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The Impact of Longevity Risks on Variable Annuities

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The Impact of Longevity Risks on Variable Annuities

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Abstract

This thesis aims to project the impact of longevity risks on variable annuities in the insurer's and retirees' point of view. Mortality rates are fitted and forecasted using the Lee-Carter method, with data from the Human Mortality Database. Data at older ages are extended using the Kannisto method. Different sources of longevity risks can be distinguished under this model, with specific focus on the macro-longevity risk. Before the longevity risk analysis, the mortality impact is incorporated in variable annuities through biometric returns in assumed interest rate. For the sake of completeness, micro-longevity risk is analyzed with different scenarios of pension fund pools and sizes. Thereafter, the focus is on macro-longevity risk, with different risk components distinguished within this risk. Two approaches are used to quantify these uncertainties, the stressed-trend method and the value-at-risk framework. Results are compared using obtained life expectancy values and variable annuity payouts.

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1 Introduction

In the last decades, mortality rates are rapidly decreasing over time. Decrease in mortality means that the life expectancy of the population increases. The Dutch mortality data shows[1] that the period life expectancy at birth for males in 1850 is 38.89 years. However this number has increased drastically to 79.88 years in 2016. For females this has risen from 40.80 years in 1850 to 83.12 years in 2016[2]. The aforementioned (period) life expectancies are based on already known mortality rates, but in reality the probabilities are unknown due to the fact that they also depends on the future. Forecasting the future comes with risk. As already mentioned, people are living longer, but actuaries are concerned due to underestimation of the life expectancy and longer lifespan than expected as a result. These risks, uncertainties about the future mortality rates or future distribution of death rates are called the longevity risks. Pension funds or pension insurers who offers (guaranteed) annuities are much affected by this risk.

However in the Netherlands, as of September 2016[3], pension funds are no more required to offer guaranteed and fixed lifetime income to retirees during their retirement phase. Pensioners can now choose between the fixed annuity or the variable annuity. Up until 2016, retirees only have the option of fixed annuity where they receive a fixed amount until death. After the new legislation in 2016, retirees can now invest a part of their accumulated pension wealth into risky assets in the decumulation phase. Investing will increase the expected pension payments of the retirees, but the benefits are now a variable due to potential financial shocks. Furthermore, the choice is made available for the pensioners to choose the assumed interest rate (AIR) for how they want to allocate their wealth among the future years [4,5]. The paper presented by Balter and Werker (2017) focuses on the wealth allocation through the AIR for variable annuities. Pension funds on the other hand, after the legislation, have the choice to pass on the longevity risk to the retirees, while the pension fund need carry the risk before. As a consequence, the retirees not only bear the investment risk but also the longevity risk. With the inclusion of longevity risk, wealth allocation will then also depends on future survival probabilities.

Not all pension funds pass the macro-longevity risk to their pensioners (Balter et al., 2018). If this is the case, it is necessary for the pension fund to adequately quantify this risk to overcome issues with the Solvency II requirements. One aspect of the pension provider's risk profile is the life expectancy of the individuals. Uncertainties about the longevity result in an increase or decrease of the individual's expected lifetime. Underestimation of the life expectancy might lead to not collecting enough premiums and the pension fund might be more prone to the solvency issues from the increased liabilities and underfunding.

On the other hand, pension funds may pass on the longevity risk to retirees and for them to carry it instead. If this is the case, their pension payouts are dependent on the uncertain development of future mortality rates.

Under the longevity risk we distinguish the two elements *micro-longevity risk* and *macro-longevity risk*. Micro-longevity risk appears when the pension fund is not large enough to follow the expected mortality. Micro-longevity might affect the *biometric return*[6] of an individual. Macro-longevity risk reflects to the shock in future mortality which is experienced by everyone in the same cohort but not all cohorts are affected the same way.

Either way, since macro-longevity risk is not diversifiable, this risk causes deviations in mortality rates no matter the amount of participants in a pool. Macro-longevity risk can further be dis-

[1] https://www.mortality.org

[2] More details of expected life expectancy are found in Section 3.1.

[3] As of September 2016, reform of the payout design lifted the requirement to provide guaranteed lifetime nominal income contracts, Bovenberg and Nijman (2019).

[4] In theory, pensioners can choose the AIR and investment mix freely. In practice however, the options are limited to what the pension provider has to offer.

[5] In the Dutch institutional setting, the wealth distribution is determined by the projected interest rate ("projectierente", which is the risk-fee interest rate) and a fixed decrease ("vaste daling"). Choosing the AIR is equivalent to choosing the projected interest rate with fixed decrease.

[6] The so-called biometric return is the capital from individuals who pass away and is allocated to the surviving individuals.

tinguished into more risk components. Because of the widely-ranging uncertainties of longevity, attention need to be given on adequately quantifying these risks, with mainly the focus on macro-longevity risk.

First of all, a stochastic mortality model is needed to forecast the mortality rates and hereafter used to outline the performance of the rates under the implementation of the risk sources. Although pension providers use their own models to forecast and obtain the best estimates, we use the well-known Lee-Carter model and base the uncertainties through this model.

Using the forecasted rates, assuming that the Lee-Carter model is correct, the (macro-)longevity risk can be estimated. In this thesis, we follow two approaches by Richards et al. (2014). The first one is the stressed-trend approach, which quantifies the risk by taking the confidence interval of the long-term trend of the mortality rates. In other words, accumulated changes from the best estimate that might appear (with a probability) over a period of time are the stress and the stress-trend is then the best-estimate of the trend minus the stress.

The second approach is the value-at-risk approach. This is a one-year approach, as it models how one-year extra data of mortality, forecasted with uncertainties, might change the best estimates of future years.

Results of the two approaches can be compared as follows. Using the expected remaining lifetime with the best estimates, it can be compared how much the change is when uncertainties are considered. This is relevant for the pension funds if they bear the risk. Another way of comparison is through the variable annuities payments. A change in survival probabilities will not change the wealth allocation that has happened at the beginning of the retirement phase, but does affect the expected payments that depends on the biometric return in the future. A change in mortality rates, changes the biometric returns.

The goal and the main purpose of this thesis is to research on:

- "What is the impact of different longevity risk components on variable annuities in the decumulation phase?".
- "How is it for the pension fund to bear the longevity risks or the retirees to share these risks instead?".

The structure of this thesis is as follows. First of all, in Section 2, a literature review about variable annuities, mortality models and longevity risks is given. In Section 3.1, the mortality model used in this thesis is explained in details, with an overview of longevity risks given in Section 3.5. Incorporating mortality into variable annuities and an numerical illustration will be given Section 4. Furthermore, micro-longevity risk is briefly discussed in Section 5. In Section 6, the risk is investigated through the stressed-trend approach in Section 6.3 and investigated through the value-at-risk in Section 6.4. Lastly, Section 7 concludes with recommendations for future research. In the Appendix, some mathematical derivations, used methods and graphs of the model analysis is given.

2 Literature Review

This chapter presents an overview of some literature concerning variable annuities, mortality and longevity risks.

Annuities are insurance products which provide retirees periodic pension benefits that last typically for the rest of their life. In case of fixed annuities, the amounts are fixed, returns are based on the risk-free rate and the benefits are paid until death. Hence, there are no financial risk or longevity risk.

2.1 Variable annuities

After the reform of September 2016, retirees under the Dutch pension system have the option to invest their pension wealth in risky assets. Retirees can choose to convert their pension wealth into life-long variable annuities. Since this type of annuity also comes with financial risks, it is interesting to know the benefits of converting into the variable annuity instead of the fixed annuity. Several authors investigated the effects variable annuity when it is implemented as a draw down instrument in the decumulation phase. Welfare effects of using the variable annuities are investigated by e.g. Koijen et al. (2011), Brown et al. (1999) or Horneff et al. (2009). Pricing of the variable annuities are researched on by for example Bacinello et al. (2011), Bauer et al. (2008). Chen and Milevsky (2003) focus their research on optimal utility-maximizing asset allocation of the variable annuities with a note by Charupat and Milevsky (2002). On the other hand, a paper by Balter and Werker (2017) studies the risk-return trade-off for variable annuities, but the focus is on the choice of the assumed interest rate (AIR) for wealth allocation to obtain constant pension payments. They also consider the setting where financial shocks are smoothed which leads to the so-called BNW discount curve.

2.2 Mortality Models

The history of mortality models started with mortality laws, a mathematical formula depending on age which describes the mortality pattern. The laws are used to parameterize the mortality patterns. The most famous one is probably the one by Gompertz (1825). The Gompertz - Makeham law is the extension of the Gompertz law which states that the death rates can be summarized with an exponentially increasing age-dependent component (Gompertz term) and an age-independent component (Makeham term, Makeham (1860)). While the Gompertz law assumes a linear increase with age for the logarithm of mortality which is relevant for ages between 30 to 80 years, the Makeham term adds the external causes of deaths[7] which improves fit for younger ages.

Since the 1950s, a new trend has started for which mortality rates at older ages has unexpectedly declined rapidly. Hence, new dynamic models are needed which are functions over age and time with the aim to be able to forecast the future mortality rates. The most influential stochastic mortality model is the model by Lee and Carter (1992). This is the mortality model used in this thesis and is further explained in Section 3.2. There follows quite a few alternatives to this model. Brouhns et al. (2002) for example propose the Poisson model. This model is a modification of the Lee-Carter model where the number of deaths conditional on the exposure-to-risk is modeled as a Poisson random variable, while on the other hand, in the Lee-Carter model, the central death rates are modeled as a random variable. Although the Lee-Carter model is one of the most well-known and used mortality model, it is not free of drawbacks. Cairns et al. (2008) mentioned few of these drawbacks. Modifications and extensions of the Lee-Carter model are proposed by e.g. Renshaw and Haberman (2003), Delwarde et al. (2007) to solve the disadvantages. Another widely used model is the two-factor 'Cairns-Blake-Dowd-model' proposed by Cairns et al. (2006). A relevant model for the insurance world is the model proposed by Li and Lee (2005). This model aims to improve mortality forecasts of an individual country by using data from different countries in Europe to estimate the model parameters.

[7] External causes of death are e.g. accidents, low-mortality rate countries, living conditions. All these factors are not age dependent.

2.3 Longevity Risk

Longevity risk arises due to the uncertainty about an individual's expected lifetime. This risk can be decomposed into the micro-longevity and macro-longevity risk. On top of that, there is parameter risk and model risk. Richards et al. (2014) listed the different components of the risks and used a Value-at-Risk framework approach, which is applicable to different stochastic mortality models, to quantify the trend risk. Hari et al. (2008) on the other hand, analyzes the solvency of portfolios with pension annuities by having the micro-longevity, macro-longevity and parameter risk. The papers by De Waegenaere et al. (2017) and De Waegenaere et al. (2018) focus on the risk sharing rules for longevity risks. Contributing to the longevity literature, Broeders et al. (2018) optimize the risk sharing rule for macro-longevity risk under welfare analyses. Furthermore, related is the group self-annuitisation schemes presented by e.g. Piggott et al. (2005) and Qiao and Sherris (2013).

In Pitacco (2002), the impact that longevity risk has on life annuities is analyzed, with particular focus on the solvency requirements and reinsurance arrangements. Furthermore, the impact of longevity uncertainties on the riskiness of pension annuity contracts are investigated by e.g. Di Lorenzo and Sibillo (2003), Olivieri and Pitacco (2003). On the other hand, the method used to quantify parameter risk depends on the stochastic mortality model used. Cairns (2000) and Cairns et al. (2006) incorporated parameter uncertainty by means of a Bayesian framework. Another approach to include parameter risk is using the bootstrap procedure by Brouhns et al. (2005) and Renshaw and Haberman (2008).

3 Mortality

In Section 3.1, we first introduce some notations related to mortality. Furthermore, Section 3.2 introduces the model for mortality rates estimation and forecast, one of the most influential statistical model of mortality by Lee and Carter (1992). The Human Mortality Database, Section 3.3, would be the source for our mortality data. In Section 3.4, the mortality patterns of the data is mentioned briefly. And lastly, in Section 3.5, the components of longevity risks are introduced.

3.1 Notations

In this section, some notations and terms used in mortality studies that are relevant for the rest of the thesis are introduced.
Available population and mortality data sets have most if not all the following:

- $p_{x,t}$: the one-year survival probability of a person with age x at time t. The one-year death probability of someone aged x at time t is denoted by $q_{x,t}$. Furthermore, there is ${}_kp_{x,t}$, the probability that an individual aged x at time t will live at least another k years. Hence,

  $$ {}_kp_{x,t} = p_{x,t} \cdot p_{x+1,t+1} \cdot \ldots \cdot p_{x+k-1,t+k-1}. $$

  For the sake of completeness, the probability that an individual aged x at time t will die within k years is denoted by ${}_kq_{x,t} = 1 - {}_kp_{x,t}$.

- $\mu_{x,t}$: the force of mortality for a person with age x at time t. This is the instantaneous rate of mortality. One can verify [8] that

  $$ p_{x,t} = \exp(-\mu_{x,t}); $$

- $D_{x,t}$: the observed number of deaths at the beginning of year t for cohort aged x in year t;

- $E_{x,t}$: the exposure-to-risk, this is the total amount of years lived at time t for individual aged x;

- $m_{x,t}$: the central death rate for age x in year t. The central death rate of an individual at age x at time t is

  $$ m_{x,t} = \frac{D_{x,t}}{E_{x,t}}; \tag{1} $$

  From Gerber (2013), the maximum likelihood estimator of the force of mortality is given by $\hat{\mu}_{x,t} = \frac{D_{x,t}}{E_{x,t}}$. Hence, $m_{x,t} = \hat{\mu}_{x,t}$.

- $T_{x,t}$: the curtate remaining lifetime at time t of an individual aged x at time t.

Conditional to survival rates up to time t, an individual aged x at time t will have an expected remaining curtate lifetime of $\mathbb{E}_t[T_{x,t}]$ years according the following equations.

$$ \mathbb{E}_t[T_{x,t}] = \sum_{k=1}^{MaxAge-x} \mathbb{E}_t\left[{}_kp_{x,t}\right] \tag{2} $$

$$ = \sum_{k=1}^{MaxAge-x} \mathbb{E}_t\left[\exp\left(-\sum_{s=0}^{k-1} \mu_{x+s,t+s}\right)\right], \tag{3} $$

where $MaxAge$ refers to the maximum age that an individual is assumed to live in a given population. $\mathbb{E}_t[T_{x,t}]$ is known as the *cohort life expectancy*.

There is also the *period* life expectancy. The remaining lifetime for an individual age x at time t is based on the survival probabilities at time t. Hence, the survival probability for a person to

[8] The force of mortality can be defined as: $\mu_{x,t} = \frac{d}{dk} \ln\left({}_kp_{x,t}\right)$. This implies that $p_{x,t} = \exp(-\int_{k=0}^{1} \mu_{x+k,t} dk)$. Furthermore it is assumed that the force of mortality is constant over the year, $\mu_{x+k,t} = \mu_{x,t}$ for $k \in [0, 1)$.

reach age $x+5$ at time $t+5$ is based on the probabilities of age $x+1$, $x+2$ etc. in year t. The period life expectancy $e_{x,t}$ for a person aged x at time t is equal to

$$e_{x,t} = \sum_{k=1}^{MaxAge-x} {}_kp_{x,t} + \frac{1}{2}.$$

More explanations can be found in Pitacco et al. (2009).

3.2 The Lee-Carter Model

In this section we focus on the model proposed by Lee and Carter (1992). This is one of the most used model nowadays for estimating the log central death rates. In this thesis we will use this model to fit and forecast the future mortality for the Dutch population data. The reason for this model is because it is simpler than other mortality models, but this does not make it less accurate (e.g. Dowd et al. 2009).

Let $x = 0, \ldots, X$ be the set of ages, where X is the $MaxAge$ considered and let $t = 1, \ldots, T$ be the set of years observed. The Lee and Carter model is given by the following expression,

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \tag{4}$$

where a_x is an age-dependent constant, κ_t is a time-varying index, the time trend, which tracks changes of the mortality rates over time. Furthermore, the age-specific constant β_x describes the sensitivity of the log central death rates to the changes in κ_t. This can be interpreted as the 'weight' gained by people aged x from the changes of the total mortality for when there is e.g. health care improvement. The error term $\epsilon_{x,t}$ is assumed to be white noise which is normally distributed with zero mean and variance $\sigma^2_{\epsilon,x}$, $\epsilon_{x,t} \sim N(0, \sigma^2_{\epsilon,x})$. Variance $\sigma^2_{\epsilon,x}$ is age-dependent but constant over time.

In the Lee and Carter (1992) model, the error term is assumed to have a constant variance over age. This implies that homoskedasticity is assumed for the error structure for all ages and time. This is one of the drawbacks of the Lee-Carter model as it is not realistic. The log of force of mortality for older ages are more variable, has more fluctuations than younger ages because the absolute number of deaths are smaller at older ages. However in this thesis, mortality rates of older ages are extrapolated and smoothed using the Kannisto model (Thatcher et al., 1998), explained in Section 3.3. This model assumes that number of deaths follows the Poisson distribution, which will make the homoskedasticity assumption not a problem if actuaries only restrict to ages after 50 and avoid the 'accident hump' (see Figure 12), Pitacco et al. (2009). To account for heteroskedasticity, (Brouhns et al., 2002) propose a Poisson log-bilinear regression approach to model death counts instead of death rates and the error term $\epsilon_{x,t}$ is replaced by a Poisson random variable. Furthermore, Fung et al. (2017) proposed a *'Lee-Carter model with Heteroskedasticity'* where they consider a relaxation of the single constant covariance assumption over time and ages. Through state-space modeling, the error terms follow an age-dependent variance. Moreover, this is in line the assumption made by Broeders et al. (2018), from which this thesis is based on. Later on in Section 6.5, a robustness check is performed when σ_ϵ is used.

Returning to the Lee-Carter model, note that this model is not fully identified because of the component of the two parameters β_x and κ_t. For any constant c and $d \neq 0$, the parameter set $(\alpha_x, \beta_x, \kappa_t)$ is equivalent to the parameter set $(\alpha^+_x, \beta^+_x, \kappa^+_t)$, where $\alpha^+_x = a_x - \beta_x c$, $\beta^+_x = \beta_x / d$ and $\kappa^+_t = d(k_t + c)$[9]. To cope with the identification problem, two constraints are introduced by the authors, where

$$\sum_{x=0}^{X} \beta_x = 1, \quad \text{and} \quad \sum_{t=1}^{T} \kappa_t = 0.$$

[9]See Nielsen and Nielsen (2014) for more details about the identification problem of the Lee-Carter model.

Parameter Estimation by Singular Value Decomposition

Lee and Carter first estimated α_x by averaging the logarithm of mortality rates over time[10]. Hence,

$$\hat{\alpha}_x = \frac{1}{T}\sum_{t=1}^{T} \ln(m_{x,t}). \tag{5}$$

Given this, α_x can be interpreted as the base shape of mortality, or the average log mortality over the time period.

Secondly, β and κ are obtained by a least squares estimation. We obtain this by minimizing $\sum_{x=0}^{X}\sum_{t=1}^{T}(\ln(m_{x,t}) - \hat{\alpha}_x - \beta_x\kappa_t)^2$. Since there are no explanatory variables, it is not possible to use ordinary least squares regression (OLS), therefore the singular value decomposition (SVD) method is used instead. SVD is performed on the matrix

$$Z = \ln(m_{x,t}) - \hat{\alpha}_x = \beta_x\kappa_t, \quad \forall x, t.$$

As a result, we achieve the following,

$$U\Sigma V' = \text{SVD}(Z) = \lambda_1 U_{x,1}V_{t,1} + \lambda_2 U_{x,2}V_{t,2} + \cdots + \lambda_k U_{x,k}V_{t,k},$$

where $k = rank(Z)$, λ_i $(i = 1, 2, \ldots, k)$ are the ordered singular values. Left singular vectors $U_{x,i}$ and right singular vectors $V_{t,i}$ $(i = 1, 2, \ldots, k)$ represent the age component and time component respectively. In the Lee-Carter model, only rank $k = 1$ is used. The first term $\lambda_1 U_{x,1}V_{t,1}$ have the biggest proportion of the total variation explained and will summarize the mortality data while ignoring the other variations. The proportion of variation explained by the i^{th} term is $\lambda_i^2 / \sum_{j=1}^{k}\lambda_j^2$. Approximation of the first term in SVD(Z) gives the estimates $\hat{\beta}_x = U_{x,1} = (u_{1,1}, u_{2,1}, \ldots, u_{X,1})^T$ and $\hat{\kappa}_t = \lambda_1 V_{t,1} = \lambda_1 \times (v_{1,1}, v_{2,1}, \ldots, v_{T,1})^T$. Because of the identification problem mentioned before, we need to implement the constraints $\sum_{x=0}^{X}\beta_x = 1$ and $\sum_{t=1}^{T}\kappa_t = 0$. The estimated parameters $\hat{\beta}_x$ and $\hat{\kappa}_t$ become,

$$\hat{\beta}_x = \frac{1}{\sum_x u_{x,1}}(u_{1,1}, u_{2,1}, \ldots, u_{X,1})^T$$

and

$$\hat{\kappa}_t = \sum_x u_{x,1} \times \lambda_1 \times (v_{1,1}, v_{2,1}, \ldots, v_{T,1})^T.$$

In the original paper of Lee and Carter, $\hat{\kappa}_t$ is re-estimated for each year such that the actual number of deaths are reproduced. Hence, $\sum_x D_{x,t} = \sum_x E_{x,t}\exp(\hat{\alpha}_x + \hat{\beta}_x\hat{\kappa}_t)$. As it is not clear whether this would lead to a more reliable estimation and this step is controversial in the sense that by re-estimating, the maximum likelihood properties of $\hat{\kappa}_t$ would be violated, we do not consider this step in our thesis. Girosi and King (2007) mentioned few drawbacks of this re-estimation step and gave alternatives to estimating the parameters in their paper.

Forecasting

One of the noteworthy feature about the Lee-Carter model is, once values of the parameters $\hat{\alpha}_x, \hat{\beta}_x$ and $\hat{\kappa}_t$ are found, only the index that represents the time dimension of mortality κ_t needs to be forecasted.
Lee and Carter (1992) used time series methods to model and forecast κ_t. They fitted a simple random walk with drift to their US population data, but they suggested to use appropriate (or best fitted) autoregressive integrated moving average (ARIMA) model depending on the population data. While other ARIMA models might be appropriate for different data sets, the random walk model with drift is used in general. The model ARIMA(0,1,0) for κ_t is as follows:

$$\kappa_t = \kappa_{t-1} + c + \eta_t, \tag{6}$$

[10]This can be proven by applying the constraint $\sum_{t=1}^{T}\kappa_t = 0$ into equation $\sum_{t=1}^{T}\ln(m_{x,t}) = \sum_{t=1}^{T}(\alpha_x + \beta_x\kappa_t + \epsilon_{x,t})$.

where c is the drift parameter, estimated by

$$\hat{c} = \frac{1}{T-1}\sum_{t=2}^{T}(\kappa_t - \kappa_{t-1}) = \frac{\kappa_T - \kappa_1}{T-1}$$

and η_t is the error term with $\eta_t \sim N(0, \sigma_\eta^2)$.
Furthermore, Brouhns et al. (2002) also found that the ARIMA(0,1,0) model is appropriate to use and they applied it on their data of Dutch death rates for both males and females. These suggest that the random walk with drift is an appropriate model to use in our thesis.

3.3 Mortality Data

Mortality data sets in this thesis are from the Human Mortality Database. Data of the Netherlands are used, with distinction between males and females. Period data of the exposure-to-risk $E_{x,t}$ and deaths $D_{x,t}$ of ages ranging from 0 to 110 and years from 1850 to 2016 are available on the website. The log mortality rates are obtained by taking the logarithm of the values given from equation 1. It should be noted that data after 1950 are provided by the Statistics Netherlands (Centraal Bureau voor de Statistiek, CBS). Since there are no death rates available for very old ages, we will only consider death rates until the age of 95 and death rates for ages after 95 are then expanded using the extrapolating method by Kannisto, Thatcher et al. (1998), see Appendix A.1. Central death rates of ages $x \in \{95, \dots 120\}$ are to be extrapolated using data of younger cohorts ($x \in \{80 \dots 95\}$). There exist a package '*MortCast*' by Sevcikova et al. (2019) in statistical software R for extrapolating older ages using the Kannisto method. With the available and smoothed data (at older ages), the Lee-Carter approach can be applied. In the next section 3.4, the mortality patterns for Dutch males and females aged 67 are given.

3.4 Mortality Pattern

In this section we look at the mortality pattern of Dutch males and females. Using the historical and smoothed data from Section 3.3, the log mortality rates for the Dutch population in the year 2016 are shown in Figure 1.

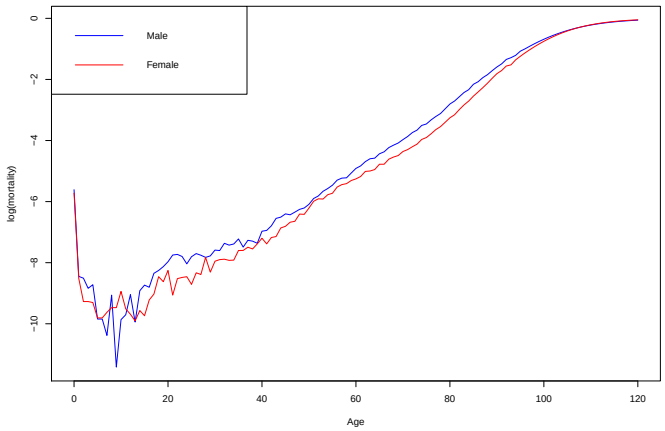

Figure 1: Log mortality rates for Dutch males and females in year 2016.

From the figure, it can be seen that the mortality are more volatile around the teenage to young adult years. Furthermore, notice at the sudden drop of mortality rate around infants, where lowest mortality is reached during the teenage years. At older ages, there is a linear increase of the log of mortality. Hence, the death rates grow exponentially. Also, the mortality rates of males are slightly higher than females for all ages, except for ages above 100, which is almost equal.

The development of mortality rates over the years for several ages (i.e. age 25, 67 and 100) is given in Figure 11 in Appendix A.2. From this figure, we can see that at age 25, the mortality has more decrease over the years than the other two older ages. On top of this, there is barely, almost to no change for the mortality at age 100. This might be due to the lack of data (population size is small) and the data is extrapolated. Around 1940, the period of World War 2, there is a sudden peak in the mortality rates for age 25, especially for men. This could be explained by "disappearance" of most men due to forced labor in Germany or they were hiding, (Ekamper et al., 2017). After 1945, mortality has dropped again, which is mostly explained by the introduction of antibiotics after the war, the penicillin.

In the introduction, it is mentioned that the (period) life expectancy has increased greatly over the years, data taken from HMD. It should be noted that the life expectancy from HMD are calculated using ages from 0 up to 109[11]. On the other hand, in Figure, 2 the (period) life expectancy are obtained using data up to the age of 120, of which for the older ages the smoothed data are used. The reason of assuming this because later on this thesis, retirees are assumed to not live past the age of 120.

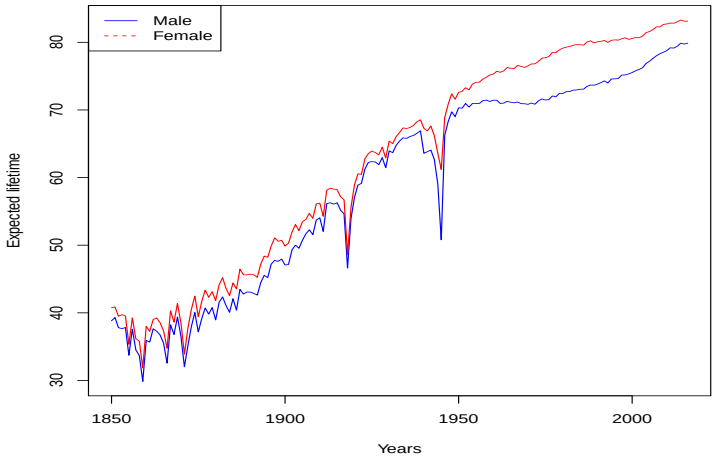

Figure 2: Period life expectancy at age 0 for males and females.

Next, we are using the Lee-Carter method, mentioned in section 3.2 to fit and forecast the mortality rates. The data sets are used are mentioned in section 3.3. Instead of using all data, only data of period 1967-2016 (50 years), with ages from 0 to 120, is used for the fitting. It is also better to not take the period before 1950 as these contains some shock in mortality rates because of the Spanish flu and World War II, seen earlier in the mortality patterns. Although a shorter period, around 30 years would correspond better to the current trend regarding improvements in mortality, the reason of not choosing this is because this will have a higher variance explained by the model[12], mentioned later on. For the fitting and forecast, there are several packages available with the code of the Lee-Carter method. The one we use in this thesis is the package '*demography*' by Hyndman (2019) in statistical software *R*. In Appendix A.2, the results of fitting the data to the Lee-Carter model for both males and females can be found.

First, in Figure 12, pattern for a_x is given. This represents the mean log of mortality for every age with 50 years of data. Mortality in the first year is high and then decreases till age 10. After the age of 10, the mortality increases almost exponentially up to the oldest ages. An interesting result is the 'hump' around age 20, which is more pronounced for males than females. This is often referred to the 'accident hump, caused by e.g. motor vehicles accidents, which happens more to males than females around this age as they live more dangerously than females.

[11] https://www.mortality.org/Public/Docs/MethodsProtocol.pdf

[12] Meanwhile with 30 years of data, this corresponds to 78.11% and 59.73% for males and females respectively.

The shape of β_x, in Figure 13, explains how sensitive the rates are to a change in κ_t. For young ages this is high, which means that mortality rates improves faster. For higher ages, improvements are still substantial, but around the highest ages, there are almost no improvements. Note that the patterns are quite volatile in the early years, this could be explained by the size of the fitting period. Fitting with more years would lead to a smoother pattern for β_x.

In Figure 14, it shows that there is a steeper trend κ_t for males than for females. This implies that during the period of 1967-2016, the drop in mortality rates for men are more than the drop for women.

On a side note, the variance explained by the model is 88.38% of the data for men. For women, this number is equal to 80.26%. These values are lower than what is typically found in the literature, which is higher than 90%, e.g. Lee and Carter (1992). Although the model explains for almost 90% of the country specific data, it does not exactly explain as much if the pension funds or insurers have a subgroup of people. Mortality data of the pension fund correspond to a more smaller group, that in general cannot be explained by the whole population data. On the other hand, pension mortality data might be to not big enough to obtain a nice fitted model. For this reason, other models are presented to relate the portfolio specific mortality to general mortality of a country. For example Brouhns et al. (2002) present their Poisson model for population mortality and also come up with a Cox Proportional Hazard model to study the relationship between Belgian annuitants' mortality and the general Belgian population. Other authors, e.g. Plat (2009), Pitacco et al. (2009) also proposed models that can deal with this problem.

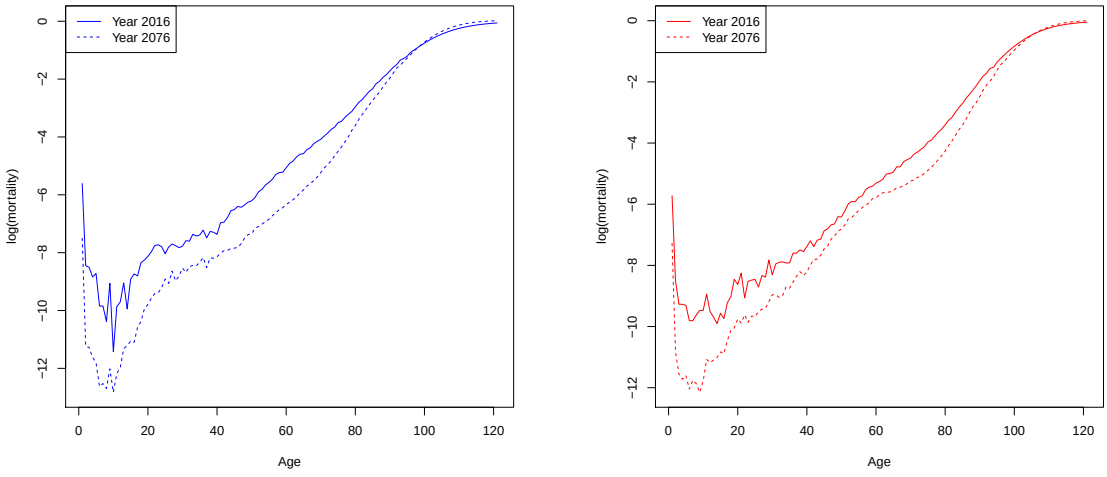

Figure 3: Graphs of mortality rates for year 2016 and 2066, with rates for males depicted on the left side and for females on the right side.

Proceeding to the model forecasts, Figure 3 illustrates the mortality patterns of Dutch males and females in 2016 and best estimates in (future) year 2076. The mortality rates after 2016 are forecasted using the past 50 years of fitted data. From both graphs, the mortality rates are forecasted to decrease in the future, with a relative higher decrease of mortality rates of males than females. After the age of 100, there is almost no more decrease, and on the contrary there is an increase of mortality for males.

In Figure 3, the historical and forecasted mortality rates are compared using specific years for all ages. To have more overview of the actual development of an individual's future mortality rates, we look at the *cohort* for an individual aged 67 in year 2016. This means that for year 2017, for the same individual, we have to look at the mortality rate at age 68. An individual aged 67 in year 2016 belongs to the cohort 1949 (they are born in this year, hence aged 0). The log mortality

rates for Dutch male and female from cohort 1949 are illustrated in Figure 15, Appendix A.2.

3.5 Longevity Risks

In this section, the various components of longevity risk are introduced. Though not all are researched on in this thesis, for the sake of completeness, most of the components will be given. In Figure 4, the different components are depicted.

First of all, notice that we refer *longevity risk* as future survival uncertainties that might arise and leads to longer or shorter length of life than the expected lifetime (Hari et al., 2008). Although in some parts of the literature, *mortality risk* refers to the uncertainty that arises randomly of an individual's lifetime and *longevity risk* refers to the risk of systematic deviations in mortality projections, we refer these risks as *micro-longevity* risk and *macro-longevity* risk respectively.

Micro-longevity risk or also known as the idiosyncratic risk in longevity or individual mortality risk. This risk is not systematic, and is resulted because of an individual's remaining lifetime is a random variable. Assuming that the survival probabilities are correct, in a pool of participants, the realized number of survivors could possibly not live up to the true probability. Furthermore, this risk is diversifiable if pool is large enough. More about this risk is elaborated in Section 5.

Macro-longevity risk on the other hand is a systematic risk. This risk is not diversifiable. Uncertainties about the future survival probabilities cause this risk. Although generally, this risk is quantified by the noise terms in the stochastic mortality model or the risk in the trend, time process, there are more sources which cause this uncertainty. We decompose the macro-longevity risk further and follow Broeders et al. (2018) and Richards et al. (2014) on distinguishing the different components of the macro-longevity risk.

First of all, there is *model risk*, the uncertainty whether the underlying mortality projection model is appropriate and correct. In our case, we assume that the Lee-Carter model is an appropriate model and hence, model risk will not be considered in this thesis. Assuming that the model is correct, there might be still other sources of risk, *parameter risk* or the *stochastic variation* (Broeders et al., 2018).

Parameter risk, or the mis-estimation risk, exists due to the true value of the parameters are uncertain no matter the stochastic mortality model used. This risk quantifies the incorrect estimation of the current biometric risk (i.e. all risks related to human life conditions). The data set used will affect the results of the parameters. The scale and the representatives of the data will determine for proper parameters estimation. It is preferred that the estimated parameters are robust to size and richness of the data. Proper use of data can minimize the mis-estimation of the parameters, hence the risk is diversifiable, Richards (2016). In this paper Richards proposes a method to assess the mis-estimation risk. Under the Lee-Carter model, the parameters are α_x, β_x and c (see equations (4) and (6)).

Since the Lee-Carter model is a stochastic model, it captures stochastic variation. This variation is caused by the two random variables in the model, $\epsilon_{x,t}$ and η_t. Macro-longevity risk is discussed in more details in Section 6.

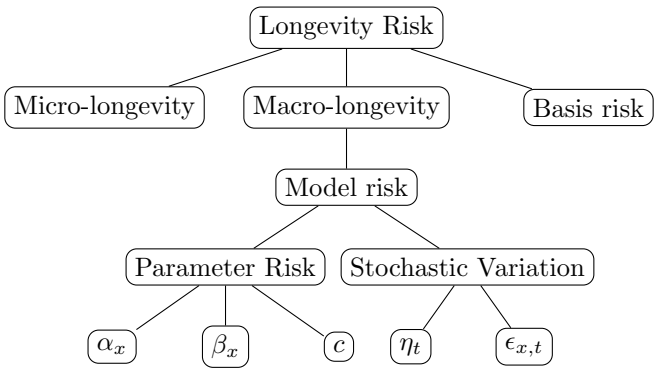

Figure 4: Longevity risk components

Furthermore, there is longevity *basis risk*, when the population data used might not fully corresponds to an insurer's own data. The population or subpopulation experiences different mortality outcomes. The paper presented by Millossovich et al. (2014) did a research on a methodology for quantifying basis risk. Brouhns et al. (2002) come up with a model to relate the general population data with an insurer's data. As we do not have insurers' data at our disposal, we will not look further into this risk.

4 Variable Annuities with Mortality

In this Section, we introduce variable annuities with mortality. First, it is explained in Section 4.1 about the wealth allocation through assumed interest rate (AIR). Then, in Section 4.2, the financial market in which the wealth would be invested in is described. In Section 4.3, we give the AIR which determines how wealth is allocated for constant expected payments in the future. And lastly, in Section 4.4, a numerical illustration for the variable annuity with mortality is given.

With variable annuities mentioned in this thesis, we are referring to lifelong variable annuities. Participants receive annuity payouts as long as they are alive till the moment of death. The difference between fixed annuities and the variable annuities is that the latter enable annuitants to invest (a part of) their pension wealth in risky assets. While annuitants have higher expected pension payments because of higher expected investment returns, they also have more risk because future payments are uncertain. As a consequence, an annuitant now carries the investment risk. Annuitants might also borne the longevity risk. Pension funds have the option to choose whether they carry the longevity risk or pass it on the retirees (Balter et al., 2018).

4.1 Wealth Allocation

In this thesis, we are only interested in the decumulation phase. This period starts at the retirement age and ends the moment the participants dies. A retiree enters the retirement phase at time $t = 0$ with an initial amount of total pension wealth W_0. This amount is to be invested in the risky assets at start of the retirement phase, and will payout at time h according to the development of the financial market.

But first, depending on the choice of the annuitant on future expected payments, the initial wealth W_0 is first to be allocated into buckets. Each bucket $W_0(h)$ allocated at time 0 has to finance the pension payment that takes place at time h. The distribution of the pension wealth is defined through the assumed interest rate (AIR). A low AIR implies that the first bucket allocated is starting at a low point, with subsequent buckets slowly decreasing over time. On the other hand, having a high AIR gives a higher allocation to bucket at retirement but also a higher probability of a decrease for subsequent buckets. More details about the AIR is elaborated in Appendix B.1. The AIR does not need to be constant, as we will see later on in the section.

We follow the setting of Balter and Werker (2017) and go into more detail about the wealth allocation based on assumed interest rate.
If retiree enters the decumulation phase, the start of retirement with an initial total pension wealth of W_0 at time 0, then this amount need to finance all future pension payments of the individual until death. In case there are no longevity risk involved, the individual is assumed to have an life expectancy of H years (i.e. dead after H years) and hence the total wealth need only to be divided for those upcoming years. If longevity risk is involved, it is not known until when an individual is actually going to live, as the remaining lifetime remains unforeseen (i.e. micro-longevity risk). It is then to our own decision to assign the maximum attainable age, meaning that an individual dies for sure at the age thereafter. It is decided to assign the maximum age of 120 assumed in this thesis. This means that the individual will not live up to age 121.

In total maximum $H(= MaxAge - RetirementAge + 1)$ pension payments (till death) need to be made, starting at horizon $h = 0$ until $h = H - 1$. Summarized mathematically, the budget constraint is as follows,

$$W_0 = \sum_{h=0}^{H-1} W_0(h), \tag{7}$$

where H is the known expected remaining lifetime if longevity risk is not assumed and H is the maximum age if the risk is considered.

Furthermore, as already explained in B.1, the assumed interest rate $a_0(h)$ at time h to deter-

mine the wealth allocation is defined through

$$\frac{W_0(h)}{W_0(0)} = \exp(-ha_0(h)) \quad \Rightarrow \quad W_0(h) = W_0(0)\exp(-ha_0(h)), \tag{8}$$

The assumed interest rate might depend on different factors, for example the financial market return (risk-free and risky returns), the investment strategy or the survival probability, as we can see later on.

4.2 Financial Market

After the allocated wealth, comes the financial market which the pension buckets are invested in. Life-cycle portfolio selection and the optimal consumption and portfolio has been studied a lot in literature, which started with the basic principles founded by Merton (1969) where Merton (1971) further studied the optimal saving and investment decisions. In this thesis we consider the financial market in the Merton (1969), equivalent to the standard Black and Scholes (1973) setting, where there is one risk-free asset M (e.g. a money market account or a bond) and one risky asset S (e.g. a stock). The risk-free asset grows at a constant risk free rate r,

$$dM_t = rM_t dt.$$

The risky asset with price S_t at time t follows the following stochastic differential equation (SDE),

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t \\ &= (r + \lambda\sigma)S_t dt + \sigma S_t dB_t \\ &= \underbrace{rS_t dt}_{risk-free} + \underbrace{\lambda\sigma S_t dt + \sigma S_t dB_t}_{risky}, \end{aligned}$$

where μ is the expected return, σ is stock volatility and λ is the Sharpe ratio $\lambda = (\mu - r)/\sigma$. Furthermore, S_t follows the geometric Brownian motion. B_t is a standard Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Assume that an individual will annuitize all the retirement wealth and there are no other assets to consider. Hence, there are no wealth set aside for consumption. Yaari (1965) showed that if an individual with no utility of bequest should annuitize all of their liquid assets. Then, for the variable annuities, each asset $W_t(h)$ is partly invested in stock S_t and the rest in risk-free account. When each $W_0(h)$ is continuously re-balanced with risk exposure w, we have the following differential equation,

$$dW_t(h) = rdt + w(\lambda\sigma W_t(h)dt + \sigma W_t(h)dB_t), \tag{9}$$

for $0 \leq t \leq h$.

This leads to the expected pension payment at horizon h of a variable annuity (see Appendix B.2 for derivation),

$$\mathbb{E}_0[W_t(h)] = W_0(h)exp(t(r + w\lambda\sigma)), \tag{10}$$

for $0 \leq t \leq h$.

The quantile at level α is then given by,

$$Q_0^{(\alpha)}(W_t(h)) = W_0(h)exp(t(r + w\lambda\sigma - \frac{1}{2}w^2\sigma^2) + w\sigma\sqrt{t}z_\alpha), \tag{11}$$

where z_α denotes quantile of the standard normal distribution. This quantile denotes the investment risk of the variable annuity.

Up until now, the focus was only on the expected payment with financial market returns. Expected payments change if longevity risk is incorporated. In this case the payments at time h need only to be made with the probability ${}_hp_{x,0}$ for someone aged x at time 0 surviving at least h years.

Consequently, each bucket $W_0(h)$ can be reduced corresponding to the chance of surviving h years. Allocating wealth based on survival probabilities is comparable to allocating wealth according to the weight of each bucket. The higher the weight, hence the higher the chance of survival, the more is allocated to the corresponding bucket. If an individual allocate his wealth in this manner at the start of retirement, then he also expects to earn the reduced amount back if he survives in a future year h. Before longevity is incorporated, the returns are based on the financial market. Now, the returns are also including the *biometric return* (or *mortality credit*) (see e.g. Donnelly et al. (2013)). This is the return, a bonus, each surviving participant will receive by staying alive. The capital of someone who dies at time h is distributed among other participants in the same pool. A deviation in (future) survival probabilities (i.e. macro longevity) or if there are more people living longer or shorter than expected affects the future payments (micro-longevity). This risk for the expected payments are then included through the biometric returns, which is denoted by

$$\frac{1}{{}_hp_{x,0}} = \exp(-\ln {}_hp_{x,0}) = \exp(hr^b_{x,0}(h)), \tag{12}$$

where $r^b_{x,0}(h)$ is the biometric return in h years.

Rewriting the expected payment from (37),

$$\mathbb{E}_0[W_t(h)] = W_0(h)\exp(t(r + w\lambda\sigma))\frac{{}_tp_{x,0}}{{}_tp_{x,0}}$$

to expected payment with biometric return at horizon h, it follows that

$$\mathbb{E}_0[W^*_t(h)] = \mathbb{E}_0\left[\frac{W_t(h)}{{}_tp_{x,0}}\right] = W_0(h)\exp(t(r + w\lambda\sigma))\frac{1}{{}_tp_{x,0}}, \tag{13}$$

which means that it is expected to receive $\frac{1}{{}_hp_{x,0}}$ extra credit if survived at time h. Furthermore, the quantile is given by

$$Q_0^{(\alpha)}\left(W^*_t(h)\right) = W_0(h)\exp(t(r + w\lambda\sigma - \frac{1}{2}w^2\sigma^2) + w\sigma\sqrt{t}z_\alpha)\frac{1}{{}_tp_{x,0}}. \tag{14}$$

4.3 Assumed Interest Rate

As mentioned before, the assumed interest rate $a_0(h)$ is determined based on how the expected payments is going to perform over the course of the retirement years. In this thesis, let the assumed interest rate $a_0(h)$ be chosen such that the expected pension payments are constant over horizon h. Furthermore, payments occur at $t = h$, hence $E_0[W_h(h)] = W_0(0)$. The expectation at time 0 for a future payment $W_h(h)$ must be equal to the first payment $W_0(0)$ which is without investment risk and longevity risk.

Deriving the horizon-dependent assumed interest rate $a_0(h)$ that leads to constant payments for the variable annuities, we have as follows:

$$a_0(h) = r + \lambda w\sigma - \frac{\ln {}_hp_{x,0}}{h}. \tag{15}$$

Note that at $h = 0$, the individual is for sure alive, hence no longevity risk and the AIR is $a_0(h) = r + \lambda w\sigma$.

Proof. (15) is derived using (8) and (13).

The expected payments is given by equation (13), $\mathbb{E}_0[W_t(h)] = W_0(h)\exp(t(r + w\lambda\sigma))\frac{1}{{}_tp_{x,0}}$, and rewriting this for $t = h$ leads to,

$$\mathbb{E}_0\left[W_h(h)\right] = W_0(h)\exp(h(r + w\lambda\sigma))\frac{1}{{}_hp_{x,0}} \tag{16}$$

$$= W_0(h)\exp\left(h\left(r + w\lambda\sigma - \frac{\ln {}_hp_{x,0}}{h}\right)\right) \tag{17}$$

$$= W_0(0)\exp(-h(a_0(h)))\exp\left(h\left(r + w\lambda\sigma - \frac{\ln {}_hp_{x,0}}{h}\right)\right). \tag{18}$$

The last line follows from (8). For $a_0(h) = r + \lambda w\sigma - \frac{\ln {}_hp_{x,0}}{h}$, we have constant expected payments, $\mathbb{E}_0 [W_h(h)] = W_0(0)$. □

How does the survival rates affect the expected payouts? First, before longevity is incorporated, the micro-longevity risk is not shared and is only individualistic, because biometric return is 1 if alive, see equation (16). Expected payment at time h will not depend on the amount of people who survives at h. In other words, there are no extra rewards being alive. Furthermore, with the constant expectation assumed interest rate, $W_0(h)$ does not depend on survival rates. Thus, a change in survival probabilities (i.e. macro-longevity risk) will not affect the allocation and in turn also does not affect the future expected benefits, with the constant expectation AIR.

Back to the micro-longevity risk, the risk lies on the fact that an individual does not know the time of death. Staying alive will not grand the individual extra return and being dead, capital is collected by the pension fund. For the pension fund on the other hand, participants not sharing the risk between each other means that the pension fund bears risk[13]. If the pool is large enough, the number of people dying at h is true to the survival probability ${}_hp_{x,0}$ and the fund needs to pay N(=number of retirees) times ${}_hp_{x,0}$ pension benefits at time h. Else, if the number of people surviving are higher, more payments need to be made than expected, or less payments if number of people dying is higher. Not only does the fund have the micro-longevity risk, but also the macro-longevity risk. If it is uncertain how future mortality rates will develop that can affect the whole cohort/pool, then it is uncertain how many pension benefits need to be paid by the fund.

Next, survival rates are considered and the biometric returns are included in future expected benefits, means that the longevity risk are shared between participants. If there is micro-longevity risk, hence if there are more people alive than expected, as a survivor yourself, realization is less than the biometric return because this needs to be shared between more participants and payment is less than the expected payout. As this risk is generally carried by the pension provider, with a large enough pool, risk will decrease to almost zero. The participants will then receive positive biometric return and can be unbothered of a payout decrease.

With macro-longevity risk, the future survival probabilities are uncertain. Although wealth is allocated with ${}_hp_{x,0}$ known at time 0, this might change later on due to some shocks that affect longevity for all participants of the same group. Viewing this situation through (13), $W_0(h)$ is allocated at time 0 such that the expected payments are constant for future years. This is already determined and will not be be adjusted later on. If mortality rates changes, the expected payments change through the biometric returns $\frac{1}{{}_hp_{x,0}}$. A deviation on the survival probabilities, reflected by ${}_h\tilde{p}_{x,0}$, causes the expectation (13) not to be constant anymore. In fact, the biometric return decreases if updated probability is higher. Macro-longevity risk can be borne by the individuals themselves, shared between participants in a pool or by the pension fund. The upside of carrying the risk yourself or sharing between participants is having higher expected benefits, but also more volatility as ${}_hp_{x,0}$ can be updated to ${}_h\tilde{p}_{x,0}$. Another advantage is that no risk premium need to be paid to insure this risk. If pension fund carry this risk, expected benefits are still high and more stable, but need to pay a premium in return.

4.4 Numerical Illustration

In this section, an numerical illustration is given on the variable annuities payouts using (8), (13), (38) and (15). First, let start with the parameters assumed. These default parameters values will assumed and used throughout this thesis:

- Total retirement wealth $W_0 = 100000$.
- Risk free rate $r = 2\%$.
- Mean stock return $\mu = 5.8\%$, 'maximaal rendement op beursgenoteerde aandelen'[14], we set it at 6%.

[13] It is obligatory in the Netherlands for this risk to be shared between participants or must be insured.

[14] This is consistent with the advice of the Comittee Parameters: 'Comissie Parameters 6 juni 2019', `https://www.rijksoverheid.nl/documenten/kamerstukken/2019/06/11/advies-commissie-parameters`

- Stock volatility $\sigma = 20\%$, 'beursgenoteerde aandelen standaarddeviatie'[14].
- Risk exposure $w = 35\%$[15].

The best-estimates of the male mortality rates with the Lee-Carter model in Section 3.2 will be used as the baseline, which will be compared with later on by incorporating longevity risk in Section 6. Since the maximum age is 120 and retirement age is assumed to be 67[16], $H = 53$.

First, equation (4.3) is used to derive the AIR for constant expectation. Hereafter, these values are plugged into equation (31) to first obtain the percentage of the total wealth W_0 allocated for each $W_0(h)$. Then, including the value of W_0, we get $W_0(h)$ values. From here, everything will be plugged into equation (13) and (38).

In Figure 5, the variable annuity payouts are given, with the constant expectation given by 'central projection' and the 5% ($z_{0.05} = -1.645$) and 95% ($z_{0.95} = 1.645$) quantile for financial risk.

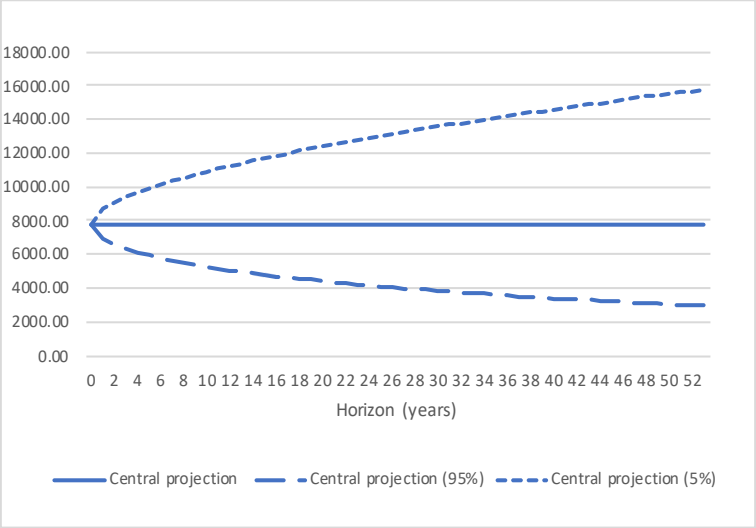

Figure 5: Variable annuities with AIR from (15).

In case of fixed annuities, when there are no risk and with constant expectation, it is a horizontal line that lies below the variable annuity line. For variable annuities, it is higher because of the extra risk taken. But there is also the risk that the realization will be lower than the fixed annuity (5% central projection).

[15] Asset allocation to risky investments is capped at 35%, Bovenberg and Nijman (2019).
[16] Retirement age in the Netherlands as of 2021.

5 Micro-longevity Risk

In this section we look at micro-longevity risk. What causes this risk and how could this risk be reduced?

When future survival probabilities are known with certainty, the expected number of people dying in a certain year is also known. However, in a group of retirees, some people might live shorter or longer than expected. As a result, the pension fund will have micro-longevity risk. If the risk-sharing pool is large enough, then the number of deaths per year will go to the expectation. A large enough pension fund will make the micro-longevity risk negligible. For a pension fund, it is necessary to take this risk into account. In the case of fixed annuity, pension funds might not collect enough premiums during the working years of individuals to finance the future payments in the retirement years. After the new legislation in 2016, micro-longevity risks are still pooled. The capital of the retirees who pass away are can now be allocated to others in an risk-sharing pool, the biometric return. By sharing this risk, not only does the pension fund diminishes the risk on an individual level, because there is extra return, but also prevents adverse selection in longevity insurance (Bovenberg and Nijman, 2019).

Pension funds can still carry the micro-longevity risk when biometric return is included. In that way, retirees receive the positive biometric return and the risk that more people are alive than expected is then borne by the fund/insurer.
Under Solvency II, longevity risks are taken into account when calculating the Solvency Capital Requirement (SCR). As micro-longevity risk is one of the sources, even though it is diversifiable, attention need to be given. There are few papers in the literature that that focus on this risk e.g. the paper by Hari et al. (2008) focuses on the micro-longevity alongside the macro-longevity and parameter risk.

Does this risk influence the solvency capital of the fund greatly depending on the size of the pool? We will not look at how the (micro-) longevity risk affects the SCR, but instead we will look at the gain/loss percentage of different pension fund scenarios. These scenarios corresponds to different collective pools in a pension fund. The paper by Piggott et al. (2005) is relevant, which they introduce Group Self Annuitization (GSA) and analyzes though this the payout implications when longevity risk is pooled. In Section 5.1 we will have a risk-sharing pool where everyone has the same investment portfolio weight and age (same survival probability). Section 5.2 focuses on the cases where everyone in the pool has: 1) the same survival probabilities, but different portfolio weights, 2) the same portfolio weights, different survival probabilities and 3) different portfolio weights and survival probabilities.

5.1 Homogeneous group

In the following scenario, we will see what kind of impact pension fund size has on the gain or loss for the pension fund regarding pension payments.

Lets consider the case where there is homogeneous pool with n retirees, everyone in this pool has the same variable annuity and has the same survival probability ${}_hp_{x,t}$, meaning everyone have the same age and belongs to the same cohort. Furthermore, assume that there are no macro-longevity risk, ${}_hp_{x,t}$ is correctly estimated and there are no financial risk. Hence, pension payments are expected to have a constant payout. We also assume that everyone has the same amount of total retirement wealth W_0.

Let the expected payment to each retiree at time h be

$$W_0(h)\exp(h(r + w\lambda\sigma))\frac{1}{{}_hp_{x,t}} = E, \tag{19}$$

where $W_0(h)$ is allocated through the AIR given in (15). That is, if we expect that everyone lives up to that survival probability, no micro-longevity risk, then the number of people that survives

are $n \times {}_hp_{x,0}$ and the total payment is

$$\begin{aligned} & n \times {}_hp_{x,t} \times E \\ =& n \times W_0(h)\exp(h(r + w\lambda\sigma)). \end{aligned}$$

In reality, the number of deaths for certain years deviate and the expected payments are not E anymore, but

$$R \times W_0(h)\exp(h(r + w\lambda\sigma))\frac{1}{{}_hp_{x,t}}, \tag{20}$$

where $R = Bin(n, {}_hp_{x,t})$, the actual number of people surviving with probability ${}_hp_{x,t}$. The difference between the expected payment and the actual payment is the gain/loss for the pension fund. The gain/loss percentage is calculated as follows,

$$Gain/Loss\% = \left|\frac{Expectation - Reality}{Expectation}\right| \times 100\%. \tag{21}$$

As illustration, Figure 6 shows the gain/loss percentage of a pension fund as the size increases. Simulations are done for pool sizes starting at $n = 2500$ until $n = 200000$ with a step size of 2500. For each pool, 1000 simulations are done for the actual numbers of survival. We simulate random numbers from a binomial distribution $Bin(n, {}_hp_{x,t})$ using $binornd(n, {}_hp_{x,t})$ in e.g. *MATLAB* or *R*. As this is a homogeneous group, there is no need to use the actual survival probability and hence ${}_hp_{x,t} = 0.90$ is assumed.
Furthermore, with the same starting wealth, allocated through the same AIR, $W_0(h)$ is also equal among all participants. Hence, it is set equal to $W_0(h) = 100$ for e.g. $h = 20$. We also assume that $r = 2\%$, $w = 30\%$, $\lambda = 0.20$, and $\sigma = 0.20$.
Following the equations (19) and (20), the expected payment and the actual payment are obtained respectively. Using (21) we have the gain/loss for each pool size and simulation. These needs to be averaged over the number of simulations and then turned into percentages. As can be seen from this figure, micro-longevity risk is negligible if the size is reasonably large enough.

5.2 Heterogeneous group, different survival probabilities and investment weights

In this section, three scenarios are simulated. First scenario is when the survival probabilities are different, hence there are different cohort groups within the pool. Second scenario corresponds to the different investment weights, but the pool consists of the same cohort. Lastly, the scenario corresponding to a pool which consists of different cohorts, which each individual having their own risk exposure.

The first scenario is illustrated in red line in Figure 6. Here, simulations are done with different survival probabilities. The pool sizes ranges from $N = 2500$ to $N = 200000$, with a step size of 2500. First, the ages x_i, $i \in (1, \ldots, n)$(between age 68 and age 120) of the participants in the pool are randomly drawn. An individual aged x_i in year t is then in the $x_i - 67 = k^{th}$ period of the payments. This individual i belongs to the $t - x_i$ cohort. Starting the retirement phase in year $t - h$, the survival probabilities ${}_hp_{x_i,t-k}$ for $h \in (0, \ldots 120 - 67)$ are needed for the calculation of AIR (15) and hence the wealth allocations $W_0(h)$ of individual x_i. With allocated wealth, which is constant in expectation, the chance of surviving in year t is randomly drawn with $Bern({}_kp_{x_i,t-k})$, resulting survival (1) or death for sure (0). This is the outcome in reality, either $W_0(k)$ or 0. In expectation the payment at time t for individual x_i is ${}_kp_{x_i,t-k} \times W_0(k)$. Expected payments of all participants and then the actual payments will be aggregated correspondingly for each fund size. In this scenario, there are 100 simulations for each fund size and the results are the average of them. For this scenario, graph is shown in Figure 6, 'Hetero 1'.

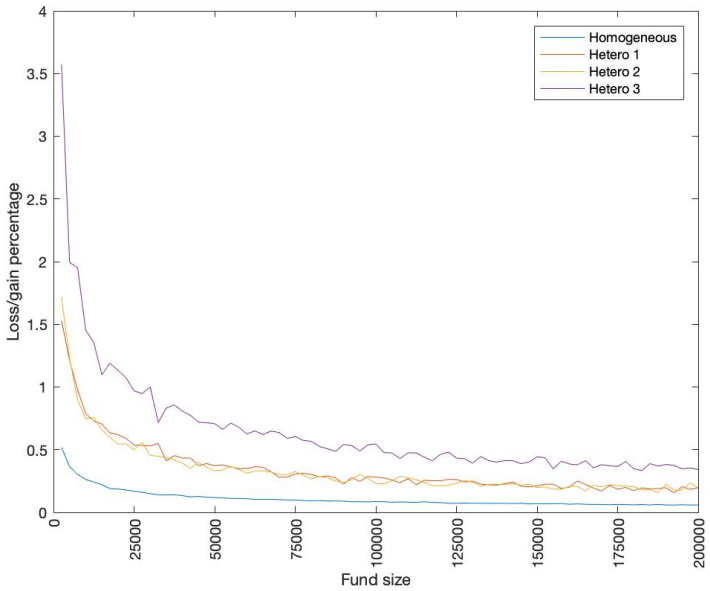

Figure 6: Simulation of the gain/loss percentage of a all scenarios mentioned in Section 5.1 and Section 5.2.

In the second scenario, Figure 6, 'Hetero 2', simulations are done on different investment weights. Survival probabilities are equal, hence we set ${}_hp_{x,t} = 0.9$. Furthermore, W_0 is equal for everyone. Only changing variable is w. Investment weights range from 5% to 35%. Hence for each participant in the pool, a random weight is assigned and hereafter the expected payment and actual payment are calculated. Apart from this, the settings are equal to the first scenario.

In the last scenario for this section, we have different investment weights and survival probabilities. The settings are the same as the previous scenarios except for simulating different investment weights and survival probabilities for each participant in the pool. The purple line illustrated in Figure 6 corresponds to this scenario.

From these scenarios in this section, it can be concluded that no matter the portfolio weight, or the survival probability, as long as if the pool is large enough, micro-longevity risk is negligible.

5.3 Comments

It can be concluded that the micro-longevity risk is almost fully negligible as pool size increase. Although for the homogeneous fund, micro-longevity risk is non-existent almost instantly as the pool size increases, the effect is not less for the heterogeneous fund. We do have to keep in mind that the graph shows the average of 100 (or 1000) simulations, hence it is more smooth. But in the case of only one simulation, the line will be more erratic, and the risk can be bigger or smaller.

Although the scenario with different investment weights and survival probabilities is considered, it is not relevant for risk-sharing. As there are different cohorts in the pool and on top of that, each individual has their own investment portfolio. This complicates the decision how much each surviving retiree will receive from the biometric return upon the death from another retiree who has a different investment portfolio. Complications caused by a too heterogeneous pool can affect the micro-longevity risk not optimally shared and hence not diversified.

6 Macro-longevity Risk

In Section 6.1 we first give an introduction to macro-longevity risk. Thereafter, the different sources to macro-longevity risk are explained in Section 6.2. Moving on to Section 6.3, the stressed-trend approach is used to quantify the risk. Another approach is also used Section 6.4, the Value-at-Risk framework. Lastly, in Section 6.7, some comments and remarks are given.

6.1 What is macro-longevity risk?

As already mentioned in Section 3.5, macro-longevity risk is a systematic risk. Meanwhile the micro-longevity risk is an idiosyncratic risk and is diversifiable if the pool in the pension fund is large enough, macro-longevity risk is not diversifiable by having a larger pool of participants of the same cohort, everyone in this cohort is affected by this risk. Macro-longevity risk appears when there are uncertainties of future mortality rates, e.g. mortality rates may change due to medical improvements, new diseases or unhealthier habits and this risk affects each cohort differently.

It is a concern for a pension fund to adequately quantify macro-longevity risk. Underestimation of the life expectancy due to this risk might not only cause the fund to not collect enough premiums in the accumulation phase or not be able to make the required payments to the retirees (in the future) if more people survives than expected, but also the funding ratio of the fund will decrease. In this case, the solvency of the fund is involved.

To overcome the solvency issue, the fund can cut pension payments if the assets are smaller than the value of the annuities (excluding indexation) (Bovenberg and Nijman, 2019). On the other hand, it will be not fair for the (future) retirees. According to Broeders et al. (2018), it may be beneficial for a pension funds to share the macro-longevity risk across cohorts since this risk affects each cohort differently, using the collective buffer. In this section, it is assumed that the risk is shared within one cohort only and biometric returns are dependent solely on the survival probabilities of this cohort.

In Section 3.5, different sources of macro-longevity risk are mentioned. In this thesis we focus on few of these uncertainties. From the point of view of the Lee-Carter model, risks can arise in the log central death rate model and can arise in the time trend model.

There is first of all, the stochastic variation (Broeders et al., 2018). A random shock captured by a stochastic mortality model. This will cause a deviation in the aggregate realized number of deaths and it is modelled as the aggregated effect of the random shocks $\epsilon_{x,t}$ and η_t in (4) and (6).

Random shock η_t in the time trend κ_t (6) is referred as the *trend volatility* (Richards et al., 2014). Another source of risk in the trend model is the *trend parameter risk*. It is the uncertainty in correctness of trend parameter c from the time trend κ_t. We will refer the uncertainties caused by trend parameter risk and trend volatility as the *longevity trend risk*.

The focus of the thesis is not on the (general) parameter risk which concerns all the parameters α_x, β_x and c, but only on the trend part, which is the uncertainty of the drift parameter in trend c, and the corresponding variance is attainable. Analytically, it is not possible to obtain a relevant prediction interval for all these parameters, due to the non-linearity of the parameters α_x, β_x and κ_t. A way to quantify this risk is using the bootstrap method by Pitacco et al. (2009). The bootstrapping method involves re-sampling of the original data and creates data sets replicates. This is briefly mentioned in Section 6.6. Although not the same, by re-calibrating the parameters for updated data, parameter risk is indirectly included in the approach taken in Section 6.4. Another more formal way of including parameter risk is using Bayesian methods by Cairns et al. (2006). On the other hand, if a data set is used is such that parameters are estimated properly and robust to the data scale and richness, then parameter risk can be diversified. In general, estimating the stochastic variation and parameter risk together will account for the macro-longevity risk involved in a given model.

6.2 Uncertainties

Before we model the macro-longevity risks, we go into more details of the uncertainties involved.

First of all, the uncertainty in death rates, given by $\epsilon_{x,t}$. It reflects the random shock occurring in the log central rate model (4). The size of this shock is given by $\sigma_{\epsilon,x}$, which is the age-dependent standard deviation of $\epsilon_{x,t}$. This risk reflects in particular some age-specific past effects/variability not captured by the model. We obtain $\sigma_{\epsilon,x}$ by

$$\sigma_{\epsilon,x} = \sqrt{\frac{\sum_{t=1}^{T}(\ln(m_{x,t}) - (\hat{\alpha_x} + \hat{\beta}_x\hat{\kappa_t}))^2}{n-1}}, \tag{22}$$

where n is the size of the time horizon and T is the last observed value. This is needed for the calculations of the stochastic variation in the following sections.

Next, under the Lee-Carter model, the trend follows a random walk with drift process (which was already mentioned in (6)),

$$\kappa_t = \kappa_{t-1} + c + \eta_t.$$

By iterating (6), we can express the future k_{t+h}, with k_T as the last observed value and h is number of years ahead in the future. The future values are expressed by,

$$\kappa_{T+h} = \kappa_T + hc + \sum_{i=1}^{h} \eta_{T+i}. \tag{23}$$

Lets first assume that c is known, and the only uncertainty is in the error term with mean 0. Given κ_t, the h years ahead central forecast is

$$\kappa^*_{T+h} = \kappa_T + hc.$$

However c is unknown, hence it is then replaced with the estimate $\hat{c}$[17]. Together with uncertainty in error term, the h years ahead central forecast of $\hat{\kappa}_{T+h}$ is given by,

$$\hat{\kappa}_{T+h} = \kappa_T + h\hat{c}.$$

As mentioned earlier, there are two components to the uncertainty of the longevity trend: trend parameter risk and trend volatility. The trend (parameter) risk lies in the uncertainty over the drift parameter c, expressed by σ_c and (trend) volatility is the noise variance risk, expressed by the σ_η.

Volatility is an additional risk to the trend uncertainty. This risk arises due to short-term fluctuations in the trend model, for example a sudden influenza outbreak, very cold winters or heatwaves (Richards et al., 2014). Volatility is in the stochastic variation mentioned in section 6.1. The volatility is estimated by the following formula[18],

$$\hat{\sigma}_\eta = \sqrt{\frac{\sum_{t=1}^{T}(\kappa_t - \kappa_{t-1} - \hat{c})^2}{T-1}}, \tag{24}$$

where T is the last observed value and $\hat{c}$ is the estimated drift value. The trend risk is the uncertainty over the parameters in the time trend model, which defines the direction that the mortality rates take. From Section 3.5, we know that trend risk is a form of parameter risk. The trend risk can be obtained (derivation in Appendix C.1.1) with

$$\hat{\sigma}_c = \sqrt{\frac{\hat{\sigma}^2_\eta}{T-1}}. \tag{25}$$

[17] Estimated by $\hat{c} = \frac{\kappa_T - \kappa_1}{T-1}$.

[18] Volatility v Trend Risk. A technical note on estimating and forecasting with the random walk with drift. By Iain Currie, Heriot-Watt University.

Under the case with trend risk and no volatility, we have from (23) (Jarner and Dengsøe, 2009),

$$\mathbb{E}[\hat{\kappa}_{T+h}|\kappa_T] = \kappa_T + hc,$$
$$Var[\hat{\kappa}_{T+h}|\kappa_T] = h^2\sigma_c^2.$$

When there is only volatility and no trend risk (parameter c is fixed),

$$\mathbb{E}[\hat{\kappa}_{T+h}|\kappa_T] = \kappa_T + hc + 0,$$
$$Var[\hat{\kappa}_{T+h}|\kappa_T] = h\sigma_\eta^2.$$

The derivation of the variance of κ_{t+h} for when there is both the trend parameter risk and volatility is given in Appendix C.1.2. There are several confidence intervals for κ_{T+h} depending on which uncertainty is involved. Analytically, the forecast of the prediction interval can be given by CLT or using simulations. Confidence interval is different for when there is only the trend risk, volatility or both. Hence, the (general) confidence interval for κ_{T+h} can be expressed as follows,

$$CI_{1-\alpha}(\kappa_{T+h}) = \kappa_T + h\hat{c} \pm \Phi^{-1}(1-\alpha/2)\sqrt{I_1(h^2\hat{\sigma}_c^2) + I_2(h\hat{\sigma}_\eta^2)}, \tag{26}$$

where $\Phi^{-1}(1-\alpha/2)$ denotes the quantile function of a standard normal distribution N(0,1) with the $(1-\alpha/2)^{th}$ percentile and $100(1-\alpha)\%$, the confidence level. I_1 and I_2 are indicators denoting whether there is trend risk, volatility or both involved. If $I_1 = 1$, there is trend risk and with $I_2 = 1$ there is volatility.

Stochastic variation on the other hand is the aggregated effect of the uncertainties in $\epsilon_{x,t}$ and η_t. Together, these two independent random variables are normally distributed (Broeders et al., 2018),

$$\beta_x\eta_t + \epsilon_{x,t} \sim N(0, \beta_x^2\sigma_\eta^2 + \sigma_{\epsilon,x}^2).$$

From (25) and (24), the standard deviation of this stochastic variation is easily obtained. Over an h-years horizon forecast, these uncertainties will be (see Appendix C.2 for the derivation),

$$\beta_{x+h}\sum_{i=1}^{h}\eta_{T+i} + \epsilon_{x+h,t+h} \sim N(0, \beta_{x+h}^2 h\sigma_\eta^2 + \sigma_{\epsilon,x+i}^2).$$

The confidence interval formula in (26) can be updated by including the possibility of stochastic variation. Instead of the confidence interval for κ_{T+h}, we now have a confidence interval for $\ln(m_{x+h,t+h})$),

$$\begin{aligned} CI_{1-\alpha}(\ln(m_{x+h,t+h})) &= \hat{\alpha}_{x+h} + \hat{\beta}_{x+h}(\kappa_T + h\hat{c}) \\ &\pm \Phi^{-1}(1-\alpha/2)\sqrt{\hat{\beta}_{x+h}^2(I_1(h^2\hat{\sigma}_c^2) + I_2(h\hat{\sigma}_\eta^2)) + I_3(\hat{\sigma}_{\epsilon,x+h}^2)}. \end{aligned} \tag{27}$$

When $I_2 = 1$ and $I_3 = 1$ we will have stochastic variation. No analysis is done further in this thesis on the case when all these uncertainties are combined ($I_1 = 1, I_2 = 1$ and $I_3 = 1$.

6.3 Stressed-trend Approach (Quantile of Trend Uncertainty)

The stress-trend approach (Richards et al., 2014) is a long-term stress projection over an accumulation of small changes. The idea of this approach is to view the small changes over the potential lifetime of the annuitant by deriving a confidence interval to the forecasted morality rates. The reason of this approach by Richards et al. is that although many insurance risks are fitted into a one-year value-at-risk framework, it is not natural for all risks. To counter the inflexibility of only measuring risks over a one-year horizon, they follow the stressed trend approach.

In the Lee-Carter model, the mortality trend κ_t is projected using the ARIMA(0,1,0) with drift model, hence the central projection/best estimate of the log mortality rates is determined from this trend model. The stress projection is then the confidence interval by including uncertainties to this best estimate, explained in Section 6.2. The stressed-trend approach can give an insurer

valuable information by quantifying the tail events. From Section 4, it is mostly interesting when the uncertainties lead to improvement in future longevity for the annuitants, hence mortality rates $m_{x,t}$ are lower than projected. The tail event is referred to the case when the survival probability is higher than the expected one. For this reason, the stress projection is the central projection deducting $\Phi^{-1}(1-\alpha/2)$ times the standard deviations, the lower bound of equations (26) and (27). It should be noted that the sensitivity due to mortality improvement is reversed around age 100, as the values of β_x goes from positive to negative, see Figure 13. This implies that mortality improvement over time will lead to an increase for old ages instead. Due to this, the sign (the direction of the sensitivity) is ignored and only the β_x in absolute value, hence the level of sensitivity, is used when longevity risk is incorporated.

Figure 7 shows the central projection ($Z = 0$) and stress projections for the different uncertainties with a confidence level of 99% (i.e. $Z = \Phi^{-1}(0.995) = 2.58$) for Dutch male at age 67. The capital requirements in Solvency II for the longevity risk are based on the 99.5% Value-at-Risk of the available capital, which corresponds to a 0.5% probability that the value is exceeded by a random event. As the risk lies on an increase of the longevity, the lower bound of the best-estimate of log mortality is taken. The projected values are based on the model fitted in Section 3.4.

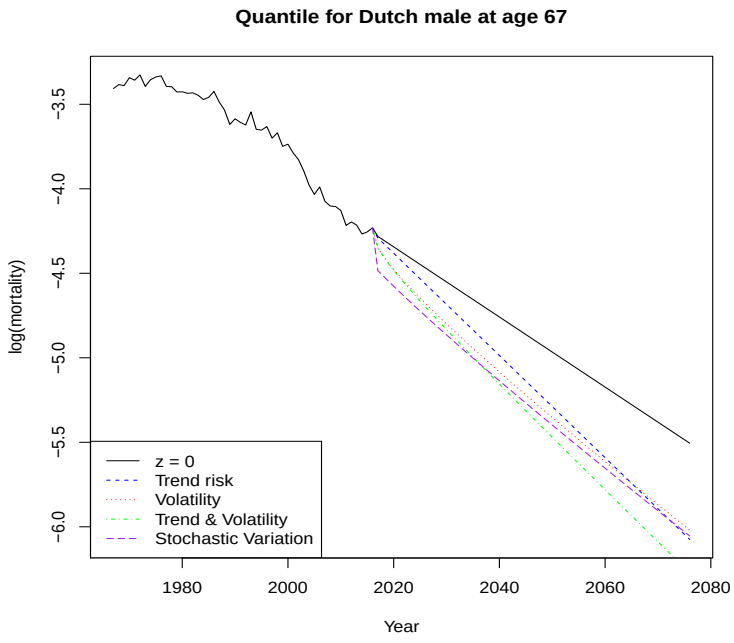

Figure 7: Stressed-trend approach by including all risk sources for Dutch male at age 67.

In Figure 7, the outcomes for mortality rates including different uncertainties are compared. First of all, the trend risk has the least effect on mortality deviations and it is constant over time. The volatility gives relatively bigger deviations in the beginning of the forecasting period compared to later when it starts the get smaller and eventually becomes less than the trend risk. As already mentioned, the volatility a short-term phenomenon, which is less of a concern here but plays an important role in the framework in Section 6.4. Because of this, volatility has less impact than trend parameter risk long term. Furthermore, for when both the trend and volatility are combined, there is a substantial deviation from the best estimate in the later period, which is the most compared to the other risks. As for the stochastic variation, there is a sudden shock in the beginning years caused by the uncertainty in death rates, $\epsilon_{x,t}$. Thereafter, volatility accounts for most part of the risk.

Note that the values given in Figure 7 are age-specific. Hence, all values corresponds to males aged 67 over all years. In Figure 8, the log mortality rates with risks for Dutch male of cohort 1949

are given. It is not clear from Figure 7 and 8 on how large the uncertainties are. To have a better idea, we calculate the (cohort) expected lifetime of an individual at the age of 67 in year 2016, when these risks are included. The expected curtate future lifetime $\mathbb{E}_t[T_{x,t}]$ at age 67 is derived using equation (3) for all the uncertainties. The values are given in the following table.

	Best-estimate	Trend Risk	Volatility	Trend & Volatility	Stochastic variation
$\mathbb{E}[T_{x,t}]$, Male	l6.60	16.94	17.32	17.40	18.05
$\mathbb{E}[T_{x,t}]$, Female	l9.80	20.59	21.26	21.46	21.47

From the table, we see that uncertainties certainly have an effect on the life expectancy, although the deviations are not substantial. This might be an helpful indication for the pension providers if the macro-longevity risks are borne by them. For instance in the Solvency II framework, not only are the insurers obliged to reserve capital for the best estimate of their future liabilities, best estimate of variable annuity payouts in this case, but also for scenarios when risk is involved. Furthermore, it should be noted that the cohort life expectancy depends on the period which the data is fitted. As mentioned in Section 3.4, a shorter fitting period will give a lower variance explained, but life expectancy is then higher. This could be explained by the the amount of information lost when a shorter time period is considered about past mortality events. The fact that a shorter period is considered, age-specific parameters are more erratic and move faster. Parameters will not fully explain the data and do not correctly correspond to the true pattern of the trend. The gain in accuracy would be larger if the fitting period was longer. If the total variance explained is higher by the fitted values, the life expectancy is lower as a result.

On the other hand, the concern lies on how the future variable annuity payments are affected for retirees if they carry the macro-longevity risk. The variable annuity payments based on the future mortality rates without longevity risk are already calculated and illustrated in Figure 8. We now update ${}_hp_{x,0}$ from the biometric returns in equation (13) by incorporating the stress projections. Note that other parameters and AIR do not change. Furthermore, the relation between mortality rates and survival probabilities is explained in Section 3.1. As a result, we have in Figure 8 the graphs of the mortality rates of Dutch male cohort of 1949 with risk given on the left hand side and on the right the corresponding graphs of variable annuity payments. For females, these are given in Appendix, Figure 18.

In the variable annuity graphs, the expected payments with the central projection of longevity is given, with the corresponding 5% and 95% quantiles due to financial risk (in blue lines). The orange lines give the scenario for the expected payments with longevity risk and the 5% and 95% quantiles for financial risk.

As can be seen from the graphs, the effect of trend parameter risk is the smallest compared to other risk components. Trend parameter risk does decrease the biometric returns of participants over the years, but the decreases is more steep at older ages. If participant is still alive around those ages, expected payment will be a lot less than expected. Relatively to the trend risk, there is barely any difference with decrease on biometric returns when volatility is involved. In the case of stochastic variation, the decrease of biometric returns are more compared to other risks.

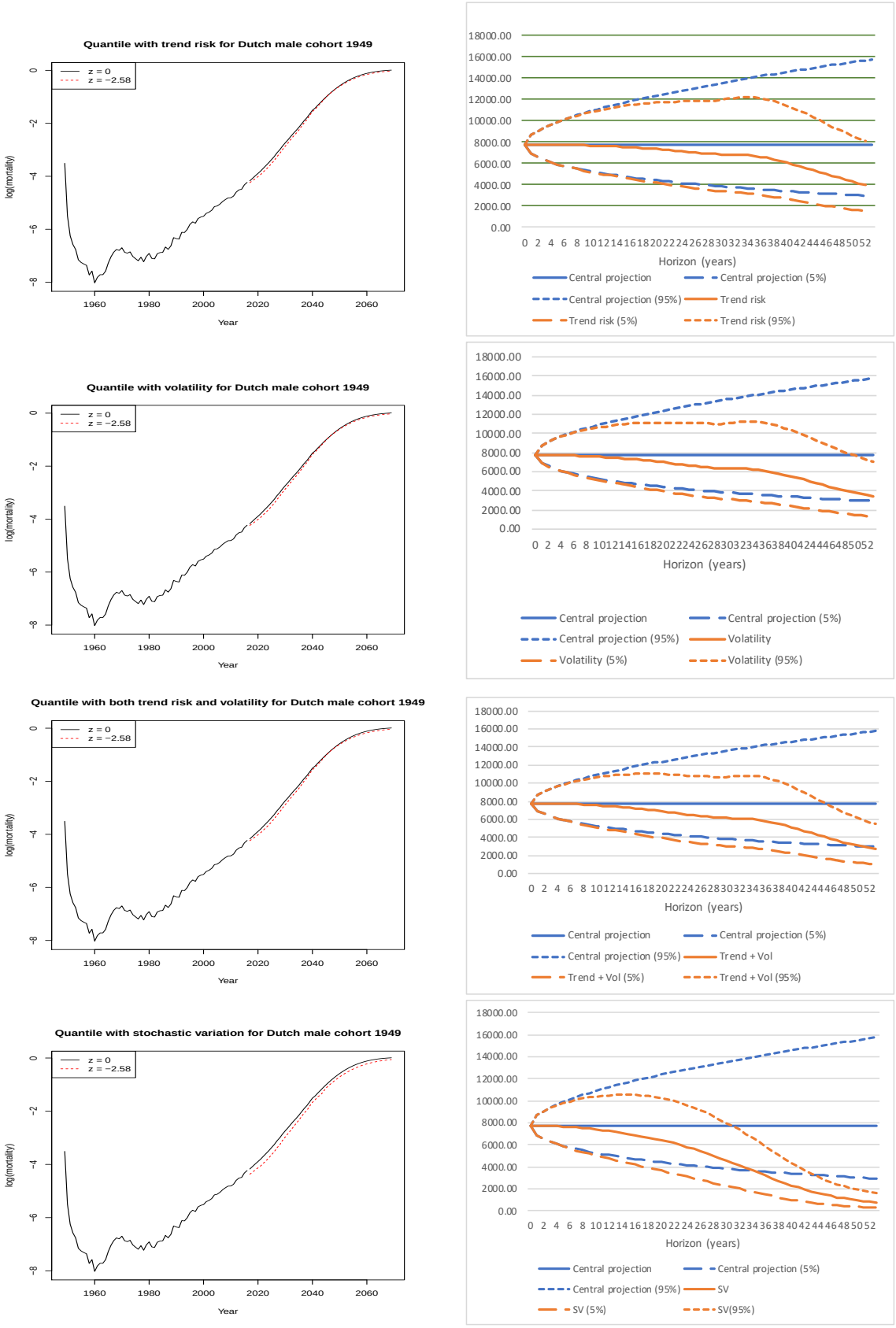

Figure 8: Graphs of mortality rates of male cohort 1949 on the left side and their corresponding variable annuity payouts on the right sides.

6.4 Value-at-Risk Framework

Quantifying longevity risk not only encompasses the calculation of uncertainties based on available data, but also on how an update on mortality data changes the previous quantified risk and how future best-estimates change based on new data.

The approach in Section 6.3 deals with long term changes of mortality rates while assuming that the best estimate of the parameter values (α_x, β_x) are correct and the future development of mortality rates depend on these parameters. Instead, the framework in this section will answer the one-year value-at-risk question (Richards et al., 2014): " What could happen over the coming year to change the best-estimate projection?". The main focus of the value-at-risk framework is the longevity trend risk, on how the central projection changes based on new data. New data will also give new parameter estimates, which is not the same as Section 6.3, where all future mortality rates depend on a fixed data set. Hence, here the parameter risk is indirectly included by parameter re-estimation. Following this framework, we can also incorporate the stochastic variation.

Although longevity risk is a long-term risk, it might be useful to consider this risk on a short horizon. This might be helpful especially under the Solvency II regulation.

The value-at-risk framework (VaR) approach in the case of using the Lee-Carter model is as follows [19]:

1. First, select the mortality rates data set with ages ranging from $x = 0$ to $x = X$ and with years ranging from $t = 1$ to $t = T$. We use the data sets mentioned in Section 3.3 from year 1967 to 2016 and age 0 to 120.

2. Fit the data to the Lee-Carter model to obtain estimates $\hat{\alpha}_x$, $\hat{\beta}_x$ and $\hat{\kappa}_t$.

3. Use the fitted model and simulate one year ahead of mortality rates, i.e. $ln(m_{x,t_{T+1}})$. Simulation can include the trend risk, trend volatility or both. Hence, $\kappa_{T+1} = \kappa_T + \hat{c} + Z\sqrt{I_1\hat{\sigma}_c^2 + I_2\hat{\sigma}_\eta^2}$, with $Z \sim N(0,1)$. Along the lines, we can also incorporate the stochastic variation, although the paper use the framework for longevity trend risk.

4. Combine simulated data from step 3 to the data in step 1. Refit using Lee-Carter on the new combined data set. Forecasts h-years ahead of mortality rates with or without risks. Without risk, $Z = 0$. Changes in these central projections shows how one year of new data affects the (former) best estimate.

5. Repeat steps 3 to 4 n times. For example, when estimating the 99.5^{th} percentile, we need at least $n = 1000$ simulations.

6. Order the n-projections based on $\ln(m_{x,t})$. If we look for the 99.5^{th} percentile, then the $(0.005 \times n)^{th}$, the $(0.5 \times n)^{th}$ and the $(0.995 \times n)^{th}$ projections are taken. Projections are ordered based on the last simulated value.

This framework however only takes one-year of new data into account and from there calculate the best-estimate for the upcoming years. Although it gives a clear view on how the best-estimate changes based on one year extra data compared to the estimates done in Section 6.3, there is still uncertainties as data will be continuously updated, hence best-estimates keeps changing year over year. A comparison for the risks is made between the method in Section 6.3 and the Value-at-Risk framework is given in Appendix C.4. The graphs for Dutch male at age 67 is depicted in Figure 19 and the graphs for the Dutch male cohort 1949 is given in Figure 20. We will not further look into the females mortality rates using this framework. Furthermore, the expected life expectancies are given in Table 1, 'VaR 1'. Comparing the expected lifetime with the stressed-trend, there are not much of a difference.

Instead of considering only a one-year data update, we consider the situation where mortality

[19] More details of the framework can be found in Richards et al. (2014). Moreover, the description of the framework in the paper is more general and can accommodate other stochastic mortality models.

data is updated every year. After several years, the best-estimate is then based on the continuously updated data. First, we update the data by forecasting one extra year. That one year is added the the old data and then refitted to a new model. After refitting the model, a one year ahead mortality forecast is projected and data will be updated again. How data is updated is based on the rolling window method. The range of periods in the data frame is constant, but when the new one year data is added, the oldest year of data is eliminated from the frame. This method is summarized in the following steps, which is still in line with the Value-at-Risk framework.

1. First, a data set of mortality rates is chosen. It is preferable to not choose a too narrow range in the periods, but one that have a significantly high total variance explained. In our case, we choose the fitting period mentioned in Section 3.4, which is 50 years.

2. Fit the data from Step 1 using a mortality model. We use the Lee-Carter method and obtain estimates $\hat{\alpha}_x$, $\hat{\beta}_x$ and $\hat{\kappa}_t$.

3. Forecast $\hat{\kappa}_t$ for one year.

4. Consider the different risk components when forecasting. What is meant here is the trend risk, volatility or stochastic variation. A standard normal value $Z \sim N(0,1)$ is randomly drawn for adding the risk in the forecast.

5. Calculate the mortality rates $m_{x,t}$ of the one year forecast (with uncertainty).

6. Note that in the beginning (step 1), fitting was done based on 50 years of past data. Instead of continuing the use of such a large horizon, we can also shorten the data period to e.g. the latest 20 years. This gives more weight to the recent events in mortality rates when the model is fitted. Results using this fitting range is given in Section 6.5. We continue with the 50 years fitting range in this section. New data is appended to the data set. The rolling window method is used, hence the oldest year data is removed from the frame and new data is added as the last column.

7. Refit the data in step 6 and then go back to step 3.

8. Step 3 until step 6 are repeated for e.g. 60 years. This implies that we consider the scenario when data is updated for 60 years with random development each year on top of the best-estimate of mortality rates because of uncertainties.

9. Step 3 till 6 is for one scenario. For n scenarios, repeat Step 2 until Step 6.

10. Order the n-projections based on $\ln(m_{x,t})$. This is the same as already mentioned in the previous framework in step 6. It should be noted that a trajectory might not be overall ranked around the $(0.995 \times n)^{th}$. In this case, the wrong trajectory can be chosen, if we solely look at the end value. It is then recommended to also look at the whole trajectory in order to choose the most reasonable one.

The results for this framework is given in Figure 9, the red dotted line, 'VaR 50'. The figure shows the morality rates pattern over the years for Dutch males at age 67. For females, this is given in the Appendix in Figure 21. First, note that the best-estimate for the stressed-trend and the VaR method is given (solid black and red line). The red dotted line shows the 99.5^{th} percentile for the VaR framework with rolling window method and this is compared with the stressed-trend method (black dotted line). Using the one-year rolling window method, it can be concluded that there is less trend risk from the best-estimate compared to the stressed-trend method. On the other hand, more uncertainty of the mortality is expected with volatility at later ages. As for the stochastic variation, the deviation is substantial, which also seems implausible.

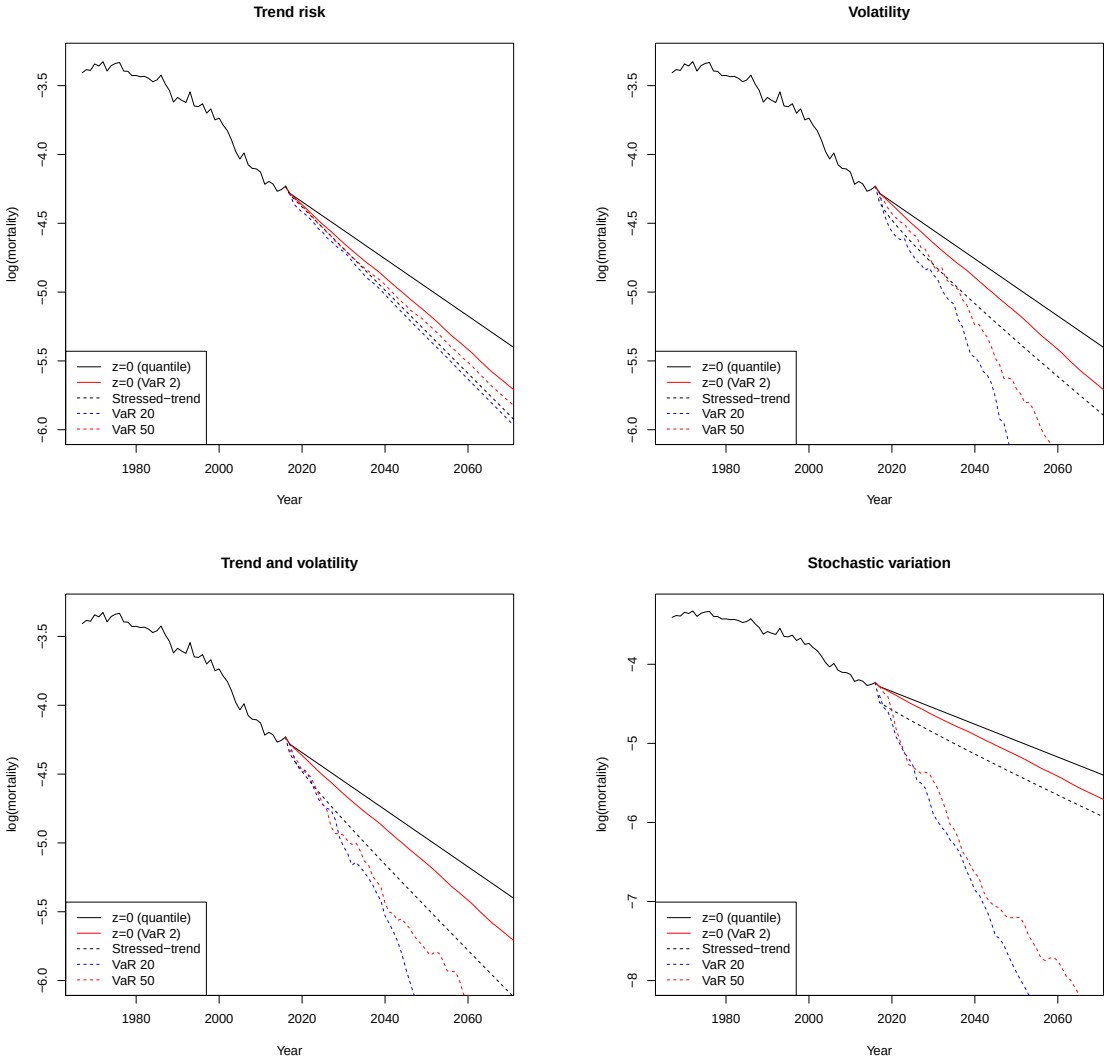

Figure 9: Graphs of mortality rates of male at age 67 with longevity uncertainties. Comparison of the stressed-trend approach and the VaR framework (with rolling window method).

A table of the cohort life expectancy for both males and females is given below. This corresponds to the 0.5^{th} percentile of the mortality rates when uncertainties are included.

		Best-estimate	Trend risk	Volatility	Trend & Vol	SV
Male	Stressed-trend	16.6	16.94	17.32	17.4	18.05
	VaR 1	16.9	16.95	17.28	17.28	18.2
	VaR 2 (50 yrs)	16.95	17.11	18.05	18.08	21.12
Female	Stressed-trend	19.8	20.59	21.26	21.46	21.47
	VaR 1	19.79	19.85	20.28	20.28	21.45
	VaR 2 (50 yrs)	19.71	19.98	21.92	21.76	24.88

Table 1: Cohort life expectancy of Dutch males and females at age 67 in 2016.

There is not much of a difference between the cohort life expectancy of the stressed-trend and the VaR 1, which could be explained by the data is only updated and fitted for only one year. With the value-at-risk framework with rolling window method ('VaR 2') on the other hand, life

expectancy is significantly different compared to stressed-trend. Also note that there is more than 4 years of difference between life expectancy of the value-at-risk method with stochastic variation and the best-estimate with stressed-trend method.

The variable annuity payments are illustrated in Figure 10. For trend risk, first graph on the right side, the pension payments with risk (range line) are close to the case of the best-estimate (blue line). This further implies that if calculation with trend risk is conducted in this way, it is expected that overall, there is barely any decrease of biometric returns except for the older ages. This result might be the cause of using a long-term risk on a short-term risk framework and also the value-at-risk framework indirectly incorporated the parameter risk, which makes the uncertainty deviating less from the best-estimate. As mentioned by Richards et al. (2014), this framework is mainly for the volatility, as short-term fluctuations on mortality rates is caused by this term. Hence, illustrated in Figure 10, the decrease of annuity payments is depicted under the case of volatility. For when trend and volatility is incorporated, the graphs is similar to almost equal to the case with volatility. It can be concluded that the trend parameter risk has almost no effects on the future expectations of the payments. As for the stochastic variation, there is barely any payments after 30 years into retirement. This implies that when stochastic variation is involved in the calculation of future mortality rates, it shows that the best-estimate of mortality rates is overly estimated. Stochastic variation will lead to a shock in the longevity which everyone in a pool will be affected, leading to a great shock in the biometric returns. Furthermore, the results for Dutch women are depicted in Figure 22.

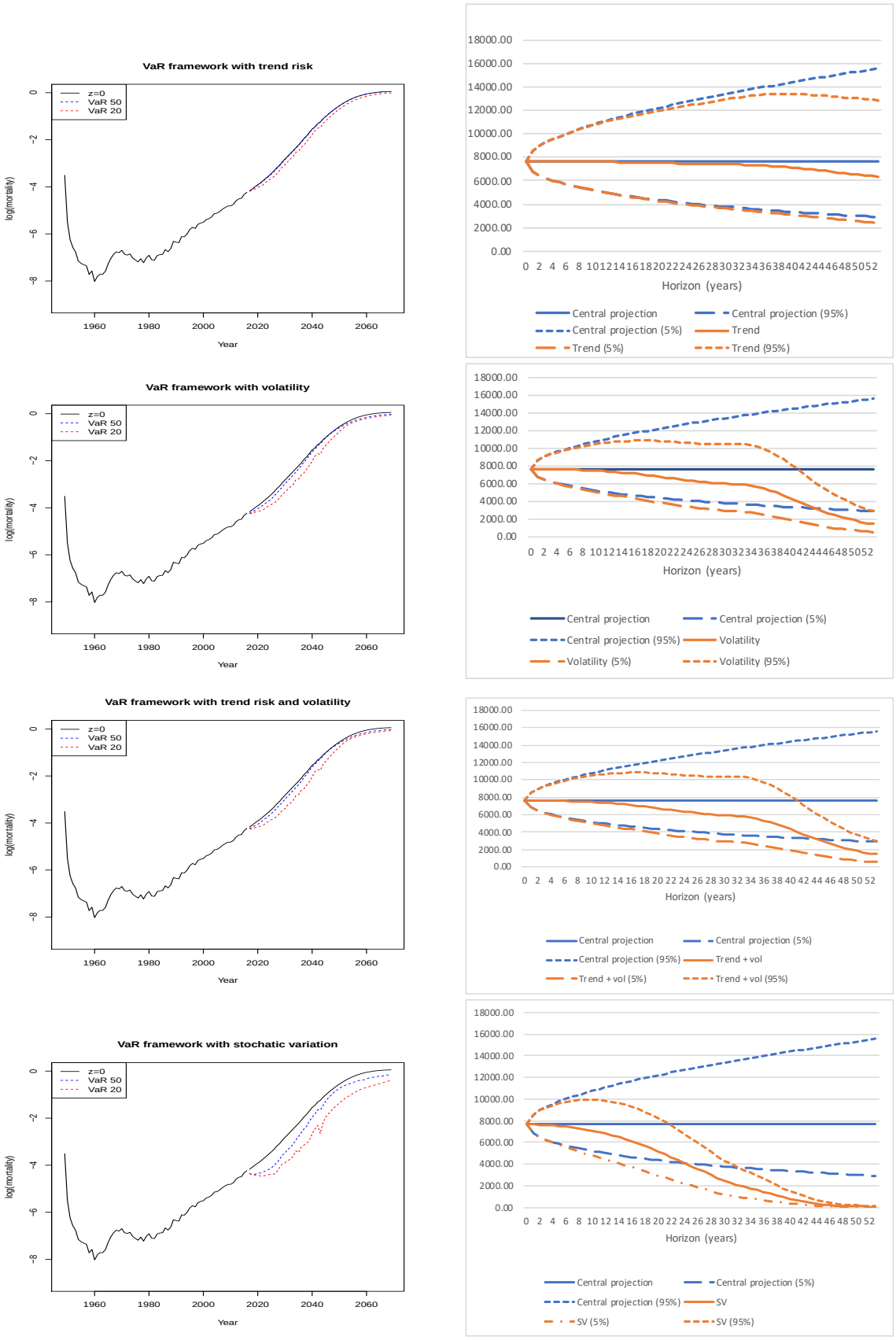

Figure 10: Graphs of mortality rates of male cohort 1949 on the left side and their corresponding variable annuity payouts on the right sides.

6.5 Robustness Check/Sensitivity Analysis

It is mentioned in Section 3.2 that the Lee-Carter model assumes a homoskedastic variance σ_ϵ for the error term $\epsilon_{x,t}$. Instead, we used the age-dependent variance $\sigma_{\epsilon,x}$. In this section, we follow the original assumption of the model and check the robustness on the outcome of mortality rates and payouts. The variance is derived from

$$\sigma_\epsilon^2 = \frac{\sum_{x=1}^{X}\sum_{t=1}^{T}(\ln(m_{x,t}) - (\hat{\alpha_x} + \hat{\beta}_x\hat{\kappa_t}))^2}{m(n-1)}, \tag{28}$$

where m is the size of the set of ages and n is the size of the time horizon. Hence, for the stochastic variation, σ_ϵ is used instead and the results are depicted in Figure 25. The uncertainty is significantly higher for women than for men. Comparing this stochastic variation with when $\sigma_{\epsilon,x}$ is used, there are more uncertainties when the homoskedastic variance is used. Annuity payments decreases faster over the horizon is this case.

Another sensitivity analysis check is done by using a smaller refitting data range for the value-at-risk framework. Before, 50 years of past data is used to fit the Lee-Carter model. Instead, a shorter period can be chosen such that recent mortality events will have more impacts on the future mortality forecasts. We start the framework with a model fitted on 50 years of data and hereafter, data is updated and only 20 years window range will be used. Results (referred as 'VaR 20') are given in the Appendix in Figure 9 for men and Figure 23 for women. In the case of trend risk, there is an adverse trend to the mortality rates. Instead of lower mortality, it is forecasted to be higher compared to the best estimates of the stressed-trend and 'VaR 50'. Other risks involved, there is a slight difference between the quantile of the 'VaR 20' and 'VaR 50' method at the age of 68 for both men and almost no difference for women. Furthermore, the variable annuity payouts for men and women are depicted in Figure 24. Using a smaller data frame for refitting will lead to bigger uncertainties. This could be explained by the variance of data explained by 20 years of data is not high and hence a lot of information is lost. This can cause an overestimation of the risk uncertainties.

The complete cohort life expectancy table including all risk mentioned and methods used is given in Appendix, Table 2.

6.6 Parameter risk

In the original Lee and Carter (1992) model, parameter risk is neglected. Parameter risk, already mentioned in 3.5, is the uncertainty of parameter estimates. If an adequate and representative data set is used, then parameters can be determined properly. Hence, uncertainty of the estimated parameters can be minimized by a proper data set (Richards, 2016). In a Bayesian setting of mortality rates modeling Czado et al. (2005), the parameter risk can be implemented. Another way to quantify the parameter risk in the original Lee-Carter method is bootstrapping. There are several ways to do bootstrapping.

First, there is the 'parametric bootstrap' method. This method samples from the empirical distribution of the error term $\epsilon_{x,t}$. First, the parameters α_x, β_x and κ_t are estimated with the given data. Then n trajectories are of mortality rates are sampled using $\hat{\alpha}_x$, $\hat{\beta}_x$, $\hat{\kappa}_t$ and $\hat{\epsilon}_{x,t}$, where $\epsilon_{x,t} \sim N(0, \sigma_\epsilon^2)$ and $\hat{\sigma}_\epsilon^2$ is given in equation (28). Problems occur when the empirical distribution of the error terms does not follow the Gaussian distribution. Analyzing the error terms of our model, it seems that it does not follow the Gaussian distribution. For this reason, the 'residual bootstrap' method can be applied. More details about these bootstrapping method can be found in e.g. Pitacco et al. (2009). We will not further analyze the implementation of the parameter risk, as this topic is not the focus of our thesis.

6.7 Comments

It is assumed that the Lee-Carter model is an appropriate and correct model to use, hence we did not look at model risk. Comments in this section is based on the results using the Lee-Carter model.

First of all, it should be noted that the point estimate of mortality in the stressed-trend approach gives a slightly different value for life expectancy compared to the VaR framework. This implies that the model used to derive future survival probabilities matters for the wealth allocation of an individual at the start of their retirement phase. This also changes the view of what is expected when longevity risk is involved. The model used can differ per pension fund or insurance company. In the Netherlands there are few methods which the projection results are often used by insurers. For example, there is the AG projection table. This table contains one year mortality probabilities forecasted for years 2018-2068 which comes from the Dutch actuarial association (Actuarieel Genootschap, AG) and values are derived using their own specified model. More importantly, the method used by the AG made it possible to incorporate uncertainties for the estimates. Another example is the projection table is given by the CBS, the latest launched is on December 19, 2017. The method used for the projection is described in Duin et al. (2013). Although these models could be used in practice, it does not necessarily mean that much attention is given to the uncertainties surrounding the best-estimates.

Moving on, it can be seen from the mortality rate graphs for the cohort 1949 that at older ages there less deviations in the mortality rates with uncertainties from the point estimate. This is visible in the graph with stochastic variation. Note that the heteroskedastic variance causes the variability of mortality to not be constant over the ages. What the graphs further suggest is the compression view for the future survival probabilities. Compression of mortality is when mortality decreases over the range of ages, but the variance of reduction is less at older ages till the age-at-death, leading to a 'rectangular' survival function (a figure is given in Broeders et al. (2018)). In this case, a natural limit for the maximum age might exists, with evidence found by e.g. Einmahl et al. (2019).

Continuing with the discussion of stochastic variation, the question lies on whether the constant variance for the error term $\epsilon_{x,t}$ should be used. The homoskedastic variance (σ_ϵ) assumption implies that the standard error is expected to be minimized and also leads to correct conclusions of the estimates. In general, there is different variability at different ages in the pattern of the mortality rates. Even after the smoothing of the mortality data at the older ages (with the Kannisto method), there is still an accident bump for example at young age for men, which makes the variance not constant over all ages. This highlights that the constant variance assumption is unrealistic in the first place. For this reason, the Poisson log-bilinear regression model is proposed (Brouhns et al., 2002). If the constant variance is used for stochastic variation as in Section 6.5, future mortality forecasts contradicts the compression view mentioned before and indicates expansion of mortality instead, Figure 25. This opposes the evidence that there exists a limit to maximum age, but coherent with the statement by e.g. Oeppen et al. (2002) that there is no maximum age.

Furthermore, the results from the experimental strategy proposed by D'Amato and Russolillo (2010) to investigate the impact of heteroskedastic residuals on mortality rates suggests that the traditional Lee-Carter method with homoskedastic residual assumption will overestimate mortality decreases.

Mortality might be overestimated with the stochastic variation, especially in the VaR framework which can be seen that the deviations are quit big and future expected payments goes to almost zero for old ages. These low payouts might be also caused by the extrapolation of mortality data for old ages from which no data are available using data of younger ages. It is not guaranteed that the observed trends of these younger ages continue in the older ages, thus might cause a high variability around these old ages.

7 Conclusion and Possible Future Research

Conclusion

The goal of this thesis was to explore the effects of different longevity risk components on the variable annuity payouts. There is first of all, not enough and highly volatile data at old ages which needs to be extrapolated. Extrapolated data makes the mortality pattern smoother but not smooth enough for the homoskedastic residual assumption in the Lee-Carter model to be valid. Hence, age-dependent heteroskedastic variances for the residuals are used in the main research of longevity uncertainties and this is later on compared with the homoskedastic constant variance.

To research the impact of longevity risk on variable annuities, mortality needs to be first incorporated in the wealth allocation of total retirement wealth for future years through the assumed interest rate. Then future developments of the mortality rates influence the outcome of the payments through the biometric returns.

Uncertainty of the longevity causes the biometric return to deviate and affects then the expected future benefits. On the micro level, uncertainty about one's time of death may cause others in the same pool to receive more or less from the biometric returns than expected. Since this micro-longevity risk is diversifiable, risk is diminished for a relatively big fund size. Hence, this risk should not be a concern for the insurers and the retirees.

Macro-longevity risk on the other hand is not diversifiable and this risk is affected by everyone in the same cohort. Conclusions are made while comparing the different risk sources mentioned for the macro-longevity risk using the two methods for predicting the future mortality rates. Starting with the best-estimates of life expectancy, these values slightly differ per method used. This already suggests that wealth allocated at the start of the retirement phase are different with different expected payments. Although the stressed-trend method used to quantify the risk is more straightforward, the value-at-risk framework might be more realistic to view future mortality developments, as the latter updates the data every year. With trend parameter risk, there is more deviations in mortality rates with the stressed trend method compared to VaR framework. This is due to the long term nature of the trend risk and the VaR framework focus on short term, which makes the effect of trend risk less relevant. Furthermore, volatility seems to have almost the same impact in both methods. As for the stochastic variation, deviations are substantial at old ages. It is a question whether this is plausible as a result. From the point of view of the insurer or pension fund, the difference in life expectancy caused by the uncertainty poses a problem for the solvency capital required. As for the retirees, in the worst-case scenario of longevity improvement, almost no payments will be received if they survive in older ages (after the age of 100). Uncertainties can be suppressed by good estimates. But with the homoskedastic variance problem, parameters may not be correctly estimated and hence further research is needed. Correct estimation of the parameters is one thing, but acknowledging the size of uncertainties that comes with it is another matter. Since we did not focus on parameter risk and model risk, no comments can be made on how these will impact the life expectancy. In conclusion, each risk component contributes differently to the mortality deviations depending on the underlying mortality model and method used to quantify risk. Thus, future research is needed on the relevance of methods for risk quantification and the model used to fit and forecast mortality data.

Possible future research

- Other stochastic mortality models could be used. For example the Poisson log-bilinear regression approach, a modification of the original Lee-Carter model that accounts for the homoskedatic residuals problem. The Li-Lee model is another possible choice, which uses data from several similar countries to estimate parameters of the model.

- Parameter risk is analyzed indirectly in this thesis, but it is possible to do some analysis on this risk using few methods mentioned in the literature, e.g. boostrap method or the Bayesian framework.

- If the pool of retirees consists of different cohorts, then risk is shared intergenerationally, biometric returns then also depends on different survival probabilities. This would be inter-

esting to research on to check the difference when compared to risk only shared within one cohort.

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A Mortality

A.1 Kannisto method

Let $D_{x,t}$ be the number of deaths and $E_{x,t}$ be the exposure-to-risk for ages $x \in \{80, \ldots, 95\}$ and all t assumed. The force of mortality/central death rates follow from equation (1). To extend mortality rates up to age 120, the force of mortality can be fitted using the Kannisto method (Thatcher et al., 1998) for old-age mortality, which is a logistic curve with an asymptote equal to one:

$$m_{x,t} = \frac{\theta_{1,t}\exp(\theta_{2,t}\chi)}{1 + \theta_{1,t}\exp(\theta_{2,t}\chi)}, \tag{29}$$

where $\chi = x - 80$. Equation (29) can be rewritten as

$$logit(_{x,t}) = \log\left(\frac{m_{x,t}}{1 - m_{x,t}}\right) = \log(\theta_{1,t}) + \theta_{2,t}\chi + \epsilon_{\chi,t}, \tag{30}$$

with $\theta_{1,t} \geq 0$ and $\theta_{2,t}0$.
Assume that $D_{x,t} \sim POI(E_{x,t}m_{x,t}(\theta_{1,t}, \theta_{2,t}))$, then $\theta_{1,t}$ and $\theta_{2,t}$ can be derived by the maximizing the following log-likelihood function,

$$logL(\theta_{1,t}, \theta_{2,t}) = \sum_{x=80}^{95} \{D_{x,t}\log[m_{x,t}(\theta_{1,t}, \theta_{2,t})] - E_{x,t}m_{x,t}(\theta_{1,t}, \theta_{2,t})\} + constant.$$

Updating (29) with $\hat{\theta}_{1,t}$ and $\hat{\theta}_{2,t}$, the mortality rates after the age of 95 can be then extrapolated.

A.2 Lee-Carter model: fitting and forecasts

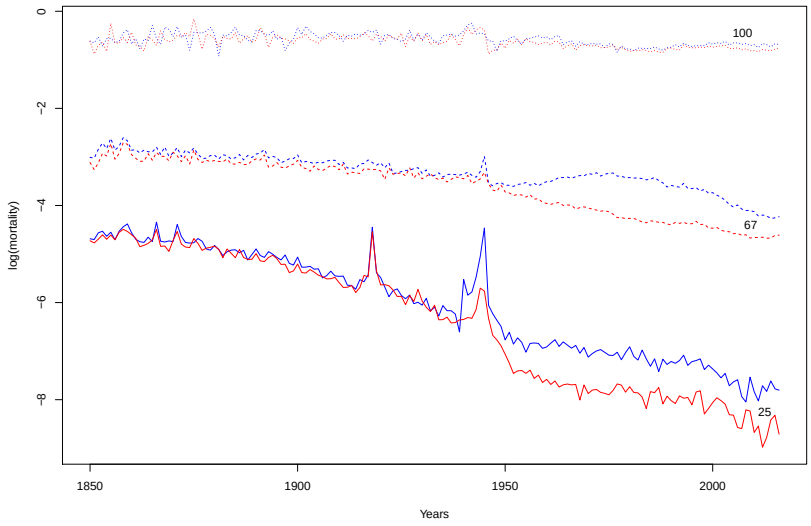

Figure 11: Log mortality rates for Dutch males and females over the years for age 25, 67 and 100.

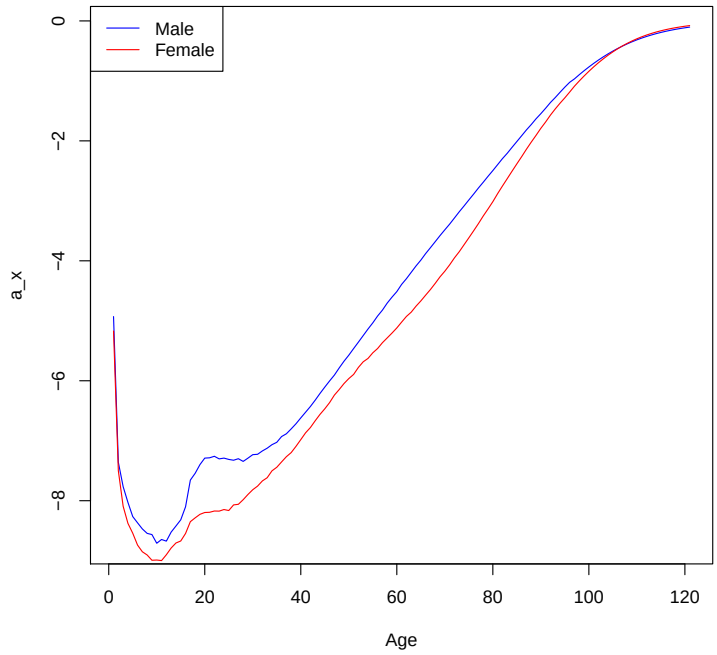

Figure 12: Estimates for α_x according to (5) for the Dutch population.

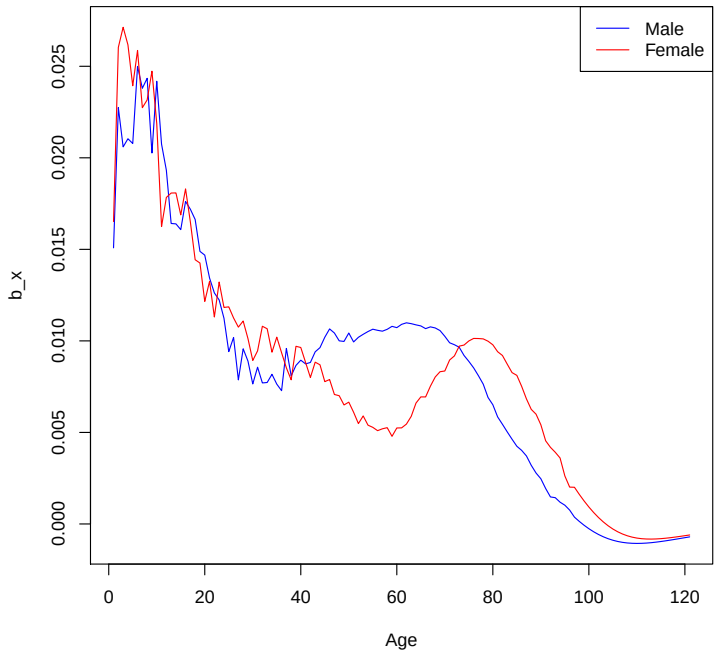

Figure 13: Estimates for β_x using SVD for the Dutch population.

Figure 14: Estimates for κ_t using SVD for the Dutch population.

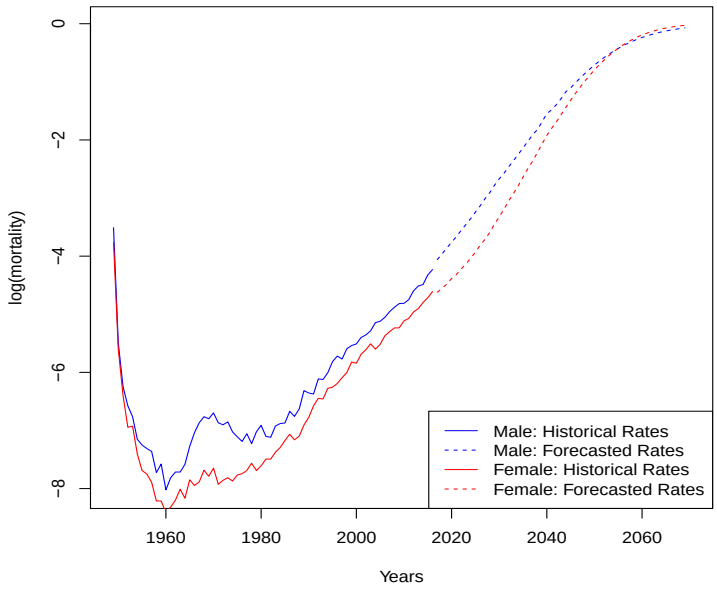

Figure 15: Lee-Carter mortality rates forecast for Dutch males and females for the cohort 1949.

B Variable Annuities with Mortality

B.1 Assumed Interest Rate

The assumed interest rate (AIR) determines the asset wealth allocation. Future pension benefits are to be discounted through this AIR, which it depends on the risk free rate and in addition, possibly a ('fixed') yearly decrease. AIR would only include the risk free rate r_f in the case of fixed annuities, as this is the only return in the future. We follow Balter and Werker (2017) on the wealth allocation through AIR. The assumed interest rate $a_0(h)$ is defined through,

$$\frac{W_0(h)}{W_0} = \frac{W_0(h)}{\sum_{k=0}^{H-1} W_0(k)} = \frac{\exp(-ha_0(h))}{\sum_{k=0}^{H-1} \exp(-ka_0(k))}, \tag{31}$$

which implies that

$$\frac{W_0(h)}{W_0(0)} = \exp(-ha_0(h)), \tag{32}$$

In case of fixed annuities, $a_0(h)$ simply equals to the constant risk free rate r_f.

In the following Figure, 16, a figure is given for the wealth allocation in the case of fixed annuities.

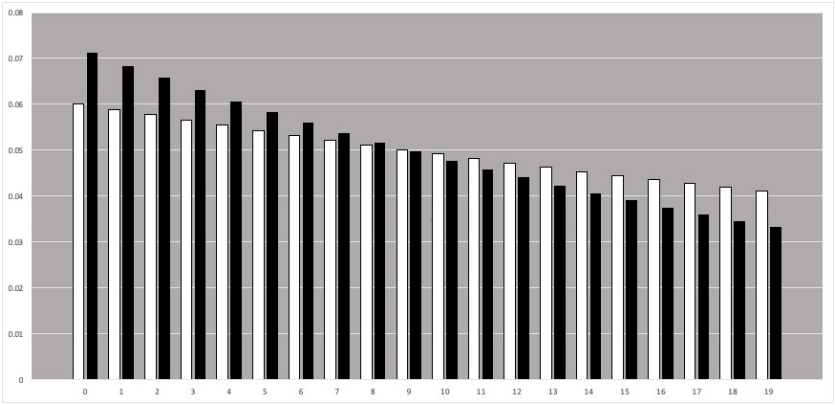

Figure 16: Wealth division with AIR = 2% (white bars) and AIR=4% (black bars).

A high AIR leads to a high initial payment level, but have more chance of an actual decrease over the years. A lower AIR on the other hand have a low initial benefit level, but it also decrease less fast over time. Take the assumed interest rate ($=r_f$) of 2.00% for example. Each pension bucket $W_0(h)$ allocated has 2% less of the initial bucket than the previous bucket, see equation (32). If each bucket grow at the risk free rate over time, the pension benefits will be constant.

B.2 Financial Market

In variable annuities, each asset $W_t(h)$ is partly invested in stock S_t and continuously re-balanced with risk exposure w. The differential equation then follows,

$$dW_t(h) = rdt + w(\lambda\sigma W_t(h)dt + \sigma W_t(h)dB_t), \tag{33}$$

for $0 \leq t \leq h$. Using the Itô's lemma, we have

$$d\log W_t(h) = (r + w\lambda\sigma - \frac{1}{2}w^2\sigma^2)dt + w\sigma dB_t. \tag{34}$$

Furthermore, using the continuous rule, we have:

$$W_t(h) = W_0(h)exp((r + w\lambda\sigma - \frac{1}{2}w^2\sigma^2)t + \sigma wB_t, \tag{35}$$

where $B_t \sim N(0, t-0)$. Rewriting this under the standard normal distribution Z, where $B_t = \sqrt{t}Z$ we have,

$$W_t(h) = W_0(h)exp((r + w\lambda\sigma - \frac{1}{2}w^2\sigma^2)t + \sigma w\sqrt{t}Z. \tag{36}$$

Note that this follows a log-normal distribution (i.e. $X = exp(m + sZ)$, where $Z \sim N(0, 1)$). For a log-normal variable X, the expectation is given by $E(X) = exp(m + \frac{1}{2}s^2)$.

$$\mathbb{E}_0[W_t(h)] = W_0(h)exp(t(r + w\lambda\sigma)), \tag{37}$$

for $0 \leq t \leq h$. The quantile at level α of a log-normal variable is given by $exp(m + sz_\alpha)$. Hence, we have

$$Q_0^{(\alpha)}(W_t(h)) = W_0(h)exp(t(r + w\lambda\sigma - \frac{1}{2}w^2\sigma^2) + w\sigma\sqrt{t}z_\alpha), \tag{38}$$

where z_α denotes quantile of the standard normal distribution.

C Macro-longevity

C.1 Trend Uncertainty

C.1.1 Variance of drift parameter

Given

$$\kappa_t = \kappa_{t-1} + c + \eta_t,$$

it can be rewritten as

$$\Delta\kappa_t = \kappa_t - \kappa_{t-1} = c + \eta_t, \quad \eta_t \sim N(0, \sigma_\eta),$$

where $\Delta\kappa_t$ are independently and identically distributed with mean c and variance σ_η.
If c is estimated by

$$\hat{c} = \frac{1}{T-1}\sum_{t=2}^{T}(\kappa_t - \kappa_{t-1}) = \frac{\kappa_T - \kappa_1}{T-1},$$

then the variance of $\hat{c}$ is

$$\begin{aligned}
\mathrm{Var}(\hat{c}) &= \mathrm{Var}\left(\frac{\kappa_T - \kappa_1}{T-1}\right) \\
&= \frac{1}{(T-1)^2}\mathrm{Var}\left(\kappa_1 + (T-1)c + \sum_{j=2}^{T}\eta_j - \kappa_1\right) \\
&= \frac{\sigma_\eta^2}{T-1} \approx \frac{\hat{\sigma}_\eta^2}{T-1}.
\end{aligned}$$

C.1.2 Variance for both drift parameter and error term

In Section 6.2, the variances of each uncertainty are separately given. Now we consider the case when κ_{T+h} included both the parameter uncertainty and the volatility, $\hat{\kappa}_{T+h}$. For simplicity, denote $\kappa_{T+h} = \kappa$, $\kappa^*_{T+h} = \kappa^*$ and $\hat{\kappa}_{T+h} = \hat{\kappa}$.
The variance for the uncertainties is the variance of the deviations of the estimated values $\hat{\kappa}_{T+h}$ from the realised values κ_{T+h}. Hence we have,

$$\mathrm{Var}(\hat{\kappa} - \kappa) = \mathbb{E}[(\hat{\kappa} - \kappa)^2] - \mathbb{E}[(\hat{\kappa} - \kappa)]^2.$$

First, note that $\mathbb{E}[(\hat{\kappa} - \kappa)]^2 = 0$,

$$\begin{aligned}
\mathbb{E}[(\hat{\kappa} - \kappa)]^2 &= \mathbb{E}[h\hat{c} - hc - \sum_{j=1}^{h}\eta_{T+j}]^2 \\
&= \left(\mathbb{E}[h\hat{c}] - hc - \mathbb{E}\left[\sum_{j=1}^{h}\eta_{T+j}\right]\right)^2 \\
&= (hc - hc - 0)^2 = 0.
\end{aligned}$$

Furthermore, $\mathbb{E}[(\hat{\kappa} - \kappa)^2]$ is given by

$$\begin{aligned}
\mathbb{E}[(\hat{\kappa} - \kappa)^2] &= \mathbb{E}[(\hat{\kappa} - \kappa^* + \kappa^*\kappa)^2] \\
&= \mathbb{E}[(\hat{\kappa} - \kappa^*)^2] + \mathbb{E}[(\kappa^* - \kappa)^2] + 2\mathbb{E}[\hat{\kappa} - \kappa^*]\mathbb{E}[\kappa^* - \kappa] \\
&= \mathbb{E}[(\hat{\kappa} - \kappa^*)^2] + \mathbb{E}[(\kappa^* - \kappa)^2] + 2\mathbb{E}[\hat{\kappa} - \kappa^*]\mathbb{E}[-\sum_{j=1}^{h}\eta_{T+j}] \\
&= \underbrace{\mathbb{E}[(\hat{\kappa} - \kappa^*)^2]}_{\text{trend parameter risk}} + \underbrace{\mathbb{E}[(\kappa^* - \kappa)^2]}_{\text{volatility}}.
\end{aligned}$$

We have $\mathbb{E}[(\hat{\kappa} - \kappa^*)^2] = h^2\sigma_c^2$ and $\mathbb{E}[(\kappa^* - \kappa)^2] = h\sigma_\eta^2$.
Hence,

$$\mathrm{Var}(\hat{\kappa} - \kappa) = h^2\sigma_c^2 + h\sigma_\eta^2.$$

C.2 Stochastic Variation

The Lee-Carter model is given as follows

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t},$$

with a trend process that follows a random walk with drift,

$$\kappa_t = \kappa_{t-1} + c + \eta_t.$$

Rewriting this we have

$$\begin{aligned} \ln(m_{x,t}) &= \alpha_x + \beta_x \kappa_t + \epsilon_{x,t} \\ &= \alpha_x + \beta_x(\kappa_{t-1} + c + \eta_t) + \epsilon_{x,t} \\ &= \alpha_x + \beta_x \kappa_{t-1} + \beta_x c + \underbrace{\beta_x \eta_t + \epsilon_{x,t}}_{\text{stochastic variation}} . \end{aligned}$$

For h-years horizon forecasts, we have

$$\kappa_{T+h} = \kappa_T + hc + \sum_{T=1}^{h} \eta_{T+i},$$

with T the last observed value. This means that for h-years ahead of forecast:

$$\ln(m_{x+h,t+h}) = \alpha_{x+h} + \beta_{x+h}\kappa_T + \beta_{x+h} hc + \beta_{x+h} \sum_{i=1}^{h} \eta_{T+i} + \epsilon_{x+h,t+h}.$$

Over h-years horizon, the two independent random variables are normally distributed and the stochastic variation follows:

$$\beta_{x+h} \sum_{i=1}^{h} \eta_{T+i} + \epsilon_{x+h,t+h} \sim N(0, \beta_{x+h}^2 h\sigma_\eta^2 + \sigma_{\epsilon,x+i}^2).$$

C.3 Stressed-trend approach

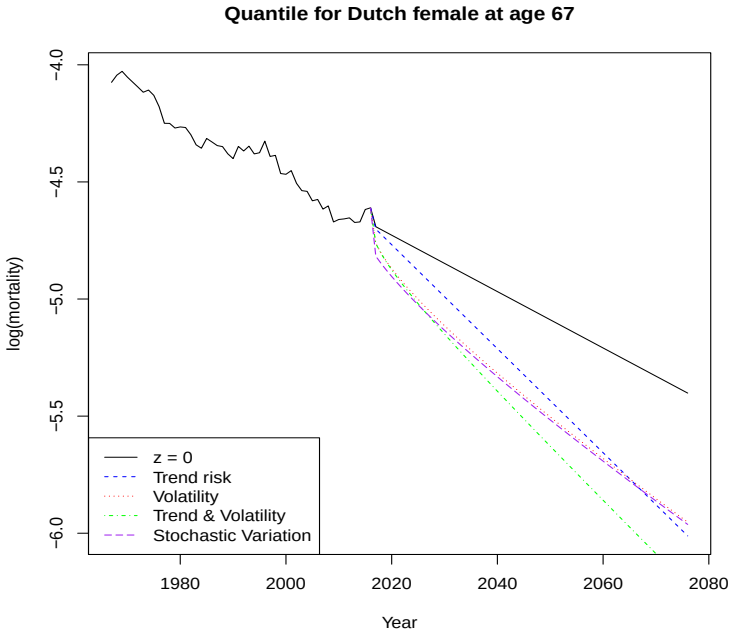

Figure 17: Stressed trend approach by including all risk sources for Dutch female at age 67.

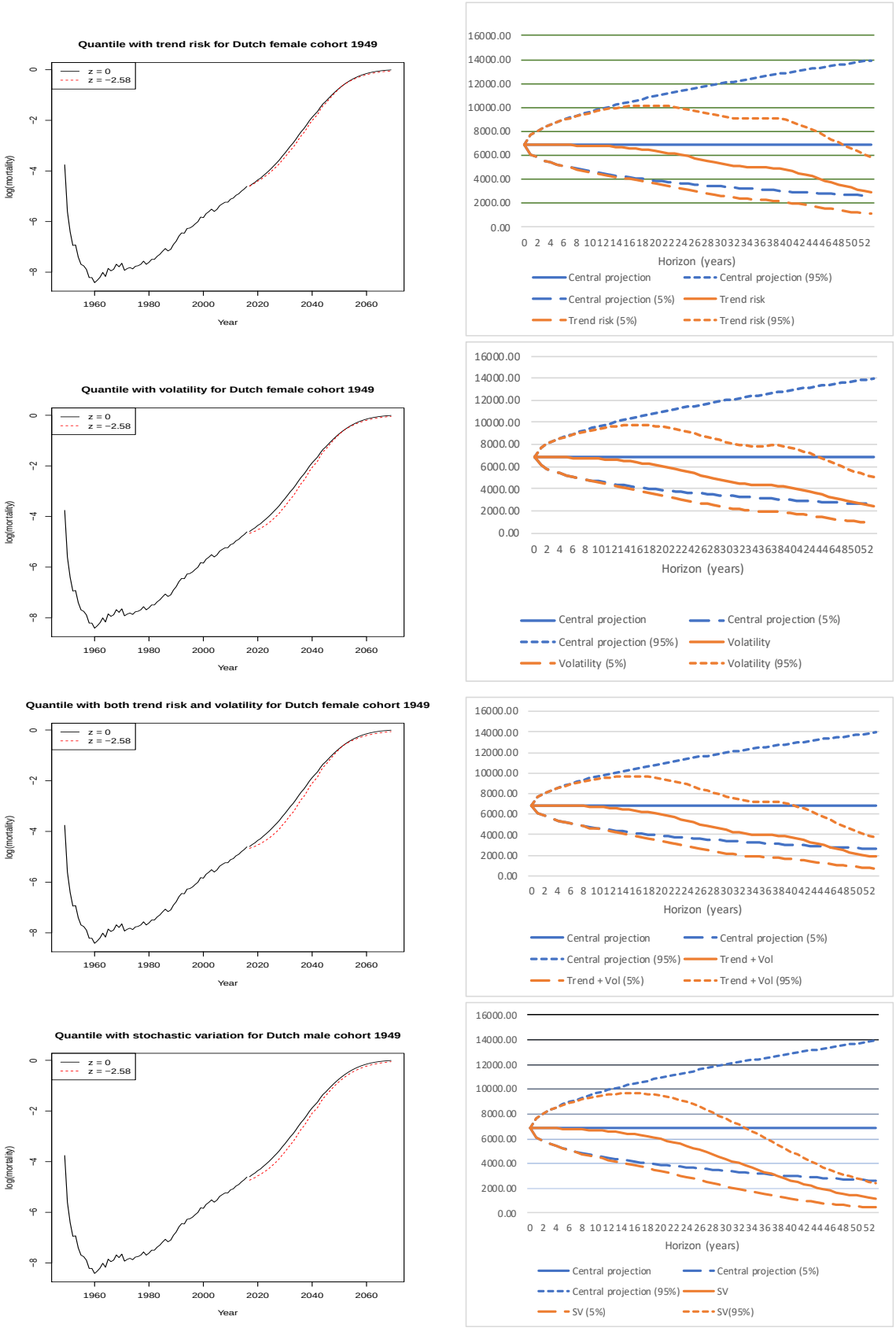

Figure 18: Graphs of mortality rates of female cohort 1949 on the left side and their corresponding variable annuity payouts on the right sides.

C.4 Value-at-Risk framework

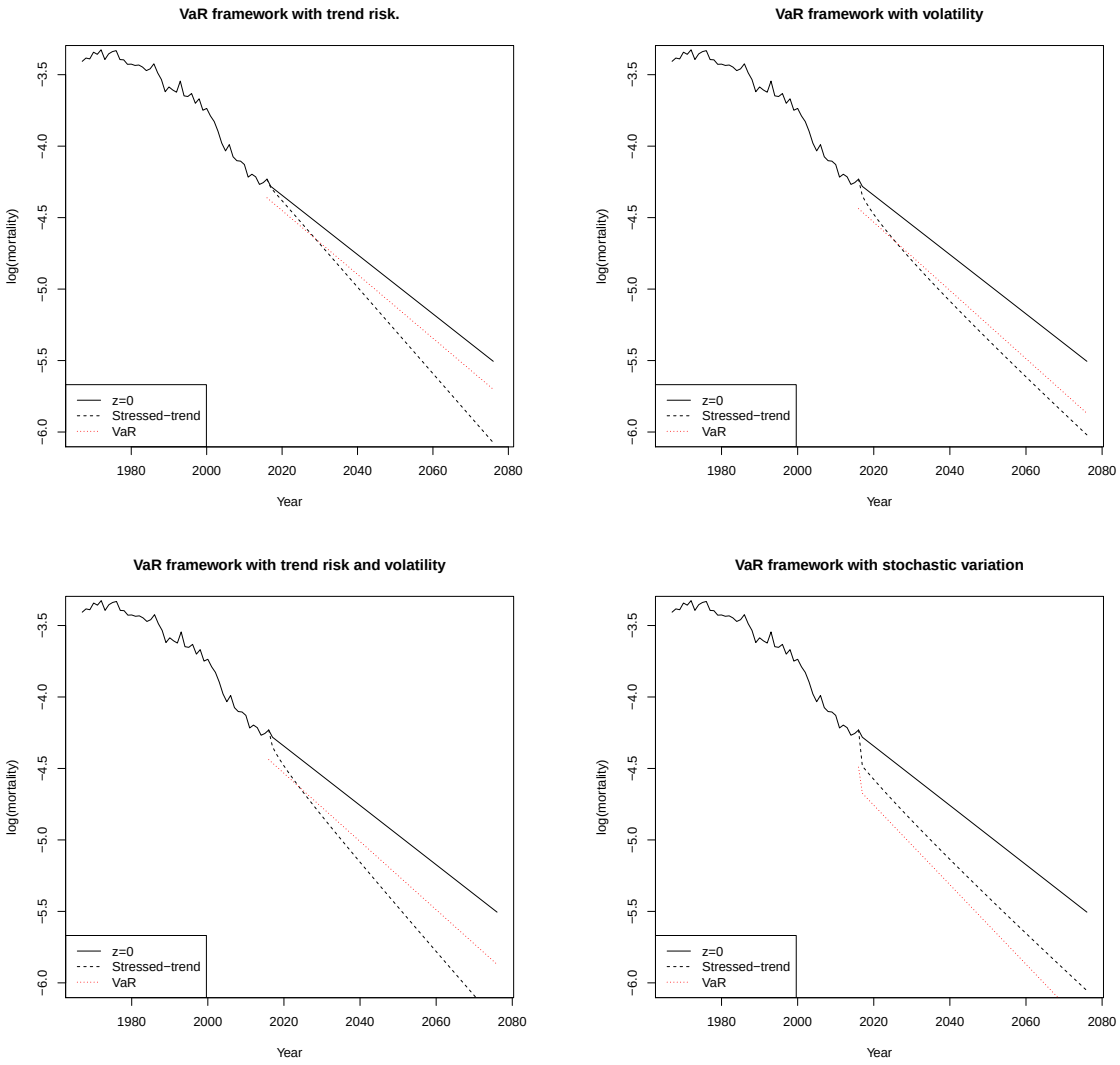

Figure 19: Graphs of mortality rates of male at age 67 with longevity uncertainties. Comparison of the stressed-trend approach and the VaR framework.

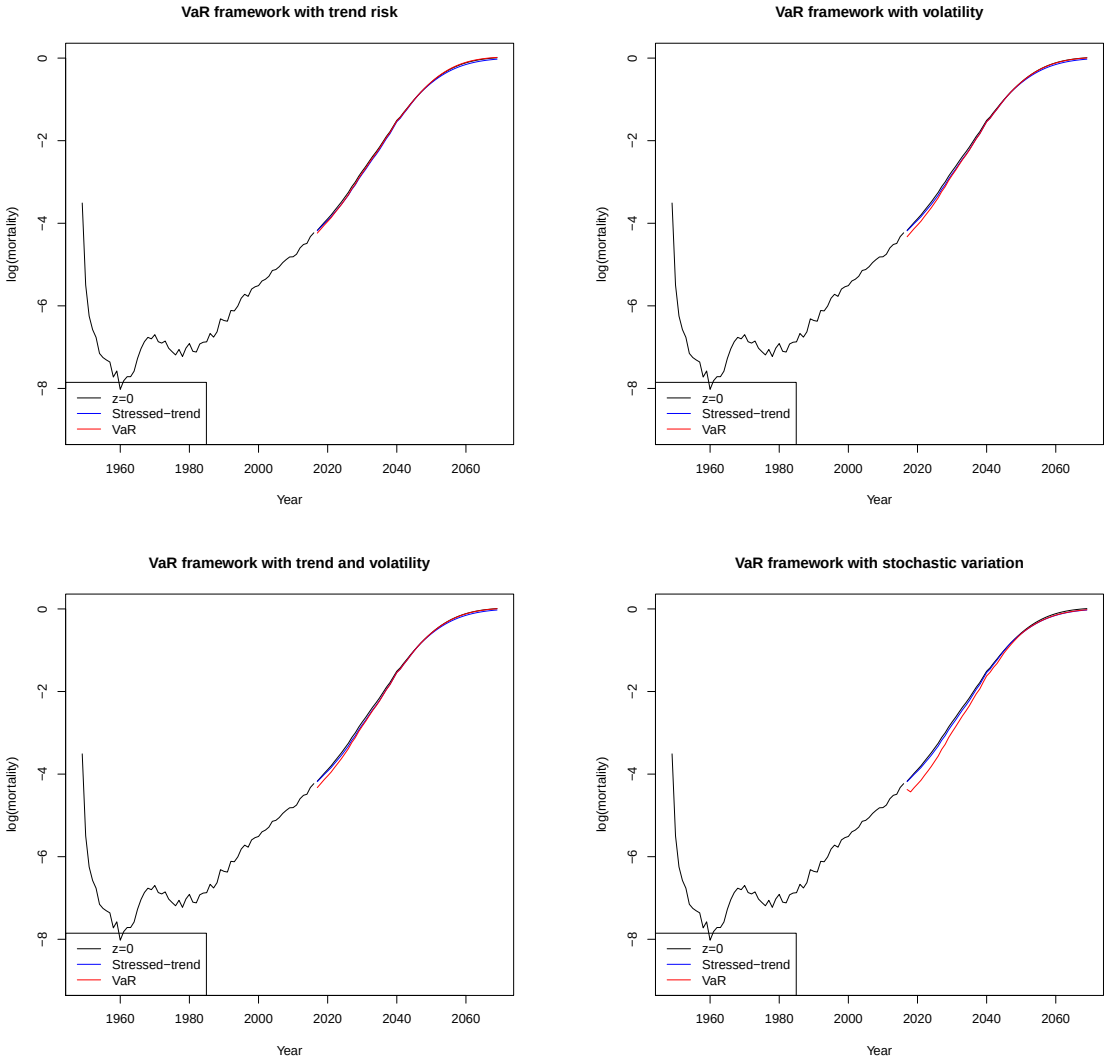

Figure 20: Graphs of mortality rates of male of cohort 1949 with longevity uncertainties. Comparison of the stressed-trend approach and the VaR framework.

C.5 Value-at-Risk framework 2

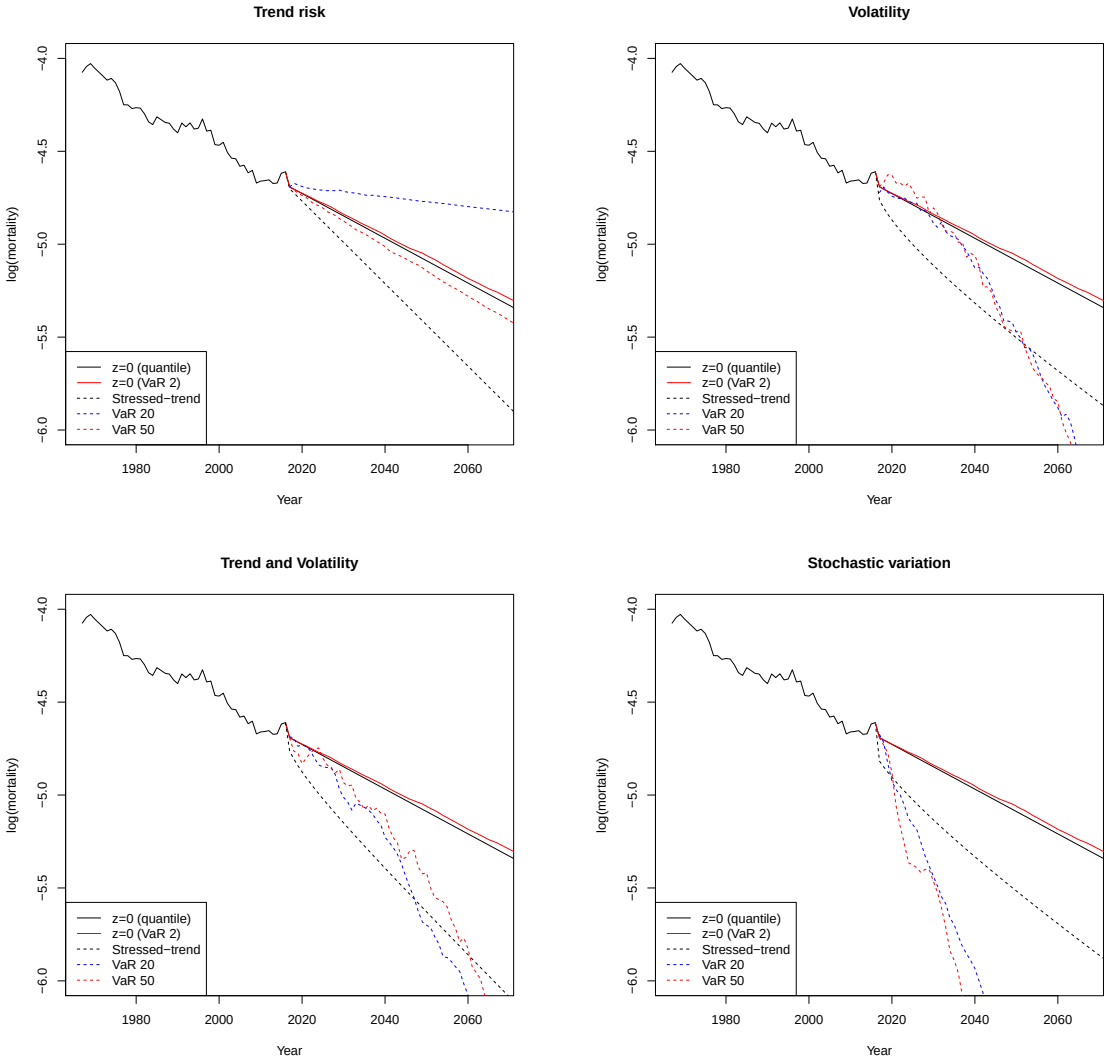

Figure 21: Graphs of mortality rates of females at age 67 with longevity uncertainties. Comparison of the stressed-trend approach and the VaR framework (with rolling window method, 50 years fitting period).

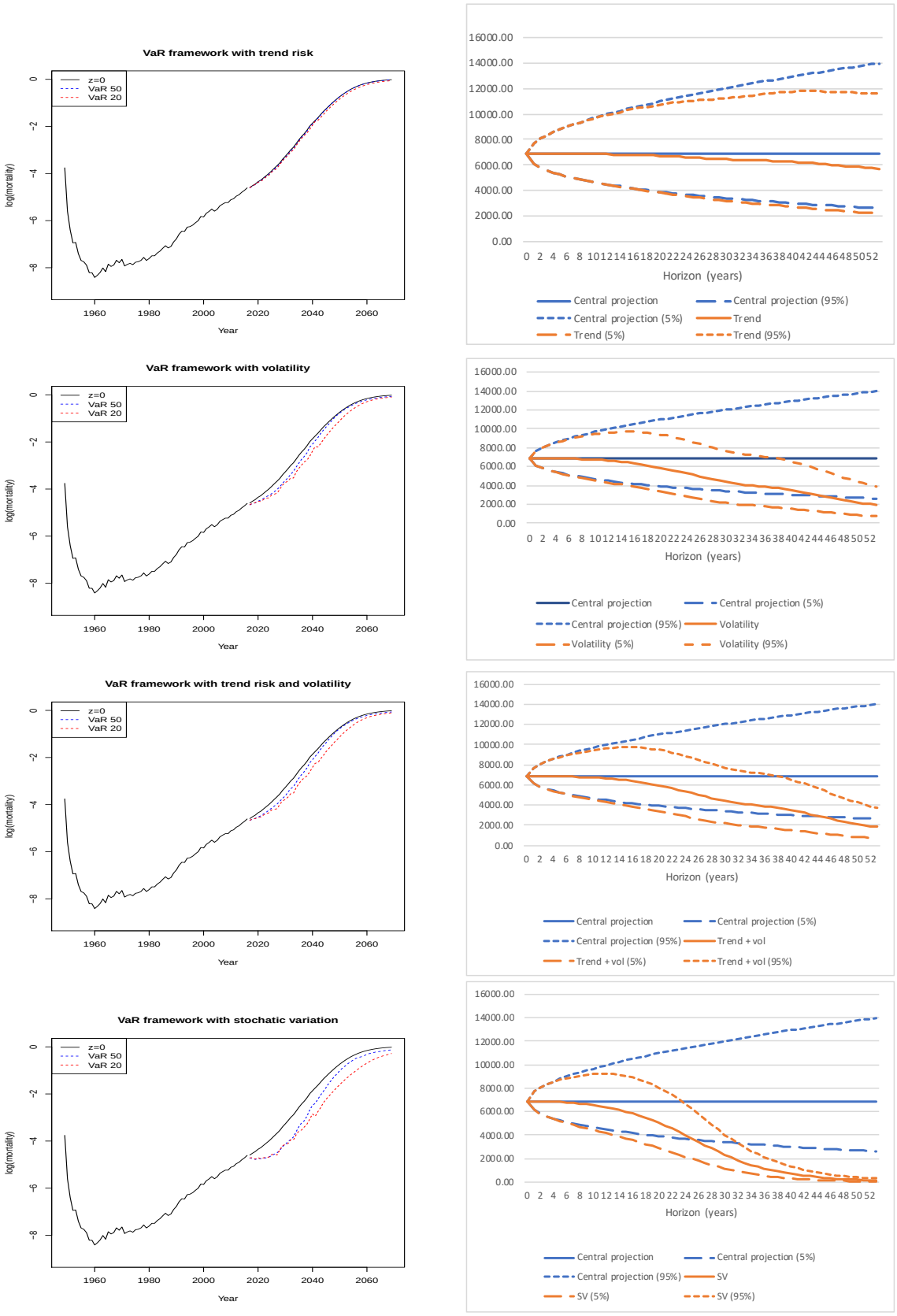

Figure 22: Graphs of mortality rates of female cohort 1949 on the left side and their corresponding variable annuity payouts (with rolling window method, 50 years fitting period) on the right sides.

C.6 Robustness check

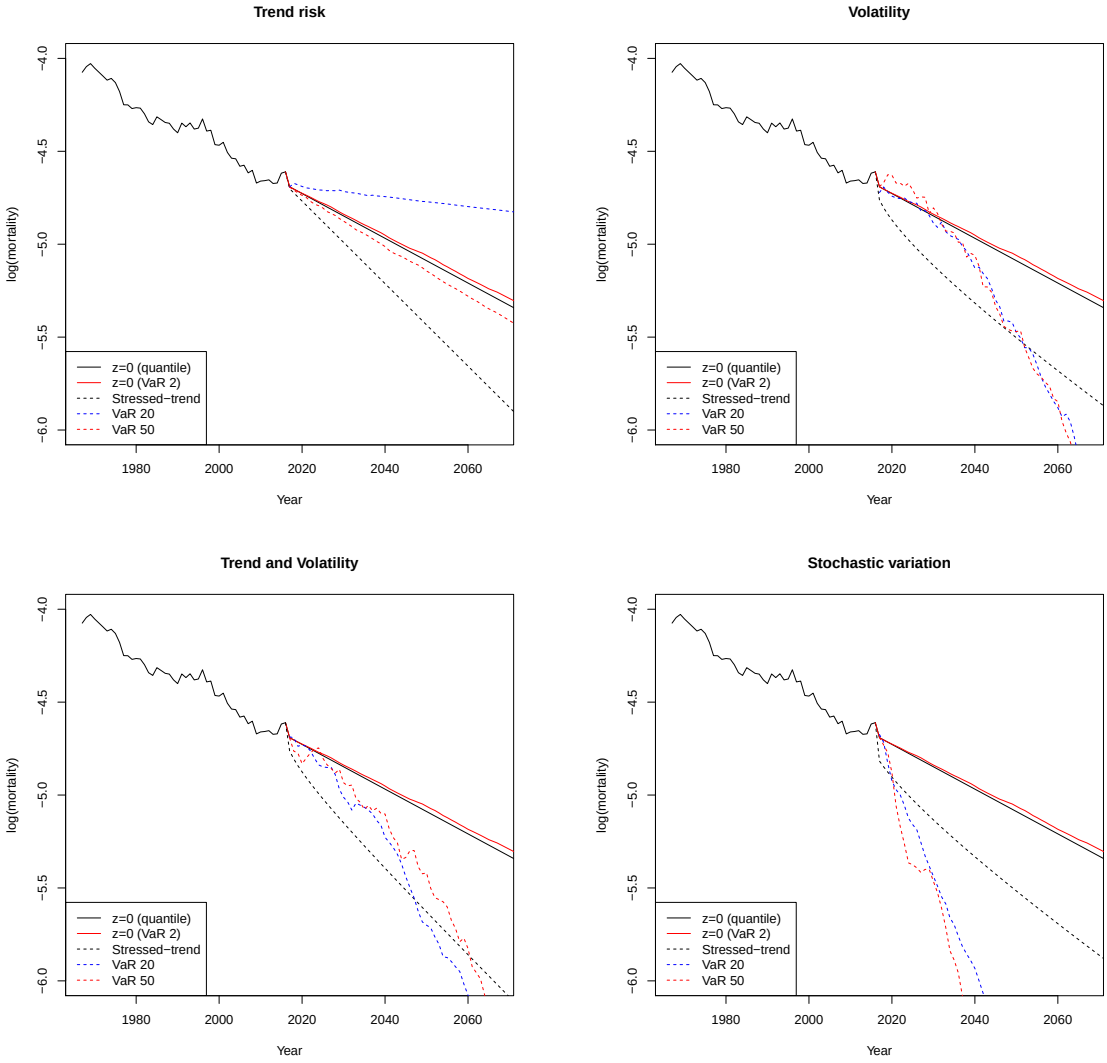

Figure 23: Graphs of variable annuity payouts for Dutch females at age 67 (with rolling window method, 20 years fitting period).

Figure 24: Graphs of variable annuity payouts (with rolling window method, 20 years fitting period) for males on the left side and females on the right sides.

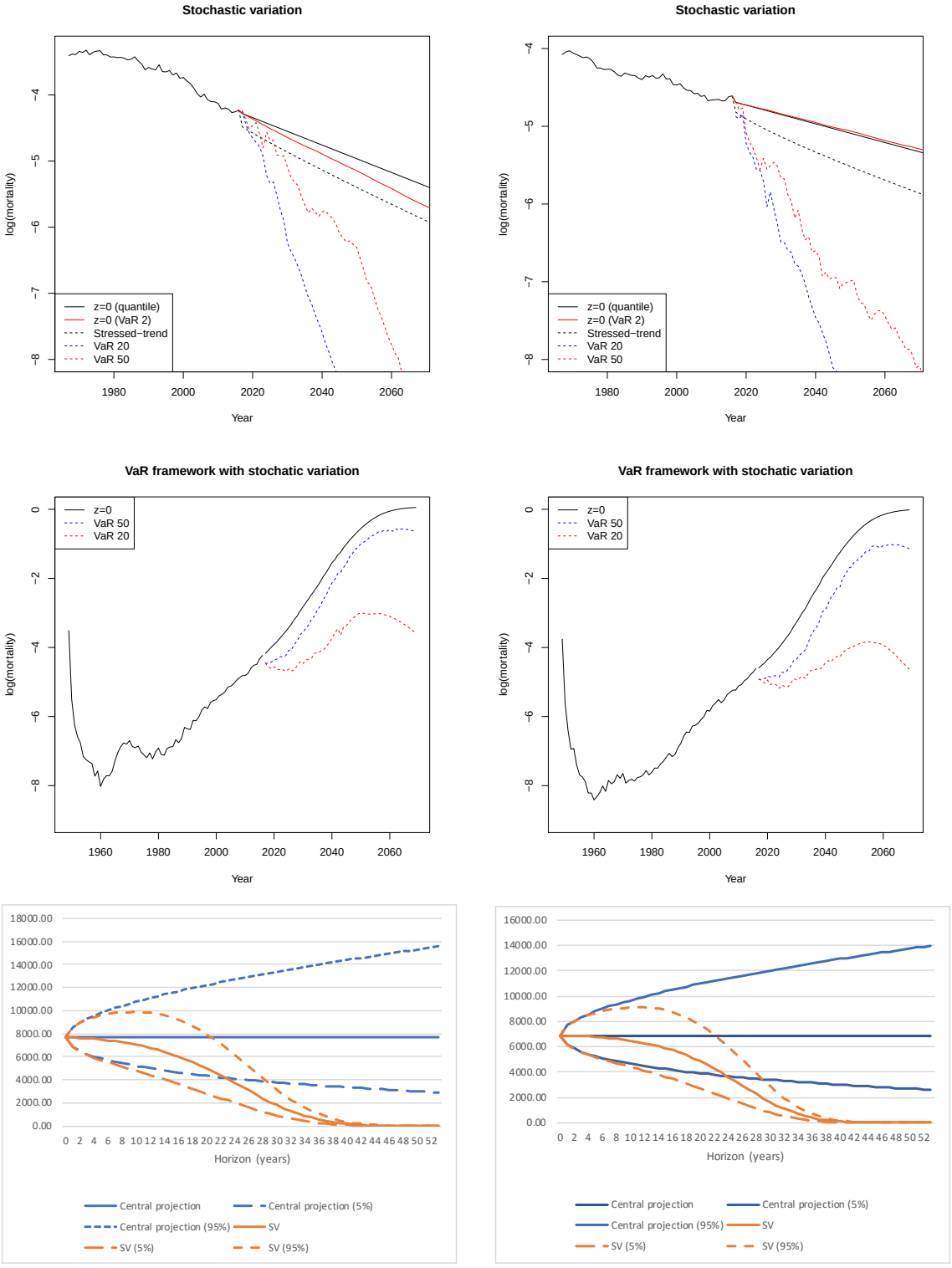

Figure 25: Graphs related to stochastic variation using σ_ϵ.

		Best-estimate	Trend risk	Volatility	Trend & volatility	Stochastic Variation	Stochastic Variation 2
Male	Stressed-trend	16.6	16.94	17.32	17.4	18.05	18.61
	VaR 1	16.9	16.95	17.28	17.28	18.2	18.45
	VaR 2 (50 yrs)	16.95	17.11	18.05	18.08	21.12	22.13
	VaR 2 (30 yrs)	17.89	18.44	20.24	20.88	24.75	36.48
Female	Stresed-trend	19.8	20.59	21.26	21.46	21.47	22.42
	VaR 1	19.79	19.85	20.28	20.28	21.45	21.66
	VaR 2 (50 yrs)	19.71	19.98	21.92	21.76	24.88	27.52
	VaR 2 (30 yrs)	19.55	20.31	23.56	24.16	26.7	43.88

Table 2: Cohort life expectancy of Dutch males and females at age 67 in 2016.