^Q(1_F X) = E_s^Q E_t^Q(1_F X) = E_s^Q(1_F E_t^Q X) \geq 0$. Therefore we have $\phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_t$, as required. Conversely, assume now that $X \in (\mathcal{A}(\phi_s))^t$. Since $E_s^Q(1_F X) = E_s^Q(1_F E_t^Q X)$ for $F \in \mathcal{F}_t$, it follows from $\phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_t$ that $\phi_s(1_F E_t^Q X)$ for all $F \in \mathcal{F}_t$. Using Lemma 2.4, we conclude from this that $E_t^Q X \geq 0$.

4.2 Compatibility

There are many possible definitions of consistency between conditional evaluations at different levels of aggregation. One possible way of looking at the "tightness" of a proposed notion of consistency is to see whether consistency between ϕ_s and ϕ_t and between ϕ_t and ϕ_u implies consistency between ϕ_s and ϕ_u (where $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_u$, and ϕ_i is an $\mathcal{F}_i$-conditional evaluation for $i \in \{s, t, u\}$), and, conversely, consistency between ϕ_s and ϕ_t and between ϕ_s and ϕ_u implies consistency between ϕ_t and ϕ_u. If these implications hold, then we say that the proposed notion of consistency satisfies the *compatibility property.* We first show two examples of notions of consistency which do not have this property.

EXAMPLE 4.4 Two conditional evaluations with respect to sub-σ-algebras $\mathcal{F}_s$ and $\mathcal{F}_t$, with $\mathcal{F}_s \subset \mathcal{F}_t$, are said to be *weakly acceptance consistent* [24] if

$$\phi_t(X) \geq 0 \;\Rightarrow\; \phi_s(X) \geq 0 \quad (X \in L^\infty) \tag{4.4}$$

where the sign convention has been adapted to our usage. Given a third sub-σ-algebra $\mathcal{F}_u$ such that $\mathcal{F}_t \subset \mathcal{F}_u$ and an $\mathcal{F}_u$-conditional evaluation ϕ_u, it is obvious that weak acceptance consistency of ϕ_s and ϕ_t and of ϕ_t and ϕ_u implies weak acceptance consistency of ϕ_s and ϕ_u. However it may happen that both ϕ_t and ϕ_u are weakly consistent updates of ϕ_s, but ϕ_t and ϕ_u are not weakly consistent. To see this, consider a standard three-step binomial tree; that is, Ω consists of eight points, $\mathcal{F}_0$ is the trivial sub-σ-algebra, $\mathcal{F}_1$ is the σ-algebra generated by $\{\omega_1, \dots, \omega_4\}$, $\mathcal{F}_2$ is the σ-algebra generated by $\{\{\omega_1, \omega_2\}, \dots, \{\omega_7, \omega_8\}\}$, $\mathcal{F}_3 = 2^\Omega$, and P is the uniform measure. The spaces L_t^∞ can be identified with $\mathbb{R}^{2^t}$ for $t = 0, 1, 2, 3$. Define

$$\begin{aligned}
\phi_0(X) &= \tfrac{1}{8}\textstyle\sum_{i=1}^{8} x_i \\
\phi_1(X) &= \big(\min(x_1, \dots, x_4), \min(x_5, \dots, x_8)\big) \\
\phi_2(X) &= \big(\tfrac{1}{2}(x_1 + x_2), \dots, \tfrac{1}{2}(x_7 + x_8)\big).
\end{aligned}$$

It is easily verified that $\mathcal{A}(\phi_1) \subset \mathcal{A}(\phi_0)$ and $\mathcal{A}(\phi_2) \subset \mathcal{A}(\phi_0)$, but we do not have $\mathcal{A}(\phi_2) \subset \mathcal{A}(\phi_1)$.

Example 4.5 Two conditional evaluations with respect to sub-σ-algebras $\mathcal{F}_s$ and $\mathcal{F}_t$, with $\mathcal{F}_s \subset \mathcal{F}_t$, are said to be *middle rejection consistent* [19, Def. 2.1.2, Prop. 2.1.6] (cf. also [24, Thm. 3.1.5]) if

$$\phi_s(X) \le \phi_s(\phi_t(X)) \quad (X \in L^\infty) \tag{4.5}$$

where again the sign convention has been adapted to our usage.[4] Consider the same situation as in the previous example with the following conditional evaluations:

$$\begin{aligned}
\phi_0(X) &= \min(x_1, \dots, x_8) \\
\phi_1(X) &= \big(\tfrac{1}{4}(x_1 + \cdots + x_4), \tfrac{1}{4}(x_5 + \cdots + x_8)\big) \\
\phi_2(X) &= \big(\min(x_1, x_2), \dots, \min(x_7, x_8)\big).
\end{aligned}$$

One easily verifies that the pairs (ϕ_0, ϕ_1) and (ϕ_0, ϕ_2) are middle rejection consistent, but the pair (ϕ_1, ϕ_2) is not.

The proposition below shows that, in contrast to the examples above, the notion of conditional consistency does satisfy the compatibility property.

Proposition 4.6 *Let ϕ_s, ϕ_t, and ϕ_u be conditional evaluations with respect to sub-σ-algebras $\mathcal{F}_s$, $\mathcal{F}_t$, and $\mathcal{F}_u$ respectively, with $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_u$. Suppose that ϕ_t is a conditionally consistent update of ϕ_s. Then ϕ_u is a conditionally consistent update of ϕ_t if and only if it is a conditionally consistent update of ϕ_s.*

[4] In [19], a sequence of conditional evaluations is said to be *prudent* if it is middle rejection consistent.

Proof In the first part of the proof, we assume that ϕ_u is a conditionally consistent update of ϕ_t and ϕ_t is a conditionally consistent update of ϕ_s. We have to show that ϕ_u is also a conditionally consistent update of ϕ_s, which means that $\phi_u(X) \geq 0$ if and only if $\phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_u$. First, take $X \in L^\infty$ such that $\phi_u(X) \geq 0$. For all $F \in \mathcal{F}_u$, we have $\phi_t(1_F X) \geq 0$ which implies that $\phi_s(1_F X) \geq 0$. Conversely, suppose that $\phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_u$. Take $F \in \mathcal{F}_u$ and $F' \in \mathcal{F}_t \subset \mathcal{F}_u$; then, since $F' \cap F \in \mathcal{F}_u$, we have $\phi_s(1_{F'} 1_F X) = \phi_s(1_{F' \cap F} X) \geq 0$. The fact that this holds for all $F' \in \mathcal{F}_t$ implies, because ϕ_t is a conditionally consistent update of ϕ_s, that $\phi_t(1_F X) \geq 0$. This inequality in its turn holds for all $F \in \mathcal{F}_u$, and so, because ϕ_u is a conditionally consistent update of ϕ_t, it follows that $\phi_u(X) \geq 0$.

In the second part of the proof, assume that both ϕ_u and ϕ_t are conditionally consistent updates of ϕ_s. We have to show that $\phi_u(X) \geq 0$ if and only if $\phi_t(1_F X) \geq 0$ for all $F \in \mathcal{F}_u$. First, take $X \in L^\infty$ such that $\phi_u(X) \geq 0$. Take $F \in \mathcal{F}_u$. For all $F' \in \mathcal{F}_t$ we have $F \cap F' \in \mathcal{F}_u$ so that $\phi_s(1_{F'} 1_F X) \geq 0$. It follows that $\phi_t(1_F X) \geq 0$. Conversely, suppose that $\phi_t(1_F X) \geq 0$ for all $F \in \mathcal{F}_u$; then we also have $\phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_u$, so that $\phi_u(X) \geq 0$. □

Suppose that we have a number of conditional evaluations $\phi_{t_0}, \dots, \phi_{t_k}$ with respect to a nested sequence of sub-σ-algebras $\mathcal{F}_{t_0} \subset \cdots \subset \mathcal{F}_{t_k}$. By the compatibility property of conditional consistency, the following two conditions are equivalent:

(i) ϕ_{t_i} is a conditionally consistent update of $\phi_{t_{i-1}}$ for all $i = 1, \dots, k$, and

(ii) ϕ_{t_i} is a conditionally consistent update of ϕ_{t_0} for all $i = 1, \dots, k$

In this case we speak of a *conditionally consistent sequence* of conditional evaluations. We may also consider a situation in which we have a continuous-parameter filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, and for every $t \in [0, T]$ an $\mathcal{F}_t$-conditional evaluation is given. We may then speak of a *conditionally consistent continuous-time family of conditional evaluations* $(\phi_t)_{0 \leq t \leq T}$ if for every sequence $(t_0, \dots, t_k)$ with $0 \leq t_0 < t_1 < \cdots < t_k \leq T$ the family $(\phi_{t_0}, \dots, \phi_{t_k})$ is a conditionally consistent sequence.

4.3 Sufficient conditions for conditionally consistent updating

The $\mathcal{F}_t$-refinement update of an $\mathcal{F}_s$-conditional evaluation may not be the acceptance set of any $\mathcal{F}_t$-conditional evaluation, as shown in Example 3.2, and therefore a conditionally consistent update may not exist. To verify whether a conditionally consistent update does exist, it suffices, by Thm. 4.2, to compute the refinement update and to check whether it satisfies the requirements (4.1). The verification can be reduced to the following. We use the notion of weak acceptance consistency as defined in (4.4).

PROPOSITION 4.7 *Let ϕ_t be the $\mathcal{F}_t$-refinement update of a given $\mathcal{F}_s$-conditional evaluation ϕ_s. Then ϕ_t is a conditionally consistent update of ϕ_s if and only if it is a weakly acceptance consistent update.*

PROOF Conditional consistency obviously implies weak acceptance consistency (use the implication from left to right in (4.1), with the particular choice $F = \Omega$). Conversely, suppose ϕ_t is the $\mathcal{F}_t$-refinement update of ϕ_s and (4.4) holds. The implication from right to left in (4.1) follows from the definition of the refinement update. Finally (as noted in [24, Kor. 3.1.8(d')]), if $\phi_t(X) \geq 0$, then for any $F \in \mathcal{F}_t$ also $\phi_t(1_F X) = 1_F \phi_t(X) \geq 0$, and we can conclude that $\phi_s(1_F X) \geq 0$ by applying (4.4) to $1_F X$. □

Computation of the refinement update in a specific case may not always be easy, and so it is of interest to have alternative conditions for the existence of conditionally consistent updates. From the results obtained above, we can compile a set of sufficient conditions as follows.

THEOREM 4.8 *An $\mathcal{F}_s$-conditional evaluation ϕ_s allows a conditionally consistent $\mathcal{F}_t$-update for given $\mathcal{F}_t \supset \mathcal{F}_s$ if it is sensitive and continuous from above, and its acceptance set has the $\mathcal{F}_t$-complementarity property.*

PROOF Let ϕ_s and $\mathcal{F}_t$ satisfy the assumptions of the theorem. Since ϕ_s is sensitive, we can define its $\mathcal{F}_t$-refinement update $\phi_s^t = \phi_{(\mathcal{A}(\phi_s))^t}$ as in Def. 3.8. By Prop. 2.11, to show that the update ϕ_s^t is conditionally consistent, it suffices to show that $(\mathcal{A}(\phi_s))^t$ is $\mathcal{F}_t$-closed and has the $\mathcal{F}_t$-local property. The $\mathcal{F}_t$-closedness follows from the assumption that ϕ_s is continuous from above by Prop. 3.4, whereas the $\mathcal{F}_t$-local property is guaranteed by the complementarity assumption and Prop. 3.5. □

By Prop. 2.11 and Thm. 4.2, a necessary condition for a conditionally consistent update to exist is that the $\mathcal{F}_t$-refinement of the acceptance set of ϕ_s has the $\mathcal{F}_t$-local property. In view of Prop. 3.6, we therefore can state the following corollary which gives necessary and sufficient conditions for generic conditionally consistent updating.

COROLLARY 4.9 *Let ϕ_0 be an unconditional evaluation (that is, ϕ is an $\mathcal{F}_0$-conditional evaluation, where $\mathcal{F}_0$ is the trivial sub-σ-algebra), and suppose that ϕ_0 is sensitive and continuous from above. The evaluation ϕ_0 admits conditionally consistent updates with respect to all sub-σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ if and only if its acceptance set has the (full) complementarity property.*

The following proposition is concerned with the preservation of several properties of interest by conditionally consistent updating.

PROPOSITION 4.10 *Let $\mathcal{F}_s$ and $\mathcal{F}_t$ be sub-σ-algebras with $\mathcal{F}_s \subset \mathcal{F}_t$. Let ϕ_s be a sensitive $\mathcal{F}_s$-conditional evaluation and let ϕ_t be a conditionally consistent update of ϕ_s. Then the following statements hold:*

(i) ϕ_t is sensitive

(ii) if ϕ_s is concave, then so is ϕ_t

(iii) if ϕ_s is continuous from above, then so is ϕ_t.

PROOF Concerning item (i), since ϕ_s is sensitive, its acceptance set has the negative cone exclusion property. This property is inherited by the $\mathcal{F}_t$-refinement of $\mathcal{A}(\phi_s)$ which is the acceptance set of ϕ_t, and it follows that ϕ_t is sensitive as well. Item (ii) follows from Prop. 2.10.

Finally, we consider the claim concerning continuity. Suppose that $(X_n)_{n\geq 1}$ is a nonincreasing sequence of elements of L^∞ that converges to $X \in L^\infty$. By the monotonicity of ϕ_t, the sequence $(\phi_t(X_n))_{n\geq 1}$ is nonincreasing as well and is bounded from below by $\phi_t(X)$, so that we can define $Z = \lim_{n\to\infty} \phi_t(X_n)$. To prove the continuity from above, we must show that $Z = \phi_t(X)$. We have $\phi_t(X_n) \geq \phi_t(X)$ for all n, which already implies that $Z \geq \phi_t(X)$. Because Z is the pointwise limit of a sequence of $\mathcal{F}_t$-measurable functions, it is itself $\mathcal{F}_t$-measurable, so the inequality $Z \leq \phi_t(X_n)$, which holds for each n, may be written as $\phi_t(X_n - Z) \geq 0$. By conditional consistency, this means that $\phi_s(1_F(X_n - Z)) \geq 0$ for all $F \in \mathcal{F}_t$. Since $1_F(X_n - Z) \searrow 1_F(X - Z)$, the assumed continuity from above of ϕ_s implies that $\phi_s(1_F(X - Z)) \geq 0$ for all $F \in \mathcal{F}_t$, which means that $\phi_t(X - Z) \geq 0$. Again using the $\mathcal{F}_t$-measurability of Z, we conclude that $\phi_t(X) \geq Z$. □

5 Sequential consistency

5.1 Definition and characterizations

The second notion of time consistency that we discuss is sequential consistency. This notion has a straightforward interpretation.

DEFINITION 5.1 Let two sub-σ-algebras $\mathcal{F}_s$ and $\mathcal{F}_t$ be given with $\mathcal{F}_s \subset \mathcal{F}_t$, and let ϕ_s and ϕ_t be conditional evaluations with respect to $\mathcal{F}_s$ and $\mathcal{F}_t$, respectively. We say that ϕ_s and ϕ_t are *sequentially consistent*, or that ϕ_t is a *sequentially consistent $\mathcal{F}_t$-update* of ϕ_s, if the following conditions hold:

$$\phi_t(X) \geq 0 \;\Rightarrow\; \phi_s(X) \geq 0 \qquad (X \in L^\infty) \tag{5.1a}$$

$$\phi_t(X) \leq 0 \;\Rightarrow\; \phi_s(X) \leq 0 \qquad (X \in L^\infty). \tag{5.1b}$$

The conditions (5.1a) and (5.1b) correspond to the notions of *weak acceptance consistency* and *weak rejection consistency* respectively as introduced by Tutsch [24, Kor. 3.1.8]. Sequential consistency was used in a study of distribution-invariant risk measures by Weber [25]. Some characterizations of sequential consistency were obtained in [21] in the setting of coherent risk measures and finite probability spaces. The following result is analogous to [21, Thm. 4.2]; cf. also [24, Kor. 3.1.8] for one-sided formulations. Recall that we use $\inf X$ ($\sup X$) to denote the essential infimum (supremum) of an element of L^∞; in particular, $\inf X$ and $\sup X$ are constants.

PROPOSITION 5.2 *The conditional evaluation ϕ_t is a sequentially consistent update of ϕ_s if and only if the following equivalent conditions hold:*

(i) $\phi_t(X) = 0 \Rightarrow \phi_s(X) = 0 \quad (X \in L^\infty)$

(ii) $\phi_s(X - \phi_t(X)) = 0 \quad (X \in L^\infty)$

(iii) $\inf \phi_t(X) \leq \phi_s(X) \leq \sup \phi_t(X) \quad (X \in L^\infty)$.

PROOF Clearly, property (i) is implied by sequential consistency. For any $X \in L^\infty$ we have $\phi_t(X - \phi_t(X)) = 0$, so that property (ii) is implied by property (i). If property (ii) holds, then for any $X \in L^\infty$ we have

$$\phi_s(X) - \inf \phi_t(X) = \phi_s(X - \inf \phi_t(X)) \geq \phi_s(X - \phi_t(X)) = 0$$

and likewise $\phi_s(X) - \sup \phi_t(X) \leq 0$, so that (iii) is satisfied. Finally, it is immediate that property (iii) implies sequential consistency. □

Under the assumption of strong sensitivity, an alternative characterization of sequential consistency can be given as follows.

PROPOSITION 5.3 *A conditional evaluation ϕ_t is a sequentially consistent update of a strongly sensitive conditional evaluation ϕ_s if and only if*

$$\phi_t(X) = 0 \;\Leftrightarrow\; \forall F \in \mathcal{F}_t : \phi_s(1_F X) = 0 \quad (X \in L^\infty). \tag{5.2}$$

PROOF It is immediate from condition (ii) of Prop. 5.2 that (5.2) implies sequential consistency. Assume now that ϕ_t is a sequentially consistent update of ϕ_s; we want to show that (5.2) holds. First, take $X \in L^\infty$ such that $\phi_t(X) = 0$. Take $F \in \mathcal{F}_t$; then $\phi_t(1_F X) = 1_F \phi_t(X) = 0$, so that the right hand side of (5.2) is satisfied. For the proof of the reverse implication, let $X \in L^\infty$ be such that $\phi_s(1_F X) = 0$ for all $F \in \mathcal{F}_t$. Take $F = \{\phi_t(X) \geq 0\} \in \mathcal{F}_t$. Making use of condition (ii) of Prop. 5.2, we can write

$$\phi_s(1_F X) = 0 = \phi_s(1_F X - \phi_t(1_F X)) = \phi_s(1_F X - 1_F \phi_t(X)).$$

Since $1_F\phi_t(X) \geq 0$ by definition of F, it follows from the strong sensitivity of ϕ_s that $1_F\phi_t(X) = 0$, or in other words, $\phi_t(X) \leq 0$. Now taking $F = \{\phi_t(X) \leq 0\}$, we can prove in the same way that the inequality $\phi_t(X) \geq 0$ holds as well. It follows that $\phi_t(X) = 0$. □

5.2 Compatibility

We prove a compatibility property analogous to Prop. 4.6.

PROPOSITION 5.4 *Let ϕ_s, ϕ_t, and ϕ_u be conditional evaluations with respect to sub-σ-algebras $\mathcal{F}_s$, $\mathcal{F}_t$, and $\mathcal{F}_u$ respectively, with $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_u$. Assume that ϕ_s be strongly sensitive and that ϕ_t is a sequentially consistent update of ϕ_s. Then ϕ_u is a sequentially consistent update of ϕ_t if and only if it is a sequentially consistent update of ϕ_s.*

PROOF If ϕ_u is a sequentially consistent update of ϕ_t, then it follows immediately from the definition that it is also a sequentially consistent update of ϕ_s. Assume now that ϕ_u is a sequentially consistent update of ϕ_s, and suppose it is not a sequentially consistent update of ϕ_t. Then there exists $X \in L^\infty$ such that $\phi_u(X) \geq 0$ and $\phi_t(X) \not\geq 0$, so that there is an $F \in \mathcal{F}_t$ with $P(F) > 0$ and an $\varepsilon > 0$ such that $1_F\phi_t(X) \leq -\varepsilon 1_F$. Take $\eta \in (0, \varepsilon)$. Because $F \in \mathcal{F}_t \subset \mathcal{F}_u$, we have $\phi_u(1_F(X+\eta)) \geq \phi_u(1_F(X)) = 1_F\phi_u(X) \geq 0$ so that $\phi_s(1_F(X+\eta)) \geq 0$ by the assumed sequential consistency of ϕ_u and ϕ_s. The conditional evaluation ϕ_t is also a sequentially consistent update of ϕ_s, so that from $\phi_t(1_F(X+\eta)) = 1_F(\phi_t(X)+\eta) \leq 0$ it follows that $\phi_s(1_F(X+\eta)) \leq 0$. We conclude that $\phi_s(1_F(X+\eta)) = 0$. Since this holds for all $0 < \eta < \varepsilon$, strong sensitivity of ϕ_s now implies that $1_F = 0$, and we have a contradiction. □

PROPOSITION 5.5 *A sequentially consistent update of a strongly sensitive conditional evaluation is itself strongly sensitive.*

PROOF Let $\mathcal{F}_s$ and $\mathcal{F}_t$ be sub-σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t$; let ϕ_s and ϕ_t be conditional evaluations with respect to $\mathcal{F}_s$ and $\mathcal{F}_t$ respectively. Suppose that ϕ_s is strongly sensitive and that ϕ_t is a sequentially consistent update of ϕ_s. To prove that ϕ_t is strongly sensitive as well, take $X, Y \in L^\infty$ such that $X \geq Y$ and $\phi_t(X) = \phi_t(Y)$. Due to sequential consistency (cf. item (ii) in Prop. 5.2), we have $\phi_s(X - \phi_t(X)) = 0$ and also $\phi_s(Y - \phi_t(X)) = \phi_s(Y - \phi_t(Y)) = 0$. Since $X - \phi_t(X) \geq Y - \phi_t(X)$, strong sensitivity of ϕ_s implies that $X - \phi_t(X) = Y - \phi_t(X)$ and therefore $X = Y$. □

5.3 Implication of conditional consistency; uniqueness

The following proposition shows that, under strong sensitivity, sequential consistency implies conditional consistency.

Proposition 5.6 *Let ϕ_s and ϕ_t be conditional evaluations with respect to sub-σ-algebras $\mathcal{F}_s$ and $\mathcal{F}_t$ respectively, with $\mathcal{F}_s \subset \mathcal{F}_t$. Assume that ϕ_s is strongly sensitive. If ϕ_t and ϕ_s are sequentially consistent, then they are conditionally consistent.*

Proof We have to prove that the equivalence (4.1) holds. First assume that $\phi_t(X) \geq 0$. Then, for any $F \in \mathcal{F}_t$, we have $\phi_t(1_F X) = 1_F \phi_t(X) \geq 0$ and hence $\phi_s(1_F X) \geq 0$ by condition (5.1a) of the definition of sequential consistency. Conversely, let $X \in L^\infty$ be such that $\phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_t$. Take $F = \{\phi_t(X) \leq 0\} \in \mathcal{F}_t$. Because $1_F \phi_t(X) \leq 0$, we can write

$$0 \leq \phi_s(1_F X) \leq \phi_s(1_F X - 1_F \phi_t(X)) = \phi_s(1_F X - \phi_t(1_F X)) = 0$$

where condition (ii) of Prop. 5.2 has been used to obtain the final equality. The strong sensitivity of ϕ_s now implies that $1_F \phi_t(X) = 0$, or equivalently, $\phi_t(X) \geq 0$. □

An immediate corollary is the following uniqueness result for sequentially consistent updating.

Corollary 5.7 *Suppose that ϕ_t is a sequentially consistent update of ϕ_s, and that ϕ_s is strongly sensitive. Then ϕ_t is the $\mathcal{F}_t$-refinement update of ϕ_s.*

Proof The claim follows from Prop. 5.6 and the uniqueness result for conditional consistency (Thm. 4.2). □

Uniqueness of sequentially consistent updating was proved for distribution-invariant risk measures under some technical conditions by Weber [25, Cor. 4.1].

5.4 Conditions for sequentially consistent updating

It follows from Cor. 5.7 that, to see whether a given strongly sensitive $\mathcal{F}_s$-conditional evaluation can be updated to an $\mathcal{F}_t$-conditional evaluation in a sequentially consistent way, it suffices to compute the $\mathcal{F}_t$-refinement update and to verify whether sequential consistency holds for this update. For this one may use for instance the definition directly or one of the conditions in Prop. 5.2. A characterization can be stated as follows.

Proposition 5.8 *A strongly sensitive $\mathcal{F}_s$-conditional evaluation ϕ_s admits a sequentially consistent $\mathcal{F}_t$-update if and only if it admits a conditionally consistent update and for each $X \in L^\infty$ there exists $C \in L_t^\infty$ such that*

$$\phi_s(1_F(X - C)) = 0 \quad (F \in \mathcal{F}_t). \tag{5.3}$$

Proof If ϕ_s has a sequentially consistent update ϕ_t, then $C = \phi_t(X)$ satisfies the requirements of the proposition. Conversely, suppose now that ϕ_s is an $\mathcal{F}_s$-conditional evaluation that has a conditionally consistent $\mathcal{F}_t$-update, say ϕ_t, and that for each $X \in L^\infty$ there exists $C \in L_t^\infty$ such that (5.3) holds. To prove that the update ϕ_t is sequentially consistent, it is sufficient, in view of Prop. 5.2, to show that the latter condition implies $C = \phi_t(X)$. Therefore, take $X \in L^\infty$, and let $C \in L_t^\infty$ be such that (5.3) holds. By conditional consistency, the condition (5.3) implies that $\phi_t(X - C) \geq 0$ and hence $C \leq \phi_t(X)$. To prove the reverse inequality, take $Y \in L_t^\infty$ and suppose that

$$\phi_s(1_F(X - C - Y)) \geq 0 \quad \text{for all } F \in \mathcal{F}_t.$$

Take in particular $F = \{Y \geq 0\}$. We then have $1_F Y \geq 0$ so that $1_F(X-C) \geq 1_F(X-C-Y)$. Using (5.3), we can write

$$0 = \phi_s(1_F(X - C)) \geq \phi_s(1_F(X - C - Y)) \geq 0.$$

The strong sensitivity of ϕ_s now implies that $1_F Y = 1_{Y \geq 0} Y = 0$ so that $Y \leq 0$. We have shown that

$$\operatorname{ess\,sup}\{Y \in L_t^\infty \mid \phi_s(1_F(X - C - Y)) \geq 0 \text{ for all } F \in \mathcal{F}_t\}. \leq 0 \tag{5.4}$$

The conditional evaluation ϕ_t must be equal to the refinement update of ϕ_s, by Thm. 4.2. In view of the expression given for the refinement update in (3.2), it follows from (5.4) that $\phi_t(X - C) \leq 0$. Therefore, we obtain the inequality $\phi_t(X) \leq C$, and the proof is complete. □

For the class of distribution-invariant convex evaluations, necessary and sufficient conditions for the existence of sequentially consistent updates were given by Weber [25, Thm. 4.3, 4.4].

6 Strong time consistency

6.1 Definition

The notion of time consistency that is used most in the literature is *strong time consistency*, also called *dynamic consistency* or just *time consistency*; see for instance [2, Def. 5.2], [11, Def. 18], [7, Def. 3.1].

Definition 6.1 Let two sub-σ-algebras $\mathcal{F}_s$ and $\mathcal{F}_t$ be given with $\mathcal{F}_s \subset \mathcal{F}_t$, and let ϕ_s and ϕ_t be conditional evaluations with respect to $\mathcal{F}_s$ and $\mathcal{F}_t$, respectively. We say that ϕ_s and ϕ_t are *strongly time consistent*, or that ϕ_t is a *strongly consistent update* of ϕ_s, if the following relation holds for all $X \in L^\infty$:

$$\phi_s(\phi_t(X)) = \phi_s(X). \tag{6.1}$$

6.2 Compatibility

We prove a compatibility property analogous to Prop. 4.6 and Prop. 5.4.

Proposition 6.2 *Let ϕ_s, ϕ_t, and ϕ_u be conditional evaluations with respect to sub-σ-algebras $\mathcal{F}_s$, $\mathcal{F}_t$, and $\mathcal{F}_u$ respectively, with $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_u$. Assume that ϕ_s is strongly sensitive and that ϕ_t is a strongly consistent update of ϕ_s. Then ϕ_u is a strongly consistent update of ϕ_t if and only if it is a strongly consistent update of ϕ_s.*

Proof First, assume that ϕ_u is a strongly consistent update of ϕ_t. Then, for any $X \in L^\infty$, we have $\phi_s(\phi_u(X)) = \phi_s(\phi_t(\phi_u(X))) = \phi_s(\phi_t(X)) = \phi_s(X)$ so that ϕ_u is a strongly consistent update of ϕ_s. Conversely, assume now that ϕ_u is a strongly consistent update of ϕ_s. Take $X \in L^\infty$. For any $F \in \mathcal{F}_t$, we have, since $\mathcal{F}_t \subset \mathcal{F}_u$,

$$\phi_s(1_F\phi_t(\phi_u(X))) = \phi_s(\phi_t(\phi_u(1_F X))) = \phi_s(\phi_u(1_F X)) = \phi_s(1_F X) = \phi_s(1_F\phi_t(X))$$

and it follows that $\phi_t(\phi_u(X)) = \phi_t(X)$ by Cor. 2.3. □

6.3 Conditions for strongly consistent updating

We prove a characterization analogous to Prop. 5.8.

Proposition 6.3 *A strongly sensitive $\mathcal{F}_s$-conditional evaluation ϕ_s admits a strongly consistent $\mathcal{F}_t$-update if and only if for each $X \in L^\infty$ there exists $C \in L_t^\infty$ such that*

$$\phi_s(1_F X) = \phi_s(1_F C) \quad (F \in \mathcal{F}_t). \tag{6.2}$$

Proof If ϕ_s admits a strongly consistent update ϕ_t, then $C = \phi_t(X)$ satisfies the requirements of the proposition; indeed, $\phi_t(X) \in L_t^\infty$ and, for all $F \in \mathcal{F}_t$, $\phi_s(1_F\phi_t(X)) = \phi_s(\phi_t(1_F X)) = \phi_s(1_F X)$. Conversely, suppose now that for each $X \in L^\infty$ there exists $C \in L_t^\infty$ such that (6.2) holds. It follows from Cor. 2.3 that for each given X there can be only one such $C \in L_t^\infty$, and so we can define a mapping $\psi : L^\infty \to L_t^\infty$ implicitly by

$$\phi_s(1_F X) = \phi_s(1_F\psi(X)) \quad (F \in \mathcal{F}_t). \tag{6.3}$$

If we can show that the mapping ψ is an $\mathcal{F}_t$-conditional evaluation, then strong consistency follows from (6.3) and the proof will be complete.

In order to prove that ψ is an $\mathcal{F}_t$-conditional evaluation, it suffices [4, Rem. 3.4] to prove that ψ is normalized and monotonic, and that it satisfies the local property as well as real translation invariance (i.e. $\psi(X + m) = \psi(X) + m$ for $X \in L^\infty$ and $m \in \mathbb{R}$). The normalization property is trivial, and monotonicity follows from an application of Lemma

2.2. Because ψ is normalized, the local property is equivalent to regularity. Take $G \in \mathcal{F}_t$ and $X \in L^\infty$. We have, for all $F \in \mathcal{F}_t$,

$$\phi_s(1_F \psi(1_G X)) = \phi_s(1_F 1_G X) = \phi_s(1_F 1_G \psi(X))$$

and moreover $1_G \psi(X) \in L_t^\infty$, so that $\psi(1_G X) = 1_G \psi(X)$ as was to be proved. To show real translation invariance, first note that $\psi(m) = m$ for all $m \in \mathbb{R}$. Now take $X \in L^\infty$ and $m \in \mathbb{R}$. Using the real translation invariance of ϕ_s as well as the regularity property of ψ which has already been proved and the property $\phi_s(X) = \phi_s(\psi(X))$ which is a special case of (6.3), we can write, for $F \in \mathcal{F}_t$,

$$\begin{aligned}\phi_s(1_F(X+m)) &= \phi_s(1_F X - 1_{F^c} m) + m = \phi_s(\psi(1_F X - 1_{F^c} m)) + m = \\ &= \phi_s(1_F \psi(X) - 1_{F^c} m) + m = \phi_s(1_F \psi(X) + 1_F m) = \\ &= \phi_s(1_F(\psi(X) + m)).\end{aligned}$$

Also, we have $\psi(X) + m \in L_t^\infty$. It follows that $\psi(X+m) = \psi(X) + m$, and this completes the proof. □

6.4 Implication of conditional and sequential consistency

PROPOSITION 6.4 *Let $\mathcal{F}_s$ and $\mathcal{F}_t$ be sub-σ-algebras, and let ϕ_s and ϕ_t be conditional evaluations with respect to $\mathcal{F}_s$ and $\mathcal{F}_t$ respectively. The following implications hold.*

(i) If ϕ_s and ϕ_t are strongly time consistent, then they are sequentially consistent.

(ii) If ϕ_s and ϕ_t are strongly time consistent and ϕ_s is sensitive, then ϕ_s and ϕ_t are conditionally consistent.

PROOF Assume that ϕ_s and ϕ_t are strongly time consistent. If $X \in L^\infty$ is such that $\phi_t(X) = 0$, then also $\phi_s(X) = \phi_s(\phi_t(X)) = 0$ so that sequential consistency follows from item (i) of Prop. 5.2. To prove the second claim, we need to show that $\phi_t(X) \geq 0$ if and only if $\phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_t$. For the "if" part, note that the validity of the relation $\phi_s(1_F \phi_t(X)) = \phi_s(\phi_t(1_F X)) = \phi_s(1_F X) \geq 0$ for all $F \in \mathcal{F}_t$ implies that $\phi_t(X) \geq 0$, by Lemma 2.4. Conversely, if $\phi_t(X) \geq 0$, then for all $F \in \mathcal{F}_t$ we also have $\phi_t(1_F X) = 1_F \phi_t(X) \geq 0$, so that $\phi_s(1_F X) = \phi_s(\phi_t(1_F X)) \geq 0$ by monotonicity. □

In particular, the above proposition implies (on the basis of Thm. 4.2) the uniqueness of strongly consistent updates of a given sensitive conditional evaluation.

The relations between the consistency types are summarized in Fig. 1. Here we also make use of Prop. 5.6 (sequential consistency implies conditional consistency, under strong sensitivity) and Prop. 4.7 (weak acceptance consistency of refinement updates implies conditional consistency).

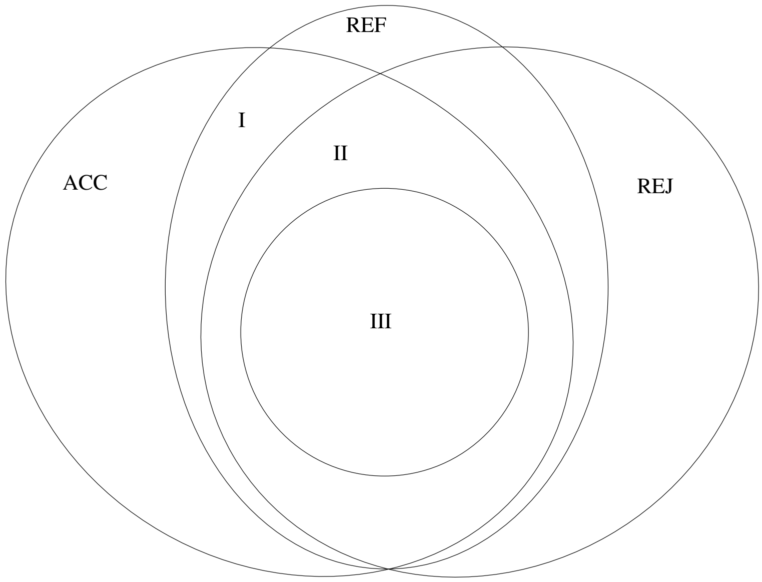

Figure 1: Relations between consistency types. Region III refers to strongly time consistent conditional evaluations, regions II and III together to sequentially consistent evaluations, and regions I, II, and III together to conditionally consistent evaluations. The ellipses refer to weak acceptance consistency, weak rejection consistency, and the class of refinement updates, as indicated. For coherent risk measures, all refinement updates are conditionally consistent. Sensitivity assumptions are as specified in the text.

7 Conclusions

The theory of dynamic risk measures (or conditional evaluations, as they are called in this paper) has many applications; it comes up whenever evaluations are to be made at different levels of aggregations and additive homogeneity is a natural requirement not only at the overall level but also at disaggregate levels. The property of conditional concavity (or convexity, depending on one's sign convention) is quite strong and may not always be natural to impose. In particular, concavity is typically lost when several risk measures are combined by means of a "best of" operation. The risk measure most frequently used in practice, Value at Risk, is not convex. Among the notions of dynamic consistency that have been studied in the literature, the notion of strong consistency (often called simply "dynamic consistency") has received most attention, but also here weaker notions can be of interest for particular applications such as the computation of regulatory capital. There are indications that, perhaps even without insisting on concavity, only few dynamic risk measures exist which are strongly consistent and which allow for a simple interpretation of overall evaluation in terms of final outcomes. The exponential family of risk measures does satisfy these properties, but this family may be less attractive for regulatory purposes because it does not focus exclusively on the tail of the distribution of outcomes.

In this paper we have discussed two notions of consistency across aggregation levels that are weaker than the standard notion of strong consistency. As we have shown, these notions are still strong enough to ensure that there can be at most one conditional evaluation at a finer level of aggregation that represents a consistent update of a conditional evaluation given at a coarser level of aggregation. We have also shown that both notions satisfy a compatibility property which again attests to the tightness of the relation expressed by these consistency notions and which makes it possible to define consistency for sequences or even continuous-time families of conditional evaluations in an unequivocal way. We have introduced the refinement update, and we have shown that this update defines the only possibly consistent way of updating conditional evaluations. Whether the refinement update is indeed consistent in either of the senses we use here depends on the particular conditional evaluation under consideration, and we have given various sufficient conditions that ensure consistency. We have established implication relations between various notions of consistency as summarized in Fig. 1.

In particular in connection with sequential consistency, we have relied in this paper for some results on the assumption of strong sensitivity. This assumption may be considered rather strong, especially in the context of risk measures that focus exclusively on the worst outcomes, and so it would be of interest to relax this condition. Also, even though existence of consistent updates can be verified by computing the refinement update or by several other methods indicated in the paper, these criteria may not always be easy to apply in concrete cases and so in future research it would be desirable to obtain alternative necessary/sufficient conditions. Under the assumption of convexity, of course one may expect to be able to state such conditions in terms of representations by means of test measures and threshold functions, and we take this as a topic for further work as well.

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