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Servaas van Bilsen, Ilja Boelaars

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Optimal Consumption and Portfolio Choice in the Presence of Risky House Prices*

Servaas van Bilsen†
Dept. of Quantitative Economics
University of Amsterdam
and NETSPAR

Ilja A. Boelaars
ING Bank N.V.

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Abstract

This paper explores optimal consumption and portfolio decisions in the presence of risky house prices. We assume that changes in real interest rates and future rents directly impact house prices. A novel aspect of our model is that rent inflation rates and consumption inflation rates are cointegrated. We show that the individual prefers to be a home owner when young and a renter when old. This motives the design of so-called reverse mortgage products. Furthermore, she invests significantly less pension wealth in inflation-linked bonds, as compared to conventional wisdom. Finally, we find that the optimal mortgage changes from fixed-rate to adjustable-rate as the individual becomes older.

JEL Classification: D15, D81, G11, R21, O18.

OR/MS Classification: Risk; Investment; Portfolio.

Keywords: Housing; Optimal Consumption Behavior; Optimal Portfolio Behavior; Real Interest Rate Risk; Rental Price Risk.

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†Mailing Address: PO Box 15867, 1001 NJ Amsterdam, The Netherlands. Phone: +31 (0) 20 525 5389.

1 Introduction

Housing represents the largest asset in household portfolios. For example, OECD (2021) estimate that for the average US household, housing is 37% of total assets. Hence, a relevant question is how an individual should consume and invest over her life-cycle in the presence of housing. Furthermore, as has been recently demonstrated, house prices drop following interest rate increases and lower expected future rents. This paper explores optimal consumption and portfolio choice for a setting in which changes in interest rates and future rents directly impact house prices.

We show that the individual prefers to be a home owner when young and a renter when old. This motives the design of so-called reverse mortgage products which allow her to convert home equity into cash and use the cash to rent back the house. We also show that she invests significantly less pension wealth into fixed-income securities, compared to conventional wisdom (Bodie, Merton, and Samuelson (1992)). For reasonable parameter values, we find that she waits until age 45 to start investing pension wealth in bonds. Finally, we show that the optimal mortgage constantly changes over her life-cycle. In particular, it changes from fixed-rate to adjustable-rate as she grows older.

The individual divides, in each period, her income between housing consumption, non-housing consumption, and savings. Her asset portfolio consists of cash, stocks, nominal and inflation-linked bonds, and housing. Hence, she can hedge against her housing costs by buying a house. We assume that the house price equals the expected discounted value of future rent payments, inspired by the dividend discount model (Gordon and Shapiro (1956) and Gordon (1959)). As a result, increases in real interest rates and decreases in future rents directly lead to lower house prices. We derive closed-form expressions for house prices, housing and non-housing consumption, and portfolio policies. A closed-form expression has several key advantages: it shows the roles played by the model parameters and it facilitates the implementation of the optimal policies.

A novel aspect of our model is that rent inflation rates and consumption inflation rates are cointegrated. This implies that, in the short-run, the price per square foot of living space can increase faster than the price of the consumption bundle. However, in the long-run, the rent inflation rate equals the consumption inflation rate. A direct implication of this cointegration is that the real rent inflation rate[1] follows a mean reverting process with a long-term mean of zero. We use Dutch data to provide empirical support for this

[1] The real rent inflation rate is defined as the difference between the rent inflation rate and the consumption inflation rate.

model implication. In particular, we find that the half-life of Dutch real rent inflation shocks is approximately equal to 1.5 years. That is, we anticipate a positive or negative Dutch real rent inflation rate to revert back to its mean of zero within 3 years. Another model implication is that the real house price is high if real interest rates are low and/or real rents are high. We find that, except for the post-financial crisis period 2009-2014, Dutch data provides empirical support for this model implication.

Our three main findings are as follows. First, we find that without bequest motives, the individual optimally decreases the degree of home ownership from 100% to 0%. When she is young, she fully owns the house she lives in. As she grows older, she becomes both a renter and a home owner. Close to the end of her life-cycle, she fully rents the house she lives in. Thus, the optimal share of wealth invested in housing drastically decreases with age, while optimal real housing consumption remains largely the same over her life-cycle. This motivates the design of so-called reverse mortgage products which allow a home owner to convert home equity into cash and use the cash to rent back the house. By buying a reverse mortgage product, she can reduce the value of her housing portfolio while, at same time, living in her home.

Second, we find that, compared to conventional wisdom (Bodie et al. (1992)), she invests less financial wealth in inflation-linked bonds. This finding holds irrespective of whether she has access to the housing market. In our benchmark case without housing, we can explain the lower demand for inflation-linked bonds by the presence of human wealth, which equals the discounted value of future labor earnings. Indeed, stable labor income can be seen as a bond. As a result, she prefers to decrease her investments in inflation-linked bonds; see also van Bilsen, Boelaars, and Bovenberg (2020). With housing, our result is much stronger since now both stable labor income and stable future rents can be seen as a bond. This observation causes her to invest substantially less financial wealth in inflation-linked bonds. Note that our finding also implies that she invests significantly less pension wealth[2] in inflation-linked bonds.

Third, we find that the optimal mortgage changes constantly over the individual's life-cycle. We observe that, in the beginning of her life-cycle, the total mortgage amount is equal to the value of her housing portfolio. That is, net housing wealth is zero. Hence, when the individual is young, she does not repay her mortgage. Furthermore, the optimal mortgage changes from fixed-rate to adjustable-rate as she grows older. Indeed, when the individual is young, she already holds large investments in the bond-like assets human wealth and gross housing wealth. Therefore, she goes short in inflation-linked bonds,

[2]Pension wealth is defined as the difference between financial wealth and net housing wealth.

which can be seen as a fixed-rate mortgage. Later in life, when human wealth and gross housing wealth are small, she goes short in cash, which can be viewed as an adjustable-rate mortgage.

1.1 Related Literature

The question of how to consume and invest wealth over the life-cycle has been extensively studied in different contexts since the seminal works of Mossin (1968), Merton (1969) and Samuelson (1969). For example, many papers[3] study how individuals should save and invest in an environment with stochastic interest rates. However, only a limited number of papers include interest rate risk in a model with housing (Campbell and Cocco (2003), van Hemert (2010) and Campbell and Cocco (2015)). Campbell and Cocco do not explore the impact of housing on households' optimal consumption and portfolio choices. Rather, they focus on how a household should choose between a fixed-rate and an adjustable-rate mortgage (Campbell and Cocco (2003)) and on households' mortgage default decisions (Campbell and Cocco (2015)). Although van Hemert (2010) studies households' asset allocation decisions in the presence of housing and stochastic real interest rates, in his setting, real interest rates do not directly impact house prices.[4] Furthermore, he assumes that the rent is a constant share of the house price.

A number of authors consider optimal consumption and portfolio choice with housing, but without real interest rate and rental price risk; see, e.g., Sinai and Souleles (2005), Cocco (2005), Yao and Zhang (2005), Stokey (2009), Corradin, Fillat, and Vergara-Alert (2014) and Chetty, Sandor, and Szeidl (2017). These authors focus on the optimal investment in risky stocks (Cocco (2005), Yao and Zhang (2005), Stokey (2009), Corradin et al. (2014) and Chetty et al. (2017)) or on the rent-versus-buy decision (Yao and Zhang (2005) and Sinai and Souleles (2005)). For example, Cocco (2005) and Yao and Zhang (2005) predict that housing leads to lower investments in risky stocks, while Corradin et al. (2014) find that the optimal stock investments depend on the expected growth rates in house prices. Our focus is on the optimal bond investments in the presence of housing.

Finally, we note that a handful papers explore optimal household behavior in a realistically calibrated life-cycle model; see, e.g., Viceira (2001), Cocco, Gomes, and

[3] See, e.g., Campbell and Viceira (2001), Brennan and Xia (2002), Sangvinatsos and Wachter (2005), Liu (2007) and Koijen, Nijman, and Werker (2010).

[4] van Hemert (2010) models house prices as a random walk with drift; see also, e.g., Flavin and Nakagawa (2008).

Maenhout (2005), Chai, Horneff, Maurer, and Mitchell (2011) and Hubener, Maurer, and Mitchell (2016).[5] These papers are successful in explaining several empirical facts such as the hump-shaped equity allocation over the life-cycle as documented by, e.g., Ameriks and Zeldes (2004). Most authors use numerical or approximation techniques to arrive at the optimal household decisions, while we present closed-form expressions. Furthermore, they do not include housing in the model specification.

The remainder of the paper is structured as follows. Section 2 introduces the model. In Section 3, we provide empirical support for two main modeling assumptions: the cointegration between rent inflation rates and consumption inflation rates, and the dependency of house prices on real interest rates and future expected rents. The optimal policies are presented in Section 4. In Section 5, we discuss the main findings. Finally, Section 6 concludes the paper. The proofs of the main theorems and the technical details are delegated to Appendix A.

2 Model

This section presents our continuous-time consumption and portfolio choice model. Denote by t adult age, which corresponds to effective age minus 20. For sake of simplicity, we assume that the individual dies at the non-random adult age $T > 0$.

2.1 Preferences

We denote by $h(t)$ and $c(t)$ real housing consumption (i.e., number of square feet of living space) and real non-housing consumption (i.e., number of consumption goods) at adult age t, respectively. The individual has CRRA preferences over time and risk and Cobb-Douglas preferences over real housing consumption and real non-housing consumption. Hence, expected lifetime utility is given by

$$U = \mathbb{E}_0 \left[\int_0^T e^{-\delta t} \frac{1}{1-\gamma} \left(h(t)^{\varphi} c(t)^{1-\varphi} \right)^{1-\gamma} \mathrm{d}t \right], \tag{2.1}$$

where $\delta \geq 0$ denotes the subjective rate of time preference, $\gamma > 0$ corresponds to the coefficient of relative risk aversion, $0 \leq \varphi \leq 1$ is the Cobb-Douglas share parameter, and $\mathbb{E}_0 [\cdot]$ represents the expectation conditional upon information available at adult age 0.

[5] See also, e.g., Gomes and Michaelides (2003), Horneff, Maurer, Mitchell, and Rogalla (2015), Horneff, Maurer, and Mitchell (2019), Horneff, Maurer, and Mitchell (2020), Maurer, Mitchell, Rogalla, and Schimetschek (2021) and Horneff, Maurer, and Mitchell (2022).

2.2 State Variables

We consider an economy with four state variables: the real stock price $S(t)$, the short-term real interest rate $r(t)$, the rent inflation rate $\pi_h(t)$, and the consumption inflation rate $\pi(t)$. The state variables $\pi_h(t)$ and $\pi(t)$ denote the price inflation rates of real housing consumption and real non-housing consumption, respectively. We assume that the real stock price, the short-term real interest rate and the consumption inflation rate follow the same dynamics as in Brennan and Xia (2002). That is, the short-term real interest rate $r(t)$ and the consumption inflation rate $\pi(t)$ are described by an Ornstein-Uhlenbeck process and the real stock price $S(t)$ evolves according to a geometric Brownian motion. Furthermore, we assume that the rent inflation rate $\pi_h(t)$ and the consumption inflation rate $\pi(t)$ are cointegrated. This captures the idea that the prices of housing consumption and non-housing consumption cannot diverge away from each other by a large distance. We model cointegration in a similar fashion as in Benzoni, Collin-Dufresne, and Goldstein (2007) who assume that labor income and dividends are cointegrated. In our setting, this means that the real rent inflation rate $\pi_h(t) - \pi(t)$ follows an Ornstein-Uhlenbeck process with a long-term mean equal to zero. The dynamics of the state variables are thus described by the following equations:

$$\mathrm{d}S(t) = (r(t) + \lambda_S \sigma_S)\, S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}Z_S(t), \tag{2.2}$$

$$\mathrm{d}r(t) = \kappa_r \left(\bar{r} - r(t)\right) \mathrm{d}t + \sigma_r \mathrm{d}Z_r(t), \tag{2.3}$$

$$\mathrm{d}\pi(t) = \kappa_\pi \left(\bar{\pi} - \pi(t)\right) \mathrm{d}t + \sigma_\pi \mathrm{d}Z_\pi(t), \tag{2.4}$$

$$\mathrm{d}\left(\pi_h(t) - \pi(t)\right) = \kappa_h \left(0 - [\pi_h(t) - \pi(t)]\right) \mathrm{d}t + \sigma_h \mathrm{d}Z_h(t). \tag{2.5}$$

Here, $\bar{r} \in \mathbb{R}$ and $\bar{\pi} \in \mathbb{R}$ denote the expected short-term real interest rate in the long-run and the expected consumption inflation rate in the long run, respectively, $\kappa_r, \kappa_\pi, \kappa_h \geq 0$ are mean reversion coefficients, $\lambda_S \geq 0$ is the Sharpe ratio of the risky stock, $Z(t) = (Z_S(t), Z_r(t), Z_\pi(t), Z_h(t))$ represents a vector of standard Brownian motions, and $\sigma = (\sigma_S, \sigma_r, \sigma_\pi, \sigma_h) \geq 0$ is a vector of diffusion coefficients.[6] We summarize the (linear) correlation coefficients between the Brownian increments in the correlation matrix ρ:

[6] For notational convenience, we often write a (column) vector in the form $y = (y_1, y_2, \dots, y_n)$, where y_i represents the ith element of y.

$$\rho = \begin{pmatrix} 1 & \rho_{Sr} & \rho_{S\pi} & \rho_{Sh} \\ \rho_{rS} & 1 & \rho_{r\pi} & \rho_{rh} \\ \rho_{\pi S} & \rho_{\pi r} & 1 & \rho_{\pi h} \\ \rho_{hS} & \rho_{hr} & \rho_{h\pi} & 1 \end{pmatrix}, \tag{2.6}$$

where $\rho_{ij} \in [-1, 1]$ ($i, j \in \{S, r, \pi, h\}$, $i \neq j$) denotes the correlation coefficient between $\mathrm{d}Z_i(t)$ and $\mathrm{d}Z_j(t)$.

The nominal rental price index $\Pi_h(t)$ (i.e., price per square foot of living space) and the consumer price index $\Pi(t)$ are defined as follows:

$$\Pi_h(t) = \exp\left\{\int_0^t \pi_h(s)\mathrm{d}s\right\}, \tag{2.7}$$

$$\Pi(t) = \exp\left\{\int_0^t \pi(s)\mathrm{d}s\right\}. \tag{2.8}$$

It follows that the relative change in the real rental price index $\widetilde{\Pi}_h(t) = \Pi_h(t)/\Pi(t)$ (i.e., price per square foot of living space in terms of real consumption goods) can diverge from zero:

$$\frac{\mathrm{d}\widetilde{\Pi}_h(t)}{\widetilde{\Pi}_h(t)} = (\pi_h(t) - \pi(t))\,\mathrm{d}t. \tag{2.9}$$

Hence, the price per square foot of living space can increase faster than the price of the consumption bundle. However, from (2.5), we observe that, in the long-run, the change of the real rental price index is zero. Hence, the prices of real housing consumption and real non-housing consumption are cointegrated. That is, in the long-run, the price per square foot of living space cannot grow faster than the price of the consumption bundle.

Denote by $m(t)$ the real stochastic discount factor at adult age t. We use this factor to determine the value (or price) of future real consumption goods in terms of current real consumption goods. We assume that $m(t)$ satisfies the following dynamics:

$$\frac{\mathrm{d}m(t)}{m(t)} = -r(t)\mathrm{d}t + \phi^\top \mathrm{d}Z(t). \tag{2.10}$$

Here, $\top$ denotes the transpose sign, and $\phi = (\phi_S, \phi_r, \phi_\pi, \phi_h) \in \mathbb{R}^4$ is a vector of factor loadings, which determines the vector of market prices of risk associated with the underlying state variables. More specifically, we can obtain the vector of market prices of risk $\lambda = (\lambda_S, \lambda_r, \lambda_\pi, \lambda_h)$ from ϕ as follows:

$$\lambda = -\rho\phi. \tag{2.11}$$

2.3 Asset Price Dynamics and Dynamic Budget Constraint

We assume that the individual can invests her total wealth, which consists of financial wealth and human wealth, into four risky assets: a nominal bond with time to maturity h_N, an inflation-linked bond with time to maturity h_I, a stock and a housing portfolio. This section derives the bond price dynamics, the dynamics of the value of the housing portfolio and the budget constraint. The stock price dynamics is given before (see (2.2)).

2.3.1 Bond Price Dynamics

Let $P_N(t, h_N)$ and $P_I(t, h_I)$ denote the real prices (i.e., in terms of real consumption goods) at adult age t of a zero-coupon nominal bond with time to maturity h_N and a zero-coupon inflation-linked bond with time to maturity h_I, respectively. Appendix A.1 derives the dynamics of $P_N(t, h_N)$ and $P_I(t, h_I)$. We find

$$\begin{aligned} \mathrm{d}P_N(t, h_N) = & \left(r(t) - \lambda_r\sigma_r B_r(h_N) - \lambda_\pi\sigma_\pi B_\pi(h_N)\right) P_N(t, h_N)\,\mathrm{d}t \\ & - P_N(t, h_N)\left[B_r(h_N)\,\sigma_r \mathrm{d}Z_r(t) + B_\pi(h_N)\,\sigma_\pi \mathrm{d}Z_\pi(t)\right], \end{aligned} \tag{2.12}$$

$$\mathrm{d}P_I(t, h_I) = \left(r(t) - \lambda_r\sigma_r B_r(h_I)\right) P_I(t, h_I)\,\mathrm{d}t - B_r(h_I)\,\sigma_r P_I(t, h_I)\,\mathrm{d}Z_r(t). \tag{2.13}$$

Here, $B_r(h) = \left(1 - e^{-\kappa_r h}\right)/\kappa_r \in [0, h]$ and $B_\pi(h) = \left(1 - e^{-\kappa_\pi h}\right)/\kappa_\pi \in [0, h]$ model the sensitivity of a bond with time to maturity h to real interest rate changes and inflation rate changes, respectively. Note that $B_r(h)$ and $B_\pi(h)$ converge to h as real interest rates and inflation rates become less predictable (i.e., as κ_r and κ_π go to zero).

2.3.2 Dynamics of Value of Housing Portfolio

Inspired by the dividend discount model (Gordon and Shapiro (1956) and Gordon (1959)), we assume that the real house price is equal to the expected discounted value of all future rents. Let $P_h(t)$ be the real price (i.e., in terms of real consumption goods) at adult age t of a house. Appendix A.2 shows that $P_h(t)$ satisfies the following dynamics under the assumption that no maintenance is conducted on the house:

$$\begin{aligned} \mathrm{d}P_h(t) = &-\widetilde{\Pi}_h(t)\mathrm{d}t + \left(r(t) - \lambda_r\sigma_r\widehat{B}_r(t) + \lambda_h\sigma_h\widehat{B}_h(t)\right)P_h(t)\mathrm{d}t \\ &- \widehat{B}_r(t)\sigma_r P_h(t)\mathrm{d}Z_r(t) + \widehat{B}_h(t)\sigma_h P_h(t)\mathrm{d}Z_h(t), \end{aligned} \tag{2.14}$$

with

$$\widehat{B}_r(t) = \frac{\widetilde{\Pi}_h(t)\int_0^\infty e^{-\delta_h h}B_r(h)P_h(t,h)\mathrm{d}h}{P_h(t)}, \tag{2.15}$$

$$\widehat{B}_h(t) = \frac{\widetilde{\Pi}_h(t)\int_0^\infty e^{-\delta_h h}B_h(h)P_h(t,h)\mathrm{d}h}{P_h(t)}, \tag{2.16}$$

$$P_h(t,h) = \exp\left\{-r(t)B_r(h) + \left(\pi_h(t) - \pi(t)\right)B_h(h) - k_h(h)\right\}. \tag{2.17}$$

Here, the expression of $k_h(h)$ is given in Appendix A.2 (see (A26)) and $\delta_h \geq 0$ models the depreciation of the house. This parameter is high if maintenance costs are high (e.g., if the house is old). Equation (2.14) shows that the return on the house depends on four terms. The first term on the right-hand side of (2.14) shows that the price of the house declines because the house becomes older. The second term models the expected financial return on the house. We observe that the individual collects two risk premiums: $-\lambda_r\sigma_r\widehat{B}_r(t)$ and $\lambda_h\sigma_h\widehat{B}_h(t)$. The first risk premium, i.e., $-\lambda_r\sigma_r\widehat{B}_r(t)$, is due to the fact that the house price is exposed to real interest rate risk. As can be seen from the third term on the right-hand side of (2.14), $\widehat{B}_r(t)$ models the real interest rate sensitivity of the house price. The second risk premium, i.e., $\lambda_h\sigma_h\widehat{B}_h(t)$, is due to the fact that the house price is exposed to real rent inflation risk. As can be seen from the fourth term on the right-hand side of (2.14), $\widehat{B}_h(t)$ models the sensitivity of the house price to unexpected shocks in the real rent inflation rate $\pi_h(t) - \pi(t)$.

The individual does not directly invest in a house. Rather, she invests in a housing portfolio that reinvests incoming rents in houses. Denote by $W_h(t)$ the real value (i.e., in terms of real consumption goods) at adult age t of an account that only invests in housing. Appendix A.2 shows that $W_h(t)$ satisfies the following dynamics:

$$\frac{\mathrm{d}W_h(t)}{W_h(t)} = \left(r(t) - \lambda_r\sigma_r\widehat{B}_r(t) + \lambda_h\sigma_h\widehat{B}_h(t)\right)\mathrm{d}t - \widehat{B}_r(t)\sigma_r\mathrm{d}Z_r(t) + \widehat{B}_h(t)\sigma_h\mathrm{d}Z_h(t). \tag{2.18}$$

2.3.3 Dynamic Budget Constraint

Let $\omega(t) = (\omega_S(t), \omega_N(t), \omega_I(t), \omega_h(t))$ be the vector of portfolio weights at adult age t. Here, $\omega_S(t)$, $\omega_N(t)$, $\omega_I(t)$, and $\omega_h(t)$ denote the shares of total wealth invested in a risky stock (with price dynamics (2.2)), a nominal bond with time to maturity h_N (with price dynamics (2.12)), an inflation-linked bond with time to maturity h_I (with price dynamics (2.13)), and a housing portfolio (with price dynamics (2.18)), respectively. As a result, the share of total wealth invested in cash at adult age t is given by $1-\omega_S(t)-\omega_N(t)-\omega_I(t)-\omega_h(t)$. Let $W(t)$ denote the investor's real total wealth (i.e., in terms of real consumption goods) at adult age t which satisfies the following dynamic budget constraint:

$$\begin{aligned} \mathrm{d}W(t) &= \left(r(t) + \omega(t)^\top \left[\mu(t) - r(t)\right]\right) W(t)\mathrm{d}t \\ &\quad + \omega(t)^\top \Sigma(t) W(t) \mathrm{d}Z(t) - \left(\widetilde{\Pi}_h(t)h(t) + c(t)\right) \mathrm{d}t. \end{aligned} \tag{2.19}$$

Here,

$$\mu(t) - r(t) = \begin{pmatrix} \lambda_S \sigma_S \\ -\lambda_r \sigma_r B_r(h_N) - \lambda_\pi \sigma_\pi B_\pi(h_N) \\ -\lambda_r \sigma_r B_r(h_I) \\ -\lambda_r \sigma_r \widehat{B}_r(t) + \lambda_h \sigma_h \widehat{B}_h(t) \end{pmatrix} \tag{2.20}$$

and

$$\Sigma(t) = \begin{pmatrix} \sigma_S & 0 & 0 & 0 \\ 0 & -B_r(h_N)\sigma_r & -B_\pi(h_N)\sigma_\pi & 0 \\ 0 & -B_r(h_I)\sigma_r & 0 & 0 \\ 0 & -\widehat{B}_r(t)\sigma_r & 0 & \widehat{B}_h(t)\sigma_h \end{pmatrix}. \tag{2.21}$$

We note that the last term on the right-hand side of the dynamic budget constraint (2.19) denotes the individual's real spending (i.e., in terms of real consumption goods) at adult age t on real housing consumption and real non-housing consumption. We also note that the individual is a renter and has to pay for real housing consumption. However, by investing in the housing portfolio, the individual reduces or eliminates real rental price risk and, in fact, becomes a home owner.

2.4 Individual's Maximization Problem

The individual faces the following dynamic maximization problem:

$$\begin{aligned} \max_{h(t),c(t),\omega(t)} \quad & \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{1-\gamma}\left(h(t)^{\varphi}c(t)^{1-\varphi}\right)^{1-\gamma}\mathrm{d}t\right] \\ \text{s.t.} \quad & \mathrm{d}W(t) = \left(r(t)+\omega(t)^{\top}\left[\mu(t)-r(t)\right]\right)W(t)\mathrm{d}t \\ & \qquad + \omega(t)^{\top}\Sigma(t)W(t)\mathrm{d}Z(t) - \left(\widetilde{\Pi}_h(t)h(t)+c(t)\right)\mathrm{d}t. \end{aligned} \tag{2.22}$$

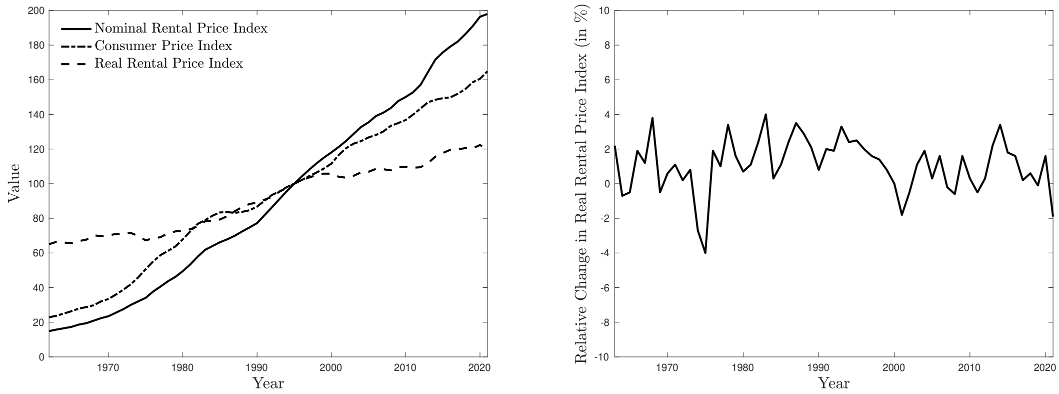
Section 4 analyzes and discusses the optimal policies over the individual's life-cycle.

3 Empirical Support

As mentioned in Section 2, we assume that the prices of real housing consumption and real non-housing consumption are cointegrated. That is, the relative change in the real rental price index $\widetilde{\Pi}_h(t)$ can diverge from zero, but, in the long-run, the price per square foot of living space cannot grow faster than the price of the consumption bundle. Figure 1 shows the Dutch nominal rental price index $\Pi_h(t)$, the Dutch consumer price index $\Pi(t)$, the Dutch real rental price index $\widetilde{\Pi}_h(t)$, and the relative change in the Dutch real rental price index, i.e., $\mathrm{d}\widetilde{\Pi}_h(t)/\widetilde{\Pi}_h(t)$. We use data from Statistics Netherlands. Our time period runs from 1962 to 2021.

We observe from the right panel of Figure 1 that the relative change in the real rental price index seems to follow a mean-reverting process. This provides empirical support for our specification of $\mathrm{d}\widetilde{\Pi}_h(t)/\widetilde{\Pi}_h(t)$; see (2.5). We now estimate the speed of mean reversion κ_h and the volatility σ_h using maximum likelihood estimation (MLE). We find the following estimated parameters: $\hat{\kappa}_h = 0.443$ and $\hat{\sigma}_h = 0.0151$. In Section 5, we assume a half-life of the real rent inflation rate of 1.5 years which roughly corresponds to $\kappa_h = 0.443$.

Another novel aspect of our model is that the real house price is equal to the expected discounted value of all future rents. Equation (A18) in Appendix A.2 shows that the real house price is high if the short-term real interest rate $r(t)$ is low and/or the real rental price index $\widetilde{\Pi}_h(t)$ is high. Figure 2 shows the Dutch real house price index, the Dutch real rental price index (see also Figure 1) and the Dutch short-term real interest rate. We use data from Statistics Netherlands and the OECD. Our time period runs from 1995 to 2021. We define the short-term real interest rate to be the difference between the short-term nominal interest rate and the inflation rate. We observe that real house



(a) indices (b) real rent inflation rate

Figure 1. Consumer Price Index, Nominal Rental Price Index, Real Rental Price Index and Change in Real Rental Price Index The left panel shows the Dutch consumer price index, the Dutch nominal rental price index and the Dutch real rental price index. The right panel shows the change in the Dutch real rental price index (i.e., the real rent inflation rate). We use data from Statistics Netherlands. Our time period runs from 1962 to 2021.

prices increase since 1995 except for the period 2009 to 2014 (post-financial crisis years). At the same time, we observe a declining trend in the short-term real interest rate and an increasing trend in real rents. Except for the post-financial crisis period 2009-2014, Dutch data thus provides support for the model implication that decreasing real interest rates and increasing real rents correspond to increasing real house prices.

4 Optimal Life-Cycle Policies

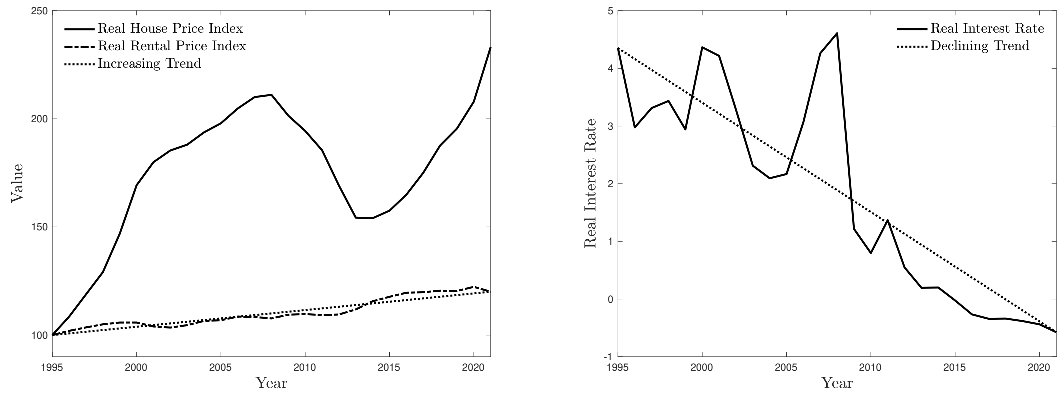
4.1 Optimal Consumption Choice

We are now ready to present the optimal real housing consumption and the optimal real non-housing consumption.

Theorem 1 (optimal consumption choice) *Consider an individual who solves the maximization problem (2.22). Then the individual's optimal real housing consumption $h^*(t)$ and the individual's optimal real non-housing consumption $c^*(t)$ are given by*

$$h^*(t) = h^*(0)e^{-\frac{\delta t}{\gamma}} m(t)^{-\frac{1}{\gamma}} \widetilde{\Pi}_h(t)^{-\frac{(1-\varphi)\gamma+\varphi}{\gamma}}, \tag{4.1}$$

$$c^*(t) = \frac{1-\varphi}{\varphi} h^*(t)\widetilde{\Pi}_h(t). \tag{4.2}$$



(a) indices (b) real rent inflation rate

Figure 2. Real House Price Index, Real Rental Price Index and Real Interest Rate The left panel shows the Dutch real house price index and the Dutch real rental price index. The right panel shows the Dutch short-term real interest rate. We use data from Statistics Netherlands and the OECD. Our time period runs from 1995 to 2021. The short-term real interest rate is equal to the difference between the short-term nominal interest rate and the inflation rate.

Here, $m(t)$ is the real stochastic discount factor and $\widetilde{\Pi}_h(t)$ is the real rental price index.

The optimal real housing consumption at adult age 0, i.e., $h^*(0)$, is chosen such that the static budget constraint holds with equality.

Proof. See Appendix A.3.

We observe that both optimal real housing consumption and optimal real non-housing consumption depend on the real stochastic discount factor $m(t)$ and the real rental price index $\widetilde{\Pi}_h(t)$. If the economy worsens, i.e., the stochastic discount factor goes up, the individual lowers optimal real housing consumption as well as optimal real non-housing consumption. Indeed, the individual's budget decreases due to bad economic times. If the real rental price index increases, optimal real housing consumption decreases, while optimal real non-housing consumption increases. Indeed, an increase in the real rental price index means that real housing consumption becomes relatively more expensive and real non-housing consumption becomes relatively cheaper.

4.2 Optimal Portfolio Choice

We are now ready to present the optimal portfolio choice.

Theorem 2 (optimal portfolio choice total wealth) *Consider an individual who solves the maximization problem (2.22). Then the individual's optimal shares of total* *wealth invested in the risky assets are given by*

$$\omega_S^*(t) = -\frac{1}{\gamma}\frac{\phi_S}{\sigma_S}, \tag{4.3}$$

$$\omega_N^*(t) = \frac{1}{\gamma}\frac{\phi_\pi}{B_\pi\left(h_N\right)\sigma_\pi}, \tag{4.4}$$

$$\omega_I^*(t) = \frac{1}{\gamma}\frac{\phi_r}{B_r\left(h_I\right)\sigma_r} + \left(1-\frac{1}{\gamma}\right)\frac{\widetilde{B}_r(t)}{B_r\left(h_I\right)} - \omega_N^*(t)\frac{B_r(h_N)}{B_r(h_I)} - \omega_h^*(t)\frac{\widehat{B}_r(t)}{B_r(h_I)}, \tag{4.5}$$

$$\omega_h^*(t) = -\frac{1}{\gamma}\frac{\phi_h}{\widehat{B}_h\left(t\right)\sigma_h} + \varphi\left(1-\frac{1}{\gamma}\right)\frac{\widetilde{B}_h\left(t\right)}{\widehat{B}_h\left(t\right)}. \tag{4.6}$$

Here, $\left(1-\frac{1}{\gamma}\right)\widetilde{B}_r(t)$ *and* $\varphi\left(1-\frac{1}{\gamma}\right)\widetilde{B}_h(t)$ *represent the sensitivity of the optimal annuity factor to unexpected changes in the real interest rate* $r(t)$ *and unexpected changes in the real rent inflation rate* $\pi_h(t)-\pi(t)$*, respectively. More specifically,*

$$\left(1-\frac{1}{\gamma}\right)\widetilde{B}_r(t) = \left(1-\frac{1}{\gamma}\right)\int_0^{T-t}\frac{V^*(t,v)}{V^*(t)}B_r(h)\,\mathrm{d}v, \tag{4.7}$$

$$\varphi\left(1-\frac{1}{\gamma}\right)\widetilde{B}_h(t) = \varphi\left(1-\frac{1}{\gamma}\right)\int_0^{T-t}\frac{V^*(t,v)}{V^*(t)}B_h(v)\,\mathrm{d}v, \tag{4.8}$$

where $V^*(t,v) = h^*(t)\widetilde{\Pi}_h(t)\exp\left\{-d^*(t,v)v\right\}/\varphi$ *with* $d^*(t,v)$ *defined in* (A43) *and* $V^*(t) = \int_0^{T-t}V^*(t,v)\,\mathrm{d}v$*.*

Proof. See Appendix A.3.

We denote by $\omega_S^*(t)$ and $\omega_N^*(t)$ the optimal shares of total wealth invested in the stock and in the nominal bond with time to maturity h_N, respectively. The individual invests in the risky stock to pick up the equity risk premium $\lambda_S\sigma_S \geq 0$ and she invests in the nominal bond to profit from the inflation risk premium $-\lambda_\pi\sigma_\pi B_\pi\left(h_N\right) \geq 0$. This is a standard result in the literature (see, e.g., Merton (1969)): under constant relative risk aversion utility, the speculative portfolio demands do not change over the individual's life-cycle.

We denote by $\omega_I^*(t)$ the optimal share of total wealth invested in the inflation-linked bond with time to maturity h_I. This share consists of four parts. The first part is the speculative demand: the individual wants to profit from the real interest rate risk premium $-\lambda_r\sigma_r B_r\left(h_I\right) \geq 0$. The second part is the hedging demand. This terms arises because the individual wants to hedge against a decline in the real interest rate. Its value depends on the real interest rate duration of the optimal annuity factor (as defined in

(4.7)): a large real interest rate duration implies a large hedging demand. The third and fourth part arise because the individual is already partly hedged against real interest rate risk through the nominal bond and the housing portfolio, respectively. The larger the investment in the nominal bond and the housing portfolio are, the smaller the investment in the inflation-linked bond will be.

We denote by $\omega_h^*(t)$ the optimal share of total wealth invested in the housing portfolio. This term consists of two parts: a speculative part and a hedging part. The individual invests part of her total wealth to pick up the real rent inflation rate premium $\lambda_h \sigma_h \widehat{B}_h(t) \geq 0$. This is not the only reason to invest in the housing portfolio: the individual also wants to hedge against real rent inflation risk.

4.3 Impact of Human Wealth

So far, we ignored human wealth. This section explores the impact of human wealth on the optimal portfolio weights. We define human wealth as the expected discounted value of future earnings, which consists of labor income and social security. Like Bodie et al. (1992), we assume that future earnings can be viewed as an asset. Furthermore, we assume that labor income and social security payments are riskless. Denote by $\bar{B}_r(t)$ the duration at adult age t of human wealth which is defined as follows:

$$\bar{B}_r(t) = \int_0^{T-t} \frac{H(t,v)}{H(t)} B_r(h)\,\mathrm{d}v, \tag{4.9}$$

where $H(t,v)$ is the value at adult age t of earnings received at time $t+v$ and $H(t) \equiv \int_0^{T-t} H(t,v)\,\mathrm{d}v$.

The individual invests her financial wealth – which equals total wealth minus human wealth – in the financial market. When making the asset allocation decision, she takes into account that she already owns an asset, i.e., future earnings. Denote by $\widetilde{\omega}(t) = (\widetilde{\omega}_S(t), \widetilde{\omega}_N(t), \widetilde{\omega}_I(t), \widetilde{\omega}_h(t))$ a vector consisting of the shares of financial wealth invested in the risky assets. We are now ready to present the following theorem.

Theorem 3 (optimal portfolio choice financial wealth) *Consider an individual who solves the maximization problem (2.22). Then the individual's optimal shares of financial wealth invested in the risky assets are given by*

$$\widetilde{\omega}^*(t) = \omega^*(t)\frac{W(t)}{F(t)} - \left(0, 0, \frac{\bar{B}_r(t)}{B_r(h_I)}\frac{H(t)}{F(t)}, 0\right). \tag{4.10}$$

Here, $F(t) = W(t) - H(t)$ denotes financial wealth at adult age t and $\omega^*(t)$ is the optimal portfolio weight vector as defined in Theorem 2.

Proof. See Appendix A.4.

The presence of human wealth has two effects on the optimal portfolio weights. First, the bond-like asset human wealth diversifies stock return risk, real interest rate risk, consumption inflation rate risk and real rent inflation rate risk. Hence, early in life, when human wealth is large, the individual can afford to invest more in the risky assets. Second, human wealth already provides a partly hedge against real interest rate risk. This causes the optimal share of financial wealth invested in the inflation-linked bond to decrease.

5 Main Findings

5.1 Parameter Values

We illustrate our main findings for an individual who starts working at age 20, retires at age 65 and passes away at age 85. Her yearly income after taxes is equal to 77,500 USD at age 20, which roughly corresponds to the average annual US household income after taxes in 2021 (US Bureau of Labor Statistics). We assume that the yearly increase in net income equals the consumption inflation rate. After retirement, she receives a social security payment which is 25% of last earned net income. This corresponds to the average social security payment in the US in 2021 (United States Social Security Administration). Her initial financial wealth is zero.

We summarize the values for the other parameters in Table 1. The parameter values for the real interest rate process and consumption inflation rate process are in line with Brennan and Xia (2002). We set the Cobb-Douglas share parameter φ equal to the average US household spending on housing in 2021 (US Bureau of Labor Statistics). The depreciation rate δ_h is set equal to the average house maintenance costs in the US in 2019 (American Housing Survey). The values for the relative risk aversion coefficient, time preference rate, stock return volatility and equity risk premium are common choices in the literature. We determine the market price of real rent inflation rate risk such that, at age 20, the price of the house the individual lives in matches the value of the individual's housing portfolio. The half-life and the volatility of the real rent inflation rate follow from maximum likelihood estimation (MLE); see Section 3 for more details. The time of maturity of the nominal bond and the inflation-linked bond are set equal to 10 years and 30 years, respectively. Finally, we assume that ρ equals the identity matrix.

Table 1. Parameter Values. This table summarizes the parameter values used in the illustrations.

Preferences		
Relative risk aversion	γ	5
Time preference	δ	4%
Cobb-Douglas share	φ	0.35
Stock return	$\mathrm{d}S(t) = (r(t) + \lambda_S \sigma_S) S(t) \mathrm{d}t + \sigma_S S(t) \mathrm{d}Z_S(t)$	
Volatility	σ_S	20%
Market price	λ_S	0.2
Equity risk premium	$\lambda_S \sigma_S$	4%
Real interest rate	$\mathrm{d}r(t) = \kappa_r(\bar{r} - r(t))\mathrm{d}t + \sigma_r \mathrm{d}Z_r(t)$	
Long-term mean	$\bar{r}$	2%
Volatility	σ_r	1%
Half-time $(\bar{r} - r(t))$	$\frac{\ln(0.5)}{-\kappa_r}$	10 years
Market price	λ_r	-0.20
Consumption inflation	$\mathrm{d}\pi(t) = \kappa_\pi(\bar{\pi} - \pi(t))\mathrm{d}t + \sigma_\pi \mathrm{d}Z_\pi(t)$	
Long-term mean	$\bar{\pi}$	5%
Volatility	σ_π	1%
Half-time $(\bar{\pi} - \pi(t))$	$\frac{\ln(0.5)}{-\kappa_\pi}$	20 years
Market price	λ_π	-0.10
Real rent inflation rate	$\mathrm{d}(\pi_h(t) - \pi(t)) = -\kappa_h(\pi_h(t) - \pi(t))\mathrm{d}t + \sigma_h \mathrm{d}Z_h(t)$	
Volatility	σ_h	1.5%
Half-time $\pi_h(t) - \pi(t)$	$\frac{\ln(0.5)}{-\kappa_h}$	1.5 years
Market price	λ_h	0.24
Depreciation rate	δ_h	0.6%
Time to maturity		
Nominal bond	h_N	10 years
Inflation-linked bond	h_I	30 years

5.2 Home Ownership over the Life-Cycle

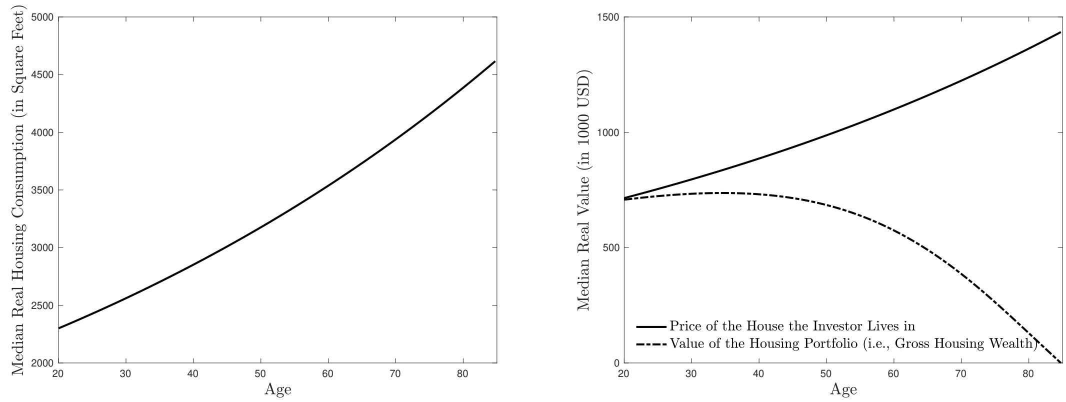
Our first finding is that the individual's optimal degree of home ownership constantly changes over her life-cycle. We first illustrate optimal median real housing consumption (in square feet of living space) as a function of age; see the left panel of Figure 3. We observe that optimal median real housing consumption increases over the individual's life-cycle. The share of the individual's total consumption spend on housing is constant.

In our example, this share is $\varphi = 35\%$; see Table 1. However, the share of the individual's income spend on total consumption is not constant but depends on the parameter values and the current state variables. We find that for our setting, the optimal median share of income spend on total consumption increases with age. This explains why the individual's optimal median real housing consumption is higher for older ages. From a practical viewpoint, it means that, in the median scenario, the individual moves to a bigger house when she becomes older.

The right panel of Figure 3 compares the value of the individual's optimal housing portfolio (i.e., gross housing wealth) with the price of the house the individual lives in. We can interpret the ratio between these two numbers as the individual's optimal degree of home ownership. As mentioned in Section 5.1, we choose the market price of real rent inflation risk λ_h such that at the start of the individual's life-cycle, the value of her optimal housing portfolio matches the price of the house she lives in. Hence, at age 20, the optimal degree of home ownership is 100%, i.e., she fully owns the house she lives in. If she becomes older, she will be both owner and renter. Indeed, we observe that the value of the individual's optimal housing portfolio is smaller than the price of the house the individual lives in, i.e., the optimal degree of home ownership is less than 100%. Towards the end of her life-cycle, the individual fully rents the house she lives in. Our conclusion is that without bequest motives, the individual optimally decreases the degree of home ownership from 100% to 0%.

5.3 Preference for Reverse Mortgage Products

Our second finding is that the individual has a preference for so-called reverse mortgage products. To illustrate our second finding, we consider the optimal median portfolio strategy in terms of total wealth; see Figure 4. The figure considers two cases: no possibility to invest in the housing portfolio (left panel) and possibility to invest in the housing portfolio (right panel). Consistent with conventional wisdom (Merton (1969)), the optimal shares of total wealth invested in the risky stock and the nominal bond are constant over the life-cycle. Also, we see that the optimal median share of total wealth invested in the inflation-linked bond decreases with age, which is also consistent with conventional wisdom (Brennan and Xia (2002)). Indeed, when the individual becomes older and her investment horizon shrinks, the hedging demand for the inflation-linked bond becomes lower. If the individual has access to the housing market, then she will decrease the optimal median share of total wealth invested in housing over her life-cycle



(a) real housing consumption (b) house price vs. value of the housing portfolio

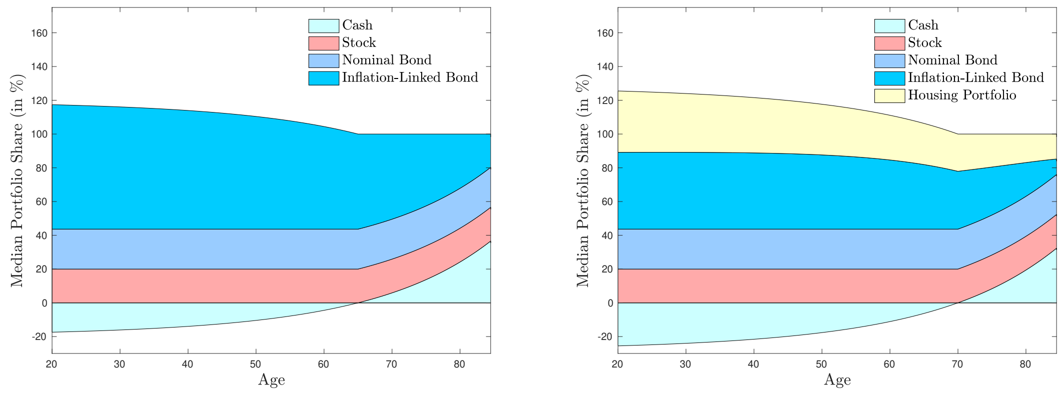
Figure 3. Real Housing Consumption, House Price and Value of the Housing Portfolio The left panel shows optimal median real housing consumption (in square feet of living space) as a function of age. The right panel compares the value of the individual's optimal housing portfolio (i.e., gross housing wealth) with the price of the house the individual lives in. The parameter values are given in Section 5.1.

(see the right panel of Figure 4 and Section 5.2). At the same time, she will increase optimal median real housing consumption as she gets older (see Figure 3). This motivates the design of reverse mortgage products which allow home owners to convert home equity into cash and use the cash to rent back the house. By buying a reverse mortgage product, she can reduce the value of her housing portfolio (i.e., gross housing wealth) while staying in her home. A reverse mortgage will thus not affect real housing consumption. Figure 5 confirms the preference for reverse mortgage products. This figure shows the optimal median composition of total real wealth as a function of age. Indeed, as we can observe in this figure, the value of the optimal housing portfolio decreases with age.

5.4 Less Demand for the Inflation-linked Bond

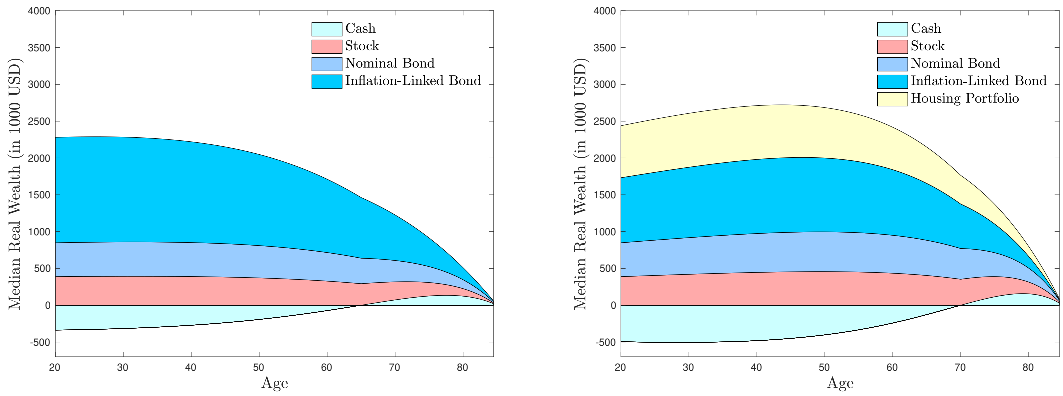
Our third finding is that, compared to conventional wisdom (Bodie et al. (1992)), the individual linvests less in the inflation-linked bond. To illustrate this finding, we explore the asset allocation of financial wealth. Hence, we include human wealth in our analysis.[7] Figure 6 takes human wealth into account and shows the median optimal shares of total wealth invested in the assets as a function of age. As in Figures 4 and 5, we consider two cases: no possibility to invest in the housing portfolio (left panel) and possibility to

[7]Figures 4 and 5 assumed that human wealth was not present.



(a) no housing (b) with housing

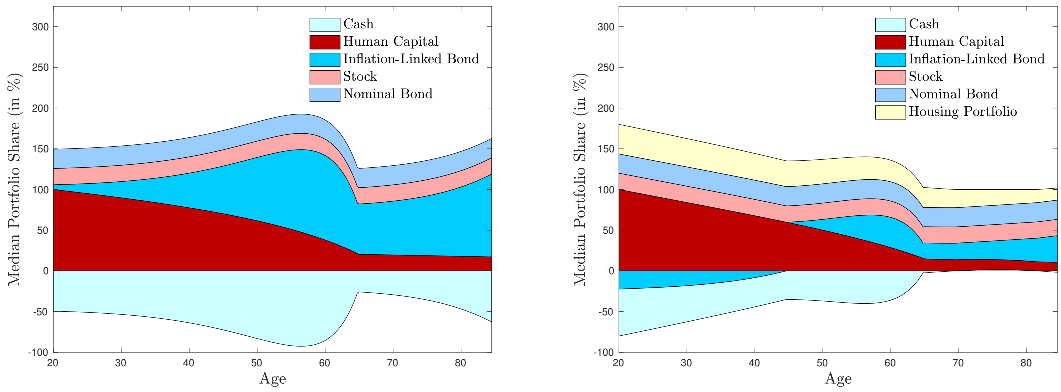
Figure 4. Optimal Portfolio Strategy without Human Wealth The figure shows the median optimal shares of total real wealth invested in the assets as a function of age. We consider two cases: no possibility to invest in the housing portfolio (left panel) and possibility to invest in the housing portfolio (right panel). The parameter values are given in Section 5.1.



(a) no housing (b) with housing

Figure 5. Optimal Composition of Total Real Wealth without Human Wealth The figure shows the median optimal composition of total real wealth as a function of age. We consider two cases: no possibility to invest in the housing portfolio (left panel) and possibility to invest in the housing portfolio (right panel). The parameter values are given in Section 5.1.

invest in the housing portfolio (right panel). We observe that human wealth causes the optimal share invested in the inflation-linked bond to decrease. This holds for both cases. Without housing, we can explain the lower demand for the inflation-linked bond by the presence of human wealth, which equals the discounted value of future labor earnings. Indeed, stable labor income can be seen as a bond. As a result, she prefers to decrease her investments in the inflation-linked bond; see also van Bilsen et al. (2020). However, with housing, our result is much stronger since now both stable labor income and stable future rents can be seen as a bond. This observation causes her to invest substantially less financial wealth in the inflation-linked bond. This finding is confirmed by Figure 5, which shows the median optimal composition of total real wealth as a function of age. Indeed, as we can observe in this figure, the size of the optimal inflation-linked bond portfolio is low or even negative for many ages, especially for the case with housing.

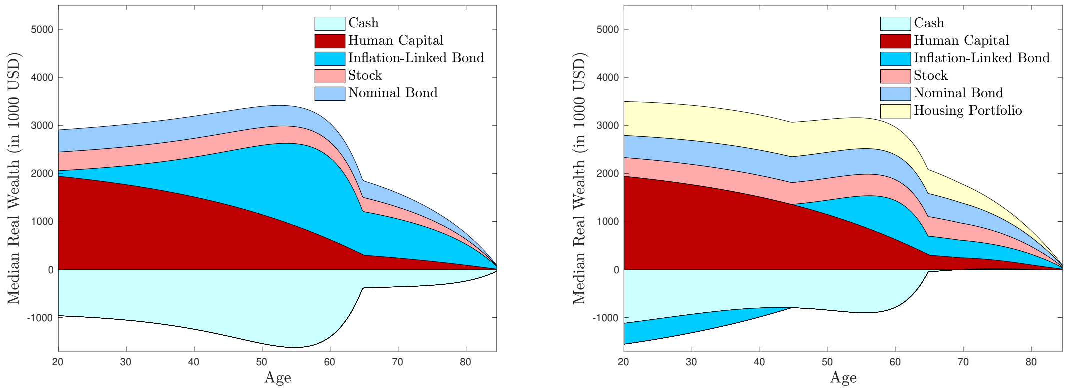


(a) no housing (b) with housing

Figure 6. Optimal Portfolio Strategy with Human Wealth The figure shows the median optimal shares of total real wealth invested in the assets as a function of age. We consider two cases: no possibility to invest in the housing portfolio (left panel) and possibility to invest in the housing portfolio (right panel). The parameter values are given in Section 5.1.

5.5 Optimal Mortgage: from Fixed-Rate to Adjustable-Rate

Our fourth finding is that the optimal mortgage changes constantly over the individual's life-cycle. Figure 8 shows the optimal mortgage. We can interpret the short positions in cash and the inflation-linked bond (see Figure 6) as an adjustable-rate mortgage and a fixed-rate mortgage, respectively. As can be seen in Figure 5, in the beginning of the life-cycle, the total loan amount is larger than the value of the individual's housing portfolio, i.e., gross housing wealth. Therefore, we assume that the mortgage amount equals the value of the housing portfolio as long as the total loan amount exceeds the value of the housing portfolio. Hence, the individual does not repay her mortgage when young. Indeed, as we observe in Figure 8, median net housing wealth, which equals median gross housing wealth minus the median mortgage amount, is zero until age 64. Furthermore,



(a) no housing (b) with housing

Figure 7. Optimal Composition of Total Real Wealth with Human Wealth The figure shows the median optimal composition of total real wealth as a function of age. We consider two cases: no possibility to invest in the housing portfolio (left panel) and possibility to invest in the housing portfolio (right panel). The parameter values are given in Section 5.1.

we observe that the optimal composition of the mortgage changes from fixed-rate to adjustable-rate as the individual becomes older. Indeed, when the individual is young, she already holds large investments in the bond-like assets human wealth and housing. Therefore, she goes short in the inflation-linked bond when young.

5.6 Asset Allocation of Defined Contribution Wealth

Our last finding is that the presence of housing leads to a change in the optimal asset allocation of defined contribution (DC) wealth. The individual uses her human wealth, her defined contribution wealth and her net housing wealth[8] to finance consumption in retirement. The sum of DC wealth and net housing wealth is equal to financial wealth $F(t) = W(t) - H(t)$. Figure 9 shows how the individual should invest her DC wealth over the life-cycle. We observe that, for most ages, the individual holds a short position in cash to finance investments in stocks and nominal bonds.[9] Furthermore, we observe that the presence of a housing portfolio has a dramatic impact on the portfolio strategy. The preference to invest in the inflation-linked bond almost completely vanishes. As already pointed out in Section 5.5, the individual already holds large investments in the bond-like

[8] See Figure 8 for the development of median net housing wealth over her life-cycle.

[9] We note that the investor also holds a short position in cash and a short position in inflation-linked bonds to finance housing investments; see Figure 8. However, we assume that these short positions are not arranged through the DC pension plan.

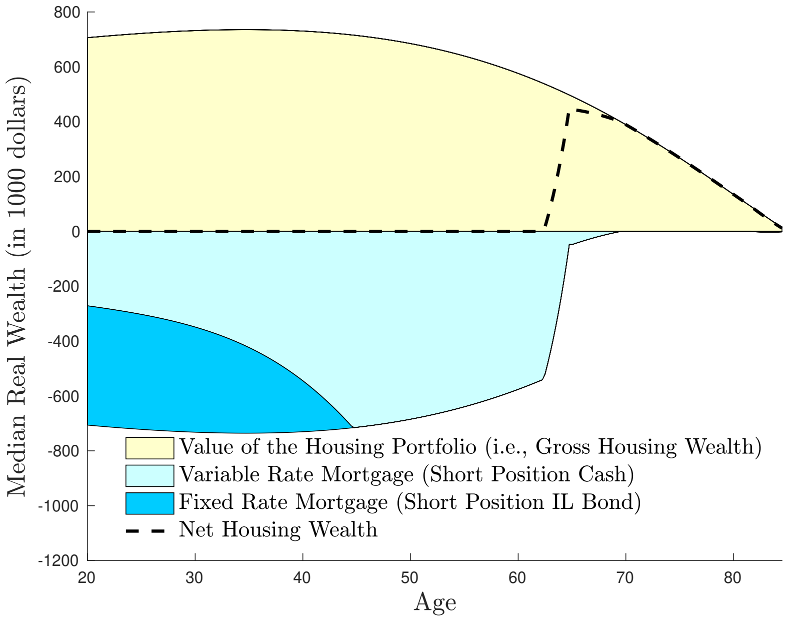


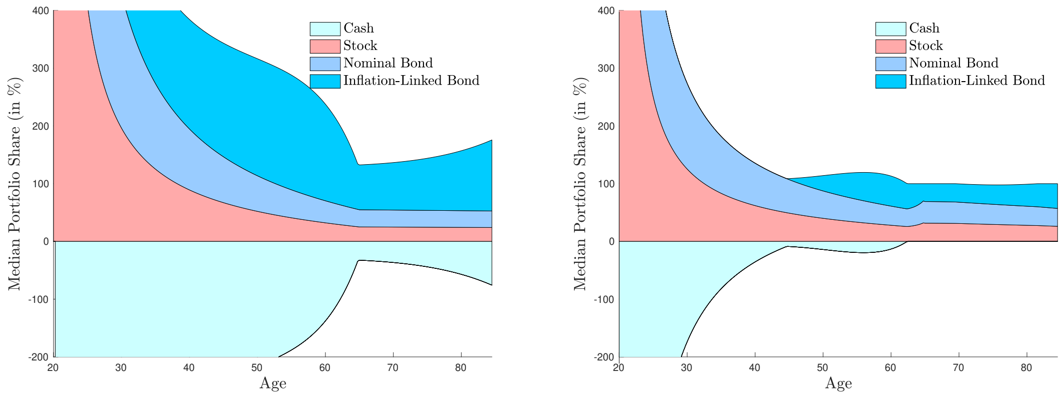
Figure 8. Optimal Mortgage The figure shows the optimal mortgage. The parameter values are given in Section 5.1.

asset housing. This explains why we observe the reduction in the optimal share of DC wealth invested in the inflation-linked bond in Figure 9. This is also confirmed by Figure 10, which shows the median optimal composition of DC wealth as a function of age.

6 Conclusion

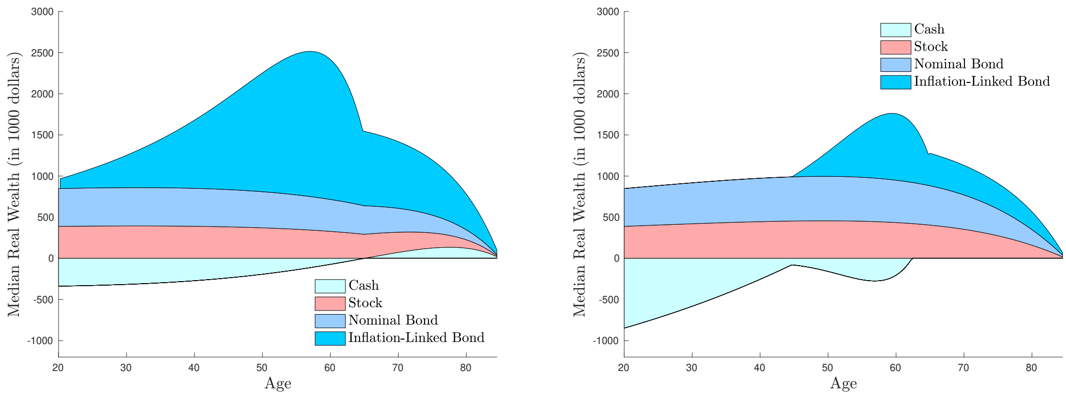
This paper explored optimal consumption and portfolio decisions in the presence of risky house prices. In our house price model, changes in real interest rates and future rents directly impact house prices. Hence, it becomes important to model rent inflation rates. We assume cointegration between rent inflation rates and consumption inflation rates. This is not only a desirable property from theoretical perspective but also backed by Dutch data. Our three main findings are as follows. First, we show that the individual prefers to be a home owner when young and a renter when old. This motives the design of so-called reverse mortgage products. Second, she invests significantly less pension wealth in inflation-linked bonds, as compared to conventional wisdom. Finally, the optimal mortgage changes from fixed-rate to adjustable-rate as the individual becomes older.

Two potential limitations of our setting are the following. First, the individual



(a) no housing (b) with housing

Figure 9. Optimal Portfolio Strategy of DC Wealth The figure shows the median optimal shares of DC wealth invested in the assets as a function of age. We consider two cases: no possibility to invest in housing portfolio (left panel) and possibility to invest in housing portfolio (right panel). The parameter values are given in Section 5.1.



(a) no housing (b) with housing

Figure 10. Optimal Composition of DC Wealth The figure shows the median optimal composition of DC wealth as a function of age. We consider two cases: no possibility to invest in housing portfolio (left panel) and possibility to invest in housing portfolio (right panel). The parameter values are given in Section 5.1.

continuously adjusts real housing consumption. This is not realistic due to high transaction costs. However, it will not significantly affect our main findings. For example, home ownership is the main reason why the individual invests less pension wealth in inflation-linked bonds. Transaction costs do not play a major role in this finding. Another potential limitation is that the current value of the individual's housing portfolio is not necessarily equal to the current value of the house the individual lives in. However, we can address this limitation by allowing the individual to buy reverse mortgage products and to invest in a real estate portfolio.

A Mathematical Proofs

A.1 Derivation of Bond Price Dynamics

This appendix derives the dynamics of the real price of a nominal bond and the dynamics of the real price of an inflation-linked bond. We assume that the economy consists of four state variables: the real interest rate $r(t)$ (with dynamics (2.3)), the consumption inflation rate $\pi(t)$ (with dynamics (2.4)), the real stock price $S(t)$ (with dynamics (2.2)), and the rent inflation rate $\pi_h(t)$ (with dynamics (2.5)).

A.1.1 Nominal Bond Price Dynamics

We can obtain the real nominal bond price $P_N(t, h_N)$ by computing the following conditional expectation:

$$\begin{aligned} P_N(t, h_N) &= \frac{1}{\Pi(t)} \mathbb{E}_t \left[\frac{m(t+h_N)}{\Pi(t+h_N)} \frac{\Pi(t)}{m(t)} \cdot 1 \right] \\ &= \frac{1}{\Pi(t)} \mathbb{E}_t \left[\exp\left\{ -\int_0^{h_N} \left(r(t+v) + \pi(t+v) + \frac{1}{2}\phi^\top \rho \phi \right) \mathrm{d}v \right.\right. \\ &\qquad \left.\left. + \phi^\top \int_0^{h_N} \mathrm{d}Z(t+v) \right\} \right]. \end{aligned} \tag{A1}$$

Here, $\mathbb{E}_t[\cdot]$ denotes the expectation conditional upon the information available at time t.

Equation (A1) shows that the aggregate real interest rate $\bar{r}(t, h_N) = \int_0^{h_N} r(t+v)\mathrm{d}v$ and the aggregate consumption inflation rate $\bar{\pi}(t, h_N) = \int_0^{h_N} \pi(t+v)\mathrm{d}v$ play a key role in determining the real nominal bond price. We find that the aggregate real interest rate

$\bar{r}(t, h_N)$ is given by

$$
\begin{aligned}
\bar{r}(t, h_N) &= \int_0^{h_N} r(t+v)\mathrm{d}v \\
&= \int_0^{h_N} \left(e^{-\kappa_r v} r(t) + \left(1 - e^{-\kappa_r v}\right)\bar{r}\right)\mathrm{d}v + \sigma_r \int_0^{h_N}\int_0^{v} e^{-\kappa_r (v-u)}\mathrm{d}Z_r(t+u)\mathrm{d}v \\
&= \int_0^{h_N} \left(r(t) + \left(1 - e^{-\kappa_r v}\right)(\bar{r} - r(t))\right)\mathrm{d}v + \\
&\quad \sigma_r \int_0^{h_N}\int_v^{h_N} e^{-\kappa_r (h_N - u)}\mathrm{d}u\mathrm{d}Z_r(t+v) \\
&= \int_0^{h_N} \left(r(t) + \kappa_r B_r(v)(\bar{r} - r(t))\right)\mathrm{d}v \\
&\quad + \sigma_r \int_0^{h_N} \frac{1}{\kappa_r}\left(1 - e^{-\kappa_r (h_N - v)}\right)\mathrm{d}Z_r(t+v) \\
&= \int_0^{h_N} \mathbb{E}_t\left[r(t+v)\right]\mathrm{d}v + \sigma_r \int_0^{h_N} B_r(h_N - v)\mathrm{d}Z_r(t+v).
\end{aligned} \tag{A2}
$$

The second equality in (A2) follows from the fact that

$$
\begin{aligned}
r(t+v) &= e^{-\kappa_r v} r(t) + \left(1 - e^{-\kappa_r v}\right)\bar{r} + \sigma_r \int_0^{v} e^{-\kappa_r (v-u)}\mathrm{d}Z_r(t+u) \\
&= \mathbb{E}_t\left[r(t+v)\right] + \sigma_r \int_0^{v} e^{-\kappa_r (v-u)}\mathrm{d}Z_r(t+u).
\end{aligned} \tag{A3}
$$

We can derive (A3) by repeated substitution. In a similar fashion, we find

$$
\bar{\pi}(t, h_N) = \int_0^{h_N} \mathbb{E}_t\left[\pi(t+v)\right]\mathrm{d}v + \sigma_\pi \int_0^{h_N} B_\pi(h_N - v)\mathrm{d}Z_\pi(t+v). \tag{A4}
$$

Substituting (A2) and (A4) into (A1) to eliminate $\int_0^{h_N} r(t+v)\mathrm{d}v$ and $\int_0^{h_N} \pi(t+v)\mathrm{d}v$, we arrive at

$$
\begin{aligned}
\Pi(t)P_N(t,h_N) = \exp\Bigg\{ & -\int_0^{h_N} \left(\mathbb{E}_t\left[r(t+v)+\pi(t+v)\right]+\frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}v\Bigg\} \\
& \mathbb{E}_t\Bigg[\exp\Bigg\{\int_0^{h_N}\left(\phi_r - B_r(h_N-v)\sigma_r\right)\mathrm{d}Z_r(t+v) \\
& \qquad + \int_0^{h_N}\left(\phi_\pi - B_\pi(h_N-v)\sigma_\pi\right)\mathrm{d}Z_\pi(t+v) \\
& \qquad + \int_0^{h_N}\phi_S \mathrm{d}Z_S(t+v) + \int_0^{h_N}\phi_h \mathrm{d}Z_h(t+v)\Bigg\}\Bigg] \qquad \text{(A5)} \\
= \exp\Bigg\{ & -\int_0^{h_N}\Big(\mathbb{E}_t\left[r(t+v)+\pi(t+v)\right] - \lambda_r\sigma_r B_r(v) - \lambda_\pi\sigma_\pi B_\pi(v) \\
& -\frac{1}{2}B_r^2(v)\sigma_r^2 - \frac{1}{2}B_\pi^2(v)\sigma_\pi^2 - \rho_{r\pi}B_r(v)B_\pi(v)\sigma_r\sigma_\pi\Big)\mathrm{d}v\Bigg\} \\
= \exp\Bigg\{ & -\int_0^{h_N} R(t,v)\mathrm{d}v\Bigg\}.
\end{aligned}
$$

Here, the instantaneous nominal forward interest rate at adult age t for horizon v, i.e., $R(t,v)$, is defined as follows:

$$
\begin{aligned}
R(t,v) = & \ \mathbb{E}_t\left[r(t+v)+\pi(t+v)\right] - \lambda_r\sigma_r B_r(v) - \lambda_\pi\sigma_\pi B_\pi(v) \\
& - \frac{1}{2}B_r^2(v)\sigma_r^2 - \frac{1}{2}B_\pi^2(v)\sigma_\pi^2 - \rho_{r\pi}B_r(v)B_\pi(v)\sigma_r\sigma_\pi.
\end{aligned} \qquad \text{(A6)}
$$

The log real nominal bond price is given by (this follows from (A5) and (A6))

$$
\begin{aligned}
\log P_N(t,h_N) = -\int_0^{h_N} & \Big(r(t) + \kappa_r B_r(v)\left(\bar{r}-r(t)\right) + \pi(t) + \kappa_\pi B_\pi(v)\left(\bar{\pi}-\pi(t)\right) \\
& - \lambda_r\sigma_r B_r(v) - \lambda_\pi\sigma_\pi B_\pi(v) - \frac{1}{2}B_r^2(v)\sigma_r^2 - \frac{1}{2}B_\pi^2(v)\sigma_\pi^2 \\
& - \rho_{r\pi}B_r(v)B_\pi(v)\sigma_r\sigma_\pi\Big)\mathrm{d}v - \log\Pi(t).
\end{aligned} \qquad \text{(A7)}
$$

Solving the integral (A7), we arrive at[10]

$$
\begin{aligned}
\log P_N\left(t, h_N\right) = & -r(t)h_N - \left(\bar{r} - r(t)\right)\left(h_N - B_r(h_N)\right) \\
& - \pi(t)h_N - \left(\bar{\pi} - \pi(t)\right)\left(h_N - B_\pi(h_N)\right) \\
& + \frac{\lambda_r \sigma_r}{\kappa_r}\left(h_N - B_r(h_N)\right) + \frac{\lambda_\pi \sigma_\pi}{\kappa_\pi}\left(h_N - B_\pi(h_N)\right) \\
& + \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\left(h_N - 2B_r(h_N) + \frac{1}{2}B_r(2h_N)\right) \\
& + \frac{1}{2}\frac{\sigma_\pi^2}{\kappa_\pi^2}\left(h_N - 2B_\pi(h_N) + \frac{1}{2}B_\pi(2h_N)\right) - \log \Pi(t) \\
& + \frac{\rho_{r\pi}\sigma_r\sigma_\pi}{\kappa_r \kappa_\pi}\left(h_N - B_r(h_N) - B_\pi(h_N) + \frac{1 - e^{-(\kappa_r + \kappa_\pi)h_N}}{\kappa_r + \kappa_\pi}\right) \\
= & -r(t)B_r(h_N) - \pi(t)B_\pi(h_N) - k_N\left(h_N\right) - \log \Pi(t).
\end{aligned} \tag{A8}
$$

Here, $k_N\left(h_N\right)$ is defined as follows:

$$
\begin{aligned}
k_N\left(h_N\right) = & \left(\bar{r} - \frac{\lambda_r \sigma_r}{\kappa_r} - \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\right)\left(h_N - B_r(h_N)\right) + \frac{1}{4\kappa_r}B_r^2(h_N)\sigma_r^2 \\
& + \left(\bar{\pi} - \frac{\lambda_\pi \sigma_\pi}{\kappa_\pi} - \frac{1}{2}\frac{\sigma_\pi^2}{\kappa_\pi^2}\right)\left(h_N - B_\pi(h_N)\right) + \frac{1}{4\kappa_\pi}B_\pi^2(h_N)\sigma_\pi^2 \\
& + \frac{\rho_{r\pi}\sigma_r\sigma_\pi}{\kappa_r \kappa_\pi}\left(h_N - B_r(h_N) - B_\pi(h_N) + \frac{1 - e^{-(\kappa_r + \kappa_\pi)h_N}}{\kappa_r + \kappa_\pi}\right).
\end{aligned} \tag{A9}
$$

To calculate how the real price of a nominal bond with a fixed maturity date $t + h_N$ develops as time proceeds (i.e., $t + h_N$ is fixed but t changes), we apply Itô's lemma to

$$
P_N\left(t, h_N\right) = \frac{1}{\Pi(t)} \exp\left\{-r(t)B_r(h_N) - \pi(t)B_\pi(h_N) - k_N(h_N)\right\}. \tag{A10}
$$

[10] The first equality follows from $B_r^2(v) = \left(1 - 2e^{-\kappa_r v} + e^{-2\kappa_r v}\right)/\kappa_r^2$ and the second equality follows from $B_r^2(h) = \left(2B_r(h) - B_r(2h)\right)/\kappa_r$.

We find

$$
\begin{aligned}
\frac{\mathrm{d}P_N\left(t,h_N\right)}{P_N\left(t,h_N\right)} &= \Big(R(t,h_N) - \kappa_r B_r(h_N)\left(\bar{r}-r(t)\right) - \kappa_\pi B_\pi(h_N)\left(\bar{\pi}-\pi(t)\right) \\
&\quad + \frac{1}{2}B_r^2(h_N)\sigma_r^2 + \frac{1}{2}B_\pi^2(h_N)\sigma_\pi^2 + \rho_{r\pi}B_r(h_N)B_\pi(h_N)\sigma_r\sigma_\pi\Big)\mathrm{d}t \\
&\quad - B_r(h_N)\sigma_r\mathrm{d}Z_r(t) - B_\pi(h_N)\sigma_\pi\mathrm{d}Z_\pi(t) - \pi(t)\mathrm{d}t \\
&= \left(r(t) - \lambda_r\sigma_r B_r(h_N) - \lambda_\pi\sigma_\pi B_\pi(h_N)\right)\mathrm{d}t \\
&\quad - B_r(h_N)\sigma_r\mathrm{d}Z_r(t) - B_\pi(h_N)\sigma_\pi\mathrm{d}Z_\pi(t).
\end{aligned}
\tag{A11}
$$

A.1.2 Inflation-linked Bond Price Dynamics

We can obtain the real inflation-linked bond price $P_I\left(t,h_I\right)$ by computing the following conditional expectation:

$$
\begin{aligned}
P_I\left(t,h_I\right) &= \mathbb{E}_t\left[\frac{m\left(t+h_I\right)}{m\left(t\right)}\right] \\
&= \mathbb{E}_t\left[\exp\left\{-\int_0^{h_I}\left(r(t+v)+\frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}v + \phi^\top\int_0^{h_I}\mathrm{d}Z(t+v)\right\}\right].
\end{aligned}
\tag{A12}
$$

Here, $\mathbb{E}_t\left[\cdot\right]$ denotes the expectation conditional upon the information available at time t.

Using (A2) and (A3), we find

$$
\log P_I\left(t,h_I\right) = -\int_0^{h_I}\left(r(t) + \kappa B_r(v)\left(\bar{r}-r(t)\right) - \lambda_r\sigma_r B_r(v) - \frac{1}{2}B_r^2(v)\sigma_r^2\right)\mathrm{d}v. \tag{A13}
$$

Solving the integral (A13), we arrive at

$$
\begin{aligned}
\log P_I\left(t,h_I\right) &= -r(t)h_I - \left(\bar{r}-r(t)\right)\left(h_I - B_r(h_I)\right) + \frac{\lambda_r\sigma_r}{\kappa_r}\left(h_I - B_r(h_I)\right) \\
&\quad + \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\left(h_I - 2B_r(h_I) + \frac{1}{2}B_r(2h_I)\right) \\
&= -r(t)B_r(h_I) - k_I\left(h_I\right).
\end{aligned}
\tag{A14}
$$

Here, $k_I\left(h_I\right)$ is defined as follows:

$$
k_I\left(h_I\right) = \left(\bar{r} - \frac{\lambda_r\sigma_r}{\kappa_r} - \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\right)\left(h_I - B_r(h_I)\right) + \frac{1}{4\kappa_r}B_r^2(h_I)\sigma_r^2. \tag{A15}
$$

To calculate how the real price of an inflation-linked bond with a fixed maturity date

$t+h_I$ develops as time proceeds (i.e., $t+h_I$ is fixed but t changes), we apply Itô's lemma to

$$P_I(t,h_I) = \exp\left\{-r(t)B_r(h_I) - k_I(h_I)\right\}. \tag{A16}$$

We find

$$\frac{\mathrm{d}P_I(t,h_I)}{P_I(t,h_I)} = \left(r(t) - \lambda_r\sigma_r B_r(h_I)\right)\mathrm{d}t - B_r(h_I)\sigma_r \mathrm{d}Z_r(t). \tag{A17}$$

A.2 Dynamics of Value of Housing Portfolio

The real price of one unit of housing (a square foot), i.e., $P_h(t)$, is equal to the expected discounted value of all future rents. Hence,

$$\begin{aligned} P_h(t) &= \mathbb{E}_t\left[\int_t^\infty e^{-\delta_h(s-t)}\frac{m(s)}{\Pi(s)}\frac{\Pi(t)}{m(t)}\frac{\Pi_h(s)}{\Pi(t)}\mathrm{d}s\right] \\ &= \widetilde{\Pi}_h(t)\mathbb{E}_t\left[\int_t^\infty e^{-\delta_h(s-t)}\frac{\widetilde{\Pi}_h(s)}{\widetilde{\Pi}_h(t)}\frac{m(s)}{m(t)}\mathrm{d}s\right] \\ &= \widetilde{\Pi}_h(t)\mathbb{E}_t\left[\int_0^\infty e^{-\delta_h h}\exp\left\{-\int_0^h\left(r(t+v) - \widetilde{\pi}_h(t+v) + \frac{1}{2}\phi^\top\rho\phi\right)\mathrm{d}v\right.\right. \\ &\qquad\left.\left. +\phi^\top\int_0^h \mathrm{d}Z(t+v)\right\}\mathrm{d}h\right]. \end{aligned} \tag{A18}$$

Here, $\mathbb{E}_t[\cdot]$ denotes the expectation conditional upon the information available at time t, $\widetilde{\Pi}_h(t) = \Pi_h(t)/\Pi(t)$, δ_h models the depreciation of the house, and $\widetilde{\pi}_h(t) = \pi_h(t) - \pi(t)$.

Equation (A18) shows that $\int_0^h r(t+v)\mathrm{d}v$ and $\int_0^h \widetilde{\pi}(t+v)\mathrm{d}v$ play a key role in determining the real house price. We find that (compare with (A2))

$$\int_0^h r(t+v)\mathrm{d}v = \int_0^h \mathbb{E}_t\left[r(t+v)\right]\mathrm{d}v + \sigma_r\int_0^h B_r(h-v)\mathrm{d}Z_r(t+v), \tag{A19}$$

$$\int_0^h \widetilde{\pi}_h(t+v)\mathrm{d}v = \int_0^h \mathbb{E}_t\left[\widetilde{\pi}_h(t+v)\right]\mathrm{d}v + \sigma_h\int_0^h B_h(h-v)\mathrm{d}Z_h(t+v). \tag{A20}$$

Substituting (A19) and (A20) into (A18) to eliminate $\int_0^h r(t+v)\mathrm{d}v$ and $\int_0^h \widetilde{\pi}_h(t+v)\mathrm{d}v$, we arrive at

$$
\begin{aligned}
\frac{P_h(t)}{\widetilde{\Pi}_h(t)} &= \int_0^\infty e^{-\delta_h h} \exp\left\{ -\int_0^h \left(\mathbb{E}_t\left[r(t+v) - \widetilde{\pi}_h(t+v) \right] + \frac{1}{2}\phi^\top \rho \phi \right) \mathrm{d}v \right\} \\
&\quad \mathbb{E}_t \left[\exp\left\{ \int_0^h \left(\phi_r - B_r(h-v)\sigma_r \right) \mathrm{d}Z_r(t+v) \right. \right. \\
&\qquad + \int_0^h \left(\phi_h + B_h(h-v)\sigma_h \right) \mathrm{d}Z_h(t+v) \\
&\qquad \left. \left. + \int_0^h \phi_\pi \mathrm{d}Z_\pi(t+v) + \int_0^h \phi_S \mathrm{d}Z_S(t+v) \right\} \right] \mathrm{d}h \\
&= \int_0^\infty e^{-\delta_h h} \exp\left\{ -\int_0^h \left(\mathbb{E}_t\left[r(t+v) - \widetilde{\pi}_h(t+v) \right] \right. \right. \\
&\qquad -\lambda_r \sigma_r B_r(v) + \lambda_h \sigma_h B_h(v) \\
&\qquad \left. \left. -\frac{1}{2} B_r^2(v)\sigma_r^2 - \frac{1}{2} B_h^2(v)\sigma_h^2 + \rho_{rh} B_r(v) B_h(v) \sigma_r \sigma_h \right) \mathrm{d}v \right\} \mathrm{d}h \\
&= \int_0^\infty e^{-\delta_h h} \exp\left\{ -\int_0^h R_h(t,v)\mathrm{d}v \right\} \mathrm{d}h.
\end{aligned} \tag{A21}
$$

Here, $R_h(t,v)$ is defined as follows:

$$
\begin{aligned}
R_h(t,v) = \mathbb{E}_t\left[r(t+v) - \widetilde{\pi}_h(t+v) \right] - \lambda_r \sigma_r B_r(v) + \lambda_h \sigma_h B_h(v) \\
- \frac{1}{2} B_r^2(v)\sigma_r^2 - \frac{1}{2} B_h^2(v)\sigma_h^2 + \rho_{rh} B_r(v) B_h(v) \sigma_r \sigma_h.
\end{aligned} \tag{A22}
$$

Define

$$
P_h(t,h) = \exp\left\{ -\int_0^h R_h(t,v)\mathrm{d}v \right\}. \tag{A23}
$$

We find (this follows from (A22) and (A23))

$$
\begin{aligned}
\log P_h(t,h) = -\int_0^h \Big(& r(t) + \kappa_r B_r(v)\left(\bar{r} - r(t) \right) - \widetilde{\pi}_h(t) + \kappa_h B_h(v) \widetilde{\pi}_h(t) \\
& - \lambda_r \sigma_r B_r(v) + \lambda_h \sigma_h B_h(v) - \frac{1}{2} B_r^2(v)\sigma_r^2 - \frac{1}{2} B_h^2(v)\sigma_h^2 \\
& + \rho_{rh} B_r(v) B_h(v) \sigma_r \sigma_h \Big) \mathrm{d}v.
\end{aligned} \tag{A24}
$$

Solving the integral (A24), we arrive at

$$
\begin{aligned}
\log P_h(t,h) &= -r(t)h - (\bar{r} - r(t))\,(h - B_r(h)) \\
&\quad + \widetilde{\pi}_h(t)h - \widetilde{\pi}_h(t)\,(h - B_h(h)) \\
&\quad + \frac{\lambda_r \sigma_r}{\kappa_r}\,(h - B_r(h)) - \frac{\lambda_h \sigma_h}{\kappa_h}\,(h - B_h(h)) \\
&\quad + \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\left(h - 2B_r(h) + \frac{1}{2}B_r(2h)\right) \\
&\quad + \frac{1}{2}\frac{\sigma_h^2}{\kappa_h^2}\left(h - 2B_h(h) + \frac{1}{2}B_h(2h)\right) \\
&\quad - \frac{\rho_{rh}\sigma_r\sigma_h}{\kappa_r\kappa_h}\left(h - B_r(h) - B_h(h) + \frac{1 - e^{-(\kappa_r+\kappa_h)h}}{\kappa_r + \kappa_h}\right) \\
&= -r(t)B_r(h) + \widetilde{\pi}_h(t)B_h(h) - k_h\,(h)\,.
\end{aligned}
\tag{A25}
$$

Here, $k_h\,(h)$ is defined as follows:

$$
\begin{aligned}
k_h\,(h) &= \left(\bar{r} - \frac{\lambda_r\sigma_r}{\kappa_r} - \frac{1}{2}\frac{\sigma_r^2}{\kappa_r^2}\right)(h - B_r(h)) + \frac{1}{4\kappa_r}B_r^2(h)\sigma_r^2 \\
&\quad + \left(\frac{\lambda_h\sigma_h}{\kappa_h} - \frac{1}{2}\frac{\sigma_h^2}{\kappa_h^2}\right)(h - B_h(h)) + \frac{1}{4\kappa_h}B_h^2(h)\sigma_h^2 \\
&\quad + \frac{\rho_{rh}\sigma_r\sigma_h}{\kappa_r\kappa_h}\left(h - B_r(h) - B_h(h) + \frac{1 - e^{-(\kappa_r+\kappa_h)h}}{\kappa_r + \kappa_h}\right).
\end{aligned}
\tag{A26}
$$

To calculate how $P_h(t,h)$ develops as time proceeds (i.e., $t+h$ is fixed but t changes), we apply Itô's lemma to

$$
P_h(t,h) = \exp\left\{-r(t)B_r(h) + \widetilde{\pi}_h(t)B_h(h) - k_h(h)\right\}. \tag{A27}
$$

We find

$$
\begin{aligned}
\frac{\mathrm{d}P_h(t,h)}{P_h(t,h)} &= \Big(R_h(t,h) - \kappa_r B_r(h)\,(\bar{r} - r(t)) - \kappa_h B_h(h)\widetilde{\pi}_h(t) \\
&\qquad + \frac{1}{2}B_r^2(h)\sigma_r^2 + \frac{1}{2}B_h^2(h)\sigma_h^2 - \rho_{rh}B_r(h)B_h(h)\sigma_r\sigma_h\Big)\mathrm{d}t \\
&\quad - B_r(h)\sigma_r \mathrm{d}Z_r(t) + B_h(h)\sigma_h \mathrm{d}Z_h(t) \\
&= (r(t) - \widetilde{\pi}_h(t) - \lambda_r\sigma_r B_r(h) + \lambda_h\sigma_h B_h(h))\,\mathrm{d}t \\
&\quad - B_r(h)\sigma_r \mathrm{d}Z_r(t) + B_h(h)\sigma_h \mathrm{d}Z_h(t).
\end{aligned}
\tag{A28}
$$

To calculate how the real house price $P_h(t)$ develops as time proceeds, we apply Itô's lemma to

$$P_h(t) = \widetilde{\Pi}_h(t) \int_0^\infty e^{-\delta_h h} P_h(t,h) \mathrm{d}h. \tag{A29}$$

We find

$$\begin{aligned} \frac{\mathrm{d}P_h(t)}{P_h(t)} &= \frac{\partial P_h(t)}{\partial t} \frac{1}{P_h(t)} \mathrm{d}t + \widetilde{\Pi}_h(t) \int_0^\infty e^{-\delta_h h} \frac{\mathrm{d}P_h(t,h)}{P_h(t,h)} \frac{P_h(t,h)}{P_h(t)} \mathrm{d}h + \frac{\mathrm{d}\widetilde{\Pi}_h(t)}{\widetilde{\Pi}_h(t)} \\ &= -\frac{\widetilde{\Pi}_h(t)}{P_h(t)} \mathrm{d}t + \left(r(t) - \lambda_r \sigma_r \widehat{B}_r(t) + \lambda_h \sigma_h \widehat{B}_h(t) \right) \mathrm{d}t \\ &\quad - \widehat{B}_r(t) \sigma_r \mathrm{d}Z_r(t) + \widehat{B}_h(t) \sigma_h \mathrm{d}Z_h(t), \end{aligned} \tag{A30}$$

with

$$\widehat{B}_r(t) = \frac{\widetilde{\Pi}_h(t) \int_0^\infty e^{-\delta_h h} B_r(h) P_h(t,h) \mathrm{d}h}{P_h(t)}, \tag{A31}$$

$$\widehat{B}_h(t) = \frac{\widetilde{\Pi}_h(t) \int_0^\infty e^{-\delta_h h} B_h(h) P_h(t,h) \mathrm{d}h}{P_h(t)}. \tag{A32}$$

Denote by $W_h(t)$ the real value at adult age t of an account that invests in housing only. The real return on the housing portfolio consists of two parts: incoming rents and capital gains/losses. Hence,

$$\frac{\mathrm{d}W_h(t)}{W_h(t)} = \widetilde{\Pi}_h(t) \frac{1}{P_h(t)} \mathrm{d}t + \frac{\mathrm{d}P_h(t)}{P_h(t)}. \tag{A33}$$

The first on the right-hand side of (A33) denotes the real return from rents, while the second term on the right-hand side of (A33) denotes the real return from capital gains/losses. Using (A30), we arrive at

$$\frac{\mathrm{d}W_h(t)}{W_h(t)} = \left(r(t) - \lambda_r \sigma_r \widehat{B}_r(t) + \lambda_h \sigma_h \widehat{B}_h(t) \right) \mathrm{d}t - \widehat{B}_r(t) \sigma_r \mathrm{d}Z_r(t) + \widehat{B}_h(t) \sigma_h \mathrm{d}Z_h(t). \tag{A34}$$

A.3 Derivation of Optimal Life-cycle Policies

This appendix derives the optimal life-cycle policies. We assume that the economy consists of four state variables: the real interest rate $r(t)$ (with dynamics (2.3)), the consumption inflation rate $\pi(t)$ (with dynamics (2.4)), the real stock price $S(t)$ (with dynamics (2.2)), and the rent inflation rate $\pi_h(t)$ (with dynamics (2.5)). We assume that the investor has the opportunity to invest in four risky assets: a nominal bond with fixed time to maturity h_N, an inflation-linked bond with fixed time to maturity h_I, a risky stock, and a housing portfolio. The dynamics of the bond prices and the housing portfolio are derived in Appendices A.1 and A.2, respectively.

We start by deriving optimal real housing services $h^*(t)$ and optimal real consumption $c^*(t)$. We can, by virtue of the martingale approach (Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989, 1991)), transform the individual's dynamic maximization problem (2.22) into the following equivalent static variational problem:

$$\begin{aligned} \max_{h(t),c(t)} \quad & \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{1-\gamma}\left(h(t)^{\varphi}c(t)^{1-\varphi}\right)^{1-\gamma}\mathrm{d}t\right] \\ \text{s.t.} \quad & \mathbb{E}\left[\int_0^T \left(m(t)\widetilde{\Pi}_h(t)h(t)+m(t)c(t)\right)\mathrm{d}t\right] \leq W(0). \end{aligned} \tag{A35}$$

Denote by $\mathcal{L}$ the Lagrangian which is given by

$$\begin{aligned} \mathcal{L} = & \mathbb{E}\left[\int_0^T e^{-\delta t}\frac{1}{1-\gamma}\left(h(t)^{\varphi}c(t)^{1-\varphi}\right)^{1-\gamma}\mathrm{d}t\right] \\ & + y\left(W(0) - \mathbb{E}\left[\int_0^T \left(m(t)\widetilde{\Pi}_h(t)h(t)+m(t)c(t)\right)\mathrm{d}t\right]\right). \end{aligned} \tag{A36}$$

Here, $y \geq 0$ denotes the Lagrange multiplier associated with the static budget constraint. We find the following first-order optimality conditions:

$$e^{-\delta t}\left(h^*(t)^{\varphi}c^*(t)^{1-\varphi}\right)^{-\gamma}\varphi h^*(t)^{\varphi-1}c^*(t)^{1-\varphi} = y m(t)\widetilde{\Pi}_h(t), \tag{A37}$$

$$e^{-\delta t}\left(h^*(t)^{\varphi}c^*(t)^{1-\varphi}\right)^{-\gamma}(1-\varphi)c^*(t)^{-\varphi}h^*(t)^{\varphi} = y m(t). \tag{A38}$$

After solving the first-order optimality conditions, we obtain the following optimal real housing services $h^*(t)$ and optimal real consumption $c^*(t)$:

$$h^*(t) = h^*(0)e^{-\frac{\delta t}{\gamma}}m(t)^{-\frac{1}{\gamma}}\widetilde{\Pi}_h(t)^{-\frac{(1-\varphi)\gamma+\varphi}{\gamma}}, \tag{A39}$$

$$c^*(t) = \frac{1-\varphi}{\varphi}h^*(t)\widetilde{\Pi}_h(t). \tag{A40}$$

Here, $h^*(0)$ is determined such that the static budget constraint holds with equality.

A standard verification that the optimal solution to the Lagrangian equals the optimal solution to the investor's maximization problem (2.22) (see, e.g., Karatzas and Shreve

(1998)) completes the proof.

Denote by $V^*(t)$ the market-consistent value at adult age t of current and future optimal real housing services and optimal real consumption. We define $V^*(t)$ as follows (the second equality follows from (A40)):

$$\begin{aligned} V^*(t) &= \int_0^{T-t} \mathbb{E}_t \left[\frac{m(t+h)}{m(t)} h^*(t+h)\widetilde{\Pi}_h(t+h) + \frac{m(t+h)}{m(t)} c^*(t+h) \right] \mathrm{d}h \\ &= \frac{1}{\varphi} h^*(t)\widetilde{\Pi}_h(t) \int_0^{T-t} \mathbb{E}_t \left[\frac{m(t+h)}{m(t)} \frac{h^*(t+h)\widetilde{\Pi}_h(t+h)}{h^*(t)\widetilde{\Pi}_h(t)} \right] \mathrm{d}h \\ &= \frac{1}{\varphi} h^*(t)\widetilde{\Pi}_h(t) A^*(t), \end{aligned} \tag{A41}$$

where $A^*(t)$ denotes the optimal annuity factor at adult age t:

$$A^*(t) = \int_0^{T-t} \mathbb{E}_t \left[\frac{m(t+h)}{m(t)} \frac{h^*(t+h)\widetilde{\Pi}_h(t+h)}{h^*(t)\widetilde{\Pi}_h(t)} \right] \mathrm{d}h = \int_0^{T-t} \exp\left\{-d^*(t,h)h\right\} \mathrm{d}h. \tag{A42}$$

Here, $d^*(t,h)$ represents the market-consistent discount rate at adult age t for horizon $h \geq 0$. Straightforward computations show that

$$\begin{aligned} d^*(t,h) = \frac{1}{h} \Bigg[& \left(1 - \frac{1}{\gamma}\right) \int_0^h \left(r(t) + \kappa_r B_r(v) \left[\bar{r} - r(t)\right] + \frac{1}{2}\phi^\top \rho \phi \right) \mathrm{d}v \\ & + \varphi \left(1 - \frac{1}{\gamma}\right) \int_0^h \left(\pi(t) - \pi_h(t) + \kappa_h B_h(v) \left[\pi_h(t) - \pi(t)\right]\right) \mathrm{d}v \\ & + \frac{\delta}{\gamma} h - \frac{1}{2} \int_0^h \nu(v)^\top \rho \nu(v) \, \mathrm{d}v \Bigg], \end{aligned} \tag{A43}$$

where $\nu(v) = \left(1 - \frac{1}{\gamma}\right)(\phi_r - B_r(v)\sigma_r, \phi_\pi, \phi_S, \phi_h + \varphi B_h(v)\sigma_h)$. It follows that $\log V^*(t)$ evolves according to (this follows from (A39), (A42) and (A43))

$$\begin{aligned} \mathrm{d}\log V^*(t) &= \mathrm{d}\log h^*(t) + \mathrm{d}\log \widetilde{\Pi}_h(t) + \mathrm{d}\log A^*(t) \\ &= (\ldots)\,\mathrm{d}t - \left(\frac{1}{\gamma}\phi_r + \left(1 - \frac{1}{\gamma}\right) \widetilde{B}_r(t)\sigma_r \right) \mathrm{d}Z_r(t) - \frac{1}{\gamma}\phi_\pi \, \mathrm{d}Z_\pi(t) \\ &\quad - \frac{1}{\gamma}\phi_S \, \mathrm{d}Z_S(t) - \left(\frac{1}{\gamma}\phi_h - \varphi \left(1 - \frac{1}{\gamma}\right) \widetilde{B}_h(t)\sigma_h \right) \mathrm{d}Z_h(t). \end{aligned} \tag{A44}$$

Here, $\left(1 - \frac{1}{\gamma}\right)\widetilde{B}_r(t)$ and $\varphi\left(1 - \frac{1}{\gamma}\right)\widetilde{B}_h(t)$ represent the sensitivity of the annuity factor to unexpected changes in the real interest rate $r(t)$ and unexpected changes in the real rent inflation rate $\pi_h(t) - \pi(t)$, respectively. We define

$$\widetilde{B}_r(t) = \int_0^{T-t} \frac{V^*(t,h)}{V^*(t)} B_r(h)\,\mathrm{d}h, \tag{A45}$$

$$\widetilde{B}_h(t) = \int_0^{T-t} \frac{V^*(t,h)}{V^*(t)} B_h(h)\,\mathrm{d}h, \tag{A46}$$

where $V^*(t,h) = h^*(t)\widetilde{\Pi}_h(t)\exp\{-d^*(t,h)h\}/\varphi$.

Log real total wealth evolves according to:

$$\begin{aligned}\mathrm{d}\log W(t) = (\ldots)\,\mathrm{d}t - \left[\omega_N(t)B_r(h_N) + \omega_I(t)B_r(h_I) + \omega_h(t)\widehat{B}_r(t)\right]\sigma_r\mathrm{d}Z_r(t) \\ - \omega_N(t)B_\pi(h_N)\,\sigma_\pi\mathrm{d}Z_\pi(t) + \omega_S(t)\sigma_S\mathrm{d}Z_S(t) + \omega_h(t)\widehat{B}_h(t)\,\sigma_h\mathrm{d}Z_h(t).\end{aligned} \tag{A47}$$

Comparing (A47) with (A44), we find

$$\omega_N^*(t) = \frac{1}{\gamma}\frac{\phi_\pi}{B_\pi(h_N)\,\sigma_\pi}, \tag{A48}$$

$$\omega_I^*(t) = \frac{1}{\gamma}\frac{\phi_r}{B_r(h_I)\,\sigma_r} + \left(1 - \frac{1}{\gamma}\right)\frac{\widetilde{B}_r(t)}{B_r(h_I)} - \omega_N^*(t)\frac{B_r(h_N)}{B_r(h_I)} - \omega_h^*(t)\frac{\widehat{B}_r(t)}{B_r(h_I)}, \tag{A49}$$

$$\omega_S^*(t) = -\frac{1}{\gamma}\frac{\phi_S}{\sigma_S}, \tag{A50}$$

$$\omega_h^*(t) = -\frac{1}{\gamma}\frac{\phi_h}{\widehat{B}_h(t)\,\sigma_h} + \varphi\left(1 - \frac{1}{\gamma}\right)\frac{\widetilde{B}_h(t)}{\widehat{B}_h(t)}. \tag{A51}$$

A.4 Impact of Human Wealth on Optimal Portfolio Weights

This appendix explores the impact of human wealth on the optimal portfolio weights. We assume the same setting as in Appendix A.3. We define human wealth as follows:

$$H(t) = \int_0^{T-t} H(t,h)\,\mathrm{d}h, \tag{A52}$$

where

$$H(t,h) = \mathbb{E}_t\left[\frac{m(t+h)}{m(t)}O(t+h)\right] \tag{A53}$$

with $O(t+h)$ outside income at adult age $t+h$ (in real consumption units), which we assume to be risk-free in real terms.

Straightforward computations show

$$\mathrm{d}H(t) = \big(r(t) - \lambda_r \sigma_r B_r^H(t)\big) H(t)\,\mathrm{d}t - B_r^H(t)\sigma_r H(t)\,\mathrm{d}Z_r(t) - O(t)\,\mathrm{d}t, \tag{A54}$$

where

$$B_r^H(t) = \int_0^{T-t} \frac{H(t,h)}{H(t)} B_r(h)\,\mathrm{d}h \tag{A55}$$

denotes the duration of human wealth.

Real financial wealth $F(t)$ evolves according to:

$$\begin{aligned}\mathrm{d}F(t) = (\ldots)\,\mathrm{d}t - \Big[\widetilde{\omega}_N(t)B_r(h_N) + \widetilde{\omega}_I(t)B_r(h_I) + \widetilde{\omega}_h(t)\widehat{B}_r(t)\Big]\,\sigma_r F(t)\mathrm{d}Z_r(t)\\ - \widetilde{\omega}_N(t)B_\pi(h_N)\,\sigma_\pi F(t)\mathrm{d}Z_\pi(t) + \widetilde{\omega}_S(t)\sigma_S F(t)\mathrm{d}Z_S(t)\\ + \widetilde{\omega}_h(t)\widehat{B}_h(t)\,\sigma_h F(t)\mathrm{d}Z_h(t).\end{aligned} \tag{A56}$$

Hence, total real wealth $W(t) = F(t) + H(t)$ satisfies

$$\begin{aligned}\mathrm{d}W(t) &= \mathrm{d}F(t) + \mathrm{d}H(t)\\ &= (\ldots)\,\mathrm{d}t - \Big[\widetilde{\omega}_N(t)B_r(h_N) + \widetilde{\omega}_I(t)B_r(h_I) + \widetilde{\omega}_h(t)\widehat{B}_r(t)\Big]\,\sigma_r F(t)\mathrm{d}Z_r(t)\\ &\quad - \widetilde{\omega}_N(t)B_\pi(h_N)\,\sigma_\pi F(t)\mathrm{d}Z_\pi(t) + \widetilde{\omega}_S(t)\sigma_S F(t)\mathrm{d}Z_S(t)\\ &\quad + \widetilde{\omega}_h(t)\widehat{B}_h(t)\,\widetilde{\sigma}_h F(t)\mathrm{d}Z_h(t) - D_H(t)\sigma_r H(t)\mathrm{d}Z_r(t).\end{aligned} \tag{A57}$$

Comparing (2.19) with (A57), we arrive at the optimal shares of financial wealth invested in the risky assets.

References

Ameriks, J., and S. Zeldes. 2004. How Do Household Portfolio Shares Vary with Age? *Working paper* .

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein. 2007. Portfolio Choice over the Life-Cycle when the Stock and Labor Markets Are Cointegrated. *Journal of Finance* 62:2123–2167.

Bodie, Z., R. C. Merton, and W. F. Samuelson. 1992. Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model. *Journal of Economic Dynamics and Control* 16:427–449.

Brennan, M. J., and Y. Xia. 2002. Dynamic Asset Allocation under Inflation. *Journal of Finance* 57:1201–1238.

Campbell, J. Y., and J. F. Cocco. 2003. Household Risk Management and Optimal Mortgage Choice. *Quarterly Journal of Economics* 118:1449–1494.

Campbell, J. Y., and J. F. Cocco. 2015. A Model of Mortgage Default. *Journal of Finance* 70:1495–1554.

Campbell, J. Y., and L. M. Viceira. 2001. Who Should Buy Long-Term Bonds? *American Economic Review* 91:99–127.

Chai, J., W. J. Horneff, R. H. Maurer, and O. S. Mitchell. 2011. Optimal Portfolio Choice over the Life Cycle with Flexible Work, Endogenous Retirement, and Lifetime Payouts. *Review of Finance* 15:875–907.

Chetty, R., L. Sandor, and A. Szeidl. 2017. The Effect of Housing on Portfolio Choice. *Journal of Finance* 72:1171–1212.

Cocco, J. F. 2005. Portfolio Choice in the Presence of Housing. *Review of Financial Studies* 18:535–567.

Cocco, J. F., F. J. Gomes, and P. J. Maenhout. 2005. Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies* 18:491–533.

Corradin, S., J. L. Fillat, and C. Vergara-Alert. 2014. Optimal Portfolio Choice with Predictability in House Prices and Transaction Costs. *Review of Financial Studies* 27:823–880.

Cox, J. C., and C. Huang. 1989. Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process. *Journal of Economic Theory* 49:33–83.

Cox, J. C., and C. Huang. 1991. A Variational Problem Arising in Financial Economics. *Journal of Mathematical Economics* 20:465–487.

Flavin, M., and S. Nakagawa. 2008. A Model of Housing in the Presence of Adjustment Costs: a Structural Interpretation of Habit Persistence. *American Economic Review* 98:474–495.

Gomes, F., and A. Michaelides. 2003. Portfolio Choice with Internal Habit Formation: A Life-Cycle Model with Uninsurable Labor Income Risk. *Review of Economic Dynamics* 6:729–766.

Gordon, M. J. 1959. Dividends, Earnings and Stock Prices. *Review of Economics and Statistics* 41:99–105.

Gordon, M. J., and E. Shapiro. 1956. Capital Equipment Analysis: The Required Rate of Profit. *Management Science* 3:102–110.

Horneff, W. J., R. H. Maurer, and O. S. Mitchell. 2019. How will persistent low expected returns shape household economic behavior? *Journal of Pension Economics and Finance* 18:612–622.

Horneff, W. J., R. H. Maurer, and O. S. Mitchell. 2020. Putting the pension back in 401(k) retirement plans: Optimal versus default deferred longevity income annuities. *Journal of Banking and Finance* 114:105783.

Horneff, W. J., R. H. Maurer, and O. S. Mitchell. 2022. How would 401(k) 'Rothification' alter saving, retirement security, and inequality? *Journal of Pension Economics and Finance* First View:1–19.

Horneff, W. J., R. H. Maurer, O. S. Mitchell, and R. Rogalla. 2015. Optimal life cycle portfolio choice with variable annuities offering liquidity and investment downside protection. *Insurance: Mathematics and Economics* 63:91–107.

Hubener, A., R. Maurer, and O. S. Mitchell. 2016. How Family Status and Social Security Claiming Options Shape Optimal Life Cycle Portfolios. *Review of Finance* 29:937–978.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve. 1987. Optimal Consumption and Portfolio Decisions for a "Small Investor" on a Finite Horizon. *SIAM Journal of Control and Optimization* 25:1557–1586.

Karatzas, I., and S. E. Shreve. 1998. *Methods of Mathematical Finance*, vol. 39. Springer.

Koijen, R. S. J., T. E. Nijman, and B. J. M. Werker. 2010. When Can Life Cycle Investors Benefit from Time-Varying Bond Risk Premia? *The review of financial studies* 23:741–780.

Liu, J. 2007. Portfolio Selection in Stochastic Environments. *Review of Financial Studies* 20:1–39.

Maurer, R. H., O. S. Mitchell, R. Rogalla, and T. Schimetschek. 2021. Optimal social security claiming behavior under lump sum incentives: theory and evidence. *Journal of Risk and Insurance* 88:5–27.

Merton, R. C. 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *Review of Economics and Statistics* 51:247–257.

Mossin, J. 1968. Optimal Multiperiod Portfolio Policies. *Journal of Business* 41:215–229.

OECD. 2021. Brick by Brick. Building Better Housing Policies. Tech. rep., Organization for Economic Cooperation and Development.

Pliska, S. R. 1986. A Stochastic Calculus Model of Continuous Trading: Optimal Portfolios. *Mathematics of Operations Research* 11:371–382.

Samuelson, P. A. 1969. Lifetime Portfolio Selection by Dynamic Stochastic Programming. *The Review of Economics and Statistics* 51:239–246.

Sangvinatsos, A., and J. A. Wachter. 2005. Does the Failure of the Expectations Hypothesis Matter for Long-Term Investors? *Journal of Finance* 60:179–230.

Sinai, T., and N. S. Souleles. 2005. Owner-Occupied Housing as a Hedge Against Rent Risk. *Quarterly Journal of Economics* 120:763–789.

Stokey, N. L. 2009. Moving costs, nondurable consumption and portfolio choice. *Journal of Economic Theory* 144:2419–2439.

van Bilsen, S., I. A. Boelaars, and A. L. Bovenberg. 2020. The Duration Puzzle in Life-Cycle Investment. *Review of Finance* 24:1271–1311.

van Hemert, O. 2010. Household Interest Rate Risk Management. *Real Estate Economics* 38:467–505.

Viceira, L. M. 2001. Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *The Journal of Finance* 56:433–470.

Yao, R., and H. H. Zhang. 2005. Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints. *Review of Financial Studies* 18:197–239.