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Robust Bond Investment Strategies under Parameter Uncertainty

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CentER

Tilburg University

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As you might know, keeping things short is not exactly my specialty. Luckily, this is the only chapter without a page limit. It becomes even harder to keep it short since my path to the PhD already began back in 2019, in a rather unexpected place: a cycling trip. There I saw Leonie working on her PhD deadlines during her holidays, but she quite enjoyed it. When I said I could see myself doing the same, she told me about a relevant vacancy at her sister's department.

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By now I have already spent more than two pages on academic life, but it was also fun to analyse something different: the cycling world, where I was lucky to find many people who shared the same passion. Yva, Noor, and Miranda, our endless voice memos about power data and team tactics grew into so much more and coloured my days. Thank you for making me laugh, keeping an eye on me in busy times, giving me new insights into life, and... letting me sleep. Frank, I enjoyed continuing our cycling conversations as former colleagues, and I appreciated having someone who understood that while hard work can be rewarding, there must also be time to recover on the bike. I am also grateful for all rides that helped me to switch off from work. Tim, Goedele, and Ilona, thank you for joining rides even at six in the morning or in the rain. Elisa, Erik, Jelmer, Tessa, and Bianca, thank you as well for all the rides together. And Goedele, we were meant to be: even when we did not ride together, we somehow kept bumping into each other.

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With many thanks,

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Contents

Introduction

This thesis focuses on portfolio allocations to nominal bonds when investors acknowledge that parameters cannot be estimated with certainty. Bond prices increase when interest rates fall and therefore provide protection against declines in the return on bank deposits due to lower interest rates. For this reason, bonds can be an important component of portfolios, especially for long-term investors such as pension funds. However, this attractive hedging property of bonds relies on estimated parameters of the interest rate dynamics, which are difficult to estimate with precision. This highlights the relevance of analysing parameter uncertainty, that is, accounting for multiple plausible parameter sets instead of a single one, and motivates the search for robust strategies that limit wealth losses arising from such uncertainty.

Therefore, this thesis shows how parameter uncertainty can expose investors to losses in portfolio value when holding nominal bonds, and how robust strategies can limit these losses. To analyse this, it examines how alternative values of certain parameters in the interest rate process affect bond allocations and the investor's utility. Building on this analysis, this thesis proposes robust strategies, referring to strategies that may be suboptimal under the initial parameter estimates, but can considerably reduce utility losses if the true parameter set of the financial market differs from the initial estimates. With these strategies, investors can protect themselves against large losses in wealth when future financial markets develop differently than expected.

Building on the existing literature, this thesis extends model-based optimal allocations that maximise expected utility at the investment horizon. The standard models are based on affine interest rate frameworks, which link bond prices to underlying risk factors and form the basis of seminal work such as Brennan and Xia (2002) and Sangvinatsos and Wachter (2005). They rely on the standard concept of risk, where investors form attitudes towards a known probability distribution of the risk factors. In this setting, the thesis considers a range of risk aversion: less risk-averse investors accept possible bond portfolio losses from rising interest rates in exchange for a risk premium, whereas very risk-averse investors prioritise protecting their wealth against such risks.

Beyond this framework, this thesis incorporates Knightian uncertainty, where the probability distribution of risks is unknown: the investor has initial beliefs about the nominal interest process, but acknowledges that these beliefs might be misspecified. Allocations under the standard risk preferences are only optimal if the interest rate process is estimated correctly. However, this assumption is fragile, as model-based bond allocations are highly sensitive to estimation errors (Martellini et al. (2015)).[1] Yet the question of how investors should allocate when acknowledging possible misspecifications has received only limited attention in the bond allocation literature.

This thesis consists of three chapters, each addressing a different aspect of parameter uncertainty and robustness. Chapter 1 shows how incorrect parameter values can cause substantial wealth losses. Moreover, strategies less sensitive to these parameter values, although not optimal under correct estimates, can prevent such losses. Chapter 2 extends the analysis by using a robust control method to incorporate parameter uncertainty and derive a formally defined max-min robust strategy that limits losses when the worst-case parameter set materialises. Chapter 3 combines these approaches by analysing the impact of estimation errors on wealth and deriving a max-min strategy across several specifications. Moreover, it introduces uncertainty about a potential climate impact on interest rates.

To provide an overarching perspective in more detail, I now discuss the aspects of parameter uncertainty and robustness that are addressed in each chapter. Chapter 1, *Robust Hedging of Terminal Wealth under Interest Rate Risk and Inflation Risk*, which is joint work with Anne Balter and Frank de Jong, demonstrates the impact of incorporating parameter uncertainty by applying an investment strategy based on incorrect parameter values. We highlight the relevance of parameter uncertainty in certain parameters of the interest rate process: the mean-reversion parameters of the nominal interest rate and inflation rate, which determine the speed at which the factors revert to their long-term means after a shock, and the feedback parameter, which captures the impact of the inflation rate level on the drift of the nominal interest rate. Small estimation errors in these parameters can cause large losses in wealth, compared to the correct beliefs.

In the context of robustness, we show the advantage of single-bond strategies, even while the optimal strategy without parameter uncertainty is to invest in two bonds. In practice, this term “bond” may also refer to a bond portfolio constructed to replicate the

[1] Chapter 3 motivates that estimation errors are likely to occur by providing an overview of different estimates, showing that the interest rate process can vary considerably across time periods and data sets.

interest rate sensitivity of a single bond, which is the more common approach.[2] Such strategies can mitigate utility losses arising from estimation errors and therefore can serve as "robust" strategies compared to two-bond strategies. The following chapter provides a more formal definition of a robust strategy.

Chapter 2, *Robust Hedging of Terminal Wealth under Interest Rate Risk with the Constraint Approach*, which is joint work with Anne Balter and Frank de Jong, builds on the nominal interest rate and the inflation rate as risk factors, similarly to Chapter 1. We now incorporate parameter uncertainty by applying the constraint approach, where the parameters of the financial market are allowed to deviate from their initial estimates within a bounded set. We model these deviations by distortions in the market price of risk parameter, where this parameter reflects the compensation for bearing interest rate risk by investing in bonds. We first apply a commonly used constant distortion. Moreover, we introduce a stochastic distortion which translates into changes in the mean-reversion parameter, where we highlighted the relevance of these changes in Chapter 1.[3]

Where Chapter 1 shows the sensitivity of strategies, we now formally determine the robust strategy by deriving max-min bond allocations. In this framework, the investor chooses the allocation that *maximises* expected utility under the assumption that "mother nature" *minimises* expected utility by selecting the worst-case parameters for the financial market. The robust strategy is suboptimal if the initial parameters are correct, but becomes optimal if these worst-case parameters materialise. Under the constant distortion, a robust investor anticipating the worst-case scenario (in which risk premia vanish), should invest like a very risk-averse investor focused on hedging interest rate risk. Under the stochastic distortion, the robust strategy depends on both the current level of the nominal interest rate and the investment horizon.

Chapter 3, *Bond Allocations under Climate Risk and Parameter Uncertainty*, which is single-authored, extends the analysis by introducing uncertainty about a potential climate impact. The investor now acknowledges the possibility that climate change can affect long-term bond returns, not yet explored in the bond allocation literature. The motivation is that an increase in actual or perceived climate risk can lower interest rates through an increased demand for bonds as safe assets. I analyse two potential climate risk factors: temperature deviations from its mean, and a climate news index derived from news coverage in the Wall Street Journal. The latter risk factor appears especially

[2]For the two-bond strategy, this would refer to two bond portfolios with different sensitivities.

[3]The stochastic distortion is applied to a one-factor model, where the feedback parameter is not relevant.

promising to consider, as it improves the estimation of the nominal interest rate compared to an estimation without a climate factor. Moreover, the model estimates indeed imply that higher climate indices are associated with lower future interest rates.

I incorporate two types of uncertainty. First, I introduce a new dimension by considering an investor who is uncertain whether the feedback parameter and risk factor correlation are equal to zero or not. These cases correspond to financial markets in which climate either does or does not affect bond returns, and can therefore be interpreted as an investor facing uncertainty about the relevant set of risk factors. Second, I analyse the impact of estimation errors, as in Chapter 1, by assuming that climate risk is present but that the parameter estimates of the climate factor process may be misspecified. I show that estimation errors in the risk factor correlation can lead to substantial utility losses.

With regard to robust strategies, I derive the robust max-min allocation as in Chapter 2. For the first type of uncertainty, the robust strategy is to invest as if climate risk affects bond returns, even when the investor initially believes that there is no climate impact. For the second type of uncertainty, I construct the robust strategy across three data-based values of the relevant parameters: the initial estimate, a smaller value, and a larger value. Finally, I propose a robust strategy that can mitigate the losses caused by estimation errors in the correlation between the nominal interest rate and the climate factor, as proxied by temperature deviations.

The overarching message of the chapters is that long-term investors should recognise that initial assumptions may be misspecified when investing in nominal bonds over long investment horizons. To mitigate the risk of substantial wealth losses due to these misspecifications, investors can consider robust strategies that may prevent these losses.

The remainder of this thesis is structured such that each chapter has its own introduction and conclusion, and the appendices and complete reference list are provided at the end. The thesis concludes with an English and a Dutch summary.

1

Robust Hedging of Terminal Wealth under Interest Rate Risk and Inflation Risk

Joint work with Anne G. Balter and Frank de Jong.[1]

[1]We are grateful for the valuable feedback of the discussants at the Netspar Pension Day (2021), the Netspar International Pension Workshop (2021,2022), and the International Pension Research Association Doctorial Tutorial (2022); Matteo Bonetti, Peter Schotman, Sijmen Duinveld, and Jonathan Ziveyi. We are also grateful for Antoon Pelsser, Kim Peijnenburg, Mathijs Cosemans, Christoph Hambel, Bart Dees, Bas Werker, Theo Nijman, Lieke Beekers, Jorgo Goossens, Henk Keffert, and other conference and seminar participants for their helpful comments and suggestions.

ABSTRACT: *Long-term investors face challenges in hedging both interest rate and inflation risks. We assume that these risks follow a bivariate mean-reverting process, and we demonstrate that optimal allocations to nominal bonds are sensitive to small changes in the parameters of the risk factors process. The investor may face significant potential losses in utility due to estimation errors in the mean-reversion parameters and the feedback parameter, which accounts for the impact of the level of the inflation rate on the drift of the nominal interest rate. We show that single-bond strategies can lead to smaller utility losses in case of estimation errors than the initially optimal two-bond strategy.*

1.1 Introduction

Long-term institutional investors typically hedge against nominal interest rate risk by holding long positions in nominal bonds. However, inflation erodes the real value of wealth over time, particularly for a long investment horizon. Thus, it is crucial for the investor to consider both interest rate and inflation risks when determining the optimal nominal bond investments. In practice, these risks are only partially hedged with correlated stocks and inflation-linked bonds (Munk et al. (2004); Lioui and Tarelli (2023)).

The model proposed by Sangvinatsos and Wachter (2005) derives an optimal strategy that hedges both nominal interest rate and inflation risks by only investing in a portfolio of nominal bonds. In this set-up, the risk factors follow a mean-reverting process and are linked through correlated shocks and a feedback parameter which accounts for the impact of the inflation rate level on the nominal interest rate drift term. However, this optimal bond strategy is sensitive to small changes in underlying parameters that are relevant for investors aiming to hedge shocks in the risk factors. For instance, the mean-reversion parameters determine the persistence of shocks in the risk factors and are especially relevant for long investment horizons, but these parameters are difficult to estimate precisely (Brennan and Xia (2002); Martellini et al. (2015)). Moreover, a different feedback parameter leads to a significantly different optimal strategy, as studied by Brennan and Xia (2002) and Munk et al. (2004).

We compute the impact on expected utility from terminal real wealth if the investor bases her bond strategy on incorrect estimations of these parameters. Moreover, we investigate whether suboptimal single-bond strategies result in smaller utility losses in the presence of estimation errors than two-bond strategies. These single-bond strategies are usually used in the literature to constrain the bond positions to be non-negative or to measure the cost of ignoring inflation risk and tend to result in smaller bond positions

(Brennan and Xia (2002); Van Bilsen et al. (2020); Munk et al. (2004)). As a result, they may produce less sensitive strategies compared to the optimal two-bond strategy, which typically involves extreme long-short positions.

Our study makes two main contributions to the existing literature. First, we analyse the impact of changes in the mean-reversion parameters and the feedback parameter on bond allocations and the investor's utility. Flor and Larsen (2014) conduct a similar analysis, but focus on the market price of risk parameter, which is relevant for more speculative investors.[2] In contrast, long-term institutional investors have a considerable hedging demand for bonds to mitigate the real interest rate risk. Second, we investigate single-bond strategies, as these are expected to be less sensitive to parameter changes. Specifically, we focus on the trade-off of single-bond strategies: while these are unable to fully hedge inflation risk, they result in a smaller potential utility loss in the presence of estimation errors compared to the optimal strategy. Feldhütter et al. (2012) study this trade-off between a single-bond and a multiple-bond strategy, but they do not incorporate inflation risk, nor the isolated effect of uncertainty regarding the mean-reversion parameters or the feedback parameter. Moreover, they consider strategies based on either one or three risk factors, and thus do not analyse the trade-off between parameter uncertainty and the ability to hedge two risk factors using a single- or two-bond strategy.

We apply the model of Sangvinatsos and Wachter (2005, henceforth SW), which is a foundational model in the optimal allocation literature to hedge the real interest rate which is assumed to be affine in risk factors. We assume a financial market without stocks or unhedgeable inflation. This choice will be further motivated in the model description, but the main idea is that these aspects are not relevant when focusing on the core trade-off in bond strategies between hedging inflation and sensitivity to parameter estimation errors. For completeness, we perform a robustness check to show that the results of our study remain valid after including a stock in the portfolio.

We assume two risk factors equal to the nominal interest rate and the inflation rate, so that the optimal strategy consists of two bonds to hedge the exposure to both risks. For the suboptimal strategy, we apply a single-bond strategy. Note that one bond in our model can alternatively be interpreted as a portfolio of long positions in bonds with several maturities that has the same nominal price elasticity to one risk factor as the bond in the single-bond strategy (see e.g., Munk et al. (2004); Brennan and Xia (2002)). Therefore,

[2] Furthermore, there is more literature exploring parameter uncertainty while focusing on speculative demands for stocks instead of bonds (e.g. Garlappi et al. (2007); Maenhout (2004); Biagini and Pınar (2017)).

the results of our study can be extended to a portfolio with multiple bond maturities. Our single-bond strategy provides an analytically tractable approach to addressing extreme weights in two-bond strategies (and, consequently, parameter sensitivity). Single-bond strategies have been applied by e.g., Brennan and Xia (2002), Van Bilsen et al. (2020), Munk et al. (2004), and Campbell and Viceira (2001). Our numerical results are based on parameter estimates from Brennan and Xia (2002), along with various sensitivity parameter sets.

We find that the investor may face a substantial loss in her certainty equivalent return (CER) on wealth when making an estimation error in the mean-reversion parameters, feedback parameter, or correlation. The hedge investor is harmed considerably by estimation errors in all those parameters, except in the case where she estimates the mean-reversion parameter of inflation too low. In the extreme case where she incorrectly believes that the mean-reversion parameters are very high, or the feedback parameter is close to zero, she faces a certainty equivalent wealth of zero at the investment horizon because she applies suboptimal strategies over a long time horizon. The speculative investor is especially harmed when she incorrectly believes that the mean-reversion parameters are very high, decreasing her CER from 4.4% to -3.8% per year.

We show that the estimation errors have such a large effect because the optimal two-bond strategy allocations are typically extreme, especially when the nominal interest rate is barely impacted by inflation or for certain bond maturities. Considering bond maturities between 1 and 50 years, we use the combination of a 10- and 50-year bond because it minimises the sum of absolute weights of the optimal two-bond strategy.[3] But even with these maturities and a considerable impact of inflation on the nominal interest rate, the weights are large in absolute value. For example, under the initial parameters the hedge investor allocates 290% and -90% of wealth, given a remaining investment horizon of 30 years. To investigate whether bond strategies with smaller absolute demands or less sensitive demands can mitigate the potential losses due to estimation errors, we consider alternative strategies.

Our findings indicate that single-bond strategies result in smaller potential utility losses due to estimation errors than the optimal two-bond strategy. For instance, consider the aforementioned decrease from 4.4% to -3.8% in CER in the case of a speculative investor

[3]The maturity of 50 years is in line with the largest available swap maturity. In practice, long-term swaps can be used to replicate long-term bonds. Long-term swaps are sometimes debated because they can lock-in a fund, and they can be costly and complex. However, this is not the focus of our research. Furthermore, investing in these long-term swaps is commonly used in practice and complies with e.g. Dutch pension fund regulations (Blommestein (2007)).

regarding parameter uncertainty about the mean-reversion parameters. In the case of applying a single-bond SW-strategy with a bond maturity of 10 years, her CER equals only 3.8% without estimation errors, but decreases only to 3.7% under the parameter uncertainty. The fact that this single-bond strategy appears to be more stable can be explained by the smaller bond weight than in the two-bond strategy, in this case equal to 140% of wealth instead of 570% and -260% of wealth in the two-bond SW-strategy.

In summary, we show that it is important for long-term investors to recognise the challenges of hedging the real interest rate. We show that while optimal two-bond strategies may fully hedge the real interest rate in a two-factor model, they can lead to significant losses in case of estimation errors. Conversely, we show that single-bond strategies might mitigate these potential losses, but are less effective in hedging inflation risk. Consequently, this results in utility losses under single-bond strategies compared to two-bond strategies in the case of no parameter uncertainty. Therefore, long-term investors should consider potential losses due to parameter uncertainty and explore robust strategies that mitigate these losses.

We now briefly discuss some remaining related literature. We apply the study of Sangvinatsos and Wachter (2005) that derives the optimal nominal bond strategy based on a nominal interest rate that is affine in the risk factors. The study of Brennan and Xia (2002) derives the optimal nominal bond strategy in a nested model which restricts the feedback parameter and assumes constant risk premiums.[4] These two papers are fundamental studies regarding optimal investment strategies to hedge the real interest rate and emphasise the relevance of (time-varying) hedging demands for bonds of long-term investors. Assuming that the nominal interest rate and expected inflation rate are two risk factors in the SW-model, leads to a long-short position in nominal bonds which is in line with observed investment behaviour of U.S. long-term investors (Baker et al. (2021)). Furthermore, Koijen et al. (2010) apply the SW-model to two latent risk factors and incorporate borrowing and short-sale constraints. Their model is used by the Dutch Central Bank to simulate scenario sets for pension fund feasibility tests in the Netherlands.

The bond pricing literature commonly employs more sophisticated models than single- or two-factor models with constant risk premiums. For example, the SW-model is based on a three-factor model with a time-varying market price of risk, referred to as an essentially affine interest rate model which has been initially studied by Duffee (2002). Another

[4]To clarify: when we apply the SW-model, we use the model of Sangvinatsos and Wachter (2005) without the restriction on the feedback parameter, but with the assumptions of the constant risk premiums, because we focus on the hedge demand.

approach is to model changes in bond returns by incorporating regime shifts such as in Hamilton (1988), Bansal and Zhou (2002), or Ang et al. (2008). However, these studies focus on excess return predictability and use latent factor models, whereas we focus on hedging the real interest rate risk with the interest rate and expected inflation as explicit risk factors, as in Brennan and Xia (2002) and Campbell and Viceira (2001).

A different approach to hedge the real interest rate in the literature is to invest in an Index Linked Bond (ILB), which protects the real value of the principal (or coupon) against inflation. Despite the increased market attention to inflation ILBs are, in practice, illiquid compared to nominal bonds (Ciocyte and Westerhout (2017); Van Bilsen et al. (2020); Lioui and Tarelli (2023)). Moreover, ILBs are found to imperfectly hedge real wealth due to mispricing relative to nominal bonds in the U.S. (Fleckenstein et al. (2014)), the limited availability of maturities (Booth and Yakoubov (2000)), the indexation lag of at least a quarter, and their link to different country-specific inflation indices (Farrugia et al. (2018)). Therefore, we focus on the nominal bond literature.

A disadvantage of the optimal nominal bond allocations based on two risk factors is that these are typically extreme and consequently sensitive to parameter estimation errors. One possible solution is to constrain the portfolio weights. Martellini et al. (2015) show that imposing constraints on the optimal two-bond strategy allocations of Brennan and Xia (2002) can lead to significant utility losses. Although the constrained strategies are not able to fully hedge the risk factors, they may perform better under parameter uncertainty. For example, Garces and Shen (2025) show that short-selling constraints on stock allocations mitigate utility losses for risk-averse investors who ignore model uncertainty about stock returns and a stochastic interest rate. Moreover, DeMiguel et al. (2009) show that constrained portfolio weights perform well under estimation errors in another context. The authors find that constraints improve the out-of-sample performance of a static portfolio optimisation problem.

A disadvantage of imposing constraints is that solving the constrained problem requires numerical optimisation. Single-bond strategies provide an analytical solution to moderate the extreme allocations, and may have the additional advantage of resulting in smaller utility losses under parameter uncertainty, compared to multiple-bond strategies (Feldhütter et al., 2012). Another analytical approach to enhance the strategy's robustness to parameter uncertainty is to include ambiguity-aversion as in Flor and Larsen (2014), which reflects aversion to unknown probabilities.

When applying a single-bond strategy to hedge the interest rate which is affine in two risk factors as in our set-up, the bond maturity affects the ability to hedge the interest rate.

Brennan and Xia (2002) determine the optimal bond maturity in the single-bond strategy, but still see a substantial loss in utility compared to the two-bond strategy. Quaedvlieg and Schotman (2020) empirically show that a naive hedging strategy that consists of a long position in a bond with the maturity equal to the Last Liquid Point (20-years) leads to less sensitive bond demands which empirically outperform bond strategies based on multi-factor interest models in terms of fit and turnover. Yet, about 50% of the interest rate risk remains unhedged based on a horizon investment of 50 years (Quaedvlieg and Schotman (2020)). Hence, even when optimising the bond maturity it remains relevant to show the inability to hedge the interest rate in a single-bond strategy.

The structure of this chapter is as follows. In Section 1.2, we explain the financial market and the corresponding optimal nominal bond demands. We also show alternative strategies that might be less sensitive to changes in parameters. In Section 1.3, we compute the impact on utility if the investor makes estimation errors in the parameters of the underlying risk factors processes. Section 1.4 performs some robustness checks and Section 1.5 concludes.

1.2 Model description

1.2.1 Financial market

We assume a standard affine model for the nominal interest rate as proposed in Sangvinatsos and Wachter (2005) with the instantaneous nominal interest rate R_t and expected inflation rate π_t as risk factors that follow an Ornstein-Uhlenbeck (OU) process:

$$d\begin{bmatrix} R_t \\ \pi_t \end{bmatrix} = -\begin{bmatrix} \kappa_R & \kappa_{R\pi} \\ 0 & \kappa_\pi \end{bmatrix}\begin{bmatrix} R_t - \bar{R} \\ \pi_t - \bar{\pi} \end{bmatrix} dt + \begin{bmatrix} \sigma_R & 0 \\ 0 & \sigma_\pi \end{bmatrix} d\begin{bmatrix} Z_t^R \\ Z_t^\pi \end{bmatrix} \tag{1}$$

which simplifies to:

$$dX_t = -KX_t dt + \sigma_X dZ_t \tag{2}$$

Here, κ_R and κ_π are defined as the mean-reversion parameters and $\kappa_{R\pi}$ as the feedback parameter that corresponds to the relation between the level of π_t and the drift adjustment of R_t. We assume that $\kappa_R > 0, \kappa_\pi > 0, \sigma_R > 0, \sigma_\pi > 0$, $\kappa_{R\pi} < 0$, and $\kappa_R \neq \kappa_\pi$. This model set-up implies that if the expected inflation rate is above its long-term average, it will increase the nominal interest rate's drift term. The economic intuition is that if the expected inflation rate is high, usually the nominal interest rate increases since the nominal interest rate can be seen as the sum of the real interest rate and the inflation rate. Furthermore, the Brownian motions Z_t^R and Z_t^π have correlation matrix ρ, expressed in terms of $\rho_{R\pi} \in (-1, 1)$

$$\rho = \begin{bmatrix} 1 & \rho_{R\pi} \\ \rho_{R\pi} & 1 \end{bmatrix} \tag{3}$$

We assume that the nominal interest rate and expected inflation are linear functions of the state variables, $R_t = \delta_{0R} + \delta_R' X_t$ and $\pi_t = \delta_{0\pi} + \delta_\pi' X_t$ with scalars $\delta_{0.}$ and $m \times 1$ vectors δ_R and δ_π. The nominal pricing kernel is given by

$$\frac{d\zeta_t}{\zeta_t} = -R_t dt - \rho^{-1} \lambda dZ_t, \; \zeta_0 = 1 \tag{4}$$

where the 2×1 vector λ contains the market prices of risk (MPoR) on the risk factors. To focus on the (hedge demand of) bond allocations, we consider a market without stocks.[5] The investor can invest in one or two nominal bonds with prices

$$\frac{dP_t(\tau_j)}{P_t(\tau_j)} = \left[R_t + \sigma_{Bj}\lambda\right] dt + \sigma_{Bj} dZ_t \tag{5}$$

where $\sigma_{Bj} = -B(\tau_j)'\sigma_X$ is a 1×2 vector of exposures to the risk factors of bond $j \in \{1, 2\}$ with maturity τ_j, and the 2×1 vector $B(\tau_j)$ is computed by solving the ODE:[6]

$$B'(\tau_j) = -K\, B(\tau_j) + \delta_R \tag{6}$$

Hence, the bond volatilities depend on the mean-reversion matrix K. The intuition is that a higher mean-reversion parameter implies a lower persistence of shocks in the nominal interest rate. The remaining wealth is allocated to a bank account with an instantaneous risk-free nominal return R_t.[7] We consider an investor who aims to optimise her real wealth, subject to the budget constraint. The optimisation problem is defined as follows:

$$\max_{W_T} \mathbb{E}_t \left[u\left(\frac{W_T}{\Pi_T} \right) \right] \qquad \text{s.t. } \mathbb{E}_t \left[\zeta_T W_T\right] = W_t \tag{7}$$

[5]We show that only the speculative demand changes due to the presence of stocks, which we analyse as part of the robustness check on bond allocations within a portfolio with stocks in Section 1.4.

[6]The 2×1 vector $B'(\tau_j)$ denotes the derivative of the vector $B(\tau_j)$ with respect to τ_j. Appendix A.1.1 derives the corresponding solution for our financial market, given in equation (A.1.8).

[7]Hence, in discrete time, this implies that the nominal return is riskless if we consider only one time step, but becomes stochastic thereafter.

where $u()$ is the investor's utility function, W_t is the nominal wealth at t, T is the investment horizon, and Π_t is the price index which equals:

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt \tag{8}$$

so we assume no unexpected inflation and hereinafter inflation refers to π_t.[8] Moreover, we assume that the investor has a constant relative risk aversion (CRRA) utility function with coefficient of relative risk aversion $\gamma \geq 1$:

$$u(w) = \begin{cases} \frac{w^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 1 \\ \log(w) & \text{if } \gamma = 1 \end{cases} \tag{9}$$

The optimal investment strategy is derived by Sangvinatsos and Wachter (2005):

$$x_t = \frac{1}{\gamma}\Omega^{-1}\sigma_B\lambda - \left(1 - \frac{1}{\gamma}\right)\Omega^{-1}\sigma_B\rho\sigma_X'\tilde{B}(T-t) \tag{10}$$

where the 2×1 vector $\tilde{B}(T-t)$ corresponds to the exposure to both the nominal interest rate risk and inflation risk for the remaining investment horizon, and is given in Appendix A.1.1 in equation (A.1.10). The notation of $\tilde{B}(T-t)$ is used to distinct from the nominal interest rate exposures in $B(\tau_j)$ of the nominal price bond process. Moreover, we have the 2×2 covariance-matrix of the bond returns $\Omega = \sigma_B\rho\sigma_B'$. In case of the two-bond strategy, x_t is a 2×1 vector and the corresponding volatility matrix equals the 2×2 matrix

$$\sigma_B = \begin{bmatrix} \sigma_{B1} \\ \sigma_{B2} \end{bmatrix} \tag{11}$$

In case of the single-bond strategy, x_t is a scalar and the volatility matrix reduces to the 1×2 matrix $\sigma_B = \sigma_{B1}$.

The optimal strategy consists of two terms, where the risk aversion coefficient γ determines the allocation between them. The first term represents the so-called speculative

[8] Sangvinatsos and Wachter (2005) derive the optimal bond allocations in the presence of correlated asset returns and unexpected inflation. However, they only find a considerable correlation between unexpected inflation and a third risk factor of the interest rate, as well as between unexpected inflation and the stock index. We do not consider either component, so including the unexpected inflation component is not relevant when determining the optimal bond allocations. Moreover, Brennan and Xia (2002) conclude that the impact on utility of not being able to hedge unexpected inflation is relatively small as they find a small volatility of unexpected inflation.

demand, which is proportional to the mean-variance optimal strategy. A speculative investor with $\gamma = 1$ would fully allocate her wealth to this speculative demand. The larger the risk aversion, the more is allocated to the second term, the hedge demand. The intuition is that a more risk-averse investor wants to invest more in the instantaneously riskless asset, but this increases the interest rate risk in the future when the interest rate would decrease. Therefore, this hedge demand is constructed such that the value of the corresponding hedge portfolio increases when the real interest rate, $R_t - \pi_t$, decreases.

By applying the two-bond strategy in which the bonds have different maturities, the market is complete, based on the Martingale pricing approach of Cox and Huang (1989), and the real interest rate can be fully hedged (Brennan and Xia (2002)). Sangvinatsos and Wachter (2005) explain that the optimal two-bond strategy consists of a short-long position in nominal bonds due to the positive correlation between bonds of different maturities. In this way one bond can be used to leverage the other bond. However, these long-short positions are typically sensitive to small estimation errors in the parameters.[9] Therefore, we also analyse single-bond strategies, which result in an incomplete market setting where the bond cannot fully hedge inflation risk, but may offer a more robust alternative. The next section will explain how we measure the impact of parameter uncertainty.

1.2.2 Parameter uncertainty

Consider a financial market that is based on the "true" parameters in parameter set $\theta^0 = \{K_X, \sigma_X, \rho, \lambda\}$. The investor applies a bond strategy that is optimal based on an alternative parameter set m, defined by $x_t(\theta^m)$. This strategy determines the evolution of wealth under the financial market based on the true parameters and is only optimal if she uses the correct parameter set, so $\theta^m = \theta^0$. We are interested in the potential utility loss if the investor makes estimation errors and bases her strategy on incorrect parameters not equal to those of the financial market, so $\theta^m \neq \theta^0$. In line with Brennan and Xia (2002), we measure the utility of terminal wealth based on the investment strategy by the certainty equivalent return (CER), defined as:

$$\text{CEW} = u^{-1}\left(\mathbb{E}_0\left[u\left(\frac{W_T}{\Pi_T}\right)\middle| x_t(\theta^m)\right]\right) \tag{12}$$

$$\text{CER} = \frac{1}{T}\log\left[\frac{\text{CEW}}{W_0/\Pi_0}\right] \tag{13}$$

[9]Numerical examples in Section 1.3 demonstrate this.

where $\mathbb{E}_0$ is the expectation under θ^0. The CER is the annualised real return on initial wealth that compounds to the certainty equivalent of wealth at the investment horizon. The certainty equivalent of wealth is the amount that makes the investor indifferent between receiving this amount for certain or obtaining the terminal wealth from investing $x_t(\theta^m)$ in a financial market based on the true parameter set θ^0. If the investor applies the optimal strategy based on the true parameters, $x_t(\theta^0)$, she receives the highest possible CER in this financial market. We are in particular interested in the case where she makes estimation errors and therefore bases her strategy on incorrect parameters, $\theta^m \neq \theta^0$, resulting in a lower CER.

For the investment strategy $x_t(\theta^m)$, we consider different types of strategies, indexed by type i. These types determine both the choice between a single- or two-bond strategy and the formulation of the feedback parameter $\kappa^m_{R\pi}$ in θ^m. To explain this, first consider two-bond strategies. The optimal SW-strategy is denoted by $i = SW2$ where $\kappa^m_{R\pi}$ is a free parameter. In contrast, under type $i = BX2$, the feedback parameter is determined as $\kappa^m_{R\pi} = \kappa^m_\pi - \kappa^m_R$.[10] The origin of this strategy is based on the optimal strategy in a nested SW-model analysed by Brennan and Xia (2002; henceforth BX). The origin of this strategy is based on the optimal strategy in a nested SW-model analysed by Brennan and Xia (2002; henceforth BX).

When we consider single-bond strategies, we apply three types of strategies. Similar to the two-bond case, $i = SW1$ denotes the strategy in which the feedback parameter is treated as a free parameter, while $i = BX1$ corresponds to the strategy that maintains the feedback structure used in the BX-strategy. Additionally, we introduce the strategy type $i = MSV$ which is based on the optimal strategy in a nested SW-model without feedback from the risk factors' level to each other's drift terms. This model is analysed by Munk et al. (2004; henceforth MSV) and corresponds to the literature about single-bond strategies aiming to hedge the nominal interest rate that is only affected by the expected inflation rate through correlated shocks.

To summarise, the type of strategy i determines whether the strategy is based on a

[10] See equation (A.1.12) in Appendix A.1.2 for the relation between the feedback parameter and the mean-reversion parameters.

transformed feedback parameter from parameter set θ^m:

$$\kappa_{R\pi}^m = \begin{cases} \kappa_{R\pi}^m & \text{if } i \in \{SW2, SW1\} \\ \kappa_{\pi}^m - \kappa_R^m & \text{if } i \in \{BX2, BX1\} \\ 0 & \text{if } i \in \{MSV\} \end{cases} \tag{14}$$

We are interested whether suboptimal strategies might lead to less sensitive investment strategies if the investor makes estimation errors such that $\theta^m \neq \theta^0$. The single-bond strategies are expected to be less sensitive because of the less extreme bond positions and based on the literature review. Moreover, the BX- and MSV-strategy could be less sensitive because it is possible that the investor incorrectly estimates a mean-reversion parameter, but she updates the feedback parameter so that the feedback structure remains similar to that of the financial market (in case of $i \in \{BX1, BX2\}$) or she assumes no feedback at all (in case of $i \in \{MSV\}$). The next section will explain how the investment strategy affects the investor's wealth process.

1.2.3 Wealth evolution and bond maturities

This section explains how we compute the terminal wealth, and elaborates on the bond maturity choice in these computations. To simulate the nominal wealth evolution, we plug the bond allocations into the Stochastic Differential Equation (SDE) of the nominal wealth process

$$\frac{dW_t}{W_t} = [R_t + x_t' \sigma_B \lambda] \;\, dt + x_t' \sigma_B dZ_t \tag{15}$$

where σ_B and the Brownian motions in Z_t are based on the financial market parameters in θ^0. If we plug in a two-bond strategy based on the correct financial market parameter estimates, $x_t(\theta^0)$, the bond exposures are cancelled out:

$$\frac{dW_t}{W_t} = [R_t + \sigma_W' \lambda] \, dt + \sigma_W' dZ_t \tag{16}$$

$$\sigma_W = \frac{1}{\gamma}\lambda - \left(1 - \frac{1}{\gamma}\right)(\sigma_X)' \, \tilde{B}(T-t) \tag{17}$$

where $\tilde{B}(T-t)$ is based on θ^0. This shows that the expected utility under the optimal strategy is independent of the bond maturities.

In contrast, when we study suboptimal strategies due to estimation errors or by applying a single-bond strategy, the impact on utility depends on the bond maturities. This is because if the exposures in $B(\tau_j)$ are based on θ^m, the bond exposures would not be cancelled out. Therefore, the assumption about the bond maturities in the computations of the wealth evolution becomes relevant. We choose the bond maturities by minimising the sum of absolute optimal bond weights in the SW2-strategy at time $t = 0$ based on the true parameters of the financial market:

$$(\hat{\tau}_1, \hat{\tau}_2) = \underset{(\tau_1,\tau_2)}{\arg} \min \|x_0(\theta^0)\|_1 \tag{18}$$

Note that the bond maturity choice is based on a fixed investment horizon, so we keep the bond maturities constant through time.[11]

1.2.4 Parameter values

We base the parameters in θ^0 on the estimations of Brennan and Xia (2002). We use the second set of their Table 1 that are based on annual nominal interest rate and CPI data from 1890 to 1985.[12] We acknowledge that the estimates are not up to date. However, to the best of our knowledge, no study provides updated estimates of an affine two-factor model that models the interest rate and expected inflation rate as explicit risk factors. Appendix A.1.2 explains how we convert the reported estimates of the joint process of the real interest rate and inflation rate to our risk factor process with the nominal interest rate and inflation rate. This results in the parameter values shown in the first column of Table 1, and long-term averages of $\bar{R} = 7.1\%$ and $\bar{\pi} = 5.4\%$.[13] We observe a larger mean-reversion parameter of the nominal interest than that of the inflation rate, which is supported by empirical evidence as the estimated mean-reversion of expected inflation tends to be low (e.g., Pennacchi (1991); Campbell and Viceira (2001)). Moreover, the parameters result in positive risk premiums $\sigma_B\lambda$.

Based on this parameter set and risk aversion coefficients that will be explained in Section 1.2.4, we determine the bond maturity choice with the minimisation function

[11] Alternatively, we could compute it for a $t > 0$, or equivalently a smaller T, which decreases the remaining horizon investment $T - t$. We verified that this did not change the bond maturity choice under the initial parameters.

[12] Brennan and Xia (2002) also estimate parameters based on monthly observations from 1970 to 1995, but show that the corresponding estimations result in an unrealistic amount of oscillations in the estimated real interest rate.

[13] We acknowledge that the long-term averages are lower based on more recent data. However, long-term averages do not directly impact the optimal bond allocation.

in equation (18). We assume an investment horizon of $T = 30$ years that will also be used in the simulations, and we use a yearly grid values in the range $\tau_j \in \{1, 2, ..., 50\}$ for $j \in \{1, 2\}$. This large maximum maturity is in line with the maximum available swap maturity data and the common practice of long-term investors that invest in very long-term bonds.[14] We find the bond maturity combination $(\hat{\tau}_1, \hat{\tau}_2) = (11, 50)$.[15] We slightly adjust the combination to $(10, 50)$ as the bond maturity of 10 years is more in line with the literature such as the considered single-bond strategy based on a bond maturity of 10 years in an example of Brennan and Xia (2002). For completeness, Figure 1 shows the optimal bond allocations based on a varying bond maturity τ_1 on the x-axis and the second bond maturity of 50 years. Based on the figure, we observe that choosing $(10, 50)$ instead of $(11, 50)$ has no significant effect on the sum of the absolute bond weights. Therefore, we choose $(\hat{\tau}_1, \hat{\tau}_2) = (10, 50)$ for the two-bond strategies for the remainder of this chapter. If we apply a one bond strategy, we choose $\tau_1 = 10$ to keep the one and two bond strategies comparable.

1.3 Numerical results

Section 1.3.1 analyses the impact of estimation errors in the mean-reversion parameters and correlation. Section 1.3.2 analyses the impact of estimation errors in the feedback parameter.[16] For readability, we denote the strategy types $i \in \{SW2, SW1, BX2, BX1, MSV\}$ by the SW2-, SW1-, BX2-, BX1-, and MSV-strategy respectively.

The results will be shown for five relevant risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, where $\gamma = 1$ represents the 'log utility' or 'speculative' investor that only has a speculative demand, and $\gamma = 25$ is the 'hedge' investor corresponding to the almost infinitely risk-averse investor who only has a hedge demand.[17] In reality, an investor will often be interested in both demands and therefore we include the moderately risk-averse investor with $\gamma = 2$ as well, which corresponds to an 'average' bond demand of the speculative and hedge investor, and the commonly used values of $\gamma \in \{5, 10\}$. We assume a long-term

[14] Note that long-term bonds are not very liquid in practice. However, as explained in Section 1.1, the long (short) position in a long-term bond can be replicated by an interest rate swap contract. These swaps are available for high maturities and therefore commonly used by long-term investors.

[15] As a robustness check, we verified that the resulted bond combination remains the same for the alternative feedback parameter values considered in the numerical analysis.

[16] As mentioned in the introduction, we leave parameter uncertainty regarding the market price of risk out of scope, as this has already been analysed in the literature.

[17] The term of 'infinitely risk-averse' is line with Campbell and Viceira (2001), and the value of 25 is in line with Sangvinatsos and Wachter (2005).

investment horizon of $T = 30$ years.

We set the initial values of the risk factors equal to their long-term averages. Regarding the simulations of wealth, we discretise the wealth process by the Euler approximation of the log wealth process so that it looks like

$$\log W_{t+\Delta} = \log W_t + \left[R_t + x_t' \sigma_B \lambda - \frac{1}{2} x_t' \Omega x_t \right] \Delta + x_t' \sigma_B \sqrt{\Delta} \epsilon_{t+1}] \tag{19}$$

with step size Δ, and $\epsilon_{t+1} \sim N(0, \Omega)$. We assume constant bond maturities through time. Furthermore, we apply 10,000 simulations with a monthly rebalancing, i.e. $\Delta = 1/12$ and we let the investor rebalance her bond allocation to the perceived optimal strategy every time step.

1.3.1 Estimation errors in mean-reversion and correlation

This section focuses on how estimation errors in the mean-reversion and correlation parameters impact utility. We consider the estimation errors as presented in one of the alternative parameter sets θ^m, shown in Table 1. The estimation errors result in different parameters than those in the initial "true" parameter set θ^0. The sets with κ_R^+ and κ_π^+ contain the doubled initial mean-reversion parameter of the nominal interest rate and inflation rate respectively, while the two sets with κ_R^- and κ_π^- contain a halved initial mean-reversion parameter. The volatilities of the risk factors are adjusted simultaneously to maintain a constant unconditional variance.[18] The alternative parameter sets $(\kappa_R^+, \kappa_\pi^+)$ and $(\kappa_R^-, \kappa_\pi^-)$ contain the simultaneous increase or decrease in both mean-reversion parameters. The final two sets consider an increase and decrease in the correlation between shocks in the risk factors, $\rho_{R\pi}^+ = 0.9$ and $\rho_{R\pi}^- = 0.6$.

The alternative values of doubling and halving the initial values were chosen because we do not have statistical measures, such as standard errors, of all parameters to determine plausible alternative values. However, Brennan and Xia (2002) do report the standard error of the inflation mean-reversion parameter, and our alternative values are in line with initial parameters plus or minus two times the standard error.[19] Moreover, we consider the values of $\kappa_R^- = 0.053$ and $\kappa_R^+ = 0.210$ to be reasonable, as two latent factor models can result in even smaller or larger mean-reversion parameter estimates for the most persistent

[18] For instance, the set with κ_R^+ doubles κ_R and increases σ_R by a factor of $\sqrt{2}$. This adjustment ensures that the unconditional variance of R_t , $\sigma_R^2/2\kappa_R$, remains consistent with the initial set.

[19] For completeness, Section 1.4 performs a robustness check of alternative values based on two times the standard error of the initial estimate of the inflation mean-reversion parameter.

factor. For instance, Bergström and Nowman (1999) estimate a value of 0.033 based on U.S. data from 1981-1995, while Babbs and Nowman (1998) find a value of at least 0.256 for five out of nine countries based on different sample periods between 1987-1997.

Table 2 shows the impact on the CER when the investor applies an investment strategy based on parameters in θ^m, while the financial market is based on θ^0. For the initially optimal SW2-strategy, estimation errors in the mean-reversion parameters and correlation can lead to substantial utility losses. The utility loss is the largest when investments are based on $(\kappa_R^+, \kappa_\pi^+)$ across all levels of risk aversion. For example, an investor with $\gamma = 2$ faces a CER of 3.8% per year in the absence of estimation errors. However, if this investor follows the SW2-strategy using mean-reversion parameters that are twice as high, the CER drops sharply to -12.1%. This happens because higher mean-reversion parameters would reduce the persistence of shocks to nominal interest rates and inflation rates. As a result, bond prices would become less sensitive to risk factors shocks, so that the investor increases her absolute bond demands to replicate the optimal exposure under her incorrect beliefs about the parameters. However, in the actual financial market the shocks are more persistent and these more extreme bond positions lead to a high risk exposure. This causes such large fluctuations in her bond portfolio that she ultimately faces a negative CER. The CER of -100% for risk aversions $\gamma \in \{10, 25\}$ means that the more risk-averse investors even face a certainty equivalent wealth of zero in this case. This extremely low outcome results from following investment strategies based on incorrect parameters that deviate significantly from the optimal strategies over a long investment horizon of 30 years.

When considering changes in only one parameter at a time, the largest utility loss for the speculative investor with $\gamma = 1$ occurs under $\rho_{R\pi}^+$, causing the CER to drop from 4.4% to 1.0%. This happens because she overestimates the correlation as strongly positive, assuming that low nominal interest rates will go together with low inflation rates and the real interest rate risk is relatively small. She increases her absolute bond demands relative to those based on the initial parameters which leads to too much wealth exposure to the risk factors. For more risk-averse investors, the largest loss due to a change in only one parameter occurs when estimating the mean-reversion of the nominal interest rate too high. This shows that the utility loss in the simultaneous change of $(\kappa_R^+, \kappa_\pi^+)$ is mainly caused by the increase in the mean-reversion parameter of the nominal interest rate. Notably, the CER under the SW2-strategy shows limited sensitivity to changes in the mean-reversion parameter of inflation and to the lower correlation.

Consecutively, we consider the impact when applying the BX2-strategy. This can result in either smaller or larger potential losses compared to the SW2-strategy, but it appears to mitigate the maximum loss in the presence of estimation errors. For example, for both strategies the largest loss for risk-averse investors with $\gamma > 1$ occurs under the alternative set with $(\kappa_R^+, \kappa_\pi^+)$. For $\gamma = 2$, the CER decreases to 1.0% when the BX2-strategy is applied, which is significantly higher than the -12.1% under the SW2-strategy. This suggests that while the BX2-strategy may still lead to losses under parameter uncertainty, it might offer better protection against extreme losses because the feedback structure between the risk factors remains the same as in the initial parameter set.

Next, we consider the impact on utility with single-bond strategies. The CER is notably lower in the absence of parameter uncertainty ($\theta^m = \theta^0$), due to the inability of these strategies to hedge against real interest rate risk. However, the single-bond strategies appear to be less sensitive to estimation errors: the maximum utility losses are smaller compared to those with two-bond strategies. For example, the largest loss in the CER with a single-bond strategy for $\gamma = 1$ is 0.3% (from 3.5% to 3.2%) when applying the MSV-strategy based on κ_R^-, whereas the loss with the SW2-strategy is much larger, namely 8.2% (from 4.4% to -3.8%). The reduction of the maximum utility loss is even stronger when the risk aversion increases.

Note that in addition to the advantage of lower sensitivity to estimation errors, the single-bond strategies also may result in a higher CER than the two-bond strategies when errors in the mean-reversion parameter of the nominal interest rate occur. For instance, for $\gamma = 25$ the SW2-strategy based on κ_R^- results in a CER of -4.4%, while the single-bond strategies result in a CER of at least -2.5%. Moreover, note that there is no distinct difference between the performance under parameter uncertainty of the SW1- and BX1-strategy. However, the MSV-strategy leads to a lower CER than the SW1- and BX1-strategy under all parameter sets. Therefore, even in a single-bond strategy the investor gains from taking into account a certain feedback effect from the inflation rate level to the nominal interest rate drift term.

The impact of estimation errors can be attributed to the high sensitivity of bond allocations across the alternative parameter sets. Table 3 presents the corresponding bond weights for the initial parameter set and three alternative sets as an example. For the set with κ_π^-, the single-bond and two-bond strategies show minimal deviations from the initial weights, which explains the limited impact on utility. In contrast, the sets $(\kappa_R^+, \kappa_\pi^+)$ and $\rho_{R\pi}^+$ result in significant deviations from the optimal strategy based on the initial parameters for the two-bond strategies. However, it results in smaller deviations from the optimal single-bond strategy based on the initial parameters for the single-bond strategies.

This reduced sensitivity of the single-bond strategies explains why estimation errors may result in smaller utility losses compared to the two-bond strategies.

Thus, although two-bond strategies typically yield higher CER values when the investor accurately estimates the parameters, they can lead to significant declines if the investor has incorrect beliefs about the mean-reversion parameters or correlation. The advantage of single-bond strategies lies in their lower sensitivity to estimation errors, especially in the mean-reversion parameter of the nominal interest rate. Under these estimation errors, single-bond strategies might even lead to a higher CER than two-bond strategies.

1.3.2 Estimation errors in feedback parameter

In Section 1.3.1, we examine the sensitivity of (sub)optimal investment strategies to changes in the mean-reversion and correlation parameters while holding other parameters constant. Another interesting analysis is to examine the impact of estimation errors in the feedback parameter, which can be seen as a special case of model uncertainty as different choices of the feedback parameter shown in equation (14) result in distinct nested SW-models such as analysed in Brennan and Xia (2002) when $\kappa_{R\pi} = \kappa_\pi - \kappa_R$, and Munk et al. (2004) when $\kappa_{R\pi} = 0$.

The initial value of the feedback parameter is based on the BX-model and equals $\kappa^0_{R\pi} = -0.078$. We also consider a lower value $\kappa^-_{R\pi} = -0.017$ and a higher value $\kappa^+_{R\pi} = -0.039$, corresponding to a decrease or increase of 50% from the initial value in absolute terms. We also include the case of a feedback parameter that is equal to or close to zero. The two-bond strategy is not defined for a feedback parameter equal to zero, because it uses each bond to hedge the exposure to one factor. With a feedback parameter of zero, however, the investor can only apply a single-bond strategy to hedge parallel shifts in the yield curve as a response to changes in one factor, namely the nominal interest rate (Brigo and Mercurio (2016)).[20] Therefore, we choose a value close to zero instead for the two-bond strategies.

As the BX-and MSV-strategy are the same as the SW-strategy based on the initial feedback parameter and a zero feedback parameter respectively, we only consider the two- and single-bond SW-strategies with $i \in \{SW2, SW1\}$. Table 4a shows the CER for the SW2-strategy when the financial market is based on the initial parameter set and hence $\kappa^0_{R\pi}$, while the investor applies the SW-strategy based on $\kappa^m_{R\pi}$. Misestimating

[20]For the mathematical explanation, see equation (A.1.8) in Appendix A.1.1, which shows that with a very small feedback parameter, bond prices do not respond to a shock in the inflation rate level.

$\kappa^m_{R\pi} = -0.001$ results in a CER of -100%. This occurs because there is still a very small effect of the inflation rate on the bond price so that the two-bond strategy replicates the optimal exposure to shocks in both risk factors. Because the feedback effect is so small, extreme positions are required to replicate the optimal exposure under the alternative feedback parameter. These extreme positions cause a large mismatch with the actual optimal bond demands under the initial feedback parameter which are less extreme. When the investor misestimates $\kappa^-_{R\pi}$ or $\kappa^+_{R\pi}$, this can result in substantial utility losses as well, especially the hedge investor. When she incorrectly believes that the feedback parameter is less negative than in the actual market, she faces a large loss in CER from 3.1% to -12.3% per year. As explained for the feedback parameter close to zero, if the investors expects it to be small, she will increase her bond positions. However, if the actual feedback parameter is larger (more negative), her wealth exposure to the risk factors becomes too large, leading to a substantial utility loss.

Moreover, consider Table 4b which shows the impact on the CER for the single-bond SW-strategy. This strategy mitigates the effect of estimation errors compared to the two-bond strategy. It is less sensitive to changes in the feedback parameter, leading to smaller maximum CER losses. The largest potential utility loss occurs when the investor applies a strategy based on a feedback parameter of zero. For instance, the speculative investor faces a CER loss of 0.3% (from 3.8% to 3.5). The higher the risk aversion, the greater the loss, with a maximum loss of 1.4% for the hedge investor. Interestingly, for more risk-averse investors the SW1-strategy actually results in a higher CER than the SW2-strategy when the investor misestimates $\kappa^+_{R\pi}$. This shows that, despite being initially suboptimal in the absence of estimation errors, a single-bond strategy may outperform a two-bond strategy in case of estimation errors because it is less sensitive to changes in the feedback parameter.

Similar as for the analysis about estimation errors in the mean-reversion and correlation parameters, the results about the potential large utility losses can be explained by the sensitivity of bond allocations to changes in the feedback parameter. Table 5 shows the optimal bond allocations in the single-and two-bond strategies for the different feedback parameters. It shows that the two-bond strategy is sensitive to variations in the feedback parameter, while the single-bond strategy is less sensitive. This is especially the case for more risk-averse investors. For instance, when the feedback parameter shifts from the initial feedback parameter to $\kappa^+_{R\pi}$, the hedge investor adjusts her two-bond strategy allocations of 2.9 and -0.9, to 5.1 and -2.4 as fraction of wealth. The single-bond strategy decreases only slightly from 0.5 to 0.1. This demonstrates that when the feedback parameter is incorrectly estimated, the resulting two-bond strategy allocations deviate substantially from the optimal strategy, while the single-bond strategy changes only slightly.

Finally, note that the SW2-strategy results in unreasonable bond weights when the feedback parameter is near zero. This explains the extreme low CER in case of a feedback parameter close to zero in the investment strategy. Therefore, if an investor assumes a feedback parameter of zero, she may choose to apply the SW1-strategy instead of the SW2-strategy. This represents an additional benefit of the SW1-strategy, beyond its reduced sensitivity to estimation errors.

As a robustness check, Section 1.4 verifies that the conclusions regarding the CER are not impacted by choosing another true feedback parameter $\kappa^0_{R\pi}$. As another robustness analysis, we repeated the computations using alternative combinations of bond maturities (τ_1, τ_2).[21] For the SW2-strategy, this does not affect the qualitative conclusions regarding the CER loss due to parameter uncertainty about the feedback parameter. However, a higher bond maturity for the SW1-strategy may lead to a lower CER without parameter uncertainty, but at the same time a greater mitigation of potential losses from estimation errors compared to the SW2-strategy.

In summary, investors face significant utility losses if they make an estimation error in the feedback parameter under both the two-bond and single-bond SW-strategy. However, potential losses due to estimation errors are smaller under the single-bond strategy compared to the two-bond strategy. It is possible that the initially suboptimal single-bond strategy could result in a higher utility than the two-bond strategy if the investor's beliefs are incorrect. Another benefit is that the single-bond strategies are well-defined when the feedback parameter is near zero.

1.4 Robustness checks

The numerical results in Section 1.3 are based on a bond portfolio without stocks and the initial and alternative parameter sets given in Table 1. For completeness, this section performs four robustness checks to assess the impact of changes in these initial assumptions on the main results of this chapter.

First, we show the optimal asset allocations when alongside two nominal bonds and

[21] The alternative combinations were based on mixtures of 1-, 10-, and 30-year maturity bonds. The results are available upon request.

the bank account, the investor includes a stock with price process:

$$\frac{dS_t}{S_t} = (R_t + \sigma_S \lambda_S)\, dt + \sigma_S dZ_t^s \tag{20}$$

where the correlation matrix is extended with the correlation between the stock and the nominal interest rate and the inflation rate, ρ_{SR} and $\rho_{S\pi}$, respectively. The optimal asset allocation formula in equation (10) remains unchanged, but the dimensions of the corresponding vectors and matrices are extended to account for the stock as additional asset. When the investor invests in two bonds, the stock market, and the bank account, the correlation matrix ρ cancels out in the hedge demand.[22] As a result, the hedge demand is unaffected by the additional stock investment. Only the speculative bond demands change due to the correlation between the stock prices and risk factors.

We base the parameter values of the stock process on the same dataset as for the interest rate and inflation rate processes from Brennan and Xia (2002). Figure 2 shows the optimal allocations over time t, where the time horizon decreases over time. For a speculative investor with $\gamma = 1$, adding correlation between the stock and risk factors decreases the absolute bond demands compared to the case without stocks. The stock earns a substantial risk premium with a market price of risk $\lambda_S = 0.343$. The stock price process is quite volatile with $\sigma_S = 0.158$, but this risk is partly diversified by the negative correlation between the stock and nominal interest rate, $\rho_{SR} = -0.272$.[23] Therefore, the investor reallocates some wealth from the bonds to the stock. However, for the higher risk aversions $\gamma = 5$ and $\gamma = 25$, the impact of introducing the stock becomes negligible, as the hedge demand remains unaffected.

Moreover, Figure 3 shows bond allocations in the total portfolio when a stock is included, both under the initial parameter set and with the higher mean-reversion parameter of the nominal interest rate. The figure shows that even with smaller absolute bond demands due to the added stock, the bond allocations remain very sensitive to estimation errors in the mean-reversion parameter of the interest rate. Therefore, the qualitative results in our study about large utility losses in the two-bond strategy due to parameter uncertainty remain relevant even while we do not include stocks in the investor's asset portfolio. This

[22] This is also the case in the setting without stocks. However, equation (10) still includes ρ in the hedge demand because this term does not cancel out in the single-bond strategy.

[23] This correlation is computed using the correlations between the stock and the real interest rate and inflation rate reported in Brennan and Xia (2002), based on matching the linearly transformed variance-covariance matrix of the real interest rate structure, as shown in Appendix A.1.2 for the case without stocks.

result holds for single-bond strategies as well, because these are already less sensitive than a two-bond strategy. Moreover, the results in our study regarding higher risk aversions remain the same.

As a second robustness check, we consider our assumption in Section 1.2.4 regarding a particular estimate from Brennan and Xia (2002), namely the correlation between the real interest rate and the inflation rate. The original estimate is $\rho_{r\pi} = -0.061$, but in our main analysis we set it to be zero when transforming the real interest rate estimates into nominal interest rate parameter values. Similar to the robustness check concerning correlation with the stock, Figure 4 presents the nominal bond allocations over time t under both the original estimated correlation and the assumption of no correlation. The figure indicates that the impact on bond demands in a two-bond strategy is negligible. This result extends to single-bond strategies as these are even less sensitive than two-bond strategies.

The third robustness check applies two different alternative values for the inflation mean-reversion parameter, because in our main analysis we use informal alternative values. The question therefore remains whether the small utility loss in two-bond strategies under different inflation mean-reversion parameters would change if we instead base the alternative values on the standard error of the estimate. To investigate this, we apply two different alternative values by subtracting or adding two times the standard error to the initial parameter value, resulting in $\tilde{\kappa}_\pi^+ = 0.027 + 2 * 0.009 = 0.045$ and $\tilde{\kappa}_\pi^- = 0.009$.[24] Table 6 shows the impact on utility when the investor bases her strategy on these alternative values. The results show that the certainty equivalent returns under these alternative values, $\tilde{\kappa}_\pi^+$ and $\tilde{\kappa}_\pi^+$, remain similar to the original results obtained with the investor's beliefs of κ_π^+ and κ_π^-, as shown in Table 2.

The fourth robustness check investigates the impact of estimation errors in the feedback parameter on utility when the financial market is based the sensitivity values of the feedback parameter, rather than the initial parameter value. Table 7 shows the effect on the CER when the investor makes an estimation error in a financial market where the true feedback parameter is more negative value than the initial value, $\kappa_{R\pi}^-$, less negative, $\kappa_{R\pi}^+$, or (a value close to) zero. The results confirm that the conclusions from the main analysis hold across these financial markets: the two-bond strategy can result in substantial utility losses when the investor misestimates the feedback parameter, while the single-bond strategy mitigates these losses. Additionally, when the financial market is based on $\kappa_{R\pi}^-$ or $\kappa_{R\pi}^+$,

[24]Similar to the initial calculations in Section 1.3.1, we adjust the inflation rate volatility to ensure that the unconditional variance of the inflation rate remains unchanged.

utility losses under the two-bond strategy are especially large when the investors estimates the feedback parameter less negative than the actual one in the financial market. This finding aligns with the main analysis as well.

Hence, the four robustness checks confirm that the qualitative results regarding parameter uncertainty remain similar when accounting for correlation with a stock or between the real interest rate and the inflation rate. Moreover, the results regarding estimation errors in the inflation mean-reversion parameter remain similar when using different alternative values, and the findings for the feedback parameter hold when we consider different true feedback parameter values.

1.5 Conclusion

We consider a long-term investor aiming to hedge against real interest rate risk. Assuming an affine interest rate model, the optimal nominal bond strategy proposed by Sangvinatsos and Wachter (2005) (SW) effectively hedges this risk. However, our main finding is that the optimal nominal bond investment strategy is significantly affected by the parameters of the mean-reversion matrix in the joint process of the short-term nominal interest rate and expected inflation, and the correlation between these risk factors. We find that incorrect investor beliefs about these parameters can lead to substantial losses in the utility of terminal real wealth compared to the situation where the correct parameters are known. The utility losses are particularly pronounced when the investor overestimates the mean-reversion parameter of the nominal interest rate or has incorrect beliefs about the feedback parameter.

To explore potential solutions for the SW-strategy's sensitivity to parameter values, we study four alternative strategies based on restricted versions of the general SW-model. The first alternative is based on the model of Brennan and Xia (2002) (BX), which restricts the feedback from expected inflation to real interest rates to zero. Additionally, we investigate three single-bond strategies: the single-bond SW-strategy, the single-bond BX-strategy, and a strategy with no feedback from the inflation rate to the nominal interest rate. The single-bond strategies lead to less extreme bond positions compared to the two-bond strategies, resulting in smaller maximum losses across various alternative parameter sets. Furthermore, the alternative strategies can result in a higher utility than the SW2-strategy when facing estimation errors.

Given the challenges associated with accurately estimating mean-reversion parameters, it is relevant for institutional investors to consider both optimal strategies and the potential impact of estimation errors, along with robust suboptimal strategies that are less sensitive to parameter uncertainty. However, in the absence of estimation errors, suboptimal strategies may result in a lower utility, especially for highly risk-averse investors. Thus, an interesting topic for future research is to formalise the trade-off between a bond strategy's sensitivity to parameter changes and its ability to hedge inflation risk. One approach could involve incorporating ambiguity aversion into the analysis. To explore how the investor's hedge demand is affected in such a framework, the next chapter computes a robust strategy for an investor who is infinitely ambiguity averse, while taking into account a pre-specified range of plausible alternative parameter values.

Table 1. Values of parameter sets. The parameter values of the joint process of the nominal interest rate R_t and expected rate of inflation π_t. The set with $m = 0$ corresponds to the "true" parameter set θ^0, where sets $m \neq 0$ represent an alternative parameter set θ^m. If the mean-reversion parameters κ_R or κ_π are adjusted compared to the one in the parameter set θ^0, the volatility of the corresponding risk factor is adjusted accordingly to keep the long-term variance equal.

m	0	κ_R^+	κ_R^-	κ_π^+	κ_π^-	κ_R^+, κ_π^+	κ_R^-, κ_π^-	$\rho_{R\pi}^+$	$\rho_{R\pi}^-$
κ_R	0.105	0.210	0.053	0.105	0.105	0.210	0.053	0.105	0.105
κ_π	0.027	0.027	0.027	0.054	0.014	0.054	0.014	0.027	0.027
σ_R	0.019	0.027	0.014	0.019	0.019	0.027	0.014	0.019	0.019
σ_π	0.014	0.014	0.014	0.020	0.010	0.020	0.010	0.014	0.014
$\rho_{R\pi}$	0.733	0.733	0.733	0.733	0.733	0.733	0.733	0.900	0.600
λ_R	-0.219	-0.219	-0.219	-0.219	-0.219	-0.219	-0.219	-0.219	-0.219
λ_π	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105	-0.105

Table 2. Certainty equivalent returns under varying parameter sets. The certainty equivalent return (CER) on wealth when the financial market is based on the "true" parameters in set θ^0. The investor bases her optimal strategy on the alternative parameter set θ^m shown in Table 1. The results are shown for five types of strategies, namely the two-bond strategies labelled by $i \in \{SW2, BX2\}$ or the single-bond strategies labelled by $i \in \{SW1, BX1, MSV\}$. The CER is expressed in annual percentages for the risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$.

m	$\mathbf{0}$					$\boldsymbol{\kappa}_{\boldsymbol{R}}^{+}$					$\boldsymbol{\kappa}_{\boldsymbol{R}}^{-}$				
γ / i	$SW2$	$BX2$	$SW1$	$BX1$	MSV	$SW2$	$BX2$	$SW1$	$BX1$	MSV	$SW2$	$BX2$	$SW1$	$BX1$	MSV
1	4.4	4.4	3.8	3.8	3.5	1.7	4.2	3.7	3.7	3.4	2.7	3.3	3.7	3.5	3.2
2	3.8	3.8	2.9	2.9	2.8	-1.8	3.6	2.8	2.8	2.7	1.4	2.3	2.8	2.8	2.7
5	3.3	3.3	1.6	1.6	1.3	-11.5	3.1	1.2	1.6	0.7	-1.1	0.5	1.4	1.6	1.5
10	3.2	3.2	0.1	0.1	-0.7	-16.7	2.8	-0.7	0.1	-2.6	-3.0	-1.5	-0.1	0.1	-0.3
25	3.1	3.1	-1.8	-1.8	-3.2	-19.6	2.1	-3.1	-1.7	-5.7	-4.4	-3.1	-1.5	-2.0	-2.5

m	$\boldsymbol{\kappa}_{\boldsymbol{\pi}}^{+}$					$\boldsymbol{\kappa}_{\boldsymbol{\pi}}^{-}$					$\left(\boldsymbol{\kappa}_{\boldsymbol{R}}^{+}, \boldsymbol{\kappa}_{\boldsymbol{\pi}}^{+}\right)$				
γ / i	$SW2$	$BX2$	$SW1$	$BX1$	MSV	$SW2$	$BX2$	$SW1$	$BX1$	MSV	$SW2$	$BX2$	$SW1$	$BX1$	MSV
1	4.2	3.3	3.7	3.7	3.5	4.4	4.4	3.7	3.7	3.5	-3.8	1.8	3.7	3.7	3.4
2	3.4	1.2	2.8	2.8	2.8	3.7	3.6	2.8	2.8	2.7	-12.1	1.0	2.8	2.8	2.8
5	2.6	-4.3	1.6	1.5	1.1	3.2	2.9	1.5	1.5	1.5	-28.4	-1.0	1.0	1.4	0.4
10	1.7	-9.2	-0.0	-0.3	-1.4	3.0	2.3	0.1	0.1	-0.2	-100	-3.6	-1.4	-0.2	-3.5
25	-0.1	-12.1	-2.0	-2.4	-4.3	2.6	1.3	-1.6	-1.6	-2.0	-100	-5.5	-4.0	-2.1	-6.9

m	$\left(\boldsymbol{\kappa}_{\boldsymbol{R}}^{-}, \boldsymbol{\kappa}_{\boldsymbol{\pi}}^{-}\right)$					$\boldsymbol{\rho}_{\boldsymbol{R\pi}}^{+}$					$\boldsymbol{\rho}_{\boldsymbol{R\pi}}^{-}$				
γ / i	$SW2$	$BX2$	$SW1$	$BX1$	MSV	$SW2$	$BX2$	$SW1$	$BX1$	MSV	$SW2$	$BX2$	$SW1$	$BX1$	MSV
1	2.6	3.1	3.7	3.6	3.2	1.0	1.0	3.7	3.7	3.5	4.3	4.3	3.7	3.7	3.5
2	1.3	2.1	2.7	2.7	2.6	2.0	2.0	2.8	2.8	2.8	3.7	3.7	2.8	2.8	2.8
5	-1.4	0.5	1.1	1.4	1.5	2.6	2.6	1.6	1.6	0.7	3.3	3.3	1.6	1.6	1.5
10	-3.4	-1.1	-0.4	0.1	0.0	2.9	2.9	0.0	0.0	-2.7	3.2	3.2	0.2	0.2	-0.1
25	-4.8	-2.2	-1.8	-1.5	-1.8	3.0	3.0	-1.9	-1.9	-5.9	3.1	3.1	-1.6	-1.6	-1.9

Table 3. Optimal bond allocations based on varying parameter sets. The optimal bond allocations $x_0(\theta^m)$ at time $t = 0$ based on different parameter sets θ^m. The results are shown for five types of strategies. The headers *SW2-1, SW2-2, BX2-1, BX2-2* correspond to the optimal bond weights in the two-bond SW- and BX-strategy with bond maturities $\hat{\tau}_1 = 10$ and $\hat{\tau}_2 = 50$. The headers *SW1, BX1, MSV* correspond to the optimal bond weight in the single-bond SW-, BX-, and MSV-strategy with bond maturity $\hat{\tau}_1$. The bond weights are shown for the (a) "true" parameters in θ^0, (b) lower mean-reversion parameter κ_π^-, (c) higher mean-reversion parameters κ_R^+ and κ_π^+, and (d) higher correlation $\rho_{R\pi}^+$. The investment horizon equals $T - t = 30$. The results are shown for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, and expressed as fractions of wealth.

(a) θ^0

γ	*SW2-1*	*SW2-2*	*BX2-1*	*BX2-2*	*SW1*	*BX1*	*MSV*
1	4.3	-1.1	4.3	-1.1	1.4	1.4	1.9
2	3.6	-1.0	3.6	-1.0	0.9	0.9	0.8
5	3.2	-1.0	3.2	-1.0	0.6	0.6	0.1
10	3.0	-0.9	3.0	-0.9	0.5	0.5	-0.1
25	2.9	-0.9	2.9	-0.9	0.5	0.5	-0.2

(b) κ_π^-

γ	*SW2-1*	*SW2-2*	*BX2-1*	*BX2-2*	*SW1*	*BX1*	*MSV*
1	4.2	-1.0	4.0	-0.9	1.5	1.4	1.9
2	3.4	-0.9	3.2	-0.8	1.0	1.1	0.9
5	3.0	-0.9	2.7	-0.7	0.8	0.8	0.3
10	2.9	-0.8	2.6	-0.7	0.7	0.8	0.1
25	2.8	-0.8	2.5	-0.6	0.6	0.7	0.0

(c) $(\kappa_R^+, \kappa_\pi^+)$

γ	*SW2-1*	*SW2-2*	*BX2-1*	*BX2-2*	*SW1*	*BX1*	*MSV*
1	5.7	-2.6	4.8	-1.9	1.4	1.1	1.9
2	5.0	-2.7	3.6	-1.4	0.6	0.7	0.6
5	4.6	-2.7	2.8	-1.2	0.1	0.4	-0.2
10	4.5	-2.8	2.6	-1.1	-0.1	0.3	-0.5
25	4.4	-2.8	2.4	-1.1	-0.2	0.3	-0.7

(d) $\rho_{R\pi}^+$

γ	*SW2-1*	*SW2-2*	*BX2-1*	*BX2-2*	*SW1*	*BX1*	*MSV*
1	10.9	-3.5	10.9	-3.5	1.3	1.3	1.9
2	6.9	-2.2	6.9	-2.2	0.8	0.8	0.6
5	4.5	-1.4	4.5	-1.4	0.5	0.5	-0.2
10	3.7	-1.2	3.7	-1.2	0.4	0.4	-0.5
25	3.2	-1.0	3.2	-1.0	0.4	0.4	-0.6

Table 4. Certainty equivalent returns under varying feedback parameters. The certainty equivalent return (CER) on wealth when the financial market is based on the feedback parameter $\kappa^0_{R\pi}$, while the investor applies the SW-strategy based on $\kappa^m_{R\pi}$. The other parameters of the financial market and the investment strategy are based on the remaining parameters in set θ^0. The results are computed for the value of $\kappa^0_{R\pi} = -0.078$, the sensitivity values $\kappa^-_{R\pi} = -0.117$ and $\kappa^+_{R\pi} = -0.039$, and a value near or equal to zero. The results are shown for (a) the two-bond SW-strategy, and (b) the single-bond SW-strategy. The bond maturities equal $(\hat{\tau}_1, \hat{\tau}_2) = (10, 50)$. The CER is shown for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, and expressed in annual percentages.

(a) SW2-strategy

γ \ $\kappa^m_{R\pi}$	$\kappa^-_{R\pi}$	$\kappa^0_{R\pi}$	$\kappa^+_{R\pi}$	-0.001
1	4.3	4.4	3.7	-100
2	3.5	3.8	1.5	-100
5	2.6	3.3	-4.3	-100
10	1.6	3.2	-9.4	-100
25	0.3	3.1	-12.3	-100

(b) SW1-strategy

γ \ $\kappa^m_{R\pi}$	$\kappa^-_{R\pi}$	$\kappa^0_{R\pi}$	$\kappa^+_{R\pi}$	0
1	3.7	3.8	3.7	3.5
2	2.8	2.9	2.8	2.8
5	1.6	1.6	1.5	1.3
10	0.2	0.1	-0.1	-0.7
25	-1.5	-1.8	-2.1	-3.2

Table 5. Optimal bond allocations based on varying feedback parameters. The optimal bond allocations $x_0(\theta^m)$ at time $t = 0$ in the SW-strategy based on different feedback parameters $\kappa^m_{R\pi}$, and the other parameters equal to those in the "true" parameter set θ^0. The results are shown for two types of strategy. The headers *SW2-1, SW2-2* correspond to the optimal bond weights in the two-bond SW-strategy with bond maturities $\hat{\tau}_1 = 10$ and $\hat{\tau}_2 = 50$. The header *SW1* corresponds to the optimal bond weight in the single-bond SW-strategy with bond maturity $\hat{\tau}_1$. The remaining investment horizon equals $T - t = 30$. The results are shown for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, and expressed as fractions of wealth.

$\kappa^m_{R\pi}$	-0.117			-0.078			-0.039			-0.001		
γ	*SW2-1*	*SW2-2*	*SW1*	*SW2-1*	*SW2-2*	*SW1*	*SW2-1*	*SW2-2*	*SW1*	*SW2-1*	*SW2-2*	*SW1*
1	3.9	-0.9	1.2	4.3	-1.1	1.4	5.2	-1.7	1.6	76	-48	1.8
2	3.0	-0.7	1.0	3.6	-1.0	0.9	5.2	-2.1	0.8	126	-81	0.8
5	2.5	-0.5	0.9	3.2	-1.0	0.6	5.2	-2.3	0.4	156	-101	0.1
10	2.3	-0.5	0.8	3.0	-0.9	0.5	5.2	-2.4	0.2	166	-107	-0.1
25	2.2	-0.4	0.8	2.9	-0.9	0.5	5.2	-2.4	0.1	172	-111	-0.2

Table 6. Certainty equivalent returns under varying feedback parameters in different strategies. Robustness check with respect to Table 2: the certainty equivalent return (CER) on wealth when the financial market is based on the parameters in set θ^0. The investor bases her optimal strategy on the parameter set θ^0, or the alternative parameter set θ^m based on $\tilde{\kappa}^+_\pi = 0.045$ or $\tilde{\kappa}^-_\pi = 0.009$, The results are shown for five types of strategies, namely the two-bond investment strategies labelled by $i \in \{SW2, BX2\}$ or single-bond investment strategies labelled by $i \in \{SW1, BX1, MSV\}$. The CER is expressed in annual percentages for the risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$.

m	**0**					$\tilde{\boldsymbol{\kappa}}^+_{\boldsymbol{\pi}}$					$\tilde{\boldsymbol{\kappa}}^-_{\boldsymbol{\pi}}$				
γ / i	$SW2$	$BX2$	$SW1$	$BX1$	MSV	$SW2$	$BX2$	$SW1$	$BX1$	MSV	$SW2$	$BX2$	$SW1$	$BX1$	MSV
1	4.5	4.5	3.8	3.8	3.5	4.3	4.1	3.7	3.7	3.5	4.4	4.4	3.7	3.3	3.5
2	3.8	3.8	2.9	2.9	2.8	3.6	2.9	2.8	2.8	2.8	3.7	3.6	2.8	2.5	2.7
5	3.3	3.3	1.6	1.6	1.3	3.0	1.1	1.6	1.5	1.1	3.2	2.7	1.4	1.1	1.4
10	3.2	3.2	0.1	0.1	-0.7	2.6	-1.7	0.0	0.0	-1.3	2.9	1.9	-0.1	-0.2	-0.2
25	3.1	3.1	-4.5	-4.5	-7.3	2.6	-1.7	0.0	0.0	-1.3	2.9	1.9	-0.1	-0.2	-0.2

Table 7. Certainty equivalent returns under varying feedback parameters in the investment strategy and financial market. Robustness check with respect to Table 4: the certainty equivalent return (CER) on wealth when the financial market is based on the feedback parameter $\kappa^0_{R\pi}$, while the investor applies the SW-strategy based on $\kappa^m_{R\pi}$. The other parameters of the financial market and the investment strategy are based on the remaining parameters in set θ^0. The results are computed for the value of $\hat{\kappa}_{R\pi} = -0.078$, the sensitivity values $\kappa^-_{R\pi} = -0.117$ and $\kappa^+_{R\pi} = -0.039$, and a value near or equal to zero. The results are shown for (a) the two-bond SW-strategy, and (b) the single-bond SW-strategy. The bond maturities equal $(\hat{\tau}_1, \hat{\tau}_2) = (10, 50)$. The CER is shown for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, and expressed in annual percentages.

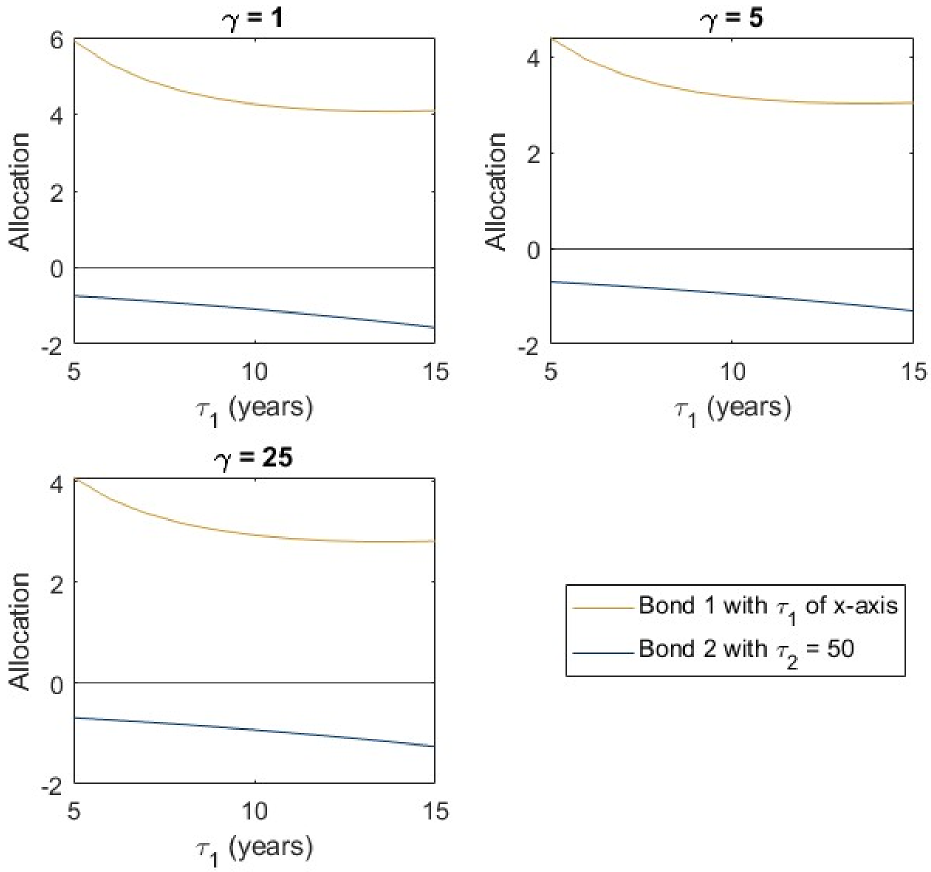
(a) SW2-strategy

$\kappa^0_{R\pi}$	$\boldsymbol{\kappa^-_{R\pi}}$				$\boldsymbol{\kappa^+_{R\pi}}$				$\mathbf{-0.001}$			
γ \ $\kappa^m_{R\pi}$	$\kappa^-_{R\pi}$	$\hat{\kappa}_{R\pi}$	$\kappa^+_{R\pi}$	-0.001	$\kappa^-_{R\pi}$	$\hat{\kappa}_{R\pi}$	$\kappa^+_{R\pi}$	-0.001	$\kappa^-_{R\pi}$	$\hat{\kappa}_{R\pi}$	$\kappa^+_{R\pi}$	-0.001
1	4.4	4.2	1.4	-100	4.1	4.2	4.4	-100	3.7	3.7	3.7	4.4
2	3.9	3.3	-5.3	-100	2.6	3.0	3.5	-100	1.0	1.0	1.0	3.2
5	3.6	1.8	-21.5	-100	-0.0	1.3	3.0	-100	-3.8	-3.8	-4.0	2.5
10	3.5	-0.5	-27.0	-100	-2.3	-0.5	2.8	-100	-6.7	-6.8	-6.9	2.2
25	3.4	-2.8	-30.7	-100	-4.1	-2.0	2.7	-100	-8.4	-8.5	-8.7	2.1

(b) SW1-strategy

$\kappa^0_{R\pi}$	$\boldsymbol{\kappa^-_{R\pi}}$				$\boldsymbol{\kappa^+_{R\pi}}$				**0**			
γ \ $\kappa^m_{R\pi}$	$\kappa^-_{R\pi}$	$\hat{\kappa}_{R\pi}$	$\kappa^+_{R\pi}$	0	$\kappa^-_{R\pi}$	$\hat{\kappa}_{R\pi}$	$\kappa^+_{R\pi}$	0	$\kappa^-_{R\pi}$	$\hat{\kappa}_{R\pi}$	$\kappa^+_{R\pi}$	0
1	3.6	3.6	3.4	3.0	3.8	3.9	3.9	3.9	3.8	4.0	4.1	4.1
2	3.0	3.0	3.0	2.9	2.6	2.6	2.6	2.6	2.1	2.2	2.2	2.2
5	2.2	2.2	1.9	1.4	0.5	0.6	0.7	0.6	-0.9	-0.7	-0.6	-0.6
10	1.3	1.0	0.2	-1.3	-1.4	-1.2	-1.4	-1.7	-3.3	-3.0	-3.0	-3.2
25	-0.4	-0.6	-2.2	-4.2	-2.9	-3.0	-3.4	-3.8	-5.0	-4.7	-4.9	-5.2

Figure 1. Optimal bond allocations for varying bond maturity combinations. The optimal bond allocations under the parameter set θ^0 of the two-bond SW-strategy, for a fixed investment horizon $T-t=30$, and risk aversion coefficients $\gamma \in \{1,5,25\}$. The allocations are shown for varying bond maturity combinations $(\tau_1, 50)$. The yellow line shows the corresponding long position in bond 1 with a varying bond maturity $\tau_1 \in \{5,6,...,15\}$ shown on the x-axis. The blue line shows the corresponding short position in bond 2 with the fixed bond maturity $\tau_2 = 50$.



[...]

Figure 2. Sensitivity optimal asset allocations to correlation between stocks and bonds. The optimal dynamic asset allocations $x_t(\theta^0)$ of the SW-strategy applied to a stock index and two nominal bonds, based on the parameter set θ^0. The solid lines are based on correlation between stock and bond prices as computed in Brennan and Xia (2002). The dashed lines are based on no correlation between the stock and bond prices. The green line shows the stock allocation. The yellow line shows the corresponding long position in bond 1 with bond maturity $\tau_1 = 10$. The blue line shows the corresponding short positions in bond 2 with bond maturity $\tau_2 = 50$. The allocations are shown for time t from 0 to the investment horizon $T = 30$, so from the left to the right the remaining investment horizon decreases to 0. The allocations are shown for risk aversions $\gamma \in \{1, 5, 25\}$ and as fraction of wealth.

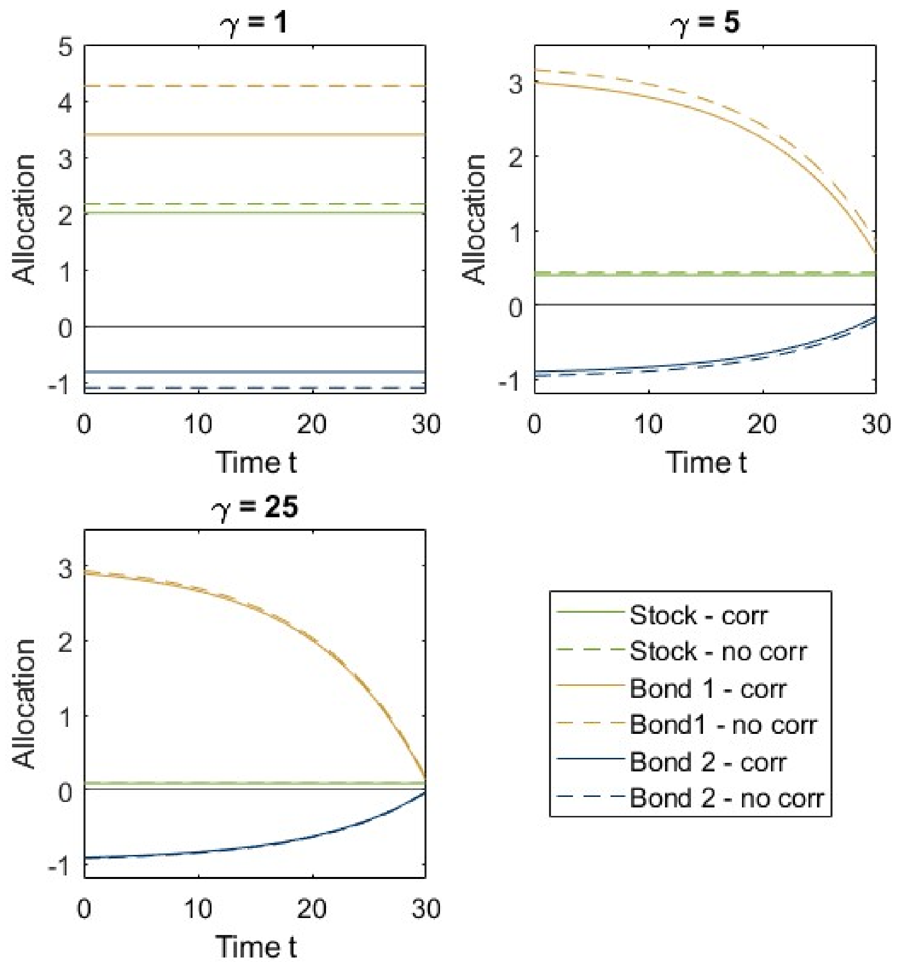


Figure 3. Sensitivity optimal stock and bond allocations to mean-reversion parameter. The optimal dynamic asset allocations $x_t(\theta^0)$ of the SW-strategy applied to a stock index and two nominal bonds where the stock and bond prices are correlated. The solid lines are based on the parameter set θ^0. The dashed lines are based on the alternative parameter set with the higher nominal interest rate mean-reversion parameter κ_R^+. The green line shows the allocation to the stock, where the solid and dashed line overlap since this allocation is not affected by κ_R. The yellow line shows the corresponding long position in bond 1 with bond maturity $\tau_1 = 10$. The blue line shows the corresponding short positions in bond 2 with bond maturity $\tau_2 = 50$. The allocations are shown for time t from 0 to the investment horizon $T = 30$, so from the left to the right the remaining investment horizon decreases to 0. The allocations are shown for risk aversions $\gamma \in \{1, 5, 25\}$ and as fraction of wealth.

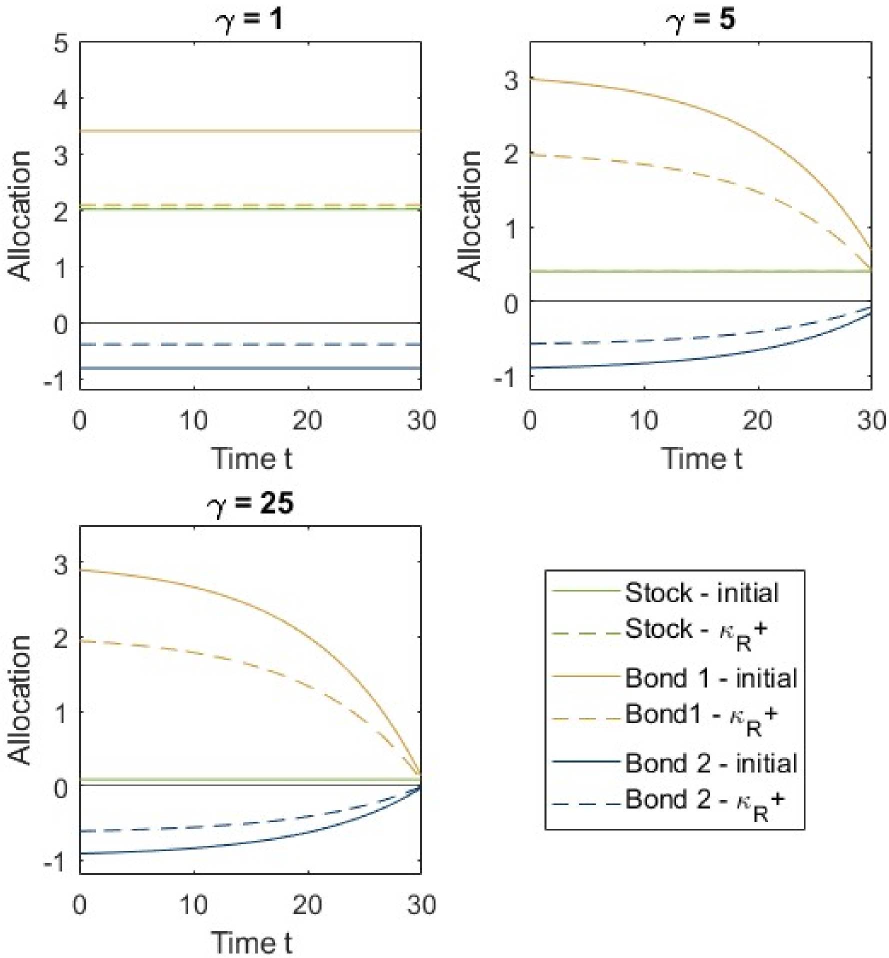
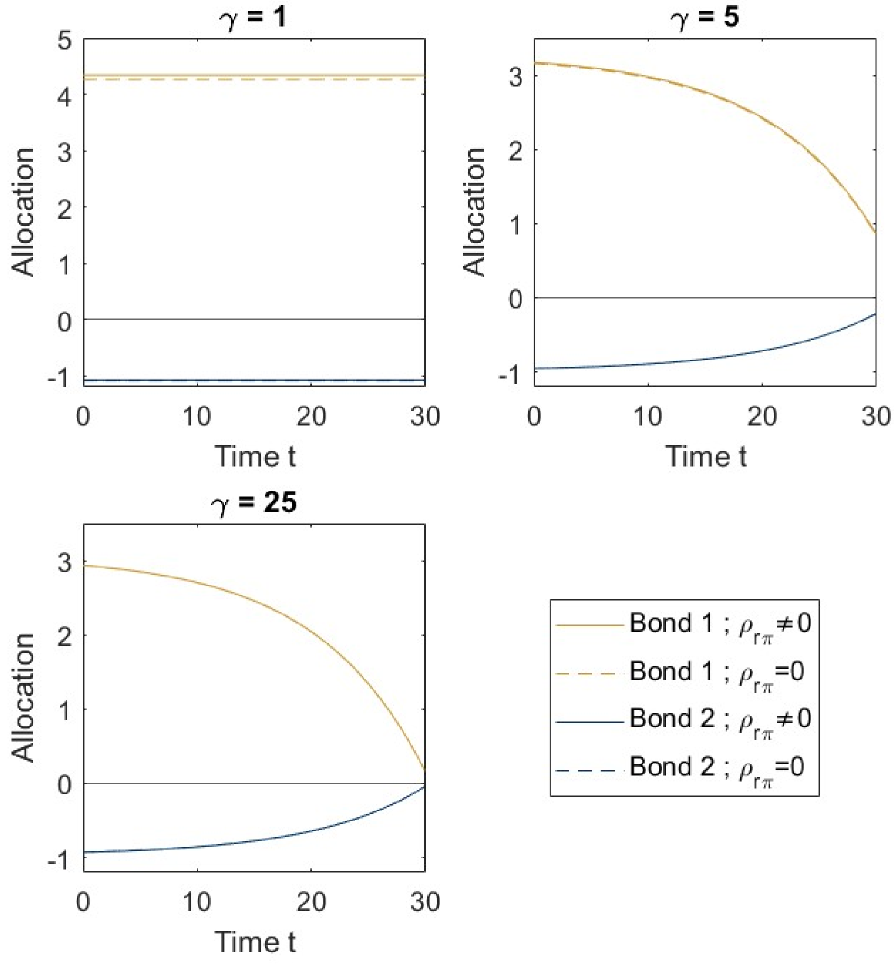


Figure 4. Sensitivity optimal bond allocations to correlation between risk factors. The optimal dynamic bond allocations $x_t(\theta^0)$ of the SW-strategy. The solid lines are based on the parameter set θ^0 and the estimated value of the correlation between the real interest rate and inflation rate of $\rho_{r\pi} = -0.061$. The dashed lines are based on the parameter set θ^0 and no correlation, $\rho_{r\pi} = 0$. The yellow line shows the corresponding long position in bond 1 with bond maturity $\tau_1 = 10$. The blue line shows the corresponding short positions in bond 2 with bond maturity $\tau_2 = 50$. The allocations are shown for time t from 0 to the investment horizon $T = 30$, so from the left to the right the remaining investment horizon decreases to 0. The allocations are shown for risk aversions $\gamma \in \{1, 5, 25\}$ and as fraction of wealth.



2

Robust Hedging of Terminal Wealth under Interest Rate Risk with the Constraint Approach

Joint work with Anne G. Balter and Frank de Jong.[1]

[1]We are grateful for the valuable feedback of Christoph Hambel, Mathijs Cosemans, Kim Peijnenburg, Antoon Pelsser, Joost Driessen, Rik Frehen, Stefano Cassella, Bas Werker, Jorgo Goossens, Nikolaus Schweizer, Rodolphe Vanderveken, and other seminar participants for their helpful comments and suggestions during the QFAS workshop Tilburg University (2022,2023), Belgian Financial Research Forum (2023), Financial Risks International Forum (2023), Finance Seminar of University of Southern Denmark (2022), Finance Seminar of Aarhus University (2022), DGPE workshop (2022), Netspar Pension Day (2022), and Finance Seminar of Tilburg University (2022).

ABSTRACT: *Long-term investors hedge their liabilities against interest rate risk by nominal bond investments. However, model-based optimal bond allocations are very sensitive to the underlying parameters, so that an investor who ignores parameter uncertainty may face a large utility loss. Therefore, we consider a robust bond investor who takes into account parameter uncertainty. We apply the constraint approach that limits the considered entropy around the market price of risk and the mean-reversion parameter. In case of only uncertainty about the market prices of risk, we find that the strategy of a speculative investor shrinks to that of a hedge investor. If we additionally allow uncertainty about the mean-reversion parameter, we find state- and horizon-dependent robust bond allocations.*

2.1 Introduction

Chapter 1 showed that long-term investors can apply the optimal nominal bond strategy to fully hedge both the nominal interest rate and inflation rate risk. However, this strategy is very sensitive to the input parameters which can result in considerable utility losses in the presence of estimation errors. Through simulations, we demonstrated that although alternative strategies yielded lower utility in the absence of estimation errors, they mitigated utility losses when such estimation errors were made. In this chapter, we explore how so-called robust control methods can be applied to derive a robust strategy under Knightian parameter uncertainty, where the investor considers alternative plausible parameter values besides the baseline estimates.

We address parameter uncertainty about the risk factor processes in an affine term structure model, by using a robust max-min objective function as proposed by Chen and Epstein (2002). In this approach, a robust investor selects the investment strategy that maximises her utility under the worst-case parameter within a range of parameters that she considers to be plausible. Since this parameter range is defined by a constraint on the set of considered probability measures, the method can be referred to as the constraint approach. The literature mainly analyses robust strategies for a given range of the market price of risk of stocks (see e.g., Biagini and Pınar (2017)). However, we focus on bonds. We first derive and analyse the robust nominal bond allocations in a one-factor and two-factor model analytically, given the considered range for the market price of risk of the nominal interest rate and inflation rate. Additionally, Chapter 1 demonstrated the considerable impact of estimation errors in the mean-reversion parameter on utility. Therefore, we subsequently analyse robust nominal bond allocations within a considered range of the mean-reversion parameter of the nominal interest rate. Since this problem appears to be challenging to solve analytically, we address it numerically with a one-factor model.

We make three main contributions to the existing literature. First, we focus on dynamic robust bond allocations under parameter uncertainty with the constraint approach. Garlappi et al. (2007) and Biagini and Pınar (2017) analyse optimal stock allocations with the constraint approach, in a static mean-variance and dynamic set-up respectively. Implications for bonds might differ from those for stocks due to the hedge demand of bonds that hedges the (real) interest rate risk. A related study by Flor and Larsen (2014) derives robust dynamic bond allocations based on a different robust control method from the penalty approach.[2] Moreover, Feldhütter et al. (2012) consider the impact of uncertainty about the underlying risk factors of the nominal interest rate in a Bayesian framework.[3] Both papers assume a distribution over alternative parameters. In contrast, by applying the constraint approach, we show the direct link between the range of possible parameters and robust bond allocations that prevent utility losses in the worst-case scenario. As a result, the constraint on the alternative parameter can be economically interpreted as not taking into account parameter values that are considered implausible, in line with the Good-Deal-Bound methodology of Cochrane and Saá-Requejo (2000).

Second, we extend the existing robust bond literature with parameter uncertainty regarding two market prices of risk, that of the nominal interest rate and the inflation rate. Flor and Larsen (2014) investigate the impact of uncertainty about the market price of risk of the nominal interest rate. We also analyse the case with uncertainty regarding two market prices of risk simultaneously, where the relation between deviations from the baseline estimates depends on the correlation between the nominal interest rate and the inflation rate. Balter and Pelsser (2020) analyse this multidimensional uncertainty for stock allocations instead of bond allocations. To the best of our knowledge, we are the first to apply the constraint approach to find robust bond allocations under inflation rate risk and interest rate risk.

Third, we extend the existing literature with parameter uncertainty regarding the market price of risk where the deviation from the baseline parameter is proportional in the state variable. We show that this can be translated to parameter uncertainty about the mean-reversion parameter. Martellini et al. (2015) show that the highly sensitive two-bond strategy may indeed perform worse than a suboptimal strategy, especially due to estimation errors of the mean-reversion parameters of the real interest rate and expected

[2]Flor and Larsen (2014) apply the penalty approach instead of the constraint approach, which will be explained further in the literature review in Section 2.2.

[3]Theoretically, a Bayesian investor assumes a single prior which is known, so there is actually no Knightian parameter uncertainty, but rather risk that parameter estimations are made.

inflation rate in practice.[4] While that study proposes to evaluate the bond strategy's sensitivity ex-post regularly, we follow the robust control method literature to find an ex-ante robust strategy that mitigates future utility losses for the worst-case estimation error in the mean-reversion parameter. This worst-case can depend on the state variables as well, in line with studies about uncertainty regarding the mean-reversion parameter of economic growth by Andrei et al. (2019) and Hansen and Sargent (2021, 2022). These studies show that the investor fears a low mean-reversion in a 'bad state', and a high mean-reversion in a 'good state'. Instead, we are interested in uncertainty about the mean-reversion in the (real) interest rate and inflation rate, along with its implications for a robust investment strategy.

In our analysis, we first derive the optimal robust allocations in a one-factor model without inflation. We show that uncertainty about the market price of interest rate risk parameter results in investment behaviour that is similar to that of a more risk-averse investor, resulting in a strong desire for the hedge demand for long-term bonds. By investing in the robust bond allocation, the more speculative investor can prevent large utility losses under the worst-case market price of risk.

Consecutively, we extend to a two-factor model where the nominal interest rate is the sum of the real interest rate and the inflation rate, and the investor is uncertain about the market prices of risk of both these risk factors. Similar to the one-factor model, parameter uncertainty appears to be relevant for a more speculative investor who behaves more like a hedge investor when parameter uncertainty increases. The robust investor takes less extreme long-short positions in terms of absolute bond weights, especially when the investment horizon is relatively short.

We extend our model to account for state-dependent uncertainty by allowing the market prices of risk to be an affine function of the state variables. In this framework, the distorted model remains affine, though the mean reversion parameters differ. For a robust speculative investor, the distorted risk premium is lower compared to a non-robust speculative investor. As risk aversion increases, the investor is concerned not only with a lower market price of risk but also with the persistence of low interest rates, especially over longer investment horizons. The corresponding robust bond weights may be smaller or larger than non-robust investments, depending on the initial interest rate and investment horizon. Our findings indicate that when the interest rate is above the long-term average, robust bond weights are smaller than the non-robust ones. In contrast, with a low interest

[4] As the real interest rate is a linear combination of the nominal interest rate and expected inflation, this translates to a fluctuating nominal interest rate mean-reversion parameter.

rate and a long investment horizon, robust bond allocations are larger, driven by the investor's fear of a high persistence in low interest rates.

In summary, we offer a robust max-min strategy framework for investors that can be directly linked to a range of parameters considered to be plausible. We show that when a speculative investor considers larger ranges for constant market prices of risk, her allocations shrink toward the strategy of a hedge investor. We moreover show that investors should consider a range of plausible mean-reversion parameters of the nominal interest rate. For this type of uncertainty, it is important to consider robust strategies that are updated based on the investment horizon and the level of the current nominal interest rate.

The structure of this chapter is as follows. Section 2.2 presents the literature review. Section 2.3 explains the model set-up. Section 2.4 analyses parameter uncertainty about the market price of risk – without uncertainty about the mean-reversion parameter – in a one-factor model for the interest rate (without inflation) and in a two-factor model where the state variables are the nominal interest rate and the expected inflation. Section 2.5 analyses uncertainty about the mean-reversion parameter in a one-factor model. Section 2.6 concludes.

2.2 Literature review

This section discusses the literature about robust asset allocations under Knightian uncertainty. For this type of uncertainty, the investor has an initial belief about the financial market's probability measure, but acknowledges that her initial beliefs may be incorrect. We apply the max-min method of Gilboa and Schmeidler (1989), where the multiple priors are defined by a set of different probability measures. Chen and Epstein (2002) and Epstein and Schneider (2003) apply this method in a dynamic setting for stock allocations. As the max-min method imposes exogenously specified bounds on deviations from the baseline probability measure, it is also known as the constraint approach. Constraints can be imposed as maximal deviations of the baseline parameters at each point in time, as in Biagini and Pınar (2017), or as bounds on the divergence from the baseline model measured by relative entropy over the entire investment horizon, as in Hansen et al. (2006).[5]

[5]Recent literature by Hansen and Sargent (2021, 2022) extends the constraint on total entropy to a more general case, where the investor considers not only so-called structured model uncertainty (where the parameters can be incorrectly estimated within the baseline model), but also unstructured models

The max-min method fits into literature about robust worst-case methods, as the minimisation operator selects the worst-case prior, while the maximisation operator optimises expected utility given this prior. Kim et al. (2014) review studies on robust worst-case asset allocation techniques and report that some numerical approaches to dynamically optimising worst-case asset allocations may help reduce large portfolio value fluctuations during financial crises.

An alternative robust control method is the penalty approach introduced by Anderson et al. (2003). This method adds a penalty term to the objective function to measure the divergence between the alternative and the baseline probability measure. Most related papers apply relative entropy (Kullback-Leibler divergence) as the divergence function due to tractability advantages, following Hansen and Sargent (2008). Maccheroni et al. (2006) design a variational preferences framework that embeds both the constraint and penalty approach.

The main difference between the penalty and constraint approach is that the former considers all equivalent probability measures, and assigns weights to them based on their divergence from the baseline measure. In contrast, the constraint approach includes only alternative measures within a specified range and hence applies a 'binary penalty' of zero within the range and infinity outside the range. Hansen et al. (2006) show that the penalty approach is related to the constraint approach through the Lagrange multiplier, when the constraint is applied on the entropy over the entire investment horizon. The penalty approach applies a constant penalty parameter reflecting the investor's aversion to deviations in the probability measure rather than that this parameter is endogenously determined as the Lagrange multiplier in the constraint approach.[6]

Klibanoff et al. (2005) introduced a smooth ambiguity model in which the impact of parameter uncertainty is modelled by both ambiguity– the investor's subjective beliefs about alternative probability measures– and ambiguity attitudes, the investor's aversion or preference toward this ambiguity. In the special case where the distribution over priors is uniform and the investor is infinitely ambiguity-averse, the smooth ambiguity model reduces to the constraint approach based on a max-min objective. Balter et al. (2021) analyse the implication of time-inconsistency of optimal investments under smooth ambiguity about stock returns.

Another strand of literature combines ambiguity attitudes with learning: the ambiguity-

that are statistically indistinguishable from structured models within a given neighbourhood. We focus on structured model uncertainty.

[6]See Hansen and Marinacci (2016) for a discussion on homothetic robustness of the penalty approach and time-consistency of the constraint approach.

averse investor considers a set of plausible priors, but updates her beliefs about an unobserved component over time. As a result, the range of plausible measures can change over time. Branger et al. (2013) and Munk and Rubtsov (2014) show the considerable utility losses for ambiguity-averse investors who do not apply Bayesian updating when unobserved state variables affect expected stock returns or the expected inflation rate, respectively. Liu (2011) assumes regime-switching expected stock returns and shows that Bayesian updating offers almost no economic value for highly ambiguity-averse investors with a long investment horizon. Campanale (2011) and Peijnenburg (2018) show that optimal stock allocations increase with age when investors learn about expected excess stock returns, as uncertainty about the parameters decreases over time.

Ambiguity attitudes can alternatively be implemented in the constraint approach by the alpha-maxmin expected utility method, introduced by Marinacci (2002). The objective function is a linear combination of the best-case and worst-case scenarios, where a more ambiguity-averse investor places larger weights on the max-min problem corresponding to the worst case. In contrast, a less ambiguity-averse investor places more weight on the baseline prior corresponding to the case without parameter uncertainty. Yu et al. (2020) and Yang et al. (2025) analyse optimal asset allocations under drift uncertainty using the alpha-maxmin model in a static mean-variance and dynamic set-up, respectively.

Kim et al. (2014) explain that estimation errors in expected asset returns have a larger impact on portfolio performance than errors in the (co)variances of asset returns, which is why most robust asset allocation studies focus on expected stock returns. They also note that the ellipsoidal uncertainty set, which captures simultaneous deviations on each asset's return, is commonly used for tractability. Epstein and Ji (2013) acknowledge the tractability advantage of uncertainty in mean returns but argue that ambiguity in asset price volatilities is also important, which is supported experimentally by Baltussen and Van der Grient (2018). Moreover, Kim et al. (2014) investigate the impact of correlation uncertainty in a mean-variance setting numerically.

However, as explained in Biagini and Pınar (2017) and Hölzermann (2021), considering ambiguous volatilities may require adjustments to the model due to the non-dominated priors which result in an incomplete market. An example applied to bond prices is the study by Hölzermann (2021) that applies the G-Brownian motion technique. Biagini and Pınar (2017) present an example of stock price volatility uncertainty while remaining within a dominated prior setting with uncertainty about the excess returns, and the range of priors depends on the considered alternative volatilities. They conclude that the worst-case priors results in a low expected return and a high volatility. This is in line with Hölzermann (2024), who applies the G-Brownian motion technique and considers upper and lower bounds on both the market price of risk and the volatilities of a bond and stock. Our study aligns with research that adjust the drift term, but we apply a state-dependent drift adjustment that results in a different mean-reversion parameter. This will be explained in the following section.

2.3 Model description

2.3.1 Financial market

We assume a standard affine model for the nominal interest rate R_t and expected inflation π_t as proposed in Sangvinatsos and Wachter (2005). Under the baseline real-world probability measure $\mathcal{P}$, the state variables X_t follow the process:

$$dX_t = -KX_t dt + \sigma_X dZ_t \tag{1}$$

where X_t is an $n \times 1$ vector, Z_t is an $n \times 1$ vector of Brownian motion processes ($N(0, \rho)$) with $n \times n$ correlation matrix ρ, K is the $n \times n$ mean-reversion matrix, and σ_X is the $n \times n$ factor volatility matrix. We assume that the nominal interest rate and expected inflation are linear functions of the state variables, $R_t = \delta_{0R} + \delta_R' X_t$ and $\pi_t = \delta_{0\pi} + \delta_\pi' X_t$ with scalars $\delta_{0(.)}$ and $m \times 1$ vectors δ_R and δ_π. The realised inflation process equals

$$\frac{d\Pi_t}{\Pi_t} = \pi_t dt \tag{2}$$

so we assume no unexpected inflation. The pricing kernel scalar ζ_t is given by

$$\frac{d\zeta_t}{\zeta_t} = -R_t dt - \lambda' \rho^{-1} dZ_t, \quad \zeta_0 = 1 \tag{3}$$

where the $n \times 1$ vector λ contains the (negative) constant market prices of risk (MPoR) on the risk factors.[7] We assume a complete market so that the investor can invest in the risk-free asset with return R_t, and n nominal bonds with prices

$$\frac{dP_t(\tau_j)}{P_t(\tau_j)} = [R_t + \sigma_{Bj}\lambda]\, dt + \sigma_{Bj} dZ_t \tag{4}$$

[7] The negative market prices of risk ensure a positive risk premium.

where $\sigma_{Bj} = -B(\tau_j)'\sigma_X$ is a $1 \times n$ vector of exposures to the risk factors of bond j with bond maturity τ_j, and the $n \times 1$ vector $B(\tau_j)$ is computed by solving the ODE:[8]

$$B'(\tau_j) = -K\, B(\tau_j) + \delta_R \tag{5}$$

Hence, the bond volatilities depend on the mean-reversion parameter matrix. The intuition is that a higher mean-reversion parameter implies a lower persistence of shocks in the nominal interest rate, causing nominal bond prices to respond less strongly to changes in the nominal interest rate. We consider an investor who aims to optimise her real wealth, subject to the budget constraint. The optimisation problem is defined as follows:

$$\max_{W_T} \mathbb{E}_t \left[u\left(\frac{W_T}{\Pi_T}\right)\right] \qquad \text{s.t. } \mathbb{E}_t\left[\zeta_T W_T\right] = W_t \tag{6}$$

where $u()$ is the investor's utility function, W_t is the nominal wealth at t, and T is the investment horizon. Moreover, we assume that the investor has a constant relative risk aversion (CRRA) utility function with coefficient of relative risk aversion $\gamma \geq 1$:

$$u(w) = \begin{cases} \frac{w^{1-\gamma}}{1-\gamma} \text{ if } \gamma > 1 \\ \log(w) \text{ if } \gamma = 1 \end{cases} \tag{7}$$

The optimal investment strategy in $n \times 1$ vector x_t is derived by Sangvinatsos and Wachter (2005), which in our notation reads as:

$$x_t = \frac{1}{\gamma}\Omega^{-1}\sigma_B\lambda - \left(1 - \frac{1}{\gamma}\right)\Omega^{-1}\sigma_B\rho\sigma_X'\tilde{B}_2(T-t) \tag{8}$$

where the $n \times 1$ vector $\tilde{B}_2(T-t)$ contains the exposure to the risk factors when hedging the real interest rate and is computed by solving the ODE in Appendix B of Sangvinatsos and Wachter (2005).[9] Similar as for the nominal interest rate exposures within the nominal bond price process $B(\tau_j)$, given in equation (5), it depends on the mean-reversion matrix of the risk factors. Moreover, we have the $n \times n$ covariance-matrix of bond returns

[8]The $n \times 1$ vector $B'(\tau_j)$ denotes the derivative of the vector $B(\tau_j)$ with respect to τ_j. Appendix A.2.1 explains the origin of the ODEs and derives the corresponding solutions for the model structures analysed in Sections 2.4 and 2.5.

[9]We analyse different model structures in Sections 2.4 and 2.5, leading to different ODEs of $\tilde{B}_2(T-t)$. Therefore, we present the corresponding ODEs and their solutions in Appendix A.2.1.

$\Omega = \sigma_B \rho \sigma'_B$ with σ_B as the $n \times n$ matrix containing the bond exposures of n bonds:

$$\sigma_B = \begin{bmatrix} \sigma_{B1} \\ \dots \\ \sigma_{Bn} \end{bmatrix} \tag{9}$$

Until now, we have considered the financial market without parameter uncertainty. The next section explains how we incorporate parameter uncertainty.

2.3.2 Parameter uncertainty

We now consider an investor who is uncertain about the real-world probability measure $\mathcal{P}$. We assume that the investor knows the exact risk-neutral probability measure $\mathcal{Q}$. The intuition is that she observes nominal bond prices and therefore can calculate the risk-neutral probabilities precisely. However, the investor is uncertain about her estimate of the market price of risk. We introduce a second player in this game, the so-called mother nature, who applies distortions on the risk factors and distorts the empirical measure from $\mathcal{P}$ to $\mathcal{P}^*$. We model these distortions by vector C_t with the corresponding distorted Brownian Motion process Z_t^*:

$$dZ_t = C_t dt + dZ_t^* \tag{10}$$

Following Hansen and Sargent (2021), their equation (20), we consider a class of affine distortion functions

$$C_t = \alpha X_t + \beta \tag{11}$$

where α is an $n \times n$ matrix and β is an $n \times 1$ vector. Substituting from equation (10) gives

$$dX_t = -K^* X_t dt + \sigma_X \beta dt + \sigma_X dZ_t^* \tag{12}$$

with the distorted mean reversion parameters:

$$K^* = K - \sigma_X \alpha \tag{13}$$

Bond prices under the distorted measure follow the process:

$$\frac{dP_t}{P_t} = [R_t + \sigma_B \lambda_t^*]\, dt + \sigma_B dZ_t^* \tag{14}$$

with

$$\lambda_t^* = \lambda + C_t = \lambda + \alpha X_t + \beta \tag{15}$$

the distorted market prices of risk (MPoR). By modelling distortions through the market price of risk, we impose a distorted mean-reversion matrix of the risk factors, shown in equation (13), and consequently a change in the bond price volatility through the bond exposure shown in equation (5). Despite the change in the bond price volatility, the distorted measure can still be interpreted as an equivalent martingale measure: the investor faces the risk of being unable to distinguish between the original measure $\mathcal{P}$ with λ and the distorted measure $\mathcal{P}^*$ with λ_t^*.[10] The intuition that a stochastic distortion in the market price of risk results in a different mean-reversion estimate will be explained in Section 2.5.

We want to distinguish between the effect of deterministic distortions in the market price of risk and the effect of stochastic distortions. Deterministic distortions are commonly studied in the robust asset allocation literature, but we are interested in stochastic distortions which result in a distorted mean-reversion matrix. Therefore, we consider two special cases. First, we consider the so-called 'deterministic distortion' with $\alpha = \mathbf{0}$:

$$\lambda_t^* = \lambda + \beta \tag{16}$$

$$K^* = K \tag{17}$$

Second, we consider the so-called 'stochastic distortion' with $\beta = \mathbf{0}$:

$$\lambda_t^* = \lambda + \alpha X_t \tag{18}$$

$$K^* = K - \sigma_X \alpha \tag{19}$$

that corresponds to a change in the mean-reversion matrix.

[10] This is in line with the change of measure in the risk-neutral measure $\mathcal{Q}$ to the real-world measure $\mathcal{P}$ involving a state-dependent market price of risk analysed by Sangvinatsos and Wachter (2005). In contrast, non-dominated prior methods or model uncertainty frameworks assume different processes for Z_t than Brownian Motions and consequently result in market incompleteness, as outlined in the literature review in Section 2.2.

2.3.2.1 Robust Optimisation problem

We consider a robust investor who aims to optimise her real wealth, subject to the budget constraint. To implement robustness, we use a max-min objective function so that the investor aims to optimise against a worst-case scenario. To determine which probability measures she considers to be plausible, we use the constraint approach with the uncertainty bound $k > 0$. The optimisation problem is defined as follows:

$$\max_{W_T} \min_{\alpha,\beta} \mathbb{E}_t^* \left[u \left(\frac{W_T}{\Pi_T} \right) \right] \tag{20}$$

$$\text{s.t. } \mathbb{E}_t^* \left[\zeta_T^* W_T \right] = W_t \tag{21}$$

$$\text{and } \begin{cases} \text{if } \alpha = \mathbf{0} \text{ then } \sqrt{\beta' \rho^{-1} \beta} \leq k \\ \text{if } \beta = \mathbf{0} \text{ then } \|\alpha\|_2 \leq k \end{cases} \tag{22}$$

where $\mathbb{E}^*$ is the expectation under the distorted measure $\mathcal{P}^*$. The first constraint in equation (21) is the budget constraint, where the pricing kernel under the alternative $\mathcal{P}^*$-measure is

$$\frac{d\zeta_t^*}{\zeta_t^*} = -R_t dt - (\lambda_t^*)' \rho^{-1} dZ_t^* \quad (\zeta_0^* = 1) \tag{23}$$

The second constraint in equation (22) imposes bounds on the distortion through a scalar $k > 0$. A larger value of k corresponds to a wider range of considered alternative plausible measures and can therefore be interpreted as more parameter uncertainty. We refer to k as the bound on uncertainty that restricts distortions at each point in time.[11]

We now provide the intuition behind the bounds for each type of distortion. In case of a constant distortion when $\alpha = \mathbf{0}$, and within a one-factor model, the bound k corresponds to the concept of Good-Deal-Bounds (GDB) introduced by Cochrane and Saá-Requejo (2000).[12] The investor does not consider market prices of risk that are considered "too good to be true". In a two-factor model, the inverse correlation matrix is based on Biagini and Pınar (2017), resulting in an ellipsoid uncertainty set corresponding to simultaneous distortions on the two factors, with the baseline measure as the centre.[13]

[11] This is in contrast to methods which constrain accumulated distortions over the entire investment horizon through an "entropy budget" at time zero (see e.g., Hansen and Sargent (2008)).

[12] Cochrane and Saá-Requejo (2000) is one of the first studies defining GDBs in a static setting, while Björk and Slinko (2006) analyse GDBs in continuous time with jump events. Moreover, Klöppel and Schweizer (2007) show dynamic properties such as time consistency when GDBs are implemented through local conditional restrictions on the market price of risk.

[13] The constraint in Biagini and Pınar (2017) is based on deviations in gross returns and the covariance

Figure 1 shows an example of the ellipsoid uncertainty set for $k \in \{\sqrt{0.01}; \sqrt{0.05}; \sqrt{0.5}\}$ with positively correlated risk factors. It shows that larger maximum distortions in one risk factor are only considered when accompanied by larger maximum distortions in the other risk factor. The ellipsoid uncertainty set falls into the category of spherical sets. An advantage of this structure is that it avoids extreme corner solutions, unlike a box set that considers two independent intervals for each market price of risk, as in Chen and Epstein (2002).[14]

In case of a stochastic distortion when $\beta = \mathbf{0}$, we are, to the best of our knowledge, not aware of directly related robust asset allocations. Therefore, we draw on insights from the literature on distorted mean-reversion parameters in economic growth models. Hansen and Sargent (2021) present an illustration that can be translated to our stochastic distortion setting.[15] Their lower and upper bound on the alternative mean-reversion parameter are based on restricting the time derivative of the conditional expectation of relative entropy at each point in time. We propose a bound on the 2-norm of the matrix α, which would reduce to an interval around the scalar in a one-factor model, $|\alpha| \leq k$.

2.3.2.2 Robust investment strategy and wealth process

The problem in equation (13) is a max-min problem. We assume a saddle point so that we can find the solution by reversing the max and min operators, which we explain in Appendix A.2.3. In this case, we can first derive the optimal wealth given the values of α and β, and consecutively minimise with respect to α and β. Therefore, we first solve the maximisation problem finding the optimal wealth based on investing optimally under a given distorted measure $\mathcal{P}^*$. Similar as for the optimal strategy based on the undistorted measure in equation (8), x_t, the optimal strategy based on the distorted measure, x_t^*, is derived by Sangvinatsos and Wachter (2005), which in our notation reads as:

$$x_t^* = \frac{1}{\gamma}\Omega^{-1}\sigma_B\lambda_t^* - \left(1 - \frac{1}{\gamma}\right)\Omega^{-1}\sigma_B\rho\sigma_X' \left[\frac{\tilde{B}_3(T-t)' + \tilde{B}_3(T-t)}{2} \cdot X_t + \tilde{B}_2(T-t)\right] \quad (24)$$

matrix. In our model, this corresponds to $(\sigma_B\lambda^*)' \, \Omega^{-1} \, (\sigma_B\lambda^*) \leq \epsilon_k$. As bond returns depend on σ_B as well, σ_B cancels out through Σ^{-1} and the constraint can be simplified as a bound on $\beta'\rho^{-1}\beta$.

[14]See their equation (3.11) for the multiple-dimensional case of their so-called κ-ignorance framework.

[15]Specifically, Hansen and Sargent (2021) show in their Illustration 4.1 the resulting constraint on the distortion parameter, as given in their equation (24). This constraint also depends on a second distortion affecting the capital growth process. However, since that distortion does not imply a distortion in a second mean-reversion parameter, it should be set to zero to align with our model.

The optimal speculative demand is now based on the distorted market price of risk λ_t^*, and the hedge term includes an additional term involving the $n \times n$ matrix $\tilde{B}_3$, compared to the optimal strategy under the undistorted measure. This additional term is derived from the ODE in Appendix B of Sangvinatsos and Wachter (2005) and captures the effect in exposure to the risk factors due to a change in the mean-reversion parameter. It equals zero in case of a deterministic distortion.[16] As we assumed that the investor is not uncertain about her nominal bond prices under $\mathcal{Q}$, the bond exposures in σ_B do not depend on the distortions. However, the empirical measure is distorted so that the distorted MPoR λ_t^*, $\tilde{B}_2$ and $\tilde{B}_3$ are dependent on the distortion parameters α and β. The corresponding optimal wealth process W_t^* under the distorted measure $\mathcal{P}^*$ is

$$dW_t^*/W_t^* = [R_t + \sigma_{W^*}' \lambda_t^*]dt + \sigma_{W^*}' dZ_t^*, \tag{25}$$

$$\sigma_{W^*} = \frac{1}{\gamma}\rho^{-1}\lambda_t^* - \left(1 - \frac{1}{\gamma}\right)\sigma_X' \left[\frac{\tilde{B}_3(T-t)' + \tilde{B}_3(T-t)}{2} \cdot X_t + \tilde{B}_2(T-t)\right] \tag{26}$$

where the $n \times 1$ vector σ_{W^*} corresponds to the optimal wealth exposure to the risk factors. After solving the inner maximisation problem of the reversed objective, we now solve the outer minimisation. This implies that we compute, given the optimal strategy based on a distorted measure $\mathcal{P}^*$, the so-called worst-case scenario. In particular, we determine the distortion parameters $\hat{\alpha}$ and $\hat{\beta}$ that minimise the expected utility over the alternative parameters:

$$\{\hat{\alpha}, \hat{\beta}\} = \arg \min_{\alpha,\beta} \mathbb{E}_t^* \left[u\left(\frac{W_T^*}{\Pi_T}\right) \middle| x_t^* \right] \tag{27}$$

In the next sections, we shall present the solutions for the optimal distortions for several special cases.

We denoted the optimal strategy under a distorted measure as x_t^*. We now define the robust strategy $\hat{x}_t^*$ as the optimal strategy under the optimally distorted measure $\mathcal{P}^*$, i.e. based on $\hat{\alpha}$ and $\hat{\beta}$. We define the robust strategy over the investment horizon as $\hat{x}_t^*$. In the numerical examples, we measure the impact on utility of the optimal distorted

[16]The solution for the ODE of $\tilde{B}_3(T-t)$ for the stochastic distortion analysis in Section 2.5 is provided in equation (A.2.19).

terminal wealth by the certainty equivalent return (CER) on wealth, defined as:[17]

$$CEW_t = u^{-1}\left(\mathbb{E}_t^*\left[u\left(\frac{W_T^*}{\Pi_T}\right)\middle|\hat{x}_t^*\right]\right) \tag{28}$$

$$CER_t = \left[\frac{CEW_t}{W_t/\Pi_t}\right]^{1/(T-t)} - 1 \tag{29}$$

The CER is the annualised real return on initial wealth that compounds to the certainty equivalent of wealth at the investment horizon. The certainty equivalent of wealth is the amount that makes the investor indifferent between receiving this guaranteed amount or obtaining the terminal wealth resulting from applying the robust strategy $\hat{x}_t^*$ in a financial market based on the optimally distorted $\mathcal{P}^*$.

2.4 Deterministic distortion

This section analyses the robust investment strategy under the deterministic distortion given in equation (16), hence all entries in α equal 0 and the entries of β are free parameters. We start our analysis with a one-factor model, and then extend the setting to a two-factor model.

2.4.1 One-factor model

Problem 1 (One-factor model with deterministic distortion)**.** *We assume that the risk factor is the nominal interest rate R_t ($n = 1; \delta_R = 1$), and there is no inflation risk ($\pi_t = 0; \delta_\pi = 0$). For readability, we suppress some subscripts and denote the bond's*

[17] For completeness, note that in case of $\gamma = 1$ the certainty equivalent of wealth CEW_t is computed as the exponential of expected utility, because the utility function reduces to the log of terminal real wealth.

time-to-maturity by τ:

$$\max_{W_T}\min_{\beta} \mathbb{E}_t^* \left[u\left(W_T\right)\right] \tag{30}$$

$$\text{s.t. } \mathbb{E}_t^* \left[\zeta_T^* W_T\right] = W_t \text{ ; } \frac{d\zeta_t^*}{\zeta_t^*} = -R_t dt - \lambda^* dZ_t^* \tag{31}$$

$$\lambda^* = \lambda + \beta \tag{32}$$

$$|\beta| \leq k \tag{33}$$

$$dR_t = -\kappa\left(R_t - \bar{R}\right) dt + \sigma\beta dt + \sigma dZ_t^* \tag{34}$$

$$\frac{dP_t(\tau)}{P_t(\tau)} = \left[R_t - \lambda^* B(\tau)\sigma\right] dt - B(\tau)\sigma dZ_t^* \tag{35}$$

Recall that ζ_t^* is the pricing kernel under the distorted market price of risk. In case of the deterministic distortion, $\alpha = 0$ and hence $\tilde{B}_3(T-t) = 0$. Moreover, since there is no inflation risk, it holds that $\tilde{B}_2(T-t) = B(T-t)$.[18] For simplicity, we use the notation of κ and σ for the mean-reversion and volatility of the interest rate in the one-factor model. The optimal wealth process from equation (25) for this one-factor model under the distorted measure equals:

$$\begin{aligned} \frac{dW_t^*}{W_t^*} = &\left[R_t + \left(\frac{1}{\gamma}\lambda^* - \left(1 - \frac{1}{\gamma}\right)\sigma B(T-t)\right)\lambda^*\right] dt \\ &+ \left[\frac{1}{\gamma}\lambda^* - \left(1 - \frac{1}{\gamma}\right)\sigma B(T-t)\right] dZ_t^* \end{aligned} \tag{36}$$

This equation illustrates the risk-return trade-off faced by mother nature. Consider a positive distortion parameter β. Since the baseline market price of risk λ is negative, this results in a less negative distorted market price of risk λ^* compared to the undistorted one. Therefore, a positive distortion decreases the returns through the lower drift term of the distorted optimal wealth process W_t^*. However, it simultaneously reduces the risk by decreasing the volatility of W_t^*.

Deriving the optimal distortion can be challenging because of the presence of the stochastic risk-free rate R_t in the drift term. This would result in a drift term which contains distorted nominal interest rates due to distorted interest rate shocks, and therefore correlates with the wealth volatility which depends on the same shocks.[19] Therefore, we

[18] For completeness, the analytical solution of $\tilde{B}_2(T-t)$ for the one-factor model is given in the appendix in equation (A.2.4).

[19] This would complicate the computations where we assume an uncorrelated drift and volatility term (see equation (45) in the proof of Theorem 2.4.1).

consider the optimal wealth process expressed in units of a nominal bond with maturity $T-t$ that solves the objective function. The next theorem summarises the corresponding worst-case scenario and corresponding robust bond strategy.

Theorem 2.4.1 (One-factor model with deterministic distortion)**.** *Consider Problem 1, and assume that $\lambda + \sigma B(T-t) < 0$.*[20] *Then, the optimal distortion and corresponding worst-case distorted market price of risk equal:*

$$\hat{\beta} = \min\{-\lambda - \sigma B(T-t); k\} \tag{37}$$

$$\hat{\lambda}^* = \min\{-\sigma B(T-t); \lambda + k\} \tag{38}$$

where we recall that the notation $(^*)$ is used for the variables under a distorted measure, and $(\hat{\ })$ to those under the optimally distorted measure. Consequently, the investor optimally applies the corresponding robust bond allocation:

$$\hat{x}_t^* = -\frac{1}{\gamma} \cdot \frac{\min\{-\sigma B(T-t); \lambda + k\}}{\sigma B(\tau)} + \left(1 - \frac{1}{\gamma}\right) \cdot \frac{B(T-t)}{B(\tau)} \tag{39}$$

$$= -\frac{1}{\gamma} \cdot \frac{\min\{0; \lambda + \sigma B(T-t) + k\}}{\sigma B(\tau)} + \frac{B(T-t)}{B(\tau)} \tag{40}$$

Proof. We first consider the process of the optimal wealth under the distorted measure, scaled by the nominal bond price $P_t(T-t)$. Note that this does not change the value of scaled wealth at the investment horizon, because $P_T(0) = 1$. This results in the process

$$\frac{d\left(W_t^*/P_t(T-t)\right)}{W_t^*/P_t(T-t)} = \left[\frac{\left(\lambda^* + \sigma B(T-t)\right)^2}{\gamma}\right] dt + \left[\frac{\lambda^* + \sigma B(T-t)}{\gamma}\right] dZ_t^* \tag{41}$$

Compared to the unscaled process W_t^* in (25), we now do not have the stochastic interest rate in the drift term of the optimal wealth process. Because the scaled wealth is log-normal, we can solve the objective function for $\gamma \geq 1$ by:

$$\min_{\beta} \mathbb{E}_t^* \left[u\left(W_T^*/P_T(0)\right)\right] = \max_{\beta} \mathbb{E}_t^* \left[\left(e^{w_T}\right)^{1-\gamma}\right] \tag{42}$$

$$\Leftrightarrow \max_{\beta} e^{(1-\gamma)\bar{\mu} + \frac{1}{2}(1-\gamma)^2 \bar{\sigma}^2} \tag{43}$$

$$= \min_{\beta} \bar{\mu} + \frac{1}{2}\left(1-\gamma\right)\bar{\sigma}^2 \tag{44}$$

[20] This assumption is based on empirical results in the literature. Due to the relatively large (negative) λ and small σ, this equation is satisfied in practice.

where $w_T = \log(W_T^*/P_T(0))$, $\bar{\mu} = \mathbb{E}_t^*[w_T]$, and $\bar{\sigma}^2 = \text{Var}_t^*(w_T)$. Therefore, we can compute the optimal distortion by:

$$\hat{\beta} = \arg\min_{\beta} \int_t^T \frac{(\lambda^* + \sigma B(T-s))^2}{\gamma} ds - \frac{\gamma}{2}\left(\frac{\lambda^* + \sigma B(T-s)}{\gamma}\right)^2 ds \tag{45}$$

$$= \arg\min_{\beta} \int_t^T (\lambda^* + \sigma B(T-s))^2 \, ds \tag{46}$$

$$= \arg\min_{\beta} \; (\lambda^* + \sigma B(T-t))^2 \tag{47}$$

$$= \arg\min_{\beta} \frac{1}{2}\beta^2 + [\sigma B(T-t) + \lambda]\,\beta \tag{48}$$

We can combine this with the constraint given in equation (33) to the following Lagrange problem:

$$\min_{\beta} \frac{1}{2}\beta^2 + [\sigma B(\tau) + \lambda]\,\beta, \text{ s.t.} \quad \beta \leq k, \quad \beta \leq k \tag{49}$$

with the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\beta^2 + [\sigma B(\tau) + \lambda]\,\beta - \ell_{1t}(\beta - k) - \ell_{2t}(-\beta - k) \tag{50}$$

Because the objective function and constraints are convex, this can be solved with the Karush Kuhn Tucker (KKT)-conditions as given below:

$$\frac{d\mathcal{L}}{d\beta} = 0 \tag{51}$$

$$\ell_1(\beta - k) = 0 \tag{52}$$

$$\ell_2(-\beta - k) = 0 \tag{53}$$

$$\ell_1 \leq 0; \ell_2 \leq 0; \tag{54}$$

This results in three possible solutions. The first two are the boundary solutions where mother nature reaches the upper bound on uncertainty and β equals either $-k$ or k. The third is the interior solution $\beta = -\lambda - \sigma B(T-t)$. Via the KKT-conditions, we can translate this to cases where $\lambda + \sigma B(T-t)$ is either positive or negative. Figure 2 shows both cases and the intuition of the corresponding optimal distortions. Mother nature aims to distort $\lambda + \sigma B(T-t) + \beta$ to zero, which can be seen in equation (47). This results in the optimal distortions and optimally distorted market prices of risk:

$$\hat{\beta} = \begin{cases} -\max\{\lambda + \sigma B(T-t); -k\} & \text{if } \lambda + \sigma B(T-t) < 0 \\ -\min\{\lambda + \sigma B(T-t); k\} & \text{if } \lambda + \sigma B(T-t) \geq 0 \end{cases} \tag{55}$$

$$\hat{\lambda}^* = \begin{cases} \min\{-\sigma B(T-t); \lambda + k\} & \text{if } \lambda + \sigma B(T-t) < 0 \\ \max\{-\sigma B(T-t); \lambda - k\} & \text{if } \lambda + \sigma B(T-t) \geq 0 \end{cases} \tag{56}$$

We make the empirical relevant assumption of $\lambda + \sigma B(T-t) < 0$, in line with our empirical parameter set. With the corresponding optimal distortion, we can compute the optimal distorted market price of risk. We can plug in this distorted market price of risk into the optimal investment that is given in equation (24). In this way, we get the robust investment strategy in equation (40). □

The intuition of the min and max-operators in the optimal distortion $\hat{\beta}$ is that mother nature distorts the market price of risk as much as possible until she reaches either the interior solution or the solution restricted by the bound on uncertainty k. In case of the interior solutions, she can minimise the utility of the investor without restrictions. In case of the bounded solution, she can decrease the utility less than in the unconstrained case. Interestingly, this worst-case distortion is independent of the investor's risk aversion coefficient γ.

The corresponding robust bond allocation in equation (40) contains a speculative term and hedge term. The speculative term is impacted by parameter uncertainty via k. In case of $k > 0$, the investor considers alternative MPoRs. In the empirical relevant case of $\lambda < 0$, the increase in the negative MPoR leads to a lower optimal distorted risk premium $-\sigma B(T-t)\hat{\lambda}^* < -\sigma B(T-t)\lambda$. Therefore, the investor lowers her bond allocation to maximise her utility in the worst-case scenario with the lowest possible risk premium.

The second term of the bond weight corresponds to the hedge term and is unaffected by parameter uncertainty. The economic intuition is that an allocation of 100% in a bond with a maturity equal to the investment horizon can fully hedge the interest rate, independent of the distortion in the MPoR. In case of an infinite bound on uncertainty, any investor will act like a hedge investor. This result is in line with Flor and Larsen (2014) who show that a very ambiguity-averse investor acts like a very risk-averse investor without ambiguity in a similar setting.

We now show a numerical example of optimal robust bond allocations over an investment horizon of $T = 30$ years, a long-term nominal bond with fixed maturity $\tau = 30$, and the baseline parameter values given in Table 1, based on the values of Chapter 1. We set $R_0 = \bar{R}$, and $\pi_0 = \bar{\pi}$. We consider risk aversions $\gamma \in \{1, 2, 25\}$ that represent a speculative investor, a moderate risk-averse investor, and a very risk-averse investor respectively. Moreover, we study several uncertainty bounds $k \in \{0; 0.01; 0.05; 0.5\}$. To interpret the values of k, recall that it can be seen as a Good-Deal-Bound (GDB). For example, the baseline MPoR $\lambda = -0.219$ and a bound on uncertainty $k = 0.05$ result in a GDB of $-0.219 - 0.05 = -0.224$ and upper bound of $-0.219 + 0.05 = -0.214$, meaning that the investor does not consider MPoRs outside these bonds to be plausible. The value $k = 0.5$ is so large that it empirically corresponds to an unbounded distortion so that the results are similar to $k = \infty$.

Figure 3 shows the robust bond allocations. Under both a distorted ($k > 0$) and undistorted ($k = 0$) probability measure, the optimal hedge demand decreases over time as the exposure to the interest rate risk over the remaining investment horizon decreases.[21] With regard to the distortions, the figure shows that an uncertainty bound $k \in \{0.01, 0.05\}$ leads to a lower robust investment compared to the undistorted strategy. More parameter uncertainty, reflected by an increase in k, expands the range of plausible MPoRs the investor considers to be plausible. Consequently, the investor reduces her bond demand to account for potential low risk premiums. When the uncertainty bound is very large ($k = 0.5$), this range of plausible MPoRs becomes so broad that the constraint with the bound on uncertainty is no longer binding, causing the investor to have no speculative demand any more and behave as a hedge investor. Consequently, a larger risk aversion diminishes the effect of parameter uncertainty. For instance, for $\gamma = 25$, the robust allocations are nearly identical for $k = 0$ and $k = 0.5$. Moreover, the impact of parameter uncertainty for a more speculative investor increases over time, because the undistorted speculative and hedge demand deviate more from each other for a shorter investment horizon.

Finally, we study the corresponding impact on the CER. To compute the optimal wealth distribution, we plug the optimal robust bond allocations into the nominal wealth process, discretised with the Euler approximation with monthly time steps. Moreover, we assume that $R_0 = \bar{R}$. Table 2a reports the CER for different values of $k \in \{0; 0.01; 0.05; 0.5\}$.

[21] See equation (36) for the optimal (distorted) wealth process. Under both a distorted and undistorted probability measure, the volatility term of optimal wealth depends on the function $B(T - t)$, which decreases over time t.

We observe two main results. First, when k increases and the uncertainty constraint is no longer binding, the CER declines. This occurs because the market price of risk can be distorted to a greater extent. For instance, a bound on uncertainty of $k = 0.01$ corresponds to an alternative market price of risk that can deviate by 1% from her baseline estimate. For the speculative investor, in the worst-case the risk premium decreases and her CER reduces from 9.9% ($k = 0$) to 9.8%. When the bound on uncertainty increases to $k = 0.05$, the investor considers market prices of risk that can deviate more from her baseline estimate, namely 5%. In the worst case, this would decrease her CER to 9.3%. Second, the impact of deterministic distortions in this one-factor model is especially relevant for a more speculative investor. A very risk-averse investor, whose portfolio is already close to the hedge portfolio, is hardly affected by mother nature's distortion.

2.4.2 Two-factor model

The last section analysed the robust investment strategy and the corresponding impact on utility under the deterministic distortion in a one-factor model. We now consider a two-factor model. We assume a standard affine two-factor model for the nominal interest rate R_t and expected inflation π_t. Under the baseline real-world probability measure $\mathcal{P}$, the two factors follow the process:[22]

$$d\begin{bmatrix} R_t \\ \pi_t \end{bmatrix} = -\begin{bmatrix} \kappa_R & \kappa_{R\pi} \\ 0 & \kappa_\pi \end{bmatrix} \begin{bmatrix} R_t - \bar{R} \\ \pi_t - \bar{\pi} \end{bmatrix} dt + \begin{bmatrix} \sigma_R & 0 \\ 0 & \sigma_\pi \end{bmatrix} d \begin{bmatrix} Z_t^R \\ Z_t^\pi \end{bmatrix} \tag{57}$$

We denote the correlation between the Brownian motions of the risk factors as

$$\rho = \begin{bmatrix} 1 & \rho_{R\pi} \\ \rho_{R\pi} & 1 \end{bmatrix} \tag{58}$$

For the remainder of this section, we refer to κ_R and κ_π as the mean-reversion parameters, and $\kappa_{R\pi}$ as the feedback parameter that corresponds to the relation between the level of π_t and the drift adjustment of R_t. We assume that the mean-reversion parameters and volatilities of the factors are positive: $\kappa_R, \kappa_\pi, \sigma_R, \sigma_\pi > 0$; and that $\kappa_R \neq \kappa_\pi$. Under the deterministic distorted MPoR $\lambda^* = \lambda + \beta$, the optimal wealth process under the distorted

[22]Compared to the general notation in Section 2, the variables are $X_t = (R_t - \bar{R},\ \pi_t - \bar{\pi})'$, $\delta_R = (1, 0)'$, $\delta_\pi = (0, 1)'$, $\delta_{0R} = \bar{R}$ and $\delta_{0\pi} = \bar{\pi}$.

measure equals:[23]

$$dW_t^*/W_t^* = [R_t + \sigma_{W^*}'\lambda^*]\,dt + \sigma_{W^*}' dZ_t^* \tag{59}$$

$$\sigma_{W^*} = \frac{1}{\gamma}\rho^{-1}\lambda^* - \left(1 - \frac{1}{\gamma}\right)\sigma_X' \tilde{B}(T-t) \tag{60}$$

Where we use the notation of $\tilde{B}(T-t)$ instead of $\tilde{B}_2(T-t)$ for readability purposes. Similar to the one-factor model, we derive the worst-case scenario and corresponding robust bond strategy. The results are summarised in the next theorem:

Theorem 2.4.2 (Two-factor model with deterministic distortion)**.**

$$\max_{W_T}\min_{\beta}\ \mathbb{E}_t^*\left[u\left(\frac{W_T}{\Pi_T}\right)\right] \tag{61}$$

$$s.t.\ \mathbb{E}_t^*\left[\zeta_T^* W_T\right] = W_t \tag{62}$$

$$and\ \beta'\rho^{-1}\beta \leq k^2 \tag{63}$$

Define $\check{\lambda} = \begin{bmatrix}\check{\lambda}_R & \check{\lambda}_\pi\end{bmatrix}' = \lambda + \rho\sigma_X'\tilde{B}(T-t)$ *and assume that* $\check{\lambda}_R < \check{\lambda}_\pi < 0$ *and* $k > 0$*, then the optimal distortion of the market price of risk equals:*

$$\hat{\beta} = -\check{\lambda}\min\{1; k \cdot D\} \tag{64}$$

where

$$D \equiv \frac{\sqrt{1-\rho_{R\pi}^2}}{\sqrt{\check{\lambda}_R^2 + \check{\lambda}_\pi^2 - 2\rho_{R\pi}\check{\lambda}_R\check{\lambda}_\pi}} \tag{65}$$

This results in the optimally distorted market price of risk:

$$\hat{\lambda}^* = \min\left\{-\rho\sigma_X'\tilde{B}(T-t); \lambda - k\cdot D\cdot\check{\lambda}\right\} \tag{66}$$

Consequently, the investor applies the corresponding robust optimal bond allocation:

$$\hat{x}_t^* = \frac{1}{\gamma}\Omega^{-1}\sigma_B \cdot \min\left\{0; (1-k\cdot D)\check{\lambda}\right\} - \Omega^{-1}\sigma_B\rho\sigma_X'\tilde{B}(T-t) \tag{67}$$

[23]For completeness, the analytical solution of the risks exposures $B(\tau_j)$ used to compute σ_B and $\tilde{B}(T-t)$ used to compute the optimal real wealth are derived in the appendix in equations (A.2.5) and (A.2.6), respectively.

Proof. See Appendix A.2.2. □

To explain the intuition of the optimal distortions and corresponding worst-case MPoRs, consider Figure 4 that visualises the optimal distortion under a positive correlation between the risk factors ($\rho_{R\pi} > 0$). The figure shows the undistorted values in the vector $\check{\lambda}$ which is defined below equation (63), and the distorted values the vector which we will define by $\check{\lambda}^*$. As can be seen from equation (66), the unconstrained optimal distortion would imply $\check{\lambda}^* = 0$. Therefore, the optimisation problem aims to find the distorted $\check{\lambda}^*$ in the range represented by a multidimensional 'confidence' interval, corresponding to an ellipse around the undistorted baseline $\check{\lambda}$. This ellipse is the result of the constraint on uncertainty in equation (22). Mother nature finds the worst-case scenario by minimising the distance between the distorted values and the origin of the axes. Therefore, this solution is on the line from the undistorted baseline $\check{\lambda}$ to the origin of the axes.[24] The two subfigures show both the bounded and unbounded solutions. The upper subfigure represents a boundary solution: the bound on uncertainty k restricts mother nature's distortion so that she cannot distort to the origin at $(0, 0)$. Second, the lower figure represents the interior solution: the bound on uncertainty k is large enough so that it includes the origin and mother nature can minimise the expected utility without restrictions.

Note that the qualitative observations for this two-factor model are similar to those of the one-factor model: the optimal distortion is independent of the risk aversion coefficient, the speculative term decreases in absolute terms in case of more parameter uncertainty, the hedge term is unaffected by parameter uncertainty, and a very large uncertainty bound k makes the investor investing like a very risk-averse investor with a large γ.

Figure 5 shows the bond weights for two nominal bonds with maturities $\tau_1 = 10$ and $\tau_2 = 30$. For both the distorted and undistorted strategies, the bond weights decrease over time in absolute value for $\gamma > 1$, reflecting the decreasing hedge demand, similar as for the one-factor model.[25] Note that the levels of the one- and two-factor models are not directly comparable. In the one-factor model that hedges the nominal interest rate, the hedge demand at time 0 would consist of investing 100% of wealth in a bond with $\tau_1 = T = 30$ years. In the two-factor model, we additionally include inflation risk as a risk factor and we need to choose two different bond maturities, which requires scaling. For example, the hedge demand based on bond maturities $\tau_1 = 10$ and $\tau_2 = 30$ consists of a long and short

[24]This is in line with the tangency points corresponding to worst-case scenarios, shown in Figure 1 of Hansen and Sargent (2021).

[25]The allocations are modelled by the baseline parameter set for the real interest rate and the inflation rate used to approximate the baseline parameters of the nominal interest rate and inflation rate in Table 1. See Appendix B in Chapter 1 for more details.

position of -180% and 400% of wealth, respectively. The long position in the 10-year bond hedges the nominal interest rate risk, which requires a larger allocation compared to a 30-year bond due to the lower interest rate sensitivity of shorter-maturity bonds. The short position in the 30-year bond accounts for decreasing bond prices associated with higher inflation rates.

The graph shows that the impact of parameter uncertainty is similar as for the one-factor model: compared to no uncertainty ($k = 0$), the (absolute values of the) robust bond allocations decrease, especially for a short investment horizon. This decrease is less strong for a more risk-averse investor. We see that the larger the parameter uncertainty (larger k), the more the investor behaves like a more risk-averse investor. Moreover, we observe that the bound on uncertainty of $k = 0.5$ is so large that mother nature can unrestrictedly distort the parameters so that the investor behaves like a hedge investor. This is explained by the fact that Figure 1 shows that the 'confidence interval' corresponding to $k = \sqrt{0.05} \approx 0.22$ already contains the origin, so a larger value than 0.22 will result in an optimally unrestricted distortion.

Finally, we study the impact of the robust bond allocations on the CER, similar as for the one-factor model. Note that the two-factor model contains inflation and therefore generates a lower CER on real wealth than the one-factor model without inflation. Table 2b shows the CER for the values $k \in \{0; 0.01; 0.05; 0.5\}$. The results are similar as for the one-factor model: if k increases and the constraint on uncertainty is not satisfied yet, the CE decreases. Moreover, the utility decrease is negligible for a high risk aversion.

Hence, the results of the deterministic distortion in the one-factor and two-factor model are similar when the correlation between the risk factors is positive. Parameter uncertainty appears to be relevant for a more speculative investor, who decreases the absolute values of her bond demands, in particular for a shorter investment horizon. The more parameter uncertainty, the more the robust investor behaves like a hedge investor. By applying the robust strategy, the speculative investor only faces a small utility loss under the worst-case measure. The impact on utility of the hedge investor is negligible.

2.5 Stochastic distortion

Up to this point, we have analysed deterministic distortions that do not affect the mean-reversion parameter. In contrast, this section analyses the stochastic distortion that introduces uncertainty regarding the mean-reversion parameter. To illustrate the relevance of this type of uncertainty, we show a wide range of mean-reversion parameter estimates from the literature, both within and across one- and two-factor interest rate models. The variation within a model can arise from differences in the underlying dataset or from assumptions about whether the market price of risk is constant or time-varying. For example, Table 3 shows that in a one-factor model, the estimates of the mean-reversion parameter for U.S. interest rates can range from nearly zero (Feldhütter et al. (2012)) to 0.191 (Babbs and Nowman (1999)), based on different datasets but under the same assumption of a constant market price of risk. Moreover, using the same dataset, the estimate varies from nearly zero under a constant market price of risk to 0.109 based on a time-varying market price of risk in Feldhütter et al. (2012).

In a two-factor model, estimates of the mean-reversion matrix can also deviate considerably. Table 4a shows the eigenvalues of mean-reversion matrices in models where the nominal interest rate is affine in two latent state variables.[26] The first and second eigenvalues correspond to the mean-reversion parameters of the first and second factor, respectively. The estimates may as well vary both across datasets and on whether the market price of risk is assumed to be constant or time-varying. However, variation across dataset appears to be most relevant. For example, based on the same dataset of U.S. interest rates, the largest eigenvalue of the mean-reversion matrix varies "only" from 0.060 under a constant market price of risk to 0.008 under a time-varying market price of risk in Cheridito et al. (2003). In contrast, under the assumption of a constant market price of risk, the largest eigenvalue may range from 0.060 (Cheridito et al. (2003)) to 1.249 (Bergström and Nowman (1999)), depending on the dataset.

Finally, some studies estimate two-factor models with one latent factor interpreted as the real interest rate, and a second observed factor, namely the (expected) inflation rate. Table 4b shows that these models also result in considerably different mean-reversion parameter estimates, both across datasets and compared to models with two latent factors. For example, the largest eigenvalue ranges from 0.105 to 0.631, both observed in Brennan and Xia (2002). The smallest eigenvalue ranges from -0.119 to 0.091, both observed in Pennacchi (1991).

Therefore, parameter uncertainty with respect to the mean-reversion parameter is relevant. To make the intuition behind the results as clear as possible, in our analysis we focus on the one-factor model without inflation risk. As a consequence, the investor invests in one instead of two bonds. Therefore, similar as for the one-factor model

[26]Latent state variables can be interpreted as economic news. We can therefore interpret the risk factors as the real or nominal interest rate and expected inflation rate, although other interpretations are possible.

with a deterministic distortion, we suppress the '1' of bond maturity τ_1, and denote the mean-reversion parameter and volatility of the interest rate by κ and σ respectively for readability purposes.

Problem 2 (One-factor model with stochastic distortion)**.** *We assume that the risk factor is the nominal interest rate R_t ($n = 1; \delta_R = 1$), and there is no inflation risk ($\pi_t = 0; \delta_\pi = 0$):*

$$\max_{W_T} \min_{\alpha} \mathbb{E}_t^* \left[u \left(W_T \right) \right] \tag{68}$$

$$\text{s.t. } \mathbb{E}_t^* \left[\zeta_T^* W_T \right] = W_t \ ; \ \frac{d\zeta_t^*}{\zeta_t^*} = -R_t dt - \lambda_t^* dZ_t^* \tag{69}$$

$$|\alpha| \leq k \tag{70}$$

$$\lambda_t^* = \lambda + \alpha \left(R_t - \bar{R} \right) \ ; \ \kappa^* = \kappa - \alpha \cdot \sigma \tag{71}$$

$$dR_t = -\kappa^* \left(R_t - \bar{R} \right) dt + \sigma dZ_t^* \tag{72}$$

$$\frac{dP_t(\tau)}{P_t(\tau)} = \left[R_t - \lambda_t^* B(\tau) \sigma \right] dt - B(\tau) \sigma dZ_t^* \tag{73}$$

This set-up shows that a distortion in the mean-reversion parameter requires a stochastic distortion in the MPoR. To explain the economic intuition of this link, consider a positive distortion parameter ($\alpha > 0$), which results in more persistent interest rate shocks under the distorted measure than under the undistorted measure ($\kappa^* < \kappa$). The risk premium compensates the investor for the risk that the accumulated bond returns over the remaining investment horizon are smaller than when the investor would have invested in the bank account. However, this risk is reduced when the interest rate is above its long-term average ($R_t > \bar{R}$), because the interest rate is expected to decrease more slowly toward its long-term average. This decreases the distorted risk premium:

$$\lambda_t^* B(\tau) \sigma = -\left(\lambda + \alpha \cdot (R_t - \bar{R}) \right) \cdot B(\tau) \sigma \tag{74}$$

where by assumption the undistorted MPoR is negative ($\lambda < 0$) and the bond exposure under the risk-neutral measure is positive ($B(\tau)\sigma > 0$). Hence, when the interest rate is high, a positive distortion parameter results in a less negative MPoR and therefore smaller risk premium than without distortion. Vice versa, when the interest rate is below its long-term average, the investor requires a higher risk premium under the distorted measure, because more persistent interest rate shocks are disadvantageous for the investor. The same reasoning for state-dependent risk premiums applies when mother nature selects a negative distortion parameter ($\alpha < 0$).

The stochastic distortion implies that the distorted optimal wealth process has a stochastic drift and volatility term, both dependent on R_t, as shown in equation (25) which for this special case reads as

$$dW_t^*/W_t^* = [R_t + \sigma_{W^*}'\lambda_t^*]dt + \sigma_{W^*}'dZ_t^*, \tag{75}$$

$$\sigma_{W^*} = \frac{1}{\gamma}\lambda_t^* - \left(1 - \frac{1}{\gamma}\right)\sigma\left[\tilde{B}_3(T-t)R_t + \tilde{B}_2(T-t)\right] \tag{76}$$

where the implemented optimal bond allocations and ODEs for $\tilde{B}_3$ and $\tilde{B}_2$ can be found in equations (A.2.19) and (A.2.20), and depend on α. This wealth process shows that the impact of α on the investor's utility is not straightforward: it affects both the market price of risk λ_t^* and the mean-reversion of the interest rate process κ^* and thus $\tilde{B}_2(T-t)$ and $\tilde{B}_3(T-t)$. We cannot derive an analytical solution for the optimal distortion. Instead, we present numerical solutions based on the ODE's for which we rely on Corollary 1 of Sangvinatsos and Wachter (2005), which derives the indirect utility function $J(t, W_t)$.[27] In case of one state variable R_t and no inflation, the indirect utility equals:

$$J(t, W_t) = \frac{W_t^{1-\gamma}}{1-\gamma}\exp\left[(1-\gamma)\left(\frac{1}{2}\tilde{B}_3(T-t)\cdot R_t^2 + \tilde{B}_2(T-t)\cdot R_t + \tilde{B}_1(T-t)\right)\right] \tag{77}$$

The scalar $\tilde{B}_1(T-t)$ can be found by solving the ODE in equation (A.2.23), and depends on α. We now use the definition of a certainty equivalent wealth (CEW), namely $u(\text{CEW}) = J(t, W_t)$, to compute the continuously compounded CER on wealth, expressed in annual returns. This results in:

$$CEW_t = W_t^{1-\gamma}\cdot\exp\left(\frac{1}{2}\tilde{B}_3(T-t)\cdot R_t^2 + \tilde{B}_2(T-t)\cdot R_t + \tilde{B}_1(T-t)\right) \tag{78}$$

$$CER_t = \frac{1}{T-t}\cdot\left[(1-\gamma)\cdot\log(W_t) + \left(\frac{1}{2}\tilde{B}_3(T-t)\cdot R_t^2 + \tilde{B}_2(T-t)\cdot R_t + \tilde{B}_1(T-t)\right)\right] \tag{79}$$

The optimal distortion parameter $\hat{\alpha}$, assuming that it remains constant over the time interval (t, T), can be computed by:

$$\hat{\alpha} = \arg\min_{\alpha}\mathbb{E}_t^*\left[u(W_T)\right] = \arg\min_{\alpha}\, CER_t \tag{80}$$

[27] For details about reconciling our parameters with the parameters in Sangvinatsos and Wachter (2005), see Appendix A.2.1.2.

We now show numerical examples of the optimal distortion and corresponding robust bond weights. We solve equation (80) numerically at $t = 0$. We choose a long-term investment horizon $T = 30$. We focus on the risk aversion coefficients $\gamma = 1$ and $\gamma = 2$. We consider values of $k \leq 5$ so that we consider a range of values of $\alpha \in (-5, 5)$. Under our baseline parameter, those lower and upper bounds on alpha would (approximately) result in a distorted mean-reversion parameter κ^* close to zero or twice as large as the baseline parameter κ, respectively.

First, we analyse the speculative investor ($\gamma = 1$), who is harmed when the distortion results in a low risk premium. Consider the case where the current interest rate is below the long-term average, $R_t < \bar{R}$. In this case, mother nature can decrease the risk premium given in equation (74) by selecting a negative distortion parameter $\alpha < 0$, given that the undistorted MPOR is negative ($\lambda < 0$). Indeed, Figure 6a shows that the optimal distortion parameter is negative when interest rates are below their long-term average. Similarly, when $R_t > \bar{R}$, mother nature optimally applies a positive distortion parameter. As a result, the optimally distorted MPoR is always smaller in absolute value than the undistorted MPoR, as shown in Figure 6b.

Next, consider the slightly more risk-averse investor ($\gamma = 2$) who additionally has a hedging motive. Consequently, she is also concerned about persistently low interest rates, which heightens the relevance of the impact on the distorted mean-reversion parameter, κ^*. To explain why, consider the case $R_t < \bar{R}$. We observed before that an $\alpha < 0$ decreases the risk premium. However, equation (71) shows that this would also result in $\kappa^* > \kappa$, making future interest rates increase faster to the long-term average rate $\bar{R}$ under the distorted measure. This means that the interest rates will stay below $\bar{R}$ for a shorter time, which is advantageous for the investor. Therefore, mother nature should make the trade-off between the advantage and disadvantage of the sign of the distortion parameter, dependent on the recent nominal interest rate. For instance, Figure 6a shows that for interest rates around $R_t \in (4.1\%, 7.1\%)$, we observe a positive optimal distortion parameter and hence a more negative MPoR, as shown in Figure 6b. This result in a higher risk premium, but this advantageous effect for the investor is offset by the simultaneous lower $\kappa^* < \kappa$, shown in 6c, and hence an interest rate that remains under the long-term average for a longer time.

This trade-off between the effects of the distortion parameter on λ_t^* and κ^* for $\gamma > 1$ depends on the investment horizon. Specifically, a shorter investment horizon reduces the size of the exposures in $\tilde{B}_3(T-t)$ and $\tilde{B}_2(T-t)$, and consequently their sensitivity to the distortion in the mean-reversion parameter. The intuition is that with a shorter investment horizon, persistent shocks become less relevant: there is less time for the effects of shocks to accumulate. As a result, the impact of the distortion on the mean-reversion parameter decreases, while the impact on the risk premium remains equally relevant regardless of the investment horizon. For instance, the right panel in Figure 7a shows the optimal distortion parameter $\hat{\alpha}$ at every $T - t$ for a current interest rate below the long-term average rate, $R_t = 5\% < \bar{R}$, and a risk aversion coefficient $\gamma = 2$. In line with our earlier observations, at $t = 0$ we observe an optimal distortion parameter which is positive, because the negative impact on wealth due to the lower mean-reversion parameter dominates the advantageous impact on the distorted MPoR. However, for a smaller investment horizon the impact on the risk premium is going to dominate the effect on the mean-reversion parameter. As a result, we observe a switch around $t = 10$ from a positive to negative $\hat{\alpha}$.

Finally, we consider the resulting robust bond allocations under the optimal distortion in Figure 7b. The left panel shows that the distorted bond demand of the speculative investor is lower than the undistorted one, which can be explained by a smaller distorted risk premium due to the fact that $-\lambda_t^* < -\lambda$, which we observed before in Figure 6a. The right panel shows the bond allocations for a risk aversion coefficient of $\gamma = 2$. The robust investor increases her bond weights compared to the non-robust investor. This increase is the result of a more negative optimal wealth exposure, shown in Figure 6d. That figure shows that for $R_t = 5\%$, the volatility of the distorted wealth process is larger in absolute terms than the volatility of the undistorted process. Consequently, the optimal bond allocation increases, reflecting the higher interest rate risk exposure of the optimal wealth process.[28] However, the increase is almost negligible at $t = 0$. For a shorter investment horizon, the effect of uncertainty about the risk premium becomes more significant, and the distortion switches from an increase to a decrease in robust bonds weights around 10 years, in line with the switch of the sign of the optimal distortion parameter which we observed before in Figure 7a.

For completeness, we include Figure 8 to show the optimal distortion parameters and bond weights for a high recent interest rate above the long-term average rate. It shows that the robust bond weights are considerably lower than the non-robust ones than in the case of the low current interest rate analysed before. This means that mother nature can apply more significant distortions in a high interest rate environment, in line with the larger $\hat{\alpha}$ for a high R_t in Figure 6a.

Hence, the sign and size of the optimal stochastic distortion depend on the interest

[28]Since bond prices are negatively correlated with interest rates, a larger negative exposure to interest rate risk requires a larger positive bond allocation.

rate, investment horizon, risk aversion, and baseline estimates of the underlying risks. In the worst case, a speculative investor is harmed by a smaller distorted risk premium than the undistorted one. If the investor is more risk averse, she is also worried about parameter uncertainty regarding the worst-case mean-reversion, especially for a larger investment horizon. The corresponding robust bond weights can be either smaller or larger than the non-robust investments.

2.6 Conclusion

We derived robust nominal bond allocations that hedge (real) interest rate risk under parameter uncertainty. In a one-factor model without inflation, uncertainty about the market price of interest rate risk parameter results in investment behaviour that shrinks the bond portfolio weights to the pure hedge demand for long-term bonds. In a two-factor model where the nominal interest rate is the sum of the real interest rate and the inflation rate, and the investor is uncertain about the market prices of interest rate and inflation risk, the robust investor takes less extreme long-short positions, especially when the investment horizon is relatively short.

When we allow for state-dependent uncertainty, the market price of risk becomes time-varying. In that case the investor fears not only about the market price of risk, but also a persistent low interest rate state, especially for a longer investment horizon. The corresponding robust bond weights can either be smaller or larger than the non-robust investments, depending on the risk aversion, the initial interest rate and investment horizon. For a low interest rate and long investment horizon, due to the strong fear of a high persistence in the interest rate, the robust allocations of a more risk-averse investor can be larger than the non-robust ones.

For future research, it is relevant to incorporate the stochastic distortion in a two-factor model with inflation. Moreover, it would be interesting to consider losses in utility if the measure is distorted, but the investor does not know this due to estimation errors. In this case, the utility loss can be larger than in our recent set-up, because the investor will base her investment strategy on the incorrect parameters.

Table 1. Baseline parameter values. The input parameters for the baseline parameters for the joint process of the nominal interest rate R_t and expected inflation rate π_t in the two-factor affine interest rate model. The values are taken from Chapter 1.

Parameter	*Value*
κ_R	0.105
κ_π	0.027
$\kappa_{R\pi}$	-0.078
σ_R	0.019
σ_π	0.014
$\rho_{R\pi}$	0.733
λ_R	-0.219
λ_π	-0.105
$\bar{R}$	0.071
$\bar{\pi}$	0.054

Table 2. Certainty equivalent returns under deterministic distortion. Certainty equivalent return per year (CER, in %) at $t = 0$ with investment horizon $T = 30$. The returns are computed under the deterministic distortion in (a) the one-factor model without inflation, and (b) the two-factor model with inflation. Mother nature distorts optimally within the bound on uncertainty k, and the investor optimises her strategy accordingly. The results are shown for the bounds on uncertainty $k \in \{0; 0.01; 0.05; 0.5\}$ and risk aversion coefficients $\gamma \in \{1; 2; 25\}$.

(a) One-factor model (without inflation)

$k \backslash \gamma$	*1*	*2*	*25*
0	9.9	9.6	9.3
0.01	9.8	9.6	9.3
0.05	9.6	9.4	9.3
0.5	9.3	9.3	9.3

(b) Two-factor model (with inflation)

$k \backslash \gamma$	*1*	*2*	*25*
0	3.9	3.5	3.1
0.01	3.8	3.5	3.1
0.05	3.5	3.3	3.1
0.5	3.1	3.1	3.1

Table 3. Literature-based mean-reversion estimates in one-factor models. Overview of the mean-reversion estimates from one-factor affine interest rate models reported in the literature. The column *CA/EA* indicates whether the market prices of risk is assumed to be completely affine (constant) or essentially affine (time-varying). The column $\hat{\kappa}_R$ reports the estimated mean-reversion parameter of the nominal interest rate under the risk-neutral measure $\mathcal{Q}$. The column *TS-data* describes the region, time range, and frequency (if available) of the time-series input data. The column *#bonds CS-data* indicates how many bond maturities are used for the cross-sectional estimation.

Study	*CA/EA*	$\hat{\kappa}_R$	*TS-data*	*# bonds CS-data*
Feldhütter et al. (2012)	CA	2.7e-7	US (1971-2006, day)	6, up to 10YR
Cheridito et al. (2003)	CA	2.8e-4	US (1947-1991)	6, up to 10YR
Cheridito et al. (2003)	EA	2.7e-4	US (1947-1991)	6, up to 10YR
De Jong (2000)	CA	0.022	US (1970-1991, month)	4, up to 10 YR
Munk et al. (2004)	CA	0.040	US (1951-2003, month)	5, up to 10 YR
Feldhütter et al. (2012)	EA	0.109	US (1971-2006, day)	6, up to 10YR
Babbs and Nowman (1998)	CA	0.124	Japan (1992-1996, week)	8, up to 10 YR
Babbs and Nowman (1999)	CA	0.191	US (1987-1996, month)	1, 1/12YR
Babbs and Nowman (1998)	CA	0.362	NL (1991-1997, week)	1, 1/12 YR

Table 4. Literature-based mean-reversion estimates in two-factor models. Overview of mean-reversion matrices $K^{\mathbb{Q}}$ from two-factor affine nominal interest rate models reported in the literature, where $R_t = \delta_{0(R)} + \iota' X_t$ and the risk factors follow $dX_t = -K^{\mathbb{Q}} X_t dt + \sigma_X dZ_t^{\mathbb{Q}}$. The row *CA/EA* indicates whether the market prices of risk are assumed to be completely affine (constant) or essentially affine (time-varying). The rows *Region*, *TS*, and *# bonds CS* describe the dataset by region, time range, and the number of bond maturities used in the cross-sectional estimations, respectively. The rows *EVs* report the corresponding eigenvalues of $K^{\mathbb{Q}}$. The overview is provided for two types of risk factors in X_t: (a) latent (unobserved) risk factors, and (b) one latent risk factor r_t and one observed risk factor π_t. For case (b), the original estimates of the studies have been transformed to correspond to values in $K^{\mathbb{Q}}$ for the nominal interest rate process, in line with the real interest rate transformations in Appendix B of Chapter 1.

(a) Latent state variables

Study	*Babbs & N (1999)*		*Babbs & N (1998)*		*Babbs & N (1998)*		*Bergstrom & N(1999)*	
CA/EA	CA		CA		CA		CA	
Region	US		Japan		Netherlands		US	
TS	1978-1996 (week)		1992-1996 (week)		1991-1997 (week)		1981-1995 (month)	
CS bonds	8, up to 10 YR		8, up to 10 YR		8, up to 10 YR		1, 1/12 YR	
K	0.553	0.000	1.792	0.000	1.393	0.000	1.249	0.000
	0.000	0.065	0.000	0.146	0.000	0.269	0.000	0.033
EVs	**0.065**		**0.146**		**0.269**		**0.033**	
	0.553		**1.792**		**1.393**		**1.249**	

Study	*CFK (2003)*		*CFK (2003)*		*De Jong (2000)*		*HJW (2016)*	
CA/EA	CA		EA		CA		CA	
Region	US		US		US		US	
TS	1947-1991 (month)		1947-1991 (month)		1947-1991 (month)		1973-2016 (week)	
CS bonds	6, up to 10 YR		6, up to 10 YR		4, up to 10 YR		4, up to 10 YR	
K	-0.005	0.000	-0.0004	0.022	0.023	0.000	0.076	0.000
	-0.043	0.063	0.060	0.060	0.000	0.842	0.000	0.307
EVs	**-0.005**		**-0.017**		**0.023**		**0.076**	
	0.063		**0.008**		**0.842**		**0.307**	

(b) Latent variable r_t and π_t

Study	*BX (2002)*		*BX (2002)*		*Penn.(1991)*		*Penn.(1991)*	
CA/EA	CA		CA		CA		CA	
Region	US		US		US		US	
TS	1970-1995(month)		1890-1985(year)		1968-1988(month)		1968-1978(month)	
CS bonds	11, up to 10 YR		11, up to 10 YR		4, up to 1 YR		4, up to 1 YR	
K	0.105	-0.078	0.631	-0.604	0.031	0.148	-0.165	0.176
	0.000	0.027	0.000	0.027	-0.110	0.363	-0.185	0.593
EVs	0.027		0.027		0.091		-0.119	
	0.105		0.631		0.303		0.547	

Figure 1. Bounded deterministic distortions in two-factor model. The intervals in the two-factor model under deterministic distortions in the vector β, centered around the baseline estimate $\check{\lambda} \equiv \lambda + \rho\sigma_X' \tilde{B}(T-t)$, where the distorted value is given by $\check{\lambda}^* = \check{\lambda} + \beta$. The distortions are bounded by the constraint $\beta'\rho^-\beta < k^2$, with $k^2 \in \{0.01; 0.05; 0.5\}$. The figure shows that $k^2 = 0.05$ is sufficiently large to apply the optimal unbounded distortions $\hat{\beta}|_{k=\infty}$ such that the entries in $\check{\lambda}^*$ would equal zero.

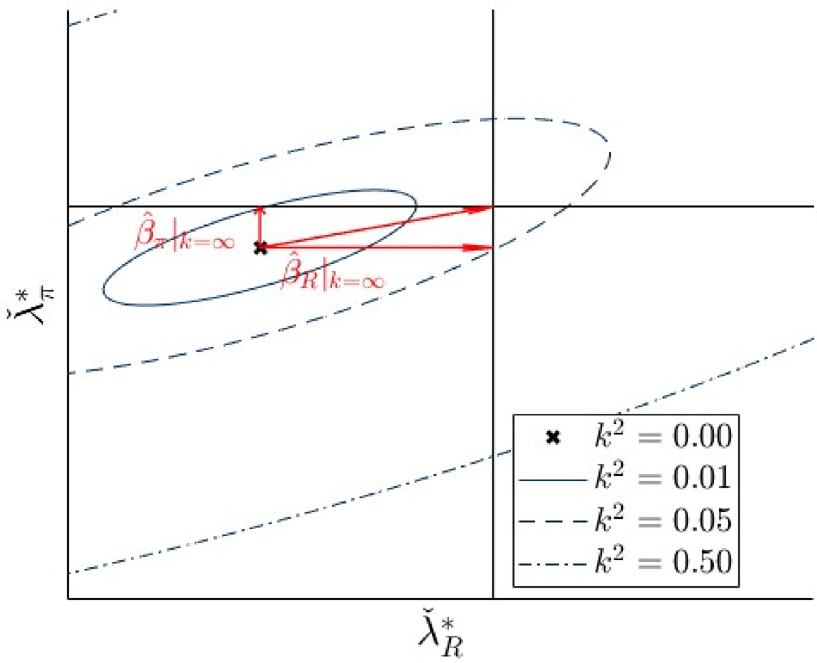


Figure 2. Deterministic distortion in one-factor model. The optimal deterministic distortion $\hat{\beta}$ in the one-factor model. The upper figure shows that when the term $\lambda + \sigma B(T-t)$ is negative, it is increased by a positive β until it either reaches the uncertainty bound k or applies the unrestricted optimal distortion. The lower figure shows that when the term $\lambda + \sigma B(T-t)$ is non-negative, it is decreased by a negative β until it reaches the bound k or applies the unrestricted optimal distortion.

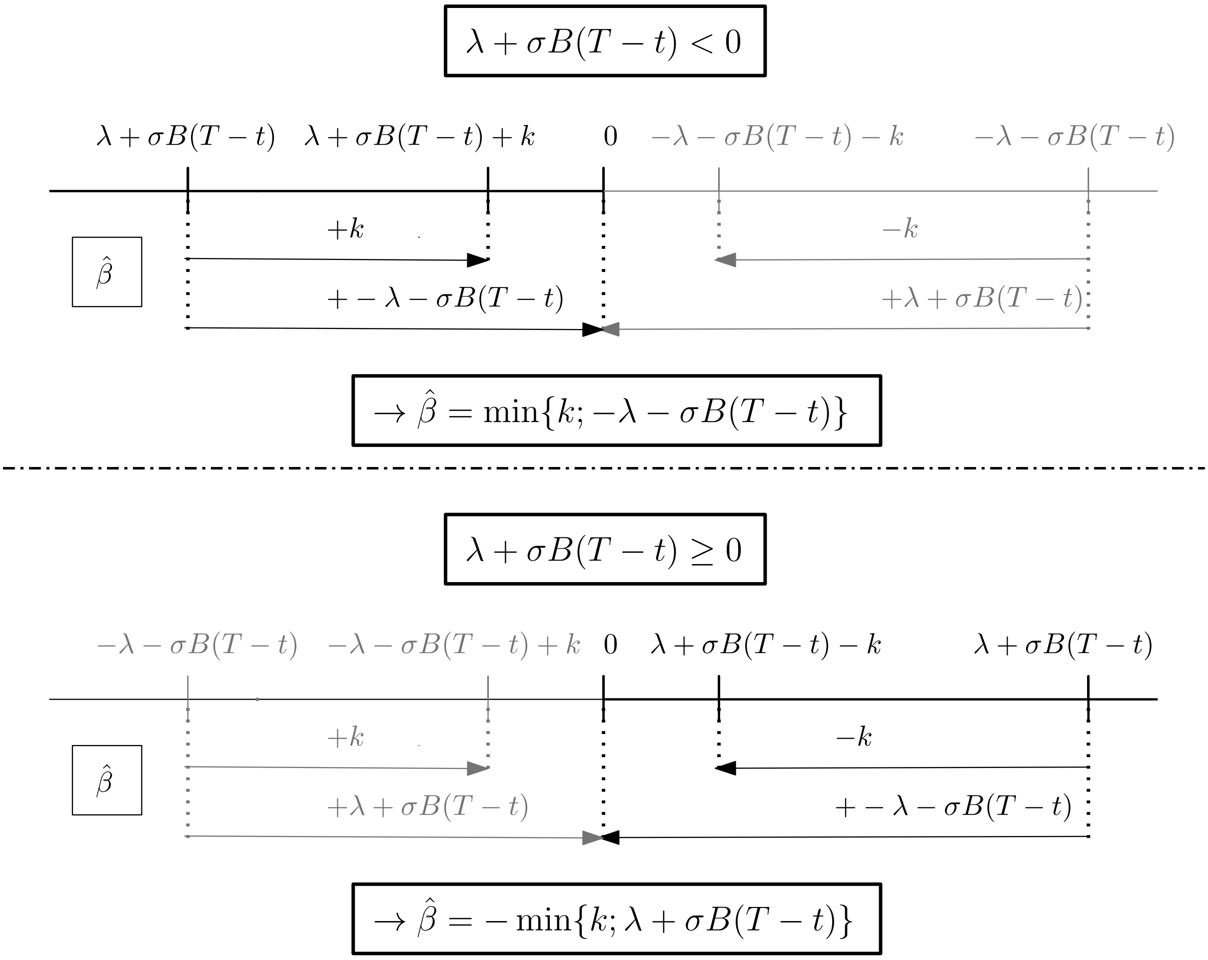


Figure 3. Bond allocations in one-factor model under deterministic distortion. The robust nominal bond allocations $\hat{x}_t^*$ in a one-factor interest rate model under a deterministic distortion, with bond maturity $\tau_1 = 30$ and risk aversion coefficients $\gamma \in \{1, 2, 25\}$. Each line corresponds to a different bound on uncertainty, ranging from the undistorted case ($k = 0$) to the unrestricted distortion case ($k = 0.5$). The allocations are shown as a fraction of wealth for times $t \in (0, T)$ with investment horizon $T = 30$, meaning the remaining investment horizon decreases from left to right.

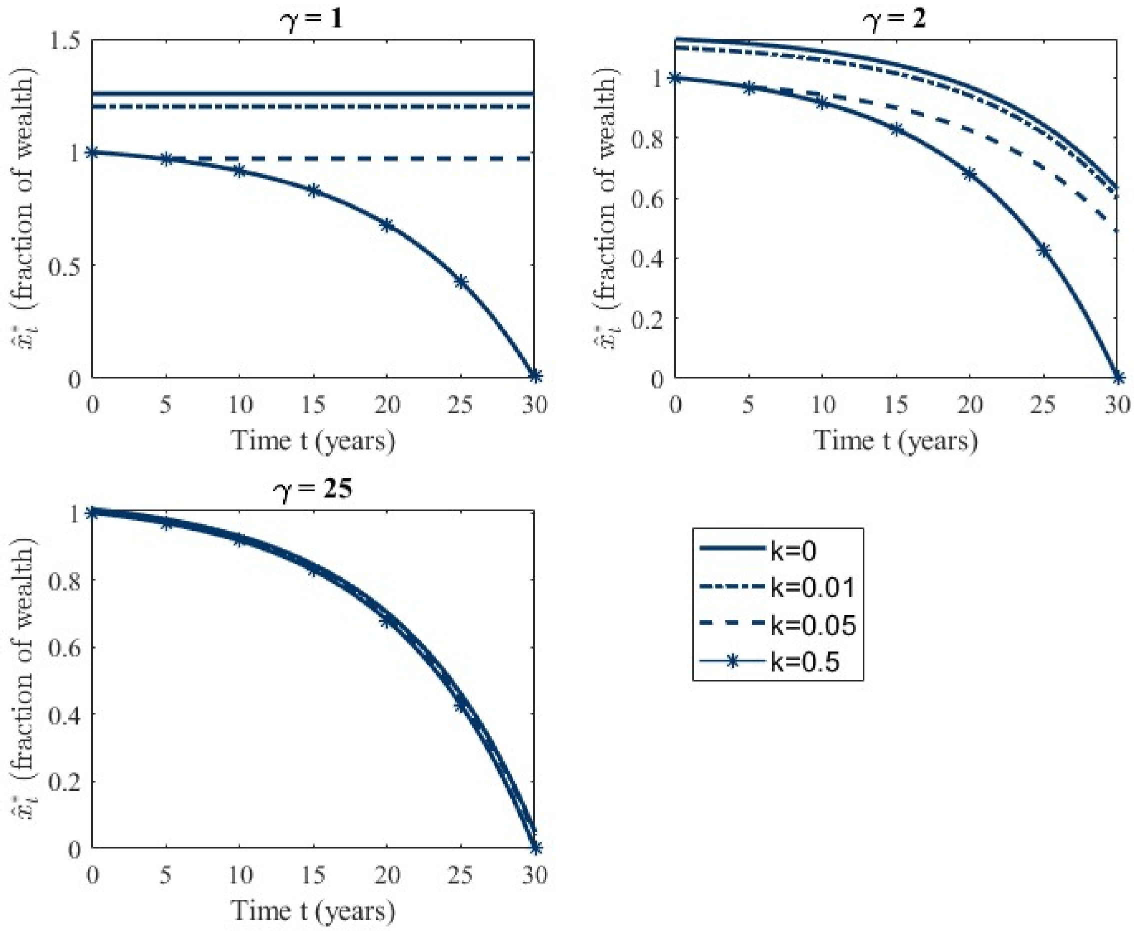
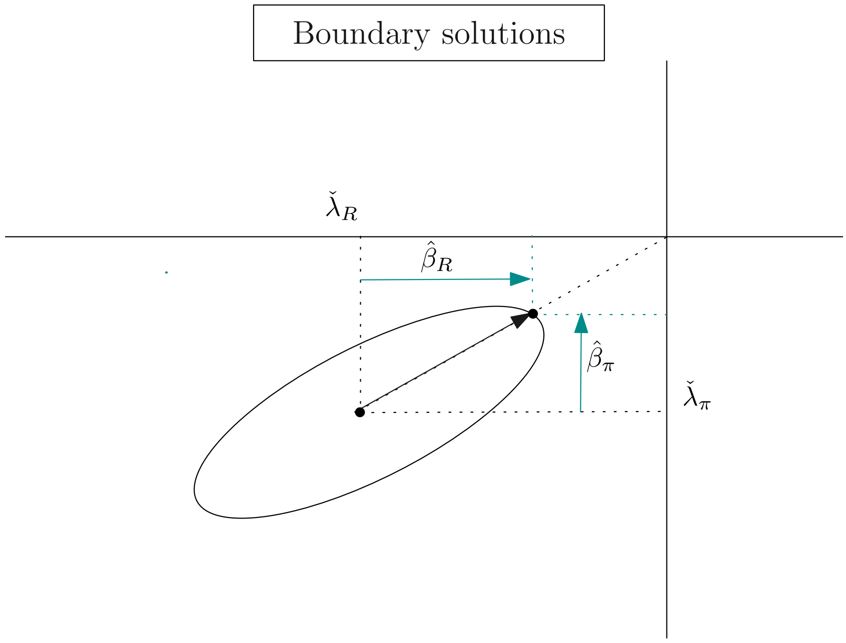


Figure 4. Visualisation boundary and interior solutions. A graphical representation of the distortions applied to the entries in $\check{\lambda} \equiv \lambda + \rho\sigma_X' \tilde{B}(T-t)$ that minimise the investor's expected utility, in case of a positive correlation between the risk factors and the assumption of $\check{\lambda}_R < \check{\lambda}_\pi < 0$. The upper figure shows the case in which the distortions in the market prices of risk are constrained by the uncertainty bound. The lower figure shows the interior solution case where the uncertainty constraint is not binding, and the optimal distortions move $\check{\lambda}$ to the origin.



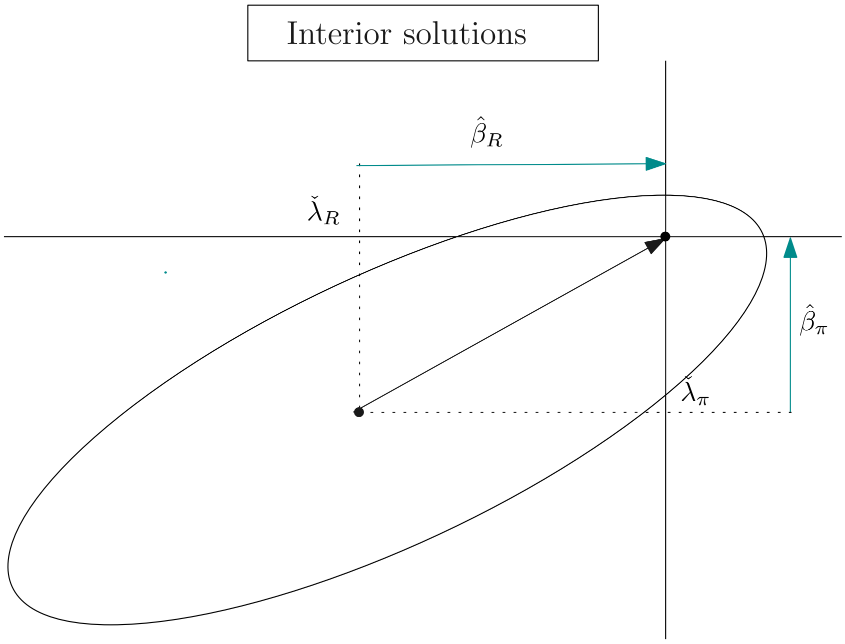


Figure 5. Bond allocations in two-factor model under deterministic distortions. The robust nominal bond allocations $\hat{x}_t^*$ in a two-factor interest rate model under a deterministic distortion with risk aversion coefficients $\gamma \in \{1, 2, 25\}$. Each line corresponds to a different bound on uncertainty, ranging from the undistorted case ($k = 0$) to the unrestricted distortion case ($k = 0.5$). The upper yellow lines refer to the allocations in the bond with maturity $\tau_1 = 10$, while the lower blue lines correspond allocations in the bond with maturity $\tau_2 = 30$. The allocations are shown as fractions of wealth for times $t \in (0, T)$ with investment horizon $T = 30$, meaning the remaining investment horizon decreases from left to right.

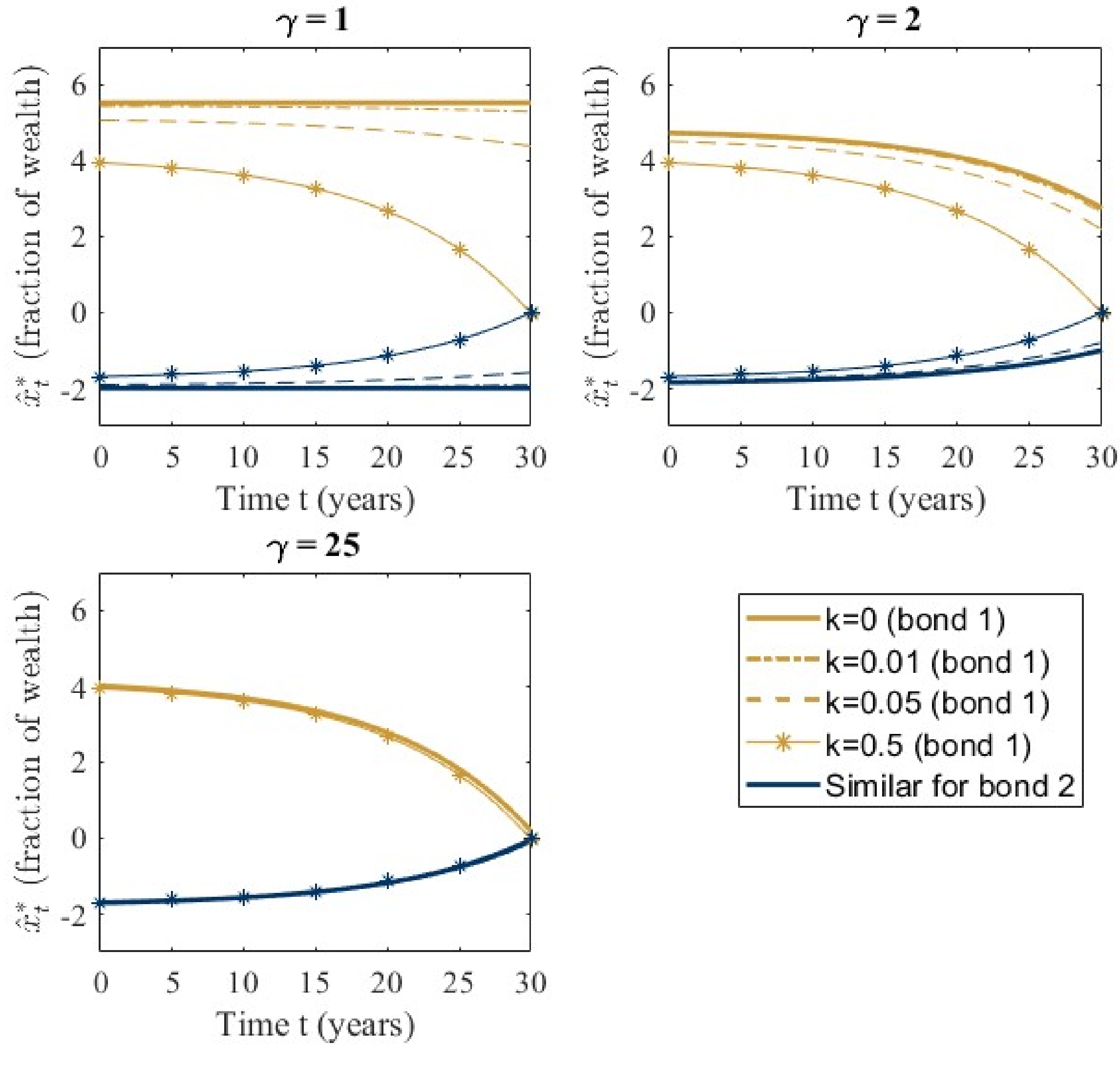
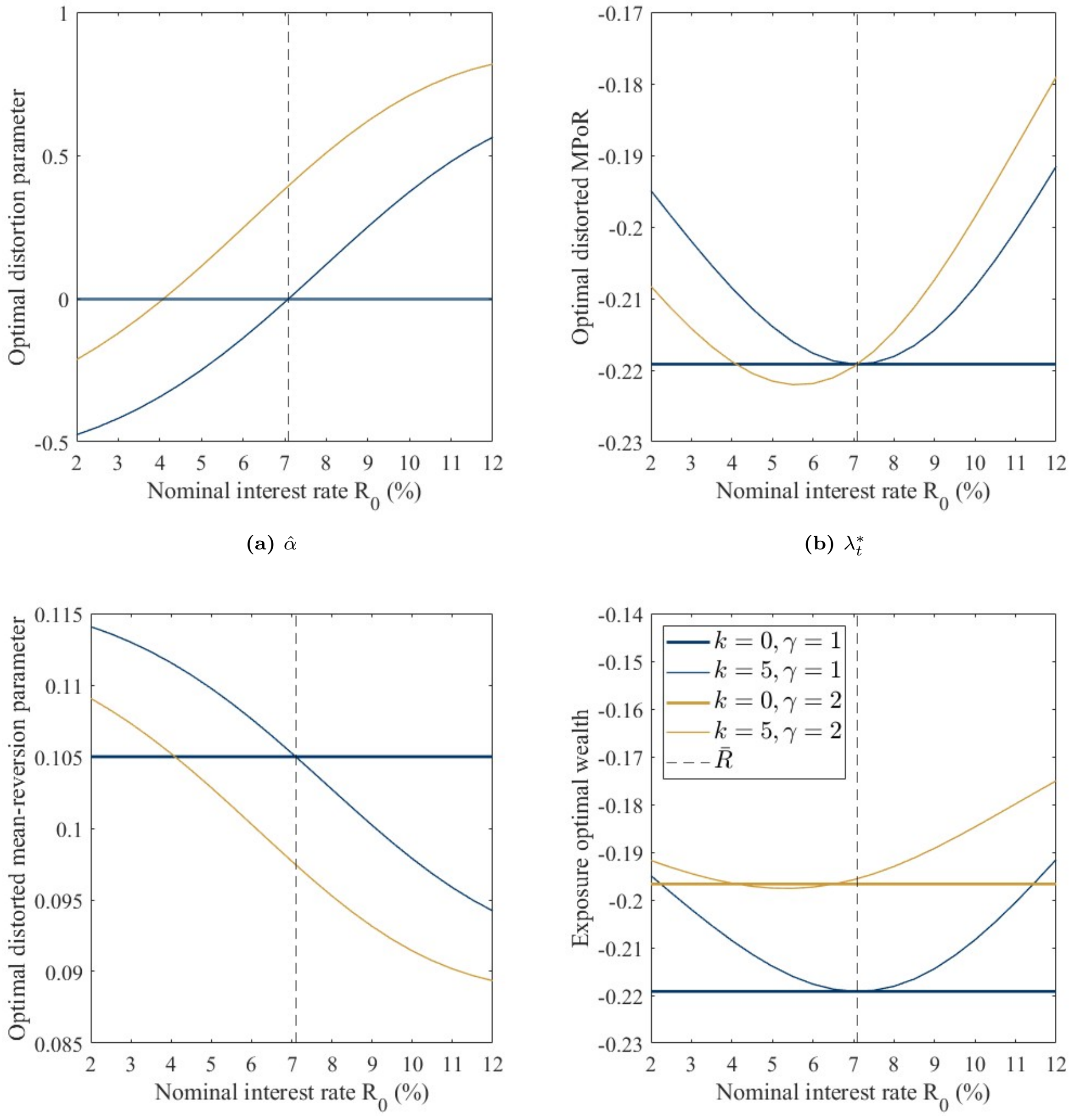
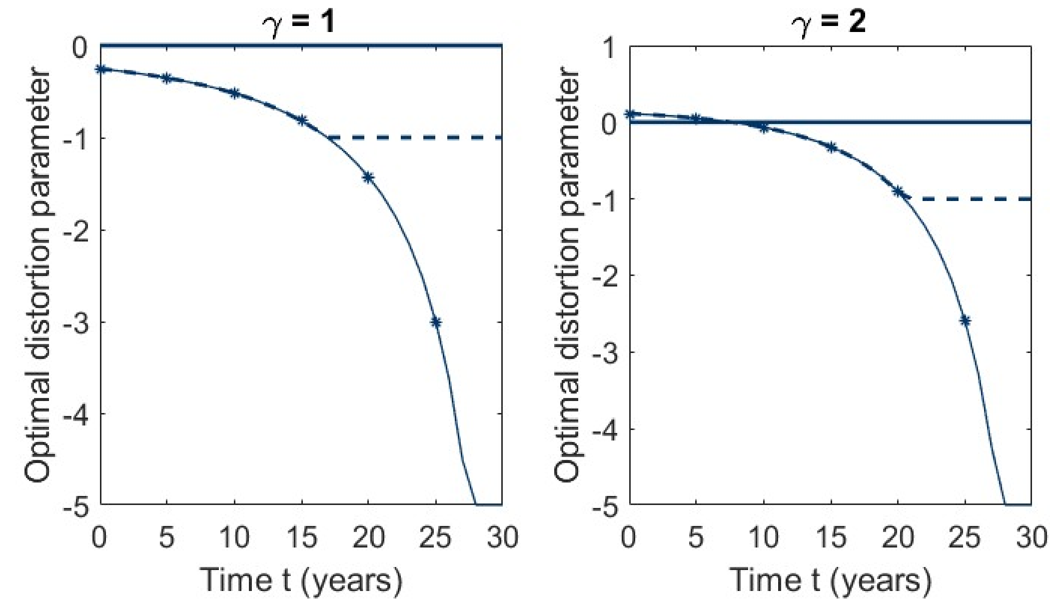


Figure 6. Optimal stochastic distortions and implications. Solutions of the optimal stochastic distortion in a one-factor model. Subfigure (a) shows the optimal distortion parameter $\hat{\alpha}$ for risk aversion coefficients $\gamma \in \{1, 2\}$, market price of risk $\lambda = -0.219$, remaining investment horizon $T - t = 30$ years, bounds on uncertainty $k \in \{0, 5\}$, and a range of initial interest rates $R_0 \in \{2\%, 3\%, ..., 12\%\}$, shown on the x-axis. The long-term average interest rate is $\bar{R} = 7.1\%$. The other subfigures show the corresponding optimally distorted (b) market price of risk (MPoR), (c) mean-reversion parameter, and (d) wealth process exposure σ_{W^*}. The legend is given in subfigure (d).

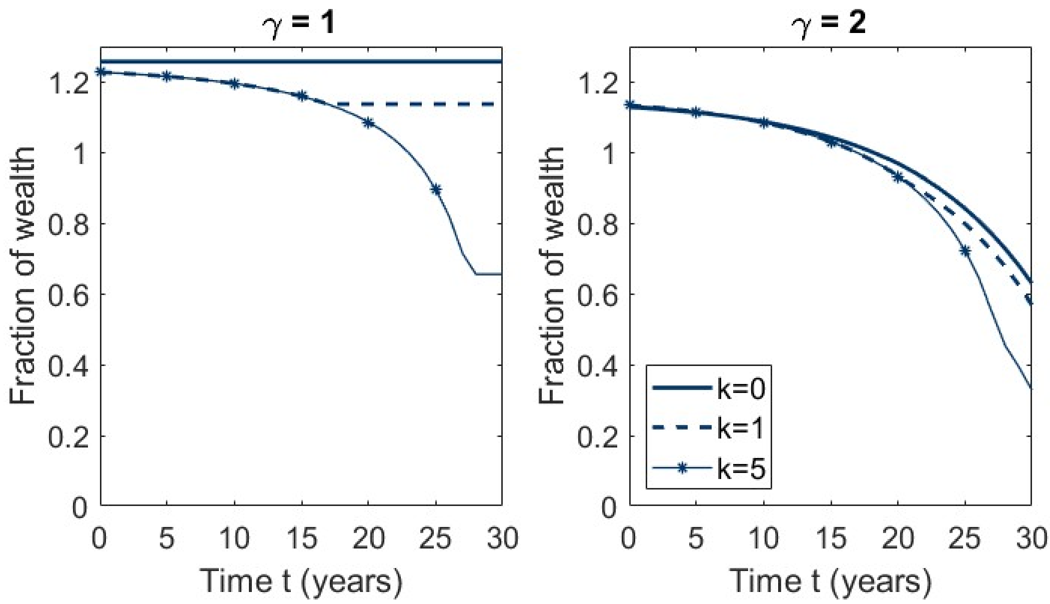


(a) $\hat{\alpha}$ **(b)** λ_t^* **(c)** κ^* **(d)** σ_{W^*}

Figure 7. Optimal stochastic distortions and bond allocations under low interest rates. Solutions of the optimal stochastic distortion in a one-factor model. Subfigure (a) shows the optimal distortion parameter $\hat{\alpha}_t$, and subfigure (b) shows the corresponding robust bond allocations. The computations are based on a constant interest rate $R_t = 5\% \ \forall t$ below the long-term average interest rate, an undistorted market price of risk $\lambda = -0.219$, bond maturity $\tau = 30$, risk aversion coefficients $\gamma \in \{1, 2\}$, and bounds on uncertainty $k \in \{0, 1, 5\}$. The allocations are shown as fractions of wealth for times $t \in (0, T)$ with investment horizon $T = 30$, meaning the remaining investment horizon decreases from left to right. The legend is given in subfigure (b).

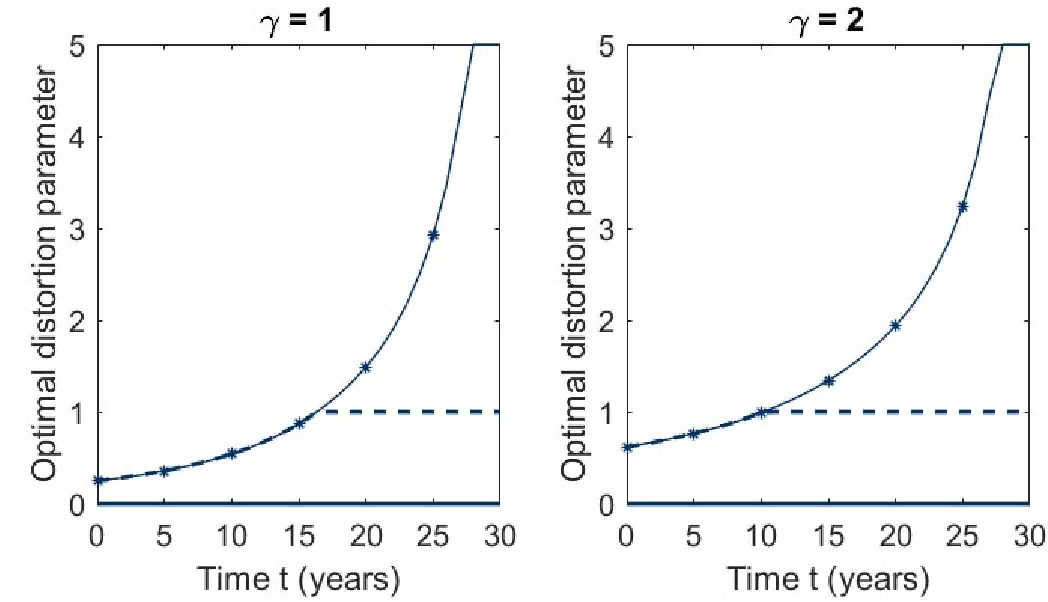


(a) $\hat{\alpha}$

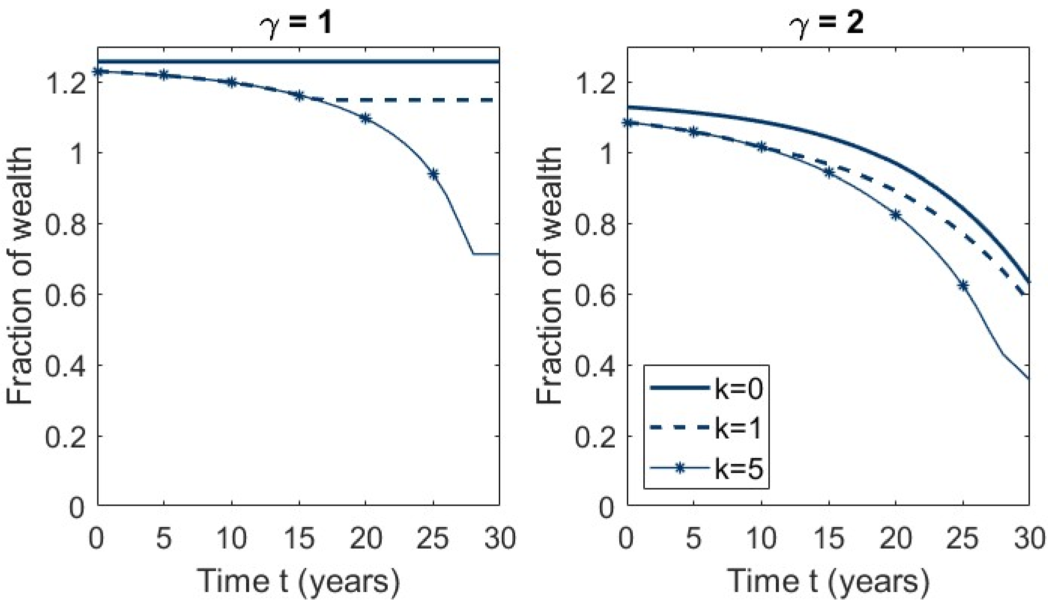


(b) $\hat{x}_t^*$

Figure 8. Optimal stochastic distortions and bond allocations under high interest rates. Solutions of the optimal stochastic distortion in a one-factor model. Subfigure (a) shows the optimal distortion parameter $\hat{\alpha}_t$, and subfigure (b) shows the corresponding robust bond allocations. The computations are based on a constant interest rate $R_t = 9\% \ \forall t$ above the long-term average interest rate, an undistorted market price of risk $\lambda = -0.219$, bond maturity $\tau = 30$, risk aversion coefficients $\gamma \in \{1, 2\}$, and bounds on uncertainty $k \in \{0, 1, 5\}$. The allocations are shown as fractions of wealth for times $t \in (0, T)$ with investment horizon $T = 30$, meaning the remaining investment horizon decreases from left to right. The legend is given in subfigure (b).



(a) $\hat{\alpha}$



(b) $\hat{x}_t^*$

3

Bond Allocations under Climate Risk and Parameter Uncertainty

Single authored.[1]

[1]I am grateful for the valuable feedback of Frank de Jong, Anne Balter, Christoph Hambel, Mathijs Cosemans, Kim Peijnenburg, Antoon Pelsser, Servaas van Bilsen, Joost Driessen, Rik Frehen, Lieven Baele, Henk Keffert, seminar attendees at the Netspar Pension Day (2024), the Brown Bag seminar (2024), the Environmental Economics seminar (2023), and the colleagues of Aarhus University during the CentER research visit for their helpful comments and suggestions.

ABSTRACT: *Climate change is expected to affect long-term bond returns, though quantifying this impact remains difficult. We therefore analyse the implications of parameter uncertainty related to the impact of climate risk on the optimal bond allocations of a long-term investor, based on an affine term structure model. If the investor is uncertain whether climate risk impact bond returns, we find that a robust strategy is to invest based on the assumption that there is a climate impact. If a speculative investor correctly assumes a climate effect but is uncertain about the underlying parameters, she should be concerned about an estimation error in the correlation between shocks in the climate risk factor and interest rates. We propose a robust strategy that can mitigate potential utility losses arising from these estimation errors.*

3.1 Introduction

Long-term investors should consider climate change because it can significantly affect asset returns (e.g., Bansal et al. (2019); Donadelli et al. (2017); Giglio et al. (2021)). Nominal bonds are a relevant asset class because they are commonly used in long-term investment strategies. However, there is still no consensus on the magnitude of climate change's impact on bond returns, due to the numerous plausible risk factors and potential future scenarios (e.g., Van Dijk (2020); Kunreuther et al. (2014)). The relatively short observation history worsens this uncertainty, leading to estimation errors. Moreover, as explained in Chapter 1, model-based optimal bond allocations are sensitive to small changes in the underlying parameters related to the risk factors. Therefore, it is important to study the impact of parameter uncertainty on the portfolio implications for long-term investors facing climate risk.

Following common practice in climate finance, we use global temperatures as a proxy for climate change (e.g., Cosemans et al. (2023); Lemoine (2021)). We analyse the impact of estimation errors regarding the (impact of) the underlying temperature process on a long-term investor's expected utility from terminal wealth. Additionally, we identify a robust bond strategy that mitigates utility losses resulting from estimation errors. We examine the impact of estimation errors and this robust strategy in relation to both uncertainty regarding the presence of the temperature impact on bond returns and uncertainty related to the underlying parameters if temperature impact is assumed to be present. Finally, we investigate how the results would change if the assumption regarding bond maturities choice would be modified.

Our contribution to the literature is threefold. First, for the baseline estimates we apply an interest rate model and hence contribute to the literature on the impact of temperatures on interest rates. To the best of our knowledge, the recent literature mainly determines the impact of temperature on implied expected interest rates via macro-economic models. In particular, we consider temperature deviations from its trend as an additional risk factor in the affine term structure model of Sangvinatsos and Wachter (2005). By modelling this impact due to temperature deviations, our approach ensures analytical tractability. To our knowledge, no existing literature has incorporated temperature as a risk factor in interest rate modelling. Thus, our model may serve as a starting point for future research incorporating the effect of the positive trend in temperatures as well.

Second, we focus on optimal bond allocations under climate change and parameter uncertainty, rather than stock allocations (e.g., Cosemans et al. (2023); Engle et al. (2020)) or green and brown capital allocations (e.g., Hambel et al. (2024); Karydas and Xepapadeas (2022)). The impact on optimal bond and stock allocations can differ significantly due to varying underlying risk factors and corresponding interest rate hedging demands in bonds. Third, we study the impact of parameter uncertainty regarding the (impact of) temperature on bond returns via an affine interest rate model, rather than limiting our analysis to correlations between shocks of the temperature and bond returns of a specific bond maturity (e.g., Shen and Rubtsov (2024); Shen et al. (2019)). Using an affine term structure model allows us to differentiate between uncertainties in various parameters of the (impact of the) underlying temperature process, and to incorporate observed bond prices across maturities.

We find that a climate denier, assuming that temperatures do not impact interest rates, faces significant utility losses if temperatures actually do impact interest rates. If this climate denier is a speculative investor with a 30-year investment horizon, she would initially receive a certainty equivalent return on wealth (CER) of 10.2% per year if temperatures do not impact interest rates. However, if temperatures do affect interest rates and the investor does not take that into account in her strategy, her CER decreases to 4.7%, due to the impact on long-term average interest rates and therefore the prices of the bonds in her investment portfolio. Applying a strategy akin to that of a climate believer appears to be a robust strategy, resulting in at least 5.9% CER. This strategy also performs well if temperatures would actually not impact interest rates, resulting in a 7.9% CER.

If we consider a climate believer in a financial market with the temperature impact, the investment strategy proves to be robust against estimation errors in the temperature-related parameters. However, there is one exception where a more speculative investor should be concerned. The correlation between temperatures and nominal interest rates appears to be hard to estimate accurately, but incorrectly estimating the correlation to be negative instead of positive results in a significant utility loss. Applying the robust strategy mitigates this large utility loss in a financial market evolving based on the worst-case parameter set with this negative correlation.

Moreover, we perform a sensitivity check on the impact of different bond maturity combinations. We find that the impact on utility depends on the maturities in the case of uncertainty about the presence of the temperature impact. However, the robust strategy remains the same.

Finally, we provide additional analyses by replacing temperature deviations as the climate risk factor with a climate news index from Engle et al. (2020), who find a link between stock returns and climate-related news. In contrast to the initial model based on temperatures, we find that this alternative climate factor improves the model's ability to capture changes in the nominal interest rate. When observing the news index, the robust strategy for the more risk-averse investor remains the same as in the temperature-based analysis, namely to invest like a climate believer. The robust strategy for a speculative investor is now to invest like a climate denier.

In summary, if long-term investors ignore climate risk when determining their bond allocations, they face considerable utility losses if climate change actually impacts interest rates. A climate news index appears to be a promising proxy for capturing this climate impact. For more risk-averse investors, a robust strategy that assumes the impact of the news index prevents these potential losses, and performs reasonably well even if climate risk ultimately does not materialise. Therefore, long-term investors should consider robust bond investment strategies that perform well in case of a climate impact.

This chapter is organised as follows. Section 3.2 reviews the literature most relevant to our research. Section 3.3 describes the financial market and the corresponding optimal bond allocations, both with and without parameter uncertainty. Section 3.4 explains the data and computes the corresponding Best Estimate parameters. Section 3.5 explores the impact on utility if the investor bases her strategy on alternative parameters and identifies strategies that are robust across different financial markets. Section 3.6 performs a sensitivity check on the assumed bond maturities. Section 3.7 repeats the analysis using a climate news index as an alternative climate factor. Section 3.8 concludes.

3.2 Literature review

Our study is related to two topics in the literature. As we mainly focus on parameter uncertainty, we first discuss the link with studies on optimal horizon-dependent investments under parameter uncertainty about climate risk. For instance, Cosemans et al. (2023) empirically show that optimal stock investments are considerably impacted by (Bayesian) parameter uncertainty about natural disasters. They show that on the short-term brown assets are more advantageous due to diversification. However, on the long-term green assets have the advantage of being less prone to future climate damages. This balance between the short- and long-term (dis)advantages of brown and green investment strategies aligns with the studies of Hambel et al. (2024) and Jung et al. (2025) that study brown and green capital and portfolios respectively. We extend this literature by investigating bonds as hedging assets.

To the best of our knowledge, there are only two papers that consider optimal government bond allocations under parameter uncertainty about climate risk. Shen et al. (2019) show that mean-variance optimal 10-year bond allocations decrease for a long-term risk-averse investor if the observed correlation of temperature changes on realised excess bond returns is taken into account. Moreover, Shen and Rubtsov (2024) apply a max-min objective including a penalty function to find robust asset allocations under uncertainty about the impact of temperature changes, where the penalty function is larger if alternative parameters are assumed to be less likely to occur. They find that uncertainty about the risk premium of the stock barely impacts the interest rate process and the corresponding bond returns, because no uncertainty about the risk premium on bonds is modelled. As mentioned before, we study the impact on bond allocations based on an affine interest rate model and alternative parameter sets instead.[2]

Second, our study contributes to the still scarce literature about the impact of temperature on government bond returns and interest rates. There is quite some literature about the impact of climate change on *municipal* (e.g., Painter (2020)) and *corporate* bonds (e.g. Le Guenedal and Tankov (2024) and Blasberg et al. (2023)), based on regressions or endogenous default models (credit risk). Moreover, there is literature observing that vulnerability to climate change correlates positively with *sovereign risk* (e.g. Boehm (2022), Cevik and Jalles (2022), Bingler (2022), Battiston and Monasterolo (2020) and Kling et al. (2018)). Finally, there are studies that measure the ability of *green bonds*

[2]We apply our model to different data as well. We will motivate in Section 3.4 why we assume trend-stationary temperatures instead, and that we use a more general dataset.

to hedge the impact of climate change (e.g. Jin et al. (2020)). However, our research deviates from these studies, since we are interested in optimal model based government bond investment strategies instead.

Kizys et al. (2024) observe an overall (small) positive correlation between temperatures and 10-year government bond returns in panel data of developed and emerging countries. However, they also observe that the relationship between temperature and bond returns is non-linear and depends on whether temperatures exceed a certain threshold. They argue that this non-linearity can be attributed to psychological factors impacting the demand for risk-free assets. Furthermore, Bingler (2022) and Shah (2022) find that the impact of climate on bond yields may vary by maturity. Moreover, Shen et al. (2019) do not find a significant effect of temperature changes on the 3-month bond return (a proxy for the interest rate) and 10-year bond return. Their study applies a log-linear approximation approach for the bond return, using the yield and duration approximation. In contrast, we use an affine factor model that is applied on several bond maturities and could capture a long-term impact of temperatures on bond returns.

Other studies measure the impact of temperature on interest rate not via observed interest rates, but derive the implied risk-free rate with an Integrated Assessment Model (IAM). An IAM is a macro-economic model where climate change is integrated, for example via the direct effect on consumption growth through natural disasters and indirect impact of stringent climate policy.[3] For example, Karydas and Xepapadeas (2022) and Hambel et al. (2024) find that climate risk reduces the risk-free rate through more natural events followed by a decrease in consumption growth and therefore increase the demand for a risk-free asset. Mongelli et al. (2023), Hambel and Van der Ploeg (2025), and Lemoine (2021) confirm that this negative impact on the risk-free rate is the most likely outcome, but also find that stringent transition policies may lead to an increase in interest rates. They attribute this to factors such as increased green investments, more green innovations, or sufficient distance from tipping points regarding climate change or natural resource depletion. Therefore, it is challenging to determine the impact on the implied risk-free rate via an IAM.[4] Modelling the temperature as a risk factor on bond returns results in

[3]Typically, climate finance literature differentiates between *transition* and *physical* risks due to the economic impact of climate policy and natural disasters respectively Giglio et al. (2021). The distinction between these two types of risks may differ among papers or can even be intertwined as in Karydas and Xepapadeas (2022). It is beyond the scope of our study to distinguish between the transition and physical risk, the climate risk in our model can be represented by both types.

[4]Moreover, observing the realised impact of temperature on economic growth, Kahn et al. (2021) conclude that the impact differs among countries and speed of temperature increases. So the impact of climate change on the consumption growth might be hard to model.

new insights based on observed interest rates.

In summary, there is no consensus yet in the literature about the link between temperatures and government bond returns (interest rates). The IAM based studies support both positive and negative relationships between temperature and implied interest rates. The studies based on correlations between observed temperature and bond returns do not agree on a positive or negative correlation either, and find that the sign may depend on the magnitude and sign of the temperature change. These findings make parameter uncertainty in optimal bond allocations under temperature risk relevant to study.

3.3 Model description

This section outlines the model set-up. Section 3.3.1 describes the financial market, Section 3.3.2 explains how we assess the impact of parameter uncertainty and aim to find a robust strategy under such uncertainty.

3.3.1 Financial market

Initially, we consider temperature as the risk factor to represent climate change. As discussed in the literature review, this approach is commonly used in the climate finance literature. For analytical tractability, we model temperature deviations from their time trend, consistent with Gadea-Rivas et al. (2024) and McKitrick et al. (2023) who find that temperatures are trend-stationary around a linear trend.[5] As discussed in the literature review as well, unanticipated hot periods can reduce productivity or increase climate risk awareness, which can lead to less consumption or to stringent climate policies that dampen economic growth on the short term. Consequently, temperature deviations from the trend can affect the nominal interest rate.[6]

We start with an affine model as proposed in Sangvinatsos and Wachter (2005), incorporating two risk factors: the instantaneous nominal interest rate, denoted as R_t, and de-trended temperatures, denoted as Y_t. Henceforth, we will refer to Y_t as 'temperature'

[5]Both studies report a structural break around 1960 in the temperature trend, which we will take into account into the empirical part of this chapter.

[6]Because these deviations capture unanticipated extreme weather rather than climate change (sustained increases in average temperatures), Section 3.7 analyses a climate news index as alternative climate risk factor that may better reflect climate risk.

for readability purposes. The state variables are defined as follows:

$$X_t = \begin{bmatrix} R_t - \bar{R} \\ Y_t \end{bmatrix} \tag{1}$$

We assume that the risk factors in X_t follow an Ornstein-Uhlenbeck (OU) process:

$$d\begin{bmatrix} R_t \\ Y_t \end{bmatrix} = -\begin{bmatrix} \kappa_R & \kappa_{RY} \\ 0 & \kappa_Y \end{bmatrix} \begin{bmatrix} R_t - \bar{R} \\ Y_t \end{bmatrix} dt + \begin{bmatrix} \sigma_R & \sigma_{RY} \\ 0 & \sigma_Y \end{bmatrix} d\begin{bmatrix} Z_t^R \\ Z_t^Y \end{bmatrix} \tag{2}$$

which simplifies to:

$$dX_t = -KX_t dt + \sigma_X dZ_t \tag{3}$$

where the Brownian motion processes in Z_t are uncorrelated. We define $\kappa_R > 0$ and $\kappa_Y > 0$ as the mean-reversion parameters, and κ_{RY} as the feedback parameter that relates the level of Y_t to the drift adjustment on R_t and therefore can capture a possible 'lagged' effect of temperatures on interest rates, compared to the direct correlation between shocks in the risk factors. We assume that the factor volatilities are positive, and $\kappa_R \neq \kappa_Y$. An upper-diagonal factor volatility matrix σ_X is assumed, to allow for shocks in temperature to impact the short-term interest rate directly, while it is economically intuitive that a shock in the interest rate might not directly lead to a shock in temperatures.

The setup in equation (2) implies an OU-process for temperature deviations from its trend, which is the most common approach in the weather derivatives literature (Tong and Liu (2020)). The model structure implies an univariate mean-reverting process around the stochastic average interest rate $\bar{R} - \frac{\kappa_{RY}}{\kappa_R} \cdot Y_t$ for the interest rate. Such a mean-reverting component has been studied for other risk factors where it represents a short-term central tendency such as the current federal fund target like the expected inflation rate or a regime shift (e.g., Brennan and Xia (2000); Brennan and Xia (2002); Bansal and Zhou (2002)). It can also represent a longer term central tendency such as investors' beliefs about future monetary policies (e.g., Jegadeesh and Pennacchi (1996); Nowak and Romaniuk (2017)).[7]

The pricing kernel of the financial market is given by

$$\frac{d\zeta_t}{\zeta_t} = -R_t dt - \lambda' dZ_t, \quad \zeta_0 = 1 \tag{4}$$

[7]Including a gradual impact component of climate change on financial markets aligns with the Financial Stability Review of the European Central Bank (2023), see their page 128.

where the 2×1 vector λ contains the constant market price of risks (MPoR) on the risk factors in equation (2).[8] The investor can invest in the risk-free asset with return R_t, and nominal bonds with prices

$$\frac{dP_t(\tau_j)}{P_t(\tau_j)} = [R_t + \sigma_{Bj}\lambda]\, dt + \sigma_{Bj} dZ_t \tag{5}$$

where $\sigma_{Bj} = -B'(\tau_j)\sigma_X$ is a 1×2 vector of exposures to the risk factors of bond j with maturity τ_j, and the 2×1 vector $B(\tau_j)$ is computed by solving the ODE:[9]

$$B'(\tau_j) = -K\, B(\tau_j) + \delta \tag{6}$$

with 2×1 vector $\delta = [1\ 0]'$. Hence, the bond volatilities depend on the mean-reversion matrix K. We consider an investor who aims to optimise her wealth, subject to the budget constraint. The optimisation problem is defined as follows:

$$\max_{W_T} \mathbb{E}_t \left[u\left(W_T \right) \right] \qquad \text{s.t. } \mathbb{E}_t \left[\zeta_T W_T \right] = W_t \tag{7}$$

where $u()$ is the investor's utility function, W_t is the nominal wealth at time t, and T is the investment horizon. Moreover, we assume the investor has a constant relative risk aversion (CRRA) utility function with a coefficient of relative risk aversion γ:

$$u(w) = \begin{cases} \frac{w^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 1 \\ \log\left(w\right) & \text{if } \gamma = 1 \end{cases} \tag{8}$$

The investor allocates her wealth between the money market account with return R_t and one or two nominal bonds. The optimal bond investment x_t is derived by Sangvinatsos and Wachter (2005) and can be found in their equation (44), which in our notation reads as:

$$x_t = \frac{1}{\gamma}\Omega^{-1}\sigma_B\lambda - \left(1 - \frac{1}{\gamma}\right)\Omega^{-1}\sigma_B\sigma_X' B(T-t) \tag{9}$$

where $\Omega = \sigma_B\sigma_B'$ is the covariance matrix of the bond returns, where σ_B is a 2×2 matrix

[8] Assuming constant market prices of risks in a two-factor model for the interest rate with a feedback from the level of the second risk factor to the drift term of the first factor is in line with Brennan and Xia (2002).

[9] The 2×1 vector $B'(\tau_j)$ denotes the derivative of the vector $B(\tau_j)$ with respect to τ_j. The corresponding solution for $B(\tau_j)$ in our financial market is given in equation (A.3.3).

containing the bond exposures of bonds 1 and 2:

$$\sigma_B = \begin{bmatrix} \sigma_{B1} \\ \sigma_{B2} \end{bmatrix} \tag{10}$$

However, this optimal investment strategy is optimal only if the parameters are estimated correctly. The next section explains how we compute the impact of estimation errors.

3.3.2 Parameter uncertainty

We now consider a robust investor who accounts for the possibility that her Best Estimates $\theta^m = \{K_X^m, \sigma_X^m, \lambda^m\}$ may differ from the actual financial market parameter set $\theta^n = \{K_X^n, \sigma_X^n, \lambda^n\}$. In that case, an investment strategy based on θ^m would be suboptimal, leading to a lower expected utility from terminal wealth.

We assess the impact of parameter uncertainty by evaluating an investor applying a strategy x_t^m based on θ^m, while the financial market is based on θ^n. The utility of terminal wealth is measured by the certainty equivalent return (CER), defined as:[10]

$$\text{CEW} = u^{-1}\left(\mathbb{E}_0^n\left[u\left(W_T\right)\middle| x_t^m, \theta^n\right]\right) \tag{11}$$

$$\text{CER} = \left[\frac{CEW}{W_0}\right]^{1/T} - 1 \tag{12}$$

where CER is the return that makes the investor indifferent between investing her initial wealth with the investment strategy x_t^m at all time $t \in (0, T)$, or receiving the yearly CER for certain at time $t = 0$.

To find a strategy that aims to mitigate large losses in the CER, we consider an investor who identifies the strategy that performs best under the so called worst-case financial market parameter set, based on the max-min methods used for ambiguous investors (e.g. Chen and Epstein (2002)). The robust investor considers M investment strategies based on θ^m and N plausible financial market parameter sets θ^n where $\theta^m, \theta^n \in \Theta$,

[10]For completeness, note that in case of $\gamma = 1$ the certainty equivalent of wealth CEW is computed as the exponential of expected utility, because the utility function reduces to the log of terminal wealth.

$m \in \{1, 2, ..., M\}$, and $n \in \{1, 2, ..., N\}$. We define a robust strategy x_t^{rob} as:

$$x_t^{rob} = \arg\max_{x_t^m} \min_{\theta^n} \mathbb{E}_t^0 \left[u\left(W_T\right)\right] \tag{13}$$

$$\text{s.t. } \mathbb{E}_t^n \left[\zeta_T^n W_T\right] = W_t \ ; \ \frac{d\zeta_t^n}{\zeta_t^n} = -R_t dt - \left(\lambda^n\right)' dZ_t^n \tag{14}$$

where the superscripts n refer to the financial market based on θ^n. By solving the inner minimisation problem, the investor evaluates each strategy x_t^m considering the worst-case parameter set of the financial market that would result in the lowest expected utility. Through the outer maximisation problem, the investor sequentially selects the robust strategy that maximises the expected utility given this worst-case parameter set in the financial market. This approach ensures that, in the event of the worst-case scenario, the investor achieves the highest possible expected utility.

3.4 Estimation

For the de-trended temperature process described in equation (2), we use average annual global temperature anomalies. Temperature anomalies are the levels below or above the average temperature of a time span of 30 years.[11] The data source is the Goddard Institute for Space Studies Surface Temperature Analysis (GISTEMP) which combines meteorological station data (NASA v4) and ocean area data (ERSST v5).[12] Incorporating ocean temperatures aligns with the IPCC's goal of providing a complete representation of global temperatures, and results in more stable global temperatures than only considering surface data. Seasonal fluctuations are minimised by incorporating high spatial resolution satellite data to 'appropriately' capture the long-term temperature trend (Hansen et al. (2010)).

Our analysis covers the period from 1960 to 2023, motivated by the detection of one structural break in the linear time trend of average global surface and ocean temperature in 1960, as identified by Gadea-Rivas et al. (2024). McKitrick et al. (2023) report varying structural breaks between 1960 and the early 1980s, depending on the unit root tests employed. Figure 1a illustrates the temperatures and corresponding linear time trends for the subsample periods before 1960, after 1960, and the full sample. The figure suggests that temperatures indeed behave like a trend-stationary process with a notably steeper trend

[11] We use data that uses the period of 1951-1980 as its baseline.
[12] `https://data.giss.nasa.gov/gistemp/`

after the break. Table 1a provides summary statistics for the original temperature data for each time range, confirming the distinct patterns for each period. For completeness, Table 1b provides summary statistics for the de-trended temperature data between 1960 and 2023.

Regarding the bond returns, we use monthly yields on U.S. zero-coupon bonds with bond maturities of 0.25, 1, 5 and 10 years to capture the interest rate process across shorter and longer maturities (De Jong (2000)). The bond return data from 1960 to 1991 are obtained from the data provided by McCulloch and Kwon (1993).[13] For the subsequent period from 1992 to 2024, we use data from the FRED database. Figure 1b and Table 1c provide a graphical representation and some summary statistics of the yields, respectively. These demonstrate a wide range of interest rates over time. The average term structure is upward sloping, and the volatility of interest rates decreases with longer bond maturities, as expected.

To estimate the model, we apply the Kalman filter, where details about the estimation procedure can be found in Appendix A.3.2.2. We assume that the nominal interest rate can be fitted by one latent factor, and simultaneously add one perfectly measured observed factor, namely the temperature deviations. Recall that the nominal interest rate is affected by these temperature deviations through the impact of temperature on the interest rate's drift term, as shown in (2). We refer to this model as the "bond yields and climate factor" (B+C)-model.

Table 2 describes the estimation results. The first column shows the results for the unrestricted model which represents the B+C-model with correlated shocks between the risk factors. The small mean-reversion parameter κ_R of 0.03 and small volatility σ_R of 1% per year are in line with the literature.[14] In contrast, the estimated temperature process has a much larger mean-reversion parameter κ_Y of 1.12, which aligns with our previous observations in Figure 1, where interest rates tend to revert more slowly to their long-term average compared after a shock compared to the faster convergence of temperatures to their trend.

We observe a positive effect of temperature deviations on the interest rate drift term due to the negative feedback parameter κ_{RY}. One standard deviation of temperature of $0.16°$ per year increases the interest rate drift term with $-(-0.009) * 0.16 = 0.1\%$ per year. The positive effect of temperature deviations on the interest rate is strengthened by

[13] `https://www.asc.ohio-state.edu/mcculloch.2/ts/mcckwon/mccull.htm`

[14] See for example Munk et al. (2004), who estimate the mean-reversion parameter and the volatility of the nominal interest rate at 0.04 and 0.02, respectively, using U.S. bond data from 1951 to 2003.

the positive σ_{RY}. However, note that the standard errors of κ_{RY} and σ_{RY} are large, so that values of the opposite sign are included in the 95% confidence interval of the baseline estimates as well, resulting in a negative impact of temperature on the interest rate. As discussed in the literature review, the impact of temperature deviations on realised interest rates has not yet been explored in the existing literature. Therefore, we do not have an expectation of either a positive or negative feedback parameter or correlation.

Moreover, Figure 2 shows the observed and fitted factors in the Kalman filter. It shows that the predicted interest rates by the Kalman filter follow the movements of the observed 3-month bond yield, which is commonly used as a proxy for the interest rate. For completeness, the figure shows the estimated factor Y_t as well.

Furthermore, the third row of Table 2 compares the estimates from the B+C-model with the one-factor model resulting from an interest rate process independently from temperatures ($\sigma_{RY} = \kappa_{RY} = 0$). We refer to this model as the "bond yields only"(B)-model. For the B-model, the mean-reversion and volatility of the interest rate process remain quite similar to those in the B+C-model. However, we see that the root mean squared error (MSE) between the fitted interest rate and the 3-month yield is smaller for the B-model at 0.007 than for the B+C-model at 0.008. This means that the modelling the interest rate with a climate impact results in a slightly worse fit of the implied interest rates than without the climate factor.

Consecutively, we consider the implied risk premiums. The risk premium components of the two risk factors, RP_R and RP_Y, are defined as follows:

$$RP(\tau_j) = -B'(\tau_j)\sigma_X\lambda = \underbrace{-B_R(\tau_j)\sigma_R\lambda_R}_{\equiv RP_R(\tau_j)} \underbrace{- \left(B_R(\tau_j)\sigma_{RY} + B_Y(\tau_j)\sigma_Y\right)\lambda_Y}_{\equiv RP_Y(\tau_j)} \quad (15)$$

where B_R and B_Y are scalars corresponding to the first and second item in the vector $B(\tau_j)$.[15] As the B-model does not take into account the temperature impact, the risk premium in the B-model consists of simply the first term, $RP_R(\tau_j)$. Table 2 shows that the B-model results in a risk premium of 3.1% on a 10-year bond, while the B+C-model results in a total risk premium of 1.3+0.4=1.7% on a 10-year bond. Hence, the model estimates imply a different risk premium if the impact of temperature deviations is included, but the order of magnitude remains similar.

Because we observe that the parameter σ_{RY} is far from significant in the B+C-model, as a sensitivity analysis we estimate a restricted version under the assumption of uncorrelated

[15]For completeness, the 2×1 vector $B(\tau_j)$ of bond exposures is given in equation (A.3.3).

shocks ($\sigma_{RY} = 0$). We restrict σ_Y to be less than or equal to the estimate obtained from the unrestricted B+C-model. As shown in the second column, this restriction reduces the model's ability to capture the impact on the fitted interest rate: the root MSE of the interest rate increases from 0.008 in the unrestricted model to 0.012 in the restricted model. Moreover, the risk premium component of the nominal interest rate is unreasonably large at 13.5%, compared to a risk premium of 3.1% in the B-model. Therefore, it makes more sense to fit the model with $\sigma_{RY} \neq 0$.

Hence, it appears to be challenging to estimate the precise impact of temperature deviations on the nominal interest rate through parameters κ_{RY} and σ_{RY}. Moreover, compared to a one-factor model based on yields only, integrating the temperature impact results in a slightly worse fit of the implied interest rates and a smaller total risk premium on bonds. Next, we will analyse how alternative parameters about the temperature impact affect the expected utility of the investor.

3.5 Results

The previous section provides the Best Estimates of the financial market. A robust investor considers alternative values to evaluate the potential impact of estimation errors in actual financial market parameters. We analyse how the investor's utility is affected when her investment strategy is based on incorrect parameters and identify the robust investment strategy across different financial market parameter sets. Section 3.5.1 investigates uncertainty regarding the presence of a temperature impact on interest rates, while Section 3.5.2 assumes such a temperature impact exists, but takes into account potential estimation errors in the temperature-related parameters.

All computations assume a long investment horizon of $T = 30$. The certainty equivalent return (CER) on wealth is computed according to equation (12), using a weekly Euler discretisation and 1,000 simulations. We assume that the initial interest rate equals the long-term average. The robust strategy is calculated as in equation (13). The results are presented for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, covering a large range from speculative to almost infinitely risk-averse (hedge) investors.[16] We use a bond maturity of $\tau_1 = 30$ years in the B-model, and a maturity combination $\tau = (\tau_1, \tau_2) = (3, 30)$ in the B+C-model.[17] The bond maturities remain constant over the investment horizon.

[16] Due to numerical considerations, we use a log-normal approximation of simulated wealth for $\gamma > 2$.
[17] Section 3.6 will provide sensitivity analyses for different bond maturities.

3.5.1 Uncertainty about presence of temperature impact

To address parameter uncertainty regarding the presence of temperature impact on the interest rate, we select the alternative parameter sets $\Theta = \{\theta^B, \theta^{B+C}\}$, reflecting the beliefs of a climate denier who relies on the B-model's best estimates, and a climate believer who relies on the B+C-model, respectively. Table 3 shows the CER if the investor applies x_t^m based on $\theta^m \in \Theta$, and the financial market is based on $\theta^n \in \Theta$, for all 2×2 combinations of strategies and financial markets. The table shows that there is a loss in her CER if the investor bases her strategy on the B-model, but in reality the financial market follows the B+C-model. For instance, for $\gamma = 1$ her CER would be 10.2% per year if the temperatures would not affect the financial market and her strategy is optimal. However, the CER declines to 7.9% if the financial market follows the B+C-model.

This potential large loss in utility is especially present for the more speculative investor, because it is caused by different estimate of the market price of risk in the B- and B+C-model as explained in Section 3.4. Consequently, the investment strategy based on the B-model overestimates the risk premium and performs poorly if the financial market is actually based on the B+C-model.

Table 3 shows that the robust strategy is to invest as a climate believer ($x_t^{rob} = x_t^{B+C}$). For both investment strategies, the worst-case financial market parameter set is the one based on the B+C-model. To maximise the expected utility in this worst-case, the robust investor applies x_t^{B+C}, even when she initially believes that temperatures do not affect interest rates in the base model. For example, for $\gamma = 1$, by investing with the robust strategy x_t^{B+C} her CER will at least be 5.9%, which prevents receiving a lower CER of 4.7% when she would apply the investment x_t^B and the financial market would follow the B+C-model.

This lower CER of 4.7% under the B+C-model is the result of an incorrect estimation of the risk premium. The B+C-model implies a smaller total risk premium than the B-model, as discussed in the estimation results in Section 3.4. In the financial market based on the B+C-model, a strategy based on the B-model overestimates the risk premium and therefore exposes the investor to more risk than would be optimal.

For more risk-averse investors, the worst-case financial market and robust strategy are based on a climate impact as well, but note that the investor's assumption about the climate factor has little effect on utility. Such investors are less sensitive to an incorrectly estimated risk premium, focusing instead on hedging bond exposures to the risk factors. For example, for $\gamma = 10$, the worst-case CER is 5.5% when she applies the strategy based on the B+C-model, versus 5.4% based on the B-model.

In summary, without parameter uncertainty, the climate denier maximises her expected utility by investing based on the B-model, and the climate believer would do so based on the B+C-model. However, when investors want to apply a robust strategy maximising utility across the considered alternative measures, both investors invest like a climate believer.

3.5.2 Uncertainty about temperature-related parameters

The results in the previous section consider uncertainty regarding whether temperatures impact the interest rate. We now focus on a climate believer who exclusively considers the B+C-model but faces parameter uncertainty about the initial Best Estimates.

We first construct 5×2 alternative parameter sets θ^n, each varying by one parameter from the initial estimates:

$$\kappa_Y^- = \max\{\widehat{\kappa}_Y - 2 \cdot \mathrm{SE}(\widehat{\kappa}_Y), 0\} \; ; \; \kappa_Y^+ = \max\{\widehat{\kappa}_Y + 2 \cdot \mathrm{SE}(\widehat{\kappa}_Y), 0\} \tag{16}$$

$$\sigma_Y^- = \max\{\widehat{\sigma}_Y - 2 \cdot \mathrm{SE}(\widehat{\sigma}_Y), 0\} \; ; \; \sigma_Y^+ = \max\{\widehat{\sigma}_Y + 2 \cdot \mathrm{SE}(\widehat{\sigma}_Y), 0\} \tag{17}$$

$$\lambda_Y^- = \widehat{\lambda}_Y - 2 \cdot \mathrm{SE}(\widehat{\lambda}_Y) \; ; \; \lambda_Y^+ = \min\{\widehat{\lambda}_Y + 2 \cdot \mathrm{SE}(\widehat{\lambda}_Y)\} \tag{18}$$

$$\kappa_{RY}^- = \widehat{\kappa}_{RY} - 2 \cdot \mathrm{SE}(\widehat{\kappa}_{RY}) \; ; \; \kappa_{RY}^+ = \widehat{\kappa}_{RY} + 2 \cdot \mathrm{SE}(\widehat{\kappa}_{RY}) \tag{19}$$

$$\sigma_{RY}^- = \widehat{\sigma}_{RY} - \mathrm{SE}(\widehat{\sigma}_{RY}) \; ; \; \sigma_{RY}^+ = \widehat{\sigma}_{RY} + \mathrm{SE}(\widehat{\sigma}_{RY}) \tag{20}$$

where $(\widehat{..})$ denotes the Best Estimate, and SE is the standard error of the estimate. A deviation of two standard errors represents plausible estimation errors, as the corresponding alternative value remains within a normal distribution 95%-confidence interval of the Best Estimate. Since we see that the alternatives σ_{RY}^- and σ_{RY}^+ would imply extremely high variances in the process of R_t, we choose to only deviate one standard deviation. However, the variance of R_t still remains high under σ_{RY}^- and σ_{RY}^+, equal to 0.055% and 0.089%, compared to the initial variance of 0.0014% in the B+C-model.[18] Therefore, these values might still not represent plausible alternatives and we form additional alternative estimates $\tilde{\sigma}_{RY}^-$ and $\tilde{\sigma}_{RY}^+$ based on the boundaries of the constraint:

$$\widehat{\sigma}_R^2 + (\tilde{\sigma}_{RY}^n)^2 \leq (\widehat{\sigma}_R + 2 \cdot SE(\widehat{\sigma}_R))^2 + \widehat{\sigma}_{RY}^2 \tag{21}$$

The intuition of this constraint is as follows. We assume that the investor believes that the

[18] i.e. the interest rate element (1,1) in the variance matrix of the risk factors' changes, $\sigma_X \sigma_X'$.

model estimates reasonably accurate estimations of the interest rate variance and standard errors of σ_R. She considers only alternative values of $\tilde{\sigma}^n_{RY}$ that result in a variance not higher than the variance resulting from an alternative $\sigma^+_R = \widehat{\sigma}_R + 2 \cdot \text{SE}(\widehat{\sigma}_R)$.

The resulting sensitivity sets for the temperature process and impact parameters are summarised in Table 4. To gain insights into the relevance of estimation errors for each parameter, we compute the CER if the financial market parameter set is the initial B+C-parameter set θ^{B+C}, but the investor applies an investment strategy x^m_t based on $\theta^m \in \Theta$ where it is possible that $\theta^m \neq \theta^{B+C}$. Table 5 presents the results of applying the strategy based on the initial estimates in the set B+C or one of the sensitivity sets. The table shows that the impact on utility from most of the considered estimation errors appears to be negligible.

However, there is one exception. The more speculative investor may face a loss in utility if she makes estimation errors about the correlation between the temperature and interest rate. For instance, without estimation errors, the speculative investor ($\gamma = 1$) achieves a CER of 5.9% per year. If she bases her strategy on the incorrect σ^-_{RY}, this results in a CER of -1.4%. As explained before, she might think that σ^-_{RY} based on the standard errors of $\widehat{\sigma}_{RY}$ is not plausible, and therefore only consider an alternative $\tilde{\sigma}^-_{RY}$ or $\tilde{\sigma}^+_{RY}$. Applying a strategy based on $\tilde{\sigma}^-_{RY}$ results in a CER of 4.6% compared to the initial 5.9% for the speculative investor. The impact of the estimation error is negligible for the very risk-averse investor with $\gamma = 25$.

The impact of estimation errors in σ_{RY} can be explained by Figure 3 that shows the optimal bond allocations based on the initial value σ_{RY} and sensitivity $\tilde{\sigma}^-_{RY}$. The figure shows that for the more speculative investor, the estimation error leads to a suboptimal strategy that deviates considerably from the optimal one based on the correct parameter, resulting in the large utility loss. For the very risk-averse investor, that deviation is much smaller and therefore has a limited impact on utility.

Because this $\tilde{\sigma}^-_{RY}$ appears to be the most relevant parameter when it comes to parameter uncertainty, we identify the robust strategy regarding σ_{RY}. Similar to the earlier analysis comparing the B- and B+C-model, Table 6 shows the CER for strategies x^m_t based on θ^m applied in financial markets based on θ^n. Let the initial strategy and financial market be denoted as x^0_t and Θ^0, respectively. The table indicates that a speculative investor with a risk aversion of $\gamma = 1$ optimises her strategy for the worst-case financial market parameter set by employing the strategy based on the initial σ^0_{RY}. The worst-case financial market is then based on $\tilde{\sigma}^-_{RY}$, resulting in a CER of 4.7%. This is a higher return in the worst case than investing based on $\tilde{\sigma}^-_{RY}(3.7\%)$ or $\tilde{\sigma}^+_{RY}(3.8\%)$. This initial estimate seems to be good on average, because applying a strategy based on a too low parameter results in an even larger loss if the financial market is based on parameter higher than the initial parameter, and vice versa. However, for $\gamma \geq 2$, the robust strategy is to invest based on the parameter $\tilde{\sigma}^{-}_{RY}$.

Hence, it is relevant for a more speculative investor to consider parameter uncertainty about the correlation between temperature shocks and interest rate shocks. We show that this parameter is hard to estimate precisely, and an estimation error can have a substantial impact on utility. The robust strategy is to invest with the optimal strategy based on the worst-case parameter.

3.6 Impact bond maturities

The previous calculations were based on a bond maturity of $\tau_1 = 30$ in a one bond strategy, and a bond maturity combination $\tau = (\tau_1, \tau_2) = (3, 30)$ in a two bond strategy. This section explains why different bond maturities may affect the results regarding the impact on utility and the robust strategy, and analyses these differences numerically.

Suppose that the optimal investment strategy shown in equation (9) is based on the initial Best Estimates in parameter set θ^m. For readability, denote the parameters of θ^m without a superscript. If there is no parameter uncertainty, we assume that this set is the correct parameter set of the financial market as well and results in the optimal wealth process:

$$dW_t/W_t = [R_t + \sigma_W' \lambda]\, dt + \sigma_W' dZ_t \tag{22}$$

$$\sigma_W = \frac{1}{\gamma}\lambda - \left(1 - \frac{1}{\gamma}\right) \sigma_X' B(T-t) \tag{23}$$

However, if the financial market evolves based on θ^n where $\theta^n \neq \theta$, she will invest suboptimally:

$$dW_t^n/W_t^n = [R_t + x_t' \sigma_B^n \lambda^n]\, dt + x_t' \sigma_B^n dZ_t \tag{24}$$

$$x_t = \frac{1}{\gamma}\Omega^{-1}\sigma_B \lambda - \left(1 - \frac{1}{\gamma}\right) \Omega^{-1}\sigma_B \left(\sigma_X\right)' B(T-t) \tag{25}$$

where the future risk factors are based on K_X^n and σ_X^n. The equation shows that the bond maturities may be relevant if the "mismatch" factor $\Omega^{-1}\sigma_B\sigma_B^n$ is larger or smaller for different τ, where we recall that the bond exposures σ_B and σ_B^n depend on the bond maturities in τ. In our analysis, this mismatch factor is dependent of τ in two situations. The first situation is when the investor believes in the B-model and only applies σ_{Bj} for one bond with maturity τ_j, but the optimal investment would be based on a B+C-model with the corresponding σ_B^n, or the other way around. The second situation occurs if both parameter sets are based on the B+C-model, but the investor applies her bond strategy on an incorrect κ_Y that differs from the financial market parameter, κ_Y^n.[19]

We repeat the analysis of the impact of parameter uncertainty when the investor considers the two parameter sets of the B and B+C-model, as discussed in Section 3.5.1. The initial computations in that section are based on $\tau = (3, 30)$ for the two bond strategy and $\tau_1 = 30$ for the one bond strategy. We now apply $\tau = (3, 10)$, $\tau = (10, 30)$, $\tau_1 = 3$ and $\tau_1 = 10$ instead. Table 7 illustrates that the impact on utility barely change w.r.t. to the initial bond maturities. The robust strategy remains the same.

Regarding the sensitivities in the B+C-model, the only sensitivities that might result in a different impact on utility are κ_Y^- and κ_Y^+. However, the bond maturity combinations $\tau = (3, 10)$ and $\tau = (10, 30)$ do not lead to considerably different results than the initial assumed $\tau = (3, 30)$. Therefore, the robust strategy results with respect to uncertainty about the parameters in the temperature process remain the same as well.

Hence, the bond maturities choice may affect the impact on utility or robust strategy. However, for various bond maturities, the differences in impact on utility are negligible, and the robust investment strategies remain the same.

3.7 Climate news index as climate factor

We initially modelled the climate factor using deviations of annual temperatures from their trend, which captures short-term weather fluctuations rather than long-term climate change.[20] However, Choi et al. (2020) show that extreme weather can make retail investors more aware of climate risk, but it has little impact on institutional investors or asset prices. Similarly, Bansal et al. (2019) find that short-term temperature changes have little effect on equity pricing, whereas long-term increases do.

To better investigate the impact of climate risk, we now assume that bond prices are

[19] To understand this, note that only the $B(\tau)$ matrix depends on τ. The B-matrix changes in the analysis where κ_{RY} or κ_Y is changed. Equation (A.3.3) shows that a change in κ_{RY} results in a linear transformation of B and therefore does not affect the "mismatch" factor.

[20] As explained in the literature review, this effect of annual temperatures on sovereign bond returns has been examined in prior work as well, as warmer weather can lead to more natural disasters (e.g., Kizys et al. (2024); Boehm (2022)).

impacted by a climate news index of Engle et al. (2020). More news coverage about topics such as physical changes of the planet or regulatory developments may reflect both more actual climate risk and perceived risk (due to more awareness). As Engle et al. (2020) find a link with financial markets, the index can be an interesting alternative risk factor.[21]

We will re-estimate our model based on this index as the climate factor instead of temperature deviations. Section 3.7.1 describes the data. Section 3.7.2 presents the Kalman filter estimates. Section 3.7.3 computes the impact of uncertainty about the climate effect on nominal interest rates.

3.7.1 Data

The climate news index of Engle et al. (2020) measures the intensity of terms related to climate risk in the Wall Street Journal (WSJ). Their monthly index is publicly available from January 1984 to June 2017. For ease of interpretation, we rescale the index by a factor of 1,000. The resulting series is shown in Figure 4. For our analysis, we transform the monthly index to quarterly data by computing the quarterly average of the index.[22] Using monthly observations can namely result in unrealistically high estimates of the mean-reversion parameter, because short time intervals make it difficult to reliably estimate the mean-reversion parameter. In fact, monthly observations could lead to mean-reversion parameter estimates exceeding 15. Quarterly observations offer a balance as they provide more information than yearly data, but still result in reliable mean-reversion parameters. Using the quarterly frequency aligns with the quarterly observed expected inflation in the affine interest rate model of Pennacchi (1991). Henceforth, we refer to this quarterly average as the WSJ-index. Table 8a presents the summary statistics of the WSJ-index.

We first test the WSJ-index for the presence of a unit root and a structural break using a 1% significance value. The Augmented Dickey-Fuller test rejects the null hypothesis of a unit root in the series. To asses structural breaks, we apply the Log-Likelihood Ratio (LLR) test under the assumption that the index follows an ARMA(1,1)-process, in line with our subsequent analysis where we assume that we cannot perfectly measure the climate index.[23] Given the limited sample size of 134 observations, the ARMA(1,1)-estimations

[21]Engle et al. (2020) report a link with stock prices: they construct a hedge portfolio of stocks that effectively hedges changes in the climate news index.

[22]Note that a quarterly average of the news index is not a true average in the statistical sense, as the index is based on a fraction. Nevertheless, we believe that the quarterly average remains a reasonable proxy for climate risk (awareness).

[23]Hamilton (1994) discusses the relationship between the Kalman filter with measurement errors in an observed factor and ARMA-processes.

on subsamples before and after a potential break produced unreliable mean-reversion estimates. Consequently, we only test for a structural break in the variance, which will be relevant in our later estimation procedure. We only apply the test for potential structural breaks before 2005, since mean-reverting processes require long time spans, and post-2005 subsamples did not result in reliable parameter estimates.

The LLR-test rejects the null hypothesis of a constant variance at several breakpoints. This means that we find one structural break, but with an uncertain timing. We focus on a breakpoint between Q1 1994 and Q4 2004, because in this time span the test consistently rejects the null hypothesis. Among three major climate-relevant news announcements in Q2 1994, Q1 1998, and Q2 2001, the test statistic is the most significant for the breakpoint in Q2 2001.[24] However, starting the post-break sample in Q2 1994 provides a longer estimation period, improving the reliability of mean-reversion parameter estimates. To balance this statistical significance and sample size, we select Q1 1998 as breakpoint, the first quarter after the adoption of the Kyoto Protocol. The alternative choices of 1994 or 2001 lead to similar qualitative results.

Table 8a presents summary statistics of the WSJ-index for the total data sample and the post-break sample. Moreover, Table 9 reports the corresponding ARMA-estimates. It shows that, compared to the full sample, the annual standard deviation of the ARMA-process increases from 2.29 to 2.61 when the process is estimated on data from Q1 1998 onwards.

Although we identify a structural break, a longer time span than 1998-2017 may still improve the model parameters due to the longer time span to estimate mean-reversion parameters.[25] Therefore, for robustness, we estimate the model on the full data sample as well. For completeness, the summary statistics of the bond yields from January 1984 and March 2001 onwards are reported in Table 8b and Table 8c, respectively. The tables show that bond yields for these periods are, on average, lower and less volatile than in our initially considered period, for which the summary statistics were shown in Table 1c.

3.7.2 Estimation

Table 10 shows the estimation results of the Kalman filter. We first consider the estimations for the B-model in the columns labelled “B”, which are estimated on bond yields only without a climate impact. Compared to our initial estimation based on interest rates

[24]These major climate-relevant news announcements are provided in Figure 2 of Engle et al. (2020).

[25]For example, typical time ranges for affine interest models are 46 years in Sangvinatsos and Wachter (2005), and 36 years in Duffee (2002).

during 1960-2024, the mean-reversion parameter remains (relatively) small at 0.030 (1984-2017) or 0.121 (1998-2017), still indicating a persistent interest rate process. Moreover, the 10-year bond risk premiums remains similar, 3.6% (1984-2017) and 2.9% (1998-2017), compared to 3.1% in our initial estimation.

Consecutively, we estimate the B+C-model, where we include the observed de-meaned WSJ-index as a climate factor Y_t. We de-mean the WSJ-index since we assumed a zero mean of the second risk factor in equation (2). Subtracting the constant mean does not affect the remaining estimates. Because the WSJ-index is a proxy for climate risk (awareness) and hence difficult to measure precisely, we allow for measurement errors in the climate factor, denoted by h_Y. In terms of notation, the form of the measurement error matrix takes the form $H = \text{diag}(h_B^2, h_B^2, h_B^2, h_B^2, h_Y^2)$, where h_B denotes the measurement error in the yields of each of the four bonds with different maturities.

Moreover, we impose two lower bounds in the B+C model. First, we set the lower bound on the long-term average interest rate $\bar{R}$ equal to the estimated value of $\bar{R}$ in the B-model. This restriction simplifies the comparison of both models. Second, we set the lower bound on the volatility of the climate factor equal to the implied volatility in the ARMA-estimations of the fitted WSJ-index, after removing measurement noise. Equation (A.3.43) in Appendix A.3.2.2 shows how to compute this implied volatility. The corresponding lower bounds are equal to $\sigma_Y \geq 1.34$ for the 1984-2017 sample and $\sigma_Y \geq 1.53$ for the 1998-2017 sample. These bounds prevent that the estimates in the Kalman filter approach zero.

We first consider the estimates based on the data from 1998 onwards. Table 10 shows the estimation results. The interest rate process remains similar to the one in the B-model, and results in a similar risk premium equal to 2.8%, compared to 2.9% in the B-model. The climate factor process is less persistent than the fitted interest rate, with a mean-reversion parameter of $\kappa_Y = 0.35$, compared to $\kappa_R = 0.12$. The climate factor's volatility parameter is equal to the imposed lower bound, $\sigma_Y = 1.53$. The fitted climate factor is shown in Figure 5, combined with the observed WSJ-index. The RMSE between the fitted and observed climate factor equals $\sqrt{MSE_Y} = 1.36$.

We then consider the impact of the fitted climate factor on the implied interest rates. The estimated error correlation is positive, so that $\sigma_{RY} = 0.008$.[26] Moreover, the feedback parameter is positive, $\kappa_{RY} = 0.005$. As a result of the positive feedback parameter, a high WSJ-index is associated with lower future interest rates. For example, if the

[26]Technically, σ_{RY} is not the correlation itself, but a positive correlation results in a positive σ_{RY}. The definition of σ_{RY} is given in equation (2).

climate factor Y_t increases with one standard deviation, the nominal interest rate's drift term changes with $-\kappa_{RY} \cdot \sigma_Y = -0.8\%$. As mentioned in the literature review, the possible economic interpretation of the negative relationship is that more climate risk increases the demand for risk-free assets due to declining productivity and increasing precautionary savings. This decreases interest rates.[27] Compared to the B-model without a climate impact, including the relationship between the interest rate and climate risk improves the fit of the model. The RMSE of the bond yields substantially reduces from $\sqrt{MSE_B} = 0.0034$ to $\sqrt{MSE_B} = 0.0017$, and the measurement error of the bond yields reduces from $h_B = 0.0040$ to $h_B = 0.0016$.

For the full data sample of 1984-2017, the qualitative results remain similar. For example, the estimated impact of one standard deviation increase in the climate factor on the interest rate's drift term is -0.9% per year, compared to -0.8% in the post-1998 sample. Therefore, the estimates in the B+C-model appear to be quite stable.

Hence, when the WSJ-index is included as a factor, the model estimates imply that a high WSJ-index is associated with decrease in the interest rate's drift term. The fit of the interest rates in the Kalman filter improves significantly when taking into account the impact of climate news. These findings are consistent among the full data sample and the post-1998 data. In the next section, we compute the impact of uncertainty about the impact of the WSJ-index process.

3.7.3 Uncertainty about climate impact

Following the same approach as in the initial analysis based on temperature data, we compute the impact of uncertainty about the impact of climate news. We apply the estimates based on the data of 1984-2017. However, employing estimates based on the post-1998 period does not change the qualitative conclusions.

First, we note that the absolute levels of the CER of the speculative investor are substantially higher than in the initial computations based on the 1960-2024 sample. This is due to the relatively large MPoR of the interest rate in the period from 1984 onwards, resulting in extreme speculative bond demands. These demands considerably increase the number of simulations paths that simulate "explosive" wealth processes over the investment horizon.[28] To address this, for the risk aversion level $\gamma = 1$ we maximise the

[27] Recall that Mongelli et al. (2023) offer an overview of literature about the impact of climate change and climate policies on model-implied interest rates. They conclude that it is most commonly to find a negative relationship between climate change and interest rates.

[28] Shortening the time intervals to daily compounding or increasing the number of simulations did not

terminal wealth at a maximum value of 1,000, while we use an initial wealth of 1. This adjustment does not affect the qualitative comparison between strategies, as the wealth is maximised only in the optimal strategies (investing based on the correct parameters of the actual financial market). We are interested in the impact of suboptimal strategies. However, it shows that the speculative bond demands are very large, especially in the two-bond strategy based on the B+C-model. For example, in case of the B-model, the upper bound was binding in 17% of the simulations, and in case of the B+C-model even in 63% of the simulations.

Table 11 presents the impact on the CER when the investor's strategy and the financial market are based on the model without or with the climate factor. For less speculative investors with $\gamma > 1$, the robust strategy is to invest based on the model with the impact of the climate factor. For example, an investor with risk aversion level $\gamma = 5$ faces a CER of 10.5% if she invests based on the B-model and the financial market indeed follows that model. However, if the actual financial market process is also based on the climate factor, her CER decreases to 4.1%. By investing in the robust strategy based on the B+C-model, she prevents this drop: even in the worst case where the financial market follows the B-model, she still faces a CER of 5.3%. In contrast, for the speculative investor with $\gamma = 1$, the robust strategy is to invest based on the B-model without impact of climate news.

Hence, if the investor considers two plausible models, namely the model without and with impact of climate news, the robust strategy depends on the risk aversion coefficient. For the more risk-averse investor ($\gamma > 1$), the robust strategy is to invest based on the model with the climate news impact. In contrast, for the speculative investor, the robust strategy is to invest like there is no climate news impact.

3.8 Conclusion

We show that it is hard to estimate the precise impact of temperature deviations on the instantaneous interest rate. This makes parameter uncertainty relevant. To investigate the impact on utility in case of this parameter uncertainty, we consider an investor who is uncertain about the impact of temperature deviations on her bond returns. If she invests like a climate denier in a financial market where the interest rate is actually impacted by temperature, the utility of a (more) speculative investor decreases compared to the case in which she would have invested like a climate believer. The robust strategy is to invest

solve this numerical issue.

like a climate believer, independent of whether the investor believes in it, and independent of the choice of bond maturities.

Moreover, we consider a climate believer who invests in a financial market that includes a temperature impact, but who is uncertain about the parameters of the underlying temperature process. In most cases, parameter uncertainty appears to have a negligible impact. However, when a more speculative investor incorrectly estimates the correlation between shocks in temperature deviations and interest rates to be negative instead of positive, this can have a large impact on utility. The robust strategy is to invest in a strategy that would be optimal under the initial estimates for the speculative investor, and under the negative correlation for the more risk-averse investor. The robust strategy is independent of the choice of bond maturities.

Finally, we consider an investor who observes the impact of climate risk via an alternative climate factor, namely a climate news index instead of temperature deviations. We show that if the index is above its long-term mean, the interest rate's drift term decreases. The robust strategy for a speculative investor is now to invest like a climate denier, but for the more risk-averse investors the robust strategy remains to invest like a climate believer.

For future research, it might be interesting to consider simultaneous changes in parameters. A change in one parameter might have a small impact, but could result in a larger impact when it occurs simultaneously with a change in another parameter. Moreover, the model could include inflation as additional risk factor to analyse the real interest rate instead of the nominal rate. However, this improvement would makes the model more complex because of interactions between the nominal interest rate, inflation, and climate risk, given that even the current model is already difficult to estimate precisely. Finally, it would be valuable to test other climate news indices, such as the one of Bua et al. (2024), which distinguishes between news about transition and physical risks.

Table 1. Summary statistics of source data. Summary statistics of the source data: (a) 64 annual average global surface and ocean temperature anomalies w.r.t. the period from 1951 to 1980, presented across different periods based on the structural break in 1960, and (b) the de-trended temperature anomalies between 1960-2024, and (c) 771 monthly observed U.S. interest rates from January 1960 to March 2024, shown per bond maturity ("Mat") in years and expressed in percent per year.

(a) Temperature

	1880-1960	**1960-2024**	**1880-2024**
Mean	-0.18	0.38	0.07
Std	0.16	0.34	0.39
Min	-0.48	-0.20	-0.48
Max	0.21	1.17	1.17

(b) De-trended temperature 1960-2024

	1960-2024
Mean	0
Std	0.10
Min	-0.21
Max	0.24

(c) Interest rates

	1960-2024			
Mat	0.25	1	5	10
Mean	4.43	4.82	5.46	5.89
Std	3.22	3.30	3.07	2.84
Min	0.01	0.02	0.25	0.70
Max	16.00	16.35	15.70	15.07

Table 2. Kalman filter results. Results of the Quasi-Maximum Likelihood estimates of the process $dX_t = -K_X dt + \sigma_X dZ_t$, obtained through the Kalman filter using monthly U.S. bond returns and yearly global temperatures over the period 1960 to 2024. The B+C-model is the initial unrestricted model based on two risk factors, a latent factor for the nominal interest rate R_t and temperature deviations Y_t. For comparison, the restricted B+C-model ($\sigma_{RY} = 0$) represents the model under the assumption of uncorrelated shocks in the risk factors and a maximum σ_Y of 0.16, and the B-model under the assumption that $\sigma_{RY} = \kappa_{RY} = 0$. The numbers in brackets show the standard errors of the parameters, determined by the finite difference method and the delta method. All yields are assumed to be measured with error, where h is the error size in measurement matrix $H = h^2 * I$. The last row shows the root mean squared errors (MSE) of the fitted interest rate compared to the observed 3-month bond yield. The last rows show the implicit bond risk premium components w.r.t. the first and second risk factor (as shown in equation (15)), and the implied change in the interest rate's drift term as a response to one standard deviation change in the temperature deviations.

	B+C	**B+C ($\boldsymbol{\sigma_{RY} = 0}$)**	**B**
$\bar{R}$	0.041 (0.041)	0.049 (0.038)	0.042 (0.033)
κ_R	0.030 (0.002)	0.035 (0.002)	0.033 (0.002)
κ_{RY}	-0.009 (0.034)	0.156 (0.099)	
κ_Y	1.12 (0.07)	2.58 (0.07)	
σ_R	0.010 (0.0006)	0.023 (0.0006)	0.011 (0.0006)
σ_{RY}	0.004 (0.02)		
σ_Y	0.160 (0.10)	0.160 (0.10)	
λ_R	-0.15 (0.14)	-0.70 (0.13)	-0.33 (0.11)
λ_Y	-0.10 (0.04)	-1.24 (0.03)	
h	0.010 (0.0001)	0.006 (0.0001)	0.006 (0.0001)
$\sqrt{\text{MSE}}$	0.008	0.012	0.007
RP_R (10YR)	1.3%	13.7%	3.1%
RP_Y (10YR)	0.4%	-9.8%	
$-\kappa_{RY} \cdot \sigma_Y$	0.1%	-2.5%	

Table 3. Impact uncertainty about climate impact. Impact on the certainty equivalent return (CER, shown in %) on wealth when the investor applies bond allocations x_t^m based on θ^m, and the financial market parameter set is θ^n, which is based on either the B-model estimated on bond yields only, or the B+C-model including the impact of temperature deviations on the nominal interest rate. The results are presented for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, with bond maturities set to $\tau = (3, 30)$ in the two bond strategy and $\tau_1 = 30$ in the one bond strategy. The robust choice *rob* of the investment strategy *m* is shown in the last column and can be found as follows: per investment strategy *m*, choose set $n \in$ {B,B+C} that results in the worst-case financial market parameter set with the lowest CER, shown in **bold**. Consecutively, determine *rob* that leads to the maximum of those two worst-case CERs.

x_t^m	m	B		B+C		
θ^n	n	B	B+C	B	B+C	*rob*
γ	1	10.2	**4.7**	7.9	**5.9**	B+C
	2	8.9	**5.1**	7.7	**5.7**	B+C
	5	8.1	**5.3**	7.6	**5.6**	B+C
	10	7.8	**5.4**	7.6	**5.5**	B+C
	25	7.7	**5.4**	7.6	**5.5**	B+C

Table 4. Alternative values w.r.t. temperature process. Alternative values of temperature-related parameters which are computed with equations (16)-(20), based on the Best Estimates and the standard errors of these estimates. The alternative values of $\tilde{\sigma}_{RY}^-$ and $\tilde{\sigma}_{RY}^+$ are based on the constraint given in equation (21).

	$(...)^-$	$(...)^+$
κ_{RY}	-0.078	0.059
κ_Y	0.977	1.263
σ_{RY}	-0.021	0.028
$\tilde{\sigma}_{RY}$	-0.007	0.007
σ_Y	0.000	0.362
λ_Y	-0.175	-0.030

Table 5. Impact alternative values w.r.t. temperature process. Impact on the certainty equivalent return (CER, shown in %) on wealth when the investor applies bond allocations x_t^m based on θ^m, and the financial market parameter set is θ^{B+C}. The different parameter sets correspond to the Best Estimates and alternative temperature-related parameter values. The CER is shown for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, with bond maturities set to $\tau = (3, 30)$.

x_t^m	m	B+C	κ_{RY}^-	κ_{RY}^+	κ_Y^-	κ_Y^+	σ_{RY}^-	σ_{RY}^+	$\tilde{\sigma}_{RY}^-$	$\tilde{\sigma}_{RY}^+$	σ_Y^-	σ_Y^+	λ_Y^-	λ_Y^+
γ	1	5.9	5.9	5.8	5.9	5.9	-1.4	-1.1	4.6	5.8	5.9	5.9	5.6	5.7
	2	5.7	5.7	5.6	5.7	5.7	2.0	2.2	5.1	5.7	5.7	5.7	5.6	5.6
	5	5.6	5.6	5.5	5.6	5.6	4.1	4.1	5.3	5.6	5.6	5.6	5.5	5.5
	10	5.5	5.5	5.5	5.5	5.5	4.8	4.8	5.4	5.5	5.5	5.5	5.5	5.5
	25	5.5	5.5	5.5	5.5	5.5	5.2	5.2	5.4	5.5	5.5	5.5	5.5	5.5

Table 6. Robust strategy w.r.t. $\boldsymbol{\sigma_{RY}}$. Impact on the certainty equivalent return (CER, shown in %) on wealth when the financial market parameter set is θ^n, and the investor applies bond allocations x_t^m based on θ^m. The different parameter sets correspond to the Best Estimate σ_{RY}^0 and alternative values $\tilde{\sigma}_{RY}^-$ and $\tilde{\sigma}_{RY}^+$. The CER is shown for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, with bond maturities set to $\tau = (3, 30)$. The robust choice *rob* of the investment strategy m is shown in the last column and can be found as follows: per investment strategy m, choose set $n \in \{\sigma_{RY}^0, \tilde{\sigma}_{RY}^-, \tilde{\sigma}_{RY}^+\}$ that results in the worst-case financial market parameter set with the lowest CER, shown in **bold**. Consecutively, determine *rob* that leads to the maximum of those three CERs.

x_t^m	m	σ_{RY}^0			$\tilde{\sigma}_{RY}^-$			$\tilde{\sigma}_{RY}^+$			
θ^n	n	σ_{RY}^0	$\tilde{\sigma}_{RY}^-$	$\tilde{\sigma}_{RY}^+$	σ_{RY}^0	$\tilde{\sigma}_{RY}^-$	$\tilde{\sigma}_{RY}^+$	σ_{RY}^0	$\tilde{\sigma}_{RY}^-$	$\tilde{\sigma}_{RY}^+$	*rob*
γ	1	5.9	**4.7**	5.8	4.6	5.9	**3.7**	5.8	**3.8**	5.9	$\tilde{\sigma}_{RY}^0$
	2	5.7	**4.5**	5.7	5.1	5.1	**4.6**	5.7	**4.0**	5.7	$\tilde{\sigma}_{RY}^-$
	5	5.6	**4.3**	5.6	5.3	**4.6**	5.2	5.6	**4.1**	5.6	$\tilde{\sigma}_{RY}^-$
	10	5.5	**4.2**	5.6	5.4	**4.4**	5.4	5.5	**4.1**	5.6	$\tilde{\sigma}_{RY}^-$
	25	5.5	**4.2**	5.5	5.4	**4.3**	5.4	5.5	**4.2**	5.5	$\tilde{\sigma}_{RY}^-$

Table 7. Impact uncertainty about climate impact for varying bond maturities. Impact on the certainty equivalent return (CER, shown in %) on wealth when the financial market parameter set is θ^n, which is based on either the B-model estimated on bond yields only, or the B+C-model including the impact of temperature deviations on the nominal interest rate. The investor applies bond allocations x_t^m based on θ^m. The results are presented for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, with bond maturities set to $\tau_1 \in \{3, 10\}$ in the one bond strategy and $\tau \in \{(3, 10); (10, 30)\}$ in the two bond strategy. The robust choice *rob* of the investment strategy m is shown in the last column and can be found as follows: per investment strategy m, choose set $n \in \{B, B + C\}$ that results in the worst-case financial market parameter set with the lowest CER, shown in **bold**. Consecutively, determine *rob* that leads to the maximum of those four worst-case CERs.

x_t^m	m	B ($\tau_1 = 3$)		B ($\tau_1 = 10$)		B+C ($\tau = (3, 10)$)		B+C ($\tau = (10, 30)$)	
θ^n	n	B	B+C	B	B+C	B	B+C	B	B+C
γ	1	10.2	**4.9**	10.2	**4.8**	8.5	**5.9**	6.0	**5.9**
	2	8.9	**5.2**	8.9	**5.2**	6.8	**5.7**	6.8	**5.7**
	5	8.1	**5.4**	8.1	**5.4**	7.3	**5.6**	7.3	**5.6**
	10	7.8	**5.5**	7.8	**5.4**	7.4	**5.5**	7.4	**5.5**
	25	7.7	**5.4**	7.7	**5.5**	7.5	**5.5**	7.5	**5.5**

Table 8. Summary statistics of source data for news index analysis. Summary statistics of the source data, expressed on an annual basis: (a) 134 observations from Q1 1984 to Q2 2017 and 78 observations from Q1 1998 to Q2 2017 of the WSJ-index. The row "Std(Δ)" presents the standard deviation of the differenced time series. The WSJ-index is the quarterly average of the climate news Wallstreet Journal index provided by Engle et al. (2020), rescaled by a factor 1,000. (b) 402 monthly observed U.S. interest rates from January 1984 to June 2017 from the FRED, shown per bond maturity ("Mat") in years and expressed in percent. (c) 195 monthly interest rates from January 1998 to June 2017.

(a) WSJ-index

	1984-2017	**1998-2017**
Mean	5.4	6.1
Std	2.3	2.5
Std (Δ)	1.8	2.1
Min	3.3	3.6
Max	12.9	12.9

(b) Interest rates (1984-2017)

Mat	**0.25**	**1**	**5**	**10**
Mean	3.66	4.00	4.98	5.53
Std	2.83	2.94	2.83	2.59
Min	0.00	0.09	0.59	1.50
Max	10.82	11.99	13.44	13.62

(c) Interest rates(1998-2017)

Mat	**0.25**	**1**	**5**	**10**
Mean	1.97	2.17	3.09	3.76
Std	2.08	2.05	1.65	1.33
Min	0.00	0.09	0.59	1.50
Max	6.38	6.37	6.71	6.66

Table 9. ARMA-model parameters for the WSJ-index. Estimates from the ARMA(1,1)-model $Y_t = \alpha + \phi Y_{t-1} + \theta \varepsilon_{t-1} + \varepsilon_t$ applied to the WSJ-index, expressed on an annual basis. The variable σ_ε denotes the standard deviation of the error term ε_t. For the subsample of 1998-2017, the parameters α, ϕ, and θ are fixed at their values from the full sample of 1984-2017. Hence, only σ_ε is re-estimated.

	1984-2017	**1998-2017**
α	0.64	0.64
ϕ	0.88	0.88
θ	-0.41	-0.41
σ_ε	2.29	2.61

Table 10. Kalman filter results based on news index. Results of the Kalman filter, with the yields only model (B) under the assumption that there is no link between the nominal interest rate R_t and the climate news index. The model B+C includes the de-meaned WSJ-index as a second risk factor. The results are shown for both January 1984 to June 2017 and January 1998 to June 2017. The numbers in brackets show the standard errors of the parameters, determined by the finite difference method and the delta method. All yields and the WSJ-index are assumed to be measured with error, where h_B is the error size of the four bond yields and h_Y the error size of the WSJ-index. These error sizes form the measurement matrix $H = \text{diag}(h_B^2, h_B^2, h_B^2, h_B^2, h_Y^2)$. The root mean squared errors (MSE) are shown for the fitted interest rate compared to the observed 3-month bond yield, and the fitted climate factor compared to the observed one. The last rows show the implicit bond risk premium components w.r.t. the first and second risk factor (as shown in equation (15)), and the implied change in the interest rate's drift term as a response to one standard deviation change in the climate factor Y_t.

	1984-2017				**1998-2017**			
	B		**B+C**		**B**		**B+C**	
$\bar{R}$	0.049	(0.009)	0.049	(0.018)	0.024	(0.008)	0.024	(0.021)
κ_R	0.030	(0.003)	0.070	(0.005)	0.121	(0.006)	0.120	(0.010)
κ_{RY}			0.007	(0.009)			0.005	(0.002)
κ_Y			0.42	(0.19)			0.35	(0.04)
σ_R	0.009	(0.001)	0.01	(0.0004)	0.008	(0.001)	0.01	(0.001)
σ_{RY}			0.008	(0.096)			0.008	(0.001)
σ_Y			1.34	(0.04)			1.53	(0.37)
λ_R	-0.48	(0.01)	-0.59	(0.14)	-0.65	(0.01)	-0.51	(0.27)
λ_Y			-0.30	(0.07)			-0.02	(0.08)
h_B	0.0053	(0.0001)	0.0017	(0.0001)	0.0040	(0.0001)	0.0016	(0.0001)
h_Y			1.70	(0.07)			1.87	(0.13)
$\sqrt{\text{MSE}_B}$	0.0037		0.0018		0.0034		0.0017	
$\sqrt{\text{MSE}_Y}$			1.32				1.36	
RP_R(10YR)	3.6%		4.6%		2.9%		2.8%	
RP_Y(10YR)			-2.2%				-0.1%	
$-\kappa_{RY} \cdot \sigma_Y$			-0.9%				-0.8%	

Table 11. Impact uncertainty about climate impact based on news index. Impact on the certainty equivalent return (CER, shown in %) on wealth when the investor applies bond allocations x_t^m based on θ^m, and the financial market parameter set is θ^n, which is based on either the bond yields only (B) or the bonds yields and climate factor (B+C). The parameters are estimated on data from 1984 to 2017. The results are presented for risk aversion coefficients $\gamma \in \{1, 2, 5, 10, 25\}$, with bond maturities set to $\tau = (10, 30)$ in the two bond strategy and $\tau_1 = 30$ in the one bond strategy. The robust choice *rob* of the investment strategy m is shown in the last column and can be found as follows: per investment strategy m, choose set $n \in \{\text{B,B+C}\}$ that results in the worst-case financial market parameter set with the lowest CER, shown in **bold**. Consecutively, determine *rob* that leads to the maximum of those two worst-case CERs.

x_t^m	m	B		B+C		
θ^n	n	B	B+C	B	B+C	*rob*
γ	1	15.9	**-2.8**	**-8.9**	20.1	B
	2	12.7	**1.4**	**-0.2**	16.7	B+C
	5	10.5	**4.1**	**5.3**	10.6	B+C
	10	9.7	**5.0**	**7.2**	8.5	B+C
	25	9.3	**5.4**	8.2	**7.2**	B+C

Figure 1. Observed temperatures and interest rates. Graphical representation of the source data: (a) average temperature anomalies w.r.t. the period from 1951 to 1980 temperatures, the time trends based on the period before and after the structural break in 1960 (dashed lines), and the time trend without a structural break (solid line); and (b) U.S. term structure yields from the McCulloch & Kwon data (1960-1991) and FRED (1992-2024). The yields are shown for the bond maturities of 0.25, 1, 5 and 10 years as indicated in the legend.

(a) Temperature **(b)** Interest rates

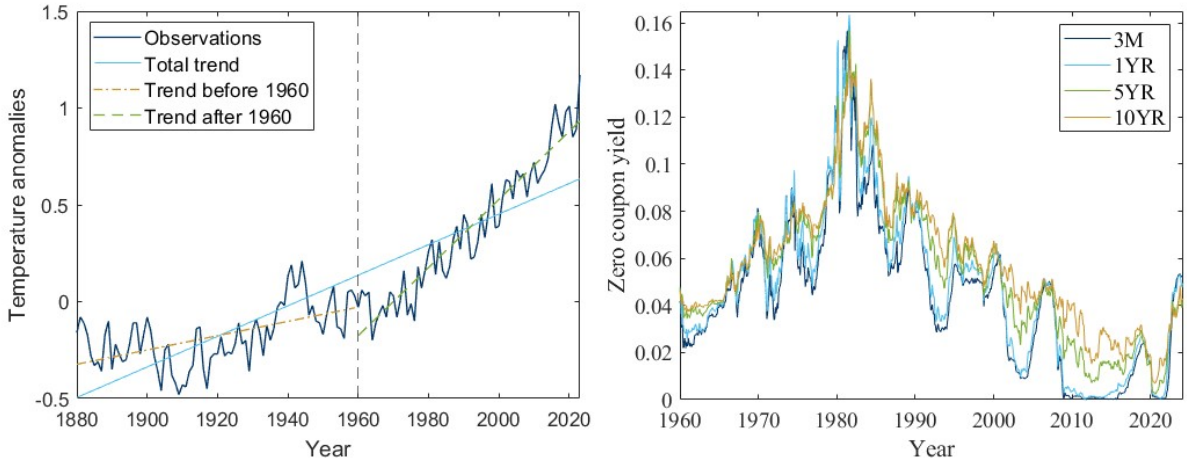


Figure 2. Fitted versus observed interest rates and temperatures. The predicted factors R_t and Y_t, estimated by the unrestricted Kalman filter, combined with the monthly observed 3M bond yields and yearly observed de-trended temperatures.

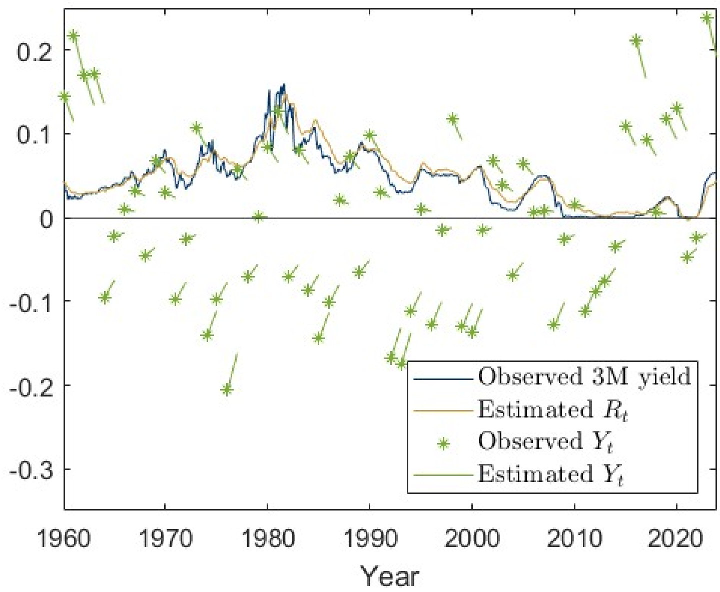


Figure 3. Optimal bond allocations for two values of $\boldsymbol{\sigma_{RY}}$. Optimal bond allocations in the B+C-model, shown for risk aversion coefficients $\gamma \in \{1, 2, 25\}$. The solid lines represent the allocations for the 3-year bond maturity, while the dashed lines represent the 30-year bond maturity. The bond allocations are shown for the initial values indicated by σ^0_{RY} (yellow lines) and the alternative value $\tilde{\sigma}^-_{RY}$ (blue lines).

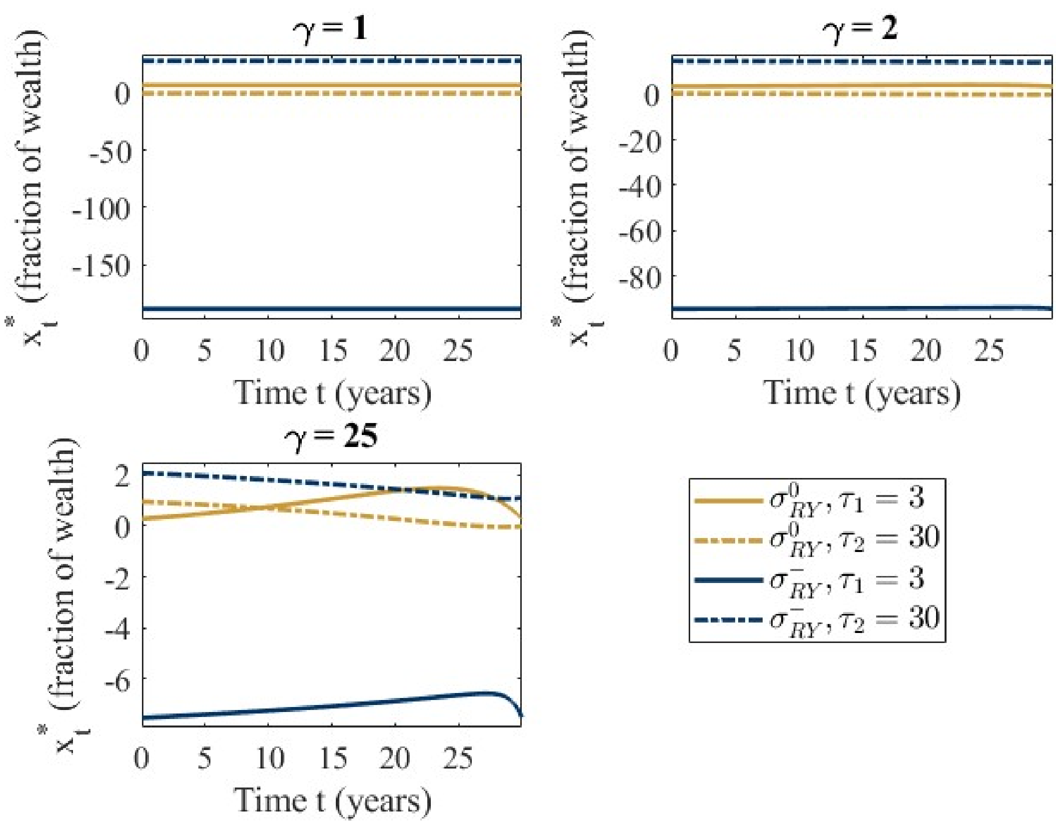


Figure 4. Climate news index. Graphical representation of the source data of the news index: 402 monthly Wallstreet Journal climate news indices from January 1984 to June 2017, provided by Engle et al. (2020), rescaled by a factor 1,000.

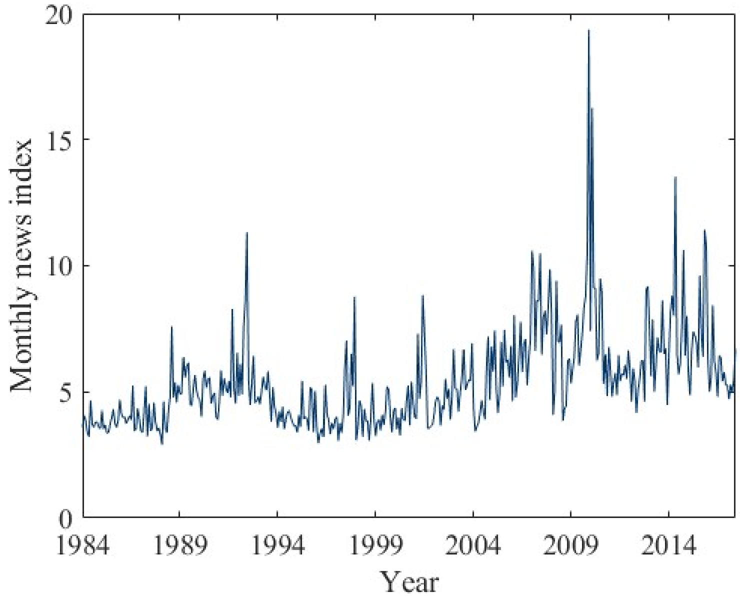
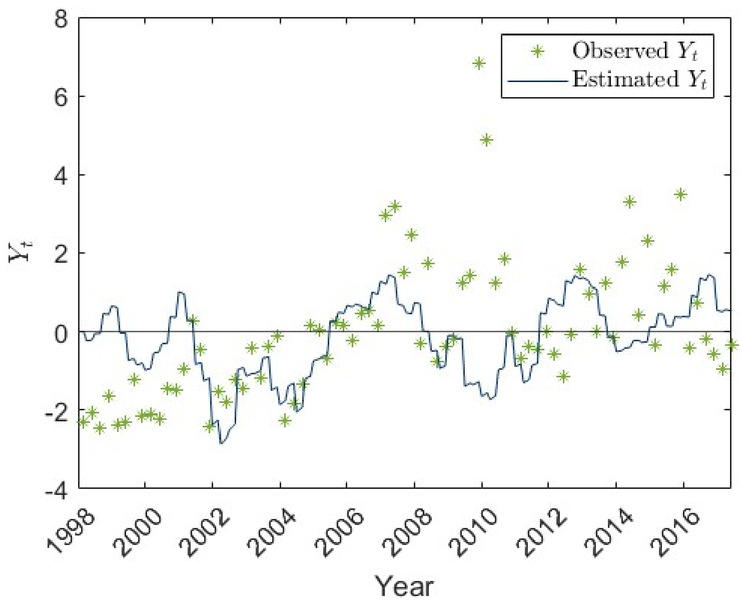


Figure 5. Predicted versus observed WSJ-index. The predicted factor de-meaned WSJ-index Y_t and the quarterly observed de-meaned WSJ-index. The predicted factor is estimated by the Kalman filter based on data from January 1998 to June 2017.



A
Appendix

A.1 Supplementary material for Chapter 1

A.1.1 Derivations $B(\tau_j)$ and $\tilde{B}(T-t)$

This appendix shows how to compute the bond and wealth exposures to the risk factors with regard to the nominal and real interest rate. Sangvinatsos and Wachter (2005) show that for bond maturity τ_j, the exposure to both risk factors with regard to the nominal interest rate can be computed based on the Ordinary Differential Equation:[1]

$$B'(\tau_j) = -K\,B(\tau_j) + \delta_R \tag{A.1.1}$$

which implies

$$B(\tau_j) = -K^{-1}(e^{-K\tau_j} - I)\delta_R \tag{A.1.2}$$

To find the analytical solution, let V be a matrix that contains the column vectors of the right eigenvectors of mean-reversion parameter matrix K, and let K_d be a diagonal matrix with the eigenvalues of K corresponding to these eigenvectors on the diagonal entries. Since K is a diagonalizable matrix, we know by linear algebra that we can express K in these matrices:

$$K = VK_dV^{-1} \tag{A.1.3}$$

$$\rightarrow -\tau_j K = V\ -\tau_j K_d V^{-1} \tag{A.1.4}$$

$$\rightarrow \exp(-\tau_j K) = V\ \exp(-\tau_j K_d)V^{-1} \tag{A.1.5}$$

Furthermore, since K is upper triangular, the eigenvalues are κ_R and κ_π. Therefore:

$$K_d = \begin{bmatrix} \kappa_R & 0 \\ 0 & \kappa_\pi \end{bmatrix}; V = \begin{bmatrix} 1 & -\frac{\kappa_{R\pi}}{\kappa_R - \kappa_\pi} \\ 0 & 1 \end{bmatrix} \tag{A.1.6}$$

And the matrix $\exp(-\tau_j K_d)$ equals

$$\exp(-\tau_j K_d) = \begin{bmatrix} e^{-\tau_j \kappa_R} & 0 \\ 0 & e^{-\tau_j \kappa_\pi} \end{bmatrix} \tag{A.1.7}$$

[1] Note that we leave out the term '$(1-\gamma)$' that is given in the equation of Sangvinatsos and Wachter (2005) in front of their term $(\delta_R - \delta_\pi)$. This is because we use this term in the optimal asset allocation formula, resulting in the same optimal asset allocation results.

For the nominal bond exposure this results in

$$B(\tau_j) = \begin{bmatrix} \frac{1}{\kappa_R}(1 - e^{-\tau_j \kappa_R}) \\ -\frac{\kappa_{R\pi}}{\kappa_R \kappa_\pi}(1 - e^{-\tau_j \kappa_R}) - \frac{\kappa_{R\pi}}{\kappa_\pi(\kappa_R - \kappa_\pi)}(e^{-\tau_j \kappa_R} - e^{-\tau_j \kappa_\pi}) \end{bmatrix} \tag{A.1.8}$$

The exposures to the risk factors regarding the real interest rate for the remaining time horizon $T - t$ are determined by the ODE:

$$\tilde{B}'(T - t) = -K\,\tilde{B}(T - t) + \delta_R - \delta_\pi \tag{A.1.9}$$

Similarly as for $B(\tau_j)$, this results in:

$$\tilde{B}(T - t) = \begin{bmatrix} \frac{1}{\kappa_R}(1 - e^{-(T-t)\kappa_R}) \\ -\frac{\kappa_{R\pi} + \kappa_R - \kappa_\pi}{\kappa_R - \kappa_\pi}\frac{1}{\kappa_\pi}(1 - e^{-(T-t)\kappa_\pi}) + \frac{\kappa_{R\pi}}{\kappa_R - \kappa_\pi}\frac{1}{\kappa_R}(1 - e^{-(T-t)\kappa_R}) \end{bmatrix} \tag{A.1.10}$$

A.1.2 Transformation from real to nominal interest rate

For the parameter set θ^0 in Table 1 we use the estimates of Brennan and Xia (2002). However, these estimates are based on the parameters $(\kappa_r; \sigma_r; \rho_{r\pi}; \lambda_r)$ corresponding to the real interest rate $r_t = R_t - \pi_t$. This section shows the transformation to the relevant estimates of the nominal interest rate process, $(\kappa_R; \sigma_R; \rho_{R\pi}; \lambda_R)$. Because the nominal interest rate equals the sum of the real interest rate and the expected inflation, we apply a linear transformation to the nominal risk factor vector with the matrix $L = [1, 1; 0, 1]$ and assume that all long-term averages $\bar{R}, \bar{r}$, and $\bar{\pi}$ are equal to zero without loss of generality:

$$\begin{aligned} d\begin{bmatrix} R_t \\ \pi_t \end{bmatrix} &= dL\begin{bmatrix} r_t \\ \pi_t \end{bmatrix} = LK_r r_t dt + L\sigma_r dZ_t \\ &= LK_r L^{-1} R_t dt + L\sigma_r dZ_t \end{aligned} \tag{A.1.11}$$

where K_r and σ_r refer to the mean-reversion matrix and volatility matrix of the real interest rate process. From matching the drift term above with the initial nominal risk factor process of equation (2), we get:

$$\kappa_R = \kappa_r \ ; \ \kappa_{R\pi} = \kappa_\pi - \kappa_r \tag{A.1.12}$$

Similarly, from matching the variance-covariance matrix, we get:

$$\sigma_R = \sqrt{\sigma_r^2 + \sigma_\pi^2 + 2\rho_{r\pi}\sigma_r\sigma_\pi} \tag{A.1.13}$$

$$\rho_{R\pi} = \frac{\rho_{r\pi}\sigma_r + \sigma_\pi}{\sigma_R} \tag{A.1.14}$$

Moreover, we match the risk premiums to compute λ_R:

$$\begin{aligned} \lambda_R\sigma_R &= \lambda_r\sigma_r + \lambda_\pi\sigma_\pi + 2\rho_{r\pi}\sigma_r\sigma_\pi\left(\lambda_r + \lambda_\pi\right) \\ \rightarrow \lambda_R &= \frac{\lambda_r\sigma_r + \lambda_\pi\sigma_\pi + 2\rho_{r\pi}\sigma_r\sigma_\pi\left(\lambda_r + \lambda_\pi\right)}{\sigma_R} \end{aligned} \tag{A.1.15}$$

Finally, in the numerical results, we set $\rho_{r\pi}$=0 in the computations above, because the estimate for $\rho_{r\pi}$ in Brennan and Xia (2002) tends to be very low.[2]

[2] As shown in Section 1.4, we do not observe considerable differences between using the original estimate or zero in the bond allocations.

A.2 Supplementary material for Chapter 2

A.2.1 Reconciliation Sangvinatsos and Wachter (2005)

Sangvinatsos and Wachter (2005) (SW) derive the optimal investment strategy and corresponding optimal wealth process in case of an interest rate affine in risk factors and no parameter uncertainty. For completeness, this appendix shows how we use their results to compute the exposure to the risk factors in case of a deterministic distortion and a stochastic distortion in Section A.2.1.1 and A.2.1.2 respectively.

A.2.1.1 Deterministic distortion

SW derive the Ordinary Differential Equations (ODEs) that should be solved to find the risk exposures of bonds and wealth. If we consider the deterministic distortion in the two-factor model where all entries of the 2×2 matrix α are equal to zero, the no-arbitrage condition results in a 2×1 bond exposure vector $B(\tau_j)$ for a bond with maturity τ_j

$$B'(\tau_j) = -K\, B(\tau_j) + \delta_R \tag{A.2.1}$$

This nominal bond exposure contains the exposure to the *nominal interest rate* risk, corresponding to the 2×1 vector $\delta_R = (1, 0)'$. If we consider the nominal wealth excess returns exposed to the *real interest rate* risk for the remaining time horizon $T - t$ instead, the ODE becomes:

$$\tilde{B}_2'(T-t) = -\tilde{B}_2(T-t)\, K + \delta_R - \delta_\pi \tag{A.2.2}$$

with the 2×1 vector $\delta_\pi = (0, 1)'$. For completeness, we show the analytical solutions in case of a *one-factor* model without inflation risk explained in Section 2.4.1:

$$B(\tau_j) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tau_j}\right) \tag{A.2.3}$$

$$\tilde{B}_2(T-t) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \tag{A.2.4}$$

In case of the *two-factor* model explained in Section 2.4.2, the bond exposure equals:

$$B(\tau_j) = \begin{bmatrix} \frac{1}{\kappa_R}(1 - e^{-\tau_j \kappa_R}) \\ -\frac{\kappa_{R\pi}}{\kappa_R \kappa_\pi}(1 - e^{-\tau_j \kappa_R}) - \frac{\kappa_{R\pi}}{\kappa_\pi(\kappa_R - \kappa_\pi)}(e^{-\tau_j \kappa_R} - e^{-\tau_j \kappa_\pi}) \end{bmatrix} \tag{A.2.5}$$

and the real interest rate exposure equals:

$$\tilde{B}_2(T-t) = \begin{bmatrix} \frac{1}{\kappa_R}(1-e^{-(T-t)\kappa_R}) \\ -\frac{\kappa_{R\pi}+\kappa_R-\kappa_\pi}{\kappa_R-\kappa_\pi}\frac{1}{\kappa_\pi}(1-e^{-(T-t)\kappa_\pi}) + \frac{\kappa_{R\pi}}{\kappa_R-\kappa_\pi}\frac{1}{\kappa_R}(1-e^{-(T-t)\kappa_R}) \end{bmatrix} \quad \text{(A.2.6)}$$

A.2.1.2 Stochastic distortion

SW2005	**Our variables**
$r(X(t),t) = \delta_0 + \delta X(t)$	R_t
$dX(t) = K(\Theta - X(t))dt + \sigma_X dz(t)$	$dR_t = -\kappa(R_t - \bar{R})dt + \sigma d_t$
$X(t)$	R_t
Λ_t	λ^*_t
δ_0	0
δ	1
ζ	0
σ_X	σ
K	$\kappa - \sigma\alpha$
λ^*_1	$\lambda - \alpha\bar{R}$
λ^*_2	α
$K + \sigma_X\lambda_2$	κ
$K\theta - \sigma_X\lambda_1$	$\kappa\bar{R} - \sigma\lambda$
$K\theta$	$(\kappa - \sigma\alpha)\bar{R}$

Table A.1. Equations and parameters of Sangvinatsos and Wachter (2005) and our study in case of a one-factor model without inflation. The left column shows the equations and parameters of Sangvinatsos and Wachter (2005), the right columns shows our parameters.

Consecutively, this section shows the ODEs to compute the risk exposures in the *one-factor* model as explained in Section 2.5. The no-arbitrage condition of bond prices remains the same as we assume that the investor estimates the $\mathcal{Q}$-measure correctly:

$$B(\tau_j) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tau_j}\right) \quad \text{(A.2.7)}$$

However, the reconciliation with SW of the real interest rate exposure for the stochastic distortion requires more explanation than the deterministic distortion. This is because in our approach we assume a constant MPoR in the undistorted empirical measure $\mathcal{P}$, whereas SW assume a stochastic MPoR in their empirical measure.

Table A.1 summarises the equations and parameters of SW and the corresponding notation in our study in the case of the one-factor model. Recall that in our financial market the distorted interest rate under $\mathcal{P}^*$ and the undistorted risk-neutral process under

$\mathcal{Q}$ follow the processes:

$$dR_t = (\kappa - \sigma\alpha)(\bar{R} - R_t)dt + \sigma dZ_t^* \tag{A.2.8}$$

$$= \kappa(\bar{R} - \frac{\sigma\lambda}{\kappa} - R_t)dt + \sigma dZ_t^{\mathcal{Q}} \tag{A.2.9}$$

SW assume an undistorted MPoR of $\Lambda(t) = \lambda_1 + \lambda_2 X(t)$ in their equation (2) and therefore implicitly include a kind of 'distortion' in their risk-neutral process well. Applying $dZ_t = dZ_t^{\mathcal{Q}} - \Lambda(t)dt$ results in:

$$dX(t) = K(\theta - X(t))dt + \sigma_X dZ_t \tag{A.2.10}$$

$$= (K + \sigma_X\lambda_2)(\theta^Q - X(t))dt + \sigma_X dZ_t^{\mathcal{Q}} \tag{A.2.11}$$

with $(K + \sigma_X\lambda_2)\theta^Q = K\theta - \sigma_X\lambda_1$. Hence, to reconcile our R_t with their $X(t)$, we match the volatility parameters $\sigma_X = \sigma$. Matching the expressions under $\mathcal{Q}$ in equations (A.2.9) and (A.2.11) results in:

$$\begin{cases} K + \sigma_X\lambda_2 = \kappa \\ K\theta - \sigma_X\lambda_1 = \kappa\bar{R} - \sigma\lambda \end{cases} \tag{A.2.12}$$

The empirical measure in SW is to be understood as our *distorted* empirical measure $\mathcal{P}^*$. Hence, matching equations (A.2.8) and (A.2.10) leads to:

$$\begin{cases} K = \kappa - \sigma\alpha \\ K\theta = (\kappa - \sigma\alpha)\bar{R} \end{cases} \tag{A.2.13}$$

Finally, note that matching (A.2.12) and (A.2.13) results in $\lambda_2 = \alpha$ and $\lambda_1 = \lambda - \alpha\bar{R}$, so that we have a distorted market price of risk:

$$\lambda_t^* = \lambda_1 + \alpha X(t) \text{ (see their eq. (3))} \tag{A.2.14}$$

$$= \underbrace{\lambda}_{\mathcal{Q}\to\mathcal{P}} + \underbrace{\alpha(R_t - \bar{R})}_{\mathcal{P}\to\mathcal{P}^*} \tag{A.2.15}$$

as imposed by our model set-up in the stochastic distortion case. Recall that the equations and parameters are summarise in Table A.1.[3]

[3]Alternatively, we could choose $X(t) = R_t - \bar{R}_t$ which would result in $-\sigma\lambda = K\Theta - \sigma_X\lambda_1$ and $\Theta = 0$ in equations (A.2.12) and (A.2.13). We normalise $X(t) = R_t$ in this appendix to focus on the effect of

Moreover, we explain our slight different notation from SW when computing the exposure to the risk factors. This difference is caused in a different term $1/\gamma$ in their hedge demand in their equation (44), compared to our term $-(1-1/\gamma)$. Using the temporary definitions $\bar{B}_1, \bar{B}_2$, and $\bar{B}_3$ that will later be transformed to our notation of $\tilde{B}_1, \tilde{B}_2$, and $\tilde{B}_3$, the relevant ODEs for the real interest rate exposure originating from Appendix B in Sangvinatsos and Wachter (2005) are:

$$\bar{B}_3'(T-t) = 2\bar{B}_3(T-t)\left[\left(\frac{1}{\gamma}-1\right)\sigma\alpha-\kappa\right]+\frac{1}{\gamma}\bar{B}_3(T-t)^2\sigma^2+\left(\frac{1}{\gamma}-1\right)\alpha^2 \quad \text{(A.2.16)}$$

$$\begin{aligned}\bar{B}_2'(T-t) &= \bar{B}_2(T-t)\left[\left(\frac{1}{\gamma}-1\right)\sigma\alpha-\kappa+\frac{1}{\gamma}\sigma^2\bar{B}_3(T-t)\right]\\ &+\bar{B}_3(T-t)\left[\bar{R}\left(\kappa-\frac{1}{\gamma}\sigma\alpha\right)+\left(\frac{1}{\gamma}-1\right)\lambda\sigma\right]\\ &+(1-\gamma)+\left(\frac{1}{\gamma}-1\right)\left(\lambda-\alpha\bar{R}\right)\alpha \end{aligned} \quad \text{(A.2.17)}$$

with the boundary conditions $\bar{B}_2(0)=\bar{B}_3(0)$.[4] Now we apply the transformation to our final notation:

$$\tilde{B}_2(T-t)=\frac{\bar{B}_2(T-t)}{1-\gamma}\ ;\ \tilde{B}_3(T-t)=\frac{\bar{B}_3(T-t)}{1-\gamma} \quad \text{(A.2.18)}$$

This results in the ODEs:

$$\tilde{B}_3'(T-t) = 2\tilde{B}_3(T-t)\left[\frac{1}{\gamma}\sigma\alpha-\kappa\right]+\left(\frac{1}{\gamma}-1\right)\tilde{B}_3(T-t)^2\sigma^2+\frac{1}{\gamma}\alpha^2 \quad \text{(A.2.19)}$$

$$\begin{aligned}\tilde{B}_2'(T-t) &= \tilde{B}_2(T-t)\left[\frac{1}{\gamma}\sigma\alpha-\kappa+\left(\frac{1}{\gamma}-1\right)\sigma^2\tilde{B}_3(T-t)\right]\\ &+\tilde{B}_3(T-t)\left[\bar{R}\left(\kappa-\frac{1}{\gamma}\sigma\alpha\right)+\left(\frac{1}{\gamma}-1\right)\lambda\sigma\right]\\ &+1+\frac{1}{\gamma}\left(\lambda-\alpha\bar{R}\right)\alpha \end{aligned} \quad \text{(A.2.20)}$$

where the boundary conditions are $\tilde{B}_2(0)=\tilde{B}_3(0)=0$. As a result, the hedge demand in

$\tilde{B}_3(T-t)\cdot R_t$ instead of $\tilde{B}_3(T-t)\cdot\left(R_t-\bar{R}\right)$ as the effect of $\bar{R}$ is irrelevant in our analysis of the impact of the distortion parameter α on utility.

[4] Sangvinatsos and Wachter (2005) use a similar boundary condition that is satisfied if $B_2(0)=B_3(0)$.

the optimal bond weight in SW would read as:

$$x_t = (...) + \frac{1}{\gamma} \cdot \Omega^{-1} \sigma_B \rho \sigma_X' \left[\bar{B}_3(T-t) \cdot R_t + \bar{B}_2(T-t) \right] \tag{A.2.21}$$

and after the transformations is in line with the term $-(1 - 1/\gamma)$ in our notation of the optimal hedge demand:

$$x_t = (...) - \left(1 - \frac{1}{\gamma}\right) \cdot \Omega^{-1} \sigma_B \rho \sigma_X' \left[\tilde{B}_3(T-t) \cdot R_t + \tilde{B}_2(T-t) \right] \tag{A.2.22}$$

For completeness, we similarly determine the ODE for the transformed $\tilde{B}_1(T-t)$ as well, required for the utility computations:

$$\begin{aligned} \tilde{B}_1'(T-t) = \tilde{B}_2(T-t) \left[\bar{R} \left(\kappa - \frac{1}{\gamma} \sigma \alpha \right) + \left(\frac{1}{\gamma} - 1 \right) \lambda \sigma \right] + \frac{\sigma^2}{2} \cdot \left(\frac{1}{\gamma} - 1 \right) \tilde{B}_2(T-t)^2 \\ + \frac{\sigma^2}{2} \tilde{B}_3(T-t) + \frac{1}{2\gamma} \left(\lambda - \alpha \bar{R} \right)^2 \end{aligned} \tag{A.2.23}$$

with the boundary condition of $\tilde{B}_1(0) = 0$.

A.2.2 Proof of Theorem 2

In the two-factor model, the investor aims to hedge the real interest rate $r_t = R_t - \pi_t$. We therefore consider an inflation-linked bond with price $\tilde{P}_t(T-t)$ for bond maturity $T-t$ with the real interest rate as return and the 1×2 real bond volatility matrix

$$\sigma_{\widetilde{B}} = -\tilde{B}(T-t)'\sigma_X \tag{A.2.24}$$

where $\tilde{B}(T-t)$ refers to the real interest rate exposure that was derived before in equation (A.2.6) as $\tilde{B}_2(T-t)$.[5] We suppress this expression to $\tilde{B}(T-t)$ for readability purposes in this section. The log process of the inflation-linked bond price under $\mathcal{Q}$ is:

$$d\log\tilde{P}_t(T-t) = \left[R_t - \frac{1}{2}\sigma_{\widetilde{B}}\rho\sigma_{\widetilde{B}}'\right]dt + \sigma_{\widetilde{B}}dZ_t^{\mathcal{Q}} \tag{A.2.25}$$

For the deterministic distortion, the optimal wealth process under the distorted measure was given in equation (59). This results in a log optimal wealth process under $\mathcal{Q}$ of:

$$d\log W_t^* = \left[R_t - \frac{1}{2}\sigma_{W^*}'\rho\sigma_{W^*}\right]dt + \sigma_{W^*}'dZ_t^{\mathcal{Q}} \tag{A.2.26}$$

where we recall the definition of the 2×1 vector with optimal wealth exposure

$$\sigma_{W^*} = \frac{1}{\gamma}\rho^{-1}\lambda^* - \left(1 - \frac{1}{\gamma}\right)\sigma_X'\tilde{B}(T-t) \tag{A.2.27}$$

Next, we examine the terminal wealth expressed in units of an index-linked bond that matures at T, $W_t^*/\tilde{P}_t(T-t)$. Optimising this process instead of the initial process W_t^* remains the same at the investment horizon, since $W_T/\tilde{P}_T(0) = W_T/\Pi_T$, maintaining its relevance for the utility from terminal wealth setting. From this, we derive:

$$\begin{aligned}
d(\log W_t^* - \log\tilde{P}_t(T-t)) &= \left(-\frac{1}{2}\sigma_{W^*}'\rho\sigma_{W^*} + \frac{1}{2}\sigma_{\widetilde{B}}\rho\sigma_{\widetilde{B}}'\right)dt + \left(\sigma_{W^*}' - \sigma_{\widetilde{B}}\right)dZ_t^{\mathcal{Q}} \\
\rightarrow \frac{d(W_t^*/\tilde{P}_t(T-t))}{W_t^*/\tilde{P}_t(T-t)} &= \left(-\frac{1}{2}\sigma_{W^*}'\rho\sigma_{W^*} + \frac{1}{2}\sigma_{\widetilde{B}}\rho\sigma_{\widetilde{B}}' + \frac{1}{2}\left[\sigma_{W^*}' - \sigma_{\widetilde{B}}\right]\rho\left[\sigma_{W^*}' - \sigma_{\widetilde{B}}\right]'\right)dt \\
&\quad + \left(\sigma_{W^*}' - \sigma_{\widetilde{B}}\right)dZ_t^{\mathcal{Q}} \\
&= -\sigma_{\widetilde{B}}\rho\left(\sigma_{W^*}' - \sigma_{\widetilde{B}}\right)'dt + \left(\sigma_{W^*}' - \sigma_{\widetilde{B}}\right)dZ_t^{\mathcal{Q}}
\end{aligned} \tag{A.2.28}$$

[5]Note that in case of $\kappa_{R\pi} = 0$, it also reconciles with the nominal index-linked bond price in Li (2019), see their equation (2.12).

where we recall that σ_{W^*} has the transposed dimension of the 1×2 vector $\sigma_{\widetilde{B}}$. Substituting the expression for σ_{W^*} from (A.2.27) gives

$$
\begin{aligned}
\rightarrow \frac{d(W_t^*/\tilde{P}_t(T-t))}{W_t^*/\tilde{P}_t(T-t)} &= \frac{1}{\gamma}\tilde{B}(T-t)'\sigma_X\rho\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]' dt \\
&\quad + \frac{1}{\gamma}\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right] dZ_t^{\mathcal{Q}} \qquad \text{(A.2.29)} \\
&= \frac{1}{\gamma}\left[(\lambda^*)'+\tilde{B}(T-t)'\sigma_X\rho\right]\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]' dt \\
&\quad + \frac{1}{\gamma}\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right] dZ_t^{*} \qquad \text{(A.2.30)} \\
&= \underbrace{\frac{1}{\gamma}\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]\rho\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]'}_{\equiv\bar{\mu}} dt \\
&\quad + \underbrace{\frac{1}{\gamma}\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]}_{\equiv\bar{\sigma}} dZ_t^{*} \qquad \text{(A.2.31)}
\end{aligned}
$$

Similar as for the one-factor model, we derive the optimal distortion by:

$$\hat{\beta} = \arg\min_{\beta}\ \mu - \frac{\gamma}{2}\bar{\sigma}\rho\bar{\sigma}' \qquad \text{(A.2.32)}$$

Plugging in the drift term and volatility and using the assumption that $\gamma > 0$ results in the minimisation problem

$$
\begin{aligned}
&\min_{\beta}\ \frac{1}{\gamma}\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]\rho\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]' \\
&\qquad - \frac{1}{2\gamma}\cdot\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]\rho\left[(\lambda^*)'\rho^{-1}+\tilde{B}(T-t)'\sigma_X\right]' \\
&= \min_{\beta}\ \frac{1}{2\gamma}\left[(\lambda^*)'\rho^{-1}\lambda^*+(\lambda^*)'\sigma_X'\tilde{B}(T-t)+\tilde{B}(T-t)'\sigma_X\lambda^*+\tilde{B}(T-t)'\sigma_X\rho\sigma_X'\tilde{B}(T-t)\right] \\
&= \min_{\beta}\ \frac{1}{2}\left[\beta'\rho^{-1}\beta+\lambda'\rho^{-1}\beta+\beta'\rho^{-1}\lambda+\beta'\sigma_X'\tilde{B}(T-t)+\tilde{B}(T-t)\right] \\
&= \min_{\beta}\ \frac{1}{2}\beta'\rho^{-1}\beta+\beta'\rho^{-1}\lambda+\beta'\sigma_X'\tilde{B}(T-t) \qquad \text{(A.2.33)}
\end{aligned}
$$

where the last two terms represent combinations of transposed terms as these transformed terms are scalars. The minimisation should be performed with respect to the constraint on uncertainty given in equation (33). Because the objective function and constraints are convex, this can be solved with the Lagrange method with Karush Kuhn Tucker

(KKT)-conditions:

$$\mathcal{L} = \frac{1}{2}\beta'\rho^{-1}\beta + \beta'\left[\rho^{-1}\lambda + \sigma_X'\tilde{B}(T-t)\right] - \ell_t(\beta'\rho^{-1}\beta - k^2) \tag{A.2.34}$$

$$\text{s.t.}\ \ k \geq 0; \lambda_R < 0; \lambda_\pi < 0; \ell_1 \leq 0; \ell_2 \leq 0 \tag{A.2.35}$$

$$\frac{d\mathcal{L}}{d\beta} = 0 \tag{A.2.36}$$

$$\ell_t\left(\beta'\rho^{-1}\beta - k^2\right) = 0 \tag{A.2.37}$$

Case 1: $\ell_t = 0$: In this case, mother nature can distort the market price of risk without being restricted by uncertainty constraints. The FOC of equation (A.2.36) results in the optimal unbounded distortion:

$$\hat{\beta}|_{\ell_t=0} = -\lambda - \rho\sigma_X'\tilde{B}(T-t) \tag{A.2.38}$$

Case 2: $\ell_t < 0$. In this case, the constraint is binding and restricts mother nature's distortion. Equation (A.2.36) results in:

$$\hat{\beta}|_{\ell_t<0} = \frac{1}{1-2\ell_t}\left[-\lambda - \rho\sigma_X'\tilde{B}(T-t)\right] \tag{A.2.39}$$

Now, for convenience we define the 2×1 vector $\check{\lambda}$ with the components:

$$\begin{bmatrix}\check{\lambda}_R \\ \check{\lambda}_\pi\end{bmatrix} \equiv \lambda + \rho\sigma_X'\tilde{B}(T-t) = \begin{bmatrix}\lambda_R + \tilde{B}_R(T-t)\sigma_R + \rho_{R\pi}\tilde{B}_\pi(T-t)\sigma_\pi \\ \lambda_\pi + \tilde{B}_\pi(T-t)\sigma_\pi + \rho_{R\pi}\tilde{B}_R(T-t)\sigma_R\end{bmatrix} \tag{A.2.40}$$

where we let $\tilde{B}_R(T-t)$ and $\tilde{B}_\pi(T-t)$ correspond to the risk exposures in $\tilde{B}(T-t)$ w.r.t. R_t and π_t. We use this definition to express an individual component from the optimal distortions, $\hat{\beta}_i(\tau)|_{\ell_t<0}$, in the other component $\hat{\beta}_j(\tau)|_{\ell_t<0}$ where $i, j \in \{R, \pi\}$ and $i \neq j$:

$$\hat{\beta}_i|_{\ell_t<0} = \frac{\check{\lambda}_i}{\check{\lambda}_j}\hat{\beta}_j|_{\ell_t<0} \tag{A.2.41}$$

Combining this with the constraint in (A.2.37) results in:

$$\hat{\beta}_i|_{\ell_t<0} = \pm D \cdot \sqrt{k^2 \cdot \check{\lambda}_i^2} \tag{A.2.42}$$

where we define

$$D = \frac{\sqrt{1-\rho_{R\pi}^2}}{\sqrt{\check{\lambda}_R^2 + \check{\lambda}_\pi^2 - 2\rho_{R\pi}\check{\lambda}_R\check{\lambda}_\pi}} \tag{A.2.43}$$

To determine whether we should apply a plus or minus for the $\pm$ sign, we consider the empirical relevance case where $\check{\lambda}_R < \check{\lambda}_\pi < 0$.[6] Figure 4 demonstrates a graphical visualisation of the optimisation problem.[7] It shows that the optimisation is represented by a multidimensional 'confidence interval', represented by an ellipse around the baseline estimates ($k = 0$) for $\check{\lambda}_R$ and $\check{\lambda}_\pi$. By incorporating the constraint, the optimal solution is required to be on the line from the original estimate $(\check{\lambda}_R, \check{\lambda}_\pi)$ to the origin of the axes. Mother nature optimises by minimising the distance between the distorted estimate $(\check{\lambda}_R^*, \check{\lambda}_\pi^*)$ and the origin of the axes. Hence, she applies a positive distortion on both market price of risk, making these less negative. Therefore, for a $k > 0$ the bounded optimal distortions equals:

$$\hat{\beta}|_{\ell_t<0,\check{\lambda}_R<\check{\lambda}_\pi<0} = -k \cdot D \cdot \check{\lambda} \tag{A.2.44}$$

Final solution: if there are no constraints on uncertainty, mother nature applies the unbounded optimal distortion shown in equation (A.2.38). The bound on uncertainty k might restrict this distortion so that the bounded optimal distortion from equation (A.2.44) is applied. We can plug in the optimal distortion into the distorted market price of risk, and assume that $k > 0$ and $\check{\lambda}_R\check{\lambda}_\pi < 0$ so that we have an optimal distorted market price of risk:

$$\hat{\lambda}^* = \min\left\{-\rho\sigma_X'\tilde{B}(T-t); \lambda - k \cdot D \cdot \check{\lambda}\right\} \tag{A.2.45}$$

[6] This is the case for our empirical parameter set for all bond maturities $\tau_1 \in (0, 30)$; $\check{\lambda}_R = -0.4078$; and $\check{\lambda}_\pi = -0.3534$. We only observe $\check{\lambda}_R \approx \check{\lambda}_\pi$ for a very large τ.

[7] The figure is based on a positive correlation between the risk factors ($\rho_{R\pi} > 0$) which is empirically relevant.

where we recall that the definitions of $\check{\lambda}$ and D were given in equations (A.2.40) and (A.2.43), respectively. Consecutively, we can plug this into the robust investment strategy:

$$\begin{aligned}
\hat{x}_t^* &= \frac{1}{\gamma}\Omega^{-1}\sigma_B\hat{\lambda}^* - \left(1 - \frac{1}{\gamma}\right)\Omega^{-1}\sigma_B\rho\sigma_X'\tilde{B}(T-t) \\
&= \frac{1}{\gamma}\Omega^{-1}\sigma_B \cdot \min\left\{-\rho\sigma_X'\tilde{B}(T-t); \lambda - k \cdot D \cdot \check{\lambda}\right\} - \left(1 - \frac{1}{\gamma}\right)\Omega^{-1}\sigma_B\rho\sigma_X'\tilde{B}(T-t) \\
&= \frac{1}{\gamma}\Omega^{-1}\sigma_B \cdot \min\left\{0; (1 - k \cdot D)\check{\lambda}\right\} - \Omega^{-1}\sigma_B\rho\sigma_X'\tilde{B}(T-t) \qquad \text{(A.2.46)}
\end{aligned}$$

A.2.3 Saddle point assumption

In the original objective function in equation (20), we formulate a max-min problem. To compute the robust investment strategy, we assume the existence of a saddle point such that we can interchange the max- and min-operators (see Section 2.3.2.2). This appendix provides the motivation for the saddle point assumption for the deterministic and stochastic distortion.

A.2.3.1 Minimax theorem

Neufeld and Nutz (2018) and Biagini and Pınar (2017) apply Von Neumann's minimax theorem to prove the existence of a saddle point in a model with a constant interest rate and uncertainty about the stock price process. To explain the minimax theorem in our context, we define the expected utility function by:

$$f(x_t, C_t) \equiv \mathbb{E}_t^* \left[u \left(\frac{W_T}{\Pi_T} \right) \right] \tag{A.2.47}$$

where we recall that terminal wealth depends on the investor's strategy x_t and mother nature's distortion C_t, as explained in Section 3.3.2. We define the considered investment strategies and distortions by $\mathcal{X}$ and $\mathcal{C}$, respectively. The minimax theorem would then state that at every point in time t:

$$\max_{x_t \in \mathcal{X}} \min_{C_t \in \mathcal{C}} f(x_t, C_t) = \min_{C_t \in \mathcal{C}} \max_{x_t \in \mathcal{X}} f(x_t, C_t) \tag{A.2.48}$$

Provided that:

$$\begin{cases} f(\cdot, C_t) \text{ is concave for every fixed } C_t \in \mathcal{C} \\ f(x_t, \cdot) \text{ is convex for every fixed } x_t \in \mathcal{X} \\ \mathcal{C} \text{ and } \mathcal{X} \text{ are compact} \end{cases} \tag{A.2.49}$$

Moreover, to find a saddle point in the original expected utility function $f(x_t, C_t)$ in equation (A.2.47), we can apply the minimax theorem to an alternative function $\tilde{f}(x_t, C_t)$ which finds the same arguments of the max-min objective function. To see why, assume that we find a saddle point in the alternative objective function. The minimising argument of the alternative objective function, $\hat{C}_t$, minimises the original expected utility function as well, which implies:

$$f(\hat{x}_t, \hat{C}_t) \leq f(\hat{x}_t, C_t) \tag{A.2.50}$$

where $\hat{x}_t$ is the maximising argument of both the alternative and original function which also implies:

$$f(x_t, \hat{C}_t) \leq f(\hat{x}_t, C_t) \tag{A.2.51}$$

so that equations (A.2.50) and (A.2.51) result in:

$$f(x_t, \hat{C}_t) \leq f(\hat{x}_t, \hat{C}_t) \leq f(\hat{x}_t, C_t) \tag{A.2.52}$$

Therefore, a saddle point in the alternative function, $(\hat{x}_t, \hat{C}_t)$, is a saddle point of the function $f(x_t, C_t)$ as well by definition. Hence, we will consider the requirements of the minimax theorem for alternative objective functions which find the same arguments as the original expected utility function. The following subsections consider the requirements for the deterministic and stochastic distortions.

A.2.3.2 Deterministic distortion

We first consider the deterministic distortion in the one-factor model as defined in Problem 1. We analyse the wealth process under a distorted probability measure $\mathcal{P}^*$, and an investment strategy x_t:

$$dW_t^*/W_t^* = [R_t - x_t B(\tau)\sigma\lambda^*]\, dt - x_t B(\tau)\sigma dZ_t^* \tag{A.2.53}$$

where we recall that τ denotes the fixed bond maturity. Note that Theorem 2.4.1 derives the wealth process under the optimal strategy x_t^*, while here we consider the wealth under a given strategy x_t and measure $\mathcal{P}^*$. Similar as in the proof of Theorem 2.4.1, we apply the maximisation after scaling wealth with the price of a nominal bond with maturity τ. This results in the following process:

$$\frac{d\left(W_t^*/P_t(\tau)\right)}{W_t^*/P_t(\tau)} = (1 - x_t) \cdot \sigma B(\tau) \cdot [\sigma B(\tau) + \lambda^*]\, dt + (1 - x_t) \cdot \sigma B(\tau) dZ_t^* \tag{A.2.54}$$

where the objective function of expected utility of (scaled) terminal wealth for risk aversion coefficient $\gamma \geq 1$ can be solved by:

$$\max_{x_t} \min_{\beta} \mathbb{E}_t^* \left[u(W_T^*/P_T(0)) \right] = \min_{x_t} \max_{\beta} \mathbb{E}_t^* \left[\left(e^{w_T} \right)^{1-\gamma} \right] \tag{A.2.55}$$

$$= \min_{x_t} \max_{\beta} e^{(1-\gamma)\mathbb{E}_t^*[w_T] + \frac{1}{2}(1-\gamma)^2 \mathrm{Var}_t^*[w_T]} \tag{A.2.56}$$

where $w_T \equiv \log\left(W_T^*/P_T(0)\right)$. We find the same arguments by solving:

$$\max_{x_t} \min_{\beta} \mathbb{E}_t^*[w_T] + \frac{1}{2}(1-\gamma)\,\mathrm{Var}_t^*[w_T] \equiv \max_{x_t} \min_{\beta} \tilde{f}(x_t, \beta) \tag{A.2.57}$$

To verify the concavity of $\tilde{f}(\cdot, \beta)$, we consider a fixed distortion β by mother nature, resulting in probability measure $\mathcal{P}^*$. We rewrite the maximisation in equation (A.2.57) as:

$$\max_{x_t} [...] + \int_t^T \left(-x_s \sigma B(\tau) \cdot \left[(1-\gamma) \cdot \sigma B(\tau) + \lambda^* \right] - \frac{\gamma}{2} x_s^2 B(\tau)^2 \sigma^2 \right) ds \tag{A.2.58}$$

where [...] contains terms independent of the investment strategy x_t. We let the investor solve the optimisation at each time t:

$$\max_{x_t} \; [...] \; + \left(-\sigma B(\tau) \cdot \left[(1-\gamma) \cdot \sigma B(\tau) + \lambda^* \right] \right) \cdot x_t - \left(\frac{\gamma}{2} B(\tau)^2 \sigma^2 \right) \cdot x_t^2 \tag{A.2.59}$$

which is a concave function in x_t due to the negative coefficient of the quadratic term.

To asses the convexity of $\tilde{f}(x_t, \cdot)$, we consider a fixed strategy x_t by the investor, and rewrite mother nature's optimisation of equation (A.2.57) as:

$$\min_{\beta} [...] + \int_t^T (1 - x_s)\, \sigma B(\tau) \cdot \beta ds \tag{A.2.60}$$

where [...] contains the terms independent of the distortion β. This is a linear and hence convex function in the distortion.

The constraint on distortions in equation (22) ensures that the distortion set $\mathcal{C}$ is compact. Moreover, in the max-min objective function the investor invests optimally based on the distortion of mother nature. Under this distorted measure, the investor applies the optimal strategy derived by Sangvinatsos and Wachter (2005), which is valid for $\lambda < \infty$ and $\gamma \geq 1$ (see equation (9)). This ensures a compact strategy set $\mathcal{X}$.

In the two-factor model, the coefficient of the squared investment strategy x_t^2 as in equation (A.2.59) would equal $-\gamma/2 \cdot x_t' \Omega x_t$, where Ω is the covariance-matrix of the bond returns. Therefore, Ω is positive semi-definite, and the objective function remains concave in the investment strategy x_t. The minimisation over β as in equation (A.2.60) involves a linear combination of the two distortions in both risk factors so that it remains convex in β.[8] The distortion and investment strategy sets remain compact due to the constraint on distortions and optimal investment strategy.

Therefore, for the deterministic distortion, we verify the saddle point.

A.2.3.3 Stochastic distortion

Proving the existence of a saddle point is more challenging for the one-factor model with a stochastic distortion, as defined in Problem 2. We analyse a wealth process under a probability measure distorted by the stochastic distortion $C_t = \alpha \cdot (R_t - \bar{R})$. For a given distortion parameter α, the wealth process scaled by the price of a nominal bond with maturity τ follows:

$$\begin{aligned} \frac{d\left(W_t^*/P_t(\tau)\right)}{W_t^*/P_t(\tau)} =& (1-x_t)\cdot \sigma B(\tau) \cdot \left[\sigma B(\tau) + \lambda + \alpha \cdot (R_t - \bar{R})\right] dt \\ &+ (1-x_t)\cdot \sigma B(\tau) dZ_t^* \end{aligned} \tag{A.2.61}$$

where future nominal interest rates follow a distribution under the distorted probability measure $\mathcal{P}^*$. Similarly as for the deterministic distortion, we can alternatively find the arguments of the objective function by solving:

$$\max_{x_t} \min_{\alpha} \mathbb{E}_t^*[w_T] + \frac{1}{2}(1-\gamma)\,\mathrm{Var}_t^*[w_T] \equiv \max_{x_t} \min_{\alpha} \tilde{f}(x_t, \alpha) \tag{A.2.62}$$

where $w_T = \log\left(W_T^*/P_T(0)\right)$. We start with verifying the concavity of $\tilde{f}(\cdot, \alpha)$. Compared to the deterministic distortion, the expected value of w_T now contains an additional term that takes into account the expected cumulative deviation of the nominal interest rate from its long-term average:

$$\begin{aligned} \max_{x_t} \tilde{f}(\cdot,\alpha) = \max_{x_t} [...] + \int_t^T -x_s \sigma B(\tau) \cdot \left[(1-\gamma)\cdot \sigma B(\tau) + \lambda + \mathbb{E}_t^*\left[\alpha \cdot (R_s - \bar{R})\right]\right] ds \\ - \int_t^T \frac{\gamma}{2} x_s^2 B(\tau)^2 \sigma^2 ds \end{aligned} \tag{A.2.63}$$

[8] In the notation of Appendix A.2.2, the function to minimise, $\tilde{f}(x_t, \beta)$, would equal $\int_t^T (1-x_s)\tilde{B}(T-t)'\sigma_X \beta ds$, where x_s is the allocation to a an inflation-linked bond with bond maturity $T-t$ with a real interest rate risk exposure $\tilde{B}(T-t)$ to both the nominal interest rate and inflation rate as risk factors.

where [...] collects terms independent of the investment strategy. Because we consider a fixed probability measure, the expected cumulative deviation of the nominal interest rate from its long-term average rate is constant. Therefore, the only term that depends on the squared investment strategy is in the second integral. We let the investor maximise at every t:

$$\max_{x_t} [...] - (...) \cdot x_t - \left(\frac{\gamma}{2} B(\tau)^2 \sigma^2\right) \cdot x_t^2 \tag{A.2.64}$$

which is concave in the investment strategy x_t because of our assumption that $\gamma \geq 1$.

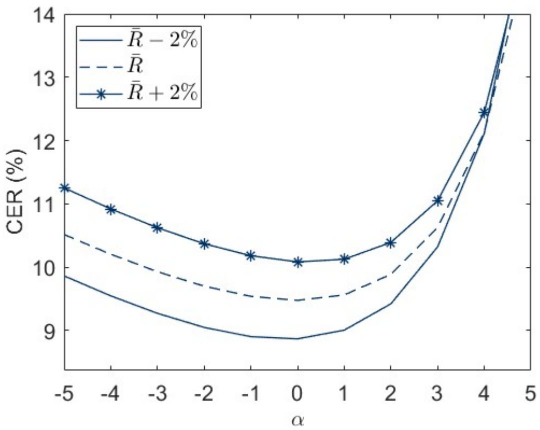
Before we analyse the second condition about convexity, we consider the third condition of compact sets of distortions and investment strategies. As the constraint in equation (70) bounds the distortion parameter α, the total distortion $C_t = \alpha \cdot R_t$ is not necessarily bounded. Nonetheless, for $R_t < \infty$, the distorted market price of risk remains finite and the optimal strategy under the distorted probability measure is bounded. Moreover, the distortion on the mean-reversion parameter remains bounded through the effect on the distorted mean-reversion parameter $\kappa^* = \kappa - \alpha\sigma$, and hence lies within an interval determined by the bound on α. Hansen and Sargent (2022) also rely on this insight. They show that specific bounds on the uncertainty bound k results in a rectangular uncertainty set around the mean-reversion parameter in which the minimax theorem can be applied.[9] Hence, we follow the intuition from Hansen and Sargent (2021) that a compact set can be constructed around the distorted mean-reversion parameter κ^*. Moreover, we assume a finite $R_t < \bar{R}$ so that we have a finite distorted market price of risk. Similarly as for the deterministic distortion, the set of investment strategies is compact based on the optimal strategy based on a distorted probability measure.

We now turn to the requirement that $\tilde{f}(x_t, \cdot)$ is convex. At the moment, we do not have a formal proof. In the case of a stochastic distortion, the drift term is impacted not only by the distortion in the market price of risk at time t, but also by distorted future interest rates via the distortion in the mean-reversion parameter. Moreover, future shocks in the bond prices are affected by these distorted interest rates.[10] Therefore, it is not straightforward to derive an analytical solution for the objective function with respect to

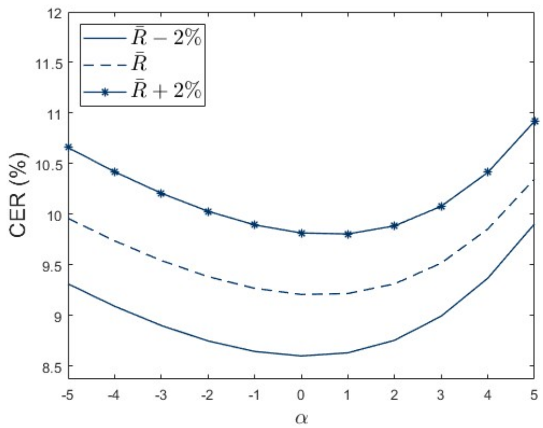
[9] Hansen and Sargent (2022) apply a bound on the time derivative of relative entropy. In a one-factor model where the risk factor X_t follows a mean-reverting process, this bound implies lower and upper bounds on the distortion factor α in the total distortion term αX_t. These bounds are derived from a relative entropy factor. Their Figure 1 shows a numerical example of bounds with bounds on two simultaneously distorted parameters, of which one is a mean-reversion parameter.

[10] See also Section 2.5, where it is explained that the optimal wealth process in (76) depends on a trade-off between the effects on the market price of risk and the mean-reversion parameter.

Figure A.2.3.1. Convex Optimisation under Stochastic Distortion. The certainty equivalent return per year (CER, in %) at $t = 0$ with investment horizon $T = 30$. The returns are computed under different distortion parameters $\alpha \in (-5, 5)$ and the assumption that the investor applies the optimal strategy based on the distorted probability measure. The solid line, dashed line, and line with star markers refer to different initial interest rates below, equal or above the long-term average. The figures are shown for the risk aversion coefficients (a) $\gamma = 1$ and (b) $\gamma = 2$.



(a) $\gamma = 1$



(b) $\gamma = 2$

the distortion parameter α.

Nevertheless, the numerical results in Section 2.5 suggest that the function is convex in α under the assumption that the investor knows the distortion applied by mother nature and therefore invests optimally under the distorted measure $\mathcal{P}^*$, as given in equation (9). As a result, the investor receives a certainty equivalent return based on the indirect utility function, as given in equation (79). Figure A.2.3.1 shows the worst-case equivalent return over different values of the distortion parameter α for the risk aversion coefficients $\gamma \in (1, 2)$. The figure confirms that the indirect utility function is convex in α, which implies the convexity of $\tilde{f}(\cdot, \alpha)$.[11]

Because we do not prove the existence of a saddle point under a stochastic distortion formally, the next subsection elaborates on related literature about saddle points in robust optimisations. Moreover, it explains why known counterexamples that would make the minimax theory invalid do not apply in our setting.

A.2.3.4 Saddle points in related literature

Yang et al. (2019) prove the existence of a saddle point in a set-up where both the drift

[11] A convex certainty equivalent return function results in a convex indirect utility function, see equation (78).

and the covariance-matrix are distorted. However, their model includes only a stock and assumes a constant interest rate. In the context of bonds, Luxenberg et al. (2024) derive robust bond portfolios under uncertainty about the yield curve based on a saddle point. However, they do not model simultaneously distorted risk-free returns on future wealth so that the worst-case probability measure can be derived explicitly as the one that reduces the bond returns the most.[12] Similarly, the worst-case mean-reversion parameter distortion in the study of Hansen and Sargent (2021) can be determined explicitly: the optimal distortion increases (decreases) the mean-reversion to zero as much as possible within the bound on uncertainty if the growth rate of the economy is negative (positive).

In contrast, the study by Hernández-Hernández and Schied (2006) considers a general framework in which the investor allocates wealth between a cash account and a risky asset, where both the return on the cash account and the asset can depend on mean-reverting risk factors, as in our set-up.[13] They verify the existence of a saddle point in the max-min objective function of maximising expected utility from terminal wealth. However, their analysis assumes a distortion that lies within a fixed and compact set so that they do not consider a change in the mean-reversion parameter of the risk factor.

Finally, we note that counterexamples mentioned in the literature that would make the saddle point assumption invalid, do not apply to our set-up. For example, Balter and Pelsser (2020) provide a counterexample in the case of linear utility. In this case, the min-max problem is unbounded because the investor would optimally invest infinitely in a stock. In contrast, in our model, the investor optimises her strategy with respect to the worst-case prior using the strategy given in equation (9), which ensures a well-defined and bounded solution for $\gamma \geq 1$.

Another counterexample is mentioned in Baltas et al. (2018), where an infinite market price of risk leads to an unbounded optimal strategy for a more risk-averse investor. This issue does not arise in our framework, as we assume a finite baseline market price of risk, and the distortion factor is bounded. As a result, for $R_t < \infty$, the distorted market price of risk remains finite and the optimal strategy under the distorted probability measure is bounded.

[12] To the best of our knowledge, the only study that applies the constraint approach to determine worst-case interest rates is Lin and Riedel (2021). However, their analysis is conducted in a financial market without stochastic bond returns.

[13] Their general example involves an incomplete market, but they distinguish between hedgeable and non-hedgeable components in the risk factor. We can align their framework with ours by setting the non-hedgeable component equal to zero.

A.3 Supplementary material for Chapter 3

A.3.1 Comparison to Sangvinatsos and Wachter (2005)

This appendix shows how to link our notation and ODE's to Sangvinatsos and Wachter (2005) for reconciliation purposes. Table A.3.1 shows the link between the notation of the Appendix E of Sangvinatsos and Wachter (2005) with our notation.[14] Using this notation results in the Ordinary Differential Equations for the $B-$ and $A-$ function in the bond prices:[15]

$$B'(\tau_j) = -K\,B(\tau_j) + \delta \tag{A.3.1}$$

$$A'(\tau_j) = \delta_0 - B(\tau_j)'\sigma_X\lambda - \frac{1}{2}B(\tau_j)'\sigma_X\sigma_X'B(\tau_j) \tag{A.3.2}$$

For completeness, we show the expression of the solution of 2×1 vector $B(\tau_j)$:

$$B(\tau_j) = \begin{bmatrix} \frac{1-e^{-\kappa_R\tau_j}}{\kappa_R} \\ \frac{\kappa_{RY}}{\kappa_R-\kappa_Y}\left(\frac{1-e^{-\kappa_R\tau_j}}{\kappa_R} - \frac{1-e^{-\kappa_Y\tau_j}}{\kappa_Y}\right) \end{bmatrix} \tag{A.3.3}$$

SW2005	**Our model**
Model settings	
δ	$\begin{bmatrix}1 & 0\end{bmatrix}'$
ζ	$\begin{bmatrix}0 & 0\end{bmatrix}'$
$\bar{\Lambda}(t) = \bar{\lambda}_1 + \bar{\lambda}_2 X_t$	λ
Our notations	
$r_t = \delta_0 + \delta X_t$	$R_t = \delta_0 + \delta' X_t$
$A_1(\tau_j)$	$-A(\tau_j)$
$A_2(\tau_j)$	$-B(\tau_j)'$
$B_2(\tau_j)$	$B(\tau_j)'$
Estimations	
$\hat{X}_t$	X_t
U	Σ^{-1}
κ_1	$-K$
d_i	Λ_i
Ω	$\Sigma\sigma_X\sigma_X'\Sigma'$
Θ	$[0;\, 0]$

Table A.3.1. Notation of Sangvinatsos and Wachter (2005) and our study in the left and right column respectively.

[14]The link of $A_2(\tau_j) = -B'(\tau_j)$ is a result of the choice for δ and not including inflation risk, and hence could not be true for different model settings than in this chapter.

[15]Note that equation (A.3.2) is not used in the main text of this chapter, but in the Kalman filter explained in Appendix A.3.2.2.

A.3.2 Estimation

This appendix shows the relevant equations to estimate the Kalman filter applied in Section 3.4. Since these equations are based on the conditional expectation and variance, we will first derive these and consequently show how to apply the Kalman filter.

A.3.2.1 Conditional expectation and variance risk factor process

We apply Itô's Lemma to the process $e^{-Kt}X_t$ where the SDE of the risk factors X_t was given in equation (2) which results in the OU-process:

$$X_{t+h} = e^{-Kh}X_t + \left(1 - e^{-Kh}\right)\bar{X} + \int_0^h e^{-K(h-s)}\sigma_X \tag{A.3.4}$$

To derive the conditional expectation and variance, note that X_t is a linear transformation from risk factors F_t that have a diagonal mean-reversion matrix:

$$dF_t = \Lambda F_t dt + \sigma_F dZ_t \tag{A.3.5}$$

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \; ; \; \sigma_F = \begin{bmatrix} \sigma_1 & \sigma_{12} \\ 0 & \sigma_2 \end{bmatrix} \tag{A.3.6}$$

where $\Lambda_1, \Lambda_2 < 0$ can be interpreted as the eigenvalues of the mean-reversion matrix. We now define and derive:

$$\Sigma \equiv \begin{bmatrix} 1 & \Sigma_{12} \\ 0 & 1 \end{bmatrix} \; ; \; X_t \equiv \Sigma^{-1}F_t \tag{A.3.7}$$

$$\rightarrow dX_t = \underbrace{\Sigma^{-1}\Lambda\Sigma}_{=-K} X_t dt + \underbrace{\Sigma^{-1}\sigma_F}_{=\sigma_X} dZ_t \tag{A.3.8}$$

So that we can determine the mean-reversion, volatility and covariance-variance matrix of the risk factors X_t by:

$$K = -\begin{bmatrix} \Lambda_1 & \Sigma_{12}\left(\Lambda_1 - \Lambda_2\right) \\ 0 & \Lambda_2 \end{bmatrix} \; ; \; \sigma_X = \begin{bmatrix} \sigma_1 & \sigma_{12} - \Sigma_{12}\sigma_2 \\ 0 & \sigma_2 \end{bmatrix} \tag{A.3.9}$$

Note that in this way, exactly $G \cdot (G+1)/2 = 3$ parameters should be estimated for the mean-reversion matrix K, where G is the number of risk factors. Therefore, the model is invariant under so-called unitary rotations of the factors and has an unique solution (De Jong (2000)[p.302]). Moreover, note that the volatility matrix σ_X is less restrictive than in Sangvinatsos and Wachter (2005) who estimate a diagonal volatility matrix. Moreover, the model set-up is less restrictive than in De Jong (2000) who assumes a diagonal mean-reversion matrix.

In our case, we have a conditional expectation of:

$$\mathbb{E}\left[X_{t+h}|X_t\right] = \Sigma^{-1}\mathbb{E}\left[F_{t+h}|F_t\right] = \Sigma^{-1}e^{h\Lambda}\Sigma X_t (= e^{-hK}X_t) \tag{A.3.10}$$

$$= \underbrace{\begin{bmatrix} e^{h\Lambda_1} & \Sigma_{12}\left(e^{h\Lambda_2} - e^{h\Lambda_2}\right) \\ 0 & e^{h\Lambda_2} \end{bmatrix}}_{\equiv\Phi} X_t \tag{A.3.11}$$

Moreover, the conditional variance equals:

$$Q \equiv \text{Var}\left(X_{t+h}|X_t\right) = \Sigma^{-1}\text{Var}\left(F_{t+h}|F_t\right)\left(\Sigma^{-1}\right)' \tag{A.3.12}$$

$$\text{where } \left[\text{Var}\left(F_{t+h}|F_t\right)\right]_{i,j} = g(\Lambda_i, \Lambda_j)\left[\Sigma\sigma_X\sigma_X'\Sigma'\right]_{i,j} \tag{A.3.13}$$

$$\text{and } g(\Lambda_i, \Lambda_j) = -\frac{1 - \exp((\Lambda_i + \Lambda_j)\cdot h)}{\Lambda_i + \Lambda_j} \tag{A.3.14}$$

where $[]_{i,j}$ refers to the element (i, j) of the matrix. These results are in line with the derivations of Sangvinatsos and Wachter (2005).[16]

A.3.2.2 Kalman filter

We apply the Kalman filter to derive the parameters for the (unobserved) interest rate process in equation (2). Define the observed zero-coupon bond yields for bond j as $y_t(\tau_j)$ and assume that there are two risk factors in vector X_t. In this case we have:

$$y_t(\tau_j) = -\frac{1}{\tau_j}\log P_t(\tau_j) = \frac{1}{\tau_j}\left(A(\tau_j) + B(\tau_j)'X_t\right) \tag{A.3.15}$$

Recall that the derivations of the scalar $A(\tau_j)$ and 2×1 matrix $B(\tau_j)$ were given in equations (A.3.2) and (A.3.1) respectively. At every time t, we collect J zero-coupon bond yields with different maturities τ_j in vector y_t, and the corresponding coefficients in

[16] See their conditional expectation and variance at page 227 when we set their $U = \Sigma^{-1}$ and $d_i = \Lambda_i$. Moreover, I verified that starting from the factors F_t and the corresponding ODEs from De Jong (2000) indeed results in the ODEs from Sangvinatsos and Wachter (2005) in their equations (A3) and (A4) with the mean-reversion matrix K and volatility matrix σ_X.

the $J \times 1$ vector A and $J \times 2$ matrix B;

$$y_t \equiv \begin{bmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ ... \\ y_t(\tau_J) \end{bmatrix}, \ A \equiv \begin{bmatrix} A(\tau_1)/\tau_1 \\ A(\tau_2)/\tau_2 \\ ... \\ A(\tau_J)/\tau_J \end{bmatrix}, \ B \equiv \begin{bmatrix} B(\tau_1)'/\tau_1 \\ B(\tau_2)'/\tau_2 \\ ... \\ B(\tau_J)'/\tau_J \end{bmatrix} \tag{A.3.16}$$

which results in the *measurement equation:*

$$y_t = A + BX_t + e_t \ ; \ \text{var}(e_t) = H \tag{A.3.17}$$

We allow the bonds to have normally distributed errors e_t with zero mean where $H = h^2 * I$ with a scalar h and the $J \times J$ identity matrix I to make H positive definite and decrease the number of parameters that should be estimated.

Besides the measurement equation, the transition equation for time interval h between two observations describes the dynamics of the (unobserved) risk factor. We normalise such that the average risk factors are set equal to zero which results in the *transition equation:*[17]

$$X_{t+h} = \Phi X_t + \eta_{t+h} \ ; \ \text{var}(\eta_{t+h}) = \text{var}(X_{t+h}|X_t) \equiv Q \tag{A.3.18}$$

Now the initial conditions become $\hat{X} = [0, ..., 0]'$ and $\hat{V}_0^x = \text{Var}(X_t)$, and the predictions at t conditional on $t - h$ are based on the measurement and transition equations:

$$X_{t|t-h} = \Phi \hat{X}_{t-h} \tag{A.3.19}$$

$$V_{t|t-h}^x = \Phi \hat{V}_{t-h}^x \Phi' + Q \tag{A.3.20}$$

where the matrices Φ and Q are shown in equations (A.3.11) and (A.3.12). Consequently, these predictions are used to compute the log-likelihood based on the prediction errors of

[17]The mean $\bar{Y}$ of the de-trended data equals 0 by construction, so the normalisation results in $\bar{R} = \delta_0$.

the yield, u_t, and the likelihood contribution to these errors, V_t^y:

$$u_t = y_t - A - BX_{t|t-h} \tag{A.3.21}$$
$$V_t^y = BV_{t|t-h}^x B' + H \tag{A.3.22}$$
$$-2\log L_t = \log |V_t^y| + u_t'(V_t^y)^{-1}u_t \tag{A.3.23}$$
$$K_t = V_{t|t-h}^x B'(V_t^y)^{-1} \tag{A.3.24}$$
$$\hat{X}_t = X_{t|t-h} + K_t u_t \tag{A.3.25}$$
$$\hat{V}_t^x = (I - K_t B)\, V_{t|t-h}^x \tag{A.3.26}$$

where L_t is the likelihood, K_t is called the Kalman gain, I is the identity matrix, and the last three equations (A.3.24) to (A.3.26) correspond to the updating equations.

A.3.2.3 One latent factor, one observed factor

We need to adjust the standard Kalman filter applied on the interest rate for only latent factors, because we assume one latent risk factor R_t and one observed risk factor Y_t. These factors are implemented in the Kalman filter by forming a state vector S_t that is a combination of the yields and temperature Y_t:

$$S_t = \begin{bmatrix} y_t \\ Y_t \end{bmatrix} = \underbrace{\begin{bmatrix} A \\ 0 \end{bmatrix}}_{(J+1)\times 1} + \underbrace{\begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}}_{(J+1)\times 2} \begin{bmatrix} X_{1t} \\ Y_t \end{bmatrix} + e_t \tag{A.3.27}$$

where the vectors A and matrix B were defined in equation (A.3.16), and:

$$\text{Var}(e_t) = \underbrace{\begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}}_{(J+1)\times(J+1)} \tag{A.3.28}$$

so that we assume no measurement error for (the observed) Y_t.

A.3.2.4 Partial observed factor

The equations above are applicable when the observed variables are observed at the same time. However, we use monthly observation bond bond yields in y_t and yearly observations of temperature Y_t (implemented in January) so that we have an unequal observation frequency. Pennacchi (1991) explain in their Appendix B that their state variable vector therefore depends on the fact if the state variables are observed or not at t, being equal to the extended vector S_t in equation (A.3.27) when an observation is missing, and being equal to the original vector y_t in equation (A.3.16) without missing observations.[18]

Suppose that the temperature is unobserved at month $t-h$. Harvey (1990) shows that for missing variables at $t-h$, the conditional mean is computed on a sequential update of the observation $t-2h$. Therefore we use this sequential update of the second element of the risk factor vector when computing the conditional means at t:[19]

$$X_{t|t-h} = \Phi \hat{X}_{t-h} \tag{A.3.29}$$

$$\hat{X}^{(1)}_{t-h} = X^{(1)}_{t-h|t-2h} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}' K u_{t-h} \tag{A.3.30}$$

$$\hat{X}^{(2)}_{t-h} = \underbrace{\Phi(2,1)}_{=0} \hat{X}^{(1)}_{t-2h} + \Phi(2,2)\hat{X}^{(2)}_{t-2h} \tag{A.3.31}$$

where $X^{(j)}$ refers to the j^{th} element of the vector X, and $\Phi^{(i,j)}$ equals the (i,j) element of Φ. Moreover, in line with Harvey (1990), the conditional variance of the risk factors at t related to the second risk factor is only based on the computed variance of the second risk factor at time $t-2h$:

$$V^x_{t|t-h} = \Phi \hat{V}^x_{t-h} \Phi' + Q \tag{A.3.32}$$

$$\hat{V}^x_{t-h} = \left(I - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} K_{t-h} B \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}' \right) V^x_{t-h|t-2h} \tag{A.3.33}$$

A.3.2.5 Standard Errors

The standard errors (SE) are determined by the inverse of the Hessian at the maximum log-likelihood function given in equation (A.3.23), where the standard errors of κ_{RY} and σ_{RY} are approximated by the delta method based on equation (A.3.9). However, the standard errors are based on the high number of observations for the nominal interest rate (#768), while the observation frequency of temperature is much lower (#64). Therefore, we adjust the initial SE from the Kalman filter, SE^{old}, to the reported SE SE^{new} for the

[18] Our method differs from Pennacchi (1991) because their approach handles missing state variable observations. We observe all state variables, but not all risk factors. However, the idea of implementing an unequal observation frequency remains the same.

[19] This is in line with Pennacchi (1991) because the residuals u_{t-h} from equation (A.3.21) are equal to 0 if $t-h$ is observed.

temperature parameters κ_Y, σ_Y, λ_Y, σ_{RY}, and κ_{RY} by:

$$SE^{new} \approx \sqrt{\frac{768}{64}} \cdot SE^{old} \tag{A.3.34}$$

A.3.2.6 Measurement errors on observed state variable

When the Kalman filter is applied on the climate news index instead of temperature deviations as in Section 3.7, we assume measurement errors in the observed news index. For the temperature deviations we do not assume measurement errors. As a result, the measurement error matrix in equation (A.3.28) changes. The matrix now includes an error size for the climate factor, h_Y, as well:

$$\text{Var}(e_t) = \underbrace{\begin{bmatrix} h_B^2 & 0 & 0 & 0 & 0 \\ 0 & h_B^2 & 0 & 0 & 0 \\ 0 & 0 & h_B^2 & 0 & 0 \\ 0 & 0 & 0 & h_B^2 & 0 \\ 0 & 0 & 0 & 0 & h_Y^2 \end{bmatrix}}_{(J+1)\times(J+1)} \tag{A.3.35}$$

Moreover, we are interested in estimating the variance of the error term in the climate index in the transition equation shown in (A.3.18), because we use this variance as a lower bound for σ_Y^2 in the Kalman filter. For that purpose, we show how to derive the corresponding variance of the state equation with an ARMA(1,1)-model applied on the time series of the observed climate factor Y_t. Consider the measurement equation to include measurement noise in the observations:

$$Y_t = \tilde{Y}_t + e_t \qquad \text{var}(e_t) = h_Y^2 \tag{A.3.36}$$

Moreover, we have the transition equation to estimate the fitted climate factor $\tilde{Y}_t$:

$$\tilde{Y}_t = \phi \tilde{Y}_{t-1} + \eta_t \qquad \text{var}(\eta_t) = \sigma_\eta^2 \tag{A.3.37}$$

Combining the measurement and transition equations implies that:

$$\tilde{Y}_t = Y_t - e_t = \phi \tilde{Y}_{t-1} + \eta_t \tag{A.3.38}$$

$$= \phi(Y_{t-1} - e_{t-1}) + \eta_t \tag{A.3.39}$$

$$\rightarrow Y_t = \phi Y_{t-1} + e_t - \phi e_{t-1} + \eta_t \tag{A.3.40}$$

We match this with the ARMA(1,1)-equation:

$$Y_t = \phi Y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \tag{A.3.41}$$

$$\rightarrow \varepsilon_t + \theta \varepsilon_{t-1} = e_t - \phi e_{t-1} + \eta_t \tag{A.3.42}$$

When we match both the variance and covariance of the left and right hand side of equation (A.3.42), we find the variance of the error term in the state equation:

$$\sigma_\eta^2 = \left(1 + \theta^2 + \frac{\theta}{\phi} + \theta \cdot \phi\right) \sigma_\varepsilon^2 \tag{A.3.43}$$

Summary

This thesis studies the role of parameter uncertainty about interest rate dynamics in long-term investment strategies that allocate wealth to nominal bonds. A key concern for long-term investors, such as institutional investors, is to safeguard portfolio values over decades. Falling interest rates reduce returns on bank deposits, but bond values rise in such times, so investors often invest in bonds for protection.

Standard models derive optimal strategies by relying on the concept of risk where the investor assumes a single estimated parameter set. However, the interest rate process is difficult to estimate with high precision. This makes parameter uncertainty relevant: the possibility that model estimates are misspecified.

Therefore, this thesis shows that uncertainty about how interest rates behave can strongly affect the outcomes for long-term investors who hold bonds. If an investor bases her optimal strategy on incorrect beliefs about certain parameters of the interest rate process, her wealth can decrease substantially compared to holding correct beliefs. To reduce this potential loss in wealth, this thesis introduces robust strategies, referring to investment strategies that may not be optimal under the initial assumptions, but perform better if those assumptions turn out to be different from the initial ones.

Chapter 1 studies the case where investors aim to hedge the real interest rate, which equals the nominal interest rate minus the inflation rate, and are uncertain about the estimations of the underlying nominal interest rate and inflation processes. Without estimation errors, the optimal strategy is to invest in two nominal bonds with a different time to maturity.[20] However, the analysis shows that the allocations to these bonds are highly sensitive to errors in the estimated parameters that determine how quickly nominal interest rates return to "normal" after a shock, and how strongly the inflation rate impacts the nominal interest rate. By contrast, a single-bond strategy is suboptimal under the initial parameter estimates, but under parameter uncertainty it can be more stable than

[20] As explained in Chapter 2, the term "bond" can correspond to a bond portfolio which is constructed to have the same sensitivity to interest rates as a single bond, which is the more common approach in practice. For the two-bond strategy, this would refer to two bond portfolios with different sensitivities.

the two-bond allocation.

Chapter 2 develops a formal method for robustness. Parameter uncertainty is now modelled by imagining a “mother nature” that distorts the financial market to the “worst case” which is the least favourable for the investor. The investor acknowledges that this worst-case distortion can take place and chooses the investment strategy that performs best under that scenario. Two types of distortions are analysed. The first type is independent of the current interest rate, and results in a robust strategy that moves to the strategy of a very risk-averse investor. This means that a robust investor who considers the worst-case scenario in which taking risks is not rewarded, should invest like a very risk-averse investor who is mainly interested in hedging the interest rate.

The second type corresponds to distortions which are dependent on the current interest rate. This appears to be a relevant type, because it corresponds to changes in the persistence of shocks in the interest rate, where Chapter 1 showed the consequence of these changes. This results in robust bond allocations that depend on the current level of interest rates and the investment horizon. This means that an investor should consider the current financial market and remaining investment horizon when determining the robust strategy.

Chapter 3 introduces a new type of parameter uncertainty: the possibility that climate change affects the nominal interest rate. More climate risk, whether actual or perceived, can make bonds more attractive for investors who view them as risk-free assets. This increases the demand for bonds and thereby lowers interest rates. Two proxies for climate change are investigated, temperature deviations and an index about climate-related news coverage in the Wall Street Journal. Especially the latter risk factor appears to be a promising proxy for climate risk (awareness). The model estimates imply that higher values of the news index are associated with lower future nominal interest rates, consistent with the intuition that climate risk tends to push interest rates down.

This last chapter combines the methods of Chapters 1 and 2, by considering the impact of estimation errors on expected utility and subsequently determining the robust strategy that performs best under the worst-case scenario with the lowest expected utility. Two types of uncertainty are considered. The first approach considers an investor who is uncertain whether there is a climate impact. In this case, the robust strategy is to invest like there is a climate impact, even if the investor initially believes that there is not. This prevents large losses in wealth when climate change actually affects the nominal interest rate in the future financial market. The second approach considers an investor who assumes a climate risk impact, but is uncertain about estimation errors in the parameters of the underlying climate factor process which is proxied by temperature deviations. The results show that the investor should be most concerned about errors in the correlation between the temperature deviations and the nominal interest rate. The robust strategy avoids large losses due to this estimation error.

In summary, the chapters in this thesis show that long-term bond strategies should not rely solely on initial estimates, because even small errors in parameters can have large consequences for the investor's wealth. They also show under which conditions robust strategies can provide valuable protection by outperforming bond allocations that are only optimal when parameter values are correctly estimated.

Samenvatting

Dit proefschrift onderzoekt de rol van onzekerheid over parameterwaarden in het renteproces bij het bepalen van langetermijninvesteringen in nominale obligaties. Een belangrijk doel van langetermijnbeleggers, zoals institutionele beleggers, is het behoud van de waarde van hun investeringen gedurende meerdere decennia. Een risico hierbij is dat de rente kan dalen, waardoor ook de rendementen op bankdeposito's afnemen. Daarom worden obligaties vaak gebruikt als bescherming, omdat hun waarde stijgt wanneer rentes dalen. Standaardmodellen zijn gebaseerd op het concept van risico: de belegger schat het renterisico in en bepaalt vervolgens de optimale strategie die het hoogste verwachte nut geeft onder deze verwachtingen. Het is echter lastig om een renteproces nauwkeurig te schatten. Dit maakt parameteronzekerheid relevant: de mogelijkheid dat de inschattingen verkeerd zijn.

Dit proefschrift laat zien dat onzekerheid over renterisico's grote gevolgen kan hebben voor langetermijninvesteerders die beleggen in obligaties. Wanneer een belegger haar strategie niet baseert op juiste aannames, kan haar vermogen aanzienlijk dalen ten opzichte van een situatie waarin zij dit wel doet. Om dat potentiële verlies te beperken, introduceert dit proefschrift robuuste strategieën: beleggingsstrategieën die mogelijk niet optimaal zijn onder de initiële aannames, maar beter presteren indien aannames afwijken van de werkelijkheid.

Hoofdstuk 1 onderzoekt de situatie waarin beleggers de reële rente (nominale rente minus inflatie) willen afdekken, maar ze onzeker zijn over de schattingen van onderliggende nominale rente- en inflatierisico's. Zonder schattingsfouten is de optimale strategie om in twee nominale obligaties met verschillende looptijden te investeren.[21] De analyse laat echter zien dat de vermogensallocaties naar deze obligaties zeer gevoelig zijn voor veranderingen in de parameterwaarden die bepalen hoe snel de rente terugkeert naar de "normale" staat na een schok, en hoe sterk de inflatie de nominale rente beïnvloedt. Als de

[21] Zoals uitgelegd in Hoofdstuk 2 kan de term "obligatie" ook verwijzen naar een obligatieportefeuille die dezelfde rentegevoeligheid heeft als een enkele obligatie, gebruikelijker in de praktijk. Een investeringsstrategie waarbij belegd wordt in twee obligaties zou dan verwijzen naar twee portefeuilles met verschillende gevoeligheden.

investeerder maar in één obligatie belegt is dit niet optimaal onder de initiële aannames, maar deze strategie kan onder parameteronzekerheid stabieler blijken dan als zij in twee obligaties belegt.

Hoofdstuk 2 ontwikkelt een formele methode voor robuustheid. Parameteronzekerheid wordt hierbij gemodelleerd door een denkbeeldige “moeder natuur” die de financiële markt verstoort naar de minst gunstige situatie voor de belegger. De belegger erkent dat dit scenario kan optreden en kiest de strategie die in dat geval het beste zou presteren. Twee typen verstoringen worden geanalyseerd. Het eerste type is onafhankelijk van de huidige rente, en leidt tot een robuuste strategie die overeenkomt met die van een zeer risico-averse belegger. Dit betekent dat een robuuste belegger, die rekening houdt met het scenario waarin risico nemen niet wordt beloond, belegt als een belegger die primair gericht is op het afdekken van renterisico.

Het tweede type betreft verstoringen die afhankelijk zijn van de huidige rente. Dit type blijkt relevant, omdat het overeenkomt met veranderingen in de persistentie van schokken in de rente, waarvan Hoofdstuk 1 de gevolgen aantoonde. Dit leidt tot robuuste obligatieallocaties die afhangen van het actuele renteniveau en de resterende beleggingshorizon. Dit impliceert dat een belegger bij het bepalen van de robuuste strategie zowel de staat van de financiële markt als de beleggingshorizon moet meenemen.

Hoofdstuk 3 introduceert een nieuw type parameteronzekerheid: de mogelijkheid dat klimaatverandering de nominale rente beïnvloedt. Meer klimaatrisico, werkelijk of waargenomen, kan obligaties aantrekkelijker maken voor beleggers die deze zien als risicovrije beleggingen. Dit verhoogt de vraag naar obligaties en drukt daardoor de rente. Twee benaderingen voor klimaatverandering worden onderzocht: temperatuurafwijkingen en een index van klimaatgerelateerd nieuws in de Wall Street Journal. Vooral de laatste blijkt een veelbelovende indicator voor klimaatrisico. De modelschattingen suggereren dat hogere waarden van deze nieuwsindex samenhangen met lagere toekomstige nominale rentevoeten, in lijn met de intuïtie dat klimaatrisico de rente verlaagt.

Dit laatste hoofdstuk combineert de methoden van Hoofdstuk 1 en 2 door zowel het effect van schattingsfouten op het verwachte nut te analyseren als de robuuste strategie te bepalen die het best presteert in de minst gunstige markt met het laagste verwachte nut. Twee vormen van onzekerheid worden onderzocht. In het eerste geval is de belegger onzeker over het bestaan van een klimaateffect. De robuuste strategie blijkt om te beleggen alsof er wel een klimaateffect is, zelfs als de belegger aanvankelijk inschat dat dit niet zo is. Dit voorkomt grote vermogensverliezen indien klimaatverandering in de toekomst toch de rente beïnvloedt. In het tweede geval is er een klimaateffect en de belegger is zich hiervan bewust. De investeerder is echter onzeker over de parameterwaardes van het onderliggende klimaatproces, gemeten aan de hand van temperatuurafwijkingen. De resultaten laten zien dat de belegger zich vooral zorgen moet maken over fouten in de correlatie tussen temperatuurafwijkingen en de nominale rente. De robuuste strategie voorkomt grote verliezen door dit type schattingsfout.

Samengevat tonen de hoofdstukken in dit proefschrift aan dat langetermijnstrategieën met obligaties niet uitsluitend gebaseerd mogen worden op initiële modelschattingen, omdat zelfs kleine parameterfouten grote gevolgen kunnen hebben voor het vermogen van de belegger. Tevens worden omstandigheden geschetst waarin robuuste strategieën waardevolle bescherming kunnen bieden door beter te presteren dan aanvankelijk optimale obligatieallocaties als parameters verkeerd zijn geschat.

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LIESKE COUMANS (Capelle aan den IJssel, 1992) received her Master's degree in Econometrics and Operations Research from Maastricht University in 2015. While working at an actuarial consultancy firm and an insurance company, she qualified as a certified AG actuary through the University of Amsterdam. In September 2019, she started her PhD at the Department of Econometrics and Operations Research at Tilburg University, partially funded by the Network for Studies on Pensions, Ageing and Retirement (Netspar). During her PhD, she conducted a research visit to Aarhus University in 2022. In September 2024, she started as a postdoctoral researcher at Tilburg University, where she works at the Department of Economics and at the Academic Collaborative Center for Governance and Management for Broad Prosperity.

Interest rate risk is a major concern for long-term investors. Standard models rely on a known distribution of this risk, while in practice the interest rate process is difficult to estimate precisely. This dissertation consists of three chapters showing that long-term investors should not rely solely on a single set of parameter estimates, but should instead consider robust strategies that perform well when financial markets evolve differently than expected.

The first chapter demonstrates that estimation errors in interest rate dynamics can strongly affect investment outcomes, sometimes making single-bond strategies more stable than two-bond strategies. The second chapter introduces parameter uncertainty and derives robust strategies within a max–min framework, in which investors account for worst-case market conditions. The third chapter analyses uncertainty about a potential climate impact on interest rates. It studies the impact of estimation errors on utility and applies the max-min approach to derive robust strategies that account for these errors.

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