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Abstract

This paper analyses an optimal investment problem, in which the agent aims to minimise a lower partial moments (LPM) criterion that depends on an exogenous reference level. The problem concerns terminal wealth alone and is specified in an affine-term structure model. We derive closed-form expressions for the optimal portfolio rules and the optimal wealth process. Moreover, we analytically disentangle the distributional features of optimal terminal wealth. In the numerical illustrations, we examine the problem in the context of a defined contribution (DC) pension scheme. The findings suggest that LPM-based investment policies can improve a pension fund's recovery potential. Despite their potentially outstanding performance, we illustrate that these policies may be difficult to implement. Furthermore, we show that the optima strongly depend on the estimates for the market prices of risk.

Keywords: Affine term structure models, lower partial moments, stochastic optimal control, reference level, utility maximisation

JEL Classification: C61, D53, G11, G22, J26, J32

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1 Introduction

Reference levels, otherwise known as benchmarks or targets, play a non-negligible role in the specification of an individual's preferences.[1] The critical nature of a reference level can be attributed to (i) its impact on preference qualifications and (ii) the fact that it constitutes a crucial part of an agent's risk profile. The impact on preferences is well-documented in studies on utility-based frameworks, see e.g. Browne (1999), Wagner (2002), Gómez and Zapatero (2003), and Berkelaar et al. (2004). The fact that it forms a pivotal element of an agent's risk profile is confirmed by e.g. Thaler (1980), Tversky and Kahneman (1991), Bateman et al. (1997), Munro and Sugden (2003), and Marzilli Ericson and Fuster (2011).[2] Due to their typical dependence on preference qualifications, the importance of a reference level carries over to optimal investment problems.

To illustrate this importance, let us visit the following stylised pension-related example.[3] Suppose that an individual receives a lump-sum of 100,000 monetary units upon retirement. We postulate that this amount is large enough to avail the individual of all resources necessary for an appropriate continuation of his/her pre-retirement life. In addition to this, assume that the individual's neighbours obtain a similar lump-sum, however, of 1,000,000 monetary units. Even though the former amount is in principle sufficient for the individual, in comparison to the neighbours' lump-sum, it appears rather small (only one tenth of the 1,000,000 monetary units). Depending on the value that this comparative magnitude has for the individual of interest, his/her experienced levels of happiness or utility will differ. In case of a strong reference-oriented individual, the 100,000 monetary units are clearly disappointing. The converse would be true for an individual who is indifferent with respect to the neighbours' financial circumstances. Note that this illustration extends to setups beyond the pension context, wherein the neighbours' retirement wealth can be identified as e.g. a life annuity, one's last earned wage or the stock index.

This example corroborates the claims above. In particular, it intuitively demonstrates that a reference level cannot be ignored in specifying an agent's preferences and, therefore, in optimising his/her utility from wealth. This intuition is scientifically supported by a great body of empirical findings dating back to a.o. Markowitz (1952), Edwards (1955), and Hershey and Schoemaker (1985).[4] These papers concretely argue that individuals tend to measure accumulated amounts of cash in relative terms, i.e. compared to a reference level. This feature gave rise to a myriad of studies on portfolio problems that explicitly incorporate such a level. We refer to section 2 of van Bilsen et al. (2020b) for an overview of such studies. In the contributions by a.o. Bernard and Ghossoub (2010), Balter et al. (2020),

[1] Throughout this paper, we use "reference level", "benchmark", "goal" and "target" interchangeably.

[2] This short outline of empirical studies on reference-dependent preferences is by far not exhaustive. We refer to O'Donoghue and Sprenger (2018) for a more comprehensive overview.

[3] This illustration builds on the "catching/keeping up with the Joneses" idea, formulated and analysed in a.o. Abel (1990), Gali (1994), and Gómez (2007).

[4] We refer to Zank (2010) for an extensive overview of these empirical studies.

and van Bilsen and Laeven (2020), the inclusion of a reference level in investment-linked frameworks is shown to have a non-negligible impact on the optimal decision variables. By virtue of the aforementioned empirical evidence and the latter non-negligible impact, research on reference-oriented or goal-based investment routines is highly relevant. Within the confines of portfolio optimisation, the literature on reference levels can roughly be distinguished into three categories: those that concentrate on (i) loss aversion, (ii) risk aversion, and (iii) preference-independent hedging criteria.

Loss aversion outlines a key concept in the domain of prospect theory, cf. Kahneman and Tversky (1979). It refers to the empirically confirmed tendency of individuals to value losses greater than equivalent gains. Losses and gains are defined with respect to a person-specific reference level. To model this phenomenon, loss aversion models ordinarily rely on S-shaped utility functions, see e.g. He and Zhou (2011). Risk aversion setups do not explicitly distinguish between losses and gains. These setups hinge on Inada-type preferences that characterise an agent's attitude towards risk. The agent's risk profile is accordingly defined across the *absolute* accumulation of capital. To model preferences around a benchmark, risk aversion frameworks redefine this absolute accumulation in relative terms. This redefinition is usually carried out by incorporating a reference level into the conventional definitions of preferences, see e.g. Detemple and Zapatero (1991), Van Binsbergen et al. (2008), and Kamma and Pelsser (2022a). Preference-independent hedging criteria ignore an individual's attitude towards risk and solely concentrate on acquiring/replicating a reference level. For these criteria, the reference level primarily serves as a goal or a target. Due to the independence from an individual's risk profile, hedging criteria involve a strong target-orientation. Models of this type are defined on a broad spectrum that includes e.g. super-replication and expected shortfall hedging, cf. Cvitanić et al. (1999) and Cvitanic (2000).

In this paper, we focus on an investment problem, in which the agent aims to minimise a lower partial moments (LPM) criterion. We assume that this criterion depends on an exogenous reference level. In terms of the three categories above, the LPM framework combines a preference-independent hedging criterion with the concept of risk aversion. Although not explicitly, as pointed out by Jarrow and Zhao (2006), aspects unique to loss aversion return in the LPM operator.[5] We stress that the LPM operator enters into the problem via the agent's objective function. Unlike most studies, e.g. Harlow and Rao (1989), Leitner (2008), and Gao et al. (2017), this implies that it does not define a downside risk constraint. Similar problems have been analysed by e.g. Föllmer and Leukert (2000), Jarrow and Zhao (2006), and Krabichler and Wunsch (2021). We emphasise that Nawrocki and Viole (2014) economically advocate the use of such partial moments for the

[5]In Jarrow and Zhao (2006), the authors demonstrate that the LPM problem can be subsumed under the umbrella of prospect theory. To this end, they employ a definition of (downside) loss aversion that relaxes the convexity requirement with regard to losses, cf. equation (1) and footnote 4 of their study.

study of utility and portfolio theory.

Mathematically speaking, the LPM operator specifies a smoothed variant of the expected shortfall criterion. The latter identifies a valid hedging criterion that is unaffected by an agent's risk profile. As a result of this smoothing procedure, the LPM operator can be interpreted as a utility function that is closely related to the Inada-family. The agent's preference qualification or attitude towards risk is correspondingly specified as follows. If wealth is equal to the benchmark, the agent becomes infinitely risk averse in an attempt to "lock in" wealth at the current level. On the contrary, if wealth falls below the benchmark, the agent becomes considerably less risk averse. The previous decline in the agent's level of risk aversion comes close to the "gamble for resurrection" behaviour specific to loss aversion frameworks. Adapted to setups involving risk aversion, Yang et al. (2021) refer to this phenomenon as "risk-taking for resurrection". These attributes of the LPM criterion imply that the agent is strongly target-oriented.

We analyse the problem in continuous-time for terminal wealth alone. Moreover, we place the problem in the financial market model proposed by Koijen et al. (2009). This model assumes an affine-term structure for the (real) interest rate and involves four risk-drivers. The market distinguishes nominal from real returns and accommodates market prices of risk that depend on a mean-reverting state variable. For the exogenous reference level, we postulate a general log-normal process. To the best of our knowledge, there are no studies available that couple the LPM objective to such a complex model specification. Due to the state-dependency of the market prices of risk, it is highly nontrivial to derive analytical solutions to the LPM problem. Nevertheless, using the Fourier transform similar to Carr and Madan (1999), we are able to derive closed-form expressions for the optimal portfolio rules and the optimal wealth process. In addition to this, we manage to disentangle an analytical formulation for the distributional properties of optimal terminal wealth. These analytical expressions form our first main contribution to the literature

In the numerical experiments, we analyse the problem in the context of a defined contribution (DC) pension scheme. For this reason, we identify the reference level as a life annuity that generates annual payments until the agent's fixed date of death. We numerically analyse the distribution of optimal retirement wealth and execute a sensitivity analysis of the optimal portfolio rules. The findings suggest that LPM-based investment policies can increase the likelihood of achieving one's pension goals, i.e. improve the pension fund's recovery potential. We demonstrate that both the recovery potential and the portfolio rules grandly depend on the estimates for the market prices of risk. This finding supports the use of policies that account for model uncertainty. These two economic takeaways constitute our second main contribution to the literature.

The remainder of the paper is structured as follows. Section 2 introduces the model setup. Section 3 presents the optimality conditions. Section 4 contains the numerical analysis. Finally, section 5 concludes. We collect all proofs in the appendix.

2 Model Setup

In this section, we spell out the model setup. First, we introduce the financial market model. Second, we provide the agent's dynamic budget constraint. Third, we specify the dynamics of a benchmark process, which models the agent's "goal" with regard to retirement. Fourth, we outline the optimal terminal wealth problem.

2.1 Financial Market Model

We make use of the financial market introduced in the paper by Koijen et al. (2009). Their model is defined in continuous-time, contains four independent risk-drivers and involves two distinct state variables. To specify this environment, we introduce a finite-valued trading horizon, $T > 0$. Correspondingly, the trading interval reads $[0, T]$. The uncertainty is described by a complete filtered probability space, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. Note that the separate components of this space live by their conventional definitions. On this space, the four factors are defined by an $\mathbb{R}^4$-valued standard Brownian motion process, $\{W_t\}_{t\in[0,T]}$, such that $W_t = [W_{1,t}, W_{2,t}, W_{3,t}, W_{4,t}]^\top$. Henceforth, we assume that all (in)equalities between stochastic processes hold in either a $\mathbb{P}$-a.s. or $\mathrm{d}t \otimes \mathbb{P}$-a.e. sense.

In line with the market specification in Koijen et al. (2009), we introduce an $\mathbb{R}^2$-dimensional state variable process, $Z_t = [Z_{1,t}, Z_{2,t}]^\top$. These state variables are assumed to evolve according to an Ornstein-Uhlenbeck process as follows:

$$\mathrm{d}Z_t = -K_Z Z_t \mathrm{d}t + \Sigma_Z \mathrm{d}W_t,\ Z_0 = 0_2, \tag{2.1}$$

where $0_2 = [0, 0]^\top$ Here, the drift and diffusion terms, K_Z and Σ_Z, naturally achieve values in $\mathbb{R}^{2\times 2}$ and $\mathbb{R}^{4\times 4}$, respectively. We postulate that K_Z constitutes a lower-triangular matrix. Moreover, we set $\Sigma_Z = [I_{2\times 2}, 0_{2\times 2}]$, where $I_{2\times 2}$ characterises a two-dimensional identity matrix and $0_{2\times 2}$ identifies an $\mathbb{R}^{2\times 2}$-valued matrix containing zeros.

Similar to the models in Brennan and Xia (2002) and Sangvinatsos and Wachter (2005), this market makes a distinction between nominal and real return dynamics. To model this distinction, we introduce three processes for (i) the nominal interest rate, (ii) the commodity price index (CPI), and (iii) the inflation rate. The first is specified as:

$$r_t = \delta_{0,r} + \delta_{1,r}^\top Z_t, \tag{2.2}$$

where $\delta_{0,r} \in \mathbb{R}_+$ and $\delta_{1,r} \in \mathbb{R}^2$. That is, the process for the instantaneous interest rate is affine in both state variables, $Z_{1,t}$ and $Z_{2,t}$. For the CPI, we assume a geometric form, such that it evolves according to the following stochastic differential equation (SDE):

$$\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t \mathrm{d}t + \sigma_\Pi^\top \mathrm{d}W_t,\ \Pi_0 = 1, \tag{2.3}$$

where π_t spells out the instantaneous expected inflation process, and σ_Π outlines an $\mathbb{R}^4$-valued vector. As r_t, the process for π_t is assumed to be affine in Z_t:

$$\pi_t = \delta_{0,\pi} + \delta_{1,\pi}^\top Z_t, \tag{2.4}$$

where $\delta_{0,\pi} \in \mathbb{R}_+$ and $\delta_{1,\pi} \in \mathbb{R}^2$.

In this financial market, $\mathcal{M}$, the agent is allowed to continuously trade in five financial instruments: a money market account, a stock, two nominal bonds, and an inflation-linked bond. We first introduce the money market account and the stocks. After determining the term structure of interest rates, we provide the dynamics of the two bonds. The money market account lives by the following ordinary differential equation (ODE):

$$\frac{\mathrm{d}B_t}{B_t} = r_t \mathrm{d}t, \; B_0 = 1. \tag{2.5}$$

Note that B_t is specified in nominal terms. The return dynamics of the non-dividend paying (nominal) stock are characterised by the following SDE:

$$\frac{\mathrm{d}S_{1,t}}{S_{1,t}} = (r_t + \eta_S)\,\mathrm{d}t + \sigma_S^\top \mathrm{d}W_t, \; S_{1,0} = 1, \tag{2.6}$$

Here, σ_S defines the $\mathbb{R}^4$-valued vector for the volatility of the stock. Additionally, η_S characterises the time-independent scalar-valued equity risk premium of the stock.

We assume that the financial market excludes frictions, i.e. (proportional) transaction costs. Moreover, we postulate that all risk factors are traded, due to which the market is complete. Therefore, by the fundamental theorem of asset pricing, cf. Delbaen and Schachermayer (1994), we know that there exists a unique equivalent martingale measure. In view of its dependency on this measure, we consequently know that there exists a unique nominal state price density process, $\{M_t\}_{t\in[0,T]}$. Under the assumption that $\{B_t\}_{t\in[0,T]}$ serves as the numéraire quantity, this process is given by the subsequent SDE:

$$\frac{\mathrm{d}M_t}{M_t} = -r_t \mathrm{d}t - \Lambda_t^\top \mathrm{d}W_t, \; M_0 = 1, \tag{2.7}$$

where Λ_t denotes the $\mathbb{R}^4$-valued process for the nominal market prices risk. Following the reasoning around equations (1)-(4) in Detemple et al. (2003), we are able to write the state price density process as follows: $M_t = \exp\big\{ -\int_0^t r_s \mathrm{d}s - \frac{1}{2}\int_0^t \Lambda_s^\top \Lambda_s \mathrm{d}s - \int_0^t \Lambda_s^\top \mathrm{d}W_s \big\}$, for all $t \in [0,T]$. This representation plays an important role in the LPM problem.

To arrive at an affine model for the term structure of interest rates, Koijen et al. (2009) assume that these nominal market prices of risk are affine in both state variables, $Z_{1,t}$ and $Z_{2,t}$. Concretely, we suppose that the process for Λ_t is given by:

$$\Lambda_t = \Lambda_0 + \Lambda_1 Z_t, \tag{2.8}$$

where $\Lambda_0 \in \mathbb{R}^4$ and $\Lambda_1 \in \mathbb{R}^{4\times 2}$. It should be noted that the specification of $\{S_t\}_{t\in[0,T]}$ in (2.6) forces restrictions upon the preceding Λ_0 and Λ_1 parameters. Namely, by martingale arguments, it must hold that $\sigma_S^\top \Lambda_t = \eta_S$ for all $t \in [0,T]$. Suppose that $\Lambda_{0(i)}$ and $\Lambda_{1(i,j)}$ denote the i^{th} entry of Λ_0 and the $(i,j)^{\text{th}}$ element of Λ_1, respectively. Throughout the remainder of this paper, we adopt this notation for all vectors and matrices. In Koijen et al. (2009), the former condition is handled by assuming that $\Lambda_{0(4)}$, $\Lambda_{1(4,1)}$, and $\Lambda_{1(4,2)}$ are defined such that $\sigma_S^\top \Lambda_0 = \eta_S$ and $\sigma_S^\top \Lambda_1 = 0_2^\top$. We assume the same:

$$\begin{aligned} \Lambda_{0(4)} &= \frac{\eta_S}{\sigma_{S(4)}} - \frac{1}{\sigma_{S(4)}} \sum_{i=1}^{3} \sigma_{S(i)} \Lambda_{0(i)}, \\ \Lambda_{1(4,j)} &= -\frac{1}{\sigma_{S(4)}} \sum_{i=1}^{3} \sigma_{S(i)} \Lambda_{1(i,j)}, \quad j = 1, 2. \end{aligned} \tag{2.9}$$

We complete the asset mix by introducing two nominal bonds and an inflation-linked bond. For a given time to maturity, $t + \tau_i$, the nominal bonds evolve according to:

$$\frac{\mathrm{d}P_{t,t+\tau_i}}{P_{t,t+\tau_i}} = \left(r_t + B(\tau_i)^\top \Sigma_Z \Lambda_t\right) \mathrm{d}t + B(\tau_i)^\top \Sigma_Z \mathrm{d}W_t, \ P_{0,\tau_i} = e^{A(\tau_i)}, \tag{2.10}$$

for $i = 1, 2$, such that $\tau_1 \neq \tau_2$. Suppose that $\widetilde{\Lambda}_0 = \Sigma_Z \Lambda_0$ and $\widetilde{\Lambda}_1 = \Sigma_Z \Lambda_1$. Then, the deterministic functions $A(x)$ and $B(x)$ are for all $x \in \mathbb{R}_+$ given by:

$$\begin{aligned} A(x) &= -\int_0^x \left[B(s)^\top \widetilde{\Lambda}_0 - \frac{1}{2} B(s)^\top B(s) + \delta_{0,r}\right] \mathrm{d}s, \\ B(x) &= \left(K_Z^\top + \widetilde{\Lambda}_1^\top\right)^{-1} \left[\exp\left\{-\left[K_Z^\top + \widetilde{\Lambda}_1^\top\right] x\right\} - I_{2\times 2}\right] \delta_{1,r}. \end{aligned} \tag{2.11}$$

Note that the return dynamics of the nominal bonds in (2.10) are based on the following identity: $P_{t,T} = \mathbb{E}\left[\frac{M_T}{M_t} \,\middle|\, \mathcal{F}_t\right] = e^{A(T-t)+B(T-t)^\top Z_t}$ for all $t \in [0,T]$. In a similar sense, we are able to derive the nominal return dynamics of an inflation-linked bond with a given time to maturity, $t + \tau$, say $P_{t,t+\tau}^R$. That is, the identity for the inflation-linked bond is predicated on: $P_{t,T}^R = \mathbb{E}\left[\frac{M_T \Pi_T}{M_t} \,\middle|\, \mathcal{F}_t\right] = e^{A^R(T-t)+B^R(T-t)^\top Z_t} \Pi_t$ for all $t \in [0,T]$. As a result, for a given time to maturity, $t + \tau$, the SDE of $P_{t,t+\tau}^R$ evolves according to:

$$\frac{\mathrm{d}P_{t,t+\tau}^R}{P_{t,t+\tau}^R} = \left(r_t + B^R(\tau)^\top \Sigma_Z \Lambda_t + \sigma_\Pi^\top \Lambda_t\right) \mathrm{d}t + \left(B^R(\tau)^\top \Sigma_Z + \sigma_\Pi^\top\right) \mathrm{d}W_t, \ P_{0,t}^R = e^{A^R(\tau)}, \tag{2.12}$$

where the deterministic functions $A^R(\tau)$ and $B^R(\tau)$ read for all $\tau \in \mathbb{R}_+$ as:

$$\begin{aligned} A^R(\tau) &= -\int_0^\tau \left[B^R(s)^\top \widehat{\Lambda}_0 - \frac{1}{2} B^R(s)^\top B^R(s) + \widehat{\delta}_{0,r}\right] \mathrm{d}s, \\ B^R(\tau) &= \left(K_Z^\top + \widetilde{\Lambda}_1^\top\right)^{-1} \left[\exp\left\{-\left[K_Z^\top + \widetilde{\Lambda}_1^\top\right] \tau\right\} - I_{2\times 2}\right] \widehat{\delta}_{1,r}. \end{aligned} \tag{2.13}$$

Here, $\widehat{\delta}_{0,r} = \delta_{0,r} - \delta_{0,\pi} + \sigma_\Pi^\top \Lambda_0$, $\widehat{\delta}_{1,r} = \delta_{1,r} - \delta_{1,\pi} + \Lambda_1^\top \sigma_\Pi$, and $\widehat{\Lambda}_0 = \widetilde{\Lambda}_0 - \widetilde{\sigma}_\Pi$, for $\widetilde{\sigma}_\Pi = \Sigma_Z \sigma_\Pi$.

We conclude the introduction of the two bonds with three remarks. First, following Pelsser (2019), we note that the Dutch government does not issue inflation-linked bonds. However, as most other countries in the euro-area do issue such contracts (e.g. France), it is realistic to include a single inflation-linked bond in the asset mix. Second, in outlining both $P_{t,t+\tau_i}$ and $P^R_{t,t+\tau}$ in (2.10) and (2.12), respectively, we have employed the notation of Chen et al. (2020). To avoid confusion, we stress that $\widetilde{\Lambda}_0 = \left[\Lambda_{0(1)}, \Lambda_{0(2)}\right]^\top$, $\widetilde{\Lambda}_1 = \left[\Lambda_{1(1,1:2)}, \Lambda_{1(2,1:2)}\right]^\top$, and $\widetilde{\sigma}_\Pi = \left[\sigma_{\Pi(1)}, \sigma_{\Pi(2)}\right]^\top$. Third and last, we observe that the three bonds are assumed to have continuously adjusted times to maturity, $t + \tau_i$ and $t + \tau$. We are obliged to make this simplifying assumption in order to ensure that the three bonds are traded throughout the entire trading interval, $[0, T]$.[6]

Finally, suppose that $S_t = \left[S_{1,t}, P_{t,t+\tau_1}, P_{t,t+\tau_2}, P^R_{t,t+\tau}\right]^\top$. Then, the SDE of S_t reads:

$$\mathrm{d}S_t = \operatorname{diag}(S_t)\left[(r_t + \sigma\Lambda_t)\,\mathrm{d}t + \sigma \mathrm{d}W_t\right], \tag{2.14}$$

with starting value $S_0 = \left[1, e^{A(\tau_1)}, e^{A(\tau_2)}, e^{A^R(\tau)}\right]^\top$. In the latter SDE, $\operatorname{diag}(S_t)$ stands for the $\mathbb{R}^{4\times 4}$-valued diagonal matrix, which contains the four distinct entries of S_t on its diagonal. To be more precise: $\operatorname{diag}(S_t) = \left(S_t 1_4^\top\right) \odot I_{4\times 4}$, where 1_4 spells out an $\mathbb{R}^4$-valued vector of 1's, $I_{4\times 4}$ characterises a four-dimensional identity matrix, and "$\odot$" denotes the Hadamard product. The matrix σ outlines the volatility process of S_t and attains values in $\mathbb{R}^{4\times 4}$. Here, σ consists of scalars alone and is defined as follows: $\sigma = \left[\sigma_S, B(\tau_1)^\top \Sigma_Z, B(\tau_2)^\top \Sigma_Z, B^R(\tau)^\top \Sigma_Z + \sigma_\Pi^\top\right]^\top$. We cast the four different risky financial instruments into the vector-format provided by S_t in order to both facilitate derivations and simplify notation. We conclude this section by pointing out that $\{S_t\}_{t\in[0,T]}$ is indeed driven by the four Brownian motions, $\{W_t\}_{t\in[0,T]}$. As a consequence, trading in S_t alone suffices to hedge all the risk present in this financial environment.

2.2 Dynamic Budget Constraint

We proceed with the introduction of the agent's dynamic budget constraint. In the utility maximisation problem of interest, this agent is solely concerned about terminal wealth. As a result, consumption does not play a role and can be excluded from the specification of the budget constraint. Taking into account that trading takes place in continuous-time over the financial instruments from section 2.1, it suffices to introduce a single endogenous process, $\{\pi_t\}_{t\in[0,T]}$, as the control variable. This process is $\mathbb{R}^4$-valued, $\mathcal{F}_t$-progressively measurable, and records the agent's decisions with regard to investment. More precisely, $\{\pi_t\}_{t\in[0,T]}$ denotes the amount of monetary units that the agent allocates to $\{S_t\}_{t\in[0,T]}$.

[6] The yields or the continuously compounded zero-coupon rates of the nominal bonds and the index-linked bond, say y_{t,τ_i} and $y^R_{t,\tau}$, are as follows: $y_{t,\tau_i} = -\frac{\log P_{t,t+\tau_i}}{\tau_i} = -\frac{A(\tau_i)}{\tau_i} - \frac{B(\tau_i)^\top}{\tau_i} X_t$ and $y^R_{t,\tau} = -\frac{\log P^R_{t,t+\tau}}{\tau} = -\frac{A^R(\tau)}{\tau} - \frac{B^R(\tau)^\top}{\tau} X_t - \frac{1}{\tau}\int_0^t \pi_s \mathrm{d}s + \frac{1}{\tau}\frac{1}{2}\sigma_\Pi^\top \sigma_\Pi - \frac{1}{\tau}\sigma_\Pi^\top W_t$.

Let us assume that the agent is at the start of the trading interval, $t = 0$, in possession of an initial endowment equal to $X_0 \in \mathbb{R}_+$. Then, given $X_0 \in \mathbb{R}_+$, the agent's dynamic budget constraint evolves according to the following SDE:

$$\begin{aligned} \mathrm{d}X_t &= \left(X_t - \sum_{i=1}^{4} \pi_{i,t} \right) \frac{\mathrm{d}B_t}{B_t} + \sum_{i=1}^{4} \pi_{i,t} \frac{\mathrm{d}S_{i,t}}{S_{i,t}} \\ &= \left(X_t r_t + \pi_t^\top \sigma \Lambda_t \right) \mathrm{d}t + \pi_t^\top \sigma \mathrm{d}W_t. \end{aligned} \tag{2.15}$$

For the purpose of clarity, we note that $S_{i,t}$ represents the i^{th} element of the $\mathbb{R}^4$-valued vector, S_t, in (2.14). Moreover, we observe that $X_t - \sum_{i=1}^{4} \pi_{i,t}$ stands for the amount of monetary units that the agent leaves in the money market account, B_t, at time $t \in [0, T]$. Note that the allocation of assets to B_t is, therefore, entirely dependent on the allocation to S_t. To prevent the agent from implementing ill-posed investment decisions, e.g. doubling strategies, we introduce an admissibility set, $\mathcal{A}_{X_0}$. This set contains all *admissible* or well-posed investment strategies: $\mathcal{A}_{X_0}$ consists of all $\{\pi_t\}_{t\in[0,T]}$, such that $X_t \geq 0$, $\int_0^T \pi_t^\top \sigma_t \sigma_t^\top \pi_t \mathrm{d}t < \infty$, and $\int_0^T \left| \pi_t^\top \sigma_t \Lambda_t \right| \mathrm{d}t < \infty$ hold for all $t \in [0, T]$. Naturally, we restrict the agent's investment decisions to those that are contained in the admissibility set, $\mathcal{A}_{X_0}$. See for instance Karatzas and Shreve (1998), van Bilsen et al. (2020a), or Kamma and Pelsser (2022b), for similar definitions of admissible trading strategies.

2.3 Specification of Benchmark

In this financial environment, the agent is assumed to retire at $t = T$. Accordingly, the agent's retirement wealth is equal to X_T. In this context, X_T can be regarded as (i) a lump-sum that is paid out to the agent at $t = T$, or (ii) as a specific amount of monetary units that is converted into an annuity, which renders annual payments until the agent's date of death. Within the confines of a pension scheme, it is reasonable to assume that participants have particular expectations with regard to their retirement wealth. That is, the agent in our model setup may have in mind a certain benchmark or reference level that he/she ideally acquires at retirement. To model this benchmark, we make use of a log-normally distributed random variable.[7] Suppose that Y_T represents the agent's person-specific (real) benchmark. Then, we postulate that Y_T lives by:

$$Y_T = Y_0 \exp\left\{ \int_0^T \alpha_s \mathrm{d}s - \frac{1}{2} \int_0^T \beta_s^\top \beta_s \mathrm{d}s + \int_0^T \beta_s^\top \mathrm{d}W_s \right\}. \tag{2.16}$$

[7] We assume log-normality of this benchmark process for two main reasons. First, due to its distributional elegance, the log-normal benchmark does not interfere with a closed-form characterisation of optimal solutions. Second, because it outlines a strictly positive process that can be identified with a.o. B_t and S_t, the log-normal specification allows for a fairly broad selection of interpretations. In the numerical illustration related to the investment problem of interest, we define the benchmark as the analytical approximation to a life annuity. In this definition of the benchmark, its log-normality plays a central role via the so-called Fenton-Wilkinson approximate method, cf. Fenton (1960).

Here, $Y_0 \in \mathbb{R}_+$ stands for the benchmark's starting value. Moreover, α_t and β_t represent two deterministic processes that attain values in $\mathbb{R}$ and $\mathbb{R}^4$, respectively.[8]

We conclude by touching upon two features relevant to the benchmark in (2.16): (i) possible interpretations of Y_T, and (ii) the funding or coverage ratio corresponding to Y_T. Concerning item (i), as mentioned in Kamma and Pelsser (2022b), there is a wide variety of interpretations available for Y_T. We should stress here that these interpretations should respect the exogeneity of Y_T, meaning that Y_T must be left unaffected by the agent's own decisions. For example, we could interpret Y_T as one's labour income, the net worth of the agent's neighbour, or a fraction of a nation's GDP. As for item (ii), we note that the funding or coverage ratio corresponding to Y_T is defined as: $F_0 = \frac{X_0}{\mathbb{E}[Y_T M_T \Pi_T]}$. If $F_0 < 1$, the agent is not in possession of enough funds (assets) at $t = 0$ to risk-neutrally cover his/her benchmark (liabilities). The converse holds true if $F_0 \geq 1$. Taking into account that F_0 is linear in Y_0, it is clear that we can alter Y_0 to modify the coverage ratio. In the sequel, we are limited to analysing the $F_0 < 1$ case

2.4 Optimal Investment Problem

The subsequent investment problem is predicated on the LPM formulation that is addressed in section 5.2 of Föllmer and Leukert (2000). To supply this problem, we cast the agent's situation into the following context. The financial market, $\mathcal{M}$, occupies an agent, who is in possession of an initial endowment $X_0 \in \mathbb{R}_+$. The agent uses this entire prefixed amount of monetary units for investment over the trading interval, $[0, T]$. Over the course of this interval, the agent is allowed to continuously modify the weights of his/her portfolio. These portfolio weights, π_t, are defined as the amounts of monetary units that the agent allocates to the four risky assets, S_t. By modifying these weights, the agent aims to minimise the so-called lower partial moment of the difference between *real* retirement wealth, $\frac{X_T}{\Pi_T}$, and the (real) benchmark Y_T. In doing so, the agent tries to acquire an amount of real retirement wealth that is as close as possible to his/her goal: the benchmark. In mathematical terms, the agent faces the following investment problem:

$$
\begin{aligned}
\sup_{\{\pi_t\}_{t\in[0,T]} \in \mathcal{A}_{X_0}} \quad & \mathbb{E}\left[-\frac{1}{p} \left[\left(Y_T - \frac{X_T}{\Pi_T} \right)^+ \right]^p \right] \\
\text{s.t.} \quad & \mathrm{d}X_t = \left(X_t r_t + \pi_t^\top \sigma \Lambda_t \right) \mathrm{d}t + \pi_t^\top \sigma \mathrm{d}W_t, \\
& X_0 < \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left[\frac{Y_T \Pi_T}{B_T} \right] = \mathbb{E}\left[Y_T M_T \Pi_T \right].
\end{aligned}
\tag{2.17}
$$

Here, $p > 1$ holds and $\mathcal{Q}$ describes the set of all equivalent martingale measures under the numéraire $\{B_t\}_{t\in[0,T]}$. Notation-wise, we let $\mathbb{E}^{\mathbb{Q}}[\cdot]$ represent the expectation operator under the $\mathbb{Q}$ measure, and $(\cdot)^+$ stands for the max-operator: $(y)^+ = \max\{0, y\}$ for all

[8]Henceforth, with a slight abuse of notation, we set Y_t equal to $Y_T\big|_{T=t}$ for all $t \in [0, T]$.

$y \in \mathbb{R}$. If $p = 1$, the formulation in (2.17) reduces to the expected shortfall minimisation problem, cf. section 4 in Föllmer and Leukert (2000). However, $p = 1$ cannot be regarded as a special case of the optimal solution to (2.17), as $x \mapsto -\frac{\left[(x)^+\right]^p}{p}$ is only continuously differentiable for $p > 1$. Although the formulation in (2.17) is specified in the canonical form of a maximisation problem, it should be noted that it is equivalent to minimising $\mathbb{E}\left[\frac{1}{p}\left[\left(Y_T - \frac{X_T}{\Pi_T}\right)^+\right]^p\right]$. The latter criterion characterises the p^{th} lower partial moment of $\frac{X_T}{\Pi_T} - Y_T$. The agent aims to minimise this criterion over all admissible trading strategies, $\{\pi_t\}_{t\in[0,T]} \in \mathcal{A}_{X_0}$, given that $X_0 < \mathbb{E}\left[Y_T M_T \Pi_T\right]$ holds true. The preceding condition states that X_0 ought to be smaller than the super-replication price of Y_T.[9] Since the market $\mathcal{M}$ is complete, this super-replication price coincides with the risk-neutral value of Y_T, given by $\mathbb{E}\left[Y_T M_T \Pi_T\right]$. Hence, $F_0 < 1$ holds here. Observe that if $F_0 \geq 1$, X_0 exceeds the super-replication price, which enables $\frac{X_T}{\Pi_T}$ to exceed Y_T in all states of the world. As this makes problem (2.17) superfluous, we solely analyse the $F_0 < 1$ case.

3 Optimality Conditions

In this section, we analytically spell out and analyse the optimality conditions corresponding to the investment problem in (2.17). First, we provide a general formulation of the optimality conditions for the LPM problem in the spirit of Föllmer and Leukert (2000). Second, we introduce the optimal solutions to the dynamic optimisation problem (2.17) in closed-form. In line with the martingale method, these solutions are two-fold: (i) one part of the solution clearly concerns the optimal trading strategy, and (ii) the remaining part pertains to the agent's optimal wealth process. In view of the fact that item (i) follows directly from item (ii), we first focus on the latter and conclude with the former. All derivations relevant to this section can be found in the appendix.

3.1 General Optimal Solutions

In deriving the optimality conditions to the formulation in (2.17), we make use of the martingale method. This method is developed in the seminal contributions by Pliska (1986), Karatzas et al. (1987), and Cox and Huang (1989, 1991), and revolves around a static alternative to the dynamic problem in (2.17). This approach slightly differs from the one that Föllmer and Leukert (2000) employ to solve (2.17). Whereas their approach treats the LPM operator as a hedging criterion, the martingale method emphasises its specification as a utility function. To identify the LPM operator as a utility function, let us introduce the mapping $U : \mathbb{R}_+^2 \to \mathbb{R}_-$ defined by: $U(x, y) = -\frac{1}{p}\left[(y - x)^+\right]^p$, for all $x, y \in \mathbb{R}_+$. Clearly, $U : \mathbb{R}_+^2 \to \mathbb{R}_-$ qualifies as a utility function and $\mathbb{E}\left[U(X_T, \Pi_T)\right]$ spells

[9]For the definition of a super-replication price, cf. El Karoui and Quenez (1995), Kramkov (1996), Föllmer and Kabanov (1997), and Example 2 of Rogers (2003).

out the objective of the problem in (2.17). We observe that the inverse of marginal utility is characterised by: $I(z,y) = y - z^{\frac{1}{p-1}} \wedge y$. In Theorem 3.1, we introduce the general optimality conditions for the investment problem in (2.17).

Theorem 3.1. *Consider the optimal dynamic LPM investment problem in* (2.17)*. The corresponding optimal wealth process, X_t^{opt}, is for all $t \in [0,T]$ characterised by:*

$$X_t^{\text{opt}} = \frac{1}{M_t}\mathbb{E}\left[Y_T M_T^R \mathbb{1}_{\{\mathcal{A}_T\}} \mid \mathcal{F}_t\right] - \frac{1}{M_t}\mathbb{E}\left[\left(\mathcal{H}^{-1}(X_0)^{\frac{1}{p}} M_T^R\right)^{\frac{p}{p-1}} \mathbb{1}_{\{\mathcal{A}_T\}} \,\middle|\, \mathcal{F}_t\right]. \tag{3.1}$$

Here, $\mathcal{H}^{-1}: \mathbb{R}_+ \to \mathbb{R}_+$ outlines the inverse function of $\mathcal{H}: \mathbb{R}_+ \to \mathbb{R}_+$, which is specified as: $\mathcal{H}(\eta) = \mathbb{E}\left[I(\eta M_T \Pi_T, Y_T)\Pi_T M_T\right] = X_0$. The event $\mathcal{A}_T$ is given by: $\mathcal{A}_T = \left\{Y_T \geq \left(\mathcal{H}^{-1}(X_0) M_T^R\right)^{\frac{1}{p-1}}\right\}$. We define M_t^R as the real pricing kernel: $M_t^R = M_t \Pi_t$, for all $t \in [0,T]$. Consistent with X_t^{opt}, there exists an $L^2([0,T])$-valued process, $\{\psi_t\}_{t\in[0,T]}$, such that the optimal investment strategy, π_t^{opt}, reads for all $t \in [0,T]$ as:

$$\pi_t^{\text{opt}} = \sigma^{\top^{-1}} \frac{\psi_t}{M_t} + \sigma^{\top^{-1}} \Lambda_t X_t^{\text{opt}}. \tag{3.2}$$

Proof. The proof is given in Appendix A. □

The specifications of X_t^{opt} and π_t^{opt} are valid for general Y_t and general return dynamics. More specifically, the presented optima in Theorem 3.1 still hold true for non-log-normal characterisations of Y_t, e.g. a semi-martingale definition. The same applies to the specification of S_t, i.e. the return dynamics. Nevertheless, for concrete setups, it is in general possible to spell out X_t^{opt} and π_t^{opt} more explicitly. In case of $\mathcal{M}$, one can indeed derive closed-form expressions for both processes. Due to the distributional features of M_t, the evaluation of the conditional expectations in (3.1) demands special care. We elaborate on the details in section 3.2. As the retrieval of ψ_t in (3.2) entirely depends on the (analytical) definition of X_t^{opt}, the distributional properties of M_t also play a role in the identification of π_t^{opt}. We discuss this at greater length in section 3.3. Note that ψ_t can be derived from X_t in closed-form by means of Itô's Lemma or Malliavin calculus. For details on the latter less well-known approach, we refer to Nualart (2006) or Appendix C.

3.2 Optimal Wealth Process

We continue with the introduction of the optimal wealth process corresponding to the dynamic investment problem in (2.17): X_t^{opt} for all $t \in [0,T]$. As addressed in the discussion of the general optimality conditions in section 3.1, the derivation of X_t^{opt} comes down to an evaluation of the conditional expectations in (3.1). To analytically evaluate these expectations, we are required to employ Fourier transforms. Namely, the stochastic processes included in this identity for X_t^{opt} are, for non-zero Λ_1, not log-normally distributed. In fact, the distributional features of these processes are not known, which encumbers a closed-form recovery of X_t^{opt} by means of standard machinery. Therefore, the Fourier transform comes in handy, because it enables us to evaluate the conditional expectations in (3.1), without requiring the exact distributional properties of the relevant processes. Details concerning the Fourier transform in application to (3.1) can be found in Appendix B.1. In Proposition 3.2, we formally introduce X_t^{opt}.

Proposition 3.2. *Consider the optimal dynamic investment problem in* (2.17)*. The corresponding optimal wealth process, X_t^{opt}, is given in* (3.1)*. After analytical evaluation of the conditional expectations in this identity, the following must hold:*

$$\begin{aligned} X_t^{\mathrm{opt}} &= Y_t \Pi_t P_1(t, Z_t) \frac{1}{2\pi} \int_{-\infty}^{\infty} f_\kappa^*(T, \omega)\, \phi_{1,T-t}(-\omega - i\kappa, h)\, \mathrm{d}\omega \\ &\quad - \left(\mathcal{H}^{-1}(X_0) M_t \Pi_t\right)^{\frac{1}{p-1}} \Pi_t P_2(t, Z_t) \frac{1}{2\pi} \int_{-\infty}^{\infty} f_\kappa^*(T, \omega)\, \phi_{2,T-t}(-\omega - i\kappa, h)\, \mathrm{d}\omega, \end{aligned} \tag{3.3}$$

for all $t \in [0, T]$. Here, $\eta^{\mathrm{opt}} = \mathcal{H}^{-1}(X_0) \in \mathbb{R}_+$ characterises the optimal Lagrange multiplier, which can be retrieved from solving $X_0^{\mathrm{opt}} = X_0$ for η^{opt}. Moreover, $P_1(t, Z_t)$ and $P_2(t, Z_t)$ are for all $t \in [0, T]$ specified as:

$$\begin{aligned} P_1(t, Z_t) &= \exp\left\{\tilde{A}(t) + \tilde{B}(t)^\top Z_t\right\}, \text{ and} \\ P_2(t, Z_t) &= \exp\left\{\widehat{A}(t) + \widehat{B}(t)^\top Z_t + Z_t^\top \widehat{C}(t) Z_t\right\}. \end{aligned} \tag{3.4}$$

Here, $t \mapsto \tilde{A}(t)$ and $t \mapsto \tilde{B}(t)$ outline two deterministic functions that are given in (B.9)*. Likewise, $t \mapsto \widehat{A}(t)$, $t \mapsto \widehat{B}(t)$ and $t \mapsto \widehat{C}(t)$ spell out three deterministic functions that jointly solve the system of ODE's in* (B.10)*. The deterministic function $f_\kappa^*(T, \omega)$ is for all $\omega \in \mathbb{R}$ and some $\kappa \in \mathbb{R}_+$ given by: $f_\kappa^*(T, \omega) - \frac{e^{(i\omega - \kappa)\frac{1}{p-1}\log \mathcal{H}^{-1}(X_0)}}{i\omega - \kappa}$. Ultimately, let $Q_j(t, Z_t, \omega) = e^{\bar{A}_j(t,\omega) + \bar{B}_j(t,\omega)^\top Z_t + Z_t^\top \bar{C}_j(t,\omega) Z_t}$ for $j = 1, 2$, all $\omega \in \mathbb{R}$ and $t \in [0, T]$. The functions, $(t, \omega) \mapsto \bar{A}_j(t, \omega)$, $(t, \omega) \mapsto \bar{B}_j(t, \omega)$ and $(t, \omega) \mapsto \bar{C}_j(t, \omega)$, jointly solve the system of ODE's in* (B.23)*, for $j = 1, 2$. Then, for $j = 1, 2$, all $\omega \in \mathbb{R}$ and $t \in [0, T]$:*

$$\phi_{j,T-t}(\omega, h) = Q_j(t, Z_t, \omega) \left(M_t^{R^{-\frac{1}{p-1}}} Y_t\right)^{i\omega}. \tag{3.5}$$

Proof. The proof is given in Appendix B.1. □

The expression for optimal wealth over the trading interval, X_t^{opt} in (3.3), can be analysed along the following technical lines. The $Y_t \Pi_t P_1(t, Z_t)$ term coincides with the risk-neutral value of $Y_T \Pi_T$ at time $t \in [0, T]$. Similarly, the $\left(\mathcal{H}^{-1}(X_0) M_t \Pi_t\right)^{\frac{1}{p-1}} \Pi_t P_2(t, Z_t)$ term identifies the risk-neutral value of $\left(\mathcal{H}^{-1}(X_0) M_T \Pi_T\right)^{\frac{1}{p-1}} \Pi_T$ at time $t \in [0, T]$. Moreover, the two integral expressions, $\frac{1}{2\pi} \int_{-\infty}^{\infty} f_\kappa^*(T, \omega)\, \phi_{j,T-t}(-\omega - i\kappa, h)\, \mathrm{d}\omega$ for $j = 1, 2$, specify the Fourier transforms of two distinct conditional probabilities. These conditional probabilities concern the $\mathcal{A}_T$ event under different (nearly-identical) probability measures that are equivalent to $\mathbb{P}$, i.e. $\mathbb{X}_1$ and $\mathbb{X}_2$ in Appendix B.1. It is clear that these conditional probabilities determine how much weight is attached to the risk-neutral value of $Y_T\Pi_T - (\mathcal{H}^{-1}(X_0) M_T\Pi_T)^{\frac{1}{p-1}} \Pi_T$, at time $t \in [0,T]$. That is, the better the state of the economy, the closer these probabilities will be to 1; for the converse case, these probabilities will attain values near 0. Hence, X_t^{opt} can be regarded as the risk-neutral value of $Y_T\Pi_T - \left(\mathcal{H}^{-1}(X_0) M_T^R\right)^{\frac{1}{p-1}} M_T$, weighted in accordance with the state of the economy.

In Corollary 3.3, we introduce X_t^{opt} for the case where Λ_1 is equal to 0.

Corollary 3.3. *Consider the optimal dynamic investment problem in* (2.17)*. The corresponding optimal wealth process, X_t^{opt}, is given in* (3.3) *of Proposition 3.2. Suppose that $\Lambda_1 = 0_{4\times 2}$, where $0_{4\times2} = [0_4, 0_4]$ and 0_4 is an $\mathbb{R}^4$-valued vector of zeros. Then,*

$$\begin{aligned} X_t^{\text{opt}} &= Y_t\Pi_t e^{\tilde{A}(t)+\tilde{B}(t)^\top Z_t}\Phi\left(d_{1,t,T}\right) \\ &\quad - \left(\mathcal{H}^{-1}(X_0) M_t\Pi_t\right)^{\frac{1}{p-1}} \Pi_t e^{\widehat{A}(t)+\widehat{B}(t)^\top Z_t}\Phi\left(d_{2,t,T}\right), \end{aligned} \tag{3.6}$$

characterises for all $t\in[0,T]$ the optimal wealth process. Here, $t\mapsto \tilde{A}(t)$ and $t\mapsto\tilde{B}(t)$ outline two deterministic functions that are given in (B.26)*. Likewise, $t\mapsto\widehat{A}(t)$ and $t\mapsto\widehat{B}(t)$ spell out two deterministic functions that are given in* (B.27)*. Additionally, $\Phi(\cdot)$ represents the cumulative distribution function (CDF) of a random variable that is standard normally distributed. The arguments inside this function, $d_{1,t,T}$ and $d_{2,t,T}$, are for $j=1,2$ and all $t\in[0,T]$ given by the following identity:*

$$d_{j,t,T} = \frac{-\log\left(\frac{\mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}}{M_t^{R^{-\frac{1}{p-1}}}Y_t}\right) + \mathbb{E}^{\mathbb{X}_j}\left[\log\frac{M_T^{R^{-\frac{1}{p-1}}}Y_T}{M_t^{R^{-\frac{1}{p-1}}}Y_t}\;\middle|\;\mathcal{F}_t\right]}{\sqrt{\mathbb{V}ar^{\mathbb{X}_j}\left[\log\frac{M_T^{R^{-\frac{1}{p-1}}}Y_T}{M_t^{R^{-\frac{1}{p-1}}}Y_t}\;\middle|\;\mathcal{F}_t\right]}}. \tag{3.7}$$

Here, $\mathbb{V}ar^{\mathbb{X}_j}\left[\log M_T^{R^{-\frac{1}{p-1}}}Y_T/M_t^{R^{-\frac{1}{p-1}}}Y_t \;\middle|\; \mathcal{F}_t\right]$ and $\mathbb{E}^{\mathbb{X}_j}\left[\log M_T^{R^{-\frac{1}{p-1}}}Y_T/M_t^{R^{-\frac{1}{p-1}}}Y_t \;\middle|\; \mathcal{F}_t\right]$ are given in (B.32) *and* (B.33)*, respectively, for $j=1,2$ and all $t\in[0,T]$.*

Proof. The proof is given in Appendix B.2. □

As addressed in the discussion preceding Proposition 3.2, an analytical evaluation of the conditional expectations in (3.1) is complicated by the distributional features of the relevant processes. In particular, for non-zero Λ_1, these distributional properties are unknown. To arrive at an analytical expression for X_t^{opt}, we therefore employed the Fourier transform, cf. Proposition 3.2. However, if Λ_1 is equal to zero, the distributional properties of both $Y_TM_T^R$ and $\left(\mathcal{H}^{-1}(X_0)^{\frac{1}{p}} M_T^R\right)^{\frac{p}{p-1}}$ are given. In precise terms, both of the latter processes are log-normally distributed. This result can be attributed to the fact that Λ_t and $\Lambda_t^R = \Lambda_t - \sigma_\Pi$ spell out non-affine constants, as a consequence of $\Lambda_1 = 0_{4\times2}$. On the grounds of this log-normality, the closed-form evaluation of the conditional expectations in (3.1) is significantly facilitated. Furthermore, the ensuing expression for X_t^{opt} is more tractable, as it neither depends on Fourier transforms nor on analytically unsolvable systems of ODE's. Note that the analysis regarding the weighted risk-neutral value of $Y_T \Pi_T - \left(\mathcal{H}^{-1}(X_0) M_T \Pi_T\right)^{\frac{1}{p-1}} \Pi_T$ from (3.3) also applies to (3.6). The sole difference consists in the known distributional features of the probability weights.

Remark 3.1. *In spite of its analytical transparency, the expression for optimal wealth (X_t^{opt}) in* (3.3) *can be computationally challenging. Namely, in addition to $P_2(t, Z_t)$'s dependence on an analytically unsolvable matrix Riccati differential equation for $t \mapsto \widehat{C}(t)$, both $\phi_{1,T-t}$ and $\phi_{2,T-t}$ depend on similar Riccati equations. As $\phi_{1,T-t}$ and $\phi_{2,T-t}$ are parts of the distinct integrals in* (3.3)*, their dependence on a.o. two separate Riccati equations may complicate computational analyses. Therefore, to facilitate numerical evaluations of the integral(s) characterising X_t^{opt}, we provide a computationally friendlier expression for X_t^{opt} (at the cost of analytical clarity). This expression is based on a direct application of the Fourier transform to X_t^{opt} in* (3.1)*. Define $\widehat{f}_\kappa^*(T, \omega) = \frac{e^{(i\omega-\kappa+1)\frac{1}{p-1}\log \mathcal{H}^{-1}(X_0)}}{(i\omega-\kappa+1)[i\omega-\kappa]}$, for some $\kappa > 1$, and $\widehat{\phi}_{T-t}(\omega, g, j) = \widehat{Q}(t, Z_t, \omega) M_t^{R^{\frac{p-i\omega}{p-1}}} Y_t^{i\omega}$, cf.* (B.38) *and* (B.39) *for the definition of $\widehat{Q}$. Then*[10]*, for all $t \in [0, T]$, X_t^{opt} is specified as follows:*

$$X_t^{\mathrm{opt}} = \frac{1}{2\pi} \frac{1}{M_t} \int_{-\infty}^{\infty} \widehat{f}_\kappa^*(T, \omega)\, \widehat{\phi}_{T-t}(-\omega - i\kappa, g, j)\, \mathrm{d}\omega. \tag{3.8}$$

3.3 Optimal Trading Strategy

In this section, we present the optimal trading strategy that solves the dynamic investment problem in (2.17): π_t^{opt} for all $t \in [0, T]$. According to the analysis in section 3.1, π_t^{opt} can be retrieved on the basis of the analytical specification for X_t^{opt}. To emphasise the link between π_t^{opt} and X_t^{opt}, let us turn to equation (3.2) for π_t^{opt}, and observe that we are already in possession of X_t^{opt}, cf. Proposition 3.2 and Corollary 3.3. As a result, the mere unknown in this equation is the process $\{\psi_t\}_{t\in[0,T]}$. Note here that $\{\psi_t\}_{t\in[0,T]}$ outlines the integrand in $X_t^{\mathrm{opt}} M_t$'s martingale representation and consequently depends on X_t^{opt}. Therefore, in order to identify π_t^{opt}, we utilise X_t^{opt} to derive the expression for this integrand by means of Itô's Lemma or Malliavin calculus. Details concerning the applications of the latter two concepts to X_t^{opt} in (3.3) and (3.6) are provided in Appendices C.1 and C.2, respectively. In Proposition 3.4, we present the ensuing final expression for the optimal dynamic trading policy, π_t^{opt}.

Proposition 3.4. *Consider the optimal dynamic investment problem in* (2.17)*. The optimal trading strategy, i.e. the solution of* (2.17)*, is given in* (3.2)*. In this identity,*

[10]For the full derivation of X_t^{opt} in (3.8), see Appendix B.3. We stress here that the expressions in (3.3) and (3.8) naturally result in the same optimal wealth processes (X_t^{opt}).

the expression for X_t^{opt} *can be found in* (3.3). *After determining* $\{\psi_t\}_{t\in[0,T]}$, π_t^{opt} *can be decomposed in the following way:* $\pi_t^{\text{opt}} = \pi_t^M + \pi_t^\Pi + \pi_t^Y + \pi_t^{FT} + \pi_t^R$, *for all* $t \in [0,T]$. *The first two weights in this portfolio decomposition,* π_t^M *and* π_t^R, *read as follows:*

$$\begin{aligned} \pi_t^M &= \frac{\sigma^{\top^{-1}}\Lambda_t}{p-1}\left(\mathcal{H}^{-1}\left(X_0\right)^{\frac{1}{p}} M_t^R\right)^{\frac{p}{p-1}} \frac{P_2\left(t,Z_t\right)}{M_t} R_{2,t}, \\ \pi_t^\Pi &= \sigma^{\top^{-1}}\sigma_\Pi X_t^{\text{opt}} - \frac{\sigma^{\top^{-1}}\sigma_\Pi}{p-1}\left(\mathcal{H}^{-1}\left(X_0\right)^{\frac{1}{p}} M_t^R\right)^{\frac{p}{p-1}} \frac{P_2\left(t,Z_t\right)}{M_t} R_{2,t}, \end{aligned} \tag{3.9}$$

for all $t \in [0,T]$. *Here,* $R_{j,t}$ *is defined as:* $R_{j,t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_\kappa^*\left(T,\omega\right)\phi_{j,T-t}\left(-\omega - i\kappa, h\right) \mathrm{d}\omega$, *for* $j = 1,2$, *and all* $t \in [0,T]$. *The deterministic function* $f_\kappa^*\left(T,\omega\right)$ *is spelled out in Proposition 3.2. Likewise, the characteristic functions,* $\phi_{j,T-t}\left(\omega,h\right)$ *for* $j = 1,2$, *are specified in* (3.5). *Now, let* $\widehat{R}_{j,t} = \frac{1}{2\pi}P_j\left(t,Z_t\right)\int_{-\infty}^{\infty}\left(f_\kappa^*\left(T,\omega\right)\phi_{1,T-t}\left(-\omega-i\kappa,h\right)\left[\Sigma_Z^\top \bar{B}_j\left(t,-\omega-i\kappa\right) + 2\Sigma_Z^\top \bar{C}_j\left(t,-\omega-i\kappa\right)Z_t + i\left(-\omega-i\kappa\right)\left(\beta_t + \frac{1}{p-1}\Lambda_t^R\right)\right]\right)\mathrm{d}\omega$, *for* $j = 1,2$, *and all* $t \in [0,T]$, *where* $(t,\omega) \mapsto \bar{B}_j\left(t,\omega\right)$ *and* $(t,\omega) \mapsto \bar{C}_j\left(t,\omega\right)$ *jointly solve the ODE's in* (B.23). *Then, the remaining weights,* π_t^Y, π_t^{FT} *and* π_t^R, *in* π_t^{opt}*'s decomposition read:*

$$\begin{aligned} \pi_t^Y &= \left(\sigma^{\top^{-1}}\beta_t\right) Y_t \Pi_t P_1\left(t,Z_t\right) R_{1,t}, \\ \pi_t^{FT} &= \sigma_t^{\top^{-1}}\left[Y_t\Pi_t\widehat{R}_{1,t} - \left(\mathcal{H}^{-1}\left(X_0\right)^{\frac{1}{p}} M_t^R\right)^{\frac{p}{p-1}} \frac{1}{M_t}\widehat{R}_{2,t}\right], \\ \pi_t^R &= \sigma^{\top^{-1}}\Sigma_Z^\top\left(\tilde{B}\left(t\right)X_t^{\text{opt}} - \tilde{D}\left(t\right)\left(\mathcal{H}^{-1}\left(X_0\right)^{\frac{1}{p}} M_t^R\right)^{\frac{p}{p-1}} P_2\left(t,Z_t\right)R_{2,t}\right), \end{aligned} \tag{3.10}$$

for all $t \in [0,T]$, *where the mapping* $t \mapsto \tilde{B}\left(t\right)$ *is given in* (B.9). *In addition to this, the mapping* $t \mapsto \tilde{D}\left(t\right)$ *is defined as follows:* $\tilde{D}\left(t\right) = \widehat{B}\left(t\right) + 2\widehat{C}\left(t\right)Z_t + \tilde{B}\left(t\right)$, *for all* $t \in [0,T]$, *in which* $t \mapsto \widehat{B}\left(t\right)$ *and* $t \mapsto \widehat{C}\left(t\right)$ *jointly solve the system of ODE's in* (B.10).

Proof. The proof is given in Appendix C.1. □

In Proposition 3.4, we decompose the optimal trading strategy (π_t^{opt}) into five distinct hedge demands. In disentangling these demands, we primarily adhere to the decomposition principles proposed by Detemple and Rindisbacher (2010) and Li et al. (2020). These papers concentrate on utility functions of the conventional Inada-family. Although the LPM function in the objective of (2.15) cannot be entirely subsumed under this family, the corresponding optimal portfolio weights live by a structure similar to theirs. We identify the demands in (3.9), π_t^M and π_t^Π, as a variant of the optimal mean-variance portfolio and a CPI hedge, respectively. As for π_t^M, it is in this regard noteworthy that $\frac{\sigma^{\top^{-1}}\Lambda_t}{\gamma}$ corresponds to the Merton portfolio for $\gamma = p-1 > 0$, cf. Merton (1969, 1971). In a similar sense, π_t^Π incorporates Merton-like weights that depend on σ_Π instead of Λ_t. The remaining demands in (3.10), π_t^Y, π_t^{FT} and π_t^R, can be identified as a benchmark hedge, a probability hedge, and a real interest rate hedge, respectively. Due to their dependence on β_t and the parameters in $R_t = r_t - \pi_t + \sigma_\Pi^\top\Lambda_t$, the identification of π_t^Y and π_t^R is straightforward. The

π_t^{FT} weight is referred to as a probability hedge, as its components follow from the Fourier transforms in (3.3) that characterise the relevant conditional probabilities.

In Corollary 3.5, we introduce π_t^{opt} given that $\Lambda_1 = 0_{4\times 2}$ holds.

Corollary 3.5. *Consider the optimal dynamic investment problem in* (2.17)*. The optimal trading strategy, i.e. the solution of* (2.17)*, is given in* (3.2)*. Suppose that* $\Lambda_1 = 0_{4\times 2}$*, Then,* $\pi_t^{\text{opt}} = \pi_t^M + \pi_t^\Pi + \pi_t^Y + \pi_t^R$ *holds, for all* $t \in [0, T]$*. Here,* π_t^M *and* π_t^Π *read:*

$$\begin{aligned} \pi_t^M &= \frac{\sigma^{\top^{-1}} \Lambda_0}{p-1} \left(\mathcal{H}^{-1} \left(X_0 \right)^{\frac{1}{p}} M_t^R \right)^{\frac{p}{p-1}} \frac{e^{\widehat{A}(t) + \widehat{B}(t)^\top Z_t}}{M_t} \Phi \left(d_{2,t,T} \right), \\ \pi_t^\Pi &= \sigma^{\top^{-1}} \sigma_\Pi X_t^{\text{opt}} - \frac{\sigma^{\top^{-1}} \sigma_\Pi}{p-1} \left(\mathcal{H}^{-1} \left(X_0 \right)^{\frac{1}{p}} M_t^R \right)^{\frac{p}{p-1}} \frac{e^{\widehat{A}(t) + \widehat{B}(t)^\top Z_t}}{M_t} \Phi \left(d_{2,t,T} \right), \end{aligned} \tag{3.11}$$

for all $t \in [0, T]$*. The mappings* $t \mapsto \tilde{A}(t)$ *and* $t \mapsto \tilde{B}(t)$ *correspond to the deterministic functions in Corollary 3.3 and are given in* (B.26)*. Similarly, the mappings* $t \mapsto \widehat{A}(t)$ *and* $t \mapsto \widehat{B}(t)$ *follow from Corollary 3.3 and are presented in* (B.27)*. Moreover, the function* $\Phi(\cdot)$ *denotes the CDF of a standard normally distributed random variable. The arguments of this function,* $d_{1,t,T}$ *and* $d_{2,t,T}$*, are defined in equation* (3.7)*. Suppose that* $\tilde{D}(t) = \widehat{B}(t) + \tilde{B}(t)$*, for all* $t \in [0, T]$*. Then, the remaining weights,* π_t^Y *and* π_t^R*, in the decomposition of* π_t^{opt} *are specified, for all* $t \in [0, T]$*, as follows:*

$$\begin{aligned} \pi_t^Y &= \left(\sigma^{\top^{-1}} \beta_t \right) Y_t \Pi_t e^{\tilde{A}(t) + \tilde{B}(t)^\top Z_t} \Phi \left(d_{1,t,T} \right), \\ \pi_t^R &= \sigma^{\top^{-1}} \Sigma_Z^\top \left(\tilde{B}(t) X_t^{\text{opt}} - \tilde{D}(t) \left(\mathcal{H}^{-1} \left(X_0 \right)^{\frac{1}{p}} M_t^R \right)^{\frac{p}{p-1}} e^{\widehat{A}(t) + \widehat{B}(t)^\top Z_t} \Phi \left(d_{2,t,T} \right) \right). \end{aligned} \tag{3.12}$$

Proof. The proof is given in Appendix C.2. □

Corollary 3.5 spells out the optimal trading strategy (π_t^{opt}) corresponding to the optimal wealth process (X_t^{opt}) in Corollary 3.5. That is, the optimal portfolio weights in (3.11) and (3.12) precisely replicate the optimal wealth process in (3.6). The analytical tractability of this wealth process correspondingly carries over to the optimal trading strategy. Concretely, unlike the portfolio in Proposition 3.4, the optimal investment rules for $\Lambda_1 = 0_{4\times 2}$ neither depend on Fourier transforms nor on analytically troublesome multi-dimensional systems of (non-)linear ODE's. Due to the interdependent links between X_t^{opt} and π_t^{opt}, it is self-explanatory that this tractability is attributable to the advantageous distributional features of X_T^{opt}. As for the economic interpretations of the separate demands, we underline that the weights in (3.11) and (3.12) represent the same hedges as those in Proposition 3.4. Note that the hedge demand for the probability weights, π_t^{FT} in (3.10), is equal to 0_4 for $\Lambda_1 = 0_{4\times 2}$. We are able to derive this result through an application of Malliavin calculus.[11] However, because of the analytical structure of X_T^{opt} in (A.3), a similar application for the

[11] This application strongly depends on the result reported in Lakner and Nygren (2006), concerning Mallivain-differentiability of piecewise continuously differentiable functions.

general $\Lambda_1 \in \mathbb{R}^{4\times 2}$ case does not infer that $\pi_t^{FT} = 0_4$ must hold.[12] Therefore, Proposition 3.4 includes this hedge demand as a possibly non-zero one.

Remark 3.2. *In the spirit of Remark 3.1, we observe that the expression for the optimal trading strategy (π_t^{opt}) in Proposition 3.4 may pose computational difficulties. Namely, the dependencies in X_t^{opt} on multiple matrix Riccati differential equations are also present in the latter specification of π_t^{opt}. Hence, consistent with* (3.8)*, in order to facilitate numerical evaluations of the integral(s) outlining π_t^{opt}, we provide a computationally friendlier expression for this process. We derive this expression on the grounds of the identity for X_t^{opt} in equation* (3.8)*. For this purpose, we employ the definitions of $\widehat{f}_\kappa^*(T,\omega)$ and $\widehat{\phi}_{T-t}(\omega, g, j)$ given in Remark 3.1. In addition to this, we fix $\widehat{D}_{\widehat{Q}}(t,\omega) = \widehat{B}_{\widehat{Q}}(t,\omega) + 2\widehat{C}_{\widehat{Q}}(t,\omega) Z_t$, for all $t \in [0,T]$ and $\omega \in \mathbb{R}$. Here, the definitions of $(t,\omega) \mapsto \widehat{B}_{\widehat{Q}}(t,\omega)$ and $(t,\omega) \mapsto \widehat{C}_{\widehat{Q}}(t,\omega)$ can be found in* (B.39)*. Moreover, we set $\widehat{\Lambda}_t^R(p,\omega) = i\omega\beta_t - \frac{p-i\omega}{p-1}\Lambda_t^R$. Then*[13]*, the optimal trading strategy, π_t^{opt}, is for all $t \in [0,T]$ given by:*

$$\begin{aligned}\pi_t^{\text{opt}} = \frac{1}{2\pi}\frac{1}{M_t}\sigma^{\top^{-1}} \int_{-\infty}^{\infty} \Big(\widehat{f}_\kappa^*(T,\omega)\,\widehat{\phi}_{T-t}(-\omega - i\kappa, g, j) \\ \times \Big[\Sigma_Z^\top \widehat{D}_{\widehat{Q}}(t, -\omega - i\kappa) + \widehat{\Lambda}_t^R(p, -\omega - i\kappa)\Big]\Big)\,\mathrm{d}\omega + \sigma^{\top^{-1}}\Lambda_t X_t^{\text{opt}}.\end{aligned} \tag{3.13}$$

4 Numerical Analysis

In this section, we provide a numerical analysis of the optimal solutions to the investment problem in (2.15). A numerical examination of the closed-form solutions (cf. section 3) can aid and deepen our understanding of a.o. their distributional properties and parameter sensitivity. Subsequently, we first elaborate on the specification of the benchmark process, $\{Y_t\}_{t\in[0,T]}$. Thereby, we aim to cast the matter into the confines of the accumulation phase of a defined contribution (DC) pension scheme. Second, we numerically investigate the probability distribution of the optimal terminal wealth process. In particular, we are interested in the likelihood of $\frac{X_T}{\Pi_T}$ attaining Y_T. Third and last, we study the behaviour of the optimal trading strategy with respect to changes in the model parameters.

4.1 Benchmark and Life Annuity

Henceforth, we assume that the investment problem in (2.17) corresponds to a participant in a DC scheme. Accordingly, we choose to model the benchmark, $\{Y_t\}_{t\in[0,T]}$, as a life

[12] Such an application ultimately requires one to evaluate a conditional expectation of the following form: $\mathbb{E}\big[N_T \mathbb{1}_{\{\mathcal{A}_T\}} \int_t^T \chi_s \mathrm{d}W_s \mid \mathcal{F}_t\big]$, where $N_T \in L_+^0(\Omega)$ and $\{\chi_t\}_{t\in[0,T]}$ represents a deterministic process satisfying $\chi_t \in L^2(0,T)$. Although this conditional expectation can be evaluated by employing the Fourier transform, it does not enable us to determine whether the ensuing expression contributes to the π_t^{FT} weight or not. Hence, we are unable to state that $\pi_t^{FT} = 0_4$ holds for general $\Lambda_1 \in \mathbb{R}^{4\times 2}$. Note that this condition, $\pi_t^{FT} = 0_4$, neither follows from the expression for π_t^{FT} in Proposition 3.4 itself.

[13] For the full derivation of π_t^{opt} in (3.13), see Appendix C.3. As in Remark 3.1, we stress here that the expressions in Proposition 3.4 and (3.13) naturally result in the same optimal portfolio (π_t^{opt}).

annuity. As a consequence, the agent or pension fund in (2.17) aims to invest in such a manner that real retirement wealth ($\frac{X_T}{\Pi_T}$) completely covers the life annuity (Y_T). Note here that the agent departs from an underfunding situation ($F_0 < 1$), similar to most pension funds. Now, we postulate that the participant lives τ_A fixed years after his/her predetermined retirement date, T. This concretely implies that the trading interval, $[0, T]$, coincides with the participant's accumulation phase. Correspondingly, the interval that aligns with the participant's decumulation phase is given by $(T, T + \tau_A]$. In the sequel, we suppose that a_T represents the *real* value of the life annuity at time $t = T$.

The formal equation for the value of this life annuity reads:

$$a_T = C \sum_{i=1}^{\tau_A} \frac{P^R_{T,T+i}}{\Pi_T} = C \sum_{i=1}^{\tau_A} \exp\left\{A^R(i) + B^R(i)^\top Z_T\right\}. \tag{4.1}$$

The specifications of $A^R(\cdot)$ and $B^R(\cdot)$ are provided in (2.13). Without loss of generality[14], we assume here that the life annuity pays $C \in \mathbb{R}_+$ monetary units per annum in real terms. We mainly employ C to adjust the coverage or funding ratio (F_0). In Donnelly et al. (2022), the authors make use of a similar definition for the value of a life annuity. Noting that Z_T outlines a normally distributed random variable, it is clear that the expression for a_T identifies a sum of τ_A log-normally distributed processes. Therefore, a_T in (4.1) is not log-normally distributed, and cannot be directly identified with Y_T in (2.16), cf. Dufresne (2008). Nevertheless, in order to make this identification possible, we assemble an approximation to a_T. For this purpose, we resort to a modified application of the Fenton-Wilkinson (FW) method developed by Fenton (1960). This method is predicated on the observation that sums of log-normally distributed random variables are *approximately* log-normal. Applied to a_T in (4.1), this method proceeds as follows.

Let $\widehat{a}_T$ denote the log-normal approximation to a_T. Then,

$$\widehat{a}_T = C \exp\left\{\bar{\alpha} + \bar{\beta}^\top Z_T\right\} = C \exp\left\{\bar{\alpha} + \bar{\beta}^\top \int_0^T e^{-K_Z(T-s)} \Sigma_Z \mathrm{d}W_s\right\}, \tag{4.2}$$

must hold according to the FW method. In this definition of $\widehat{a}_T$, the scalar $\bar{\alpha} \in \mathbb{R}$ and the vector $\bar{\beta} \in \mathbb{R}^2$ are to be determined. Ordinarily, the FW method characterises both $\bar{\alpha}$ and $\bar{\beta}$ by matching the first and second moments of $\widehat{a}_T$ and a_T. However, as $\bar{\beta}$ is two-dimensional, this procedure is not appropriate. Therefore, to be able to specify the preceding unknowns, we slightly modify the FW approach. In particular, we still match the first moments. Yet, instead of the second moments, we also match the following two expectations: $\mathbb{E}[a_T Z_T]$ and $\mathbb{E}[\widehat{a}_T Z_T]$. The idea underscoring the latter operation is that it

[14] Deterministic variable payments can be incorporated into a_T's definition by replacing $A^R(i)$ with $\widehat{A}^R(i) = A^R(i) + \log \frac{C_i}{C}$, given $C_i \in \mathbb{R}_+$, for all $i = 1, \ldots, \tau_A$. In a similar manner, stochastic payments can be included in a_T's specification, as long as these respect the log-normality of $\exp\left\{A^R(i) + B^R(i)^\top Z_T\right\}$. For instance, $C = \alpha S_T$, with $\alpha \in \mathbb{R}_+$, would be an appropriate candidate.

is identical to fixing: $\frac{\partial}{\partial \beta}\mathbb{E}\left[\widehat{a}_T\right] = \sum_{i=1}^{\tau_A} \frac{\partial}{\partial B^R(i)}\mathbb{E}\left[\exp\left\{A^R(i) + B^R(i)^\top Z_T\right\}\right]$. That is, we match in expectation the variations of a_T and $\widehat{a}_T$ with respect to the loadings of Z_T. Based on this mildly adjusted FW method, we are able to determine $\bar{\alpha}$ and $\bar{\beta}$ in closed-form.

In precise terms, we find that $\bar{\alpha}$ and $\bar{\beta}$ are characterised as follows:

$$
\begin{aligned}
\bar{\alpha} &= \log \frac{\mathbb{E}\left[a_T\right]}{C} - \frac{1}{2}\int_0^T \bar{\beta}^\top e^{-K_Z(T-s)}\left[e^{-K_Z(T-s)}\right]^\top \bar{\beta}\mathrm{d}s, \\
\bar{\beta} &= \left(\int_0^T e^{-K_Z(T-s)}\left[e^{K_Z(T-s)}\right]^\top \mathrm{d}s\right)^{-1} \frac{\mathbb{E}\left[a_T Z_T\right]}{\mathbb{E}\left[a_T\right]}.
\end{aligned}
\tag{4.3}
$$

Completely analytical expressions for $\mathbb{E}\left[a_T\right]$ and $\mathbb{E}\left[a_T Z_T\right]$ can be found in (D.3) and (D.5), respectively. For the full derivation of $\bar{\alpha}$ and $\bar{\beta}$ in (4.3), we refer the reader to Appendix D.1. To illustrate the accuracy of this approximation, we examine the magnitude of the $L^2(\Omega)$-distance between $\widehat{a}_T$ and a_T: $\mathbb{E}\left[\left(a_T - \widehat{a}_T\right)^2\right]^{\frac{1}{2}}$. Additionally, we assess the size and sign of the correlation between $\widehat{a}_T$ and a_T: $\mathbb{C}\mathrm{orr}\left(\widehat{a}_T, a_T\right)$. We compute these quantities by means of simulations, to avoid presenting additional complicated expressions. For this reason, we make use of the baseline parameter initialisation provided in Table 1. The subsequent numbers are based on 10,000 simulated paths and an Euler scheme with 10 yearly equidistant time-points. Regarding the $L^2(\Omega)$-distance, we find: $\|a_T - \widehat{a}_T\|_{L^2(\Omega)} = 0.0923$.[15] For the correlation, we have: $\mathbb{C}\mathrm{orr}\left(a_T, \widehat{a}_T\right) = 0.9994$. Considering that this coefficient is nearly equal to 1, and that the corresponding $L^2(\Omega)$-distance is negligibly small, we conclude that the approximation to a_T is rather accurate. To finalise this section, we identify $\widehat{a}_T$ with Y_T in (2.16). The following unique specifications ensure that $Y_T = \widehat{a}_T$ holds: $Y_0 = Ce^{\bar{\alpha}}$, $\alpha_t = \frac{1}{2}\beta_t^\top \beta_t$ and $\beta_t^\top = \bar{\beta}^\top e^{-K_Z(T-t)}\Sigma_Z$, for all $t \in [0, T]$.

4.2 Distribution of Retirement Wealth

In this section, we derive the distributional properties of optimal retirement wealth, X_T^{opt}. From section 2.4, we know that the agent in the LPM problem is strongly target-oriented. This focus on a person-specific target or benchmark can be summarised as follows.[16] As long as $\frac{X_T}{\Pi_T} < Y_T$ holds, the agent is able to draw additional non-negative utility by increasing the magnitude of $\frac{X_T}{\Pi_T}$. However, if $\frac{X_T}{\Pi_T} = Y_T$ is true, no additional utility can be derived by enlarging $\frac{X_T}{\Pi_T}$. That is, the agent has reached a maximal level of utility once the desired target has been obtained. These preference-related features translate into the following behaviour. Provided that wealth drifts away from the benchmark, the agent is willing to engage in riskier trades so as to increase the odds of ultimately securing the target. On the contrary, if wealth covers the benchmark, the agent becomes more prudent

[15] We observe that $\inf_{\bar{\alpha},\bar{\beta}} \|a_T - \widehat{a}_T\|_{L^2(\Omega)} = 0.0897$ holds. As this value is very close to the one in the main text (0.0923), we can conclude that our approximation behaves similar to the $L^2(\Omega)$-optimal variant.

[16] The subsequent analysis is based on U's coefficient of absolute risk aversion (ARA). Recall that $U(x, y) = -\frac{1}{p}\left[(y - x)^+\right]^p$. ARA for U is equal to: $-\frac{U_x''(x,y)}{U_x'(x,y)} = \frac{p-1}{(y-x)^*}$ for all $x, y \in \mathbb{R}_+$.

with respect to risky investments in an attempt to "lock in" his/her wealth at the current target level. Since the agent's preferences are explicitly modelled around this target, we confine ourselves to an analysis of $\frac{X_T}{\Pi_T}$'s distribution relative to the one of Y_T.

With this end in view, we introduce Proposition 4.1

Proposition 4.1. *Consider optimal terminal wealth, X_T^{opt}, provided in* (A.3). *The corresponding benchmark, Y_T, is presented in* (2.16). *Suppose that $x \mapsto F_{X/Y}(x)$ represents the CDF of $\frac{X_T}{\Pi_T}\frac{1}{Y_T}$, i.e. $F_{X/Y}(x) = \mathbb{P}\left(\frac{X_T}{\Pi_T}\frac{1}{Y_T} \leq x\right)$ for all $x \in \mathbb{R}$. Then,*

$$F_{X/Y}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}_\kappa^*(T, x, \omega)\,\phi_T(-\omega - i\kappa, h)\,\mathrm{d}\omega, \tag{4.4}$$

holds for all $x \in [0,1)$.[17] *Moreover, $F_{X/Y}(x) = 0$ for all $x \in (-\infty, 0)$, and $F_{X/Y}(x) = 1$ for all $x \in [1, \infty)$. The deterministic function $\tilde{f}_\kappa^*(T, x, \omega)$ is for all $\omega \in \mathbb{R}$, all $x \in [0,1)$ and some $\kappa \in \mathbb{R}_-$ given by: $\tilde{f}_\kappa^*(T, x, \omega) = \frac{1}{i\omega - \kappa}e^{(i\omega-\kappa)\left[\frac{1}{p-1}\log \mathcal{H}^{-1}(X_0) - \log(1-x)\right]}$. Additionally, the characteristic function, $\phi_T(\omega, h)$, is specified as follows: $\phi_T(\omega, h) = e^{\tilde{A}(0,\omega)}Y_0^{i\omega}$ for all $\omega \in \mathbb{R}$. Here, $(t, \omega) \mapsto \tilde{A}(t, \omega)$ represents the deterministic function provided in* (D.13), *through the system of ODE's in* (D.10). *Now, let $x \mapsto f_{X/Y}(x)$ denote the density of $\frac{X_T}{\Pi_T}\frac{1}{Y_T}$ on the domain $(0,1)$, i.e. $f_{X/Y}(x) = \frac{\partial}{\partial x}F_{X/Y}(x)$ for all $x \in (0,1)$. Then, for all $x \in (0,1)$:*

$$f_{X/Y}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i\omega - \kappa}{1-x}\tilde{f}_\kappa^*(T, x, \omega)\,\phi_T(-\omega - i\kappa, h)\,\mathrm{d}\omega. \tag{4.5}$$

Proof. The proof is given in Appendix D.2. □

Proposition 4.1 analytically characterises the distribution of $X_T^{Y,\Pi} = \frac{X_T}{\Pi_T}\frac{1}{Y_T}$ using the following two functions: (i) the CDF of $X_T^{Y,\Pi}$ in (4.4), and (ii) the density of $X_T^{Y,\Pi}$ on the domain $(0,1)$ in (4.4). Note that the CDF, $x \mapsto F_{X/Y}(x)$, delivers the likelihood that $X_T^{Y,\Pi}$ attains values in an interval $(-\infty, x]$. The density, $x \mapsto f_{X/Y}(x)$, can be interpreted as a function that constitutes the continuous analogue of a histogram for $X_T^{Y,\Pi}$. We choose to analyse the distributional features of $X_T^{Y,\Pi}$, as this random variable immediately infers how well $\frac{X_T}{\Pi_T}$ performs relative to Y_T. In fact, $X_T^{Y,\Pi}$ can be identified with the so-called replacement ratio, see e.g. Balter et al. (2020). Concretely, $X_T^{Y,\Pi}$ takes on values in the half-open unit interval, $[0,1)$, and measures the degree up to which real retirement wealth is able to cover or replicate the benchmark. For example, if $X_T^{Y,\Pi}$ achieves a value of 0.75, real retirement wealth is able to cover 75% of the benchmark's value. Great performance

[17] Note here that $F_{X/Y}(x) = 1 - \mathbb{P}\big(M_T^{R^{\frac{1}{p-1}}}{Y_T}^{-1} \leq (1-x)\,{\eta^{\mathrm{opt}}}^{-\frac{1}{p-1}}\big)$ holds for all $x \in \mathbb{R}$, cf. (D.16). Whereas we were able to disentangle more explicit terms for the optimal control processes in Propositions 3.2 and 3.4, this is not possible for the case at hand. This is entirely attributable to the analytical structure of the indicator function, $x \mapsto \mathbb{1}_{\{x \in \mathcal{A}\}}$, which is not multiplicative in its sole argument. Alternative, indirect applications of the Fourier transform (used for Propositions 3.2 and 3.4) would consequently not contribute to the analytical transparency of the ensuing identities. Therefore, $x \mapsto F_{X/Y}(x)$ in (4.4) is the most analytical expression that we can engender for the CDF. On account of the immediate derivation of the density from the CDF, it is self-explanatory that the same applies to $x \mapsto f_{X/Y}(x)$.

is accordingly associated with $X_T^{Y,\Pi} \approx 1$; poor performance with $X_T^{Y,\Pi} \approx 0$. Therefore, the shape of $X_T^{Y,\Pi}$'s distribution explicitly indicates whether and up to what extent the pension fund is able to meet the financial expectations of the participant. Evidently, in particular from the participant's point of view, a highly left-skewed distribution is preferred.

In Corollary 4.2, we provide $x \mapsto F_{X/Y}(x)$ and $x \mapsto f_{X/Y}(x)$ for $\Lambda_1 = 0_{4\times 2}$.

Corollary 4.2. *Consider optimal terminal wealth, X_T^{opt}, provided in* (A.3). *The corresponding benchmark, Y_T, is presented in* (2.16). *Suppose that $\Lambda_1 = 0_{4\times 2}$. Then, the CDF of $\frac{X_T}{\Pi_T}\frac{1}{Y_T}$, i.e. $F_{X/Y}(x) = \mathbb{P}\left(\frac{X_T}{\Pi_T}\frac{1}{Y_T} \leq x\right)$ for all $x \in \mathbb{R}$, is for all $x \in [0,1)$ given by:*

$$F_{X/Y}(x) = \Phi\left(\frac{\log \frac{\mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}}{1-x} - \mathbb{E}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right]}{\sqrt{\mathbb{V}ar\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right]}}\right). \tag{4.6}$$

Furthermore, $F_{X/Y}(x) = 0$ for all $x \in (-\infty, 0)$, and $F_{X/Y}(x) = 1$ for all $x \in [1, \infty)$. Here, $\mathbb{E}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right]$ and $\mathbb{V}ar\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right]$ are given in (D.18). *In addition to this, $\Phi(\cdot)$ denotes the CDF of a standard normally distributed random variable. Now, define $x \mapsto d_{0,T}(x)$ as the argument of the CDF in* (4.6). *Let $x \mapsto f_{X/Y}(x)$ denote the density of $\frac{X_T}{\Pi_T}\frac{1}{Y_T}$ on the domain $(0,1)$, i.e. $f_{X/Y}(x) = \frac{\partial}{\partial x}F_{X/Y}(x)$ for all $x \in (0,1)$. Moreover, let $\phi(\cdot)$ denote the PDF of a random variable that is standard normally distributed: $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ for all $x \in \mathbb{R}$. Then, for all $x \in (0,1)$:*

$$f_{X/Y}(x) = \frac{\phi\left(d_{0,T}(x)\right)}{(1-x)\sqrt{\mathbb{V}ar\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right]}}. \tag{4.7}$$

Proof. The proof is given in Appendix D.3.[18] □

Following sections 3.2 and 3.3, we distinguish the $\Lambda_1 = 0_{4\times 2}$ case from the general $\Lambda_1 \in \mathbb{R}^{4\times 2}$ case. As $X_T^{Y,\Pi}$'s distributional features are unknown for $\Lambda_1 \neq 0_{4\times 2}$, we relied on Fourier transforms to derive the results in Proposition 4.1. However, provided that $\Lambda_1 = 0_{4\times 2}$ holds, $X_T^{Y,\Pi}$ solely depends on log-normal random variables. As a consequence, the expression for $X_T^{Y,\Pi}$ allows for more explicit specifications of both $x \mapsto F_{X/Y}(x)$ and $x \mapsto f_{X/Y}(x)$. These specifications are presented in (4.6) and (4.7), respectively. Indeed, neither of the two aforementioned identities depends on Fourier transforms or analytically burdensome systems of ODE's. Compared to the definition of $F_{X/Y}$, the characterisation of $f_{X/Y}$ is specifically tractable, due to the involvement of $x \mapsto \phi(x)$ rather than $x \mapsto \Phi(x)$.

[18]To avoid confusion, we emphasise that both Proposition 4.1 and Corollary 4.2 are valid for the general specification of the benchmark process, Y_T, provided in (2.16). As a consequence, the special case implied by $Y_T = \widehat{a}_T$ in section 4.1 is covered by the presented results.

Namely, whereas the latter function concerns an integral expression[19], the former is completely available in closed-form. Considering the fact that $f_{X/Y}$ can be regarded as the continuous variant of a histogram, it directly tells something about the frequency with which X_T/Π_T comes near Y_T. In the context of our DC scheme, the analytical nature of $f_{X/Y}$ is therefore highly advantageous. Notwithstanding, we must emphasise that $f_{X/Y}$ does not coincide with the formal PDF of $X_T^{Y,\Pi}$, cf. Remark 4.1. In spite of technicality, the relation between $X_T^{Y,\Pi}$'s distribution and $f_{X/Y}$ stands.

Remark 4.1. *The Borel probability measure corresponding to the CDF of $X_T^{Y,\Pi}$ admits atoms. That is, positive mass is assigned to the singleton $\{X_T^{Y,\Pi} = 0\}$. In particular, we have: $F_{X/Y}(0) - F_{X/Y}(0^-) = F_{X/Y}(0)$, which is strictly positive. As a consequence, $F_{X/Y}$ is not absolutely continuous and there does not exist a corresponding probability density function (PDF).[20] Nevertheless, although $x \mapsto f_{X/Y}(x)$ in both (4.5) and (4.7) is no formal PDF, as part of $X_T^{Y,\Pi}$'s mixed density function, it does enable us to (numerically) examine the shape of $X_T^{Y,\Pi}$'s distribution on the practically relevant domain, $(0,1)$. To get an idea of what happens at the extreme value, $x = 0$, we can in turn apply numerical machinery to approximate $\lim_{x\downarrow 0} f_{X/Y}(x)$. Furthermore, the exact probability for this extreme event, $\{X^{Y,\Pi} = 0\}$, can be directly obtained from (4.4) and (4.6) as follows: $\mathbb{P}(X^{Y,\Pi} = 0) = F_{X/Y}(0)$. By virtue of these reasons, we provide the analytical expressions in (4.5) and (4.7) for the densities of $X_T^{Y,\Pi}$ on the subdomain $(0,1)$.*

4.3 Analysis of Retirement Wealth

We continue with a numerical evaluation of $X_T^{Y,\Pi}$'s distributional features presented in Proposition 4.1 and Corollary 4.2. For this reason, we employ the parameter estimates reported in Table 1. The values for these estimates are based on calibrations to recent data. For details on this calibration procedure, we refer to Pelsser (2019). As we wish to emphasise the salient dependence of the optimal solutions on Λ_t, we vary the parameter estimates over Λ_0 and Λ_1. To this end, we introduce two additional sets of parameter estimates defined by the following values for Λ_0 and Λ_1: (i) $\Lambda_{0(i)} = 0.1$ and $\Lambda_{1(i,j)} = 0.075$, and (ii) $\Lambda_{0(i)} = 0.1$ and $\Lambda_1 = 0_{4\times 2}$, for all $i = 1, \ldots, 4$ and $j = 1, 2$. These values are comparable to the estimates for the market prices of risk provided in table 1 of Brennan and Xia (2002). Note that we modify the value for η_S in conformity with the adjustments

[19] In exact terms, $\Phi(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{1}{\sqrt{2}}x\right)\right]$, with $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t$, for all $x \in \mathbb{R}$.

[20] Suppose that $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}_+$ represents a Borel measure, such that $F_{X/Y}(x) = \mu((-\infty, x])$ holds for all $x \in \mathbb{R}$. Clearly, μ identifies the probability measure associated with $F_{X/Y}$. Then, following the Lebesgue Decomposition Theorem: $\mu(\mathcal{A}) = \int_{\mathcal{A}} f_{X/Y}(t)\,\mathrm{d}t + F_{X/Y}(0)\,\mathbb{1}_{\{0\in\mathcal{A}\}}$, where we extend $f_{X/Y}$'s definition to ensure that $f_{X/Y}(x) = 0$ holds for all $x \in (-\infty, 0] \cup [1, \infty)$. Using this expression, it is tempting to argue that the PDF reads: $f_{X/Y}(x) + F_{X/Y}(0)\,\delta(x)$, in which $x \mapsto \delta(x)$ outlines the Dirac delta function. Note that we discard the absolute non-continuity of $x \mapsto \delta(x)$ here. However, $F_{X/Y}(0) - F_{X/Y}(0^-) = F_{X/Y}(0) > 0$, which contradicts the fact that the PDF should coincide with the derivative of $F_{X/Y}$. Hence, the former expression for the PDF is invalid. Aside from this technical invalidity, at $x = 0$ it results in ∞, rendering the expression meaningless.

Parameter	Value	Parameter	Value	Parameter	Value	Parameter	Value
$S_{1,t}$		Λ_t		Π_t, Z_t		r_t, π_t	
η_S	0.0451	$\Lambda_{0(1)}$	0.6420	$\sigma_{\Pi(1)}$	−0.0010	$\delta_{0,r}$	0.0097
$\sigma_{S(1)}$	−0.0483	$\Lambda_{0(2)}$	−0.0240	$\sigma_{\Pi(2)}$	0.0013	$\delta_{1,r(1)}$	−0.0094
$\sigma_{S(2)}$	0.0078	$\Lambda_{1(1,1)}$	0.1710	$\sigma_{\Pi(3)}$	0.0055	$\delta_{1,r(2)}$	−0.0024
$\sigma_{S(3)}$	0.0010	$\Lambda_{1(1,2)}$	0.3980	$K_{Z(1,1)}$	0.0479	$\delta_{0,\pi}$	0.0158
$\sigma_{S(4)}$	0.1335	$\Lambda_{1(2,1)}$	−0.5140	$K_{Z(2,1)}$	1.2085	$\delta_{1,\pi(1)}$	−0.0028
		$\Lambda_{1(2,2)}$	−1.1470	$K_{Z(2,2)}$	0.5440	$\delta_{1,\pi(2)}$	−0.0014

Table 1. Baseline parameter input. This table contains the baseline parameter input on which we rely to compute the numerical results in section 4. The displayed values are derived from the second column of Table 1 in Pelsser (2019). Note that the estimates documented in the latter paper are based on calibrations to recent data. In fact, these estimates are employed by the Dutch Central Bank (DNB). As in both Koijen et al. (2009) and Pelsser (2019), we define $\Lambda_{0(3)} = \Lambda_{1(3,1)} = \Lambda_{1(3,2)} = \sigma_{\Pi(4)} = 0$. In addition to this, we set the trading horizon (retirement date) equal to 40, i.e. $T = 40$. Moreover, we assume that the agent lives 20 fixed years after his/her retirement, i.e. $\tau_A = 20$. The funding or coverage ratio is held fixed at 100%, i.e. $F_0 = 1$, unless stated differently. The times to maturity of the three bonds are: $\tau_1 = 5$, $\tau_2 = 20$ and $\tau = 20$. Last, we set the agent's initial endowment equal to: $X_0 = 10$. Observe that C in Y_T's definition, cf. section 4.1, follows from the parameter input. That is, $C = \frac{X_0}{F_0} e^{-\bar{\alpha}-\tilde{A}(0)}$, where $\bar{\alpha}$ is shown in (4.3), and $t \mapsto \tilde{A}(t)$ is given in (B.9).

in Λ_0 and Λ_1. We label the baseline input in Table 1 as "P^0"; the aforementioned two inputs as "P^1" and "P^2", respectively. Although all relevant details are given in the table, we note that the subsequent results correspond to an accumulation phase of $T = 40$ years. We assume that the related decumulation phase lasts for $\tau_A = 20$ years. The agent endows the pension fund at $t = 0$ with $X_0 = 10$ monetary units, faces a funding ratio of $F_0 = 80\%$, and has a risk profile characterised by $p = 2$. We solely consider the $p = 2$ case. Our primary findings on the subject of robustness do not change for higher values of p. Observe that $p = 2$ comes close to the situation for the celebrated expected shortfall criterion ($p = 1$). In Figures 1 and 2, we present the CDF of $X_T^{Y,\Pi}$ and the density function of $X_T^{Y,\Pi}$, respectively. In Figure 3, we display the success probability of $X_T^{Y,\Pi}$ for different values of Λ_0 and Λ_1. The success probability is defined as the likelihood that $X_T^{Y,\Pi}$ exceeds 95%. For a funding ratio of 80%, this definition of "success" seems acceptable.

4.3.1 Technical Discussion: Figures 1 and 2

We proceed with a technical discussion of Figures 1 and 2. From both figures, one can infer that $X_T^{Y,\Pi}$ is equal to approximately 100% with a probability that approaches 1. This result implies that retirement wealth is in almost all states of the world able to *completely* cover the agent's life annuity. Considering that $F_0 = 80\%$, this outcome seems highly unrealistic. Nevertheless, we should take into account that: (i) the market prices of risk (Λ_t) are fairly large, and (ii) the diffusion coefficient of Y_T (β_t) is small compared to Λ_t. The market prices of risk, Λ_t, play a dominant role in the specifications of both $x \mapsto F_{X/Y}(x)$ and $x \mapsto f_{X/Y}(x)$. This is particularly visible in the definition of $\mathcal{H}^{-1}(X_0)$.

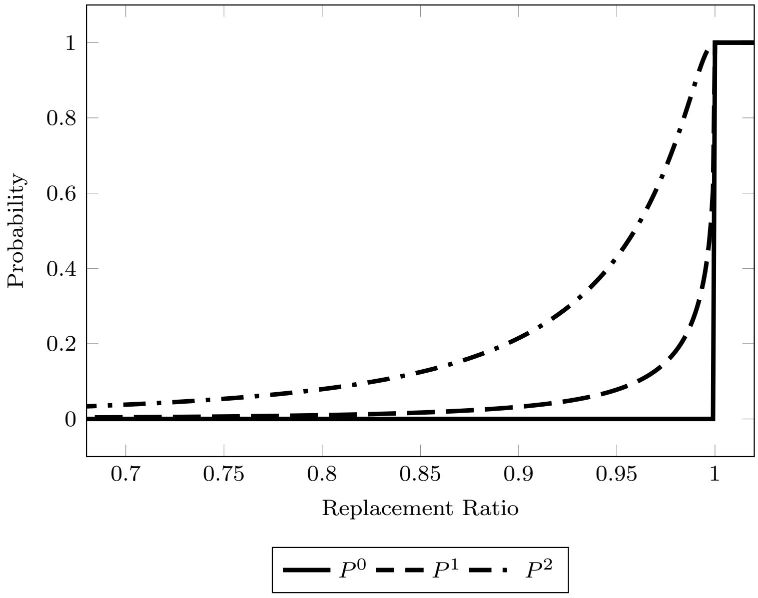

Figure 1. CDF of replacement ratio. This figure depicts the CDF of the replacement ratio ($X_T^{Y,\Pi}$) for three different inputs of parameter estimates. The black line corresponds to the P^0 input; the dashed line to the P^1 input; and the dash-dotted line to the P^2 input. For the baseline input of estimates (P^0), we refer to Table 1. The remaining inputs are equal to P^0's, where (i) $\Lambda_{0(i)} = 0.1$ and $\Lambda_{1(i,j)} = 0.075$ (P^1), and (ii) $\Lambda_{0(i)} = 0.1$ and $\Lambda_1 = 0_{4\times 2}$ (P^2), for all $i = 1, \ldots, 4$ and $j = 1, 2$. The horizontal axis represents the value of the replacement ratio. The vertical axis represents the value of the probability rendered by the relevant CDF. For this graph, we relied on $X_T^{Y,\Pi}$'s CDF provided in Proposition 4.1. Moreover, we fixed $p = 2$ and $F_0 = 80\%$. Although we solely plot the three trajectories on the practically relevant subdomain $[0.7, 1)$, we note that the behaviour of the CDFs on the remaining domains follows from what is presented here. In fact, for $X_T^{Y,\Pi} \geq 1$, the value of the CDF is equal to 1; for $X_T^{Y,\Pi} < 0.7$, all three trajectories tend towards 0 and stay there once $X_T^{Y,\Pi}$ has passed 0.

The identity for X_t^{opt} in (3.3) depends on exponentially compounded quadratic versions of Λ_0 and Λ_1. If $|\Lambda_0|$, $|\Lambda_1|$ and/or T grow, this will (exponentially) increase the value of the second term in (3.3). Note that $\mathcal{H}^{-1}(X_0)$ solves $X_0^{\text{opt}} = X_0$. Consequently, large values for $|\Lambda_0|$, $|\Lambda_1|$ and/or T will considerably drive down the magnitude of $\mathcal{H}^{-1}(X_0)$. From (4.4) and (4.6), we know that declines in $\mathcal{H}^{-1}(X_0)$ result in smaller values for the CDF. Hence, increases in $|\Lambda_0|$ and/or $|\Lambda_1|$ negatively influence the CDF and, by extension, the density function. Note that the opposite holds for increases in β_t. As the substantial impact of Λ_t on both $F_{X/Y}$ and $f_{X/Y}$ is not offset by an equal impact of β_t, we find the shapes in Figures 1 and 2.[21] In section 4.3.3, we comment on the implications of this finding.

These claims are verified by the trajectories for the P^1 and P^2 cases in Figures 1 and 2. Due to the smaller values for $|\Lambda_0|$ and $|\Lambda_1|$ in P^1, the CDF of $X_T^{Y,\Pi}$ is indeed driven away from zero. The density function correspondingly lives by a more natural shape. The latter still displays that $X_T^{Y,\Pi}$ is likely to achieve values in a region near 95%. This

[21]To get an idea of the magnitudes, consider (4.6) and let $\Lambda_1 = 0_{4\times 2}$ in P^0. Then, $\mathcal{H}^{-1}(X_0) = 2.8668 \times 10^{-06}$. Moreover, $\mathbb{E}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right] = 18.9333$ and $\mathbb{V}\mathrm{ar}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right] = 27.1862$. For all $x \in [0, 1)$, this will clearly steer $F_{X/Y}(x)$ towards values near 0.

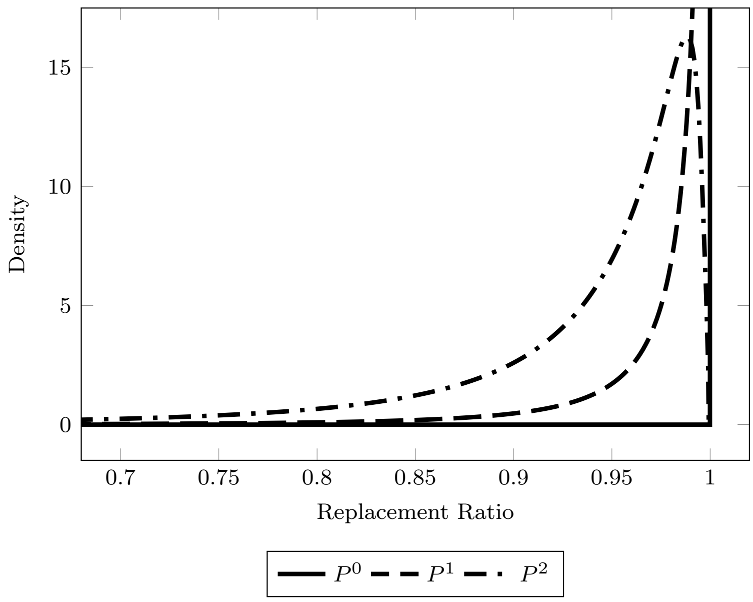

Figure 2. Density function of replacement ratio. This figure depicts the density function of the replacement ratio ($X_T^{Y,\Pi}$) for three different inputs of parameter estimates. Similar to Figure 1, the black line corresponds to the P^0 input; the dashed line to the P^1 input; and the dash-dotted line to the P^2 input. For the definitions of P^0, P^1 and P^2, we refer the reader to the main text of section 4.3 and the description of Figure 1. The horizontal axis represents the value of the replacement ratio. The vertical axis represents the value of the density function. For this graph, we relied on $X_T^{Y,\Pi}$'s density function provided in Proposition 4.1. Moreover, we fixed $p = 2$ and $F_0 = 80\%$. Although we solely plot the three trajectories on the practically relevant subdomain $[0.7, 1)$, we note that the corresponding behaviour on $(0, 0.7)$ follows from what is presented here. In fact, for $X_T^{Y,\Pi} < 0.7$, all three trajectories tend towards 0. Observe that the density function is only defined on the open unit interval $(0, 1)$.

is partially attributable to the comparatively small values for β_t. However, the agent now also faces notable positive odds of obtaining $X_T^{Y,\Pi}$ below the level of his/her starting position ($F_0 = 80\%$). For the P^2 case, the graphical results appear even more plausible. The density function indicates that $X_T^{Y,\Pi}$ is likely to achieve values in a much wider region around 95%. Additionally, the agent faces significantly higher odds of obtaining $X_T^{Y,\Pi}$ below 80%. For a funding ratio of 80%, such outcomes seem reasonable. In our discussion preceding this analysis of P^1 and P^2, we did not distinguish between the impact of Λ_0 and Λ_1 on $F_{X/Y}$ and $f_{X/Y}$. The technical reason for this indifference is that $|\Lambda_0|$ and $|\Lambda_1|$ play similar roles in the characterisation of $\mathcal{H}^{-1}(X_0)$, cf. Proposition 3.2. As shown in the two plots for P^1 and P^2, the impact is indeed comparable in terms of its sign.

4.3.2 Technical Discussion: Figure 3

Figure 3 confirms the previous claim. For a fixed value of Λ_0, the success probability positively depends on Λ_1. The same holds with respect to Λ_0, for a fixed value of Λ_1. Note here that the elements in Λ_0 and Λ_1 are assumed to be identical: $\Lambda_0 = \Lambda_{0(i)}$ and $\Lambda_1 = \Lambda_{0(i,j)}$, for all $i = 1, \dots 4$ and $j = 1, 2$ (with a slight abuse of notation). Despite the former similarities, the impact of Λ_1 on the success probability is greater than the impact

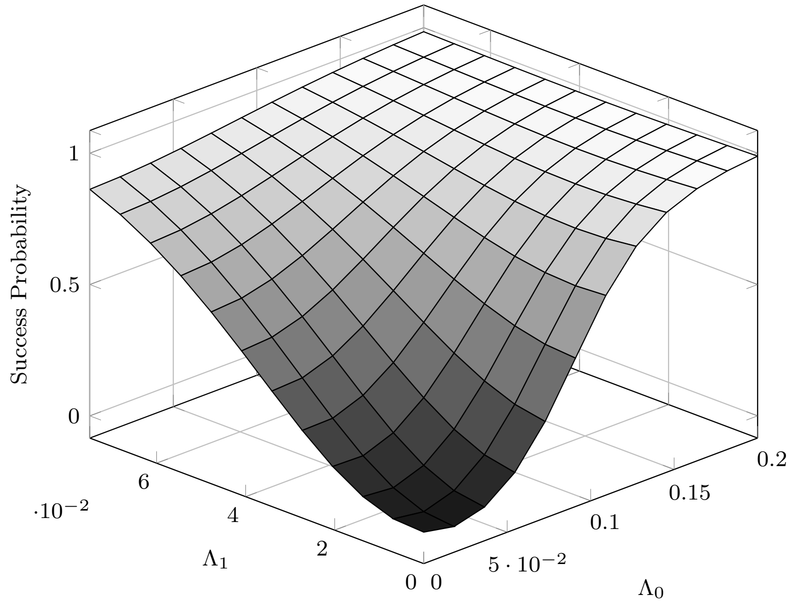

Figure 3. Success probability for replacement ratio. This figure depicts the success probability for the replacement ratio ($X_T^{Y,\Pi}$) varied with respect to Λ_0 and Λ_1. The success probability is defined as the likelihood that $X_T^{Y,\Pi}$ exceeds 95%: $\mathbb{P}\big(X_T^{Y,\Pi} \geq 0.95\big)$. Moreover, the elements in Λ_0 and Λ_1 are assumed to be identical: $\Lambda_0 = \Lambda_{0(i)}$ and $\Lambda_1 = \Lambda_{0(i,j)}$, for all $i = 1, \dots 4$ and $j = 1, 2$. The output corresponds to the baseline collection of parameter estimates (P^0) reported in Table 1. The vertical axis represents the success probability. The axes labelled Λ_0 and Λ_1 represent the values for Λ_0 and Λ_1, respectively. For this graph, we relied on $X_T^{Y,\Pi}$'s CDF provided in Proposition 4.1. In addition to this, we fixed $p = 2$ and $F_0 = 80\%$. Similar to Figures 1 and 2, we solely plot the three-dimensional trajectories on two distinct subdomains for Λ_0 and Λ_1: $[0, 0.2]$ and $[0, 0.075]$, respectively. The graph's dynamics on the remainder of $\mathbb{R}^2$ namely follow from what is presented here. In fact, for larger values of Λ_0 and Λ_1, the success probability converges to 1; for negative values, the graph behaves as displayed.

of Λ_0. This difference can be explained by the fact that Λ_t depends on Λ_1 through Z_t. The variance of Z_t grows in time. Therefore, upon retirement, small non-zero values for Λ_1 are capable of inflating the market prices of risk. We stress that $\Lambda_1 Z_t$ can be regarded as a time-dependent analogue of Λ_0. As a result, small values for Λ_1 have an impact on the success probability that resembles the impact of comparatively large values for Λ_0. We underline that Figure 3 substantiates our claims regarding the outcomes' sensitivity to Λ_0 and Λ_1. In fact, for values of these parameters in rather small subdomains, the success probability varies between 0 and 1. In spite of this sensitivity, we observe that the replacement ratio is potentially very likely to exceed a level of 95%. Given that $\Lambda_0 > 0.125$ and/or $\Lambda_1 > 0.050$ hold true, the success probability tends towards 100%.

4.3.3 Economic Takeaways

The set of economic takeaways corresponding to the output in Figures 1, 2 and 3 is twofold. First, we find that LPM-based investment strategies can increase the likelihood of achieving one's pension goals. This finding is robust to *some* uncertainty around Λ_0 and Λ_1. Second, the displayed outcomes are highly dependent on the estimates for Λ_0

and Λ_1. This result supports the use of approaches that account for model/parameter uncertainty. We elaborate on these takeaways in the following two summaries:

Takeaway 1. The first takeaway is a straightforward consequence of a.o. the displayed trajectories for $F_{X/Y}$ and $f_{X/Y}$. Based on the plots for $f_{X/Y}$, we can conclude that real retirement wealth is very likely to achieve values in the neighborhood of the life annuity. These high odds are coupled to fairly low odds of the replacement ratio falling below the agent's funding ratio. Note that the pension fund departs from a funding ratio of 80%. In terms of hedging, this means that the fund is able to arrive at a replacement ratio of likewise 80% with a probability of 1. Without sacrificing too much of this certainty, the LPM-framework is able to significantly improve this recovery potential of 80%. Note that these qualitative results hold true for different values of Λ_0 and Λ_1, despite their relatively large impact on the shapes of $F_{X/Y}$ and $f_{X/Y}$. Figure 3 indeed demonstrates that the replacement ratio exceeds a level of 95% with a probability near 1, provided that $\Lambda_0 > 0.125$ and/or $\Lambda_1 > 0.050$ hold true The conclusions that we draw here are consequently quite robust to *some* uncertainty around Λ_0 and Λ_1.

Takeaway 2. The second takeaway also follows from the graphs for $F_{X/Y}$ and $f_{X/Y}$. Nevertheless, the sensitivity of the outcomes to the estimates for Λ_0 and Λ_1 is most distinct in Figure 3. The pronounced impact of Λ_t on these graphs highlights the strong dependence of the optimality conditions on the market prices of risk. As pointed out in section 4.3.1, particular values for Λ_0 and Λ_1 may even lead to nonsensical outcomes. This dependence has an important implication for the way in which one ordinarily treats parameter estimates. Small estimation errors can namely generate meaningless outcomes and/or imply enormous policy changes.[22] In order to avoid such adverse events, it is wise to be careful concerning the parameter estimates and account for parameter/model uncertainty. The latter refers to doubts that one may have about the veracity of particular parameter estimates or model specifications. An agent can account for this by cautiously preparing him- or herself for a worst-case scenario. Accordingly, one becomes less sensitive to estimation-related errors, i.e. robust. This robustness naturally returns in the optima and the corresponding policy rules. For applications of parameter/model uncertainty in similar frameworks, we refer to Balter (2016) and references therein.

4.4 Analysis of Portfolio Rules

In this section, we present a numerical analysis of the optimal portfolio rules provided in Proposition 3.4 and Corollary 3.5. We mainly aim to inspect the implications of the LPM-mechanism for the optimal trading behaviour. For this analysis, we make use of the

[22]It is noteworthy that the standard errors corresponding to the Λ_t-related estimates reported in Pelsser (2019) are fairly large compared to those for the other estimates.

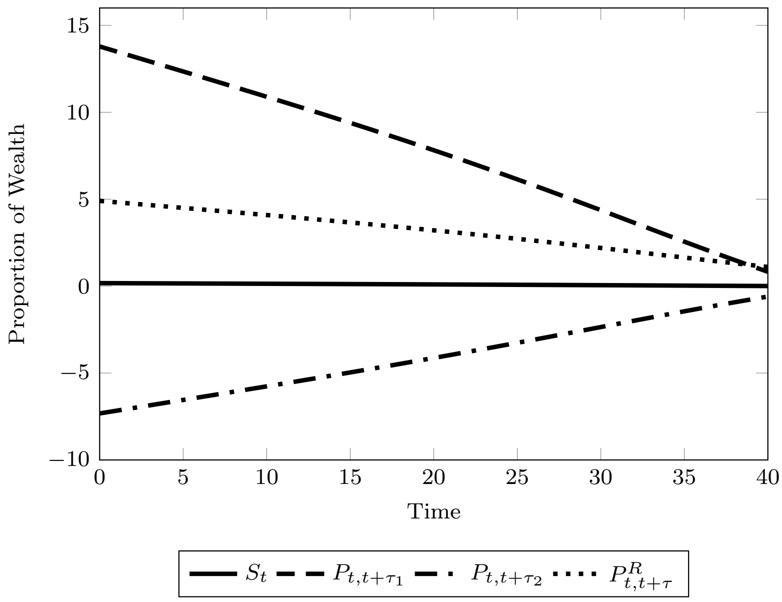

Figure 4. Optimal portfolio for progressively improving replacement ratio. This figure depicts the optimal allocation of assets varied with respect to time ($t \in [0,T]$). The risk-neutral value of the replacement ratio progressively grows to a level of 100% at retirement ($T = 40$). The black line corresponds to the demand for the stock (S_t); the dashed line to the demand for the 5-year nominal bond ($P_{t,t+\tau_1}$); the dash-dotted line to the demand for the 20-year nominal bond ($P_{t,t+\tau_2}$); and the dotted line to the demand for the 20-year inflation-linked bond ($P^R_{t,t+\tau}$). The output is based on the set of parameter estimates labelled as P^2, cf. section 4.3. The horizontal axis represents the time-dimension. The vertical axis represents the proportion of wealth. For this graph, we relied on the analytical expression for π_t^{opt} in (3.13). Moreover, we set $p = 2$ and $F_0 = 80\%$. Throughout the time-dimension, we hold the observable quantities fixed as follows: $r_t = \delta_{0,r}$, $\pi_t = \delta_{0,\pi}$, $Y_t = Y_0 e^{\delta_{0,r} t}$, $\Pi_t = e^{\delta_{0,\pi} t}$ and $X_t = X_0 e^{(\delta_{0,r} + \delta_{0,\pi} + \frac{1}{T} \log Y_0/X_0)t}$, for all $t \in [0,T]$. As the demands are calculated on a yearly basis, we used cubic-spline interpolation to compute and accordingly smoothen the demands over the entirety of $[0,T]$.

parameter estimates defined by P^2. This collection of estimates is spelled out in section 4.3 and the description of Figure 1. Due to the nature of the analysis, it is not necessary to employ the P^0 and P^1 collections. Note that all subsequent results correspond to a situation wherein $T = 40$, $\tau_A = 20$, $X_0 = 10$, $F_0 = 80\%$ and $p = 2$. The pension-related interpretation of this initialisation is given at the beginning of section 4.3. Observe that the asset mix consists of a stock, two τ_i-year nominal bonds and a τ-year inflation-linked bond. Henceforth, we assume that $\tau_1 = 5$, $\tau_2 = 20$ and $\tau = 20$.

The optimal portfolio rules can be expressed in terms of observable quantities. To this end, consider e.g. (3.13) and note that the right-hand side depends on the following processes: M_t, Π_t, Z_t, Y_t and X_t. The values for Y_t, Π_t and X_t are directly observed. The value for M_t uniquely depends on the ones for Z_t, Y_t, Π_t and X_t, cf. (3.8). Moreover, Z_t can be expressed in terms of r_t and π_t, both of which are observed. Hence, given the values of r_t, π_t, Y_t, Π_t and X_t, the agent knows precisely how he/she should optimally invest. This is an outstanding advantage of the closed-form nature of the optimality conditions. To emphasise this advantage and to examine the implications of the LPM-mechanism for the optimal exposures, we present Figures 4 and 5. These figures depict the optimal allocation

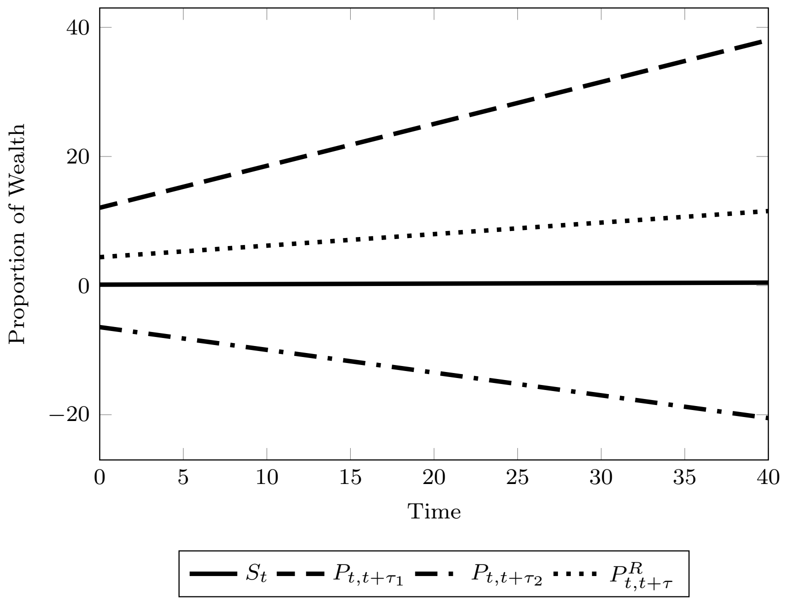

Figure 5. Optimal portfolio for progressively worsening replacement ratio. This figure depicts the optimal allocation of assets varied with respect to time ($t \in [0, T]$). The risk-neutral value of the replacement ratio progressively declines to a level of 60% at retirement ($T = 40$). In line with Figure 4, the black line corresponds to the demand for the stock (S_t); the dashed line to the demand for the 5-year nominal bond ($P_{t,t+\tau_1}$); the dash-dotted line to the demand for the 20-year nominal bond ($P_{t,t+\tau_2}$); and the dotted line to the demand for the 20-year inflation-linked bond ($P^R_{t,t+\tau}$). Note that these line patterns coincide with those shown in Figure 4. The displayed trajectories are predicated on the set of parameter estimates labelled as P^2, cf. section 4.3. The horizontal axis represents the time-dimension. The vertical axis represents the proportion of wealth. As this graph presents the same as Figure 4 but for $X_t = X_0 e^{(\delta_{0,r}+\delta_{0,\pi}+\frac{1}{T}\log 0.6\times Y_0/X_0)t}$, we refer to the description of Figure 4 for further details on (i) LPM-specific parameters, (ii) the observable quantities and (iii) the interpolation technique.

of assets (π_t^{opt}) varied with respect to time ($t \in [0, T]$). In Figure 4, the risk-neutral value of the replacement ratio progressively *grows* to a level of 100% at retirement. Figure 5 displays the same, but for a risk-neutral value of the replacement ratio that progressively *declines* to a level of 60% at retirement. The risk-neutral value of the replacement ratio lives by: $F_t = \frac{X_t}{\mathbb{E}[Y_T \Pi_T M_T/M_t \mid \mathcal{F}_t]}$, for all $t \in [0, T]$. Observe that F_t evaluated at $t = 0$ renders the funding ratio (F_0). For Figures 4 and 5, we fix: $r_t = \delta_{0,r}$, $\pi_t = \delta_{0,\pi}$, $Y_t = Y_0 e^{\delta_{0,r} t}$, $\Pi_t = e^{\delta_{0,\pi} t}$ and $X_t = X_0 e^{(\delta_{0,r}+\delta_{0,\pi}+\frac{1}{T}\log F_T\times Y_0/X_0)t}$, for all $t \in [0, T]$. It is clear that Π_t is initialised at its approximate expected value. We predicate the specification of Y_t on the assumption that it moves together with the cash account. The definition of X_t therefore ensures that $X_T^{Y,\Pi} = F_T$ holds. Figures 4 and 5 accordingly set $F_T = 100\%$ and $F_T = 60\%$, respectively. Note that we hold r_t and π_t fixed at their expected values.

4.4.1 Technical Discussion: Figure 4

We continue with a technical discussion of Figure 4. The depicted allocations correspond to a participant who enters the pension scheme with a funding ratio of 80%. Over the course of the accumulation phase, his/her wealth moves closer to the reference level. Upon retirement, the replacement ratio is equal to 100%. For this situation, the optimal portfolio exhibits a clear life-cycle pattern. That is, the percentages of wealth allocated to the risky assets *decrease* as the individual ages. This trading behaviour can be attributed to the LPM-mechanism. From section 4.2, we recall that an LPM-agent's level of risk aversion positively depends on the risk-neutral value of the replacement ratio (F_t). Due to the inverse relation between levels of risk aversion and optimal portfolios, progressive growth of F_t generates life-cycle strategies as shown in the graph. We emphasise that the LPM-agent is willing to bear significant risk as long as the target is not achieved. For instance, at $t = 0$ when $F_0 = 80\%$, the fund is required to invest nearly 15 times their accumulated wealth in the 5-year nominal bond. This extreme behaviour comes close to the "gamble for resurrection" phenomenon unique to loss aversion frameworks. The opposite can be observed when the target is more or less reached. Indeed, for F_t close to 100% (at $t = T$), the fund "de-risks" their portfolio so as to "lock in" wealth at the current level.

4.4.2 Technical Discussion: Figure 5

Let us now turn to Figure 5. As in the previous case, the participant enters the pension scheme with a funding ratio of 80%. However, during the accumulation phase, his/her wealth gradually moves away from the reference level. Upon retirement, the replacement ratio now equals 60%. Consequently, in contrast to Figure 4, the risk-neutral value of the replacement ratio decreases over time. This leads to optimal allocations that behave as reversed life-cycle strategies. In particular, the percentages of wealth allocated to the risky assets *increase* as the individual ages. Strategies of this form are in stark contrast with conventional wisdom, cf. Cocco et al. (2005). Nevertheless, using the same arguments as before, one is able to explain the progressively increasing exposure to risk. As a result of the declining value of F_t, the agent namely becomes increasingly less risk averse. This directly leads to trading patterns involving more risk as time passes. In fact, near $t = T$ when F_t approaches 60%, the agent is so worried about achieving the target that the pension fund is required to invest almost 40 times their wealth in the 5-year nominal bond. Note that the overall exposure to risk was already relatively large at $t = 0$. This behaviour confirms that the LPM operator accommodates features specific to loss aversion setups. Although these strategies are highly impractical, we point out that they are potentially capable of rendering great outcomes for retirement wealth, cf. section 4.3.

4.4.3 Economic Takeaways

As in section 4.3.3, the set of economic takeaways corresponding to Figures 4 and 5 is twofold. First, we find that LPM-based investment strategies strongly depend on the risk neutral value of the replacement ratio. This dependency highlights the core mechanism of the LPM operator. Second, we show that the optimal portfolio rules can be difficult to implement in practice. For this purpose, it is advisable to account for trading/solvency constraints. We expand on these takeaways in the following two summaries:

Takeaway 1. In section 4.2, we addressed the dependence of the agent's attitude towards risk on the difference between the reference level and accumulated wealth. This dependence gives rise to a positive relation between risk aversion and the risk-neutral value of the replacement ratio (F_t). The latter relation constitutes the central feature of the LPM mechanism. As a consequence of this relation, the optimal LPM-based investment policies negatively depend on F_t. In particular, for values of F_t near 100%, the LPM-agent implements a notably prudent investment strategy. In this way, he/she aims to "lock in" wealth at the desired reference level. On the contrary, if F_t decreases in value, the LPM-agent allocates strikingly larger proportions of wealth to the risky assets. Thereby, he/she tries to improve the likelihood of ultimately achieving the target. This phenomenon approaches the "gamble for resurrection" behaviour specific to loss aversion frameworks. Although the agent does not become risk-loving for $F_t < 1$, Figures 4 and 5 show indeed that he/she is considerably more willing to bear risk. The graphs accordingly support the theoretical treatment by Jarrow and Zhao (2006). In this study, the authors demonstrate that the LPM operator can be nested under the aegis of prospect theory.

Takeaway 2. The second takeaway ties in with the previous one. In line with the LPM mechanism, declines in F_t namely force the agent to increase his/her exposure to risk. Particular values of F_t may even result in unrealistically leveraged and/or large positions in the financial instruments. This is saliently visible in Figure 5. For values of F_t between 60% and 80%, the agent may be required to invest nearly 40 times his/her wealth in the 5-year nominal bond. Similar extreme positions should be taken in the other three assets. In reality, pension funds must deal with solvency requirements. These requirements involve specific restrictions concerning a.o. borrowing and short-selling. The LPM-based investment policies are correspondingly difficult to adopt by most pension funds. Due to the very extreme nature of the optimal allocations, this extends to a broader set of agents. Therefore, in order to arrive at more practical portfolio rules, it is recommendable to account for trading/solvency constraints. We refer to Cvitanić and Karatzas (1992) and Basak and Shapiro (2001) for analyses involving such constraints. By the same token, it can be advantageous to take parameter/model uncertainty into account, cf. section 4.3.3. The worst-case preparation naturally leads to more conservative investment policies.

5 Conclusion

This paper has studied an optimal terminal wealth problem, in which the agent aims to minimise an LPM criterion. This criterion incorporates a log-normal exogenous reference level. We have placed the problem in the complex market model proposed by Koijen et al. (2009). In this continuous-time framework, the market prices of risk depend on a mean-reverting state variable. As a result, it is highly nontrivial to derive closed-form solutions to the LPM problem. Nevertheless, using Fourier machinery, we have been able to deduce analytical expressions for the optimal portfolio rules and the optimal wealth process. Moreover, we have managed to disentangle a closed-form specification for the distributional features of optimal terminal wealth. In the numerical illustrations, we have cast the LPM problem into the context of a DC pension scheme. The ensuing results have demonstrated that LPM-based investment policies can improve a pension fund's recovery potential. In spite of their possibly outstanding performance, we have exemplified that these policies may be difficult to implement in reality. Furthermore, we have shown that these outcomes strongly depend on the estimates for the market prices of risk.

A Proof of Theorem 3.1

The dynamic problem in (2.17) is equivalent to:

$$\sup_{X_T \in L^0_+(\Omega) \text{ s.t. } \mathbb{E}[X_T M_T] \leq X_0} \mathbb{E}\left[-\frac{1}{p} \left[\left(Y_T - \frac{X_T}{\Pi_T} \right)^+ \right]^p \right]. \tag{A.1}$$

Here, $L^0_+(\Omega)$ represents the Lebesgue space of all non-negative random variables.

The Lagrangian for (A.1), $\mathcal{L} : L^0_+(\Omega) \times \mathbb{R}_+ \to \mathbb{R}$, is given by:

$$\begin{aligned} \mathcal{L}(X_T, \eta) &= \mathbb{E}\left[U(X_T, Y_T) - \eta X_T M_T \right] + \eta X_0 \\ &= \mathbb{E}\left[-\frac{1}{p} \left[\left(Y_T - \frac{X_T}{\Pi_T} \right)^+ \right]^p - \eta X_T M_T \right] + \eta X_0, \end{aligned} \tag{A.2}$$

where $\eta \in \mathbb{R}_+$ represents the strictly positive scalar-valued Lagrange multiplier.

Appropriate optimisation of this Lagrangian results in:

$$X_T^{\text{opt}} = I\left(\mathcal{H}^{-1}(X_0) M_T \Pi_T, Y_T \right) \Pi_T. \tag{A.3}$$

Due to the fact that $\mathbb{E}\left[I\left(\mathcal{H}^{-1}(X_0) M_T \Pi_T, Y_T \right) \Pi_T M_T \right] = X_0$ holds, $\left\{ X_t^{\text{opt}} M_t \right\}_{t \in [0,T]}$ spells out a $\mathbb{P}$-martingale process with respect to $\{\mathcal{F}_t\}_{t \in [0,T]}$. This results in (3.1).

Moreover, by the martingale representation theorem: $X_t^{\text{opt}} M_t = X_0 + \int_0^T \psi_s^\top \mathrm{d}W_s$, for all $t \in [0,T]$ and some $L^2([0,T])$-valued process $\{\psi_t\}_{t \in [0,T]}$. Then, (3.2) follows from:

$$\begin{aligned} X_t^{\text{opt}} M_t &= X_0 + \int_0^t \left(\pi_s^{\text{opt}^\top} \sigma - \Lambda_s^\top X_s^{\text{opt}} \right) M_s \mathrm{d}W_s \\ &= X_0 + \int_0^T \psi_s^\top \mathrm{d}W_s, \quad \forall\, t \in [0,T]. \end{aligned} \tag{A.4}$$

B Proofs I

B.1 Proof of Proposition 3.2

Define the following two processes:

$$\begin{aligned}
\left.\frac{\mathrm{d}\mathbb{X}_1}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} &= \frac{\mathbb{E}\left[Y_T M_T^R \mid \mathcal{F}_t\right]}{\mathbb{E}\left[Y_T M_T^R\right]} = e^{-\frac{1}{2}\int_0^t \lambda_{1,s}^\top \lambda_{1,s}\mathrm{d}s + \int_0^t \lambda_{1,s}^\top \mathrm{d}W_s}, \\
\left.\frac{\mathrm{d}\mathbb{X}_2}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} &= \frac{\mathbb{E}\left[\left(M_T^R\right)^{\frac{p}{p-1}} \;\middle|\; \mathcal{F}_t\right]}{\mathbb{E}\left[\left(M_T^R\right)^{\frac{p}{p-1}}\right]} = e^{-\frac{1}{2}\int_0^t \lambda_{2,s}^\top \lambda_{2,s}\mathrm{d}s + \int_0^t \lambda_{2,s}^\top \mathrm{d}W_s}.
\end{aligned} \tag{B.1}$$

Both $\left.\frac{\mathrm{d}\mathbb{X}_1}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}$ and $\left.\frac{\mathrm{d}\mathbb{X}_2}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}$ qualify as valid Radon-Nikodym derivatives, for all $t \in [0,T]$.[23] That is, $\left.\frac{\mathrm{d}\mathbb{X}_1}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}$ corresponds to change of measure from $\mathbb{P}$ to $\mathbb{X}_1$; likewise, $\left.\frac{\mathrm{d}\mathbb{X}_1}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}$ corresponds to change of measure from $\mathbb{P}$ to $\mathbb{X}_2$. Note that $\mathbb{X}_i \sim \mathbb{P}$, for $i = 1, 2$. Here, the two processes $\{\lambda_{1,t}\}_{t\in[0,T]}$ and $\{\lambda_{2,t}\}_{t\in[0,T]}$ are to be determined. Under the $\mathbb{X}_i$ measure, we have that the following two processes are standard Brownian motions, for $i = 1, 2$ and all $t \in [0,T]$:

$$W_t^{\mathbb{X}_i} = W_t - \int_0^t \lambda_{i,s}\mathrm{d}s. \tag{B.2}$$

Using the changes of measure implied by the Radon-Nikodym derivatives in (B.1), we are able to rewrite the conditional expectation in (3.1) as follows, for all $t \in [0,T]$:

$$\begin{aligned}
X_t^{\mathrm{opt}} &= \frac{1}{M_t}\mathbb{E}\left[Y_T M_T^R \mathbb{1}_{\{\mathcal{A}_T\}} \mid \mathcal{F}_t\right] - \frac{1}{M_t}\mathbb{E}\left[\left(\mathcal{H}^{-1}\left(X_0\right)^{\frac{1}{p}} M_T^R\right)^{\frac{p}{p-1}} \mathbb{1}_{\{\mathcal{A}_T\}} \;\middle|\; \mathcal{F}_t\right] \\
&= Y_t \Pi_t \mathbb{E}\left[\frac{Y_T M_T^R}{Y_t M_t^R} \;\middle|\; \mathcal{F}_t\right] \mathbb{X}_1\left(\mathcal{A}_T \mid \mathcal{F}_t\right) \\
&- \left(\mathcal{H}^{-1}\left(X_0\right) M_t^R\right)^{\frac{1}{p-1}} \Pi_t \mathbb{E}\left[\left(\frac{M_T^R}{M_t^R}\right)^{\frac{p}{p-1}} \;\middle|\; \mathcal{F}_t\right] \mathbb{X}_2\left(\mathcal{A}_T \mid \mathcal{F}_t\right).
\end{aligned} \tag{B.3}$$

In the last line, we use that $\mathbb{X}_i\left(\cdot \mid \mathcal{F}_t\right) = \mathbb{E}\Big[\left.\frac{\mathrm{d}\mathbb{X}_i}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} \left.\frac{\mathrm{d}\mathbb{X}_i}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}^{-1} \mathbb{1}_{\{\cdot\}} \mid \mathcal{F}_t\Big]$, for $i = 1, 2$.

We start by evaluating the expectations in (B.3). Relying on the results in Duffie and Kan (1996), we note that the following holds for all $t \in [0,T]$:

$$\begin{aligned}
P_1\left(t, Z_t\right) &= \mathbb{E}\left[\frac{Y_T M_T^R}{Y_t M_t^R} \;\middle|\; \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_1}\left[\exp\left\{-\int_t^T F\left(s, Z_s\right)\mathrm{d}s\right\} \;\middle|\; Z_t\right], \\
P_2\left(t, Z_t\right) &= \mathbb{E}\left[\left(\frac{M_T^R}{M_t^R}\right)^{\frac{p}{p-1}} \;\middle|\; \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}_2}\left[\exp\left\{-\int_t^T G\left(s, Z_s\right)\mathrm{d}s\right\} \;\middle|\; Z_t\right].
\end{aligned} \tag{B.4}$$

[23] A process qualifies as a Radon-Nikodym derivative if it satisfies the following two conditions. First, the process must be a strictly positive $\mathbb{P}$-martingale with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$. Second, its unconditional expectation ought to equate to 1. Both conditions are clearly fulfilled by the processes $\left.\frac{\mathrm{d}\mathbb{X}_1}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}$ and $\left.\frac{\mathrm{d}\mathbb{X}_2}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}$ in (B.1). See e.g. Karatzas and Shreve (1991, 1998) for details on Radon-Nikodym derivatives.

Here, the measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are induced by: $\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{-\frac{1}{2}\int_0^t \widehat{\lambda}_{i,s}^\top \widehat{\lambda}_{i,s}\mathrm{d}s + \int_0^t \widehat{\lambda}_{i,s}^\top \mathrm{d}W_s}$, for $i = 1, 2$, where $\widehat{\lambda}_{1,t} = -\Lambda_t^R + \beta_t$ and $\widehat{\lambda}_{2,t} = -\frac{p}{p-1}\Lambda_t^R$, for all $t \in [0, T]$. Note that $R_t = r_t + \pi_t - \sigma_\Pi^\top \Lambda_t$ and $\Lambda_t^R = \Lambda_t - \sigma_\Pi$, for all $t \in [0, T]$. Furthermore, the continuously differentiable (quadratic-)affine functions, $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $G : [0, T] \times \mathbb{R} \to \mathbb{R}$, read:

$$
\begin{aligned}
F(t, Z_t) &= R_t - \alpha_t + \Lambda_t^{R^\top} \beta_t = a_t + b_t^\top Z_t, \\
G(t, Z_t) &= \frac{p}{p-1} R_t - \frac{1}{2}\frac{p}{(p-1)^2} \Lambda_t^{R^\top} \Lambda_t^R = \tilde{a} + \tilde{b}^\top Z_t + Z_t^\top \tilde{c} Z_t,
\end{aligned}
\tag{B.5}
$$

where $a_t = \widehat{\delta}_{0,r} - \alpha_t + \beta_t^\top (\Lambda_0 - \sigma_\Pi)$, and $b_t = \widehat{\delta}_{1,r} + \Lambda_1^\top \beta_t$; as well as $\tilde{a} = \frac{p}{p-1}\widehat{\delta}_{0,r} - \frac{1}{2}\frac{p}{(p-1)^2}(\Lambda_0 - \sigma_\Pi)^\top (\Lambda_0 - \sigma_\Pi)$, $\tilde{b} = \frac{p}{p-1}\widehat{\delta}_{1,r} - \frac{p}{(p-1)^2}\Lambda_1^\top (\Lambda_0 - \sigma_\Pi)$, and $\tilde{c} = -\frac{1}{2}\frac{p}{(p-1)^2}\Lambda_1^\top \Lambda_1$. Hence, the function F is affine in Z_t, and the function G is affine-quadratic in Z_t.

Now, let us note that the SDE's of $Y_t M_t^R$ and $Y_t M_t^{R^{\frac{p}{p-1}}}$ are given by:

$$
\begin{aligned}
\frac{\mathrm{d}Y_t M_t^R}{Y_t M_t^R} &= -\left(R_t - \alpha_t + \beta_t^\top \Lambda_t^R\right)\mathrm{d}t - \left(\Lambda_t^R - \beta_t\right)^\top \mathrm{d}W_t, \\
\frac{\mathrm{d}M_t^{R^{\frac{p}{p-1}}}}{M_t^{R^{\frac{p}{p-1}}}} &= -\left(\frac{p}{p-1}R_t - \frac{1}{2}\frac{p}{(p-1)^2}\Lambda_t^{R^\top}\Lambda_t^R\right)\mathrm{d}t - \frac{p}{p-1}\Lambda_t^{R^\top}\mathrm{d}W_t.
\end{aligned}
\tag{B.6}
$$

As a result, the following must be true:

$$
\begin{aligned}
P_{1,t} - P_{1,Z}^\top K_Z Z + \frac{1}{2}\mathrm{tr}\left(P_{1,ZZ}\right) - \left(R - \alpha + \beta^\top \Lambda^R\right) P_1 - P_{1,Z}^\top \Sigma_Z \left(\Lambda^R - \beta\right) &= 0, \\
P_{2,t} - P_{2,Z}^\top K_Z Z + \frac{1}{2}\mathrm{tr}\left(P_{2,ZZ}\right) - \left(\frac{pR}{p-1} - \frac{1}{2}\frac{p}{(p-1)^2}\Lambda^{R^\top}\Lambda^R\right) P_2 - \frac{p}{p-1} P_{2,Z}^\top \Sigma_Z \Lambda^R &= 0.
\end{aligned}
\tag{B.7}
$$

Observe here that $\mathrm{tr}(\cdot)$ spells out the trace operator.

Inspired by Duffie and Kan (1996), Sangvinatsos and Wachter (2005), and Koijen et al. (2009), we define the subsequent two *ansatz* functions for P_1 and P_2:

$$
\begin{aligned}
P_1(t, Z_t) &= \exp\left\{\tilde{A}(t) + \tilde{B}(t)^\top Z_t\right\}, \text{ and} \\
P_2(t, Z_t) &= \exp\left\{\widehat{A}(t) + \widehat{B}(t)^\top Z_t + Z_t^\top \widehat{C}(t) Z_t\right\}.
\end{aligned}
\tag{B.8}
$$

Here, we postulate that $\tilde{A} : [0, T] \to \mathbb{R}$, $\tilde{B} : [0, T] \to \mathbb{R}^2$, $\widehat{A} : [0, T] \to \mathbb{R}$, $\widehat{B} : [0, T] \to \mathbb{R}^2$, and $\widehat{C} : [0, T] \to \mathbb{R}^{2\times 2}$, are deterministic functions of time, $t \in [0, T]$, alone. Now, we proceed in the spirit of Dai and Singleton (2002) and Koijen et al. (2009), and insert the (ansatz) definitions for P_1 and P_2 into (B.7). Let $c_t = \Lambda_0 - \sigma_\Pi - \beta_t$. For P_1, we then find:

$$
\begin{aligned}
\tilde{A}(t) &= -\int_t^T \left[\tilde{B}(s)^\top \Sigma_Z c_s - \frac{1}{2}\tilde{B}(s)^\top \tilde{B}(s) + a_s\right]\mathrm{d}s, \\
\tilde{B}(t) &= -\int_t^T \exp\left\{-\left[K_Z^\top + \Lambda_1^\top \Sigma_Z^\top\right](s - t)\right\} b_s \mathrm{d}s.
\end{aligned}
\tag{B.9}
$$

For P_2, we derive the subsequent system of ODE's:

$$\begin{aligned}\widehat{A}'(t) &= \frac{p}{p-1}\widehat{B}(t)^{\top}\Sigma_Z[\Lambda_0-\sigma_{\Pi}] - \frac{1}{2}\widehat{B}(t)^{\top}\widehat{B}(t) + \tilde{a}_t,\\ \widehat{B}'(t) &= \left(\frac{p}{p-1}\Lambda_1^{\top}\Sigma_Z^{\top} + K_Z^{\top} - 2\widehat{C}(t)^{\top}\right)\widehat{B}(t) + \tilde{b}_t,\\ \widehat{C}'(t) &= 2\left(K_Z^{\top} + \frac{p}{p-1}\Lambda_1^{\top}\Sigma_Z^{\top}\right)\widehat{C}(t) - 2\widehat{C}(t)^{\top}\widehat{C}(t) + \tilde{c},\end{aligned} \tag{B.10}$$

where we define $\tilde{a}_t = -\mathrm{tr}\left(\widehat{C}(t)\right) + \tilde{a}$ and $\tilde{b}_t = 2\frac{p}{p-1}\widehat{C}(t)^{\top}\Sigma_Z[\Lambda_0-\sigma_{\Pi}] + \tilde{b}$, for all $t\in[0,T]$. Note that: $\widehat{A}(T)=0$, $\widehat{B}(T)=0_2$, and $\widehat{C}(T)=0_{2\times 2}$, where $0_2=[0,0]^{\top}$ and $0_{2\times 2}=[0_2,0_2]^{\top}$. This system of ODE's involves a matrix Riccati differential equation, for which no closed-form solution is available. Likewise, we cannot analytically solve the second ODE in (B.10). We are only able to state that the following holds for all $t\in[0,T]$:

$$\widehat{A}(t) = -\int_t^T\left[\frac{p}{p-1}\widehat{B}(s)^{\top}\Sigma_Z[\Lambda_0-\sigma_{\Pi}] - \frac{1}{2}\widehat{B}(s)^{\top}\widehat{B}(s) + \tilde{a}_s\right]\mathrm{d}s. \tag{B.11}$$

Let us return to (B.1) and derive the following SDE's:

$$\begin{aligned}\mathrm{d}\frac{\mathrm{d}\mathbb{X}_1}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} &= -\left(\Lambda_t^{R^{\top}} - \beta_t^{\top} - \tilde{B}(t)^{\top}\Sigma_Z\right)\mathrm{d}W_t,\\ \mathrm{d}\frac{\mathrm{d}\mathbb{X}_2}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} &= -\left(\frac{p}{p-1}\Lambda_t^{R^{\top}} - \left[\widehat{B}(t)^{\top} + 2Z_t^{\top}\widehat{C}(t)^{\top}\right]\Sigma_Z\right)\mathrm{d}W_t,\end{aligned} \tag{B.12}$$

where we use that $\frac{\mathrm{d}\mathbb{X}_1}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t} = C_1P_1(t,Z_t)Y_tM_t^R$ and $\frac{\mathrm{d}\mathbb{X}_2}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t} = C_2P_2(t,Z_t)M_t^{R^{\frac{p}{p-1}}}$, for $C_1=\mathbb{E}\left[Y_TM_T^R\right]^{-1}$ and $C_2=\mathbb{E}\left[\left(M_T^R\right)^{\frac{p}{p-1}}\right]$. As a consequence, for all $t\in[0,T]$

$$\begin{aligned}W_t^{\mathbb{X}_1} &= W_t + \int_0^t\left(\Lambda_t^R - \beta_t - \Sigma_Z^{\top}\tilde{B}(t)\right)\mathrm{d}s,\\ W_t^{\mathbb{X}_2} &= W_t + \int_0^t\left(\frac{p}{p-1}\Lambda_t^R - \Sigma_Z^{\top}\left[\widehat{B}(t) + 2\widehat{C}(t)Z_t\right]\right)\mathrm{d}s,\end{aligned} \tag{B.13}$$

outline the standard Brownian motions under $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively.

To facilitate the application of the Fourier transform to the two conditional probabilities of interest, note that: $\mathcal{A}_T = \left\{\left(M_T^R\right)^{-\frac{1}{p-1}}Y_T \geq \mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}\right\}$. Therefore, $\mathbb{1}_{\{\mathcal{A}_T\}} = f(T,H_T)$, where we define $H_T=\log\left[\left(M_T^R\right)^{-\frac{1}{p-1}}Y_T\right]$. This implies that: $f_j(t,h) = \mathbb{X}_j(\mathcal{A}_T\mid\mathcal{F}_t) = \mathbb{E}^{\mathbb{X}_j}[f(T,H_T)\mid H_t=h]$, for $j=1,2$ and all $t\in[0,T]$. Via an application of the Fourier transform, we consequently have for $j=1,2$ and all $t\in[0,T]$:

$$\begin{aligned}f_j(t,h) &= \mathbb{E}^{\mathbb{X}_j}[f(T,H_T)\mid H_t=h]\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_\kappa^*(T,\omega)\,\phi_{j,T-t}(-\omega-i\kappa,h)\,\mathrm{d}\omega,\end{aligned} \tag{B.14}$$

where $f^*_\kappa(T,\omega)$ is for some $\kappa > 0$ and all $\omega \in \mathbb{R}$ given by the following:

$$\begin{aligned} f^*_\kappa(T,\omega) &= \int_{-\infty}^{\infty} e^{(i\omega-\kappa)H} f(T,H)\,\mathrm{d}H \\ &= \int_{-\infty}^{\infty} e^{(i\omega-\kappa)H} \mathbb{1}_{\left\{H \geq \frac{1}{p-1}\log\mathcal{H}^{-1}(X_0)\right\}} \mathrm{d}H = -\frac{e^{(i\omega-\kappa)\frac{1}{p-1}\log\mathcal{H}^{-1}(X_0)}}{i\omega-\kappa}. \end{aligned} \tag{B.15}$$

Clearly, $f^*_\kappa(T,\omega)$ specifies the Fourier transform of $e^{-\kappa H_T} f(T,H_T)$ with respect to H_T. Observe here that we have: $\phi_{j,T-t}(\omega,h) = \int_{-\infty}^{\infty} e^{i\omega H}\phi_j(H,h)\,\mathrm{d}H = \mathbb{E}^{\mathbb{X}_j}\left[e^{i\omega H_T} \mid H_t = h\right] = \mathbb{E}^{\mathbb{X}_j}\left[\left(M_T^{R^{-\frac{1}{p-1}}} Y_T\right)^{i\omega} \mid \mathcal{F}_t\right]$, for $j = 1,2$ and all $t \in [0,T]$. Note that $\phi_j(H,h)$ characterises the conditional density function (under the $\mathbb{X}_j$ measure) corresponding to H_T.

To determine $\phi_{j,T-t}(\omega,h)$ in (B.14), we derive:

$$\begin{aligned} \frac{\mathrm{d}M_t^{R^{-\frac{i\omega}{p-1}}} Y_t^{i\omega}}{M_t^{R^{-\frac{i\omega}{p-1}}} Y_t^{i\omega}} &= \Bigg(i\omega\alpha_t + \frac{1}{2} i\omega[i\omega-1]\beta_t^\top\beta_t + \frac{i\omega}{p-1}R_t \\ &\quad -\frac{1}{2}\frac{i\omega}{p-1}\left[-\frac{i\omega}{p-1}-1\right]\Lambda_t^{R^\top}\Lambda_t^R + \frac{(i\omega)^2}{p-1}\beta_t^\top\Lambda_t^R \\ &\quad + i\omega\left[\beta_t^\top + \frac{1}{p-1}\Lambda_t^{R^\top}\right]\lambda_{j,t}\Bigg)\mathrm{d}t + i\omega\left[\beta_t^\top + \frac{1}{p-1}\Lambda_t^{R^\top}\right]\mathrm{d}W_t^{\mathbb{X}_j}, \end{aligned} \tag{B.16}$$

for $j = 1,2$, where we define the processes $\lambda_{1,t}$ and $\lambda_{2,t}$ as follows: $\lambda_{1,t} = -\big(\Lambda_t^R - \beta_t - \Sigma_Z^\top\tilde{B}(t)\big)$ and $\lambda_{2,t} = -\big(\frac{p}{p-1}\Lambda_t^R - \Sigma_Z^\top\big[\widehat{B}(t) + 2\widehat{C}(t)Z_t\big]\big)$ for all $t \in [0,T]$. Then, in the sense of Duffie and Kan (1996), we note that the following holds for all $t \in [0,T]$:

$$Q_j(t,Z_t,\omega) = \mathbb{E}^{\mathbb{X}_j}\left[\frac{M_T^{R^{-\frac{i\omega}{p-1}}} Y_T^{i\omega}}{M_t^{R^{-\frac{i\omega}{p-1}}} Y_t^{i\omega}} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{C}_j}\left[\exp\left\{-\int_t^T R_j(s,Z_s)\,\mathrm{d}s\right\} \,\middle|\, Z_t\right]. \tag{B.17}$$

The measures $\mathbb{C}_j$ are induced by: $\frac{\mathrm{d}\mathbb{C}_j}{\mathrm{d}\mathbb{X}_j}\Big|_{\mathcal{F}_t} = e^{-\frac{1}{2}\int_0^t \bar{\lambda}_s^\top\bar{\lambda}_s\mathrm{d}s + \int_0^t \bar{\lambda}_s^\top \mathrm{d}W_s^{\mathbb{X}_j}}$, where $\bar{\lambda}_t = i\omega\big[\beta_t + \frac{1}{p-1}\Lambda_t^R\big]$. Furthermore, $R_1 : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $R_2 : [0,T]\times\mathbb{R}\to\mathbb{R}$ read:

$$\begin{aligned} -R_1(t,Z_t) &= a_{1,t}(\omega) + b_{1,t}(\omega)^\top Z_t + Z_t^\top c_{1,t}(\omega)Z_t, \\ -R_2(t,Z_t) &= a_{2,t}(\omega) + b_{2,t}(\omega)^\top Z_t + Z_t^\top c_{2,t}(\omega)Z_t. \end{aligned} \tag{B.18}$$

The deterministic function $a_{j,t}(\omega)$ is given by:

$$\begin{aligned} a_{j,t}(\omega) &= i\omega\alpha_t + \frac{1}{2}i\omega[i\omega-1]\beta_t^\top\beta_t + \frac{i\omega}{p-1}\widehat{\delta}_{0,r} \\ &\quad -\frac{1}{2}\frac{i\omega}{p-1}\left[-\frac{i\omega}{p-1}-1\right](\Lambda_0-\sigma_\Pi)^\top(\Lambda_0-\sigma_\Pi) \\ &\quad + \frac{(i\omega)^2}{p-1}\beta_t^\top(\Lambda_0-\sigma_\Pi) + i\omega\left[\beta_t^\top + \frac{1}{p-1}(\Lambda_0-\sigma_\Pi)^\top\right]\lambda_{j,0,t}, \end{aligned} \tag{B.19}$$

for $j=1,2$, all $t\in[0,T]$ and $\omega\in\mathbb{R}$, where we define $\lambda_{1,0,t}=-\big(\left(\Lambda_0-\sigma_\Pi\right)-\beta_t-\Sigma_Z^\top\tilde{B}\left(t\right)\big)$ and $\lambda_{2,0,t}=-\left(\frac{p}{p-1}\left[\Lambda_0-\sigma_\Pi\right]-\Sigma_Z^\top\widehat{B}\left(t\right)\right)$, for all $t\in[0,T]$. Note that $a_t\left(\omega\right)\in\mathbb{R}$ holds. Similarly, $b_t\left(\omega\right)\in\mathbb{R}^2$ and $c_t\left(\omega\right)\in\mathbb{R}^{2\times2}$ hold. Now, define: $\lambda_{1,1,t}=-\Lambda_1$ and $\lambda_{2,1,t}=-\left(\frac{p}{p-1}\Lambda_1-2\Sigma_Z^\top\widehat{C}\left(t\right)\right)$, for all $t\in[0,T]$. Then, $\lambda_{j,t}=\lambda_{j,0,t}+\lambda_{j,1,t}Z_t$ for all $t\in[0,T]$. The definitions of $b_{j,t}\left(\omega\right)$ and $c_{j,t}\left(\omega\right)$ are for $j=1,2$, all $t\in[0,T]$ and $\omega\in\mathbb{R}$ given by:

$$
\begin{aligned}
b_{j,t}\left(\omega\right)&=\frac{i\omega}{p-1}\widehat{\delta}_{1,r}-\frac{i\omega}{p-1}\left[-\frac{i\omega}{p-1}-1\right]\Lambda_1^\top\left(\Lambda_0-\sigma_\Pi\right)+\frac{\left(i\omega\right)^2}{p-1}\Lambda_1^\top\beta_t\\
&\quad+i\omega\lambda_{j,1,t}^\top\left(\beta_t+\frac{1}{p-1}\left[\Lambda_0-\sigma_\Pi\right]\right)+i\omega\frac{1}{p-1}\Lambda_1^\top\lambda_{j,0,t},\\
c_{j,t}\left(\omega\right)&=-\frac{1}{2}\frac{i\omega}{p-1}\left[-\frac{i\omega}{p-1}-1\right]\Lambda_1^\top\Lambda_1+i\omega\frac{1}{p-1}\Lambda_1^\top\lambda_{j,1,t}.
\end{aligned}
\tag{B.20}
$$

As in (B.8), we postulate the following ansatz for $Q_j\left(t,Z_t,\omega\right)$:

$$
Q_j\left(t,Z_t,\omega\right)=\exp\left\{\bar{A}_j\left(t,\omega\right)+\bar{B}_j\left(t,\omega\right)^\top Z_t+Z_t^\top\bar{C}_j\left(t,\omega\right)Z_t\right\}. \tag{B.21}
$$

Here, $\bar{A}_j:[0,T]\times\mathbb{R}\to\mathbb{R}$, $\bar{B}_j:[0,T]\times\mathbb{R}\to\mathbb{R}^2$, and $\bar{C}_j:[0,T]\times\mathbb{R}\to\mathbb{R}^{2\times2}$, for $j=1,2$, are deterministic functions. Following e.g. Koijen et al. (2009), we know that:

$$
\begin{aligned}
&Q_{j,t}-Q_{j,Z}^\top K_ZZ+\frac{1}{2}\mathrm{tr}\left(Q_{j,ZZ}\right)\\
&\quad-R_j\left(t,Z\right)Q_j+i\omega Q_{j,Z}^\top\Sigma_Z\left[\beta+\frac{1}{p-1}\Lambda^R\right]=0.
\end{aligned}
\tag{B.22}
$$

Then, we derive the following system of ODE's:

$$
\begin{aligned}
\bar{A}_j'\left(t\right)&=-i\omega\bar{B}_j\left(t\right)^\top\Sigma_Z\bar{\Lambda}_{0,t}^R-\frac{1}{2}\bar{B}_j\left(t\right)^\top\bar{B}_j\left(t\right)+\bar{a}_{j,t}\left(\omega\right),\\
\bar{B}_j'\left(t\right)&=\left(-\frac{i\omega}{p-1}\Lambda_1^\top\Sigma_Z^\top+K_Z^\top-2\bar{C}_j\left(t\right)^\top\right)\bar{B}_j\left(t\right)+\bar{b}_{j,t}\left(\omega\right),\\
\bar{C}_j'\left(t\right)&=2\left(K_Z^\top-\frac{i\omega}{p-1}\Lambda_1^\top\Sigma_Z^\top\right)\bar{C}_j\left(t\right)-2\bar{C}_j\left(t\right)^\top\bar{C}_j\left(t\right)-c_{j,t}\left(\omega\right),
\end{aligned}
\tag{B.23}
$$

for $j=1,2$, where we define $\bar{a}_{j,t}\left(\omega\right)=-a_{j,t}\left(\omega\right)-\mathrm{tr}\left(\bar{C}_j\left(t\right)\right)$, $\bar{b}_{j,t}\left(\omega\right)=-b_{j,t}\left(\omega\right)-2i\omega\bar{C}_j\left(t\right)^\top\Sigma_Z\bar{\Lambda}_{0,t}^R$, and $\bar{\Lambda}_{0,t}^R=\frac{1}{p-1}\left[\Lambda_0-\sigma_\Pi\right]+\beta_t$, for all $t\in[0,T]$ and $\omega\in\mathbb{R}$. Observe that: $\bar{A}_j\left(T,\omega\right)=0$, $\bar{B}_j\left(T,\omega\right)=0_2$ and $\bar{C}_j\left(T,\omega\right)=0_{2\times2}$, for $j=1,2$ and all $\omega\in\mathbb{R}$. As for the system of ODE's in (B.10), $\bar{B}_j\left(t\right)$ and $\bar{C}_j\left(t\right)$ cannot be solved in closed-form. Hence, we are solely able to state that the following[24] holds, for $j=1,2$:

$$
\bar{A}_j\left(t\right)=-\int_t^T\left[-i\omega\bar{B}_j\left(s\right)^\top\Sigma_Z\bar{\Lambda}_{0,s}^R-\frac{1}{2}\bar{B}_j\left(s\right)^\top\bar{B}_j\left(s\right)+\bar{a}_{j,s}\left(\omega\right)\right]\mathrm{d}s. \tag{B.24}
$$

[24]Note that the three deterministic functions, $\bar{A}_j$, $\bar{B}_j$, and $\bar{C}_j$, in a.o. (B.23), depend for $j=1,2$ on $\omega\in\mathbb{R}$. We have suppressed this dependency for notational simplicity.

Then, we conclude by observing that, for all $t \in [0,T]$, $\omega \in \mathbb{R}$, and $j = 1, 2$:

$$\begin{aligned}\phi_{j,T-t}(\omega,h) &= \int_{-\infty}^{\infty} e^{i\omega H}\phi_j(H,h)\,\mathrm{d}H \\ &= \mathbb{E}^{\mathbb{X}_j}\left[\left(M_T^{R^{-\frac{1}{p-1}}}Y_T\right)^{i\omega} \,\middle|\, \mathcal{F}_t\right] = Q_j(t,Z_t,\omega)\left(M_t^{R^{-\frac{1}{p-1}}}Y_t\right)^{i\omega}.\end{aligned} \tag{B.25}$$

B.2 Proof of Corollary 3.3

For the $\Lambda_1 = 0_{4\times 2}$ case, we have that $\mathbb{E}\left[\frac{Y_T M_T^R}{Y_t M_t^R} \,\middle|\, \mathcal{F}_t\right] = e^{\tilde{A}(t)+\tilde{B}(t)^\top Z_t}$, where

$$\begin{aligned}\tilde{A}(t) &= -\int_t^T \left[\tilde{B}(s)^\top \Sigma_Z c_s - \frac{1}{2}\tilde{B}(s)^\top \tilde{B}(s) + a_s\right]\mathrm{d}s, \\ \tilde{B}(t) &= K_Z^{\top^{-1}}\left[\exp\left\{-K_Z^\top (T-t)\right\} - I_{2\times 2}\right]\widehat{\delta}_{1,r},\end{aligned} \tag{B.26}$$

for all $t \in [0,T]$, in which $a_t = \widehat{\delta}_{0,r} - \alpha_t + \beta_t^\top(\Lambda_0 - \sigma_\Pi)$ and $c_t = \Lambda_0 - \sigma_\Pi - \beta_t$. Note that $\tilde{B}(t) = B^R(T-t)$. These identities follow from equations (B.4), (B.8), and (B.9). Likewise, we have that the following holds true: $\mathbb{E}\big[\big(\frac{M_T^R}{M_t^R}\big)^{\frac{p}{p-1}} \mid \mathcal{F}_t\big] = e^{\widehat{A}(t)+\widehat{B}(t)^\top Z_t}$. Suppose that $\tilde{a} = \frac{p}{p-1}\widehat{\delta}_{0,r} - \frac{1}{2}\frac{p}{(p-1)^2}(\Lambda_0 - \sigma_\Pi)^\top(\Lambda_0 - \sigma_\Pi)$. Then, from equations (B.4), (B.8), and (B.10), we know that $\widehat{A}(t)$ and $\widehat{B}(t)$ read for all $t \in [0,T]$ as follows:

$$\begin{aligned}\widehat{A}(t) &= -\int_t^T \left[\frac{p}{p-1}\widehat{B}(s)^\top \Sigma_Z[\Lambda_0 - \sigma_\Pi] - \frac{1}{2}\widehat{B}(s)^\top \widehat{B}(s) + \tilde{a}\right]\mathrm{d}s, \\ \widehat{B}(t) &= \frac{p}{p-1}K_Z^{\top^{-1}}\left[\exp\left\{-K_Z^\top(T-t)\right\} - I_{2\times 2}\right]\widehat{\delta}_{1,r}.\end{aligned} \tag{B.27}$$

Now, from (B.16), let us note that the following holds:

$$\begin{aligned}\frac{\mathrm{d}M_t^{R^{-\frac{1}{p-1}}}Y_t}{M_t^{R^{-\frac{1}{p-1}}}Y_t} &= \Bigg(\alpha_t + \frac{\widehat{\delta}_{0,r}}{p-1} + \frac{1}{2}\frac{p}{(p-1)^2}\Lambda_0^{R^\top}\Lambda_0^R + \frac{1}{p-1}\beta_t^\top\Lambda_0^R \\ &\quad + \left[\beta_t^\top + \frac{1}{p-1}\Lambda_0^{R^\top}\right]\lambda_{j,t} + \frac{1}{p-1}\widehat{\delta}_{1,r}^\top Z_t\Bigg)\mathrm{d}t + \left[\beta_t^\top + \frac{1}{p-1}\Lambda_0^{R^\top}\right]\mathrm{d}W_t^{\mathbb{X}_j},\end{aligned} \tag{B.28}$$

where $\Lambda_0^R = \Lambda_0 - \sigma_\Pi$, and in which $\lambda_{1,t} = -\big(\Lambda_0^R - \beta_t - \Sigma_Z^\top\tilde{B}(t)\big)$ and $\lambda_{2,t} = -\big(\frac{p}{p-1}\Lambda_0^R - \Sigma_Z^\top\widehat{B}(t)\big)$ for all $t \in [0,T]$. Suppose that we write the instantaneous drift term of the preceding SDE in (B.28) as follows: $\nu_{j,t} + \frac{1}{p-1}\widehat{\delta}_{1,r}^\top Z_t$, where $\nu_{j,t}$ is defined as $\nu_{j,t} = \alpha_t + \frac{\widehat{\delta}_{0,r}}{p-1} + \frac{1}{2}\frac{p}{(p-1)^2}\Lambda_0^{R^\top}\Lambda_0^R + \frac{1}{p-1}\beta_t^\top\Lambda_0^R + \left[\beta_t^\top + \frac{1}{p-1}\Lambda_0^{R^\top}\right]\lambda_{j,t}$, for $j = 1, 2$ and all $t \in [0,T]$.

Then, let us derive that the following holds true for all $t \in [0,T]$:

$$\begin{aligned}\int_t^T \frac{1}{p-1}\widehat{\delta}_{1,r}^\top Z_s\mathrm{d}s &= \frac{1}{p-1}\widehat{\delta}_{1,r}^\top K_Z^{-1}\left(I_{2\times 2} - e^{-K_Z[T-t]}\right)Z_t \\ &\quad + \frac{1}{p-1}\widehat{\delta}_{1,r}^\top \int_t^T K_Z^{-1}\left(I_{2\times 2} - e^{-K_Z[T-s]}\right)\Sigma_Z\mathrm{d}W_s.\end{aligned} \tag{B.29}$$

As $t \mapsto K_Z^{-1}\left(I_{2\times 2} - e^{-K_Z[T-t]}\right)\Sigma_Z$ clearly characterises a deterministic function of time, the latter integral is normally distributed – conditional on $\mathcal{F}_t$. Hence, we find that:

$$\log \frac{M_T^{R^{-\frac{1}{p-1}}} Y_T}{M_t^{R^{-\frac{1}{p-1}}} Y_t} = \int_t^T \left(-\frac{1}{2}\left[\beta_s + \frac{1}{p-1}\Lambda_0^R\right]^\top \left[\beta_s + \frac{1}{p-1}\Lambda_0^R\right] \right. \\ \left. + \nu_{j,s} + \frac{1}{p-1}\widehat{\delta}_{1,r}^\top Z_s \right) \mathrm{d}s + \int_t^T \left(\beta_s^\top + \frac{1}{p-1}\Lambda_0^{R^\top}\right) \mathrm{d}W_s^{\mathbb{X}_j}, \tag{B.30}$$

holds for all $t \in [0,T]$. As for the stochastic process in (B.29), we note that $t \mapsto \beta_t^\top + \frac{\Lambda_0^{R^\top}}{p-1} + \frac{\widehat{\delta}_{1,r}^\top}{p-1} K_Z^{-1}\left(I_{2\times 2} - e^{-K_Z[T-t]}\right)\Sigma_Z$ is a deterministic function of time, to conclude that the process in (B.30) is normally distributed conditional on $\mathcal{F}_t$. This concretely means that the $M_T^{R^{-\frac{1}{p-1}}} Y_T / M_t^{R^{-\frac{1}{p-1}}} Y_t$ process is log-normally distributed, conditional on $\mathcal{F}_t$, for all $t \in [0,T]$. Let us recall that: $\mathcal{A}_T = \left\{ M_T^{R^{-\frac{1}{p-1}}} Y_T / M_t^{R^{-\frac{1}{p-1}}} Y_t \geq \mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}} / M_t^{R^{-\frac{1}{p-1}}} Y_t \right\}$. Hence, we can evaluate $\mathbb{X}_1(\mathcal{A}_T \mid \mathcal{F}_t)$ and $\mathbb{X}_2(\mathcal{A}_T \mid \mathcal{F}_t)$ explicitly. That is:

$$\mathbb{X}_j(\mathcal{A}_T \mid \mathcal{F}_t) = \Phi\left(\frac{ -\log\left(\frac{\mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}}{M_t^{R^{-\frac{1}{p-1}}} Y_t} \right) + \mathbb{E}^{\mathbb{X}_j}\left[\log \frac{M_T^{R^{-\frac{1}{p-1}}} Y_T}{M_t^{R^{-\frac{1}{p-1}}} Y_t} \,\middle|\, \mathcal{F}_t \right] }{ \sqrt{ \mathbb{V}\mathrm{ar}^{\mathbb{X}_j}\left[\log \frac{M_T^{R^{-\frac{1}{p-1}}} Y_T}{M_t^{R^{-\frac{1}{p-1}}} Y_t} \,\middle|\, \mathcal{F}_t \right] } } \right). \tag{B.31}$$

In this identity, the variance term equates for $j = 1,2$ and all $t \in [0,T]$ to:

$$\mathbb{V}\mathrm{ar}^{\mathbb{X}_j}\left[\log \frac{M_T^{R^{-\frac{1}{p-1}}} Y_T}{M_t^{R^{-\frac{1}{p-1}}} Y_t} \,\middle|\, \mathcal{F}_t \right] = \int_t^T \left\| \beta_s^\top + \frac{\Lambda_0^{R^\top}}{p-1} + \frac{\widehat{\delta}_{1,r}^\top}{p-1} K_Z^{-1}\left(I_{2\times 2} - e^{-K_Z[T-s]}\right)\Sigma_Z \right\|_{\mathbb{R}^4}^2 \mathrm{d}s. \tag{B.32}$$

Here, $\|\cdot\|_{\mathbb{R}^n}$ is the ordinary n-dimensional Euclidean norm. Moreover, the expectation on the right-hand side of (B.31) reads for $j = 1,2$ and all $t \in [0,T]$ as:

$$\mathbb{E}^{\mathbb{X}_j}\left[\log \frac{M_T^{R^{-\frac{1}{p-1}}} Y_T}{M_t^{R^{-\frac{1}{p-1}}} Y_t} \,\middle|\, \mathcal{F}_t \right] = \frac{1}{p-1}\widehat{\delta}_{1,r}^\top K_Z^{-1}\left(I_{2\times 2} - e^{-K_Z[T-t]}\right) Z_t \\ + \int_t^T \left(\nu_{j,s} - \frac{1}{2}\left\| \beta_s + \frac{1}{p-1}\Lambda_0^R \right\|_{\mathbb{R}^4}^2 \right) \mathrm{d}s \\ + \frac{1}{p-1}\widehat{\delta}_{1,r}^\top \int_t^T K_Z^{-1}\left(I_{2\times 2} - e^{-K_Z[T-s]}\right)\Sigma_Z \lambda_{j,s} \mathrm{d}s. \tag{B.33}$$

B.3 Derivation of (3.8)

Let us define $\widehat{f}(T, G_T, J_T) = M_T^{R^{\frac{p}{p-1}}}\left(M_T^{R^{-\frac{1}{p-1}}} Y_T - \mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}\right)\mathbb{1}_{\{\mathcal{A}_T\}}$, where $G_T = \log M_T^{R^{-\frac{1}{p-1}}} Y_T$ and $J_T = \log M_T^{R^{\frac{p}{p-1}}}$. As in Appendix B.1, we can therefore state that

$X_t^{\mathrm{opt}} M_t = \widehat{f}(t,g,j) = \mathbb{E}\left[\widehat{f}(T,G_T,J_T) \,\middle|\, G_t = g,\ J_t = j\right]$ holds for all $t \in [0,T]$. By an application of the Fourier transform to this function, we find:

$$\begin{aligned}
\widehat{f}(t,g,j) &= \mathbb{E}\left[\widehat{f}(T,G_T,J_T) \,\middle|\, G_t = g,\ J_t = j\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{f}(T,G,J)\,\phi(G,J,g,j)\,\mathrm{d}G\mathrm{d}J \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}_\kappa^*(T,\omega)\,\widehat{\phi}_{T-t}(-\omega - i\kappa, g, j)\,\mathrm{d}\omega,
\end{aligned} \tag{B.34}$$

for all $t \in [0,T]$, where $\widehat{f}_\kappa^*(T,\omega)$ is for some $\kappa > 1$ and all $\omega \in \mathbb{R}$ given by:

$$\begin{aligned}
\widehat{f}_\kappa^*(T,\omega) &= \int_{-\infty}^{\infty} e^{(i\omega-\kappa)G}\,\widehat{f}(T,G,J)\,\mathrm{d}G \\
&= \int_{-\infty}^{\infty} e^{(i\omega-\kappa)G}\left(e^G - \mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}\right)\mathbb{1}_{\left\{G \geq \frac{1}{p-1}\log\mathcal{H}^{-1}(X_0)\right\}}\mathrm{d}G \\
&= -\frac{e^{(i\omega-\kappa+1)\frac{1}{p-1}\log\mathcal{H}^{-1}(X_0)}}{i\omega-\kappa+1} + \mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}\,\frac{e^{(i\omega-\kappa)\frac{1}{p-1}\log\mathcal{H}^{-1}(X_0)}}{i\omega-\kappa}.
\end{aligned} \tag{B.35}$$

Obviously, $\widehat{f}_\kappa^*(T,\omega)$ represents the Fourier transform of $e^{-\kappa G_T}e^{-J_T}\widehat{f}(T,G_T,J_T)$ with respect to G_T. Moreover, we have: $\widehat{\phi}_{T-t}(\omega,g,j) = \int_{\mathbb{R}^2} e^J e^{i\omega G}\phi(G,J,g,j)\,\mathrm{d}G\mathrm{d}J = \mathbb{E}\left[e^{J_T}e^{i\omega G_T} \,\middle|\, G_t = g,\ J_t = j\right] = \mathbb{E}\left[M_T^{R\,\frac{p-i\omega}{p-1}} Y_T^{i\omega} \,\middle|\, \mathcal{F}_t\right]$, for all $t \in [0,T]$, in which $\phi(G,J,g,j)$ characterises the conditional joint density corresponding to G_T and J_T.

As the characterisation of $\widehat{\phi}_{T-t}(\omega,g,j)$ is similar to the one of $\phi_{T-t}(\omega,g,j)$ in Appendix B.1, we omit an elaborate derivation. Define the deterministic function $(t,\omega) \mapsto \widehat{a}_t(\omega)$:

$$\begin{aligned}
\widehat{a}_t(\omega) &= i\omega\alpha_t + \frac{1}{2}i\omega[i\omega - 1]\beta_t^\top\beta_t - \frac{p-i\omega}{p-1}\widehat{\delta}_{0,r} \\
&\quad + \frac{1}{2}\frac{p-i\omega}{p-1}\left[\frac{p-i\omega}{p-1} - 1\right]\Lambda_0^{R\top}\Lambda_0^R - \frac{(p-i\omega)\,i\omega}{p-1}\beta_t^\top\Lambda_0^R.
\end{aligned} \tag{B.36}$$

Note that $\widehat{a}_t(\omega) \in \mathbb{R}$. In addition to this, introduce the following two deterministic functions, $(t,\omega) \mapsto \widehat{b}_t(\omega)$ and $(t,\omega) \mapsto \widehat{c}_t(\omega)$, with $\widehat{b}_t(\omega) \in \mathbb{R}^2$ and $\widehat{c}_t(\omega) \in \mathbb{R}^{2\times 2}$:

$$\begin{aligned}
\widehat{b}_t(\omega) &= \frac{p-i\omega}{p-1}\left(-\widehat{\delta}_{1,r} + \left[\frac{p-i\omega}{p-1} - 1\right]\Lambda_1^\top\Lambda_0^R - i\omega\Lambda_1^\top\beta_t\right), \\
\widehat{c}_t(\omega) &= \frac{1}{2}\frac{p-i\omega}{p-1}\left[\frac{p-i\omega}{p-1} - 1\right]\Lambda_1^\top\Lambda_1.
\end{aligned} \tag{B.37}$$

Then, postulate the next ansatz for $\widehat{Q}(t,Z_t,\omega) = \widehat{\phi}_{T-t}(\omega,g,j)$:

$$\widehat{Q}(t,Z_t,\omega) = \exp\left\{\widehat{A}_{\widehat{Q}}(t,\omega) + \widehat{B}_{\widehat{Q}}(t,\omega)^\top Z_t + Z_t^\top\widehat{C}_{\widehat{Q}}(t,\omega)\,Z_t\right\}, \tag{B.38}$$

for all $t \in [0,T]$ and $\omega \in \mathbb{R}$. We assume that $\widehat{A}_{\widehat{Q}} : [0,T]\times\mathbb{R} \to \mathbb{R}$, $\widehat{B}_{\widehat{Q}} : [0,T]\times\mathbb{R} \to \mathbb{R}^2$, and $\widehat{C}_{\widehat{Q}} : [0,T] \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ are deterministic functions of time $t \in [0,T]$ and $\omega \in \mathbb{R}$ alone. As $\widehat{Q}(t, Z_t, \omega) M_t^{R^{\frac{p-i\omega}{p-1}}} Y_t^{i\omega}$ is a $\mathbb{P}$-martingale, we derive the following system of ODE's:

$$\begin{aligned}
\widehat{A}'_{\widehat{Q}}(t) &= -\widehat{B}_{\widehat{Q}}(t)^\top \Sigma_Z \widehat{\Lambda}^R_{\widehat{Q},t}(\omega) - \frac{1}{2}\widehat{B}_{\widehat{Q}}(t)^\top \widehat{B}_{\widehat{Q}}(t) + \bar{a}_{\widehat{Q},t}(\omega), \\
\widehat{B}'_{\widehat{Q}}(t) &= \left(\frac{p-i\omega}{p-1}\Lambda_1^\top \Sigma_Z^\top + K_Z^\top - 2\widehat{C}_{\widehat{Q}}(t)^\top\right)\widehat{B}_{\widehat{Q}}(t) + \bar{b}_{\widehat{Q},t}(\omega), \\
\widehat{C}'_{\widehat{Q}}(t) &= 2\left(K_Z^\top + \frac{p-i\omega}{p-1}\Lambda_1^\top \Sigma_Z^\top\right)\widehat{C}_{\widehat{Q}}(t) - 2\widehat{C}_{\widehat{Q}}(t)^\top \widehat{C}_{\widehat{Q}}(t) - \widehat{c}_t(\omega),
\end{aligned} \tag{B.39}$$

where we define $\bar{a}_{\widehat{Q},t}(\omega) = -\widehat{a}_t(\omega) - \operatorname{tr}\left(\widehat{C}_{\widehat{Q}}(t)\right)$, $\bar{b}_{\widehat{Q},t}(\omega) = -\widehat{b}_t(\omega) - 2\widehat{C}_{\widehat{Q}}(t)^\top \Sigma_Z \widehat{\Lambda}^R_{\widehat{Q},t}(\omega)$, $\widehat{\Lambda}^R_{\widehat{Q},t}(\omega) = -\frac{p-i\omega}{p-1}[\Lambda_0 - \sigma_\Pi] + i\omega\beta_t$, for all $t \in [0,T]$ and $\omega \in \mathbb{R}$. Note here that we suppress the dependencies of $\widehat{A}_{\widehat{Q}}$, $\widehat{B}_{\widehat{Q}}$, and $\widehat{C}_{\widehat{Q}}$ on $\omega \in \mathbb{R}$. Moreover, we have that: $\widehat{A}_{\widehat{Q}}(T,\omega) = 0$, $\widehat{B}_{\widehat{Q}}(T,\omega) = 0_2$, and $\widehat{C}_{\widehat{Q}}(T,\omega) = 0_{2\times 2}$, for all $\omega \in \mathbb{R}$. Note the similarity between this system and the system in (B.23). Therefore, we only know that:

$$\widehat{A}_{\widehat{Q}}(t) = -\int_t^T \left[-\widehat{B}_{\widehat{Q}}(s)^\top \Sigma_Z \widehat{\Lambda}^R_{\widehat{Q},s}(\omega) - \frac{1}{2}\widehat{B}_{\widehat{Q}}(s)^\top \widehat{B}_{\widehat{Q}}(s) + \bar{a}_{\widehat{Q},s}(\omega)\right] \mathrm{d}s. \tag{B.40}$$

Then, we are able to conclude by deriving that, for all $t \in [0,T]$ and $\omega \in \mathbb{R}$:

$$\begin{aligned}
\widehat{\phi}_{T-t}(\omega, g, j) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^J e^{i\omega G}\phi(G,J,g,j)\,\mathrm{d}G\mathrm{d}J \\
&= \mathbb{E}\left[M_T^{R^{\frac{p-i\omega}{p-1}}} Y_T^{i\omega} \,\middle|\, \mathcal{F}_t\right] = \widehat{Q}(t, Z_t, \omega) M_t^{R^{\frac{p-i\omega}{p-1}}} Y_t^{i\omega}.
\end{aligned} \tag{B.41}$$

C Proofs II

C.1 Proof of Proposition 3.4

Let us start by noting that $X_t^{\mathrm{opt}} M_t$ is for all $t \in [0,T]$ given by:

$$\begin{aligned}
X_t^{\mathrm{opt}} M_t = {} & Y_t M_t^R P_1(t, Z_t) \frac{1}{2\pi}\int_{-\infty}^{\infty} f_\kappa^*(T,\omega)\,\phi_{1,T-t}(-\omega - i\kappa, h)\,\mathrm{d}\omega \\
& - \left(\mathcal{H}^{-1}(X_0)^{\frac{1}{p}} M_t^R\right)^{\frac{p}{p-1}} P_2(t, Z_t) \frac{1}{2\pi}\int_{-\infty}^{\infty} f_\kappa^*(T,\omega)\,\phi_{2,T-t}(-\omega - i\kappa, h)\,\mathrm{d}\omega.
\end{aligned} \tag{C.1}$$

To identify ψ_t in (3.2), we must find the diffusion coefficients for the latter process $X_t^{\mathrm{opt}} M_t$. Making use of a simple application of Itô's Lemma, we find the subsequent SDE's:

$$\begin{aligned}
\frac{\mathrm{d}Y_t M_t^R P_1}{Y_t M_t^R P_1} &= \left(-\left[\Lambda_t^R - \beta_t\right]^\top + \tilde{B}(t)^\top \Sigma_Z\right)\mathrm{d}W_t, \\
\frac{\mathrm{d}M_t^{R^{\frac{p}{p-1}}} P_2}{M_t^{R^{\frac{p}{p-1}}} P_2} &= \left(-\frac{p}{p-1}\Lambda_t^{R^\top} + \left[\widehat{B}(t) + 2\widehat{C}(t) Z_t\right]^\top \Sigma_Z\right)\mathrm{d}W_t.
\end{aligned} \tag{C.2}$$

Now, define: $R_j\big(t, Z_t, M_t^{R^{\frac{p}{p-1}}} Y_t\big) = R_j\big(t, G_t\big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_\kappa^*(T,\omega)\,\phi_{j,T-t}(-\omega - i\kappa, h)\,\mathrm{d}\omega$, for $j = 1, 2$, and all $t \in [0, T]$. The SDE's of the latter processes are given by:

$$\begin{aligned}
\mathrm{d}R_j &= \frac{\partial R_j}{\partial t} + (\nabla_G R_j)^\top \mathrm{d}G_t + \frac{1}{2}(\mathrm{d}G_t)^\top (H_G R_j)\,\mathrm{d}G_t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(f_\kappa^*(T,\omega)\,\phi_{j,T-t}(-\omega - i\kappa, h)\Big[\bar{B}_j(t)^\top \Sigma_Z \\
&\quad + 2Z_t^\top \bar{C}_j(t)^\top \Sigma_Z + i(-\omega - i\kappa)\Big(\beta_t^\top + \frac{1}{p-1}\Lambda_t^{R^\top}\Big)\Big]\Big)\,\mathrm{d}\omega\mathrm{d}W_t.
\end{aligned} \tag{C.3}$$

Note that $t \mapsto \bar{B}_j(t)$ and $t \mapsto \bar{C}_j(t)$ are assumed to incorporate the $-\omega - i\kappa$ argument.

Combining terms, we consequently find that the following holds:

$$\begin{aligned}
\mathrm{d}Y_t M_t^R P_1 R_1 &= Y_t M_t^R P_1 \Big[\Big(-\big[\Lambda_t^R - \beta_t\big]^\top + \tilde{B}(t)^\top \Sigma_Z\Big) R_1 \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(f_\kappa^*(T,\omega)\,\phi_{1,T-t}(-\omega - i\kappa, h)\Big[\bar{B}_1(t)^\top \Sigma_Z \\
&\quad + 2Z_t^\top \bar{C}_1(t)^\top \Sigma_Z + i(-\omega - i\kappa)\Big(\beta_t^\top + \frac{1}{p-1}\Lambda_t^{R^\top}\Big)\Big]\Big)\,\mathrm{d}\omega\Big]\,\mathrm{d}W_t.
\end{aligned} \tag{C.4}$$

Likewise, we are able to derive the subsequent SDE:

$$\begin{aligned}
\mathrm{d}M_t^{R^{\frac{p}{p-1}}} P_2 R_2 &= M_t^{R^{\frac{p}{p-1}}} P_2 \Big[\Big(-\frac{p}{p-1}\Lambda_t^{R^\top} + \Big[\widehat{B}(t)^\top + 2Z_t^\top \widehat{C}(t)^\top\Big]\Sigma_Z\Big) R_2 \\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(f_\kappa^*(T,\omega)\,\phi_{2,T-t}(-\omega - i\kappa, h)\Big[\bar{B}_2(t)^\top \Sigma_Z \\
&\quad + 2Z_t^\top \bar{C}_2(t)^\top \Sigma_Z + i(-\omega - i\kappa)\Big(\beta_t^\top + \frac{1}{p-1}\Lambda_t^{R^\top}\Big)\Big]\Big)\,\mathrm{d}\omega\Big]\,\mathrm{d}W_t.
\end{aligned} \tag{C.5}$$

The sum of the SDE's in (C.4) and (C.5) is identical to the SDE of $X_t^{\mathrm{opt}} M_t$. To conclude the proof of Corollary 3.5, we observe that ψ_t in (3.2) is straightforwardly identical to the diffusion coefficient of the SDE for $X_t^{\mathrm{opt}} M_t$. After appropriately re-arranging terms, one is able to disentangle the analytical portfolio demands that are reported in Proposition 3.4. As for the former, we have the following, on the basis of which ψ_t can be retrieved:

$$\begin{aligned}
\psi_t^\top \mathrm{d}W_t &= \mathrm{d}Y_t M_t^R P_1(t, Z_t)\, R_1\Big(t, Z_t, M_t^{R^{\frac{p}{p-1}}} Y_t\Big) \\
&\quad - \mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}\, \mathrm{d}M_t^{R^{\frac{p}{p-1}}} P_2(t, Z_t)\, R_2\Big(t, Z_t, M_t^{R^{\frac{p}{p-1}}} Y_t\Big).
\end{aligned} \tag{C.6}$$

C.2 Proof of Corollary 3.5

According to the Clark-Ocone formula, we have:

$$X_T^{\mathrm{opt}} M_T = X_0 + \int_0^T \mathbb{E}\left[\mathcal{D}_t^W X_T^{\mathrm{opt}} M_T \mid \mathcal{F}_t\right]^\top \mathrm{d}W_t, \tag{C.7}$$

cf. Karatzas et al. (1991) and Ocone and Karatzas (1991). Here, $\mathcal{D}_t^W : \mathbb{D}^{1,2}([0,T]) \to L^2(\Omega \times [0,T])^4$ represents the Mallivain derivative kernel, where $\mathbb{D}^{1,2}([0,T])$ stands for the Sobolev-Watanabe space of all $L^2(\Omega \times [0,T])$-valued Mallivain differentiable processes. Hence, from (C.7): $\psi_t = \mathbb{E}\left[\mathcal{D}_t^W X_T^{\text{opt}} M_T \mid \mathcal{F}_t\right]$ holds for all $t \in [0,T]$. Then, we note that:

$$\begin{aligned}\mathcal{D}_t^W X_T^{\text{opt}} M_T &= M_T \mathcal{D}_t^W \left(Y_T \Pi_T - \left(\mathcal{H}^{-1}(X_0) M_T \Pi_T\right)^{\frac{1}{p-1}} \Pi_T\right) \mathbb{1}_{\{\mathcal{A}_T\}} \\ &\quad + \left(Y_T \Pi_T - \left(\mathcal{H}^{-1}(X_0) M_T \Pi_T\right)^{\frac{1}{p-1}} \Pi_T\right) \mathbb{1}_{\{\mathcal{A}_T\}} \mathcal{D}_t^W M_T.\end{aligned} \tag{C.8}$$

We are able to derive that the following holds true:

$$\begin{aligned}\mathcal{D}_t^W M_T &= M_T \left(-\int_t^T \mathcal{D}_t^W r_s \mathrm{d}s - \int_t^T \left(\mathcal{D}_t^W \Lambda_s\right) \Lambda_s \mathrm{d}s - \mathcal{D}_t^W \int_0^T \Lambda_s^\top \mathrm{d}W_s\right) \\ &= M_T \left(-\left[\delta_{1,r}^\top K_Z^{-1}\left[I_{2\times 2} - \exp\{-K_Z(T-t)\}\right]\Sigma_Z\right]^\top - \Lambda_0\right).\end{aligned} \tag{C.9}$$

Along the same vein, we have:

$$\begin{aligned}\mathcal{D}_t^W Y_T \Pi_T &= Y_T \Pi_T \left(\int_t^T \mathcal{D}_t^W \left[\alpha_s + \pi_s + \sigma_\Pi^\top \beta_s\right] \mathrm{d}s\right. \\ &\quad \left. -\frac{1}{2}\mathcal{D}_t^W \int_0^T \|\sigma_\Pi + \beta_s\|_{\mathbb{R}^4}^2 \,\mathrm{d}s + \mathcal{D}_t^W \int_0^T \left[\sigma_\Pi + \beta_s\right]^\top \mathrm{d}W_s\right) \\ &= Y_T \Pi_T \left(\left[\delta_{1,\pi}^\top K_Z^{-1}\left[I_{2\times 2} - \exp\{-K_Z(T-t)\}\right]\Sigma_Z\right]^\top + \sigma_\Pi + \beta_t\right).\end{aligned} \tag{C.10}$$

Ultimately, we combine all preceding arguments to derive the relevant Malliavin derivative of $\left(\mathcal{H}^{-1}(X_0) M_T \Pi_T\right)^{\frac{1}{p-1}} \Pi_T$. After re-arranging terms, we find the portfolio weights provided in Corollary 3.5. The former reads for all $t \in [0,T]$:

$$\begin{aligned}\frac{\mathcal{D}_t^W M_T^{R^{\frac{1}{p-1}}} \Pi_T}{M_T^{R^{\frac{1}{p-1}}} \Pi_T} &= \mathcal{D}_t^W \log \mathcal{D}_t^W M_T^{R^{\frac{1}{p-1}}} \Pi_T = \int_t^T \mathcal{D}_t^W \left[\pi_s - \frac{1}{p-1} R_s\right] \\ &\quad + \mathcal{D}_t^W \int_0^T \left[\sigma_\Pi - \frac{1}{p-1}\Lambda_s^R\right]^\top \mathrm{d}W_t = \sigma_\Pi - \frac{1}{p-1}\left(\Lambda_0 - \sigma_\Pi\right) \\ &\quad + \left[\left(\delta_{1,\pi} - \frac{1}{p-1}\widehat{\delta}_{1,r}\right)^\top K_Z^{-1}\left[I_{2\times 2} - \exp\{-K_Z(T-t)\}\right]\Sigma_Z\right]^\top.\end{aligned} \tag{C.11}$$

C.3 Derivation of (3.13)

The closed-form specification of $X_t^{\text{opt}} M_t$ implicit in (3.8) reads as:

$$\begin{aligned}X_t^{\text{opt}} M_t &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}_\kappa^*(T,\omega)\, \widehat{\phi}_{T-t}(-\omega - i\kappa, g, j)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}_\kappa^*(T,\omega)\, \widehat{Q}(t, Z_t, -\omega - i\kappa)\, M_t^{R^{\frac{p-i(-\omega-i\kappa)}{p-1}}} Y_t^{i(-\omega-i\kappa)}\mathrm{d}\omega,\end{aligned} \tag{C.12}$$

for all $t \in [0,T]$. An application of Itô's Lemma to $X_t^{\text{opt}} M_t$ in (C.12) suffices to conclude the proof. Applying Itô's Lemma to (C.12) yields us the following:

$$\begin{aligned} \mathrm{d}X_t^{\text{opt}} M_t = & \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}_\kappa^* (T,\omega) \widehat{Q} (t, Z_t, -\omega - i\kappa) M_t^{R^{\frac{p-i(-\omega-i\kappa)}{p-1}}} Y_t^{i(-\omega-i\kappa)} \\ & \left[\left(\widehat{B}_{\widehat{Q}} (t, -\omega - i\kappa) + 2\widehat{C}_{\widehat{Q}} (t, -\omega - i\kappa) Z_t \right)^\top \Sigma_Z \right. \\ & \left. + \left(i [-\omega - i\kappa] \beta_t - \frac{p - i[-\omega - i\kappa]}{p-1} \Lambda_t^R \right)^\top \right] \mathrm{d}\omega \mathrm{d}W_t. \end{aligned} \tag{C.13}$$

D Proofs III

D.1 Derivation of (4.3)

We first determine the relevant expectations for $\widehat{a}_T$. Its first moment reads:

$$\begin{aligned} \mathbb{E}[\widehat{a}_T] &= C \exp\{\bar{\alpha}\} \mathbb{E}\left[\exp\left\{ \bar{\beta}^\top \int_0^T e^{-K_Z(T-s)} \Sigma_Z \mathrm{d}W_s \right\} \right] \\ &= C \exp\left\{ \bar{\alpha} + \frac{1}{2} \bar{\beta}^\top \left(\int_0^T e^{-K_Z(T-s)} \left[e^{-K_Z(T-s)} \right]^\top \mathrm{d}s \right) \bar{\beta} \right\}. \end{aligned} \tag{D.1}$$

Now, define the change of measure from $\mathbb{P}$ to $\mathbb{W}$ (with $\mathbb{W} \sim \mathbb{P}$), induced by the following Radon-Nikodym derivative: $\frac{\mathrm{d}\mathbb{W}}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t} = e^{-\frac{1}{2}\bar{\beta}^\top \int_0^t e^{K_Z(T-s)} \left(e^{K_Z(T-s)}\right)^\top \mathrm{d}s + \bar{\beta}^\top \int_0^t e^{K_Z(T-s)} \Sigma_Z \mathrm{d}W_s}$, for all $t \in [0,T]$. Straightforwardly, under the $\mathbb{W}$ measure, the following process is a standard Brownian motion: $W_t^{\mathbb{W}} = W_t - \int_0^t \Sigma_Z^\top \left(e^{K_Z(T-s)}\right)^\top \bar{\beta} \mathrm{d}s$, for all $t \in [0,T]$. Using this change of measure, we find that $\mathbb{E}[\widehat{a}_T Z_T]$ reduces to the following:

$$\begin{aligned} \mathbb{E}[\widehat{a}_T Z_T] &= C\mathbb{E}[\widehat{a}_T] \underbrace{\mathbb{E}\left[\frac{\widehat{a}_T}{\mathbb{E}[\widehat{a}_T]} Z_T \right]}_{=\mathbb{E}^{\mathbb{W}}[Z_T]} \\ &= C\mathbb{E}[\widehat{a}_T] \left(\int_0^T e^{-K_Z(T-s)} \left[e^{K_Z(T-s)} \right]^\top \mathrm{d}s \right) \bar{\beta}. \end{aligned} \tag{D.2}$$

Then, we turn to the relevant expectations for a_T. Similar to the identity in (D.1), we are able to make use of the log-normality of $\frac{P_{T,T+i}}{\Pi_T}$ to derive its first moment. In concrete terms, its first moment is given by:

$$\mathbb{E}[a_T] = C \sum_{i=1}^{\tau_A} \mathbb{E}\left[\frac{P_{T,T+i}}{\Pi_T} \right], \tag{D.3}$$

in which the incorporated expectation reads for all $i = 1, \dots, \tau_A$:

$$\mathbb{E}\left[\frac{P_{T,T+i}}{\Pi_T} \right] = \exp\left\{ A^R(i) + \frac{1}{2} B^R(i)^\top \left(\int_0^T e^{-K_Z(T-s)} \left[e^{-K_Z(T-s)} \right]^\top \mathrm{d}s \right) B^R(i) \right\}. \tag{D.4}$$

Following the derivation in (D.2), introduce the subsequent Radon-Nikodym derivative: $\frac{\mathrm{d}\mathbb{W}_i}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t} = \frac{\mathrm{d}\mathbb{W}}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t,\bar{\beta}=B^R(i)}$, for all $t \in [0,T]$ and $i = 1,\ldots,\tau_A$. This process corresponds to a change of measure from $\mathbb{P}$ to $\mathbb{W}_i$ (with $\mathbb{W}_i \sim \mathbb{P}$), such that $W^{\mathbb{W}_i} = W_t - \int_0^t \Sigma_Z^\top \left(e^{K_Z(T-s)}\right)^\top B^R(i)\,\mathrm{d}s$ is for all $t \in [0,T]$ and $i = 1,\ldots,\tau_A$ a $\mathbb{W}_i$-standard Brownian motion. Then, we have that $\mathbb{E}[a_T Z_T]$ is specified as follows:

$$\begin{aligned}\mathbb{E}[a_T Z_T] &= C\sum_{i=1}^{\tau}\mathbb{E}\left[\frac{P_{T,T+i}}{\Pi_T}\right]\underbrace{\mathbb{E}\left[\frac{P_{T,T+i}/\Pi_T}{\mathbb{E}[P_{T,T+i}/\Pi_T]}Z_T\right]}_{=\mathbb{E}^{\mathbb{W}_i}[Z_T]} \\ &= C\sum_{i=1}^{\tau}\mathbb{E}\left[\frac{P_{T,T+i}}{\Pi_T}\right]\left(\int_0^T e^{-K_Z(T-s)}\left[e^{K_Z(T-s)}\right]^\top \mathrm{d}s\right)B^R(i).\end{aligned} \tag{D.5}$$

Finally, (4.3) follows from solving the following system:

$$\begin{aligned}\bar{\alpha} + \frac{1}{2}\int_0^T \bar{\beta}^\top e^{-K_Z(T-s)}\left[e^{-K_Z(T-s)}\right]^\top \bar{\beta}\mathrm{d}s &= \log\frac{\mathbb{E}[a_T]}{C}, \\ \left(\int_0^T e^{-K_Z(T-s)}\left[e^{K_Z(T-s)}\right]^\top \mathrm{d}s\right)\bar{\beta} &= \frac{\mathbb{E}[a_T Z_T]}{\mathbb{E}[a_T]}.\end{aligned} \tag{D.6}$$

D.2 Proof of Proposition 4.1

Define $\tilde{f}(T,H_T) = \frac{X_T^{\mathrm{opt}}}{\Pi_T}\frac{1}{Y_T} = \left(1 - \mathcal{H}^{-1}(X_0)^{\frac{1}{p-1}}\frac{M_T^{R\,\frac{1}{p-1}}}{Y_T}\right)\mathbb{1}_{\mathcal{A}_T}$, where $H_T = \log\left(M_T^{R^{-\frac{1}{p-1}}}Y_T\right)$. Then, we know that $F_{X/Y}(x) = \mathbb{E}\big[\mathbb{1}_{\{\tilde{f}(T,H_T)\le x\}}\;\big|\; H_0 = h\big]$ holds for all $x \in \mathbb{R}$. Resorting to an application of the Fourier transform, we derive for all $x \in [0,1)$:

$$\begin{aligned}F_{X/Y}(x) &= \mathbb{E}\left[\mathbb{1}_{\{\tilde{f}(T,H_T)\le x\}}\;\middle|\; H_0 = h\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}_\kappa^*(T,x,\omega)\,\phi_T(-\omega - i\kappa, h)\,\mathrm{d}\omega.\end{aligned} \tag{D.7}$$

Here, $\tilde{f}_\kappa^*(T,\omega)$ is for some $\kappa < 0$, all $\omega \in \mathbb{R}$ and all $x \in [0,1)$ given by:

$$\begin{aligned}\tilde{f}_\kappa^*(T,x,\omega) &= \int_{-\infty}^{\infty} e^{(i\omega-\kappa)H}\mathbb{1}_{\{\tilde{f}(T,H)\le x\}}\mathrm{d}H \\ &= \int_{-\infty}^{\frac{\log\mathcal{H}^{-1}(X_0)}{p-1}} e^{(i\omega-\kappa)H}\mathrm{d}H + \int_{\frac{\log\mathcal{H}^{-1}(X_0)}{p-1}}^{\frac{\log\mathcal{H}^{-1}(X_0)}{p-1}-\log(1-x)} e^{(i\omega-\kappa)H}\mathrm{d}H \\ &= \frac{1}{i\omega-\kappa}e^{(i\omega-\kappa)\left[\frac{1}{p-1}\log\mathcal{H}^{-1}(X_0)-\log(1-x)\right]}.\end{aligned} \tag{D.8}$$

Clearly, $\tilde{f}_\kappa^*(T,x,\omega)$ is the Fourier transform of $e^{-\kappa H_T}\mathbb{1}_{\{\tilde{f}(T,H)\le x\}}$ with respect to H_T.

Moreover, we have that ϕ_T is for all $\omega \in \mathbb{R}$ specified as follows:

$$\phi_T(\omega,h) = \int_{-\infty}^{\infty} e^{i\omega H}\phi(H,h)\,\mathrm{d}H = \mathbb{E}\left[M_T^{R^{-\frac{i\omega}{p-1}}}Y_T^{i\omega}\right]. \tag{D.9}$$

Note the similarity between ϕ_T and $\phi_{j,T-t}$ in (B.25). By virtue of this similarity, we omit an elaborate derivation for the evaluation of the preceding expectation.

As in (B.23), we then introduce the following system of ODE's:

$$\begin{aligned}\bar{A}'(t) &= -i\omega\bar{B}(t)^\top \Sigma_Z \bar{\Lambda}^R_{0,t} - \frac{1}{2}\bar{B}(t)^\top \bar{B}(t) + \bar{a}_t(\omega),\\ \bar{B}'(t) &= \left(-\frac{i\omega}{p-1}\Lambda_1^\top\Sigma_Z^\top + K_Z^\top - 2\bar{C}(t)^\top\right)\bar{B}(t) + \bar{b}_t(\omega),\\ \bar{C}'(t) &= 2\left(K_Z^\top - \frac{i\omega}{p-1}\Lambda_1^\top\Sigma_Z^\top\right)\bar{C}(t) - 2\bar{C}(t)^\top\bar{C}(t) - c_t(\omega).\end{aligned} \tag{D.10}$$

Here, $\bar{a}_t(\omega) = -a_t(\omega) - \operatorname{tr}\left(\bar{C}(t)\right)$ holds for all $t \in [0,T]$ and $\omega \in \mathbb{R}$, with:

$$\begin{aligned}a_t(\omega) = {} & i\omega\alpha_t + \frac{1}{2}i\omega\left[i\omega - 1\right]\beta_t^\top\beta_t + \frac{i\omega}{p-1}\widehat{\delta}_{0,r}\\ & - \frac{1}{2}\frac{i\omega}{p-1}\left[-\frac{i\omega}{p-1} - 1\right]\Lambda_0^{R^\top}\Lambda_0^R + \frac{(i\omega)^2}{p-1}\beta_t^\top\Lambda_0^R.\end{aligned} \tag{D.11}$$

In addition to this, $\bar{b}_t(\omega) = -b_t(\omega) - 2i\omega\bar{C}(t)^\top \Sigma_Z\bar{\Lambda}^R_{0,t}$ and $\bar{\Lambda}^R_{0,t} = \frac{1}{p-1}\left[\Lambda_0 - \sigma_\Pi\right] + \beta_t$ hold for all $t \in [0,T]$ and $\omega \in \mathbb{R}$, where we have the following:

$$\begin{aligned}b_t(\omega) &= \frac{-i\omega}{p-1}\left(-\widehat{\delta}_{1,r} + \left[\frac{-i\omega}{p-1} - 1\right]\Lambda_1^\top\Lambda_0^R - i\omega\Lambda_1^\top\beta_t\right),\\ c_t(\omega) &= -\frac{1}{2}\frac{i\omega}{p-1}\left[-\frac{i\omega}{p-1} - 1\right]\Lambda_1^\top\Lambda_1.\end{aligned} \tag{D.12}$$

The system of ODE's in (D.10) is given for the deterministic functions, $\bar{A} : [0,T]\times\mathbb{R} \to \mathbb{R}$, $\bar{B} : [0,T]\times\mathbb{R} \to \mathbb{R}^2$, and $\bar{C} : [0,T]\times\mathbb{R} \to \mathbb{R}^{2\times 2}$. Note that we suppress the dependencies of these functions on $\omega \in \mathbb{R}$ in (D.10). Furthermore, we emphasise that $\bar{A}(T) = 0$, $\bar{B}(T) = 0_2$ and $\bar{C}(T) = 0_{2\times 2}$ hold. Observe the similarity between this system and the system in (B.39). As a consequence, we are merely able to infer that $\bar{A}(t)$ reads:

$$\bar{A}(t) = -\int_t^T \left[-i\omega\bar{B}(t)^\top \Sigma_Z\bar{\Lambda}^R_{0,t} - \frac{1}{2}\bar{B}(t)^\top\bar{B}(t) + \bar{a}_t(\omega)\right]\mathrm{d}s. \tag{D.13}$$

Hence: $\phi_T(\omega,h) = \int_{-\infty}^{\infty} e^{i\omega H}\phi(H,h)\,\mathrm{d}H = e^{\tilde{A}(0,\omega)}Y_0^{i\omega}$, where we use that $M_0^R = 1$ holds.

We conclude the proof with the derivation of $f_{X/Y}(x)$ for all $x \in (0,1)$:

$$\begin{aligned}f_{X/Y}(x) &= \frac{\partial}{\partial x}F_{X/Y}(x)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\tilde{f}^*_\kappa(T,x,\omega)\,\phi_T(-\omega - i\kappa,h)\,\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\omega-\kappa}{1-x}\tilde{f}^*_\kappa(T,x,\omega)\,\phi_T(-\omega - i\kappa,h)\,\mathrm{d}\omega.\end{aligned} \tag{D.14}$$

D.3 Proof of Corollary 4.2

By the law of total probability for continuous functions, we have:

$$\begin{aligned}\mathbb{P}\left(\frac{X_T}{\Pi_T}\frac{1}{Y_T} \leq x\right) &= \int_{-\infty}^{\frac{1}{p-1}\log\eta^{\mathrm{opt}}} \mathbb{P}\left(0 \leq x\right) f_H\left(h\right)\mathrm{d}h \\ &+ \int_{\frac{1}{p-1}\log\eta^{\mathrm{opt}}}^{\infty} \mathbb{P}\left(1 - \eta^{\mathrm{opt}\frac{1}{p-1}} e^{-h} \leq x\right) f_H\left(h\right)\mathrm{d}h.\end{aligned} \tag{D.15}$$

Here, we define $h \mapsto f_h(h)$ as the PDF of $H_T = \log M_T^{R^{-\frac{1}{p-1}}} Y_T$. Observe that this PDF is known for $\Lambda_1 = 0_{4\times 2}$. Moreover, for the sake of notational elegance, we make use of the fact that $\eta^{\mathrm{opt}} = \mathcal{H}^{-1}(X_0)$. Suppose that $h \mapsto F_H(h)$ denotes the CDF corresponding to the latter random variable. Then, $F_{X/Y}(x)$ reads for all $x \in [0,1)$ as follows:

$$F_{X/Y}(x) = 1 - F_H\left(\log\frac{1-x}{\eta^{\mathrm{opt}\frac{1}{p-1}}}\right) = 1 - \mathbb{P}\left(e^{H_T} \leq \frac{1-x}{\eta^{\mathrm{opt}\frac{1}{p-1}}}\right). \tag{D.16}$$

Using the fact that $M_T^{R^{-\frac{1}{p-1}}} Y_T$ is log-normally distributed (cf. Appendix B.2), we can immediately evaluate the latter probability as follows for all $x \in [0,1)$:

$$F_{X/Y}(x) = \Phi\left(\frac{\log\frac{\eta^{\mathrm{opt}\frac{1}{p-1}}}{1-x} - \mathbb{E}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right]}{\sqrt{\mathbb{V}\mathrm{ar}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right]}}\right). \tag{D.17}$$

In this identity, we have that the following is true:

$$\begin{aligned}\mathbb{E}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right] &= \log Y_0 + \int_0^T \left(\widehat{\nu}_s - \frac{1}{2}\left\|\beta_s + \frac{1}{p-1}\Lambda_0^R\right\|_{\mathbb{R}^4}^2\right)\mathrm{d}s, \\ \mathbb{V}\mathrm{ar}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right] &= \int_0^T \left\|\beta_s^\top + \frac{\Lambda_0^{R^\top}}{p-1} + \frac{\widehat{\delta}_{1,r}^\top}{p-1} K_Z^{-1}\left(I_{2\times 2} - e^{-K_Z[T-t]}\right)\Sigma_Z\right\|_{\mathbb{R}^4}^2 \mathrm{d}s.\end{aligned} \tag{D.18}$$

Here, we define $\widehat{\nu}_t = \alpha_t + \frac{\widehat{\delta}_{0,r}}{p-1} + \frac{1}{2}\frac{p}{(p-1)^2}\Lambda_0^{R^\top}\Lambda_0^R + \frac{1}{p-1}\beta_t^\top\Lambda_0^R$ for all $t \in [0,T]$.

Concerning $x \mapsto f_{X/Y}(x)$, we are able to derive the following:

$$\begin{aligned}f_{X/Y}(x) &= \frac{\partial}{\partial x}F_{X/Y}(x) \\ &= \frac{\partial}{\partial x}\Phi\left(d_{0,T}(x)\right) = \phi\left(d_{0,T}(x)\right)\frac{\partial}{\partial x}d_{0,T}(x) \\ &= \frac{\phi\left(d_{0,T}(x)\right)}{(1-x)\sqrt{\mathbb{V}\mathrm{ar}\left[\log M_T^{R^{-\frac{1}{p-1}}} Y_T\right]}}, \quad \forall\, x \in (0,1).\end{aligned} \tag{D.19}$$

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