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Georgios Petropoulos

The Effects of Rare Economic Disasters on Investment Decisions and Pension Funds

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Research MSc. Thesis in Economics: The Effects of Rare Economic Disasters on Investment Decisions and Pension Funds

Georgios Petropoulos
Supervisors:Lex Meijdam and Bas van Groezen
Chairman: Eduard Ponds
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Abstract[1]

This paper attempts to develop an insight on how economic disasters affect long horizon investment and saving decisions of individuals and what the optimal response of social security should be to counterbalance such major economic shocks. For this purpose, I develop a two overlapping generations model in which asset returns are assumed to be log-normally distributed. I take into account the low probability economic disasters of the 20th Century as they defined in Barro's (2006) Rare Disasters and Asset Markets in the Twentieth Century seminal contribution. I find that, in the absence of Pay As You Go scheme, when the disaster's probability or size increases, individuals reduce their consumption during the working period of their life while they increase the share of their portfolio they invest on government bonds. Interestingly, I conclude that a change in the disaster's probability has more severe effects on individual's investment and savings decisions than a change in the size of the disaster. The introduction of Pay As You Go system seems to partially cancel out the effects of economic disasters and under particular conditions it can be welfare improving.

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1 Introduction

Nobody can deny that the last years we have experienced one of the most influential financial crises of all times. Almost all fields and disciplines of our society were affected to a great extent. The sharp changes in unemployment and growth rates led us to a new reality in which issues related to economics and finance monopolize our daily life. Investors became more risk averse trying to recover their losses while the budget of multinational companies for various activities decreased to a noticeable extent.

As for the academia, the current financial crisis became the new hot topic that is examined from various of aspects. The question of what were the causes of the crisis seems to be in the first line of research as it can be easily understood from the numerous working papers emerged and deal with this topic. Another big branch of literature focuses on the effects of the financial turmoil on the world economy.

However, due to the fact that the recent financial crisis is still ongoing and we are not yet in the position to estimate with safety its real economic effects, it is necessary to look back into historical data, on the effects of comparable recorded crises from the 20th Century. Through the study of such economic disasters Barro (2006) developed a framework that provides information about the possibility of a crisis in the future and its major macro-economic effects.

This paper attempts to develop an insight on how economic disasters affect long horizon investment and saving decisions of individuals and what the optimal response of social security should be to counterbalance such major economic shocks. Specifically, the research questions I address here are: **How do the main parameters of an economic disaster like those we experienced in the past affect consumers and investors' behavior? How do these effects depend on the specific pension scheme of the economy (Safe Pay As You Go and fully funded scheme)?**.

For this purpose, I develop a two overlapping generations model in which each individual lives, at most, two periods, each of which corresponds to 30 years. In this way, I abandon Barro's fundamental assumption that the length of each period converges to zero and I study individual's financial decisions considering a life-cycle model with longer investment horizons. The key assumption of this paper is that asset returns are log-normally distributed. The motivation for the consideration of such an assumption comes directly from the empirical work summarized in Cont[2] (2001) and the new method for the determination of individual's portfolio developed by Campbell and Viceira (2002).

The remaining of the paper is organized as follows. In sections (1.1) and (1.2) I define economic disasters and I present their empirical foundations. Inspired by the large pool of references given by Barro (2006) I combine different data sets and empirical works in order to calibrate the main parameters of the model. The calibration of the disaster parameters is based on Maddison (2003)[3] who provides long term international GDP data for 35 countries

[2]Cont presents the stylized facts of asset return. Our consideration is in agreement with the fact of "Aggregational Gaussianity".

[3]I am sceptical about Barro and Ursua's (2008) opinion that Maddison's data is problematic. Their main argument is that "Maddison tends to fill in missing data with doubtful assumptions. As examples of

from 1900 to 2000, while the mean and the variance of asset returns are taken from Global Financial Data discussed in Taylor (2005). In section (1.3), I deal with the case of long horizon investments. In particular, I specify the conditions under which the assumption of log-normality is valid and the horizon effects are negligible. In section 2, I set up the model with the underlying assumptions and we study the effects of disaster parameters on portfolio choice, consumption and savings in the fully funded case. In section 3 I extend the model by considering that there is not any risk-free asset in the market and I reproduce the main results of disaster parameters on macro-variables. I proceed by investigating the effects of the introduction of PAYG system and its welfare implications by constructing a partial and general equilibrium model.

1.1 Economic Disasters and Empirical Foundations

As a starting point we have to define what is an economic disaster and introduce the basic parameters that characterize it, namely the probability that a disaster occurs and the size of its effect on GDP. Following Barro (2006), we can define economic disasters as events that result sudden declines in real per capita GDP and in real consumer expenditures[4] by 15% or more[5]. The distinction between falls in per capita GDP and real consumer expenditures is important especially in the case of wars. Fall in GDP does not necessarily imply a drop in real consumer expenditure. For example, in the United Kingdom during the two world wars, GDP increased while real consumer expenditure fell sharply due to the increased military spending by the UK government. On the contrary in the world wars' aftermaths while some countries experienced large drops in GDP, the real consumer expenditure did not decrease.

Economic disasters can be purely economic (like the Great Depression and the current financial crisis), war time destructions (world wars), natural disasters (for example, an

the problems, Maddison assumed that Belgium's GDP during world wars moved with France's; Mexico's GDP between 1910 and 1919, the period including the Revolution and Civil War, followed a smooth trend (with no crisis). GDP of Colombia moved over more than a decade with the average of Brazil and Chile; GDP in Germany for the crucial years 1944-46 followed a linear trend... " As there is high probability that missing data corresponds to periods of economic disasters, they conclude that Maddison's work should be expanded to estimating long-term GDP for more countries. For this reason, they proceed by modifying Maddison's work including updated long-term national accounts for many countries. I agree that Maddison introduced oversimplified assumptions to solve the missing data problem, however, by comparing the calibration results for disaster parameters of the two empirical studies we see that they are almost the same! Thus, in macroscopic level Maddison's assumptions seem to be "innocent" and not problematic.

[4]Changes in real consumer expenditures are used to capture the effects of disasters on consumption due to the difficulties in breaking consumption to durables and non-durables.

[5]Barro and Ursua (2008) change this definition by including in the economic disasters, events that trigger the decline of real per capita GDP and real consumer expenditure by 10% or more. In their study, as more events are counted as economic disasters the calibrated values of the probability and the size of a disaster are affected. Here, I choose to proceed by adopting the initial definition. Even if we adopt the Barro and Ursua approach the main results of this paper related to the effects of the probability and the size of a disaster on portfolio choice, consumption and saving decisions remain the same.

impact with a near Earth cosmic object), and epidemics of disease (for instance, the avian flu). Table 1 presents the actual effects of economic disasters on real per capita GDP for 20 OECD countries for the period 1900-2000 as reported in Maddison (2003)[6].

There are 10 contractions of GDP that were associated with World War II, 8 with World War I, 8 with the Great Depression and 2 with Spanish Civil War. In all these cases, decline is real per capita GDP was accompanied by a sufficient decrease in real consumer expenditures. Notice that the most of economic disasters of the previous century were occurred during the major wars and that only one was caused because of purely economic reasons (the Great Depression)[7].

As for the non-OECD countries the major contractions of GDP by 15% or more during the wars and depressions of 20th Century are presented in Table 2. In total 15 countries, 8 in Latin America and 7 in Asia experienced economic disasters. These countries had 11 sharp contractions in the post World War II, 8 during the Great Depression, 8 in World War II and around 5 in World War I[8]. Notice that the key different between Table 1 and Table 2 is that non-OECD countries experienced post-World War II depressions while OECD countries did not. In OECD countries the major cause of disasters were major wars, while non-OECD countries were affected more by depressions. This empirical fact seems reasonable as the OECD countries were the main participants to the world wars and the course of their economy affects the economies of non-OECD countries (with some lag).

Table 1 and Table 2 show that, in total, there were 60 at least 15% contractions for 35 countries. This implies that the probability p per year for the occurrence of a disaster equals the ratio of the number of at least 15% contractions and the number of countries experienced at least once such a contraction divided by the number of years that were

[6]Notice that Table 1 measures declines in per capita GDP over consecutive years due to the fact that the duration of the effect of an economic disaster is unknown and typically more than one calendar year. This is in agreement with Barro (2006) who supports that when a disaster occurs, countries enter a regime in which the probability of downward jump in their GDP increases. In this study, for calibration purposes, we consider that the average duration of an economic disaster is 5 years.

[7]Large contractions above 15% were observed also for the periods that follow the two world wars. For example, Canada experienced a decline in real per capita GDP of 30% for the period 1917-1921. In the case of Italy the decline was 25% for the period 1918-1921. Moreover, the United Kingdom experienced such a decline after both world wars, 19% for the period 1918-1921 and 15% for the period 1943-1947. At the latter period the decrease in per capita GDP for the United States was 28%. Nevertheless, empirical evidence has shown that these contractions were not accompanied with declines in real consumer expenditure (with the possible exception of Canada). Analytically, for the United States, data from the Bureau of Economic Analysis show that the real consumer expenditure did not decline from 1944 to 1947. The same applies for the United Kingdom for both 1918-1921 and 1944-1947 periods (as Feinstein (2002) reports) and for Italy for the period 1918-1921 (Rossi et. al. (1993)). The case of Canada seems to be more puzzling as there is not any clear empirical conclusion about its real consumer expenditure for the above mentioned period.

[8]Data, available for a few additional countries starting in the 1920s and for many countries after World War II shows that 6 more countries experienced fall in their per capita GDP by at least 15% during the Great Depression (Costa Rica, Cuba, El Salvador, Guatemala, honduras and Nicaragua), and 4 during World War II (Costa Rica, Guatemala, Burma and China). Moreover other 31 post-World War II depressions (about half war related) were reported.

Table 1 : Declines of 15% or More in Real Per Capita GDP For 20 OECD Countries in Maddison [2003]

Event	Country	Years	% fall in real per capita GDP
World War I	Austria	1913-19	35
	Belgium	1916-18	30
	Denmark	1914-18	16
	Finland	1913-18	35
	France	1916-18	31
	Germany	1913-19	29
	Netherlands	1913-18	17
	Sweden	1913-18	18
Great Depression	Australia	1928-31	20
	Austria	1929-33	23
	Canada	1929-33	33
	France	1929-32	16
	Germany	1928-32	18
	Netherlands	1929-34	16
	New Zealand	1929-32	18
	United States	1929-33	31
Spanish Civil War	Portugal	1934-36	15
	Spain	1935-38	31
World War II	Austria	1944-45	58
	Belgium	1939-43	24
	Denmark	1939-41	24
	France	1939-44	49
	Germany	1944-46	64
	Greece	1939-45	64
	Italy	1940-45	45
	Japan	1943-45	52
	Netherlands	1939-45	52
	Norway	1939-44	20

Table 1 uses data from Maddison (2003) for 20 OECD countries for 1900-2000: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom, and United States. Switzerland (-11% in 1915-18 and -10% in 1929-32) had no 15% events. The United Kingdom also lacked 15% events, and the largest contraction was 7% in 1929-31. Satisfactory data for Ireland are unavailable until after World War II. Data for Greece are missing around World War I, 1914-20, and also for 1901-12.

Table 2 : Eight Latin American & Seven Asian Countries in Maddison [2003]

Event	Country	Years	% fall in real per capita GDP
World War I	Argentina	1912-17	29
	Chile	1912-15	16
	Chile	1917-19	23
	Uruguay	1912-15	30
	Venezuela	1913-16	17
Great Depression	Argentina	1929-32	19
	Chile	1929-32	33
	Mexico	1926-32	31
	Peru	1929-32	29
	Uruguay	1930-33	36
	Venezuela	1929-32	24
	Malaysia	1929-32	17
	Sri Lanka	1929-32	15
World War II	Peru	1941-43	18
	Venezuela	1939-42	22
	Indonesia*	1941-49	36
	Malaysia**	1942-47	36
	Philippines***	1940-46	59
	South Korea	1938-45	59
	Sri Lanka	1943-46	21
	Taiwan	1942-45	51
Post-WWII Depressions	Argentina	1979-85	17
	Argentina	1998-02	21
	Chile	1971-75	24
	Chile	1981-83	18
	Peru	1981-83	17
	Peru	1987-92	30
	Uruguay	1981-84	17
	Uruguay	1998-02	20
	Venezuela	1977-85	24
	Indonesia	1997-99	15
	Philippines	1982-85	18

Table 2 covers eight Latin American and seven Asian countries that have nearly continuous data from Maddison [2003] at least from before World War I. The sample is Argentina, Brazil, Chile, Colombia, Mexico, Peru, Uruguay, Venezuela, India, Indonesia, Malaysia, Philippines, South Korea, Sri Lanka, and Taiwan. Data for Argentina and Uruguay after 2001 are from Economist Intelligence Unit, Country Data. Countries that had no 15% events are Brazil (-13% in 1928-31, -13% in 1980-83), Colombia (-9% in 1913-15), and India (-11% in 1916-20, -12% in 1943-48). Data for Peru appear to be unreliable before the mid 1920s. Adjustments were made by Maddison to account for changes in country borders.

*No data available for 1942-48.

**No data available for 1941-45.

***No data available for 1943-46.

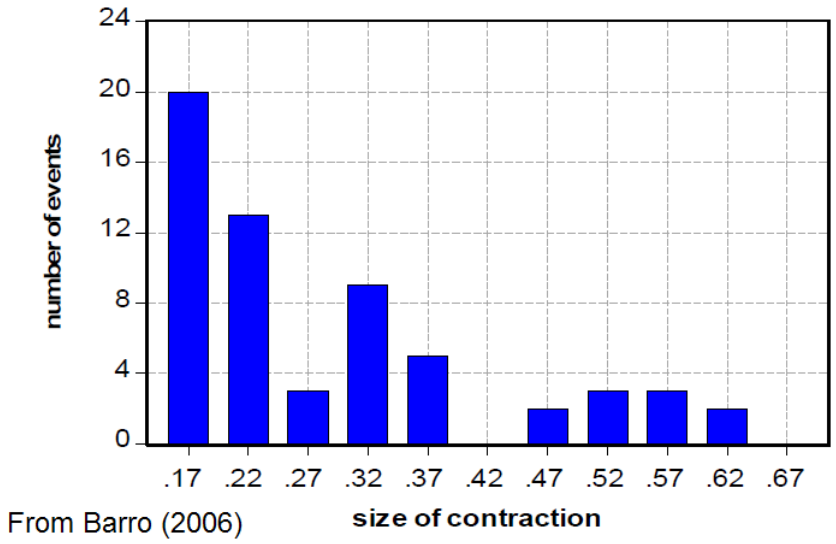


Figure 1: Frequency distribution of economic disasters, taken from Barro (2006). The histogram apply to all the 35 countries covered over the twentieth century in Table 1 and Table 2. The horizontal axis has intervals for declines in real per capita GDP. The vertical axis shows the number of economic contractions in each interval.

included in the data set:

$$p = \frac{60}{35} \times \frac{1}{2000 - 1900} = 0.017$$

As for the size of contractions, which hereafter we denote with the parameter b, according to Table I and Table II, it lies between 15% and 64%. The cumulative decline in per capita GDP during each disaster provide a reasonable estimate of b. Figure 1 presents the histogram from the contractions reported in Table 1 and Table 2. The weighted mean of contraction corresponds to[9]

$$b = 0.29$$

It becomes clear from the calibration of the disaster parameters that economic disasters are rare but result large contractions in per capita GDP.

1.2 Asset Returns and Economic Disasters

As empirical evidence has shown asset returns have been influenced from rare economic disasters. Almost in all the cases, the real rate of return on stocks fell down during such disasters, while the real rate of return on government bonds seems to have been affected to a noticeable extent, only in the cases that the rare disaster was not purely economic. Table 3 presents the real returns on stocks and on bonds for the countries that were affected by

[9]In the case of trend adjusted values of contraction sizes and for a trend growth rate for real per capita GDP of 2.5% per year, the mean becomes $b = 0.35$.

the major economic disasters of the previous century. Data is taken from Global Financial Data, except the case of Indonesia (see the notes of Table 3).

Out of the 60 cases of rare disasters, full data on asset returns are available for 21 cases. In the Great Depression and the post-World War II depressions of non-OECD countries, returns are measured up to the year before the rebound of the economy. In the former the average real rate of return was -15.3% per year for stocks and 7.5% per year for bonds, while in the latter the average real rate of returns was -27.8% per year for stocks and 6.7% per for bonds. Hence, stocks performed very poorly in comparison with bonds.

For the world wars returns are measured for the period 1914-1918 and 1939-1942 except where data is unreliable. In World War I the average real rate of returns were -16.0% per year for stocks and -9.0% per for bonds while for World War II, averages were -13.5% and -18.0%, respectively.

Portugal is the only country that despite the sharp drop in its per capita GDP during the Spanish Civil War, had positive real rate of returns of 13.4% for stocks and 3.8% for bonds.

Table 3 shows that government bonds outperformed stocks in most economic crises unrelated to war. Nevertheless, bonds were not systematic better or worse than stocks during economic contractions related to major wars.

The low return on government bonds during wartime contractions is related to partial default or loss property rights. The most common way for partial default was the depreciation of the real value of nominal debt through high inflation, which occurred shortly after some wars. For example, France experienced high inflation during and after World War I while, Austria, Belgium, Finland and Greece were some of the countries that had high inflation during or after World War II.

This paper, neglects the probability of default. Our motivation is to model disasters that are primary related to economic performance. Besides, it is difficult to find a calibrated value for the probability of default taking into account that it is strictly connected with the probability to have a major war. As the probability of a major future war is directly correlated with the stability of international (and domestic) political environment (which is by nature a dynamic rather than a static parameter), it is difficult to specify the range of values that default probability can have. Thus, we consider only the case that economic disasters result a downward jump on stock returns but they do not affect government bonds.

Annual returns for the period 1880-2004 and 1954-2004 are presented in Table 4 for the G7 countries. The long samples (1880-2004) include the disaster-type events (e.g. world wars and the Great Depression) while the period 1954-2004 corresponds to normal times after the major disasters of the previous century. In the long samples, the average annual real rate of return on stocks is 0.071, whereas, for the 1954-2004 period is 0.087. Average bond returns are -0.001 for the long samples and 0.017 for the 1954-2004 period. Notice that long samples are relatively more volatile than the normal times - samples.

Table 3 : Stock and Bond Returns during Economic Crises

	real stock return (% per year)	real bond return (% per year)
World War I		
Austria, 1914-18	--	-4.1
Denmark, 1914-18	--	-6.9
France, 1914-18	-5.7	-9.3
Germany, 1914-18	-26.4	-15.6
Netherlands, 1914-18	--	-5.2
Sweden, 1914-18	-15.9*	-13.1
Great Depression		
Australia, 1928-30	-3.6	8.2
Austria, 1929-32	-17.3*	7.1
Canada, 1929-32	-23.1*	7.1
Chile, 1929-31	-22.3*	--
France, 1929-31	-20.5	1.4
Germany, 1928-31	-14.8	9.3
Netherlands, 1929-33	-14.2*	5.7
New Zealand, 1929-31	-5.6*	11.9
United States, 1929-32	-16.5	9.3
Spanish Civil War		
Portugal, 1934-36	13.4*	3.8
World War II		
Denmark, 1939-45	-3.7*	-2.2
France, 1943-45	-29.3	-22.1
Italy, 1943-45	-33.9	-52.6
Japan, 1939-45	-2.3	-8.7
Norway, 1939-45	1.7*	-4.5
Post-WWII Depressions		
Argentina, 1998-01	-3.6	9.0
Chile, 1981-82	-37.0*	14.0
Indonesia, 1997-98	-44.5	9.6
Philippines, 1982-84	-24.3	-5.0
Thailand, 1996-97**	-48.9	6.0
Venezuela, 1976-84	-8.6*	--

Real rates of return on stocks and bonds during rare economic disasters of 20th century. Real rates of return on stocks and bonds are for periods corresponding to the economic contractions in Table 1 and Table 2. For the Great Depression and post-World War II depressions, the periods are from the start of the depression to the year before the rebound in the economy. For the world wars, the periods are 1914-18 and 1939-45, except for France and Italy where data for 1939-42 are problematic. The returns are averages of arithmetic annual real rates of return based on total return indexes, except the entries marked by an asterisk used stock-price indexes. Real values were computed by using consumer price indexes. Data are from Global Financial Data, except that the nominal return on bonds for Indonesia in 1997-98 is for money-market rates from EIU Country Data.

*Based on stock-price indexes, rather than total-return indexes.

**Thailands contraction of real per capita GDP by 14% for 1996-98 falls just short of the 15% criterion used in Table 1 and Table 2.

Table 4 : Stock and Bond Returns for G7 Countries

(averages of arithmetic annual returns, standard deviations in parentheses)

Country & time period	real stock return	real bond return	spread
1. Long samples			
Canada, 1934-2004	0.074 (0.160)	0.010 (0.036)	0.063 (0.163)
France, 1896-2004	0.070 (0.277)	-0.018 (0.095)	0.088 (0.279)
Germany, 1880-2004*	0.052 (0.306)	-0.009 (0.142)	0.061 (0.269)
Italy, 1925-2004	0.063 (0.296)	-0.009 (0.128)	0.072 (0.283)
Japan, 1923-2004	0.092 (0.296)	-0.012 (0.138)	0.104 (0.271)
U.K., 1880-2004	0.063 (0.183)	0.016 (0.055)	0.047 (0.179)
U.S., 1880-2004	0.081 (0.190)	0.015 (0.047)	0.066 (0.191)
Means	0.071 (0.240)	-0.001 (0.092)	0.072 (0.234)
2. 1954-2004			
Canada	0.074 (0.165)	0.024 (0.024)	0.050 (0.168)
France	0.091 (0.254)	0.019 (0.029)	0.072 (0.252)
Germany	0.098 (0.261)	0.018 (0.015)	0.080 (0.260)
Italy	0.067 (0.283)	0.016 (0.034)	0.051 (0.279)
Japan	0.095 (0.262)	0.012 (0.037)	0.083 (0.253)
U.K.	0.097 (0.242)	0.018 (0.033)	0.079 (0.250)
U.S.	0.089 (0.180)	0.014 (0.021)	0.076 (0.175)
Means	0.087 (0.235)	0.017 (0.028)	0.070 (0.234)

Real annual returns for stocks and government bonds for the period 1880-2004 and 1954-2004. Indexes of cumulated total nominal returns on stocks and government bonds are from Global Financial Data. See Taylor (2005) for a discussion. Nominal values for December of each year are converted to real values by dividing by consumer price indexes. Annual real returns are computed arithmetically based on December-to-December real values. CPI data since 1970 are available online from Bureau of Labor Statistics and OECD. Earlier data are from Bureau of Labor Statistics, U.S. Department of Commerce (1975), Mitchell (1980, 1982, 1983), and Mitchell and Deane (1962). The German data for the CPI have breaks corresponding to the hyperinflation in 1923-24 and the separation into East and West in 1945. Therefore, real returns are unavailable for these years. German data on dividends are missing for 1942-52. Hence, total stock returns for these years are understated.

*Data missing for Germany in 1923-24 and 1945.

1.3 Log-Normality Assumption and Long Investment Horizon

The basic principle of mean-variance analysis is that investors trade off mean value against variance in their portfolio return. Following the discussion of Campbell and Viceira (2002) we can conclude that the most tractable model that produce results in agreement with the mean variance analysis is the one in which investors have power utility and asset returns are log-normally distributed. Power utility consideration implies that absolute risk aversion is declining with consumption (or wealth), while relative risk aversion is constant. The power utility property of constant relative risk aversion is inherently attractive and is required to explain the stability of financial variables in the face of secular economic growth. The assumption of log-normality of asset returns requires both that the investor's portfolio return has a constant variance and that the returns on individual assets have constant variances and a constant covariance with the portfolio return. These conditions are consistent with one another only if the composition of the portfolio is constant, which in turn is optimal only if the expected excess returns on all assets are constant.

Notice that as a portfolio is a linear combination of individual assets and each asset return is log-normal distributed, the portfolio return is the weighted average of two log-normal variables which is not in general log-normal. Specifically, portfolio returns are log-normally distributed only if we consider short time intervals. As the time interval shrinks the non log-normality of the portfolio return diminishes, while in the limit of continuous time it totally disappears. However, the property of log-normality of the portfolio return can be conserved in longer investment horizons, as in our case, if the horizon effects[10] are negligible. Then, long term investors with horizon say, 30 years act myopically and make exactly the same portfolio choice as short term investors with horizon of one year or less.

To move further, Samuelson (1969) and Merton (1969, 1971) give two set of conditions under which long-term investors act myopically, choosing the same portfolio as a short-term investor. Specifically, portfolio choice will be myopic if investors have power utility and returns are identically and independently distributed (IID). Under the IID assumption, no new information arrives between two consecutive periods so, there is no reason for portfolio choice to change over time in a random fashion[11].

In addition, Barberis (2000), using historical data on asset returns and predictor variables estimates, calculates the magnitude of horizon effects by computing optimal asset allocations for both static buy and hold and dynamic optimal rebalancing strategies[12]. For this purpose he analyzes the portfolio choice in discrete time for an investor with power

[10] See Merton (1973) for a discussion about how variation in expected returns over time can potentially introduce horizon effects.

[11] The predictability of asset returns can be considered as one of the most popular debates in finance. From the one side, Malkiel (1973) supports that "a blindfolded chimpanzee throwing darts of the Wall Street Journal could select a portfolio that would do as well as the experts". On the other side, Shiller (2003) proves the predictability of asset returns but in his analysis he ignores the involving bubble term.

[12] In principle, the long-term investor's optimal portfolio depends not only on his objective, but also on what he is allowed to do each period. In particular, it depends on whether the investor is allowed to rebalance his portfolio each period, or must choose an allocation at time t without any possibility of asset sales or purchases during the 30 years that follow.

utility over terminal wealth in world with two different kinds of assets, treasury bills and stocks. He finds that in the buy and hold case, if returns are IID and investors ignore uncertainty about the mean and the variance of asset returns[13], they allocate the same amount to stocks regardless to their investment horizon. In particular investors use a normal distribution for their forecast of log-cumulative returns where the mean and the variance grow linearly with the investor's horizon.

In the case that parameter uncertainty is taken into account, the stock allocation falls as horizon increases. This is because the introduction of uncertainty makes stocks look riskier to long-horizon investors. In this case, the variance of the distribution of cumulative returns increase faster than linearly with the horizon as returns are not anymore IID from the point of view of the investors. Thus, horizon effects play an important role in the determination of portfolio choice.

If we relax the assumption of IID by allowing some degree of predictability over asset returns, then, in the no uncertainty case, the optimal allocation of stocks for long-horizon investor is much higher than in the short-horizon case. When we take the uncertainty into account, the long horizon stock allocation is again higher than the short horizon allocation but nearly as much higher as when we ignore estimation risk due to the fact that uncertainty increases the variance of log cumulative stock returns. Under predictability the variance of cumulative log stock returns may grow slower than linearly with the investor's horizon, making stocks look relatively less risky at longer horizons and hence leading to higher allocations to stocks in the optimal portfolio. It should be clear that the effect of predictability and that of uncertainty are in the opposite direction. The former reduces the risk of stock returns while the latter increases it. As Barberis shows, in the long horizon, the effect of predictability wins out and the allocation of stocks rises.

In conclusion, the assumption of log-normality over portfolio return requires both the exclusion of parameter uncertainty and predictability from our analysis.

Apart from log-normality of asset returns, another necessary assumption for our approach is that asset returns and consumption are jointly log-normally distributed variables. Hansen and Singleton (1983) determines what are the main forces act on consumption assuming that it is log-normally distributed. We summarize their main findings for the better understanding of the analysis that follows:

- A patient investor with high time discount factor is willing to postpone consumption raising consumption in the future relative to consumption today.
- High interest rate gives the incentive to investors to postpone consumption.
- However, diminishing marginal utility of consumption limits the willingness of the investor to deviate from a flat consumption path.
- Under power utility specification, the investor's willingness to tilt consumption in

[13]In this case, the mapping from single period log-returns to 30 year period log-returns requires that all means and variances for individual assets are scaled up by the same factor of 30.

response to incentives and the reciprocal of risk aversion coincides. Thus, under high risk aversion, investor is unwilling to postpone consumption for the future.

- Uncertainty over future rewards leads the risk averse investor to save a higher part of his income (precautionary saving).

2 The Model

2.1 Description of the Model

We consider a stochastic two-period overlapping-generations (OLG) model of closed economy with a Pay As You Go (PAYG) pension scheme. There are two generations present in each period t, the young generation that is born in period t and participates in the competitive labor market and the old generation that is born in period $t-1$ and is retired. The probability of young individuals born in period t to live throughout the old age (period $t+1$) is denoted by ε_{t+1}.

At any period t, each agent can invest either on stocks with return R^s_{t+1} and on government bonds with risk free return R^f. Thus, the return on portfolio is the weighted average of the return on stocks and the return on bonds:

$$R^P_{t+1} = y_t R^s_{t+1} + (1 - y_t) R^f \tag{1}$$

where y_t corresponds to the share of the savings that is invested on stocks at period t.

The production per worker is given by a Cobb-Douglas production function[14]:

$$F(A_t, K_t) = A_t K_t^{\alpha} \tag{2}$$

where A_t denotes the total factor productivity (TFP) at time t, K_t corresponds to the amount of capital per young individual at time t while α is the constant production elasticity.

Perfect competition and profit maximization (over capital) implies the two following equilibrium conditions:

$$W_t = (1 - \alpha) A_t K_t^{\alpha} \tag{3}$$

and

$$1 + R^s_t = 1 - \delta + \alpha A_t K_t^{\alpha - 1} \tag{4}$$

where W_t is the wage per young worker at period t received from its participation to the labor market and δ is the depreciation rate.

[14]Recall that the Cobb-Douglas is a typical neoclassical production function which exhibit constant returns to scale, positive and diminishing returns to private inputs and satisfies the Inada condition. It is one of the simplest production functions that is considered to provide a reasonable description of real world economies. It was firstly proposed by the Swedish economist Knut Wicksell.

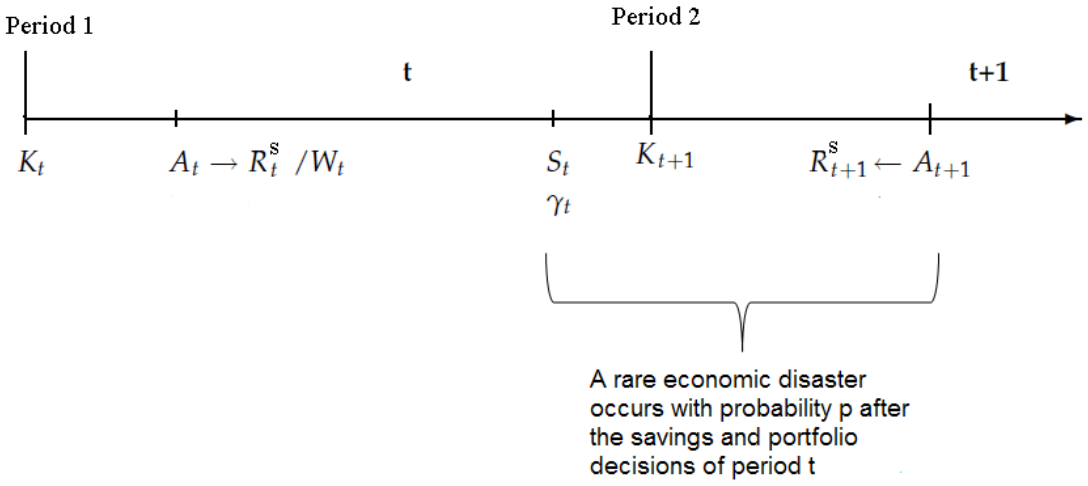


Figure 2: Time line of our model

2.2 Timing of the Model

The time line of the model is presented in Figure 2:

- Period t: At the beginning of period t, the capital stock is inherited from the last period as it is determined by the saving and portfolio decisions made in the period $t-1$ (so, it is considered to be a predetermined variable). Then, the TFP is revealed. Based on this knowledge, firms choose factor employment and consumers make their saving and portfolio decisions.

- After the saving and portfolio decisions of period t and before A_{t+1} is realized an economic disaster occurs with probability p (while the no disaster/normal times case corresponds to probability $1-p$). The major effect of such a disaster is that the A_{t+1} jumps down sharply[15] affecting the wage and the stock returns of period $t+1$. Thus, there are two possible realizations of A_{t+1}, say, $\hat{A}_{t+1}$ that corresponds to normal times (no disaster occurs) and $\check{A}_{t+1}$ that corresponds to the reduced TFP due to the occurrence of an economic disaster.

- Period t+1: The TFP is realized ($\hat{A}_{t+1}$ or $\check{A}_{t+1}$) and consequently real rate of return on stocks R_{t+1}^s is revealed.

[15]The downward jump in per capita GDP due to the disaster is fully captured by the TFP because K_{t+1} is predetermined variable. GDP per capita and TFP have the same distribution.

2.3 Assumptions and Their Implications

The main assumptions of the model are the following:

- Assumption 1: Perfect annuity markets where the assets of those who diseased at the end of period t are distributed among people who survived at period $t+1$.
- Assumption 2: TFP is log-normally distributed. This implies (through (3) and (4)) that both W_t and R_t^s are stochastic. However, people do not form expectations about W_t because A_t is already known before W_t is paid. On the contrary, due to the fact that individuals base their saving and portfolio decisions on the future return of capital, they form expectations about R_{t+1}^s.
- Assumption 3: There is 100% depreciation ($\delta = 1$).
- Assumption 4: There is no link between PAYG pension benefits and wages because in this model wages are assumed perfectly correlated with stock returns. Therefore, pension benefits are considered to be safe lump-sum benefits[16].
- Assumption 5: Consumption when old is a log-normal distributed variable.
- Assumption 6: Asset returns are IID over time.
- Assumption 7: Investors do not have the opportunity to rebalance optimally their portfolio. They buy and hold market assets for the period of 30 years (which corresponds to the first period of our 2 OLG model).

Assumption 1 is a common assumption in OLG models when there is a probability for the young to die at the end of the first period and not to enter to the retirement period. Assumptions 2 and 3 certify that R_t^s is a log-normally distributed variable. Analytically, by adopting these two assumptions, log-linearization of expression (4) for stock returns at period t+1 gives:

$$\begin{aligned} 1 + R_{t+1}^s &= 1 - \delta + \alpha A_{t+1} K_{t+1}^{\alpha-1} \\ \Rightarrow \ln[1 + R_{t+1}^s] &= \ln\alpha + \ln A_{t+1} + (\alpha - 1)\ln K_{t+1} \\ \Rightarrow r_{t+1}^s &= \ln\alpha + a_{t+1} + (\alpha - 1)k_{t+1} \end{aligned} \tag{5}$$

where, $r_{t+1}^s = \ln[1 + R_{t+1}^s]$, $a_{t+1} = \ln A_{t+1}$ and $k_{t+1} = \ln K_{t+1}$.

[16] This is a simplifying assumption that will be considered in this first state of art model. The case of not perfect correlation between wages and stock market returns seems to be more realistic and much more interesting. The reason is that in this case pension benefits are not safe but they depend on the aggregate wage growth rate. Thus, low probability economic disasters do not only affect stock market return but also pension benefits. My motivation is to develop firstly a simplified model which give reasonable and easy interpreted results before I proceed to extensions as the one discussed here.

Recall from the timing of the model that k_{t+1} is determined from the investment decisions of period t. Then, the distribution of stock returns is the same with the distribution of TFP A_{t+1} (but with different mean):

$$\begin{aligned} E_t[r^s_{t+1}] &= \ln\alpha + E_t[a_{t+1}] + (\alpha-1)k_{t+1} \\ Var_t[r^s_{t+1}] &= Var_t[a_{t+1}] \end{aligned} \tag{6}$$

Thus, indeed, stock returns are log-normally distributed. Combining this result with expression (1) we conclude that R^P_{t+1} is also log-normal distributed variable[17].

Assumption 4 implies that PAYG benefits are independent from aggregate wage growth rate. Thus, they are not affected by the disaster parameters. Assumption 5, as we will see, is necessary in order to derive the explicit solutions for portfolio choice and savings following Campbell and Viceira (2002) approach. Assumption 6, as we discussed above is connected with the validity of log-normality assumption. Finally, assumption 7 is introduced mainly for finding the calibration value of parameter b over a 30 year period. Notice that by allowing for rebalancing, we can distinguish two cases. The one where investors are sophisticated and they can rebalance optimally their portfolio when a disaster occurs. Thus, the size b of the effect of a disaster to the 30 period rate of return on stocks cannot be determined by the model as it depends on how sophisticated the investors are. The other case refers to not optimal rebalancing. As investors behave naively they are not in the position to reoptimize their investment decisions. Thus, effect b will be higher than in the optimal balancing case but again, its mean value depends on the specific intermediate decisions of investors. On the contrary, in the buy and hold case, where rebalancing is not possible, the long horizon effect of disaster of size b can be calculated, as we will see, from its short term (annual) average value considering that returns are log-normal.

A rare economic disaster as we mentioned affects the TFP and (through TFP) real rate of stock return. We model this effect as an IID shock that results a downward jump in the rate of stock return and that does not affect the shape of the probability density function (pdf) of R^s_{t+1}. The only effect of the disaster is that it shifts the pdf to lower values of R^s_{t+1}. Thus, when a disaster occurs, the stock rate of return is still a log-normal distributed variable, but with lower mean value. If R_{t+1} is the log-normal distributed rate of return on stocks that is realized during normal times, we have:

$$R^s_{t+1} = \begin{cases} R_{t+1} & \text{if no disaster occurs} \\ (1-b)R_{t+1} & \text{if a disaster occurs} \end{cases} \tag{7}$$

where b represents the size of the downward jump.

[17] Analytically, the expected value and the variance of R^P_{t+1} will be:

$$\begin{aligned} E_t[R^P_{t+1}] &= y_t E_t[R^s_{t+1}] + (1-y_t)R^f \\ Var_t[R^P_{t+1}] &= y_t^2 Var_t[R^s_{t+1}] \end{aligned}$$

which implies that if R^s_{t+1} is a log-normal distributed variable, so is R^P_{t+1}.

Hence, the expected value of the real rate of return on stocks is:

$$\begin{aligned} E_t[R^s_{t+1}] &= p(1-b)E_t[R_{t+1}] + (1-p)E_tR_{t+1} \\ &= (1-pb)E_tR_{t+1} \end{aligned} \tag{8}$$

while its variance takes the following expression:

$$\begin{aligned} Var_t[R^s_{t+1}] &= E_t([R^s_{t+1}])^2 - \{E_t[R^s_{t+1}]]\}^2 \\ &= pE_t[(1-b)^2R^2_{t+1}] + (1-p)E_t[R^2_{t+1}] - (1-pb)^2\{E_t[R_{t+1}]]\}^2 \\ &= (1-2pb+pb^2)E_t[R^2_{t+1}] - (1-2pb+p^2b^2)\{E_t[R_{t+1}]]\}^2 \\ &= (1-2pb+pb^2)Var_t[R_{t+1}] + pb^2(1-p)\{E_t[R_{t+1}]]\}^2 \end{aligned} \tag{9}$$

2.4 Short Intermezzo: Log-normal Distribution

It is useful to revise the basic properties of log-normal distributed variables before we proceed to the solution of the model. Log-normal distribution is the probability distribution of a random variable whose logarithm is normally distributed[18].

If a variable X is log-normal distributed, then its expected value and its variance are:

$$\begin{aligned} E[X] &= e^{E[\ln X]+\frac{1}{2}Var[\ln X]} \\ Var[X] &= (e^{Var[\ln X]}-1)e^{2E[\ln X]+Var[\ln X]} \end{aligned} \tag{10}$$

If we solve equations (10) for the mean and the variance of $\ln X$ we find:

$$\begin{aligned} E[\ln X] &= \ln E[X] - \frac{1}{2}\ln(1+\frac{Var[X]}{(E[X])^2}) \\ Var[\ln X] &= \ln(1+\frac{Var[X]}{(E[X])^2}) \end{aligned} \tag{11}$$

2.5 Solution of the Model

In the analysis below, the low case letters refer to the logarithm of the respective variable (for the market returns: $\ln(1+R^J_t) = r^J_t$, $J = P, s, f$, while for all the other variables: $x = \ln X$). In order to solve the model we need to define the expected utility for a representative agent of the economy. The expected lifetime utility of a representative individual born at period t is given by the CRRA utility function:

$$E_tU(C^Y_t, C^O_{t+1}) = \frac{(C^Y_t)^{(1-\theta)}-1}{1-\theta} + \frac{1}{1+\rho}\varepsilon_{t+1}E_t[\frac{(C^O_{t+1})^{(1-\theta)}-1}{1-\theta}] \tag{12}$$

[18]Note that the base of the logarithmic function does not matter for its distribution.

where C_t^Y stands for the consumption when young (period t), C_{t+1}^O is the consumption in the second period of life, θ is the coefficient of relative risk aversion and the reciprocal of the intertemporal elasticity of substitution for consumption (IES) and $\rho > 0$ represents the rate of time preference.

The lifetime budget constraint is:

$$\begin{aligned} W_t - T_t^P - T_t^B - C_t^Y &= \frac{\varepsilon_{t+1}}{1+R_{t+1}^P}(C_{t+1}^O - Z_{t+1}^P) \\ \Rightarrow C_{t+1}^O &= \frac{1+R_{t+1}^P}{\varepsilon_{t+1}}[W_t - T_t^P - T_t^B - C_t^Y + \frac{\varepsilon_{t+1}}{1+R_{t+1}^P} Z_{t+1}^P] \end{aligned} \tag{13}$$

where T_t^P the lump-sum PAYG pension contributions, Z_{t+1}^P the pension benefit received when old. The fraction $\frac{1+R_{t+1}^P}{\varepsilon_{t+1}}$ denotes the total return on savings while T_t^B represents the interest obligations on the government debt.

By maximizing the lifetime utility we derive the following Euler equation:

$$1 = \frac{1}{1+\rho}(C_t^Y)^\theta E_t[(C_{t+1}^O)^{-\theta}(1+R_{t+1}^J)] \tag{14}$$

where $J = P, s, B$.

By log-linearizing expressions (6) and (7) following Campbell and Viceira (2002) approach and taking into account (8), (9) and (11) we conclude to the following portfolio choice:

$$y_t = \frac{\ln(1+(1-pb)E_t[R_{t+1}]) - \ln(1+R_{t+1}^f)}{\theta(1-z_t)\ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]\}^2}{1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})} \tag{15}$$

where z_t is the part of old age consumption financed by PAYG.

Proof:

For $J = P$ and taking the logarithms of the two sides of (14) we have:

$$0 = \ln\frac{1}{1+\rho} + \theta \ln C_t^Y + \ln E_t[(C_{t+1}^O)^{-\theta}(1+R_{t+1}^P)] \tag{16}$$

Due to assumption 5 and because R_{t+1}^P is log-normal distributed, we can apply Jensen's lemma (which is given by the first equation of (10))[19]:

$$\begin{aligned} \ln E_t[(C_{t+1}^O)^{-\theta}(1+R_{t+1}^P)] &= E_t[\ln C_{t+1}^O)^{-\theta}] + E_t[\ln(1+R_{t+1}^P)] + \frac{1}{2}Var_t[-\theta \ln C_{t+1}^O + \ln(1+R_{t+1}^P)] \\ &= -\theta E_t[c_{t+1}^O] + E_t[r_{t+1}^P] + \frac{1}{2}Var_t[-\theta c_{t+1}^O + r_{t+1}^P] \end{aligned} \tag{17}$$

[19]For a brief summary of the applications of Jensen's inequality in finance, see Azar (2008).

Thus, (16) becomes:

$$E_t[c_{t+1}^O)] - c_t^Y = \frac{1}{\theta}\ln\frac{1}{1+\rho} + \frac{1}{\theta}E_t[r_{t+1}^P] + \frac{1}{2}\theta Var_t[c_{t+1}^O] + \frac{1}{2\theta}Var_t[r_{t+1}^P] - Cov_t[c_{t+1}^O, r_{t+1}^P] \quad (18)$$

Repeating again the calculations for $J = f$ we derive a similar expression for the risk free rate:

$$E_t[c_{t+1}^O)] - c_t^Y = \frac{1}{\theta}\ln\frac{1}{1+\rho} + \frac{1}{\theta}r^f + \frac{1}{2}\theta Var_t[c_{t+1}^O] \quad (19)$$

By subtracting (19) from (18):

$$E_t[r_{t+1}^P] - r^f + \frac{1}{2}Var_t[r_{t+1}^P] = \theta Cov_t[c_{t+1}^O, r_{t+1}^P] \quad (20)$$

The left hand side of (20) can be simplified if we take into account (1). Specifically, equation (1) can be written as:

$$\frac{1+R_{t+1}^P}{1+R^f} = 1 + y_t(\frac{1+R_{t+1}^S}{1+R^f} - 1)$$

or if we take the logarithm of both sides:

$$r_{t+1}^P - r^f = \ln(1 + y_t(e^{r_{t+1}^s - r^f} - 1)) \quad (21)$$

Taking the second order Taylor expansion of the right hand side of (21) around the point $r_{t+1}^S - r^f = 0$ we find:

$$\ln(1 + y_t(e^{r_{t+1}^s - r^f} - 1)) = y_t(r_{t+1}^s - r^f) + \frac{1}{2}y_t(1-y_t)(r_{t+1}^s - r^f)^2$$

Notice that when time interval shrinks, the deviation of the realized squared excess return $(r_{t+1}^s - r^f)^2$ from its expectation $Var_t[r_{t+1}^s - r^f]$ becomes negligible. Longer holding periods yield some horizon effects that result a deviation of $(r_{t+1}^s - r^f)^2$ from $Var_t[r_{t+1}^s - r^f]$ which is proportional on the magnitude of the horizon effects. As we have discussed above, in our consideration horizon effects are negligible, thus,

$$(r_{t+1}^s - r^f)^2 = Var_t[r_{t+1}^s - r^f] = Var_t[r_{t+1}^s] + Var_t[r^f] - 2Cov_t[r_{t+1}^s, r^f] = Var_t[r_{t+1}^s]$$

and therefore, (21) becomes:

$$r_{t+1}^P = r^f + y_t(r_{t+1}^s - r^f) + \frac{1}{2}y_t(1-y_t)Var_t[r_{t+1}^s]$$

which implies that

$$\begin{aligned} E_t[r^P_{t+1}] &= r^f + y_t(E_t[r^s_{t+1}] - r^f) + \frac{1}{2}y_t(1-y_t)Var_t[r^s_{t+1}] \\ Var_t[r^P_{t+1}] &= y_t^2 Var_t[r^s_{t+1}] \end{aligned} \tag{22}$$

Thus, the left hand side of (20) can be expressed with respect to $E_t[r^s_{t+1}]$ and $Var_t[r^s_{t+1}]$.

As for the right hand side of (20), we can derive an expression for the covariance by the log-linearization of budget constraint (13):

$$c^O_{t+1} = r^P_{t+1} - \ln \varepsilon_{t+1} + \ln[e^{w_t} - e^{\tau^P_t} - e^{\tau^B_t} - e^{c^Y_t} + e^{\ln \varepsilon_{t+1} + z^P_{t+1} - r^P_{t+1}}] \tag{23}$$

By taking the first order Taylor approximation of $\ln[e^{w_t} - e^{\tau^P_t} - e^{\tau^B_t} - e^{c^Y_t} + e^{\ln \varepsilon_{t+1} + z^P_{t+1} - r^P_{t+1}}]$ around $r^P_{t+1} = E_t[r^P_{t+1}]$ we have:

$$\begin{aligned} \ln[e^{w_t} - e^{\tau^P_t} - e^{\tau^B_t} - e^{c^Y_t} + e^{\ln \varepsilon_{t+1} + z^P_{t+1} - r^P_{t+1}}] &= \ln[e^{w_t} - e^{\tau^P_t} - e^{\tau^B_t} - e^{c^Y_t} + e^{\ln \varepsilon_{t+1} + z^P_{t+1} - E_t[r^P_{t+1}]}] \\ &\quad - \frac{e^{\ln \varepsilon_{t+1} + z^P_{t+1} - E_t[r^P_{t+1}]}}{e^{w_t} - e^{\tau^P_t} - e^{\tau^B_t} - e^{c^Y_t} + e^{\ln \varepsilon_{t+1} + z^P_{t+1} - E_t[r^P_{t+1}]}}(r^P_{t+1} - E_t[r^P_{t+1}]) \\ &= \ln[e^{w_t} - e^{\tau^P_t} - e^{\tau^B_t} - e^{c^Y_t} + e^{\ln \varepsilon_{t+1} + z^P_{t+1} - E_t[r^P_{t+1}]}] - z_t(r^P_{t+1} - E_t[r^P_{t+1}]) \end{aligned} \tag{24}$$

where,

$$\begin{aligned} z_t &= \frac{e^{\ln \varepsilon_{t+1} + z^P_{t+1} - E_t[r^P_{t+1}]}}{e^{w_t} - e^{\tau^P_t} - e^{\tau^B_t} - e^{c^Y_t} + e^{\ln \varepsilon_{t+1} + z^P_{t+1} - E_t[r^P_{t+1}]}} \\ &= \frac{\frac{\varepsilon_{t+1} Z^P_{t+1}}{e^{E_t[r^P_{t+1}]}}}{W_t - T^P_t - T^B_t - C^Y_t + \frac{\varepsilon_{t+1} Z^P_{t+1}}{e^{E_t[r^P_{t+1}]}}} \\ &= \frac{Z^P_{t+1}}{\frac{e^{E_t[r^P_{t+1}]}}{\varepsilon_{t+1}}[W_t - T^P_t - T^B_t - C^Y_t] + Z^P_{t+1}} \\ &= \frac{Z^P_{t+1}}{\frac{E_t[1+R^P_{t+1}]e^{-\frac{1}{2}Var_t[r^P_{t+1}]}}{\varepsilon_{t+1}}[W_t - T^P_t - T^B_t - C^Y_t] + Z^P_{t+1}} \end{aligned} \tag{25}$$

Note that in the last equality of (25) we used again the Jensen's lemma (10) for log-normal distributed variables. Variable z_t expresses the part of old age consumption that is financed by pension benefits Z^P_{t+1}.

Combining (23) and (24) we conclude that:

$$c^O_{t+1} = r^P_{t+1} - \ln \varepsilon_{t+1} + \ln[e^{w_t} - e^{\tau^P_t} - e^{\tau^B_t} - e^{c^Y_t} + e^{\ln \varepsilon_{t+1} + z^P_{t+1} - E_t[r^P_{t+1}]}] - z_t(r^P_{t+1} - E_t[r^P_{t+1}]) \tag{26}$$

so, the variance of c_{t+1}^O and the covariance of $Cov_t[c_{t+1}^O$ and $r_{t+1}^P]$ will be:

$$\begin{aligned} Var_t[c_{t+1}^O] &= (1-z_t)^2 Var_t[r_{t+1}^P] = (1-z_t)^2 y_t^2 Var_t[r_{t+1}^S] \\ Cov_t[c_{t+1}^O, r_{t+1}^P] &= (1-z_t) Var_t[r_{t+1}^P] = (1-z_t) y_t^2 Var_t[r_{t+1}^S] \end{aligned} \tag{27}$$

Hence, by substituting (22) and (27) to (20) we can derive an expression for the portfolio choice y_t with respect to $E_t[1+R_{t+1}^s]$ and $Var_t[r_{t+1}^s]$:

$$\begin{aligned} y_t(\ln E_t[1+R_{t+1}^s] - r^f) &= \theta(1-z_t) y_t^2 Var_t[r_{t+1}^s] \\ \Rightarrow y_t &= \frac{\ln E_t[1+R_{t+1}^s] - \ln(1+R^f)}{\theta(1-z_t) Var_t[r_{t+1}^s]} \end{aligned} \tag{28}$$

But, according to (11):

$$Var_t[r_{t+1}^s] = \ln(1 + \frac{Var[R_{t+1}^s]}{(1+E[R_{t+1}^s])^2})$$

so, by substituting (8) and (9) on this expression we get:

$$\begin{aligned} Var_t[r_{t+1}^s] &= \ln(1 + \frac{(1-2pb+pb^2)Var_t[R_{t+1}] + pb^2(1-p)\{E_t[R_{t+1}]]\}^2}{(1+(1-pb)E_t R_{t+1})^2}) \\ &= \ln(1 + \frac{(1-2pb+pb^2)Var_t[R_{t+1}] + pb^2(1-p)\{E_t[R_{t+1}]]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2 + 2(1-pb)E_t[R_{t+1}]}) \end{aligned} \tag{29}$$

Plugging (8) and (29) into (28) we derive portfolio choice (15).

End of Proof

The consumption as young will be:

$$\begin{aligned} C_t^Y &= \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + e^{(\frac{1}{2}\theta z_t^2 + (\frac{1}{2}-z_t)(\theta-1))y_t^2 \ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})} (1+R^f+y_t((1-pb)E_t R_{t+1}-R^f))^{\frac{1}{\theta}-1}\varepsilon_{t+1}} \\ &\times \left[W_t - T_t^P - T_t^B + \frac{\varepsilon_{t+1} Z_{t+1}^P}{(1+R^f+y_t((1-pb)E_t R_{t+1}-R^f))e^{-\frac{1}{2}y_t^2 \ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})}}\right] \end{aligned} \tag{30}$$

Thus, the savings of individuals will be:

$$
\begin{aligned}
S_t &= [e^{(\frac{1}{2}\theta z_t^2+(\frac{1}{2}-z_t)(\theta-1))\ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})}(1+R^f+y_t((1-pb)E_tR_{t+1}-R^f))^{\frac{1}{\theta}-1}\varepsilon_{t+1}] \\
&\times \frac{W_t-T_t^P-T_t^B}{e^{(\frac{1}{2}\theta z_t^2+(\frac{1}{2}-z_t)(\theta-1))\ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})}(1+R^f+y_t((1-pb)E_tR_{t+1}-R^f))^{\frac{1}{\theta}-1}\varepsilon_{t+1}+(1+\rho)^{\frac{1}{\theta}}} \\
&- \frac{(1+\rho)^{\frac{1}{\theta}}}{e^{(\frac{1}{2}\theta z_t^2+(\frac{1}{2}-z_t)(\theta-1))\ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})}(1+R^f+y_t((1-pb)E_tR_{t+1}-R^f))^{\frac{1}{\theta}-1}\varepsilon_{t+1}+(1+\rho)^{\frac{1}{\theta}}} \\
&\times \frac{\varepsilon_{t+1}Z_{t+1}^P}{(1+R^f+y_t((1-pb)E_tR_{t+1}-R^f))e^{-\frac{1}{2}\ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})}}
\end{aligned} \tag{31}
$$

Proof:
The expected value of c_{t+1}^O according to (26) will be:

$$E_t[c_{t+1}^O] = E_t[r_{t+1}^P] - \ln\varepsilon_{t+1} + \ln[e^{w_t} - e^{\tau_t^P} - e^{\tau_t^B} - e^{c_t^Y} + e^{\ln\varepsilon_{t+1}+z_{t+1}^P-E_t[r_{t+1}^P]}] \tag{32}$$

Thus, by equalizing (18) that was derived from Euler equation with (32) that comes from the budget constraint we have:

$$
\begin{aligned}
&c_t^Y + \frac{1}{\theta}\ln\frac{1}{1+\rho} + \frac{1}{2}\theta Var_t[c_{t+1}^O] + \frac{1}{2\theta}Var_t[r_{t+1}^P] - Cov_t[c_{t+1}^O, r_{t+1}^P] \\
&= (1-\frac{1}{\theta})E_t[r_{t+1}^P] - \ln\varepsilon_{t+1} + \ln[e^{w_t} - e^{\tau_t^P} - e^{\tau_t^B} - e^{c_t^Y} + e^{\ln\varepsilon_{t+1}+z_{t+1}^P-E_t[r_{t+1}^P]}]
\end{aligned}
$$

and by substituting the first equation (10), (27) into (32)

$$
\begin{aligned}
(32) \quad &\Rightarrow \frac{1}{2}\theta(1-z_t)^2Var_t[r_{t+1}^P] + \frac{1}{2\theta}Var_t[r_{t+1}^P] - (1-z_t)Var_t[r_{t+1}^P] + \frac{1}{2}(1-\frac{1}{\theta})Var_t[r_{t+1}^P] \\
= \quad &(1-\frac{1}{\theta})\ln E_t[1+R_{t+1}^P] - \ln\varepsilon_{t+1} + \ln[e^{w_t} - e^{\tau_t^P} - e^{\tau_t^B} - e^{c_t^Y} + e^{\ln\varepsilon_{t+1}+z_{t+1}^P-E_t[r_{t+1}^P]}] \\
- \quad &c_t^Y - \frac{1}{\theta}\ln\frac{1}{1+\rho} \\
\Rightarrow \quad &[\frac{1}{2}\theta z_t^2 + (\frac{1}{2}-z_t)(\theta-1)]Var_t[r_{t+1}^P] = (1-\frac{1}{\theta})\ln E_t[1+R_{t+1}^P] - \ln\varepsilon_{t+1} \\
+ \quad &\ln[e^{w_t} - e^{\tau_t^P} - e^{\tau_t^B} - e^{c_t^Y} + e^{\ln\varepsilon_{t+1}+z_{t+1}^P-E_t[r_{t+1}^P]}] - c_t^Y - \frac{1}{\theta}\ln\frac{1}{1+\rho}
\end{aligned}
$$

which can alternatively be written as:

$$e^{[\frac{1}{2}\theta z_t^2+(\frac{1}{2}-z_t)(\theta-1)]Var_t[r_{t+1}^P]} = \{E_t[1+R_{t+1}^P]\}^{(1-\frac{1}{\theta})}\frac{1}{\varepsilon_{t+1}}[W_t - T_t^P - T_t^B - C_t^Y + \frac{\varepsilon_{t+1}Z_{t+1}^P}{e^{E_t[r_{t+1}^P]}}]\frac{1}{C_t^Y}(1+\rho)^{\frac{1}{\theta}}$$

so, by using again (10) to substitute $E_t[r^P_{t+1}]$ and solving the resulting equation as for C^Y_t we conclude that the consumption when young is:

$$C^Y_t = \frac{(1+\rho)^{\frac{1}{\theta}}[W_t - T^P_t - T^B_t + \frac{\varepsilon_{t+1} Z^P_{t+1}}{E_t[1+R^P_{t+1}]e^{-\frac{1}{2}Var_t[r^P_{t+1}]}}]}{e^{[\frac{1}{2}\theta z_t^2 + (\frac{1}{2}-z_t)(\theta-1)]Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1} + (1+\rho)^{\frac{1}{\theta}}}$$

Thus, by substituting (22) and (29) we derive (30).

As for the savings, the expression (31) can be derived if we plug (30) into $S_t = W_t - T^P_t - T^B_t - C^Y_t$.

End of Proof

2.6 Effects of Disaster Parameters Under a Fully Funded Scheme

The main purpose of this section is to provide an illustration on how rare economic disasters affect portfolio choice, consumption and saving decisions of individuals. For favor of illustration, we assume that the financing of individuals in the second period of their life is exclusively based on their portfolio choice. Therefore, there is not a safe PAYG scheme, so, young do not pay any pension contribution and old do not receive any pension benefit ($z_t = Z^P_{t+1} = T^P_t = 0$).

In this case, the portfolio share on risky assets will be:

$$y_t = \frac{\ln(1 + (1-pb)E_t[R_{t+1}]) - \ln(1+R^f_{t+1})}{\theta \ln(1 + \frac{(1-2pb+pb^2)Var_t[R_{t+1}] + pb^2(1-p)\{E_t[R_{t+1}]\}^2}{1+(1-pb)^2\{E_t[R_{t+1}]\}^2 + 2(1-pb)E_t[R_{t+1}]})} \tag{33}$$

To capture the effects of probability p and size b of a rare economic disaster on portfolio choice we need to calibrate the main parameters of the model. As we have discussed, a realistic value for the disaster probability per year is $p = 0.017$. But, we should not ignore that for a typical two OLG model the return on portfolio is realized after around 30 years from the time of investment. Thus, the probability of the occurrence of an economic disaster in these 30 years is

$$p = 1 - 0.983^{30} = 0.4$$

As for the size of the disaster, the value of b, in one year period, it now represents a downward jump in the real rate of return on stocks. Given the definition (7) of parameter b and Table 3, we conclude that the weighted mean of b[20] is 3.092[21]. So, we should restrict our attention in values $b > 3.092$ as due to the diminishing marginal utility of consumption larger contractions count more. Thus, in the analysis below we should consider a higher

[20] if the annual rate return on stocks in normal times is 0.087 (see Table 4).

[21] In the calculation of this value we exclude the case of Portugal during the Spanish Civil War.

value for b, say, 3.3[22]. Assuming that the duration of a disaster's effects is 5 years[23], the size of the downward jump in the 30 year stock return will be equal to the expected stock return over the 30 year horizon in the case that no disaster occurs minus the expected rate of return in the case that a disaster occurs. Thus, if R_{30} is the stock return over a 30 year period in the no disaster case while $(1-b_{30})R_{30}$ is the 30 year stock return when a disaster of 5 year duration occurs, then we will have:

$$1 + R_{30} = (1 + R_1)^{30}$$

and

$$1 + (1 - b_{30})R_{30} = (1 + R_1)^{25}(1 + (1 - b_1)R_1)^5$$

where, R_1 is the annual return in the no disaster case and b_1 is the one period downward jump of the annual return conditional on the occurrence of a disaster. From the above discussion and Table 4 we have that $R_1 = 0.087$, $b_1 = 3.3$. Thus,

$$R_{30} - (1 - b_{30})R_{30} = (1 + R_1)^{30} - (1 + R_1)^{25}(1 + (1 - b_1)R_1)^5 \Rightarrow b_{30} = 0.854$$

Using the annual average values of the mean and the variance of rates of return in normal times we can calculate their values over the period of 30 years. The buy and hold specification allows the relatively easy derivation of the value of the 30 year rate of return on stocks in normal times which was used for the calibration of parameter b over the period of 30 year. As for the variance, it can be derived by applying the appropriate formula of Goodman (1962)[24].

In Figure 3 we see the effects of disaster parameters p and b on portfolio share y_t. As the probability of the disaster increases (for a given size of the disaster), individuals move their investments from risky assets (stocks) to risk free assets (government bonds). In the same way, as the size of the downward jump occurred due to a possible disaster (with a given probability) increases, individuals invest less in stocks. The direct comparison between the two curves (for the calibrated values of all the parameters) reveals that agents react more intensively when the probability of a disaster rises rather than when its size increases. This implies that individuals value more the possibility to loose some part of their investment return than the size of their actual losses[25]:

$$|\frac{\vartheta y_t}{\vartheta p}|_{b=0.854} > |\frac{\vartheta y_t}{\vartheta b}|_{p=0.4}$$

[22]Any reasonable value of b above the threshold of 3.092 does not affect the main conclusions of this study, so, there is no loss of generality to proceed assuming that $b = 3.3$.

[23]This number can be justified by Madison's data sets and here, it is only considered for calibration purposes.

[24]An alternative way to derive the variance over the period of 30 years is through the properties of log-normal distributed random variables described by (10) and (11)

[25]This result has several implications that could be tested from experimental point view. In fact, I believe that an experimental extension can contribute to behavioral finance literature. The long investment horizon of 30 years should not be considered as a drawback for such analysis. As horizon effects are negligible, the results are maintained in shorter investment periods.

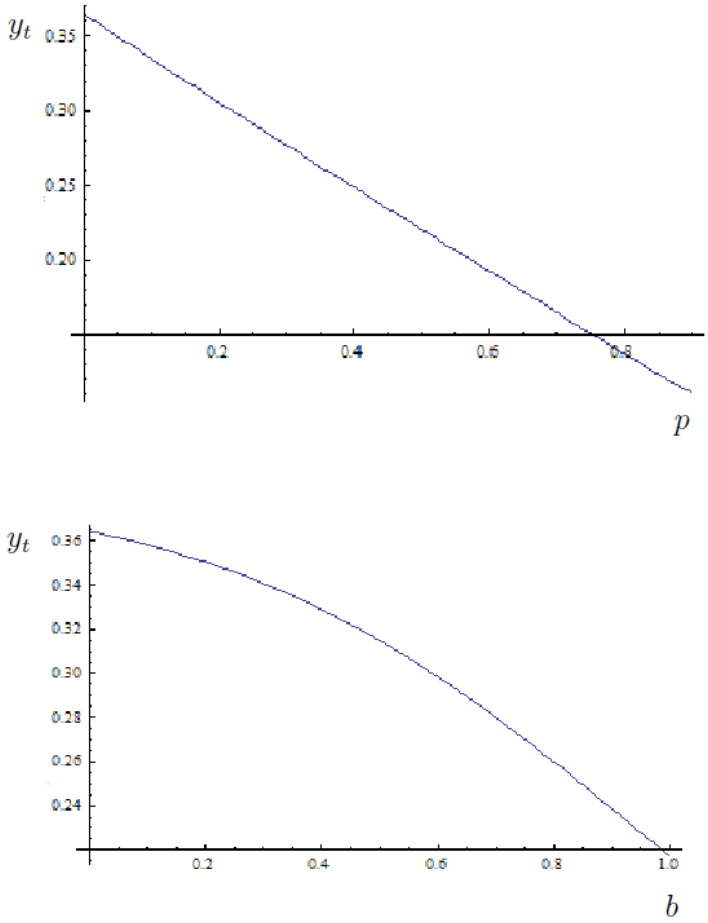


Figure 3: The effect of probability and the of a rare economic disaster to portfolio choice for $\theta = 4$, $b = 0.854$, $E_t[R_{t+1}] = (1 + 0.087)^{30}$, $Var_t[R_{t+1}] = 438.181$ and $E[R^f] = 0.017$.

Moreover, curve $y_t - b$ become steeper and steeper as b increases. This is direct implication of the diminishing marginal utility according to which we can conclude that larger contractions count more. On the other hand, an infinitesimally increase of the probability of the occurrence of a disaster from a starting value of $p = p_1$ will result a lower drop of the portfolio share on stocks than a respective increase in the probability from a starting value $p_2 < p_1$. Thus, for higher values of p, the curve $y_t - p$ becomes flatter and flatter.

The consumption C_t^Y in the case of fully funded system is given by:

$$C_t^Y = \frac{(1+\rho)^{\frac{1}{\theta}}(W_t - T_t^B)}{(1+\rho)^{\frac{1}{\theta}} + e^{\frac{1}{2}(\theta-1)y_t^2 \ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})}(1+R^f+y_t((1-pb)E_tR_{t+1}-R^f))^{\frac{1}{\theta}-1}\varepsilon_{t+1}} \tag{34}$$

while the savings become:

$$S_t = (W_t - T_t^B) \times [1 - \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + e^{\frac{1}{2}(\theta-1)y_t^2 \ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}]+pb^2(1-p)\{E_t[R_{t+1}]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2+2(1-pb)E_t[R_{t+1}]})}(1+R^f+y_t((1-pb)E_tR_{t+1}-R^f))^{\frac{1}{\theta}-1}\varepsilon_{t+1}}] \tag{35}$$

The response of consumption and savings to changes of the disaster parameters are presented in Figure 4. The y-axis in all the graphs of Figure 4 is expressed in W_t units. As the probability for the occurrence of an economic disaster increases, individuals use a smaller part of their income for consumption as young and increase their savings. The same applies when parameter b increases. Hence, higher riskiness over the future, makes it necessary to save more in order to protect oneself against very low levels of future consumption. As the parameter b increases, the curves $C_t^Y - b$ and $S_t - b$ becomes steeper due to the diminishing marginal utility of consumption. On the contrary the effect of parameter p on the slope of these curves is not clear as in the case of portfolio choice. In the neighborhood of the mean an increase in p rises the slope of the curves while for higher values of p (above 0.55) an increase in p has the opposite result. Notice that again the change in probability of a disaster in the neighborhood of its mean value seems to have a more severe effect than the respective change in b:

$$|\frac{\vartheta C_t^Y}{\vartheta p}|_{b=0.854} > |\frac{\vartheta C_t^Y}{\vartheta b}|_{p=0.4}$$

and

$$|\frac{\vartheta S_t}{\vartheta p}|_{b=0.854} > |\frac{\vartheta S_t}{\vartheta b}|_{p=0.4}$$

To sum up, as individuals know that their retirement income will entirely be based on their savings, when they face the risk of a disaster on equity, they increase their savings by investing larger part of their wage on risk free government bonds. Thus, while their savings

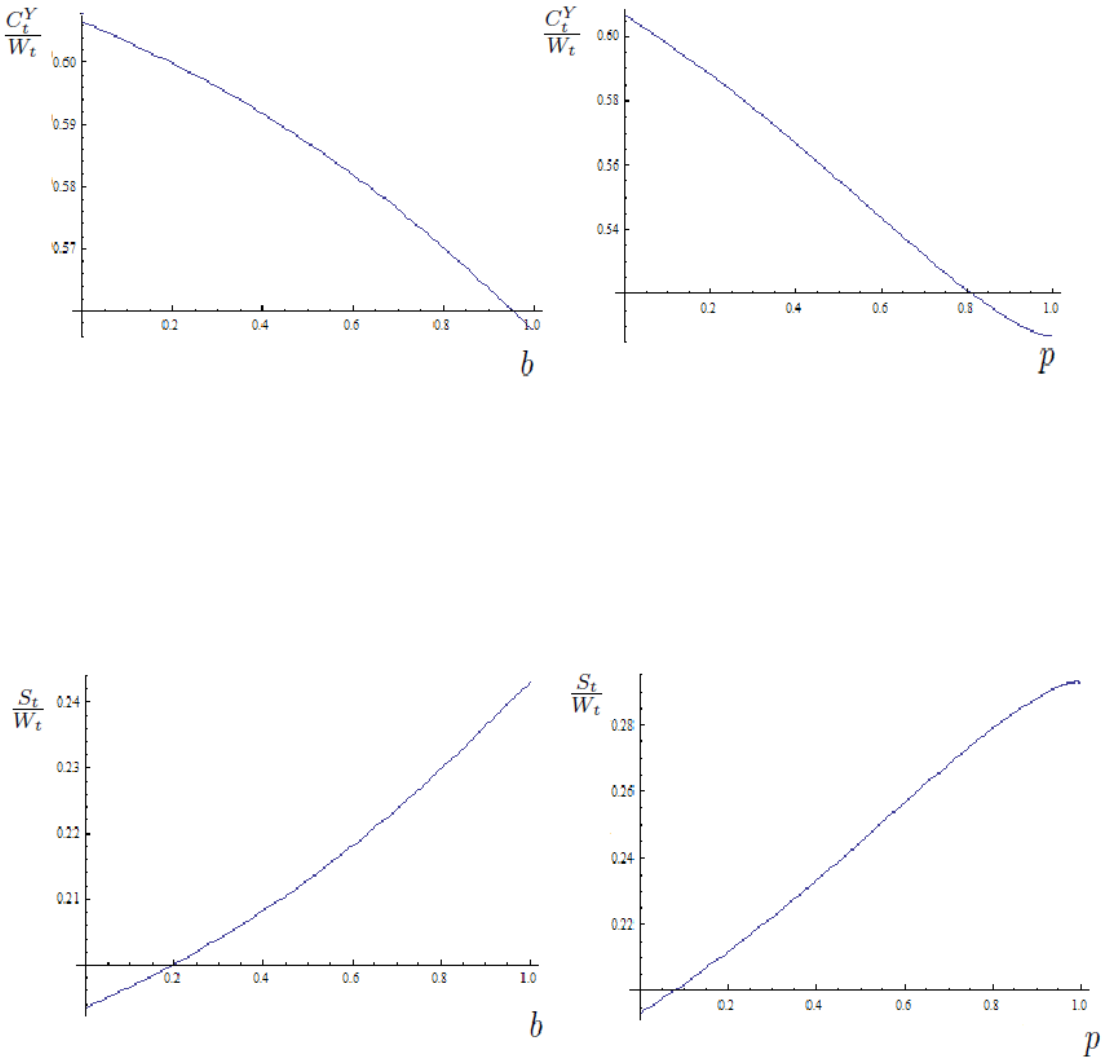


Figure 4: The effect of the probability and the size of a rare economic disaster on consumption and on savings, for $T_t^B = 0.2W_t$ and $\varepsilon_{t+1} = 0.94$.

increase, the part of their savings that they invest on equity shrinks. Interestingly, portfolio choice, savings and consumption when young are affected more severely by changes in parameter p than changes in parameter b (in the neighborhood of the mean values of p and b)[26]

Furthermore, change in disaster parameters, apart from the obvious effects on the mean of stock return ("the mean effect": as p or b increases, the average of stock return drops), affects its variance. Specifically, an increase in the probability of a disaster results the decrease of $Var_t[R^S_{t+1}]$ while an increase in b around the neighborhood of its mean value[27] rises $Var_t[R^S_{t+1}]$. Figure 5 illustrates this "variance effect" with respect to probability p and size b.

As for the lifetime utility, the effects of rare economic disasters are presented in Figure 6. As it is anticipated the utility decreases with disaster parameters. Again in the neighborhood of mean values for disaster parameters, the magnitude of the effect of an increase in p is higher than that of an increase in b.

Last but not least, we should note the two simplifications we used in order to derive these results. First of all, we assumed that government bonds are risk free. In other words, we assumed that inflation does not alter the rate of return on bonds. This is, obviously, a simplified assumption and the model can be extended in this direction by allowing the government bonds to be stochastic. Secondly, as we assumed in the beginning of this section, we are in the fully funded case. That implies, that the only source of income in the retirement period is the investments of individuals. Agents depend on their choices and consumption smoothing between the two periods can be realized only in the case that savings are sufficient. Note that the introduction of safe PAYG scheme is not going to change the above results as in the eyes of investors the safe PAYG scheme is equivalent to risk free government bonds. So, the introduction of PAYG system will only result the reduction of the amount of government bonds hold by each investor while the effect of rare economic disasters on portfolio choice, consumption and savings will remain the same.

3 Inflation Risk on Government Bonds

By considering the government bonds as risky assets due to inflation risk there is not any risk free asset in the market (in the fully funded case). This consideration will allow us to illustrate the effects of the introduction of a safe PAYG scheme to the fully funded case. We derive again the general equations of portfolio choice, consumption when young and savings before we proceed to the analysis on the effects of disaster parameters on these variables. Then, we consider the fully funded case and we study what are the effects of the

[26]This can be justified by increasing probability p and at the same time decreasing size b in such a way that the factor pb remains constant. In this case, portfolio choice and consumption decreases while savings rises. If the change move to the opposite direction, namely, if b increases and p decreases in a way that pb remains constant we find the opposite results, savings decrease while portfolio choice and consumption of young rise. Thus, indeed, the change b dominates over the change of p.

[27]For values of b below 0.83, as b increases variance of stock return drops.

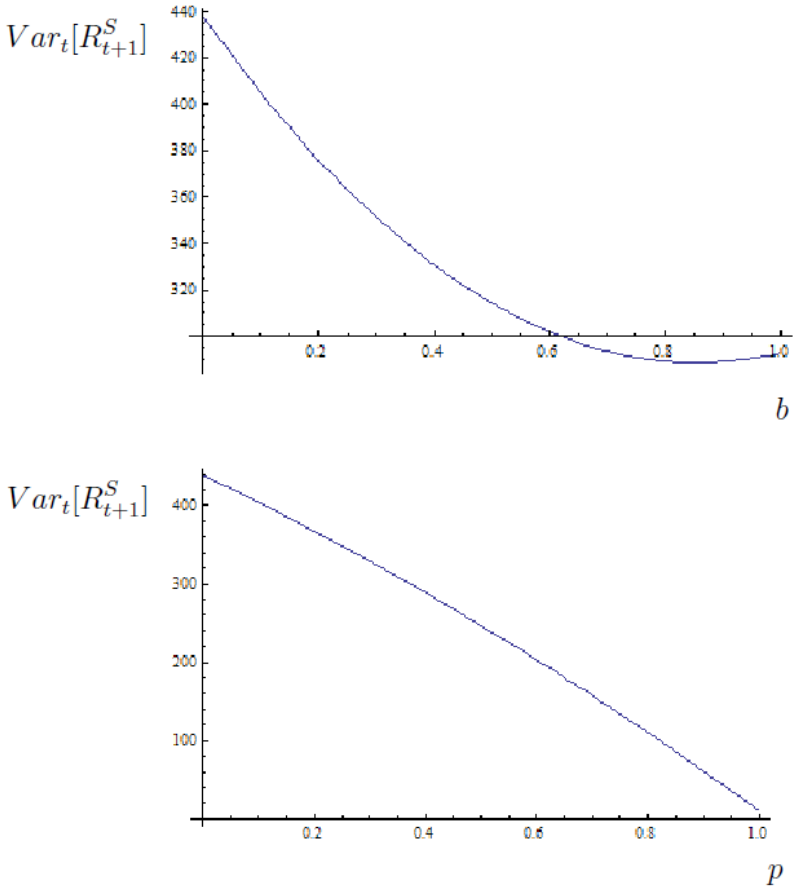


Figure 5: The "variance effect" of disaster parameters.

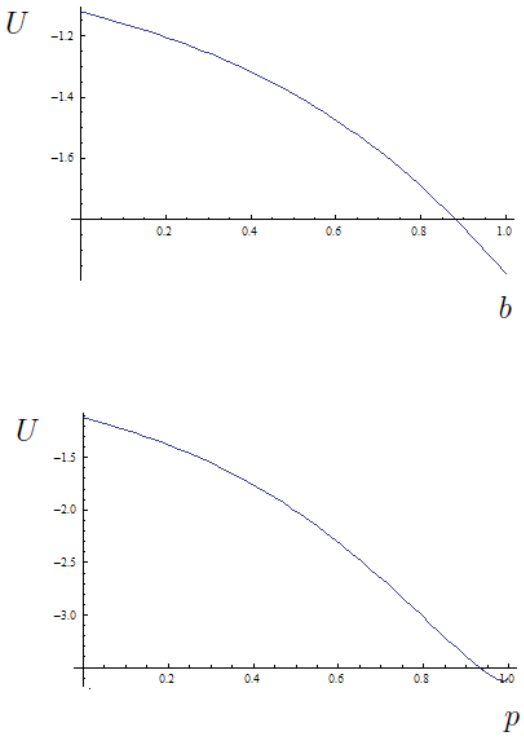


Figure 6: The effects of disaster parameters on lifetime utility.

introduction of a safe PAYG.

The safe PAYG scheme provides the retirees with a certain minimum amount of income. Notice that as people cannot invest in any risk-free asset, the introduction of safe PAYG scheme partly removes the market incompleteness, as it offers the possibility to transfer income from the working period to the retirement at a non-stochastic rate of return (which corresponds to the safeness of the system). This financial incompleteness is resulted due to the fact that there cannot be trade with unborn generations, so, human capital is not traded in financial markets. Thus, young generation is exposed to wage risk and retirees bear financial risk. The introduction of PAYG system allows the better diversification of risk between the generations by endowing retired households with a claim to labor income and therefore, it partly removes this incompleteness.

The idea that incomplete asset markets provide a justification for the PAYG social security system dates back to Diamond (1977). He points out that the absence of certain investment opportunities may lead to inefficient risk allocations. Additionally, according to Merton's (1983) point of view, PAYG system can remove the inefficiencies resulted by the non-tradeability of human capital in a two OLG framework with stochastic production. Matsen and Thogersen (2004) develop a partial equilibrium model and they derive the optimal share invested in PAYG asset. They find that the introduction of PAYG is Pareto improving reform if households are highly risk averse and they have a fairly high intertemporal elasticity of substitution[28] and physical capital is not too important in the production.

In a general equilibrium framework (see Storesletten et al. (1999), Sánchez-Marcos and Sánchez-Martin (2006), Krueger and Kubler (2006), Adema (2009)), PAYG system crowds out private savings and thus capital formation. It therefore leads to lower wages for future generations.

In what follows, I define the extended model with risky government bonds in Section (3.1) and then I present analytic expressions for portfolio choice, consumption and saving decisions. In Section (3.2) I discuss the the effects of disaster parameters on these macrovariables and in Sections (3.3) and (3.4) I study the effects of the introduction of safe PAYG system under partial and general equilibrium consideration, respectively.

3.1 The Extended Model

The real return on government bonds is:

$$1 + R_t^b = \frac{1 + R_t^N}{1 + \pi_t}$$

where, R_t^N is the nominal return on government bonds and π_t the inflation rate between periods $t-1$ and t. The nominal return is considered predetermined variable and the term

[28]Notice that the high risk aversion and high intertemporal elasticity of substitution cannot be satisfied at the same time in the power utility consideration. In this case, high risk aversion implies low intertemporal elasticity of substitution. I believe that in order to test Matsen and Thogersen's findings in our framework it is necessary to extend our model by considering an Epstein-Zin utility function.

$\frac{1}{1+\pi_t}$ is assumed to be log-normally distributed. Thus, R_t^b is also a log-normal distributed variable:

$$\begin{aligned} E_t[r_{t+1}^b] &= \ln(1+R_t^N) + E_t[\ln\frac{1}{1+\pi_t}] \\ Var_t[r_{t+1}^b] &= Var_t[\ln\frac{1}{1+\pi_t}] \end{aligned} \tag{36}$$

Adopting the framework of the state of art model presented above, the expressions for the portfolio choice becomes:

$$y_t = \frac{\ln E_t[1+R_{t+1}^s] - \ln E_t[1+R_{t+1}^b]}{\theta(1-z_t)Var_t[r_{t+1}^S - r_{t+1}^b]} - \frac{Cov_t[r_{t+1}^S - r_{t+1}^b, r_{t+1}^b]}{Var_t[r_{t+1}^S - r_{t+1}^b]} \tag{37}$$

while the expressions for consumption and savings remain the same as before. I rewrite them in a more convenient way:

$$C_t^Y = \frac{(1+\rho)^{\frac{1}{\theta}}[W_t - T_t^P - T_t^B + \frac{\varepsilon_{t+1}Z_{t+1}^P}{E_t[1+R_{t+1}^P]e^{-\frac{1}{2}Var_t[r_{t+1}^P]}}]}{e^{[\frac{1}{2}\theta z_t^2 + (\frac{1}{2}-z_t)(\theta-1)]Var_t[r_{t+1}^P]}\{E_t[1+R_{t+1}^P]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1} + (1+\rho)^{\frac{1}{\theta}}} \tag{38}$$

and

$$\begin{aligned} S_t &= \frac{e^{[\frac{1}{2}\theta z_t^2 + (\frac{1}{2}-z_t)(\theta-1)]Var_t[r_{t+1}^P]}\{E_t[1+R_{t+1}^P]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}[W_t - T_t^P - T_t^B]}{e^{[\frac{1}{2}\theta z_t^2 + (\frac{1}{2}-z_t)(\theta-1)]Var_t[r_{t+1}^P]}\{E_t[1+R_{t+1}^P]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1} + (1+\rho)^{\frac{1}{\theta}}} \\ &- \frac{(1+\rho)^{\frac{1}{\theta}}\frac{\varepsilon_{t+1}Z_{t+1}^P}{E_t[1+R_{t+1}^P]e^{-\frac{1}{2}Var_t[r_{t+1}^P]}}}{e^{[\frac{1}{2}\theta z_t^2 + (\frac{1}{2}-z_t)(\theta-1)]Var_t[r_{t+1}^P]}\{E_t[1+R_{t+1}^P]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1} + (1+\rho)^{\frac{1}{\theta}}} \end{aligned} \tag{39}$$

where $E_t[1+R_{t+1}^P] = 1 + \gamma_t(1-pb)E_t[R_{t+1}] + (1-\gamma_t)E_t[R_{t+1}^b]$, while,

$$\begin{aligned} Var_t[r_{t+1}^P] &= \gamma_t^2 Var_t[r_{t+1}^s] + (1-\gamma_t)^2 Var_t[r_{t+1}^b] + 2Cov_t[r_{t+1}^s, r_{t+1}^b] \\ Var_t[r_{t+1}^S - r_{t+1}^b] &= Var_t[r_{t+1}^S] + Var_t[r_{t+1}^b] - 2Cov_t[r_{t+1}^S, r_{t+1}^b] \end{aligned} \tag{40}$$

and

$$Cov_t[r_{t+1}^S - r_{t+1}^b, r_{t+1}^b] = Cov_t[r_{t+1}^S, r_{t+1}^b] - Var_t[r_{t+1}^b] \tag{41}$$

where $Var_t[r_{t+1}^S]$ is given by (29)

Moreover, notice that:

$$
\begin{aligned}
Cov_t[1+R^S_{t+1}, 1+R^b_{t+1}] &= E_t[(1+R^S_{t+1})(1+R^b_{t+1})] - E_t[(1+R^S_{t+1}]E_t[1+R^b_{t+1})] \\
&= e^{E_t[r^S_{t+1}]+E_t[r^b_{t+1}]+\frac{1}{2}Var_t[r^S_{t+1}]+\frac{1}{2}Var_t[r^b_{t+1}]+Cov_t[r^S_{t+1},r^b_{t+1}]} \\
&- E_t[(1+R^S_{t+1}]E_t[1+R^b_{t+1})] \\
&= e^{\ln E_t[1+R^S_{t+1}]+\ln E_t[1+R^b_{t+1}]+Cov_t[r^S_{t+1},r^b_{t+1}]} - E_t[(1+R^S_{t+1}]E_t[1+R^b_{t+1})] \\
&= E_t[1+R^S_{t+1}]E_t[1+R^b_{t+1}][e^{Cov_t[r^S_{t+1},r^b_{t+1}]}-1] \\
&\Rightarrow \; Cov_t[r^S_{t+1}, r^b_{t+1}] = \ln[1+\frac{Cov_t[1+R^S_{t+1}, 1+R^b_{t+1}]}{E_t[1+R^S_{t+1}]E_t[1+R^b_{t+1}]}]
\end{aligned}
$$

but,

$$
\begin{aligned}
Cov_t[1+R^S_{t+1}, 1+R^b_{t+1}] &= Cov_t[R^S_{t+1}, R^b_{t+1}] \\
&= E_t[R^S_{t+1}R^b_{t+1}] - E_t[(R^S_{t+1}]E_t[R^b_{t+1})] \\
&= p(1-b)E_t[R_{t+1}R^b_{t+1}] + (1-p)E_t[R_{t+1}R^b_{t+1}] - (1-pb)E_t[R_{t+1}]E_t[R^b_{t+1})] \\
&= (1-pb)Cov_t[R_{t+1}, R^b_{t+1}]
\end{aligned}
$$

Hence,

$$
Cov_t[r^S_{t+1}, r^b_{t+1}] = \ln[1+\frac{(1-pb)Cov_t[R_{t+1}, R^b_{t+1}]}{(1+(1-pb)E_t[R_{t+1}])E_t[1+R^b_{t+1}]}] \tag{42}
$$

So, from (29), (40) and (41) we have:

$$
\begin{aligned}
Var_t[r^S_{t+1} - r^b_{t+1}] &= \ln(1+\frac{(1-2pb+pb^2)Var_t[R_{t+1}] + pb^2(1-p)\{E_t[R_{t+1}]\}^2}{(1+(1-pb)^2\{E_t[R_{t+1}]\}^2 + 2(1-pb)E_t[R_{t+1}]}) + Var_t[r^b_{t+1}] \\
&- \ln[1+\frac{(1-pb)Cov_t[R_{t+1}, R^b_{t+1}]}{(1+(1-pb)E_t[R_{t+1}])E_t[1+R^b_{t+1}]}]^2
\end{aligned} \tag{43}
$$

and

$$
Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}] = \ln[1+\frac{(1-pb)Cov_t[R_{t+1}, R^b_{t+1}]}{(1+(1-pb)E_t[R_{t+1}])E_t[1+R^b_{t+1}]}] - Var_t[r^b_{t+1}] \tag{44}
$$

Using (43) and (44) we can derive the effects of disaster parameters on portfolio choice (37) for the fully funded case ($z_t = Z^P_{t+1} = T^P_t = 0$). For the value of covariance between rates of return on stocks and bonds we use the approach described in Bohrnstedt and Goldberger (1969)[29]. These effects are presented in Figure 7.

[29]Recall that the log-normality assumption requires that stock and bonds have constant covariance over time.

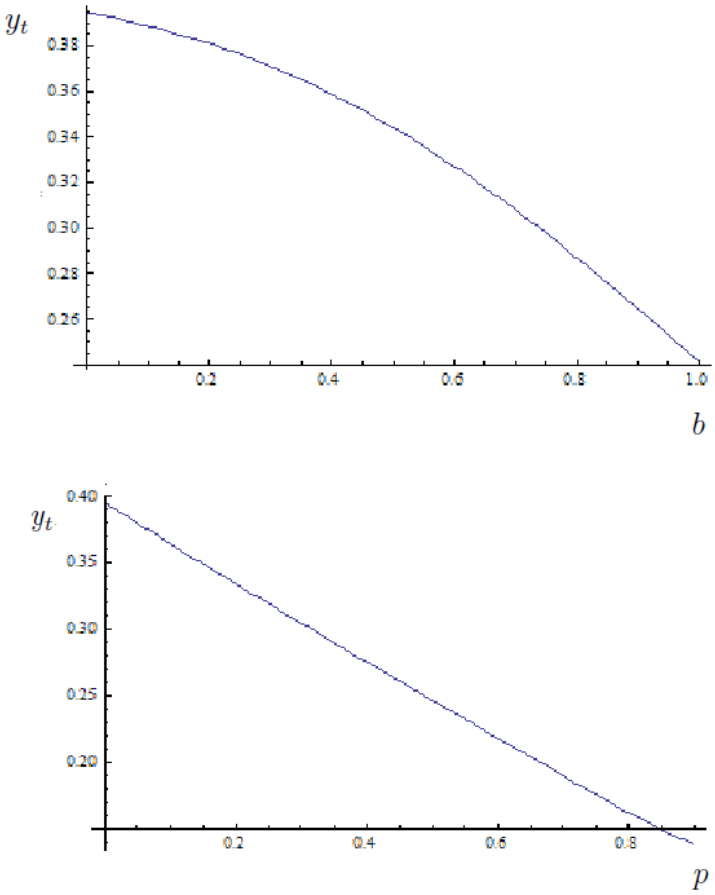


Figure 7: The effect of probability and the size of a rare economic disaster to portfolio choice for $\theta = 4$, $E_t[R_{t+1}] = 1.087^{30} - 1$, $Var_t[R_{t+1}] = 438.181$, $Var_t[R^b_{t+1}] = 0.0632$, $Cov_t[R_{t+1}, R^b_{t+1}] = 0.344355$ and $E_{R^b_{t+1}} = 1.017^{30}$.

Again, as the disaster parameters increase, agents invest a lower part of their savings on stocks. Notice that due to the riskiness of government bonds, the share of portfolio that is invested on stocks for given values of disaster parameters is higher than in the case of risk free government bonds (for example, for $p = 0.4$, $b = 0.854$, households invest the 27.5% of their portfolio on stocks when the government bonds incorporate inflation risk while they invest only 25.1% of their portfolio on stocks when government bonds are risk free assets). Moreover, from the curvature of the two curves, we conclude again that a small change in disaster probability for a given mean value of b affects to higher extent the investment behavior of households than a proportionate change in the size of the disaster given the mean value of probability p. Moreover, portfolio choice appears to be more sensitive to changes in disaster parameters when there is not any risk free asset in the market.

The effects of disaster parameters on consumption as young and savings are presented in Figure 8. As it can be easily seen the effects are similar to the risk-free bonds case. The only difference is that because of the absence of any risk-free asset, an increase of a disaster parameter has a higher effect on saving decisions than under risk-free bonds consideration. Investors have to save more in order to smooth consumption as inflation risk is taken into account. Thus, they prefer to postpone consumption for their retirement period in order to be insured against poverty.

As for the covariance between returns on stocks and bonds, it decreases, as disaster probability (given size b) or size b (given probability p) become larger (see Figure 9). Thus, as we expected, the correlation between stocks and bonds decreases as we move to higher values of p or b due to the fact that the resulting increase in expected loss affects only stocks, as bonds do not incorporate such disaster parameters[30].

Finally, the lifetime utility again decreases as disaster parameters increase. Notice that the curve $U - b$ is less steeper than in the case that bonds are considered risk-free. Additionally, for given value of disaster parameters, the utility level realized in the case that bonds are risk free is lower than the case where bonds incorporate inflation risk. This difference comes from the positive correlation between stocks and bonds that affects the utility in the latter case but is absent in the former one. In other words, this positive correlation between the two rates of return restricts the effects of p and b on lifetime utility.

In conclusion, the effects of low probability disasters follow the same direction as before but the magnitude of the effects is influenced by the inflation risk that characterizes government bonds. It seems that the conclusion that the parameter b influences less the variables of the model than the probability of a disaster is also applicable here (in the sense it was described above). This can be verified by changing the disaster parameters but keeping the expected loss constant.

[30]Similarly to the previous findings, an increase in b around its mean value has a lower impact on covariance than a respective increase in probability with which a disaster occurs.

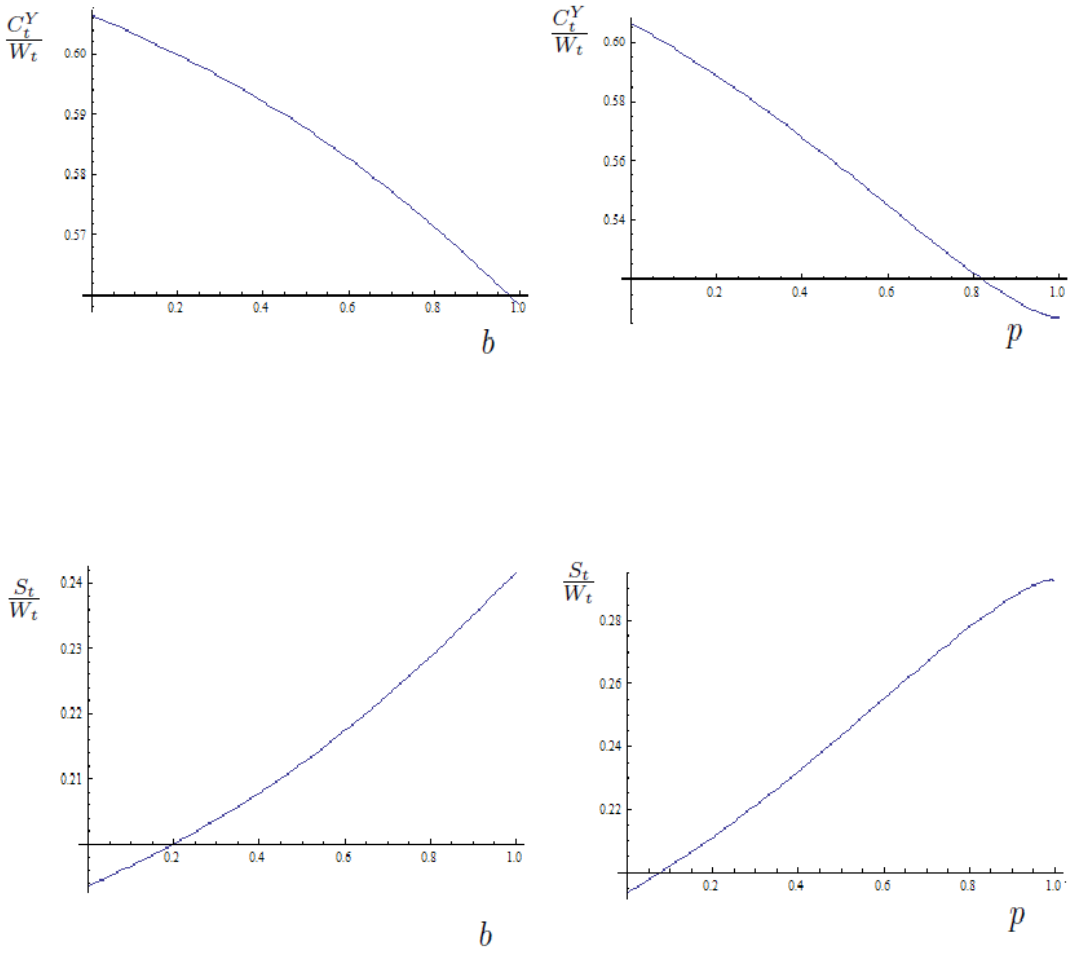
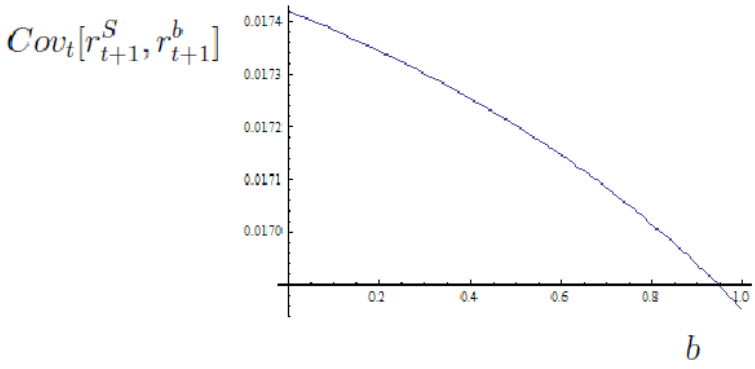


Figure 8: The effect of probability and the size of a rare economic disaster on consumption as young and savings.



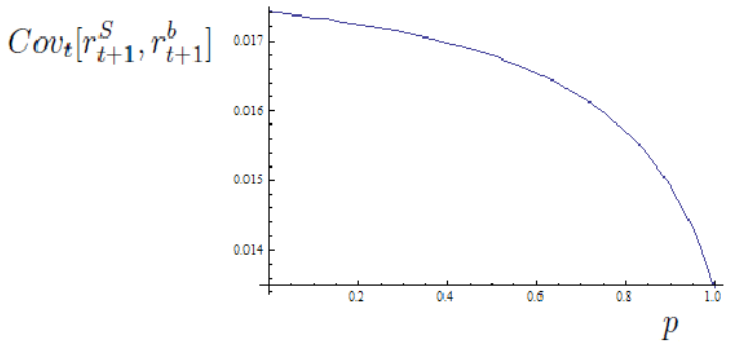


Figure 9: The effect of probability and the size of a rare economic disaster on covariance between returns on stock and government bonds.

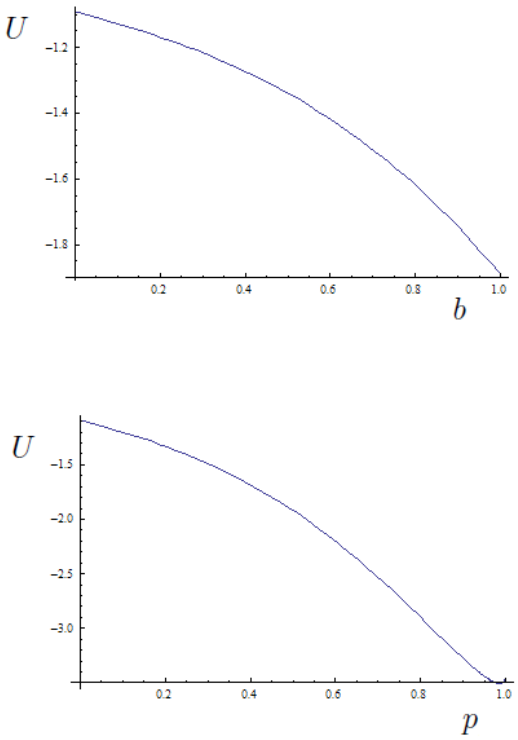


Figure 10: The effects of disaster parameters on lifetime utility in the case that bonds incorporate inflation risk.

3.2 Effects of the Introduction of PAYG Scheme

In this section, considering as a starting point the fully funded case discussed above, we investigate what are the main effects of the introduction of a safe PAYG. For this purpose we develop two different approaches. The partial equilibrium approach and the general equilibrium approach. The key difference between the two is that in the latter capital stock and rates of return are endogenously determined and therefore are affected from the introduction of PAYG while in the former they are not. Recall that with the term "safe" we refer to a lump sup pension contribution to the pension benefit (DB) of old. In the case that the pension system is defined benefit[31], if the PAYG system is totally balanced,:

$$Z_{t+1}^P = \frac{1+n_t}{\varepsilon_t} T_{t+1}^P$$

where n_t is the population growth rate between periods $t-1$ and t[32].

3.2.1 A Partial Equilibrium Approach

To begin with, we assume that the introduction of PAYG does not affect the rate of returns R_{t+1}^S and R_{t+1}^b as well as the capital stock. We assume that the safe PAYG system is introduced at period t and we analyze the effects it generates on individuals who entered the labor market at period t. Then, the introduction of a safe PAYG system affects the part of old age consumption financed by the the pension benefit. If we rewrite (25) with the help of (1) and (22) as[33]:

$$z_t = \frac{Z_{t+1}^P}{\frac{(1+E_t[R_{t+1}^P])e^{-\frac{1}{2}Var_t[r_{t+1}^P]}}{\varepsilon_{t+1}}[W_t - T_t^P - C_t^Y] + Z_{t+1}^P} \tag{45}$$

The motivation in order to use (45) instead of (25) for the analysis below is that while the safe PAYG affects the mean and the variance of return on portfolio, it does not influence the return on stocks and bonds (in this partial equilibrium analysis). Thus, it is easy to calculate the effect of Z_{t+1}^P on z_t:

$$\frac{\vartheta z_t}{\vartheta Z_{t+1}^P}|_{Z_{t+1}^P=0} = \frac{\varepsilon_{t+1}}{(1+E_t[R_{t+1}^P])e^{-\frac{1}{2}Var_t[r_{t+1}^P]}[W_t - C_t^Y]} > 0 \tag{46}$$

[31] An introduction of PAYG in a fully funded world targets mainly to alleviate individuals that are below the red line of poverty and to correct high income inequalities. Alternatively, we can consider a defined contribution system (DC). In fact, an interesting extension can be to investigate how out main results in this subsection are affected by each of those two considerations.

[32] In the case of DC this relation becomes:

$$Z_{t+1}^P = \frac{1+n_{t+1}}{\varepsilon_{t+1}} T_t^P$$

[33] Of course, the equation (45) corresponds to the case that the safe PAYG is already present

As we expected, the introduction of the safe PAYG system results the financing of a part of the old age consumption by the lump sum transfer.

Moreover, considering (37), the effect of safe PAYG on the share of portfolio that is invested on stocks will be:

$$\frac{\vartheta y_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} = \frac{\ln E_t[1+R^s_{t+1}] - \ln(1+R^f)}{\theta Var_t[r^S_{t+1} - r^b_{t+1}]} \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} > 0 \tag{47}$$

Thus, PAYG through the minimum benefit it provides to individuals that are in the retirement period, induces investors to behave in a riskier way and invest larger part of their portfolio on stocks.

In addition, the introduction of safe PAYG affects the expected value and the variance of return on portfolio. As agents invest larger part of their portfolio of stocks, the expected portfolio return increases. As for the variance, the overall effect is unclear and depends on how negative the term $Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}]$ is[34]. From (22) we have:

$$\begin{aligned}
\frac{\vartheta E_t[R^P_{t+1}]}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} &= (E_t[R^S_{t+1}] - E_t[R^b_{t+1}]) \frac{\vartheta y_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} > 0 \\
\frac{\vartheta Var_t[r^P_{t+1}]}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} &= 2(y_t Var_t[r^S_{t+1} - r^b_{t+1}] + Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}]) \frac{\vartheta y_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0}
\end{aligned} \tag{48}$$

As for the saving, taking into account (39) we can find that:

$$\begin{aligned}
\frac{\vartheta S_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} &= \frac{(W_t - T^B_t) e^{[\frac{1}{2}(\theta-1)]Var_t[r^P_{t+1}]} \{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)} \varepsilon_{t+1}}{e^{[\frac{1}{2}(\theta-1)]Var_t[r^P_{t+1}]} \{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)} \varepsilon_{t+1} + (1+\rho)^{\frac{1}{\theta}}} \\
&\times \quad [-(\theta-1) Var_t[r^P_{t+1}] \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} \\
&+ \quad \frac{1}{2}(\theta-1) \frac{\vartheta Var_t[r^P_{t+1}]}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} - (1-\frac{1}{\theta}) \{E_t[1+R^P_{t+1}]\}^{-1} \frac{\vartheta E_t[R^P_{t+1}]}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} - \frac{\varepsilon_t}{1+n_t}] \\
&- \quad \frac{(1+\rho)^{\frac{1}{\theta}} \frac{\varepsilon_{t+1}}{E_t[1+R^P_{t+1}] e^{-\frac{1}{2} Var_t[r^P_{t+1}]}}}{e^{[\frac{1}{2}(\theta-1)]Var_t[r^P_{t+1}]} \{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)} \varepsilon_{t+1} + (1+\rho)^{\frac{1}{\theta}}}
\end{aligned} \tag{49}$$

In this expression the term in the last line is clearly negative while the term in the first line is positive. Thus, if the long term that covers the second and the third line is negative, then, the effect of the introduction of PAYG is that individuals lower their savings as PAYG provides an additional source of income at the retirement period. Notice that the negative effect of safe PAYG is certified if:

$$-Var_t[r^P_{t+1}] \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} + \frac{1}{2} \frac{\vartheta Var_t[r^P_{t+1}]}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} < 0 \tag{50}$$

[34]It is $Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}] < 0$ because r^S_{t+1} and r^b_{t+1} are not perfectly correlated. For further comments on this issue see Campbell and Viceira (2007).

But, the left hand side of (50) can be written as:

$$
\begin{aligned}
- \quad & \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0}\{Var_t[r^b_{t+1}] + y_t^2 Var_t[r^S_{t+1} - r^b_{t+1}] + 2y_t Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}]\} \\
+ \quad & \frac{\vartheta y_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0}\{y_t Var_t[r^S_{t+1} - r^b_{t+1}] + Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}]\} \\
= \quad & -y_t \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0}\{y_t Var_t[r^S_{t+1} - r^b_{t+1}] + 2Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}]\} \\
+ \quad & \frac{\vartheta y_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0}\{y_t Var_t[r^S_{t+1} - r^b_{t+1}] + Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}]\} - \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} Var_t[r^b_{t+1}]
\end{aligned}
$$

Notice that by comparing (46) and (47) we can conclude that:

$$\frac{\vartheta y_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} < y_t \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0}$$

Hence, the above expression is negative if:

$$-y_t \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} Cov_t[r^S_{t+1} - r^b_{t+1}, r^b_{t+1}] - \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} Var_t[r^b_{t+1}] < 0$$

or

$$-y_t \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} Cov_t[r^S_{t+1}, r^b_{t+1}] + [y_t \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} - \frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0}] Var_t[r^b_{t+1}] < 0$$

which is true as $Cov_t[r^S_{t+1}, r^b_{t+1}] > 0$ (see Campbell and Viceira, 2007). Thus, indeed (50) is negative and

$$\frac{\vartheta S_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} < 0$$

The effect of the introduction of PAYG on consumption at the first period of agents' life can be found by differentiating (38):

$$
\begin{aligned}
\frac{\vartheta C^Y_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} = \quad & \frac{(1+\rho)^{\frac{1}{\theta}}[-\frac{\varepsilon_t}{1+n_t} + \frac{\varepsilon_{t+1}}{E_t[1+R^P_{t+1}]e^{-\frac{1}{2}Var_t[r^P_{t+1}]}}]}{e^{[\frac{1}{2}(\theta-1)]Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1} + (1+\rho)^{\frac{1}{\theta}}} \\
- \quad & \frac{(1+\rho)^{\frac{1}{\theta}}\varepsilon_{t+1} W_t \{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)} e^{-\frac{1}{2}Var_t[r^P_{t+1}]}}{[e^{[\frac{1}{2}(\theta-1)]Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1} + (1+\rho)^{\frac{1}{\theta}}]^2} \\
\times \quad & [-(\theta-1)Var_t[r^P_{t+1}]\frac{\vartheta z_t}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} + \frac{1}{2}(\theta-1)\frac{\vartheta Var_t[r^P_{t+1}]}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0} - (1-\frac{1}{\theta})\{E_t[1+R^P_{t+1}]\}^{-1}\frac{\vartheta E_t[R^P_{t+1}]}{\vartheta Z^P_{t+1}}|_{Z^P_{t+1}=0}]
\end{aligned}
\tag{51}
$$

The term in the last two lines is positive due to the fact that (50) is negative. Moreover notice that for $\varepsilon_t \simeq \varepsilon_{t+1}$:

$$[-\frac{\varepsilon_t}{1+n_t} + \frac{\varepsilon_{t+1}}{E_t[1+R^P_{t+1}]e^{-\frac{1}{2}Var_t[r^P_{t+1}]}}] < 0$$

due to the fact that in a dynamic efficient economy where Aaron condition is satisfied (see Aaron, 1966) the value of an individual's pension benefit at period $t+1$ is lower than her contribution to the scheme at period t. Thus, the overall effect of the introduction of PAYG depends on the magnitude of the two countervailing forces involved. On the one side, we have the increase of pension contribution T_t^P while, on the other the increase of pension benefit Z_{t+1}^P, portfolio return R_{t+1}^P and z_t. Notice that the negativeness of the above term depends on the increase in longevity from period t to period t+1. The higher the longevity is, the lower the effect of the above term will be. Moreover, for the calibrated values used here even for the case that $\varepsilon_{t+1} = 1$ the above term remains negative.

The results are quite intuitive. The introduction of PAYG can offer to each individual a minimum retirement income. Thus, agents, reduce the part of their wage they save in the first period of their life in order to pay part of their pension contribution while their consumption can increase or decrease depending on their retirement income expectations. Moreover, the safe benefit that they receive when old makes them feeling more secure about their future, so they are willing to undertake higher risks in their investment strategies. The introduction of safe PAYG pension scheme moves the investment strategies to the opposite direction from the economic disaster parameters resulting better risk diversification. In the former case, agents undertake riskier investments while in the latter, the main objective is a guaranteed retirement income in the second period, so, savings increase.

Notice that in this framework, cohorts that enter the labor market at period $t-1$ belong to the initial generation which receives PAYG benefit without paying any contribution. In this way, it is considered the only benefited generation as the subsequent generations have to pay additional taxes in order to make sure that the public debt is financed[35]. In this partial equilibrium approach and provided that the introduction of PAYG is announced at least one period before its implementation the effect of the introduction of PAYG on z_t, y_t and S_t on the initial generation are in the same direction as the effects for the subsequent generations described above. Specifically, as far as the part of consumption in the retirement period that is financed by PAYG, portfolio choice and portfolio returns are concerned, the effects remain the same as those described by (46),(47) and (48). For the savings, as the last negative term in line three vanishes, the effect is less negative than before. Cohorts of this generation reduce their savings to less extent and increase their consumption as young, due to the fact that they are relieved from pension contribution and have additional income to spend. Notice that this generation is the only one that its decisions are not affected whether the system is DB or BC. For the subsequent generations savings and consumption decisions are affected by what scheme is used. This is reasonable as such decisions are based on the pension contribution individuals have to pay. In DC, the above term that determines the consumption decision becomes more negative (if we take as a fact that longevity increases over time, a consideration that is not far away from the empirical findings), so, individuals respond to the introduction of PAYG through a change

[35]According to Sinn (2001), the present values of the benefit of the initial generation equals the present value of the additional taxes need to be paid by the subsequent generations for government debt financing purposes.

in their consumption less severely than in the DB case where they feel more secure about their retirement period. As for the savings, the evolution of population growth rate over time becomes an important factor of the comparison. If we adopt that n_t is declining[36], then in DC the magnitude of the effect on savings is higher as individuals reduce their savings to higher extent than in DB.

In order to develop a further insight on how the introduction of PAYG affects the main macroeconomic variables we present the following numerical application (Table 5):

T_t^P	$y_t\|_{p=b=0}$	y_t	$C_{t+1}^Y\|_{p=b=0}$	C_{t+1}^Y	$U\|_{p=b=0}$	U
0	0.376318	0.256694	0.603206	0.562689	-1.42817	-1.81189
0.05	0.397483	0.275721	0.581603	0.547205	-1.63559	-2.00923
0.1	0.425358	0.300862	0.559477	0.531382	-1.88647	-2.24103
0.15	0.463732	0.335859	0.536514	0.515005	-2.19641	-2.51952
0.2	0.519531	0.388264	0.512108	0.497605	-2.5899	-2.86529
0.25	0.605498	0.475001	0.485034	0.477974	-3.1068	-3.31523
0.3	0.741361	0.63334	0.453066	0.45276	-3.81845	-3.93625

Table 5: The effects of the introduction of safe PAYG on portfolio choice, consumption as young and lifetime utility both when disaster parameters are taken into account and when they are not.

The first column of Table 5 refers to the pension contribution while the second, fourth and sixth columns present the share of portfolio invested on stocks the consumption as young and the lifetime utility for given values of T_t^P in the case that no disaster parameters are taken into account ($p = b = 0$). The the remaining columns refer to the respective values of share investment on stocks, consumption and lifetime utility when diaster parameters are taken into account ($p = 0.4, b = 0.854$).

First of all, we see that the numerical results justify our theoretical conclusions about the portfolio choice. The introduction of safe PAYG results the increase of the share that is invested on stocks. At the same time consumption as young drops. This implies that the effect of increasing pension contribution dominates and the young generation has less available income for consumption[37]. Moreover, it is worth noting that while the effect of disaster parameters on y_t remains unaffected by the introduction of PAYG, in the case of C_{t+1}^Y and U_t the picture is different. PAYG seems to cancel out the effects of disaster parameters as the values of C_{t+1}^Y and U_t gradually approaches their "no disaster" values (as T_t^P increases[38]). Finally, notice the fact that utility decreases when PAYG is introduced in both "disaster" and "no disaster" case.

[36]Due to the decline in fertility rates.

[37]Keep also in mind the above stated remark about the power utility specification and the intertemporal elasticity of substitution.

[38]Obviously the same applies for the savings S_t.

3.2.2 General Equilibrium Approach

In general equilibrium, the capital stock and the rates of return are considered endogenously determined. In this subsection we study the effects of the introduction of PAYG in a closed economy framework. As the government debt and capital stock become the main driving forces that lead to the new steady state after the introduction of PAYG, it is necessary to introduce the following government budget constraint and capital market equilibrium condition:

$$(1+n_{t+1})B_{t+1} = (1+R_t^b)B_t + \frac{\varepsilon_t}{1+n_t}Z_t^P - T_t^P - T_t^B \tag{52}$$

where B_t is the government debt per young. Here, we consider that B_t is not affected by the introduction of the safe PAYG ($\frac{\partial B_{t+i}}{\partial Z_{t+1}^P} = 0$ for all i=0,1,2,...). So, if the PAYG system is totally balanced, we can conclude that in the steady state:

$$T_t^B = (R_t^b - n_{t+1})B \tag{53}$$

Moreover, capital market equilibrium is defined as:

$$S_t = (1+n_{t+1})(B_{t+1} + K_{t+1}) \tag{54}$$

It implies that savings at period t are sufficient to finance the capital stock and the government debt of the next period. Specifically, the share of savings that is invested to stocks can finance the capital stock of the next period while the part that is invested on government bonds is able to finance government debt:

$$\begin{aligned} y_t S_t &= (1+n_{t+1})K_{t+1} \\ (1-y_t)S_t &= (1+n_{t+1})B_{t+1} \end{aligned} \tag{55}$$

The total effect of the introduction of PAYG to the share of stocks can be written as:

$$\frac{dy_t}{dZ_{t+1}^P}|_{Z_{t+1}^P=0} = \frac{\partial y_t}{\partial E_t[R_{t+1}^S]}\frac{\partial E_t[R_{t+1}^S]}{\partial Z_{t+1}^P}|_{Z_{t+1}^P=0} + \frac{\partial y_t}{\partial E_t[R_{t+1}^b]}\frac{\partial E_t[R_{t+1}^b]}{\partial Z_{t+1}^P}|_{Z_{t+1}^P=0} + \frac{\partial y_t}{\partial z_t}\frac{\partial z_t}{\partial Z_{t+1}^P}|_{Z_{t+1}^P=0}$$

where:

$$\begin{aligned} \frac{\partial y_t}{\partial E_t[R_{t+1}^s]} &= \frac{1}{\theta E_t[1+R_{t+1}^s]Var_t[r_{t+1}^s - r_{t+1}^b]} \\ \frac{\partial E_t[R_{t+1}^s]}{\partial Z_{t+1}^P}|_{Z_{t+1}^P=0} &= -\frac{(1-a)E_t[1+R_{t+1}^s]}{K_{t+1}}\frac{\partial K_{t+1}}{\partial Z_{t+1}^P}|_{Z_{t+1}^P=0} \\ \frac{\partial y_t}{\partial E_t[R_{t+1}^b]} &= -\frac{1}{\theta E_t[1+R_{t+1}^b]Var_t[r_{t+1}^s - r_{t+1}^b]} \\ \frac{\partial E_t[R_{t+1}^b]}{\partial Z_{t+1}^P}|_{Z_{t+1}^P=0} &= \frac{E_t(1+R_{t+1}^b)}{1+R_{t+1}^N}\frac{\partial R_{t+1}^N}{\partial Z_{t+1}^P}|_{Z_{t+1}^P=0} \\ \frac{\partial y_t}{\partial z_t} &= \frac{\ln E_t[1+R_{t+1}^s] - \ln E_t[1+R_{t+1}^b]}{\theta Var_t[r_{t+1}^S - r_{t+1}^b]} \\ \frac{\partial z_t}{\partial Z_{t+1}^P}|_{Z_{t+1}^P=0} &= \frac{\varepsilon_{t+1}}{E_t[1+R_{t+1}^P]e^{-\frac{1}{2}Var_t[r_{t+1}^P]}[W_t - T_t^B - C_t^Y]} \end{aligned}$$

as it can be seen from expressions (37), (4) and (25).

Thus, the effect of safe PAYG on portfolio choice can be summarized in the relationship:

$$\begin{aligned}\frac{dy_t}{dZ^P_{t+1}}|_{Z^P_{t+1}=0} &= -\frac{1}{\theta Var_t[r^s_{t+1}-r^b_{t+1}]}[\frac{1-a}{K_{t+1}}\frac{\partial K_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}+\frac{1}{1+R^N_{t+1}}\frac{\partial R^N_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}] \\ &+ \frac{\ln E_t[1+R^s_{t+1}]-\ln E_t[1+R^b_{t+1}]}{\theta Var_t[r^S_{t+1}-r^b_{t+1}]}\frac{\varepsilon_{t+1}}{E_t[1+R^P_{t+1}]e^{-\frac{1}{2}Var_t[r^P_{t+1}]}[W_t-T^B_t-C^Y_t]}\end{aligned} \tag{56}$$

In the same way, the change of expected portfolio return (1) will be given by:

$$\begin{aligned}\frac{dE_t[R^P_{t+1}]}{dZ^P_{t+1}}|_{Z^P_{t+1}=0} &= (E_t[R^s_{t+1}]-E_t[R^b_{t+1}])\frac{\partial y_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}+y_t\frac{\partial E_t[R^s_{t+1}]}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\ &+ (1-y_t)\frac{\partial E_t[R^b_{t+1}]}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\ &= \{\frac{(1+n_{t+1})(1-y_t)(E_t[R^s_{t+1}]-E_t[R^b_{t+1}])}{S_t}+\frac{y_t}{K_{t+1}}E_t[1+R^s_{t+1}]\}\frac{\partial K_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\ &+ (1-y_t)\frac{E_t[1+R^b_{t+1}]}{1+R^N_{t+1}}\frac{\partial R^N_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\ &= -\frac{\gamma_t}{K_{t+1}}\{(y_t-a)E_t[R^s_{t+1}]+(1-\gamma_t)E_t[R^b_{t+1}]+(1-a)\}\frac{\partial K_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\ &+ (1-y_t)\frac{E_t[1+R^b_{t+1}]}{1+R^N_{t+1}}\frac{\partial R^N_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}\end{aligned} \tag{57}$$

where in the second equality we used the second equation of (55) which leads to the relation $\frac{\partial y_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}=\frac{(1+n_{t+1})(1-y_t)}{S_t}\frac{\partial K_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}$ and in the third equality the first equation of (55). The sign of the effect of the first term of the third equality depends on the exact relationship between the y_t and the a.

As for the effects of PAYG on savings, we can write:

$$\frac{dS_t}{dZ^P_{t+1}}|_{Z^P_{t+1}=0}=\frac{\partial S_t}{\partial W_t}\frac{\partial W_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}+\frac{\partial S_t}{\partial T^B_t}\frac{\partial T^B_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}+\frac{\partial S_t}{\partial T^P_t}\frac{\partial T^P_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}+\frac{\partial S_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}$$

notice that from (39), (3) and (53) we have:

$$\begin{aligned}
\frac{\partial S_t}{\partial W_t} &= \frac{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}}{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}+(1+\rho)^{\frac{1}{\theta}}} \\
\frac{\partial W_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} &= \frac{aW_t}{K_t}\frac{\partial K_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\
\frac{\partial S_t}{\partial T^B_t} &= -\frac{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}}{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}+(1+\rho)^{\frac{1}{\theta}}} \\
\frac{\partial T^B_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} &= \frac{B_t}{1+\pi_t}\frac{\partial R^N_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\
\frac{\partial S_t}{\partial T^P_t} &= -\frac{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}}{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}+(1+\rho)^{\frac{1}{\theta}}} \\
\frac{\partial T^P_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} &= \frac{\varepsilon_t}{1+n_t} \\
\frac{\partial S_t}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} &= -\frac{(1+\rho)^{\frac{1}{\theta}}\frac{\varepsilon_{t+1}}{E_t[1+R^P_{t+1}]e^{-\frac{1}{2}Var_t[r^P_{t+1}]}}}{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}+(1+\rho)^{\frac{1}{\theta}}}
\end{aligned}$$

so, we can conclude that:

$$\begin{aligned}
\frac{dS_t}{dZ^P_{t+1}}|_{Z^P_{t+1}=0} = \frac{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}}{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}+(1+\rho)^{\frac{1}{\theta}}}[\frac{aW_t}{K_t}\frac{\partial K_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\
-\frac{B_t}{1+\pi_t}\frac{\partial R^N_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}-\frac{\varepsilon_t}{1+n_t}-\frac{(1+\rho)^{\frac{1}{\theta}}}{e^{(\frac{1}{2}\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{\frac{1}{\theta}}}] \quad (58)
\end{aligned}$$

Moreover, by differentiating (54):

$$\begin{aligned}
\frac{dK_{t+1}}{dZ^P_{t+1}}|_{Z^P_{t+1}=0} &= \frac{1}{1+n_{t+1}}\frac{dS_t}{dZ^P_{t+1}}|_{Z^P_{t+1}=0} \\
&= \frac{1}{1+n_{t+1}}\frac{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}}{e^{\frac{1}{2}(\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{(\frac{1}{\theta}-1)}\varepsilon_{t+1}+(1+\rho)^{\frac{1}{\theta}}}[\frac{aW_t}{K_t}\frac{\partial K_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\
&- \frac{B_t}{1+\pi_t}\frac{\partial R^N_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0}-\frac{\varepsilon_t}{1+n_t}-\frac{(1+\rho)^{\frac{1}{\theta}}}{e^{(\frac{1}{2}\theta-1)Var_t[r^P_{t+1}]}\{E_t[1+R^P_{t+1}]\}^{\frac{1}{\theta}}}] \quad (59)
\end{aligned}$$

Notice that by differentiate the equations (55) we conclude that:

$$\frac{dK_{t+1}}{dZ^P_{t+1}}|_{Z^P_{t+1}=0} = \frac{S_t}{(1+n_{t+1})(1-y_t)}\frac{dy_t}{dZ^P_{t+1}}|_{Z^P_{t+1}=0}$$

By substituting (56) and rearranging we can derive the effect of safe PAYG on nominal return on government bonds:

$$\begin{aligned}\frac{dR^N_{t+1}}{dZ^P_{t+1}}|_{Z^P_{t+1}=0} &= -\frac{\theta(1+n_{t+1})(1-y_t)K_t Var_t[r^s_{t+1}-r^b_{t+1}]+(1-a)S_t}{S_t K_t}(1+R^{t+1}_N)\frac{\partial K_{t+1}}{\partial Z^P_{t+1}}|_{Z^P_{t+1}=0} \\ &+ \{\ln E_t[1+R^s_{t+1}]-\ln E_t[1+R^b_{t+1}]\}\frac{\varepsilon_{t+1}}{E_t[1+R^P_{t+1}]e^{-\frac{1}{2}Var_t[r^P_{t+1}]}[W_t-T^B_t-C^Y_t]}\end{aligned} \tag{60}$$

Expressions (59) and (60) can be used to analyze the general equilibrium effects of the introduction of the safe PAYG. The introduction of PAYG has as a result that the first generation that receives the PAYG benefit is privileged as it does not pay any pension contributions. All the subsequent generations pay contributions when young and receive the PAYG benefit in the second period of their life. The general effects of the introduction of PAYG for both the first and the subsequent generations are summarized in the following lines:

- The introduction of PAYG make people willing to undertake more risky investments. Thus, they have the tendency to buy more stocks than in the fully funded case. This implies that the government cannot be fully financed, so, the nominal rate of return on bonds increases in order to make government bonds more attractive and secure the financing of the public debt. This is the effect denoted by the last term of (60).

- The introduction of PAYG induce individuals to save less amount of their income as part of their consumption at their retirement period is guaranteed through the PAYG benefit they will receive (the subsequent generations have to pay a pension tax that reduces the part of their income that is available for consumption and savings). Hence, less money is available for capital stock and capital-labor ratio drops. This effect is captured by the last term within the brackets of (59).

- Lower savings and the resulting decrease in capital-labor ratio imply that the share of portfolio invested on stocks decreases and the nominal rate of return on bonds increases as the share invested on bonds should be sufficient for the financing of the government debt. This effect is reflected by the first term of (60).

Especially for the subsequent generations who pay pension contribution there are some additional effects of the introduction of PAYG (those effects are absent in the case of the first generation)

- As employees pay taxes through which the PAYG benefit of retirees is financed, their savings decrease even more. This leads to further decrease of capital-labor ratio and higher increase of the nominal rate of return on bonds. This effect is represented by the third term within the brackets of equation (59).

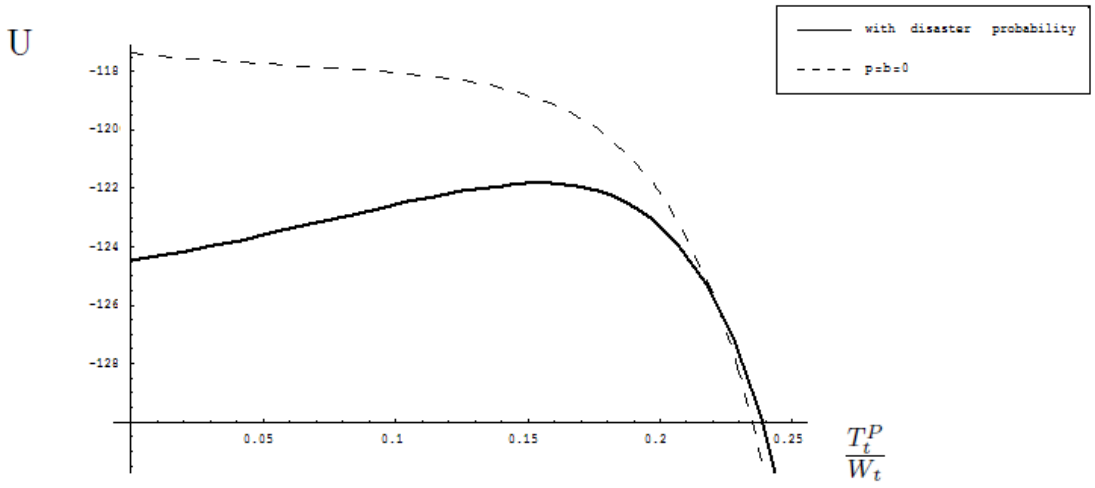


Figure 11: The effect of the introduction of PAYG system on life time utility for the "disaster" and the "no disaster" case.

- Nominal rate continues to increase over the number of periods as the economy moves to the new steady state. The second term within the brackets, in (59) captures this time evolving effect
- Capital-labor ratio decreases as number of periods increases. This effect is captured by the first term in the bracket of (59).

In conclusion, in general equilibrium consideration, the rates of return adjust in such a way that both government debt and capital stock are financed. Thus, the introduction of safe PAYG results an decrease in savings and in the share of portfolio choice invested in stocks. Notice that in the partial equilibrium analysis we found that y_t increases when the PAYG system is introduced. Here, the inclusion of the government budget constraint and capital market equilibrium leads to the opposite result. Thus, while for savings the treatment of rates of return as exogenous seems to be innocent, for portfolio choice lead to the wrong conclusion about y_t when PAYG is introduced. These effects of general equilibrium approach were verified numerically.

As for the lifetime utility of a representative agent of the subsequent generations, Figure 11 presents how the utility is affected by the introduction of PAYG in the "disaster" and the "no disaster" case for the typical values of the parameters used throughout this study. As we can see from the two curves, in the "disaster" case, the introduction of PAYG leads to an increase in the utility of the agent that faces the possibility of an economic disaster. In other words, PAYG can be used as a buffer against the risk introduced by the possibility of such violent events. The intuition behind this result is that given the aggregate shock of economic disasters, the reallocation the impact of such aggregate shocks

across generations which reduces the consumption risk in old age (intergenerational risk-sharing effect) dominates the declining aggregate capital stock (crowding out effect) only when in the aggregate shocks we include the rare economic disasters.

3.3 Summary-Conclusions

Economic disasters can be modeled as i.i.d shocks that result a downward jump on per capita GDP, TFP and consequently real rate of return on stocks. The possibility for the occurrence of such disasters has several implications for investment, consumption and savings of individuals. In the fully funded case where the retirement income depends exclusively on the investment decisions of individuals, the additional risk that is introduced by an increase in the probability or size of an economic disaster has as an effect the reduction of the share of agents' portfolios that is invested on stocks. Furthermore, consumption (at the first period) of individuals fall while their savings on government bonds increase in order to smooth their consumption in both periods of their life.

A very interesting result of this model is that an increase in the probability that a disaster occurs seems to have more severe effects on portfolio choice, consumption and savings than an increase in the size of the downward jump b.

The introduction of PAYG seems to partially cancel out the effects of economic disasters and to be beneficial for representative agent when capital stock and nominal return on bonds are treated as endogenous variables.

4 Other Remarks

During the last months I had the opportunity to discuss with a lot of people, colleagues and professors about this paper. I found the most of our discussions quite challenging and some of their questions right to the point. This section provides some remarks for the better establishment of my model that were motivated by such discussions.

It should be noted the disaster shocks modeled above as i.i.d. shocks with long term effects on the economy. That is a necessary consideration as each period lasts for around 30 years. Therefore it is assumed that the output fluctuations resulted by economic disasters are permanent rather than transitory disturbances to the level of GDP. But, is such a consideration in agreement with empirical findings? Cochrane (1988) studied the nature of such fluctuations and examined if economy had the tendency to revert to a deterministic trend value. In particular, using annual data for real per capita U.S. GNP from 1869 to 1986, he measured the relative stability of long-term growth by a variance ratio statistic which is proportional to the variance of cumulative growth over a horizon of many years divided by the variance of 1-year growth. His conclusion was that U.S. real growth rates over a horizon of a decade or more are stable relative to yearly growth rates. Thus, fluctuations in output have only temporary effects.

Cogley (1990) by using data for real per capita GDP from 1870 to 1985 for 7 OECD countries as well as real per capita GDP and GNP for the United States, showed that the

relative stability of long-term growth found by Cochrane is unique to the United States[39]. In general, for the rest of OECD Cogley showed that fluctuations in per capita GDP have permanent effects. Moreover, Cogley reported that long growth cycles are highly correlated across countries[40]. In particular, long term growth rates were found more highly correlated than 1-year growth rates.

Another issue that drew my attention through the rewarding discussions I had is the possibility of booms of the economy and whether they should be incorporated in the study as the counterpart of disasters[41]. My impression is that the possibility of booms can be neglected from the analysis. From empirical point of view, long term data on annual growth rates of per capita GDP from Maddison (2003) tend to exhibit negative skewness. Moreover, for reasonable levels of risk aversion, booms do not count as much as disasters. Furthermore, recall that in our model specification the marginal utility of consumption is diminishing. This is a theoretical figure that supports that booms do not count as much as disasters and therefore they can be neglected.

There is not doubt that the assumption of log-normality seems to be restrictive. But, our specification and the main results that can be found using this assumption is in agreement with the real experience. Notice that for the annual average values of parameters the portfolio choice seems to be affected by the economic disaster in the same way as in our study over the 30 year period. This result of our model is in agreement with Barberis (2000) and other seminal contributions in finance literature. Moreover, the fact that using the log-normality assumption we can derive a value for the 30 year period variance of rate of return on stocks which is in agreement with Goodman's formula is an additional sign that log-normality holds in long horizons.

5 Extensions

I have in mind several extensions of this model. Here are some of my thoughts:

1) Although power utility specification seems quite attractive because it is simple and its results describe in a satisfying way the macroeconomic world, it has a weak point of equating the coefficient of risk aversion and the IES (the parameter θ of our model). Empirically there are not any reasons for such an restrictive assumption. The above analysis can be extended by using an Epstein & Zin (1989) and Weil (1990) specification. I anticipate that the main results regarding the effects of the disaster parameters in the macroeconomic variable of of this study will remain the same. However, I except that the conditions under which the introduction of safe PAYG improves the utility of the representative agent will be different (maybe even in intuition).

[39] According to the findings of the paper, the United States had the smallest variance ratio among all the countries included in the study.

[40] This result is relevant with Maddison's method to treat the missing data. The assumptions he imposed seem to have the empricical support of Cogley's findings (see also footnote 1)

[41] The possibility of a boom can be considered as the possibility of an upward jump for at least, say, 15% in per capita GDP.

2) The incorporation of default probability in the analysis, which is related with government bonds and pension benefit can lead the agents to invest more in stocks. Therefore, an increase in disaster parameters may not generate so steep responses by the share of portfolio invested in stocks.

3) By linking pension benefits with the growth rate of aggregate wage we allow the economic disaster parameters to influence directly the PAYG benefits. I am not sure how this will affect the effects of the introduction of PAYG to the economy but it seems to me that it is very interesting and more realistic question.

5) Treating the PAYG pension system as a "quasi-asset" following Merton (1983), Dutta et al. (2000), Persson (2002) and Matsen & Thogersen (2004) we can estimate the optimal size on PAYG system. In all the frameworks discussed above, the optimal size of PAYG will depend of disaster parameters b and p. I expect that as those parameters increase, the optimal size of PAYG system should also increases. This can be considered as an theoretical extension of our numerical findings about the optimal size of PAYG.

6) It is interesting to see how the distinction of DB and DC affects the welfare implications of the model.

7) I defined parameter b as fixed parameter. But, from Table 3 we notice that the standard deviation of b is quite high. Alternatively, b can be defined as a random variable and the above steps can be reproduced in order to see if we will conclude to the same results.

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