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An analysis of individual welfare

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Abstract

We study the demand for retirement products given that individuals have access to innovative plans depending on the realized survival probabilities, like tontines, in addition to traditional annuities. Preferences of agents are modeled by a generalized life-cycle utility function allowing for temporal risk aversion, i.e. agents are risk averse about their lifetime. We identify conditions for pricing bounds under which agents exhibiting temporal risk aversion prefer partial tontinization combined with partial annuitization to full annuitization. In an extended model with differential mortality and wealth, we analyze a utilitarian social planner's retirement product demand, focusing on wealth transfers between the groups.

Keywords: Temporal risk aversion, retirement planning, innovative retirement products, annuities and tontines, wealth transfer

JEL: G22, H55, H23, I31, J32

1 Introduction

Due to low interest rates, tightening solvency regulations and increasing life expectancies across OECD countries (cf. OECD (2016a)), guaranteed pension benefits have become harder to sustain (cf. OECD (2016b)). Academics and practitioners are analyzing the resulting consequences along with different solutions.[1] One out of many promising possibilities to tackle these issues is developing risk-sharing schemes, i.e. innovative retirement products which allow to share longevity risk among providers and/or individuals (cf. OECD (2020)). While the economics literature on retirement products (e.g. Yaari (1965), Davidoff et al. (2005), Hu and Scott (2007), Bommier et al. (2011)) mainly discusses regular annuities, in the actuarial literature, quite some innovative retirement products whose benefits depend on the realized survival probabilities and the mortality experienced have been discussed in the last years.[2] In such a retirement plan, a pool of individuals shares the mortality/longevity risk, whereas the "providers" play exclusively an administrative role and carry no risk. It is pointed out in OECD (2020) that adding risk-sharing components to retirement plans is an important tool for potential increases in retirement benefits, while at the same time guaranteed benefits must be ensured. Markets containing *both* annuities and innovative products like tontines ensure that individuals can optimally benefit from the advantages of both products. Therefore, we aim to combine the above two streams of literature and study the demand for guaranteed and mortality-linked retirement benefits if preferences of agents are modeled by a generalized life-cycle utility function allowing for temporal risk aversion, i.e. agents are not only risk averse about consumption, but also risk averse about their lifetimes. We consider the demand of a single agent and that of a social planner acting on behalf of different cohorts with differential mortality and wealth levels.

Ever since Yaari's pioneering article (Yaari (1965)), the fixed income generated by an annuity has been recognized as the optimal source of retirement income. However, empirically, a low demand has been observed for annuities (see e.g. Hu and Scott (2007) and Inkmann et al. (2010)), a phenomenon known as the "annuity puzzle". One possible reason for this discrepancy between theory and practice might be the model setup in

[1]For example, Cai et al. (2006) analyze how well young adults in urban China insure their older family members against poverty at old ages in case pension systems fail. Galasso (2008) analyzes the postponement of retirement as a possible solution to aging societies. Brown (2013) analyzes the potential reactions of California State teachers to pension reforms caused by the changing demographics.

[2]Such products are typically known as group self annuitization, pooled annuity funds and tontines (see e.g. Piggott et al. (2005), Sabin (2010), Milevsky and Salisbury (2015) and Chen et al. (2019)).

Yaari (1965), which, although analytically convenient, imposes rather strong assumptions like temporally separable utility functions and perfect annuity markets. Davidoff et al. (2005) find that, in a complete market providing annuities which can serve any desired consumption profile, full annuitization remains optimal, even if utility functions are no longer additively separable and annuities are no longer fair. In incomplete markets, i.e. assuming that only some annuities, like constant annuities, are offered, full annuitization is no longer optimal, but partial annuitization mostly is. Bommier (2006, 2010) and Bommier et al. (2011), on the other hand, consider again a complete annuity market setting and emphasize the importance of temporal risk aversion (people being risk averse about their lifetimes), pointing out that Yaari's assumption of temporal risk neutrality is rather restrictive and comes with strong consequences.[3] Most importantly, under temporal risk aversion, the optimal annuitization profile turns out to be decreasing over time since temporal risk aversion along with mortality risk leads to a stronger discounting over time (see also Bommier (2006)). Bommier and Schernberg (2021) find that agents exhibiting temporal risk aversion prefer a retirement income dynamically adjusted to most recent information about life expectancy to an ex-ante fixed income. Staying with an agent exhibiting temporal risk aversion, in this paper, we study the optimal retirement decision when individuals have access to mortality-linked retirement products providing random retirement benefits, *in addition to* traditional annuities. In contrast to Bommier and Schernberg (2021), we focus on determining the optimal fractions invested in both traditional annuities and longevity-linked products like tontines. In the social planner's optimization problem, our emphasis lies in finding out how heterogeneous initial wealth levels of various groups determine the wealth transfer between the groups.

Assuming a "priceless life context" as introduced in Bommier (2006), which means that agents would be willing to give up most of their consumption in exchange for a longer life, we study the demand for retirement products both from an agent's and a social planner's viewpoint. Relying on the approximation technique introduced in Bommier et al. (2011), we are able to approximate the expected lifetime felicity of an agent using an additive utility function and consequently derive explicit solutions for the consumption profiles if annuities and tontines are traded exclusively in the market. If both tontines and annuities are simultaneously tradable for individuals, numerical procedures are required to work out the consumption profile of each retirement product, and consequently to determine the optimal fractions of wealth invested in annuities and tontines. Despite

[3]In particular, experimental studies confirm that individuals do not exhibit temporal risk neutrality but temporal risk aversion, see e.g. Andersen et al. (2018).

the non-existence of explicit solutions to the optimal consumption patterns, we are able to determine an upper and lower bound for the safety loading charged for the annuity, between which it is optimal to invest a positive amount in annuities and tontines.

We find that the more temporal risk averse individuals are, the more they would prefer to invest at least a fraction of wealth in tontines, even in perfect annuity markets. This differs from the optimal decision of individuals showing temporal risk neutrality (i.e. who are risk neutral with respect to their future lifetime), which is investing 100% of their wealth in annuities (cf. Yaari (1965) and Chen et al. (2020)). On the other hand, we find that it is never optimal to invest *all* wealth in a tontine, i.e. at least partial annuitization remains optimal. This suggests that the investment in tontines would not crowd out the existence of traditional annuities. This paper therefore provides a solution to temporal risk averse individuals, who are not fond of buying annuities exclusively. In particular, we find that agents with only a moderate degree of temporal risk aversion already tend to invest relatively high fractions of wealth in a tontine. Economically, these results are plausible since temporal risk aversion leads agents to behave less patiently in their consumption behavior compared to agents with temporal risk neutrality. Therefore, such agents do not enjoy the guaranteed benefits of annuities as much as agents with temporal risk neutrality. Similarly, agents with temporal risk neutrality, who desire higher consumption rates at older ages than agents with temporal risk aversion, suffer more strongly under the volatile future benefits of tontines. More intuitively, agents exhibiting temporal risk aversion might believe that they have a better chance to outlive their peers in a tontine than to live long enough to receive a profit from the annuity.

Inspired by e.g. Attanasio and Hoynes (2000) and Bommier et al. (2011), we also consider an extended model with two groups of individuals, allowing for differential mortality, wealth and safety loadings. In addition to the analysis of demand, in the social planner's problem, we look into wealth transfers among various groups of agents. We find that, mostly, the living-longer-type benefits from the living-shorter-type, a result which has already been found by Weil and Fisher (1974), Brown (2002) and Bommier et al. (2011) for annuities. To be precise, this direction of wealth transfer holds, if the wealth level of the living-longer-type does not exceed that of the the living-shorter-type. If more wealth is assigned to the living-shorter type, the wealth transfer can be reversed. In this sense, the individual wealth levels mainly determine which group transfers wealth to the other group. Depending on the precise parameter setup, temporal risk aversion can then either increase or decrease the wealth transfer.

The remainder of this paper is structured in the following way. In Section 2, we introduce the base model and solve the optimal retirement planning problem for a single agent in an annuity market, a tontine market and a market containing both products. In Section 3, we consider a utilitarian social planner in an extended model with differential mortality, and solve the corresponding optimal retirement planning problems in the annuity and the tontine market as well as the market containing both products and analyze wealth transfers between groups with differential mortality. Section 4 concludes the article and is followed by a technical appendix.

2 Individual retirement planning

Let T_x be the random remaining lifetime of an agent aged x years and $\{\mu_{x+t}\}_{t\geq 0}$ be the (possibly stochastic) force of mortality process of the agent. Denoting by $\mathcal{E}_t := \sigma\left(\{\mu_{x+s}\}_{\{0\leq s\leq t\}}\right)$ the filtration containing the information about the mortality rates μ_{x+t}, we define the survival curve:

$$s(t) = \mathbb{E}\left[\mathbb{1}_{\{T_x>t\}}\right] = \mathbb{E}\Big[\mathbb{E}\left[\mathbb{1}_{\{T_x>t\}}\,\middle|\,\mathcal{E}_t\right]\Big] = \mathbb{E}\Big[e^{-\int_0^t \mu_{x+s}\mathrm{d}s}\Big]. \tag{1}$$

Further, we define the conditional survival probabilities $S(t) := e^{-\int_0^t \mu_{x+s}\mathrm{d}s}$.

Assumption 2.1. *We assume that* $S'(t) = \mathrm{d}S(t)/\mathrm{d}t < 0$ *and that* $S(\infty) = 0$.

Assumption 2.1 simply means that under each scenario, the probability of survival (conditional on that scenario) decreases over time and that under no scenario, individuals can live forever. Note that Assumption 2.1 implies that $\frac{\mathrm{d}s(t)}{\mathrm{d}t} < 0$ and $s(\infty) = 0$ as well. It can, for example, be achieved by assuming $\mu_{x+t}(\omega) > 0$ for all t and single events ω.

In the following, we generalize the utility preferences from Yaari's pioneering article Yaari (1965) to preferences allowing for temporal risk aversion. Let $\alpha(t)$ be a subjective discount factor, u be a strictly increasing and concave utility function and $\boldsymbol{C} = \{C(t)\}_{t\geq 0}$ be a (possibly stochastic) random consumption profile. Following Bommier et al. (2011), the lifetime felicity of an agent with a remaining lifetime of T_x years is given by

$$U(\boldsymbol{C}, T_x) = \Phi\left(\int_0^{T_x} \alpha(t)u(C(t))\mathrm{d}t\right).$$

In the typical expected lifetime felicity framework (see e.g. Yaari (1965)), agents have a linear lifetime felicity. Each moment of life gives them an instantaneous felicity

$\alpha(t)u(C(t))$, which is then additively aggregated in order to get the lifetime felicity. However, as Bommier et al. (2011) point out, agents cannot know ex-ante what their lifetime felicity will be, given the uncertainty about life duration (and about consumption) and might at most know the distribution of lifetime felicity. Using a general function of Φ enables us to also consider risk aversion with respect to the future lifetime. In the typical expected lifetime felicity framework where a linear Φ is used, the agent is risk neutral about the lifetime, while a concave function Φ would indicate a positive risk aversion about the lifetime.

Assumption 2.2. *We assume that* Φ *is twice differentiable satisfying* $\Phi'' \leq 0$ *and* $\Phi(0) \in (-\infty, \infty)$.

We allow for linear and concave functions Φ. Note that we may additionally assume $\Phi(0) = 0$ without loss of generality.[4]

The expected lifetime felicity function is then given by

$$EU(\boldsymbol{C}) = \mathbb{E}\left[\Phi\left(\int_0^{T_x} \alpha(t)u(C(t))\mathrm{d}t\right)\right].$$

To simplify this expression further, we need to make assumptions regarding the consumption process $\boldsymbol{C}$. In this paper, we are particularly interested in considering a combined retirement income consisting of an annuity and a tontine. We start off with the single products and the corresponding optimal consumption profiles before coming to the combined problem.

Let us introduce one last assumption, after which we will move to the individual optimization problems.

Assumption 2.3. *We assume that* $\frac{u''(c)c}{u'(c)}$ *is independent of* c, *i.e. we assume a constant inter-temporal elasticity of substitution. This is equivalent to assuming a constant relative risk aversion, resulting in*

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, & \gamma \neq 1, \gamma > 0, \\ \ln(c), & \gamma = 1. \end{cases}$$

[4] In principle, we can assume any finite real number for $\Phi(0)$, i.e. $\Phi(0) \in (-\infty, \infty)\backslash\{0\}$. Then, instead of considering $\Phi(c)$, we can consider $\tilde{\Phi}(c) := \Phi(c) - \Phi(0)$ which resembles identical preferences as Φ and fulfills $\tilde{\Phi}(0) = 0$.

Furthermore, throughout the paper, we disregard financial market risk to focus exclusively on mortality risk.

2.1 Tontines

We consider a tontine contract in which a homogeneous pool of infinitely many agents, who are assumed to be identical copies of each other, shares the mortality risk. In such a pool, the idiosyncratic risk is fully diversified. The aggregate mortality risk, however, affects all the agents in the same direction and can therefore not be diversified. At time 0, the agents specify a deterministic payout function $d(t)$. As time passes, agents who die early pass their benefits on to agents dying at older ages. In order to make our model stay tractable, we have assumed an infinitely large pool size,[5] which leads to the result that the mortality of the agents evolves exactly according to the conditional survival curve $S(t)$, resulting in the consumption rate $C(t) = d(t)/S(t)$ of a single individual. As the conditional survival probability $S(t)$ is a random process, the future consumption stream is random, compared to regular annuity products with deterministic consumption profiles.

Now consider an agent endowed with an initial wealth W_0 which will be invested in the above tontine with an infinite pool size. To determine the single up-front premium, we assume that the insurer relies on the expected value principle with a proportional loading δ_d. In total, the premium charged for the tontine is therefore given by

$$P_0^d = (1+\delta_d)\mathbb{E}\left[\int_0^\infty e^{-\int_0^t r(s)ds} \cdot S(t) \cdot \frac{d(t)}{S(t)} \mathrm{d}t\right] = (1+\delta_d)\int_0^\infty e^{-\int_0^t r(s)ds} \cdot d(t)\mathrm{d}t,$$

where we assume that $r(t)$ is a deterministic interest rate curve. We arrive at the following optimization problem for the agent at time 0:

$$\begin{aligned} &\max_{d(t)} EU(\boldsymbol{C}) = \mathbb{E}\left[\Phi\left(\int_0^{T_x} \alpha(t) u\left(\frac{d(t)}{S(t)}\right)\mathrm{d}t\right)\right] \\ &\text{subject to } \int_0^\infty e^{-\int_0^t r(s)ds} \cdot d(t)\mathrm{d}t \le \frac{W_0}{1+\delta_d}. \end{aligned}$$

To achieve analytical solutions, we approximate the expected lifetime felicity in a similar

[5] According to the findings of the existing literature, e.g. Chen et al. (2019), with a pool size of 200, the unsystematic longevity risk is almost negligible.

way as Bommier et al. (2011). The overall expected lifetime felicity is given by

$$\begin{aligned} EU(\boldsymbol{C}) &= \mathbb{E}\left[\Phi\left(\int_0^{T_x} \alpha(t)u(C(t))\mathrm{d}t\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\Phi\left(\int_0^{T_x} \alpha(t)u\left(\frac{d(t)}{S(t)}\right)\mathrm{d}t\right)\Big|\{\mu_{x+t}\}_{t\geq 0}\right]\right] \\ &= \mathbb{E}\left[\int_0^{\infty} \mu_{x+t}\cdot S(t)\cdot\Phi\left(\int_0^{t} \alpha(s)u\left(\frac{d(s)}{S(s)}\right)\mathrm{d}s\right)\mathrm{d}t\right]. \end{aligned} \tag{2}$$

Conditioning on the path of the force of mortality process allows us to rearrange the expected lifetime felicity using the conditional density of the remaining lifetime T_x. Once a path of μ_{x+t} is given, the density of T_x is given by $\mu_{x+t}S(t)$. The expression $\mu_{x+t}S(t)$ can then be interpreted as the infinitesimal probability of dying at age $x+t$, resulting in the lifetime felicity of $\Phi\left(\int_0^t \alpha(s)u\left(\frac{d(s)}{S(s)}\right)\mathrm{d}s\right)$.

Lemma 2.4. *Assuming a priceless life context as introduced in Bommier (2006), we obtain the following additive approximation of* (2)*:*

$$EU(\boldsymbol{C}) \approx \int_0^{\infty} \kappa_\gamma(t)\alpha(t)u\left(d(t)\right)\mathrm{d}t, \tag{3}$$

where $\kappa_\gamma(t) := \mathbb{E}\left[S(t)^\gamma\beta(t)\right]$, *and*

$$\beta(t) := -\frac{1}{S(t)}\int_t^{\infty} S'(s)\cdot\Phi'\left(\int_0^{s}\alpha(\tau)d\tau\right)ds. \tag{4}$$

Proof: See Appendix A.1.

The assumption of a “priceless life context” means that agents would be willing to give up most of their consumption in exchange for a longer life. Note the following remarks about the term $\beta(t)$:

- Equation (3) tells us that we can transfer the expected lifetime felicity of an agent exhibiting temporal risk aversion and owning a subjective discount factor $\alpha(t)$ to the expected lifetime felicity function of an agent exhibiting temporal risk neutrality, but owning now a different subjective discount factor $\beta(t)\alpha(t)$. In particular, assuming that $\Phi' \geq 0$, it holds $\beta(t) \geq 0$ for all $t \geq 0$, since $S'(t) < 0$.
- If Φ is a linear function, $\beta(t)$ is equal to a constant. This corresponds to the case in which the agent has a linear lifetime felicity.

- If $\Phi'' < 0$, we find that

$$\beta'(t) = -\frac{S'(t)}{S(t)} \int_t^\infty \frac{S(s)}{S(t)} \Phi'' \left(\int_0^t \alpha(\tau) \mathrm{d}\tau \right) ds < 0,$$

  because of Assumption 2.1. Therefore, if $\Phi'' < 0$, $\beta(t)$ is decreasing in t.

We now consider the individual optimization problem assuming that the individual has an exclusive access to tontines. Using the approximation 3, we arrive at the following optimization problem:

$$\begin{aligned} &\max_{d(t)} \int_0^\infty \kappa_\gamma(t) \alpha(t) u\left(d(t)\right) \mathrm{d}t \\ &\text{subject to } (1+\delta_d) \int_0^\infty e^{-\int_0^t r(s) ds} \cdot d(t) \mathrm{d}t \leq W_0. \end{aligned} \tag{5}$$

Theorem 2.5 provides the solution to Problem (5).

Theorem 2.5. *The solution to the above optimization problem* (5) *is*

$$d^*(t) = \left(\frac{(1+\delta_d)\, \lambda_d e^{-\int_0^t r(s) ds}}{\kappa_\gamma(t) \alpha(t)} \right)^{-\frac{1}{\gamma}}, \tag{6}$$

where the Lagrangian multiplier λ_d is, for all $\gamma > 0$, given by

$$\lambda_d = \left(\frac{1+\delta_d}{W_0} \int_0^\infty e^{-\int_0^t r(s) ds} \cdot \left(\frac{(1+\delta_d)\, e^{-\int_0^t r(s) ds}}{\kappa_\gamma(t) \alpha(t)} \right)^{-\frac{1}{\gamma}} \mathrm{d}t \right)^\gamma.$$

For $\gamma \neq 1$, the expected discounted lifetime felicity is then given by

$$U_d = \frac{\lambda_d}{1-\gamma} \cdot \frac{W_0}{1+\delta_d}.$$

Proof: See Appendix A.2.

- We have seen that $\beta(t)$ is decreasing in t, assuming that $\Phi'' < 0$. Furthermore, $S(t)$ is decreasing in t. This implies that $\kappa_\gamma(t)$ is decreasing in t as well. Therefore, if we further assume that $\alpha(t) = e^{-\int_0^t r(s) ds}$, we can conclude that the optimal payout function is decreasing in t.

- The optimal payout function under temporal risk neutrality is given by

$$d^*(t) = \left(\frac{(1+\delta_d)\lambda_d e^{-\int_0^t r(s)ds}}{\mathbb{E}\left[S(t)^\gamma\right]\alpha(t)} \right)^{-\frac{1}{\gamma}},$$

where the Lagrangian multiplier λ_d is, for all $\gamma > 0$, given by

$$\lambda_d = \left(\frac{1+\delta_d}{W_0} \int_0^\infty e^{-\int_0^t r(s)ds} \cdot \left(\frac{(1+\delta_d)e^{-\int_0^t r(s)ds}}{\mathbb{E}\left[S(t)^\gamma\right]\alpha(t)} \right)^{-\frac{1}{\gamma}} \mathrm{d}t \right)^\gamma .$$

Assuming that $\alpha(t) = e^{-\int_0^t r(s)ds}$, this payout function is also decreasing in t. However, since $\beta(t)$ is decreasing in t, $\kappa_\gamma(t)$ decreases more steeply than $\mathbb{E}[S(t)^\gamma]$, i.e. temporal risk aversion leads to a more steeply decreasing payout function compared to temporal risk neutrality.

Below, we provide a numerical example, where we rely on a shocked Gompertz model. For details on the (deterministic) Gompertz model, see Gompertz (1825) and Gumbel (1958). Following e.g. Lin and Cox (2005), we apply a random shock ϵ to the Gompertz mortality rates assuming that ϵ is a continuous random variable with density $f_\epsilon(\cdot)$, support on $(-\infty, 1)$ and moment generating function $m_\epsilon(t) := \mathbb{E}[e^{t\epsilon}]$. Mortality rates are then given by

$$\mu_{x+t} = (1-\epsilon)\frac{1}{b}e^{\frac{x+t-m}{b}}, \tag{7}$$

where b is the dispersion coefficient and m is the modal age at death. This model is motivated by the fact that, in many countries, insurers are required by regulation to test their balance sheets against a variety of shock scenarios. This is for example the case in the U.S., Canada and the European Union. For $\epsilon \equiv 0$, we obtain the (deterministic) Gompertz model as special case. Furthermore, we rely on the parameters summarized in Table 1.

We rely on these parameters due to the following reasons:

- We assume that Φ is an exponential utility function with constant absolute risk aversion $\theta > 0$. Following Bommier et al. (2011), we choose θ in such a way that a plausible discounting rate is achieved.[6]

[6]For simplicity, in the following, we will refer to θ as the degree of temporal risk aversion. Note

Initial wealth $W_0 = 100$	Temporal risk aversion $\Phi(y) = \frac{1}{\theta} - \frac{1}{\theta}e^{-\theta y}$, $\theta = 0.035$	Utility function $u(y) = \frac{y^{1-\gamma}}{1-\gamma}$, $\gamma = 3$
Risk-free rate $r(t) = r = 0.01$	Subjective discount rate $\alpha(t) = e^{-rt}$	Safety loading $\delta_d = 0$
Initial age $x = 65$	Gompertz law $m = 88.721$, $b = 10$	Longevity shock $\epsilon \sim \mathcal{N}_{(-\infty,1)}\left(-0.0035, 0.0814^2\right)$

Table 1: Base case parameter setup.

- We assume a constant risk-free interest rate close to zero, based on the prevailing low interest rates in many developed countries. The subjective discount factor $\alpha(t)$ is equal to the risk-free interest rate, which allows us to focus exclusively on the effects of temporal risk aversion.

- For the Gompertz parameters m and b, we follow Milevsky and Salisbury (2015). Concerning the longevity shock ϵ, we follow Chen et al. (2019).

In Figure 1, we show the optimal payout functions under temporal risk neutrality and temporal risk aversion. By varying the parameter θ, we can study the consumption

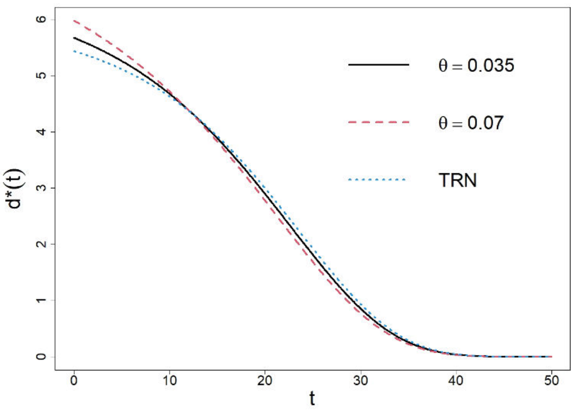


Figure 1: Optimal payout function $d^*(t)$ over time under temporal risk aversion and temporal risk neutrality (TRN). The parameters are chosen as in Table 1.

profiles of different agents who do not have an additive lifetime felicity. A θ-value approaching zero (consequently, Φ' going to 1) makes the lifetime felicity function closer to the additive felicity function. During the first ten years, the payout function under temporal risk aversion is slightly above that obtained under temporal risk neutrality. After ten years, this relation is reversed. If θ grows larger, the resulting lifetime felicity

that this definition differs from the risk aversion with respect to the age at death as defined in Bommier (2006).

moves further away from the linear lifetime felicity, and the optimal consumption deviates more from the optimal payout function under temporal risk neutrality. Risk aversion about the lifetime works similarly to a larger subjective discounting rate, in line with the remarks made about $\beta(t)$ below equation (4).

2.2 Annuities

In this section, we consider a deterministic consumption profile $c(t)$. In the following, we review and extend some results of Bommier et al. (2011), with the difference that more general stochastic, instead of deterministic, mortality rates are applied. As an additional extension, we include general safety loadings in all our derivations. Without systematic mortality risk and setting the safety loadings equal to zero, we are back in the setting of Bommier et al. (2011).

Denoting by δ_a the proportional loading charged for the annuity, the premium charged for an annuity is given by

$$P_0^a = (1+\delta_a)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s(t)\cdot c(t)\mathrm{d}t.$$

Mitchell et al. (1999) find "that the average annuity policy available to a 65-year-old man in 1995 delivered payouts valued at between 80 and 85 cents per dollar of annuity premium." Furthermore, Sabin (2010) writes that annuities are typically 14% higher than fair.

Assuming that an agent endowed with the initial wealth W_0 invests exclusively in annuities, we arrive at the following optimization problem:

$$\begin{aligned}
&\max_{c(t)} EU(\boldsymbol{c}) = \mathbb{E}\left[\Phi\left(\int_0^{T_x}\alpha(t)u(c(t))\mathrm{d}t\right)\right]\\
&\text{subject to } (1+\delta_a)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s(t)\cdot c(t)\mathrm{d}t \le W_0.
\end{aligned}$$

To approximate the expected lifetime felicity function, we define the term

$$\overline{\beta}(t) = -\frac{1}{s(t)}\int_t^\infty s'(s)\cdot\Phi'\left(\int_0^s \alpha(\tau)d\tau\right)ds, \tag{8}$$

which is related to $\beta(t)$ by Lemma A.1. Following analogous steps as in Appendix A.1, and applying Lemma A.1, we end up with the following additive approximation of the expected lifetime felicity function:

$$EU(\boldsymbol{c}) \approx \int_0^\infty s(t) \cdot \overline{\beta}(t)\alpha(t)u(c(t))\mathrm{d}t. \tag{9}$$

Note the following remarks about $\overline{\beta}(t)$:

- We can transfer the expected lifetime felicity of an agent showing temporal risk aversion and owning a subjective discount factor $\alpha(t)$ to an expected lifetime felicity of an agent showing temporal risk neutrality, but owning now a different subjective discount factor $\overline{\beta}(t)\alpha(t)$. In this sense, analogous conclusions can be drawn as for the tontine.
- Note that $\overline{\beta}(t)$ is equal to a constant if Φ is a linear function.
- Similarly as in Bommier et al. (2011), we find that

  $$\overline{\beta}'(t) = -\frac{s'(t)}{s(t)} \int_t^\infty \frac{s(s)}{s(t)} \Phi'' \left(\int_0^t \alpha(\tau)\mathrm{d}\tau \right) ds,$$

  which is smaller than zero if we assume $\Phi'' < 0$. Hence, in this case, $\overline{\beta}(t)$ is decreasing in t.

Given the approximation (9), we now arrive at the following optimization problem:

$$\begin{aligned} &\max_{c(t)} \int_0^\infty s(t) \cdot \overline{\beta}(t)\alpha(t)u(c(t))\mathrm{d}t \\ &\text{subject to } (1+\delta_a) \int_0^\infty e^{-\int_0^t r(s)ds} \cdot s(t) \cdot c(t)\mathrm{d}t \leq W_0. \end{aligned} \tag{10}$$

We are able to solve this optimization problem of an individual agent explicitly.

Theorem 2.6. *The solution to optimization problem* (10) *is*

$$c^*(t) = \left((1+\delta_a) \frac{\lambda_a e^{-\int_0^t r(s)ds}}{\overline{\beta}(t)\alpha(t)} \right)^{-\frac{1}{\gamma}}, \tag{11}$$

where the Lagrangian multiplier λ_a *is, for all* $\gamma > 0$, *given by*

$$\lambda_a = \left(\frac{1+\delta_a}{W_0} \int_0^\infty e^{-\int_0^t r(s)ds} \cdot s(t) \cdot \left(\frac{(1+\delta_a)\, e^{-\int_0^t r(s)ds}}{\overline{\beta}(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} \mathrm{d}t \right)^{\gamma}.$$

For $\gamma \neq 1$, the expected discounted lifetime felicity is then given by

$$U_a = \frac{\lambda_a}{1-\gamma} \cdot \frac{W_0}{1+\delta_a}.$$

Proof: See Appendix A.4.

There are the following special cases:

- In the special case where $\alpha(t) = e^{-\int_0^t r(s)ds}$ and Φ is a linear function, the optimal payout function $c^*(t)$ is constant. This is the famous result stated in Yaari (1965): Constant annuitization is optimal if there is no bequest motive.
- If $\alpha(t) = e^{-\int_0^t r(s)ds}$ and $\Phi'' < 0$, the optimal consumption profile is decreasing over time, because $\overline{\beta}(t)$ decreases in t. This differs from the optimal solution for an agent with a linear lifetime felicity. Similarly to the tontine, being risk averse about the lifetime makes agents less patient: They prefer to consume more in earlier retirement ages.

In Figure 2, we show the optimal consumption profiles for $\delta_a = 0$. We observe that, under

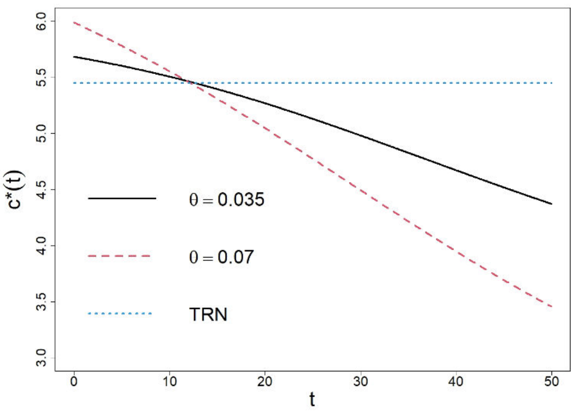


Figure 2: Optimal consumption profile $c^*(t)$ over time under temporal risk aversion and temporal risk neutrality (TRN). The parameters are chosen as in Table 1 and $\delta_a = 0$.

temporal risk aversion, consumption is decreasing over time. Note that, if Φ is a linear function, the optimal consumption is constant and equal to $W_0 / \int_0^\infty e^{-rt} s(t) \mathrm{d}t = 5.45$. Hence, we can draw similar observations as from Figure 1, i.e. the presence of temporal risk aversion leads agents to behave less patiently.

To sum up, a temporal risk averse individual does not consider the most frequently traded annuity products with constant benefits as optimal. This result can be considered as one possible interpretation why people do not trade extensively in annuities. In the following, we aim to find out whether the lifetime felicity of temporal risk averse individuals can be improved, when they can additionally trade in tontines.

2.3 Combined investment

In contrast to the existing literature in this field, we are interested in finding out whether annuities can be supplemented by a mortality-linked product for an individual showing temporal risk aversion. In other words, we assume that agents have access to both tontines and annuities and aim to find out whether and how these individuals can benefit from a combined investment in both plans.

A portfolio consisting of an annuity and a tontine results in the following consumption rate of a living agent at each time $t \geq 0$:

$$\mathcal{C}(t) = c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}.$$

Below, we present the optimization problem for an agent endowed with W_0 who aims to invest in such a portfolio:

$$\begin{aligned} &\max_{c_{ad}(t), d_{ad}(t)} \mathbb{E}\left[\Phi\left(\int_0^{T_x} \alpha(t) u\left(c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}\right) \mathrm{d}t\right)\right] \\ &\text{subject to } (1+\delta_a)\int_0^\infty e^{-\int_0^t r(s)ds} s(t) c_{ad}(t) \mathrm{d}t + (1+\delta_d)\int_0^\infty e^{-\int_0^t r(s)ds} \cdot d_{ad}(t)\mathrm{d}t \leq W_0. \end{aligned}$$

Following the steps in Appendix A.1, we obtain the following additive approximation of the expected lifetime felicity of the portfolio:

$$EU(\boldsymbol{\mathcal{C}}) \approx \mathbb{E}\left[\int_0^\infty S(t)\beta(t)\alpha(t) u\left(c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}\right) \mathrm{d}t\right]. \tag{12}$$

Therefore, we now arrive at the following optimization problem:

$$\begin{aligned} &\max_{c_{ad}(t), d_{ad}(t)} \mathbb{E}\left[\int_0^\infty S(t)\beta(t)\alpha(t) u\left(c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}\right) \mathrm{d}t\right] \\ &\text{subject to } (1+\delta_a)\int_0^\infty e^{-\int_0^t r(s)ds} s(t) c_{ad}(t) \mathrm{d}t + (1+\delta_d)\int_0^\infty e^{-\int_0^t r(s)ds} \cdot d_{ad}(t)\mathrm{d}t \leq W_0. \end{aligned} \tag{13}$$

Since the utility function of a sum is unequal to the sums of the separate utility levels for power utility, we are unable to derive an explicit optimal consumption profile to optimization problem (13). Nevertheless, depending on the magnitude of the safety loadings, we can still derive theoretical results on the comparison of annuities, tontines and a portfolio of the two, and learn when it is optimal to invest in a real portfolio, i.e. a positive amount in both tontines and annuities. In Theorem 2.7, we summarize the results.

Theorem 2.7.

1. *If and only if*

$$\delta_a \leq \underline{\delta}_a := \inf_{t \geq 0} \frac{\overline{\beta}(t)}{\mathbb{E}\left[\beta(t)\right]}(1+\delta_d) - 1, \tag{14}$$

the optimal solution to Problem (13) *is given by* $c_{ad}(t) = c^*(t)$ *,* $d_{ad}(t) = 0$ *, i.e. it is optimal to invest all the initial wealth in the optimal annuity.*

2. *If and only if*

$$\delta_a \geq \overline{\delta}_a := \sup_{t \geq 0} \frac{\kappa_{\gamma+1}(t)}{\kappa_\gamma(t)s(t)}(1+\delta_d) - 1, \tag{15}$$

the optimal solution to Problem (13) *is given by* $d_{ad}(t) = d^*(t)$ *,* $c_{ad}(t) = 0$ *, i.e. it is optimal to invest all the initial wealth in the optimal tontine.*

3. *If both conditions* (14) *and* (15) *are not fulfilled, it is optimal for the agent to invest positive fractions of wealth in both the annuity and the tontine.*

Proof: See Appendix A.5.

Remark 2.8. *In the special case where* Φ *is a linear function,* $\mathbb{E}\left[\beta(t)\right]$ *and* $\overline{\beta}(t)$ *are equal to the same constant. Therefore,* (14) *can be rearranged to* $\delta_a \leq \delta_d$ *.*[7] *In the presence of temporal risk aversion, the demand for annuities and tontines each can either be increased or decreased, depending on the mortality law and the resulting structure of* $\beta(t)$ *,* $\overline{\beta}(t)$ *and* $\kappa_\gamma(t)$*.*

In Table 2, we analyze the critical bounds of the annuity loading (14) and (15) numerically. We observe that $\underline{\delta}_a$ is below 0 (δ_d is assumed to be zero in this example) and that

[7]Thus, Theorem 2.7 is a natural extension to Proposition 3.3 in Chen et al. (2020) who consider finite pool sizes and linear lifetime felicity functions.

θ	TRN	0.035	0.07	0.105	0.14
$\underline{\delta}_a$	0	-0.0014	-0.0027	-0.0039	-0.0051
$\bar{\delta}_a$	∞	∞	∞	∞	∞

Table 2: Analysis of the bounds (14) and (15) depending on the temporal risk aversion θ. The parameters are chosen as in Table 1.

it decreases in the degree of temporal risk aversion θ. That fact that the critical value of $\underline{\delta}_a$ is mostly negative means that investing exclusively in annuities is not optimal, as along as a non-negative safety loading is charged for the annuity. In other words, even in a perfect annuity market, it is not optimal to invest all the initial wealth in the annuity. Furthermore, $\bar{\delta}_a = \infty$ tells us that no agent is willing to invest exclusively in tontines, regardless of the temporal risk aversion. These results are economically plausible since, under our parameters, we have observed that an agent with temporal risk aversion desires a decreasing consumption profile. Therefore, this agent does not enjoy guaranteed annuity benefits as much as an agent with temporal risk neutrality. Similarly, an agent with temporal risk neutrality desiring a flat consumption profile suffers more strongly under the volatile future benefits of the tontines. Another explanation might be that agents with temporal risk aversion believe that they have a higher chance to outlive their peers and thus to benefit from a tontine than to live long enough to receive a profit from the annuity.

Based on this numerical analysis, we can conclude that tontines are more attractive to individuals who are subject to temporal risk aversion and tontines become more attractive as the degree of temporal risk aversion increases. In total, we can interpret our results such that tontines provide a retirement income solution to temporal risk averse individuals, who are not fond of buying annuities exclusively. If individuals are risk neutral about their lifetime, the condition for investing exclusively in annuities (14) is fulfilled more easily, depending only on the magnitudes of the loadings δ_a and δ_d. In contrast, if individuals are risk averse about their lifetime, the introduction of a retirement market for innovative products like tontines does not crowd out annuities, but leads individuals to prefer a combination of the two to an annuity.

To illustrate the increased attractiveness of tontines under temporal risk aversion further, Figure 3 provides the optimal payout functions in the portfolio for an agent with temporal risk aversion. We observe that the agent receives no annuity but only tontine payments during the first years of retirement. Then, the agent receives a mixture of both, where the payout function in the tontine approaches zero and rather high payments from the

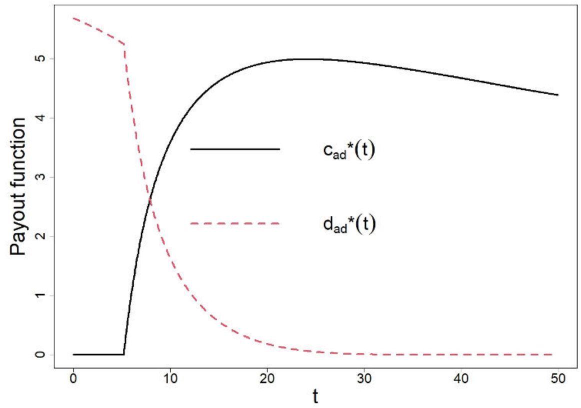


Figure 3: Optimal payout functions in the portfolio for an agent exhibiting temporal risk aversion. The parameters are chosen as in Table 1.

annuity are received. Based on the optimal payments $c^*_{ad}(t)$ and $d^*_{ad}(t)$, we are able to compute the present values of the annuity and of the tontine, respectively. The overall fraction of wealth invested in the annuity is given by 52%, meaning that the rather moderate level of temporal risk aversion $\theta = 0.035$ along with $\gamma = 3$ already reduces the fraction of wealth invested in an annuity by 48%.

To quantify the utility gains generated by the inclusion of the tontine, we introduce the wealth equivalent WE as the initial wealth level required in an exclusive annuity market to achieve the same level of expected lifetime felicity as in the combined optimization problem with initial wealth W_0. Let U_{ad} be the optimal discounted lifetime felicity resulting from the combined investment problem (13). Then, for $\gamma \neq 1$, the wealth equivalent is defined by

$$\frac{1}{1-\gamma}\left(\frac{1+\delta_a}{\mathrm{WE}}\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s(t)\cdot\left(\frac{(1+\delta_a)\,e^{-\int_0^t r(s)ds}}{\overline{\beta}(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right)^{\gamma}\frac{\mathrm{WE}}{1+\delta_a} = U_{ad}.$$

Solving for WE delivers

$$\mathrm{WE} = \left(\frac{(1+\delta_a)(1-\gamma)U_{ad}}{\left((1+\delta_a)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s(t)\cdot\left(\frac{(1+\delta_a)e^{-\int_0^t r(s)ds}}{\overline{\beta}(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right)^{\gamma}}\right)^{\frac{1}{1-\gamma}}.$$

In Table 3, we provide the wealth equivalents dependent on the standard deviation of the shock ϵ.

Standard deviation	WE	Present value of annuity payments
$\sigma = 0.0814$	100.002	51.96
$\sigma = 2 \cdot 0.0814$	100.010	51.63
$\sigma = 4 \cdot 0.0814$	100.057	49.36
$\sigma = 6 \cdot 0.0814$	100.121	48.43

Table 3: Wealth equivalents dependent on the standard deviation of the shock ϵ. The remaining parameters are taken from Table 1.

For a deterministic shock $(\sigma = 0)$, the tontine and the annuity are identical, which explains the relatively low wealth equivalents for standard deviations closer to 0. As the standard deviation increases, the difference between the tontine and the annuity becomes more pronounced and, consequently, the wealth equivalents increase. The present value of the annuity payments decreases slightly in the standard deviation, as the potential for higher benefits increases.

3 Government's problem

It is well-known that many individuals purchase lower retirement benefits than optimally suggested by the economic literature. Therefore, it should be in a government's interest to achieve retirement income structures which are in all the individuals' best interest. In addition to the individual optimization problem, we therefore consider a utilitarian government whose objective is to maximize the sum of the individual utility functions. We distinguish agents with differential mortality and wealth level, and study the impact of these two factors on the individuals' welfare, in addition to safety loadings and cohort sizes.

Assumption 3.1. *We consider two groups of agents $i = H, L$, where we assume*

$$\mu_{x+t}^{H}(\omega) < \mu_{x+t}^{L}(\omega)$$

for all single events ω, i.e. agents of type H have (on average) a longer remaining lifetime than type-L-agents.

Furthermore, we introduce the survival probabilities for a member of the groups as $s_i(t) = \mathbb{E}[S_i(t)] = \mathbb{E}\left[e^{-\int_0^t \mu_{x+s}^{(i)} \mathrm{d}s}\right]$, $i = H, L$. Again, we will first focus on the single plans before coming to the combined investment in both annuities and tontines.

3.1 Tontines

We consider a social planner maximizing the sum of the individual utility functions, taking both groups into account simultaneously. Let n_H and n_L be the percentages of agents in the two groups. The problem of the social planner is then given by

$$\begin{aligned}&\max_{d_L(t),d_H(t)} n_L \int_0^\infty \kappa_\gamma^L(t)\alpha(t)u\left(d_L(t)\right)\mathrm{d}t + n_H \int_0^\infty \kappa_\gamma^H(t)\alpha(t)u\left(d_H(t)\right)\mathrm{d}t \\ &\text{subject to } n_L(1+\delta_d^L)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot d_L(t)\mathrm{d}t + n_H(1+\delta_d^H)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot d_H(t)\mathrm{d}t \\ &\leq n_L W_0^L + n_H W_0^H.\end{aligned} \tag{16}$$

Theorem 3.2. *The solution to optimization problem* (16) *is*

$$d_i^*(t) = \left(\frac{\left(1+\delta_d^{(i)}\right)\lambda_d^G e^{-\int_0^t r(s)ds}}{\kappa_\gamma^{(i)}(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}, \tag{17}$$

where the Lagrangian multiplier λ_d^G *is given by*

$$\begin{aligned}\lambda_d^G = &\left(\left(n_L W_0^L + n_H W_0^H\right)^{-1}\left(n_L\left(1+\delta_d^H\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot\left(\frac{\left(1+\delta_d^L\right)e^{-\int_0^t r(s)ds}}{\kappa_\gamma^L(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right.\right.\\ &\left.\left.+n_H\left(1+\delta_d^H\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot\left(\frac{\left(1+\delta_d^H\right)e^{-\int_0^t r(s)ds}}{\kappa_\gamma^H(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right)\right)^{\gamma}.\end{aligned}$$

For $\gamma \neq 1$, *the collective expected discounted lifetime felicity is then given by*

$$U_d^G = \frac{\lambda_d^G}{1-\gamma}\left(n_H W_0^H + n_L W_0^L\right).$$

Proof: See Appendix A.6.

Note that the optimal consumption profiles obtained from the government's problem (16) differ from the individually optimal profile (6) only in the Lagrangian multiplier λ_d^G which is typically different from λ_d.

Remark 3.3. *If* Φ *is a linear function and* $\delta_d^L = \delta_d^H$, *we get*

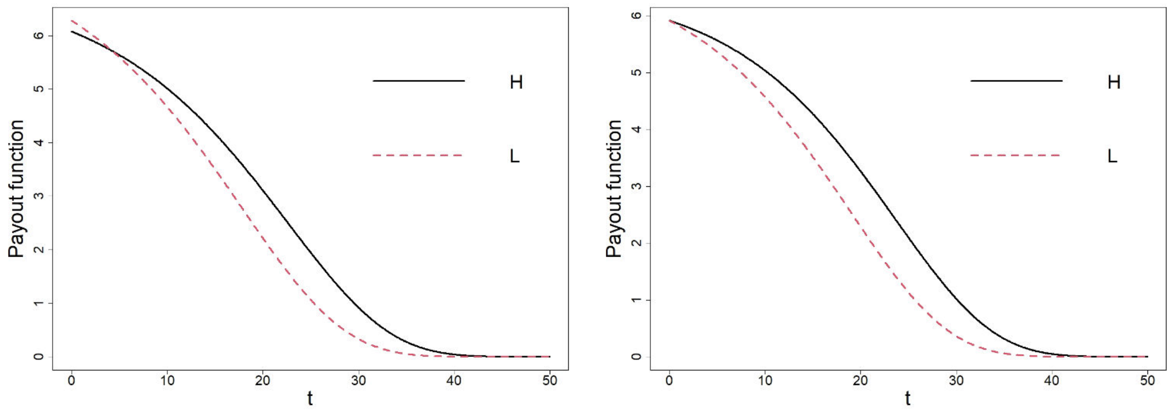
$$d_L^*(t) < d_H^*(t), \tag{18}$$

since $\mathbb{E}[S_L(t)^\gamma] < \mathbb{E}[S_H(t)^\gamma]$. *Hence, if we assume additionally that* $W_0^L \geq W_0^H$, *this implies that each type-* L *-individual transfers some wealth to type-* H *-individuals. However, if* $W_0^L < W_0^H$, *type-* L *-individuals can benefit from type-* H *-individuals, as we will show below.*

In the following, we consider a numerical example. We rely on the base case parameters summarized in Table 1 and consider the additional parameters summarized in Table 4. In Figure 4, we show the optimal payout functions $d_L^*(t)$ and $d_H^*(t)$ over time. In Panel

Pool size	Initial wealth	Safety loadings	Modal ages
$n_H = n_L = 0.5$	$W_0^H = W_0^L = 100$	$\delta_d^{(i)} = \delta_a^{(i)} = 0$	$m_H = 88.721$, $m_L = 84$

Table 4: Additional base case parameters to Table 1.



(a) Temporal risk aversion

(b) Temporal risk neutrality

Figure 4: Optimal tontine payoff over time for the two groups $i = H, L$. The parameters are chosen as in Table 1 and 4.

(a), we observe that $d_L^*(t)$ and $d_H^*(t)$ intersect. In Panel (b) however, we directly observe inequality (18).

To quantify the wealth transfers, we consider the present values of consumption for different agents under different parameter setups in Table 5.

- **Base case:** Type- L -agents suffer losses and type- H -agents benefit from the social planner joining the two groups. Under temporal risk neutrality, the wealth transfers increase compared to the case with temporal risk aversion.

	H	L
$W_0^H = W_0^L = 100,\ n_H = n_L = 0.5$		
$\theta = 0.035$	53.60	46.40
TRN	54.41	45.59
$W_0^H = 200, W_0^L = 100,\ n_H = n_L = 0.5$		
$\theta = 0.035$	80.40	69.60
TRN	81.61	68.39
$W_0^H = W_0^L = 100,\ n_H = 0.75, n_L = 0.25$		
$\theta = 0.035$	77.61	22.39
TRN	78.17	21.83
$W_0^H = W_0^L = 100,\ n_H = 0.25, n_L = 0.75$		
$\theta = 0.035$	27.80	72.20
TRN	28.46	71.54

Table 5: Present values of consumption for agents of the two groups. The parameters are chosen as in Tables 1 and 4.

- **Impact of the initial wealth:** Moving from the setting $W_0^H = W_0^L = 100$ to $W_0^H = 200$ and $W_0^L = 100$, we observe that type-L-agents now benefit from type-H-agents. Furthermore, we observe that $69.60/150 = 46.40/100$ and $80.40/150 = 53.60/100$, i.e. the optimal solution to problem (19) does not take into account individual wealth levels, but only the total collective wealth. Additionally, we observe that temporal risk neutrality now lowers the wealth transfer compared to temporal risk aversion. Similarly, we observe that $68.39/150 = 45.59/100$ and $81.61/150 = 54.41/100$.

- **Impact of the cohort size:** For both cases $n_H = 0.75$ and $n_L = 0.25$ and $n_H = 0.25$ and $n_L = 0.75$, type-H-agents benefit again from type-L-agents. Note that $72.20/75 > 22.39/25$ and $27.80/25 > 77.61/75$, i.e. if there are less type-H-agents, these gain higher benefits from the type-L-agents, which, due to their relatively high pool size, suffer relatively smaller losses. On the other hand, if there are less type-L-agents, these suffer a relatively larger loss but the gain to type-H-agents becomes smaller, because they have to share it across a larger pool. In both cases, temporal risk neutrality increases the wealth transfers. Note that it also holds $71.54/75 > 21.83/25$ and $28.46/25 > 78.17/75$.

All in all, we have seen that the individual wealth levels mainly determine which group transfers wealth to the other group. Depending on the precise parameter setup, temporal risk aversion can then either increase or decrease the wealth transfer. For our analysis, we have used n_H and n_L as weighting factors in the social planner's utility function. If it is the social planner's goal to eliminate wealth transfers between the groups, the planner can either use weighting factors taking account of the wealth levels, or use individual budget constraints instead of a collective one.

3.2 Annuities

The social planner again determines the retirement benefits paid to the two groups such that the overall level of expected lifetime felicity is maximized:

$$\begin{aligned} &\max_{c_L(t),c_H(t)} n_L \int_0^\infty s_L(t)\overline{\beta}_L(t)\alpha(t)u\left(c_L(t)\right)\mathrm{d}t + n_H \int_0^\infty s_H(t)\overline{\beta}_H(t)\alpha(t)u\left(c_H(t)\right)\mathrm{d}t \\ \text{subject to } & n_L(1+\delta_a^L)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s_L(t)\cdot c_L(t)\mathrm{d}t + n_H(1+\delta_a^H)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s_H(t)\cdot c_H(t)\mathrm{d}t \\ &\leq n_L W_0^L + n_H W_0^H. \end{aligned} \tag{19}$$

Theorem 3.4. *The solution to optimization problem* (19) *is*

$$c_i^*(t) = \left(\frac{\left(1+\delta_a^{(i)}\right)\lambda_a^G e^{-\int_0^t r(s)ds}}{\overline{\beta}_i(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}, \tag{20}$$

where the Lagrangian multiplier λ_a^G *is given by*

$$\begin{aligned} \lambda_a^G = &\left(\left(n_L W_0^L + n_H W_0^H\right)^{-1}\left(n_L\left(1+\delta_a^L\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s_L(t)\cdot\left(\frac{\left(1+\delta_a^L\right)e^{-\int_0^t r(s)ds}}{\beta_L(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right.\right. \\ &\left.\left.+n_H\left(1+\delta_a^H\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s_H(t)\cdot\left(\frac{\left(1+\delta_a^H\right)e^{-\int_0^t r(s)ds}}{\beta_H(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right)\right)^{\gamma}. \end{aligned}$$

For $\gamma \neq 1$*, the collective expected discounted lifetime felicity is then given by*

$$U_a^G = \frac{\lambda_a^G}{1-\gamma}\left(n_H W_0^H + n_L W_0^L\right).$$

Proof: See Appendix A.7.

Again, the optimal consumption profiles obtained from the government's problem (19) differ from the individually optimal profile (11) only in the Lagrangian multiplier λ_a^G.

Remark 3.5. *There are the following special cases:*

- *If* $\alpha(t) = e^{-\int_0^t r(s)ds}$ *and* Φ *is a linear function, the optimal payout function* $c_i^*(t)$ *is constant.*
- *If* Φ *is a linear function, it holds* $c_L^*(t) = c_H^*(t)$. *This is particularly different from the tontine, where* $d_L^*(t) < d_H^*(t)$ *for a linear* Φ. *Hence, similarly to the annuity, if* $W_0^H \leq W_0^L$, *type-*L*-individuals transfer some of their wealth to type-*H*-individuals. Conversely, if* $W_0^H > W_0^L$, *it is possible that the wealth transfer is reversed.*

Although there are some differences in the optimal consumption profiles of the annuity and the tontine, the numerical results regarding wealth transfers are qualitatively identical for both products. In order to make the paper focused, we therefore leave out the numerical analysis in this place.

3.3 Combined investment

Given the optimal solutions to the social planner's problem for the single products, we now consider again a social planner having access to both annuities and tontines at the same time. It is of particular importance for a government to determine the conditions under which annuities and under which tontines should be preferred. The optimization problem is give by:

$$\begin{aligned}
\max_{c_{ad}^L(t), d_{ad}^L(t), c_{ad}^H(t), d_{ad}^H(t)} \; & n_L \mathbb{E}\left[\int_0^\infty S_L(t)\beta_L(t)\alpha(t) u\left(c_{ad}^L(t) + \frac{d_{ad}^L(t)}{S_L(t)}\right) \mathrm{d}t\right] \\
& + n_H \mathbb{E}\left[\int_0^\infty S_H(t)\beta_H(t)\alpha(t) u\left(c_{ad}^H(t) + \frac{d_{ad}^H(t)}{S_H(t)}\right) \mathrm{d}t\right] \\
\text{subject to } & n_L(1+\delta_a^L)\int_0^\infty e^{-\int_0^t r(s)ds} s_L(t) c_{ad}^L(t)\mathrm{d}t + n_L(1+\delta_d^L)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot d_{ad}^L(t)\mathrm{d}t \\
& + n_H(1+\delta_a^H)\int_0^\infty e^{-\int_0^t r(s)ds} s(t) c_{ad}^H(t)\mathrm{d}t + n_H(1+\delta_d^H)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot d_{ad}^H(t)\mathrm{d}t \\
& \leq n_L W_0^L + n_H W_0^H.
\end{aligned} \tag{21}$$

Again, we are unable to derive a generally optimal consumption profile to optimization problem (13). Nevertheless, we can still derive theoretical results on the comparison of annuities, tontines and a portfolio of the two. In Theorem 3.6, we summarize the results.

Theorem 3.6.

1. *If and only if*

$$\delta_a^{(i)} \leq \underline{\delta}_a^{(i)} := \inf_{t \geq 0} \frac{\overline{\beta}_i(t)}{\mathbb{E}\left[\beta_i(t)\right]}(1+\delta_d^{(i)}) - 1, \;\; i = H, L \tag{22}$$

the optimal solution to Problem (13) *is given by* $c_{ad}^{(i)}(t) = c_i^*(t)$ *,* $d_{ad}^{(i)}(t) = 0$ *, i.e. it is optimal to invest all the initial wealth in the optimal annuity.*

2. *If and only if*

$$\delta_a^{(i)} \geq \overline{\delta}_a^{(i)} := \sup_{t \geq 0} \frac{\kappa_{\gamma+1}^{(i)}(t)}{\kappa_\gamma^{(i)}(t) s_i(t)}(1+\delta_d^{(i)}) - 1, \;\; i = H, L \tag{23}$$

the optimal solution to Problem (13) *is given by* $d_{ad}^{(i)}(t) = d_i^*(t)$ *,* $c_{ad}^{(i)}(t) = 0$ *, i.e. it is optimal to invest all the initial wealth in the optimal tontine.*

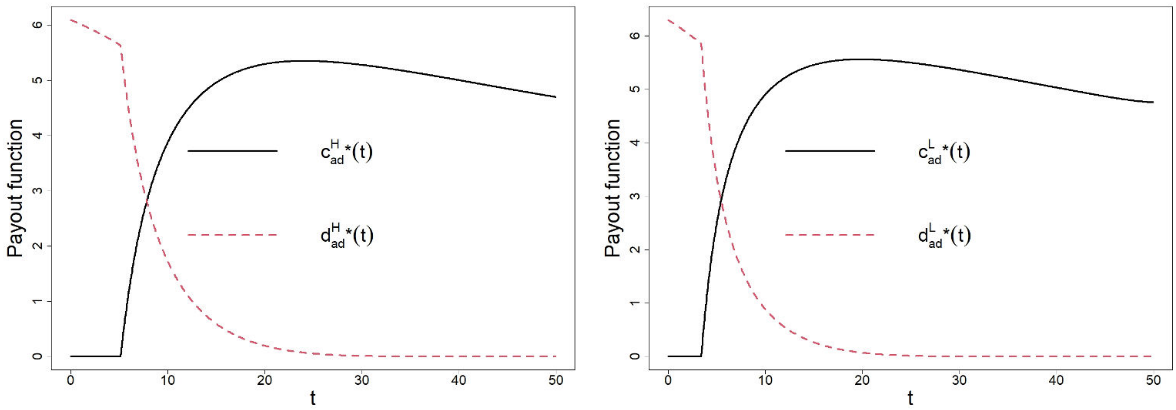
3. *If both condition* (22) *and* (23) *are not fulfilled, it is optimal for the agent to invest positive fractions of wealth in both the annuity and the tontine.*

Proof: See Appendix A.8.

Note that the conditions (22) and (23) exactly coincide with (14) and (15), respectively.

In Figure 5, we provide the optimal payout functions for both groups. We observe similar patterns as in Figure 3. The present values of consumption resulting from the annuity and the tontine for the living-longer-type agents are 59.89 and 54.38. The present values of consumption of the living-shorter-type agents are given by 54.01 and 31.71 for the annuity and the tontine. In particular, the wealth transfer is almost identical to the separate products.

In Table 6, we provide the wealth equivalents for type-H-agents, type-L-agents and the collective along with the corresponding present values. All in all, we observe similar patterns as in the individual case.



(a) Group H (b) Group L

Figure 5: Optimal payout functions in the portfolio for an agent exhibiting temporal risk aversion. The parameters are chosen as in Table 1.

Standard deviation	WE	Present values of annuity and tontine payments
	Type-H-individuals	
$\sigma = 0.0814$	100.0023	(55.85, 51.37)
$\sigma = 2 \cdot 0.0814$	100.0103	(55.27, 51.92)
$\sigma = 4 \cdot 0.0814$	100.0508	(52.90, 54.12)
$\sigma = 6 \cdot 0.0814$	100.1245	(51.70, 55.11)
	Type-L-individuals	
$\sigma = 0.0814$	100.0023	(53.57, 39.21)
$\sigma = 2 \cdot 0.0814$	100.0101	(53.08, 39.73)
$\sigma = 4 \cdot 0.0814$	100.0500	(51.14, 41.83)
$\sigma = 6 \cdot 0.0814$	100.1232	(50.39, 42.80)
	Collective	
$\sigma = 0.0814$	100.0023	(109.42, 90.58)
$\sigma = 2 \cdot 0.0814$	100.0102	(108.35, 91.65)
$\sigma = 4 \cdot 0.0814$	100.0504	(104.05, 95.95)
$\sigma = 6 \cdot 0.0814$	100.1239	(102.09, 97.91)

Table 6: Wealth equivalents dependent on the standard deviation of the shock ϵ. The remaining parameters are taken from Table 1.

4 Conclusion

This paper studies the optimal demand for retirement products in the presence of temporal risk aversion in a market offering both annuities and mortality-linked products like tontines. We consider the optimal retirement decision of a single agent as well as of a social planner maximizing the sum of the individual utility functions under differential mortality and wealth. We consider the optimal retirement decision in an exclusive annuity market, an exclusive tontine market and a market containing both types of products.

Our theoretical and numerical results suggest that agents' demand for tontines increases under temporal risk aversion. While it has been shown that no tontinization and full annuitization is optimal in the absence of safety loadings (e.g. Chen et al. (2020)), we determine conditions for pricing bounds under which agents subject to temporal risk aversion prefer to invest positive fractions of wealth in both annuities and tontines.

Similarly to individual agents, a social planner might prefer to both annuitize and tontinize some wealth instead of full annuitization. Furthermore, in most cases, we observe wealth transfers from living-shorter-type agents to living-longer-type agents in both annuities and tontines to a similar extent. The wealth ratio of the two groups turns out to be the main driving factor for the direction of the wealth transfers. If the living-longer-type owns more initial wealth than the living-shorter-type, it is possible that the wealth transfer goes from the living-longer-type to the living-shorter-type. Additionally, we find that temporal risk aversion and the group sizes can increase and decrease wealth transfers, while they do not have a major impact on the direction of the wealth transfer.

As an extension to this article, unit-linked products like variable annuities could be added to the markets. It would be interesting to see whether such products can outperform annuities and/or tontines under temporal risk aversion.

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Declaration of interest

Declaration of interest: none

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A Technical appendix

A.1 Proof of Lemma 2.4

By integration by parts and using Assumption 2.1, we rearrange (2) to:

$$EU(\boldsymbol{C}) = \mathbb{E}\left[\int_0^\infty S(t)\cdot\Phi'\left(\int_0^t \alpha(s)u\left(\frac{d(t)}{S(t)}\right)\mathrm{d}s\right)\alpha(t)u\left(\frac{d(t)}{S(t)}\right)\mathrm{d}t\right]. \tag{24}$$

We can now follow similar steps as in Appendix A.1 in Bommier et al. (2011): In particular, we assume a "priceless life context" as defined in Bommier (2006). In other words, we assume that $u(y) = 1 + \psi w(y)$, where ψ is a constant close to zero and the function $w(\cdot)$ is bounded. Then, we can rewrite (24) to

$$\begin{aligned} EU(\boldsymbol{C}) = \mathbb{E}\Bigg[&\int_0^\infty S(t)\cdot\Phi'\left(\int_0^t \alpha(s)\left(1+\psi w\left(\frac{d(t)}{S(t)}\right)\right)\mathrm{d}s\right)\alpha(t)\mathrm{d}t \\ &+\psi\int_0^\infty S(t)\cdot\Phi'\left(\int_0^t \alpha(s)\left(1+\psi w\left(\frac{d(t)}{S(t)}\right)\right)\mathrm{d}s\right)\alpha(t)w\left(\frac{d(t)}{S(t)}\right)\mathrm{d}t\Bigg]. \end{aligned}$$

Next, we assume that agents would accept giving up most of their consumption to live longer, which translates into $\psi \to 0$. We can then approximate the function Φ' using a Taylor expansion. Ignoring the terms of order greater than 1 in ψ, this delivers

$$\begin{aligned} EU(\boldsymbol{c}) \approx \mathbb{E}\Bigg[&\int_0^\infty S(t)\cdot\Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right)\alpha(t)\mathrm{d}t + \psi\int_0^\infty S(t)\cdot\Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right)\alpha(t)w\left(\frac{d(t)}{S(t)}\right)\mathrm{d}t \\ &+\psi\int_0^\infty S(t)\cdot\left(\int_0^t \alpha(s)w\left(\frac{d(s)}{S(s)}\right)\mathrm{d}s\right)\Phi''\left(\int_0^t \alpha(s)\mathrm{d}s\right)\alpha(t)\mathrm{d}t\Bigg]. \end{aligned}$$

Here, we change the order of integration in the third term to obtain the following approximation of the expected lifetime felicity:

$$
\begin{aligned}
EU(\boldsymbol{C}) \approx \mathbb{E}\Bigg[&\int_0^\infty S(t) \cdot \Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right)\alpha(t)\mathrm{d}t + \psi \int_0^\infty S(t) \cdot \Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right)\alpha(t) w\left(\frac{d(t)}{S(t)}\right)\mathrm{d}t \\
&+ \psi \int_0^\infty w\left(\frac{d(t)}{S(t)}\right) \cdot \left(\int_t^\infty \alpha(s)S(s)\Phi''\left(\int_0^s \alpha(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right)\alpha(t)\mathrm{d}t\Bigg] \\
= \mathbb{E}\Bigg[&\int_0^\infty S(t) \cdot \Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right)\alpha(t)\mathrm{d}t + \psi \int_0^\infty w\left(\frac{d(t)}{S(t)}\right)\alpha(t) \\
&\cdot \left(S(t) \cdot \Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right) + \int_t^\infty \alpha(s)S(s)\Phi''\left(\int_0^s \alpha(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right)\mathrm{d}t\Bigg].
\end{aligned}
$$

Applying the integration by parts technique, we obtain

$$
S(t) \cdot \Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right) + \int_t^\infty \alpha(s)S(s)\Phi''\left(\int_0^s \alpha(\tau)\mathrm{d}\tau\right)\mathrm{d}s = -\int_t^\infty S'(s)\Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right).
$$

This delivers the following approximation for the expected lifetime felicity:

$$
EU(\boldsymbol{C}) \approx \mathbb{E}\left[\int_0^\infty S(t) \cdot \Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right)\alpha(t)\mathrm{d}t + \psi \int_0^\infty w\left(\frac{d(t)}{S(t)}\right)\alpha(t)S(t) \cdot \beta(t)\mathrm{d}t\right],
$$

where $\beta(t)$ is defined as in (4). Using again $u(y) = 1 + \psi w(y)$ delivers

$$
\begin{aligned}
EU(\boldsymbol{C}) \approx \mathbb{E}\Bigg[&\int_0^\infty S(t) \cdot \Phi'\left(\int_0^t \alpha(s)\mathrm{d}s\right)\alpha(t)\mathrm{d}t \\
&+ \int_0^\infty u\left(\frac{d(t)}{S(t)}\right)\alpha(t)S(t) \cdot \beta(t)\mathrm{d}t - \int_0^\infty \alpha(t)S(t)\beta(t)\mathrm{d}t\Bigg].
\end{aligned}
$$

Dropping the constants in this expression, we end up with the additive lifetime felicity function

$$
EU(\boldsymbol{C}) \approx \mathbb{E}\left[\int_0^\infty u\left(\frac{d(t)}{S(t)}\right)\alpha(t)S(t) \cdot \beta(t)\mathrm{d}t\right],
$$

which is equal to (3). □

A.2 Proof of Theorem 2.5

The first order condition is given by

$$\kappa_\gamma(t)\alpha(t)u'(d(t)) = (1+\delta_d)\,\lambda_d e^{-\int_0^t r(s)ds}.$$

Hence, we obtain the optimal payout function (6). Plugging (6) back into the budget constraint delivers

$$\begin{aligned}
W_0 &= (1+\delta_d)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot\left(\frac{(1+\delta_d)\,\lambda_d e^{-\int_0^t r(s)ds}}{\kappa_\gamma(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t \\
\Leftrightarrow\quad (\lambda_d)^{\frac{1}{\gamma}} &= \frac{1+\delta_d}{W_0}\int_0^\infty e^{-\int_0^t r(s)ds}\cdot\left(\frac{(1+\delta_d)\,e^{-\int_0^t r(s)ds}}{\kappa_\gamma(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t.
\end{aligned}$$

For $\gamma \neq 1$, the expected discounted lifetime felicity is then given by

$$\begin{aligned}
\int_0^\infty \kappa_\gamma(t)\alpha(t)u(d^*(t))\mathrm{d}t &= \int_0^\infty \kappa_\gamma(t)\alpha(t)u\left(\left(\frac{(1+\delta_d)\,\lambda_d e^{-\int_0^t r(s)ds}}{\kappa_\gamma(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\right)\mathrm{d}t \\
&= \frac{(\lambda_d)^{1-\frac{1}{\gamma}}}{1-\gamma}\int_0^\infty e^{-\int_0^t r(s)ds}\left(\frac{(1+\delta_d)\,e^{-\int_0^t r(s)ds}}{\kappa_\gamma(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t \\
&= \frac{(\lambda_d)^{1-\frac{1}{\gamma}}}{1-\gamma}\cdot(\lambda_d)^{\frac{1}{\gamma}}\cdot\frac{W_0}{1+\delta_d} \\
&= \frac{\lambda_d}{1-\gamma}\cdot\frac{W_0}{1+\delta_d}.
\end{aligned}$$

□

A.3 Lemma A.1

Lemma A.1. *It holds* $\mathbb{E}\left[S(t)\beta(t)\right] = s(t)\overline{\beta}(t)$.

Proof. A straightforward calculation delivers:

$$\begin{aligned}
\mathbb{E}\left[S(t)\beta(t)\right] &= \mathbb{E}\left[-\int_t^\infty S'(s)\cdot\Phi'\left(\int_0^s \alpha(\tau)d\tau\right)\mathrm{d}s\right] \\
&= -\int_t^\infty \mathbb{E}\left[S'(s)\right]\cdot\Phi'\left(\int_0^s \alpha(\tau)d\tau\right)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= -\int_t^\infty s'(s)\cdot\Phi'\left(\int_0^s \alpha(\tau)d\tau\right)\mathrm{d}s \\
&= \overline{\beta}(t)s(t).
\end{aligned}
$$

□

A.4 Proof of Theorem 2.6

The first order condition is given by

$$
\overline{\beta}(t)\alpha(t)u'(c(t)) = (1+\delta_a)\,\lambda_a e^{-\int_0^t r(s)ds}.
$$

Hence, we obtain the solution to Problem (10) as (11). Plugging the optimal consumption (11) back into the budget constraint delivers

$$
\begin{aligned}
W_0 &= (1+\delta_a)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s(t)\cdot\left(\frac{(1+\delta_a)\,\lambda_a e^{-\int_0^t r(s)ds}}{\overline{\beta}(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t \\
\Leftrightarrow\quad (\lambda_a)^{\frac{1}{\gamma}} &= \frac{1+\delta_a}{W_0}\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s(t)\cdot\left(\frac{(1+\delta_a)\,e^{-\int_0^t r(s)ds}}{\overline{\beta}(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t.
\end{aligned}
$$

For $\gamma \neq 1$, the expected discounted lifetime felicity is then given by

$$
\begin{aligned}
\int_0^\infty s(t)\overline{\beta}(t)\alpha(t)u(c(t))\mathrm{d}t &= \int_0^\infty s(t)\overline{\beta}(t)\alpha(t)u\left(\left(\frac{(1+\delta_a)\,\lambda_a e^{-\int_0^t r(s)ds}}{\overline{\beta}(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\right)\mathrm{d}t \\
&= \frac{(\lambda_a)^{1-\frac{1}{\gamma}}}{1-\gamma}\int_0^\infty e^{-\int_0^t r(s)ds}s(t)\left(\frac{(1+\delta_a)\,e^{-\int_0^t r(s)ds}}{\overline{\beta}(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t \\
&= \frac{(\lambda_a)^{1-\frac{1}{\gamma}}}{1-\gamma}\cdot(\lambda_a)^{\frac{1}{\gamma}}\cdot\frac{W_0}{1+\delta_a} \\
&= \frac{\lambda_a}{1-\gamma}\cdot\frac{W_0}{1+\delta_a}.
\end{aligned}
$$

□

A.5 Proof of Theorem 2.7

The proof can be carried out in a similar way as in Chen et al. (2020).

1. The objective function to Problem (13) can be written as

$$\mathcal{F} = \int_0^\infty \mathbb{E}\left[\alpha(t)S(t)\beta(t)u\left(c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}\right)\right] \mathrm{d}t$$
$$+ \lambda\left(W_0 - (1+\delta_a)\int_0^\infty e^{-\int_0^t r(s)ds} s(t)c_{ad}(t)\mathrm{d}t - (1+\delta_d)\int_0^\infty e^{-\int_0^t r(s)ds} \cdot d_{ad}(t)\mathrm{d}t\right).$$

Setting the first-order conditions equal to zero, we obtain:

$$\frac{\partial \mathcal{F}}{\partial c_{ad}(t)} = \mathbb{E}\left[\alpha(t)S(t)\beta(t)u'\left(c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}\right)\right] - \lambda(1+\delta_a)e^{-\int_0^t r(s)ds}s(t) = 0$$
$$\frac{\partial \mathcal{F}}{\partial d_{ad}(t)} = \mathbb{E}\left[\alpha(t)\beta(t)u'\left(c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}\right)\right] - \lambda(1+\delta_d)e^{-\int_0^t r(s)ds} = 0.$$

Now we set $c_{ad}(t) = c^*(t)$, $d_{ad}(t) = 0$ and $\lambda = \lambda_a$ to obtain a condition under which the annuity is optimal, using Lemma A.1:

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial c_{ad}(t)} &= \mathbb{E}\left[\alpha(t)S(t)\beta(t)u'\left(c^*(t)\right)\right] - \lambda_a(1+\delta_a)e^{-\int_0^t r(s)ds}s(t) \\ &= \lambda_a(1+\delta_a)s(t)e^{-\int_0^t r(s)ds} - \lambda_a(1+\delta_a)e^{-\int_0^t r(s)ds}s(t) = 0.\end{aligned}$$

Further, we obtain

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial d_{ad}(t)} &= \mathbb{E}\left[\alpha(t)\beta(t)u'\left(c^*(t)\right)\right] - \lambda_a(1+\delta_d)e^{-\int_0^t r(s)ds} \\ &= \mathbb{E}\left[\frac{\beta(t)}{\overline{\beta}(t)}\lambda_a(1+\delta_a)e^{-\int_0^t r(s)ds}\right] - \lambda_a(1+\delta_d)e^{-\int_0^t r(s)ds} \\ &= \lambda_a e^{-\int_0^t r(s)ds}\left((1+\delta_a)\mathbb{E}\left[\frac{\beta(t)}{\overline{\beta}(t)}\right] - (1+\delta_d)\right) \overset{!}{\leq} 0.\end{aligned}$$

Now let us take a look at the second-order derivative:

$$\frac{\partial^2 \mathcal{F}}{\partial d_{ad}(t)^2} = \mathbb{E}\left[\alpha(t)\beta(t)u''\left(c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}\right)\frac{1}{S(t)}\right] < 0$$

since $u'' < 0$. If the second-order derivative is strictly negative, this implies that the first-order derivative is strictly decreasing in $d_{ad}(t)$. If $\partial\mathcal{F}(\alpha)/\partial d_{ad}(t) < 0$, the expected lifetime felicity could be increased by choosing the payout function $d_{ad}(t)$ even lower, which is not possible because we exclude negative retirement benefits. If $\partial\mathcal{F}(\alpha)/\partial d_{ad}(t) = 0$, $d_{ad}(t) = 0$ is the optimal solution and, therefore, the annuity is the optimal retirement product. Therefore, the annuity is optimal if and only if

$$(1+\delta_a)\mathbb{E}\left[\frac{\beta(t)}{\overline{\beta}(t)}\right] \leq (1+\delta_d) \quad \text{for all } t \geq 0.$$

2. Now we set $c_{ad}(t) = 0$, $d_{ad}(t) = d^*(t)$ and $\lambda = \lambda_d$ to obtain a condition under which the annuity is optimal:

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial c_{ad}(t)} &= \mathbb{E}\left[\alpha(t)S(t)\beta(t)u'\left(\frac{d^*(t)}{S(t)}\right)\right] - \lambda_d(1+\delta_a)e^{-\int_0^t r(s)ds}s(t) \\
&= \kappa_{\gamma+1}(t)\alpha(t)u'\left(d^*(t)\right) - \lambda_d(1+\delta_a)e^{-\int_0^t r(s)ds}s(t) \\
&= \frac{\kappa_{\gamma+1}(t)}{\kappa_\gamma(t)}\lambda_d(1+\delta_d)e^{-\int_0^t r(s)ds} - \lambda_d(1+\delta_a)e^{-\int_0^t r(s)ds}s(t) \\
&= \lambda_d e^{-\int_0^t r(s)ds}\left(\frac{\kappa_{\gamma+1}(t)}{\kappa_\gamma(t)}(1+\delta_d) - (1+\delta_a)s(t)\right).
\end{aligned}$$

Further, we obtain

$$\begin{aligned}
\frac{\partial \mathcal{F}}{\partial d_{ad}(t)} &= \mathbb{E}\left[\alpha(t)\beta(t)u'\left(\frac{d^*(t)}{S(t)}\right)\right] - \lambda_d(1+\delta_d)e^{-\int_0^t r(s)ds} \\
&= \alpha(t)\kappa_\gamma(t)u'\left(d^*(t)\right) - \lambda_d(1+\delta_d)e^{-\int_0^t r(s)ds} \\
&= 0.
\end{aligned}$$

Now let us take a look at the second-order derivative:

$$\frac{\partial^2 \mathcal{F}}{\partial c_{ad}(t)^2} = \mathbb{E}\left[\alpha(t)S(t)\beta(t)u''\left(c_{ad}(t) + \frac{d_{ad}(t)}{S(t)}\right)\right] < 0$$

since $u'' < 0$. If the second-order derivative is strictly negative, this implies that the first-order derivative is strictly decreasing in $c_{ad}(t)$. If $\partial\mathcal{F}(\alpha)/\partial c_{ad}(t) < 0$, the expected lifetime felicity could be increased by choosing $c_{ad}(t)$ even lower, which is not possible because we exclude negative retirement benefits. If $\partial\mathcal{F}(\alpha)/\partial c_{ad}(t) = 0$, $c_{ad}(t) = 0$ is the optimal solution and, therefore, the tontine is the optimal retirement product. Therefore, the tontine is optimal if and only if

$$\frac{\kappa_{\gamma+1}(t)}{\kappa_\gamma(t)}(1+\delta_d) - (1+\delta_a)s(t) \leq 0.$$

□

A.6 Proof of Theorem 3.2

The first order condition is given by

$$\kappa_\gamma^{(i)}(t)\alpha(t)u'(d_i(t)) = \left(1+\delta_d^{(i)}\right)\lambda_d^G e^{-\int_0^t r(s)ds}.$$

Hence, we obtain the optimal payout function (17). Plugging (17) back into the budget constraint delivers

$$\begin{aligned} n_L W_0^L + n_H W_0^H = n_L\left(1+\delta_d^L\right)&\int_0^\infty e^{-\int_0^t r(s)ds}\cdot\left(\frac{\left(1+\delta_d^L\right)\lambda_d^G e^{-\int_0^t r(s)ds}}{\kappa_\gamma^L(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t \\ &+ n_H\left(1+\delta_d^H\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot\left(\frac{\left(1+\delta_d^H\right)\lambda_d^G e^{-\int_0^t r(s)ds}}{\kappa_\gamma^H(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \left(\lambda_d^G\right)^{\frac{1}{\gamma}} = \left(n_L W_0^L + n_H W_0^H\right)^{-1}&\left(n_L\left(1+\delta_d^L\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot\left(\frac{\left(1+\delta_d^L\right)e^{-\int_0^t r(s)ds}}{\kappa_\gamma^L(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right. \\ &\left.+n_H\left(1+\delta_d^H\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot\left(\frac{\left(1+\delta_d^H\right)e^{-\int_0^t r(s)ds}}{\kappa_\gamma^H(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right). \end{aligned}$$

For $\gamma \neq 1$, the collective expected discounted lifetime felicity is then given by

$$\begin{aligned} U_d^G &= n_H\int_0^\infty \kappa_\gamma^H(t)\alpha(t)u\left(d_H^*(t)\right)\mathrm{d}t + n_L\int_0^\infty \kappa_\gamma^L(t)\alpha(t)u\left(d_L^*(t)\right)\mathrm{d}t \\ &= n_H\int_0^\infty \kappa_\gamma^H(t)\alpha(t)u\left(\left(\frac{\left(1+\delta_d^H\right)\lambda_d^G e^{-\int_0^t r(s)ds}}{\kappa_\gamma^H(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\right)\mathrm{d}t \\ &\quad + n_L\int_0^\infty \kappa_\gamma^L(t)\alpha(t)u\left(\left(\frac{\left(1+\delta_d^L\right)\lambda_d^G e^{-\int_0^t r(s)ds}}{\kappa_\gamma^L(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\right)\mathrm{d}t \\ &= \frac{(\lambda_d^G)^{1-\frac{1}{\gamma}}}{1-\gamma}\left(n_H\int_0^\infty \kappa_\gamma^H(t)\alpha(t)\left(\frac{\left(1+\delta_d^H\right)e^{-\int_0^t r(s)ds}}{\kappa_\gamma^H(t)\alpha(t)}\right)^{1-\frac{1}{\gamma}}\mathrm{d}t\right. \\ &\quad \left.+n_L\int_0^\infty \kappa_\gamma^L(t)\alpha(t)\left(\frac{\left(1+\delta_d^L\right)e^{-\int_0^t r(s)ds}}{\kappa_\gamma^L(t)\alpha(t)}\right)^{1-\frac{1}{\gamma}}\mathrm{d}t\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{(\lambda_d^G)^{1-\frac{1}{\gamma}}}{1-\gamma}\left(n_H\left(1+\delta_d^H\right)\int_0^\infty e^{-\int_0^t r(s)ds}\left(\frac{\left(1+\delta_d^H\right)e^{-\int_0^t r(s)ds}}{\kappa_\gamma^H(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right.\\
&\quad\left.+n_L\left(1+\delta_d^L\right)\int_0^\infty e^{-\int_0^t r(s)ds}\left(\frac{\left(1+\delta_d^L\right)e^{-\int_0^t r(s)ds}}{\kappa_\gamma^L(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right)\\
&= \frac{(\lambda_d^G)^{1-\frac{1}{\gamma}}}{1-\gamma}(\lambda_d^G)^{\frac{1}{\gamma}}\left(n_H W_0^H + n_L W_0^L\right) = \frac{\lambda_d^G}{1-\gamma}\left(n_H W_0^H + n_L W_0^L\right).
\end{aligned}
$$

□

A.7 Proof of Theorem 3.4

The first order condition is given by

$$
\overline{\beta}_i(t)\alpha(t)u'(c_i(t)) = \left(1+\delta_a^{(i)}\right)\lambda_a^G e^{-\int_0^t r(s)ds}, \quad i = H, L,
$$

which is equivalent to (20). Plugging this expression back into the budget constraint delivers

$$
\begin{aligned}
n_L W_0^L + n_H W_0^H = n_L\left(1+\delta_a^L\right)&\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s_L(t)\cdot\left(\frac{\left(1+\delta_a^L\right)\lambda_a^G e^{-\int_0^t r(s)ds}}{\overline{\beta}_L(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\\
+\, n_H\left(1+\delta_a^H\right)&\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s_H(t)\cdot\left(\frac{\left(1+\delta_a^H\right)\lambda_a^G e^{-\int_0^t r(s)ds}}{\overline{\beta}_H(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\left(\lambda_a^G\right)^{\frac{1}{\gamma}} &= \left(n_L W_0^L + n_H W_0^H\right)^{-1}\left(n_L\left(1+\delta_a^L\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s_L(t)\cdot\left(\frac{\left(1+\delta_a^L\right)e^{-\int_0^t r(s)ds}}{\overline{\beta}_L(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right.\\
&\quad\left.+n_H\left(1+\delta_a^H\right)\int_0^\infty e^{-\int_0^t r(s)ds}\cdot s_H(t)\cdot\left(\frac{\left(1+\delta_a^H\right)e^{-\int_0^t r(s)ds}}{\overline{\beta}_H(t)\alpha(t)}\right)^{-\frac{1}{\gamma}}\mathrm{d}t\right).
\end{aligned}
$$

For $\gamma \neq 1$, the collective expected discounted lifetime felicity is then given by

$$
U_a^G = n_H\int_0^\infty s_H(t)\overline{\beta}_H(t)\alpha(t)u\left(c_H^*(t)\right)\mathrm{d}t + n_L\int_0^\infty s_L(t)\overline{\beta}_L(t)\alpha(t)u\left(c_L^*(t)\right)\mathrm{d}t
$$

$$
\begin{aligned}
&= n_H \int_0^\infty s_H(t)\overline{\beta}_H(t)\alpha(t) u \left(\left(\frac{\left(1+\delta_a^H\right) \lambda_a^G e^{-\int_0^t r(s)ds}}{\overline{\beta}_H(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} \right) \mathrm{d}t \\
&\quad + n_L \int_0^\infty s_L(t)\overline{\beta}_L(t)\alpha(t) u \left(\left(\frac{\left(1+\delta_a^L\right) \lambda_a^G e^{-\int_0^t r(s)ds}}{\overline{\beta}_L(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} \right) \mathrm{d}t \\
&= \frac{(\lambda_a^G)^{1-\frac{1}{\gamma}}}{1-\gamma} \left(n_H \int_0^\infty s_H(t)\overline{\beta}_H(t)\alpha(t) \left(\frac{\left(1+\delta_a^H\right) e^{-\int_0^t r(s)ds}}{\overline{\beta}_H(t)\alpha(t)} \right)^{1-\frac{1}{\gamma}} \mathrm{d}t \right. \\
&\quad \left. + n_L \int_0^\infty s_L(t)\overline{\beta}_L(t)\alpha(t) \left(\frac{\left(1+\delta_a^L\right) e^{-\int_0^t r(s)ds}}{\overline{\beta}_L(t)\alpha(t)} \right)^{1-\frac{1}{\gamma}} \mathrm{d}t \right) \\
&= \frac{(\lambda_a^G)^{1-\frac{1}{\gamma}}}{1-\gamma} \left(n_H \left(1+\delta_a^H\right) \int_0^\infty e^{-\int_0^t r(s)ds} s_H(t) \left(\frac{\left(1+\delta_a^H\right) e^{-\int_0^t r(s)ds}}{\overline{\beta}_H(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} \mathrm{d}t \right. \\
&\quad \left. + n_L \left(1+\delta_a^L\right) \int_0^\infty e^{-\int_0^t r(s)ds} s_L(t) \left(\frac{\left(1+\delta_a^L\right) e^{-\int_0^t r(s)ds}}{\overline{\beta}_L(t)\alpha(t)} \right)^{-\frac{1}{\gamma}} \mathrm{d}t \right) \\
&= \frac{(\lambda_a^G)^{1-\frac{1}{\gamma}}}{1-\gamma} (\lambda_a^G)^{\frac{1}{\gamma}} \left(n_H W_0^H + n_L W_0^L \right) = \frac{\lambda_a^G}{1-\gamma} \left(n_H W_0^H + n_L W_0^L \right).
\end{aligned}
$$

□

A.8 Proof of Theorem 3.6

1. The objective function to Problem (21) can be written as

$$
\begin{aligned}
\mathcal{F} = n_L &\int_0^\infty \mathbb{E}\left[\alpha(t) S_L(t) \beta_L(t) u \left(c_{ad}^L(t) + \frac{d_{ad}^L(t)}{S_L(t)} \right) \right] \mathrm{d}t \\
&+ n_H \int_0^\infty \mathbb{E}\left[\alpha(t) S_H(t) \beta_H(t) u \left(c_{ad}^H(t) + \frac{d_{ad}^H(t)}{S_H(t)} \right) \right] \mathrm{d}t \\
&+ \lambda \Bigg(n_H W_0^H + n_L W_0^L - n_L(1+\delta_a^L) \int_0^\infty e^{-\int_0^t r(s)ds} s_L(t) c_{ad}^L(t) \mathrm{d}t \\
&\quad - n_L(1+\delta_d^L) \int_0^\infty e^{-\int_0^t r(s)ds} \cdot d_{ad}^L(t) \mathrm{d}t \\
&\quad - n_H(1+\delta_a^H) \int_0^\infty e^{-\int_0^t r(s)ds} s_H(t) c_{ad}^H(t) \mathrm{d}t \\
&\quad - n_H(1+\delta_d^H) \int_0^\infty e^{-\int_0^t r(s)ds} \cdot d_{ad}^H(t) \mathrm{d}t \Bigg).
\end{aligned}
$$

Setting the first-order conditions equal to zero, we obtain for $i = H, L$:

$$\frac{\partial \mathcal{F}}{\partial c_{ad}^{(i)}(t)} = \mathbb{E}\left[\alpha(t)S_i(t)\beta_i(t)u'\left(c_{ad}^{(i)}(t) + \frac{d_{ad}^{(i)}(t)}{S_i(t)}\right)\right] - \lambda(1+\delta_a^{(i)})e^{-\int_0^t r(s)ds}s_i(t) = 0$$

$$\frac{\partial \mathcal{F}}{\partial d_{ad}^{(i)}(t)} = \mathbb{E}\left[\alpha(t)\beta_i(t)u'\left(c_{ad}^{(i)}(t) + \frac{d_{ad}^{(i)}(t)}{S_i(t)}\right)\right] - \lambda(1+\delta_d^{(i)})e^{-\int_0^t r(s)ds} = 0.$$

Now we set $c_{ad}^{(i)}(t) = c_i^*(t)$, $d_{ad}^{(i)}(t) = 0$ and $\lambda = \lambda_a^G$ to obtain a condition under which the annuity is optimal, using Lemma A.1:

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial c_{ad}^{(i)}(t)} &= \mathbb{E}\left[\alpha(t)S_i(t)\beta_i(t)u'\left(c_i^*(t)\right)\right] - \lambda_a^G(1+\delta_a^{(i)})e^{-\int_0^t r(s)ds}s_i(t) \\ &= \lambda_a^G(1+\delta_a^{(i)})s_i(t)e^{-\int_0^t r(s)ds} - \lambda_a^G(1+\delta_a^{(i)})e^{-\int_0^t r(s)ds}s_i(t) = 0.\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial d_{ad}^{(i)}(t)} &= \mathbb{E}\left[\alpha(t)\beta_i(t)u'\left(c_i^*(t)\right)\right] - \lambda_a^G(1+\delta_d^{(i)})e^{-\int_0^t r(s)ds} \\ &= \mathbb{E}\left[\frac{\beta_i(t)}{\overline{\beta}_i(t)}\lambda_a^G(1+\delta_a^{(i)})e^{-\int_0^t r(s)ds}\right] - \lambda_a^G(1+\delta_d^{(i)})e^{-\int_0^t r(s)ds} \\ &= \lambda_a^G e^{-\int_0^t r(s)ds}\left((1+\delta_a^{(i)})\mathbb{E}\left[\frac{\beta_i(t)}{\overline{\beta}_i(t)}\right] - (1+\delta_d^{(i)})\right) \overset{!}{\leq} 0.\end{aligned}$$

Analogously to Theorem 2.7, the annuity is therefore optimal if and only if

$$(1+\delta_a^{(i)})\mathbb{E}\left[\frac{\beta_i(t)}{\overline{\beta}_i(t)}\right] \leq (1+\delta_d^{(i)}) \quad \text{for all } t \geq 0, i = H, L.$$

2. Now we set $c_{ad}^{(i)}(t) = 0$, $d_{ad}^{(i)}(t) = d_i^*(t)$ and $\lambda = \lambda_d^G$ to obtain a condition under which the annuity is optimal:

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial c_{ad}^{(i)}(t)} &= \mathbb{E}\left[\alpha(t)S_i(t)\beta_i(t)u'\left(\frac{d_i^*(t)}{S_i(t)}\right)\right] - \lambda_d^G(1+\delta_a^{(i)})e^{-\int_0^t r(s)ds}s_i(t) \\ &= \kappa_{\gamma+1}^{(i)}(t)\alpha(t)u'\left(d_i^*(t)\right) - \lambda_d^G(1+\delta_a^{(i)})e^{-\int_0^t r(s)ds}s_i(t) \\ &= \frac{\kappa_{\gamma+1}^L(t)}{\kappa_\gamma^{(i)}(t)}\lambda_d^G(1+\delta_d^{(i)})e^{-\int_0^t r(s)ds} - \lambda_d^G(1+\delta_a^{(i)})e^{-\int_0^t r(s)ds}s_i(t) \\ &= \lambda_d^G e^{-\int_0^t r(s)ds}\left(\frac{\kappa_{\gamma+1}^{(i)}(t)}{\kappa_\gamma^{(i)}(t)}(1+\delta_d^{(i)}) - (1+\delta_a^{(i)})s_i(t)\right) \overset{!}{\leq} 0.\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial d_{ad}^{(i)}(t)} &= \mathbb{E}\left[\alpha(t)\beta_i(t)u'\left(\frac{d_i^*(t)}{S_i(t)}\right)\right] - \lambda_d^G(1+\delta_d^{(i)})e^{-\int_0^t r(s)ds} \\ &= \alpha(t)\kappa_\gamma^{(i)}(t)u'\left(d_i^*(t)\right) - \lambda_d^G(1+\delta_d^{(i)})e^{-\int_0^t r(s)ds} = 0.\end{aligned}$$

Analogously to Theorem 2.7, the tontine is therefore optimal if and only if

$$\frac{\kappa_{\gamma+1}^{(i)}(t)}{\kappa_\gamma^{(i)}(t)}(1+\delta_d^{(i)}) - (1+\delta_a^{(i)})s_i(t) \leq 0 \quad \text{for all } t \geq 0, i = H, L.$$

$\square$