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Optimal Retirement for Heterogeneous Individuals Determinants of the Retirement Age and Consumption Pattern

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Optimal Retirement for Heterogeneous Individuals

Determinants of the Retirement Age and Consumption Pattern

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Abstract

In the past most people's timing of pension and retirement was fixed. Currently people live longer and are allowed flexibility in retirement and savings allocation during the working period. While the cost of retirement is increasing with extended life-expectancies and worsened economic climate, there still is an observed desire for early retirement. As a result the topic has become increasingly discussed in politics. There are however few related works which provide a model for the retirement age. In this work we consider what factors determine the ideal retirement age, by presenting a quantitative life-cycle model for the voluntary optimal retirement problem. To obtain a solution the life-cycle optimization problem is reformulated as a continuous Markov decision problem. This is then solved by an intuitive and flexible dynamic programming strategy, which determines the ideal retirement age, ideal consumption path, and corresponding ideal (dis)saving path. The flexibility of this approach allows it to be applied within various pension regulations, individual specific preferences and parameters. In this work it has been applied within a Dutch context to simulate how a Dutch citizen would ideally like to resolve their optimal retirement moment and consumption pattern if they were given considerable flexibility. Factors as the investment return, longer (healthy) lifespan, income and state pension levels, the marginal utility of consumption, and the desire for leisure, all affect this decision.

We find that the slope of consumption with age is determined by the relative height of the return on investment, compared to the rate of time preference. As a consequence this also determines when and how much should be saved. When the return on investment is high the optimizing agent starts saving earlier, does not have to save much for their pension benefits, and chooses to retire later. When the return on investment is very low, the agent chooses to postpone retirement a little, however retirement savings start much later. For the retirement moment, we find that a 20% ceteris-paribus increase in factors as the income level, public pension level, and desire for leisure, alter the ideal retirement age on the order of 0.5 to 2 years each. The size of these effects however depend on how the disutility of labour increases with age.

Contents

1 Introduction

What determines when someone should ideally retire? How does this ideal retirement age change for different individuals? How should someone save to allow for early retirement? With increasing life-expectancies and lower economic prospects this topic has become increasingly discussed. While the literature on optimal saving, investment and consumption in the context of retirement is vast, only few present a model for the optimal retirement moment. In this work we describe such a model. Here we specifically consider the voluntary retirement moment of an individually funded pension in a Dutch retirement setting. To determine their retirement age, a person has to trade-off utility from consumption with the value of free-time. For this both the retirement moment has to be decided, as well as the optimal consumption and corresponding (dis)saving path. We present an intuitive and flexible dynamic programming approach which finds both this retirement transition moment and life-cycle path simultaneously.

The Dutch pension system consists of two main pillars. The public pension (AOW) is referred to as the *first pillar*. It's height is related to the minimum wage, marital status, and the number of years of residence. As it is Pay-As-You-Go financed, past contributions do not affect it. Instead benefits are payed by taxation, which transfers wealth from working generations to those eligible. These public pension benefits start as soon as one reaches the statutory retirement age (sra). The *second pillar* is an occupational pension, where generally a minimum participation level is mandatory during employment. This is a more typical pension where benefits are funded by investing pension premiums. As such, bad market performance may lower the pension holder's funding ratio, requiring a reduction in pension benefits. In recent years this second pillar has become more flexible. Allowing a choice in both the saving rate and early retirement moment. Our model considers the optimal resolution of this *second pillar* if one is free to choose the desired (dis)saving rate and retirement age in an individually funded pension. As such this work is especially relevant for the increasing share of the population who are self employed, and have more direct control over their pension.

To find the ideal life-cycle, the model maximizes lifetime utility for the agent by choosing a consumption path and retirement age. As we aim to produce a life-cycle which can feasibly be implemented by an average citizen, we impose that retirement is irreversible. Furthermore financial leverage is not allowed. This means that loans cannot be made using future income. We consider inflation-adjusted post-taxation financial wealth, where one unit of consumption can buy the same basket of goods at each time point. The first pillar AOW pension is included as an exogenous additional income stream, which starts at the statutory retirement age (sra), irrelevant of the retirement status. Agents are blessed with perfect foresight but do not know their future date of death.

Within this framework for retirement, there are three factors which determine the optimal life-cycle. Firstly different agents may have different desires. Here we consider the marginal utility of consumption, impatience in future utility gains, and the height of disutility from labour. Secondly the budget considerations may be different, for example different wage levels, return on investment, statutory retirement age, or public pension height. Lastly the lifespan expectations may be different, which will affect health and the valuation of old-age consumption. To find how the ideal retirement age is affected by these factors, a benchmark case is considered. This benchmark case is then altered one parameter at a time, yielding a different ideal life-cycle and retirement age.

The disutility of labour, or value of free time, is an unknown but important factor for the ideal retirement age. It seems reasonable that disutility of labour should increase with either age or worsening health, however it is not an easily observable or quantifiable characteristic. Related literature for such retirement age models is limited, and assume some pattern for the unknown changing disutility with age. In these papers, it is typically linked with health. Health data is used to provide a parametric form for the disutility with age. These parameters are then calibrated under the assumption that some observed behaviour is optimal.

In our work we model disutility of labour in a similar fashion as Bloom et al. (2014). Here disutility is assumed to be proportional to the one-year risk of death. This is motivated by the assumption that the burden of labour is larger with bad health, and that less healthy individuals have a larger risk of death. The proportionality constant is then calibrated by finding the particular value for which the benchmark agent retires at the statutory retirement age. It is therefore assumed that the benchmark agent ideally retires at this statutory age.

As is often the case in life-cycle research, the considered model is essentially an alteration on the seminal Merton consumption model, Merton (1969). These models are however known to require specific parametric forms for the internal mathematics to have an arithmetic solution. In this work we did not wish to make to such simplifying assumptions. While a full arithmetic solution is unavailable, a necessary *first-order-condition* for the optimal retirement age can be found. This states that at the optimal retirement moment, the expected disutility of labour intersects the expected marginal utility of increased optimal life-cycle consumption, which is allowed by wages from extended employment. For a better intuition, the wealth generated from an additional day of labour is used to optimally increase consumption over the entire life-cycle. The additional utility that this generates, is equal to the disutility of labour at retirement. In a simplified example this means that; the disutility of labour at retirement is equal to the marginal utility of consumption multiplied by labour income. A similar arithmetic equation for the ideal retirement moment and consumption path cannot be found without the strict simplifying assumptions. We instead solve the model numerically. For this we reformulate the it as a continuous Markov decision problem, which can then be solved by an intuitively simple but flexible dynamic programming approach.

1.1 Related Literature

The previously mentioned work by Bloom et al. (2014) is perhaps most similar to ours, though their focus is specifically on the effects of increasing longevity. Like in our model, the agent chooses their retirement age and a consumption path as to maximize lifetime utility, taking into account the disutility of labour, uncertain mortality, and wage growth. Their solution approach is of a more analytical nature, which as previously mentioned, requires strict assumptions to describe the retirement moment. Specifically, it is assumed that the money market is perfectly competitive and actuarially fair, where the interest rate on a loan takes into account the probability of dying before the loan is paid off. In this context leverage is allowed, and wealth is transferable between periods, allowing an individual to take loans on future wages. Disutility of labour, wages, and mortality, are determined by simple exponential models. Utility gains with wage growth cannot exceed the growth of disutility of labour with age, as to ensure unretiring is never desirable. Lastly, the return on investment must be exactly equal to the subjective rate of time preference. Apart from such simplifying assumptions, their model differs in that it does not include a public state pension system. As is required with any such voluntary retirement model, they have to calibrate the disutility of labour. For this, they calibrate it such that the observed retirement behavior for the 1901 birth cohort is optimal. With this calibration, they find that ceteris paribus longer (healthy) life expectancy leads to later retirement, with an elasticity less than unity. Further, over the last century the effect of rising incomes, which cause earlier retirement, has dominated the effect of rising lifespans. As such, they predict continuing declines in the optimal retirement age despite rising life expectancy, provided the rate of wage growth remains as high as in the previous century.

Other notable works which consider the voluntary retirement age problem in a quantitative manner, are the works by French (2005), and Heijdra and Romp (2009). Here they however both focus on the influence of taxation and retirement related policies, instead of individual aspects. Heijdra and Romp (2009) analyzes the effect of demographic shocks and pension system changes on the performance of a small open economy. For this they use an overlapping generations model, in which individual agents choose their optimal retirement age. Similar to our work, an agent in this small scale economy maximizes a lifetime utility function subject to a budget constraint. Agents can however make loans on future labour income. Where money can be traded according to a perfectly competitive annuity market. Furthermore while their mathematics allow for an increasing disutility from employment with age, their results are produced using a constant disutility level. They do however also model a public pension system. They find that the early retirement provisions prevalent in many pension systems induce a kink in the lifetime income function, which causes most workers to retire at this age, which is typically well before the "normal" retirement age. Here the normal retirement age is considered the age at which people retire if there was no public pension system, with corresponding taxation structure to pay for said pension. They find that moderate tax reform is not sufficient to get people to move their retirement age away from this kink. Postponing the early eligibility age for the public pension system, does change the location of the kink, and thereby the retirement age.

The work by French (2005) includes many real world taxation rules in their retirement model. They consider labour participation by the amount of FTE's worked. Retirement is seen as no labour participation, as such ending retirement and reentering the labour force is allowed. They also maximize a similar lifetime utility target, however disutility from labour is set constant during the entire life, but does depend on health. They consider a binary health state, determined by an age dependent transition matrix. A pension system is included where the height of the pension is determined based on the average salary of the highest earning 35 years. Pension participation is voluntary, though pension payments are actuarially unfairly adjusted past the age of 65. Furthermore benefits can only be acquired after the early retirement age of 62. According to their model these disincentives for pension admission before 62 or past 65 is a main determinant for the observed drop in labour participation around these ages. The taxation structure has a large impact on the optimal behaviour.

In the recent work of ter Rele (2019), they do not consider the optimal retirement age for an individual. Instead a similar problem is considered, where the statutory retirement age (sra) for a public pension is optimized by a social welfare maximizing planner instead. The paper develops a framework to analyze the optimal change of the statutory retirement age in Pay As You Go (PAYG) type public pension systems, such as the Dutch first pillar system. In particular it determines the optimal rise of the sra in response to the increasing longevity and labour productivity. The authors do not assess whether the current sra is optimal, instead they implicitly assume a certain base level is optimal, and then compute how the sra should change to retain optimality with demographic shifts. They link the disutility curve not to remaining life expectancy or risk of death, but to observed increase of disability with age. The authors find that the optimal change in the sra is highly sensitive to the assumed utility valuation of increases of consumption, as this has a very large effect on the trade-off between consumption and leisure. As such, they give their results for a range of values. For the value which best fits the realized/projected development between 1957 and 2021, the optimal sra increase is less than the legislated policy line of linking it to life expectancy at 65.

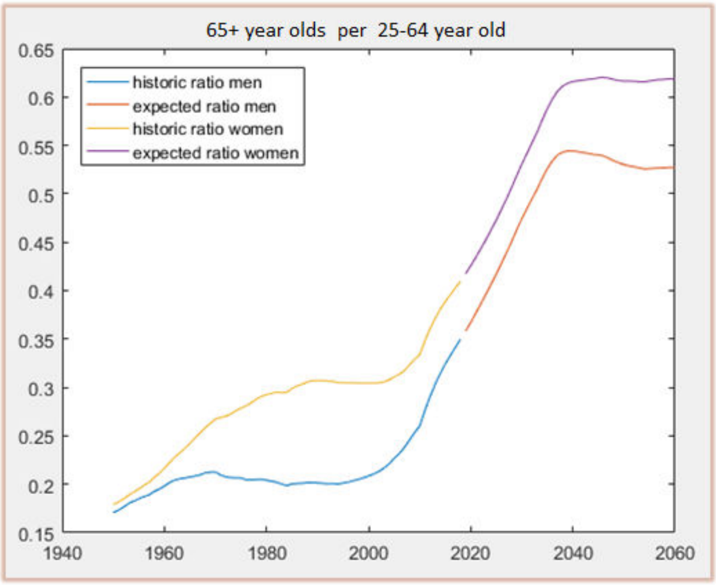
Lastly we note on the the work by Bodie et al. (1992). Their work builds on the classical Merton consumption model in a more direct manner, extending it by introducing a continuous labour participation choice, not a direct binary retirement decision. It does however allow for an analytically well behaved solution. Bodie et al. (1992) consider how the labour participation choice affects ideal investment. Here they find that the possibility to vary labour during the life cycle effectively allows one to value human capital as if it were a tradeable asset, and induces one to invest more riskily early in life, as to compensate for the relatively low risk of human capital. The original paper exclusively considers the implication for investment behaviour. Our focus is instead on the labour participation component. In the initial research for this thesis we computed the implications on labour participation In this exact model. The full documentation for this is added in Appendix C. Like the original paper this appendix considers Cobb-Douglas utility. This is an alternative utility function, where the gain from consumption and the value of free time are multiplicative instead of additive. This multiplicative utility function however yields unintuitive optimal behaviour. In particular, ideal consumption and leisure are exactly proportional. If consumption should be twice as high, then also leisure should be twice as high. This is not in line with typical retirement behaviour, as consumption does not normally increase as a result of the additional leisure. While utility function selection is a largely philosophical endeavour, this counter-intuitive result leads us to use additive utility instead, similar to most related work. Such a continuously variable labour participation choice also does not form a good proxy for the irreversible binary retirement choice.

The remainder of the paper is organised as follows. In Section 2 we provide more context on the dichotomous relation of both increasing costs and desire for retirement, which has motivated this work. In Section 3 we describe the model and how it can be solved with dynamic programming. In Section 4 the results are presented, where we consider the ideal life-cycle and its sensitivity to changes in the benchmark case. Section 5 concludes and makes some recommendation for future research, which will aid in the practicality of voluntary retirement life-cycle models.

2 Dutch Pension Developments

Our work considers the current pension system, where retirement has become a "normal" good for which people are willing to sacrifice wealth. Retirement however has only become a (partially) voluntary decision in recent years. Before the second world war some old-age pension provisions existed, however elderly who were no longer employable were typically taken care of by families. During the second world war a growing sense of solidarity allowed for a publicly funded old-age pension. As a result many allied-aligned nations ended up with similar systems, including the Dutch first pillar Algemene OuderdomsWet (AOW). As AOW is linked to the minimum income, many still would experience a drop in feasible living standards post-retirement. The second pillar consisting of investment based occupational pensions is designed to compensate for this. Combined these two have historically yielded a post-retirement income of approximately 70% of the last earned salary.

Figure 2: Demographic developments of elderly population vs. working age population. This figure is based on best estimate population distributions by Statistics Netherlands (CBS).



At its introduction in 1957 the statutory retirement age (sra) for the public pension was set at 65. Since then both true and healthy life expectancy have increased. This has had a significant effect on the age distribution among the population, see Figure 2. While in the 1950's there were more than 5 working age persons per elderly, this ratio has currently risen to approximately 3:1, and is expected to fall below 2:1 in the coming decades. As the public pension is PAYG funded, this will increase the fraction of taxes spent on the pension. Though this may be mitigated if economic growth outpaces state pension increases.

In a 2015 report the Sociale VerzekeringsBank (SVB), which is responsible for materializing AOW payments, states that it expects costs to increase from 5.5% of GDP to 6.5% in the coming 3 decades. This is driven by the increase in elderly population as well as relatively poor recent economic growth, and it is mitigated by the plans to increase the sra to counter this exact problem. Since this report, recent pension reforms have however reduced the rate at which the sra grows, which are expected to raise future costs above 6.5%.

Apart from these demographic changes, which increase the cost of the PAYG funded public pension, the current economic climate also presents worse growth projections. As a result the return on investment has reduced to the point that government bonds of very stable economies sell for more than they pay at maturity. This has inflated the present value of future benefits, requiring increased savings for the same future consumption level, and makes the occupational pension a less attractive proposition.

While the statutory age at which people retire is being increased, a wider retirement age range is introduced for occupational pensions. For example ABP, the largest dutch pension fund which covers all civil servants and educational personnel, now allows retirement from 60 years old to 5 years beyond the statutory retirement age. Furthermore when one chooses to retire early, pension benefits can be brought forward to bridge the gap between retirement and the statutory retirement age. Lastly a higher pension benefit level may be chosen in the early stages, to for example pay off a mortgage.

This flexibility has revealed a significant desire for early retirement. In a survey of approximately 6 000 ABP pension plan participants we find that 60% of retirees chose to retire before the sra. Similarly Kalwij et al. (2016) shows that while both absolute and healthy life expectancies have increased in recent decades, labour participation at old ages has barely increased. Thus while the costs of retirement are going up, this does not seem to be matched with continued old-age employment.

3 Merton Consumption Model with Retirement Age Decision

In this section we describe both the model and solution approach, which are used to obtain the main results in this work. Models for the voluntary retirement age consider the trade-off between utility from consumption and leisure. This trade-off is made during the entire life-cycle, requiring a comparison of utility between different time points. As such our work, the classic Merton Consumption model Merton (1969), and related work on retirement age optimization Bloom et al. (2014), French (2005), Heijdra and Romp (2009), all use similar life-time utility goals.

In the model agents are blessed with perfect foresight, and maximize life-time utility by choosing a consumption level C_t at each time moment t, and a retirement age R. Leisure at each time moment L_t, is a binary variable which has value 1 when the agent is currently retired: $L_t \equiv \mathbb{1}_{\{t>R\}}$. Unlike in French (2005) un-retiring is not allowed. We do not model the number of hours worked, as we would simultaneously have to include factors to explain both the roughly constant labour participation level before retirement McGrattan et al. (2004), as well as the sharp drop at retirement. The ideal consumption path and retirement age is affected by the available budget, mortality pattern, and utility function. These three elements are described in the following subsections.

3.1 Mortality Pattern

The mortality pattern determines the planning horizon of the agent, and is assumed to be exogenous. The life-cycle model starts with an agent of age age_0 at time $t_0 = 0$. This moment signifies the end of education and the start of employment. The planning horizon runs to time $t = T$, which is the highest possible age one could reach. At interior time moments the exogenous mortality pattern gives the probability of being alive $S(t)$. Before 2018 this mortality pattern is determined by the historic survival probabilities from the Human Mortality Database (HMD) for the Dutch population. From 2018 on wards, best estimate predicted survival data is used, which was released in 2018 by the Dutch Royal Association of Actuaries (AG2018).

These datasets contain values for q_x^y, which denote the probability of dying at age x for someone of age x in year y. With this the probability for someone of age age_0 in year y to survive to age x, is computed as:

$$\begin{aligned} S_{age_0}^y(age_0) &= 1, \\ S_{age_0}^y(x) &= S_{age_0}^y(x-1) \cdot (1 - q_x^{y+x-age_0}). \end{aligned} \tag{1}$$

The discrete probabilities in (1), are then interpolated in time to obtain the continuous survival function $S(t)$. In Figure 7 in Appendix A, $S(t)$ is displayed for an agent with starting age $age_0 = 25$ in year $y = 2018$.

3.2 Agent Preferences

The agent's preferences consist of a utility function to be maximized. Here disutility of labour is increasing with age, where it is linked with the short term survival prospects (hazard rate). The assumption is that a person with higher risk of death is less healthy, making employment more of a burden. In Bloom et al. (2014) disutility from labour is similarly determined by the hazard rate, where they use a Gompertz-Makeham Mortality model. This has a simple functional form for the hazard rate, which allows for a nicer solution for their analytical approach. We however find that such a fitted Gompertz-Makeham model underestimates mortality compared to the more complex model used in AG2018. As we do not aim to find an analytical solution, we instead opt to link disutility with 1 year risk of death q_x^y from the AG2018 and HMD datasets. Interpolating these values for continuous time, disutility is given by:

$$d(t) = \beta \left[(1-\theta) \cdot q^y_{\lfloor t \rfloor + age_0} + \theta \cdot q^y_{\lfloor t \rfloor + age_0 + 1} \right] \text{ where } \theta = t - \lfloor t \rfloor. \tag{2}$$

Figure 8 in Appendix A displays the development of disutility from labour with age. The variable β in (2) scales the absolute desire for leisure. In a voluntary retirement model, the sole reason for ending employment is the desire for leisure. As such the value for β critically determines the ideal retirement moment. This value for β is unobserved however, and is therefore found by calibrating the model under the assumption that some observed behaviour is optimal. After such calibration we may compute ceteris paribus responses of optimal behaviour for changing parameters, however the true "global" optimality of any particular retirement age remains unknown. In Bloom et al. (2014) this parameter is calibrated such that the optimal retirement age of the 1900 birth cohort is 65 years. French (2005) considers labour participation as a variable component with a fixed minimum cost for non-zero labour participation. As such they use a different disutility function, however the parametric values are similarly calibrated by assuming observed behaviour from birth cohorts 1922 to 1940 is optimal.

At any particular time moment we have to quantify the trade-off of consumption and leisure. For utility from consumption we use power utility, where disutility from labour is additive. The moment specific utility is given by:

$$\begin{aligned} u_t(C_t, L_t) &= u(C_t) - d(t) \cdot (1 - L_t) \\ &= \frac{C_t^{1-\eta} - 1}{1-\eta} - d(t) \cdot (1 - L_t) \end{aligned} \tag{3}$$

The choice of this utility function has a large impact on optimal retirement behaviour. Disutility from labour is implemented in an additive manner. A multiplicative relation would make consumption during retirement more favourable, causing a discontinuous positive jump in optimal consumption at retirement. More complex utility functions may mitigate such issues, however as utility function selection is an arbitrary endeavour, simplicity is preferred. In Appendix C a labour participation model with multiplicative utility from leisure is described. In this model labour participation is not binary but instead flexible.

To quantify life-time gain, the utility between different time points are to be related with each other. For this we use the common approach popularized by the classic paper Yaari (1965). Here future utility gains are discounted to the present moment, and uncertain life expectancy is handled by multiplying these utility gains by the probability of being alive. The rate of time preference used for discounting, is a subjective discount parameter for the specific individual δ. Unlike in Bloom et al. (2014) the discount factor does not need to be identical to the return on investment. In essence δ determines how shortsighted the agent is. An agent with a higher rate of time preference will be referred to as more impatient, and vice versa. From the perspective of time t, we obtain the following life-time utility goal:

$$\int_t^T e^{-\delta s} \cdot S(s) \cdot u_s(C_s, L_s) \, \mathrm{d}s. \tag{4}$$

While it is common practice is to use such separable intertemporal utility functions, this is not without consequences. For consumption it implies that the coefficient of relative risk aversion for consumption η is the inverse of the intertemporal elasticity of substitution. Agents with higher relative risk aversion will smooth consumption more over time. More importantly however, as is pointed out in Bommier (2006), this common practice also implicitly assumes risk neutrality with respect to the length of life. They find that time-consistency either implies risk-neutrality with respect to the length of life, or a constant (non-zero) absolute risk aversion in life years and non-additive utility between time points. The former risk-neutral case contains the model in (4). With an exogenous mortality scheme, the non-risk-neutral case requires a recursive utility function, where the utility valuation depends on future expected utility levels. Such recursive utility functions are certainly interesting as they may also allow for habit formation. However due to the associated complexity this will be left out of scope.

In general Frederick et al. (2002) critically reviews the literature on such discount factor based models for intertemporal utility. They note that economists should refrain from trying to find the correct singular discounting factor or discounting function. Intertemporal utility reflects many distinct considerations and involves the interplay of competing motives. Thus an emphasis on including psychological insights for intertemporal choice should be preferred, though finding sufficiently descriptive models is not easy. While this displays the implicit consequences and potential shortcomings of the chosen life-time utility model (4), there appears to be little literary consensus for a recommended and especially practical alternative.

3.3 Budget Considerations

We will now consider the budget equation subject to which utility (4) is maximized. Wealth $W_t \in \mathbb{R}_{\geq 0}$ in our model denotes the total expendable financial capital of the individual at time t. The agent will have to save in order to fund their consumption after the retirement date R. Wealth does not contain human capital, so combined with the non-negative wealth constraint, we cannot fund current consumption from discounted future labour income. This is the zero-leverage assumption. Before retirement, income is generated from labour at a rate Y_t at time t. Further after the statutory retirement age (sra) they receive a constant income rate Z from the public old age pension. Lastly income is generated from wealth with a constant rate of return r, where we assume no stochasticity in the market. With these considerations, wealth develops according to the following deterministic differential equation:

$$dW_t = (rW_t + \mathbb{1}_{\{t<R\}}Y_t + \mathbb{1}_{\{t\geq sra\}}Z - C_t)dt. \tag{5}$$

Here $\mathbb{1}_{\{t<R\}}$ is an identifier for when the agent is employed, that is leisure $L_t = 0$, where we have previously defined $L_t \equiv \mathbb{1}_{\{t>R\}}$. The other identifier indicates when the sra has been reached and the agent receives public pension. The agent is allowed to continue employment while receiving the state pension.

In this context all monetary units are inflation adjusted, and the rate of return r is the real market return. This implies that one unit of consumption C_t represents the same amount of goods regardless of time. Similarly Y_t and Z denote disposable income after taxes, and we assume no taxation on capital gains.

The wage path Y_t over time is based on promotion-indices from the 2014-2016 ABP foundation-research. This research describes the demographic characteristics of the ABP clientele. The promotion-indices denote for each age, what the average corresponding increase in salary is. It does not consider altered promotion paths for different education level or other factors. Based on this foundation-research, we assume a 4% annual income growth for ages below 25 years. From ages 25 to 45 income growth linearly decreases from 4% to 0%, and after 45 years income is constant. This income growth is less than the 5.5% reported in the foundation-research. The lower growth level is chosen to compensate for the Dutch progressive tax system, thus being closer to net income growth.

See Appendix B for a more detailed description of Y_t.

3.4 The Optimal Retirement Age

From the perspective of time t, let $C^P \equiv \{C_s\}_{s\in[t,T]}$ be a consumption path with $\{C_s \in \mathbb{R}_{\geq 0} : \forall s \in [t,T]\}$, and let $R \in [t_0, T]$ be a retirement age. Then formally the set $\Theta_{L_t,W_t,t} \equiv \left\{\left\{C^P, R\right\} : \{W_s \geq 0 : \forall s \in [t,T]\}\right\}$ contains all consumption path and retirement age pairs for which wealth is never negative, given one starts at time t with an initial wealth W_t and leisure status L_t. This set contains all feasible points in our problem, and thus defines the search space for the maximization.

Now from the perspective of time t, it is the agent's goal to find the particular consumption path and retirement age pair $\{C^P, R\}_t^* \in \Theta_{L_t,W_t,t}$ which maximizes remaining life-time utility (4):

$$V_t\left(\{C^P, R\}_t^*\right) \equiv \max_{\Theta_{L_t,W_t,t}} \left\{\int_t^T e^{-\delta s} \cdot S(s) \cdot u_s(C_s, L_s)\,\mathrm{d}s\right\}, \tag{6}$$

where the valuation function $V_t(.) : \Theta_{L_t,W_t,t} \to \mathbb{R}$ assigns the life-time utility value to each feasible solution. With the defined life-time utility maximization problem, the remainder of this subsection will analyze the effect of delaying retirement, finding a necessary first-order-condition for the ideal retirement moment.

Consider an agent who has chosen to retire at some retirement age R_1, with corresponding optimal consumption path $C^{P*} = \{C_s^*\}_{s\in[0,T]}$. If this person decides to delay retirement by some amount to R_2, then they obtain disutility from labour for a larger period of time. However labour income is also accrued for longer, allowing a (higher) optimal consumption path $\overline{C}^{P*} = \{\overline{C}_s^*\}_{s\in[0,T]}$. In this case using equation (4), we find that the change in utility from delaying retirement is given by:

$$\begin{aligned}&\int_0^{R_2} e^{-\delta s} \cdot S(s) \cdot u_s(\overline{C}_s^*, 0)\,\mathrm{d}s + \int_{R_2}^T e^{-\delta s} \cdot S(s) \cdot u_s(\overline{C}_s^*, 1)\,\mathrm{d}s \\ &- \left[\int_0^{R_1} e^{-\delta s} \cdot S(s) \cdot u_s(C_s^*, 0)\,\mathrm{d}s + \int_{R_1}^T e^{-\delta s} \cdot S(s) \cdot u_s(C_s^*, 1)\,\mathrm{d}s\right].\end{aligned} \tag{7}$$

With focus on the different time segments, this can be rewritten as:

$$\int_0^{R_1} e^{-\delta s} \cdot S(s) \cdot \Big[u_s(\overline{C}_s^*, 0) - u_s(C_s^*, 0)\Big] \; \mathrm{d}s + \int_{R_1}^{R_2} e^{-\delta s} \cdot S(s) \cdot \Big[u_s(\overline{C}_s^*, 0) - u_s(C_s^*, 1)\Big] \; \mathrm{d}s$$
$$+ \int_{R_2}^{T} e^{-\delta s} \cdot S(s) \cdot \Big[u_s(\overline{C}_s^*, 1) - u_s(C_s^*, 1)\Big] \; \mathrm{d}s. \tag{8}$$

With additive utility such as for example in equation (3), this is further simplified to:

$$\int_0^{T} e^{-\delta s} \cdot S(s) \cdot \Big[u(\overline{C}_s^*) - u(C_s^*)\Big] \; \mathrm{d}s - \int_{R_1}^{R_2} e^{-\delta s} \cdot S(s) \cdot d(s) \, \mathrm{d}s. \tag{9}$$

Here the first component is the utility gained from changing (increasing) the consumption path, and the second component is the utility lost due to longer employment. Now if $\Delta R = R_2 - R_1$ is infinitesimally small and optimal consumption at any particular time varies smoothly with R, then using a first order Taylor approximation:

$$\overline{C}_s^* \approx C_s^* + \Delta R \frac{\partial C_s^*}{\partial R} \tag{10}$$

Using an additional Taylor approximation for utility from consumption, we find:

$$u(\overline{C}_s^*) \approx u\left(C_s^* + \Delta R \frac{\partial C_s^*}{\partial R}\right) \approx u(C_s^*) + \Delta R \frac{\partial u(C_s^*)}{\partial C} \frac{\partial C_s^*}{\partial R} \tag{11}$$

Now with the mean value theorem for integrals, the change in life-time utility (9) by changing the retirement age an infinitesimal amount ΔR, can be written as:

$$\Delta R \int_0^{T} e^{-\delta s} \cdot S(s) \cdot \frac{\partial u(C_s^*)}{\partial C} \frac{\partial C_s^*}{\partial R} \, \mathrm{d}s - \Delta R \cdot e^{-\delta R_1} \cdot S(R_1) \cdot d(R_1). \tag{12}$$

This is the change in life-time utility (with optimal consumption given the retirement age) for an infinitesimally small change in the retirement age. Thus for some retirement age R^* to be optimal, we find the first-order-condition:

$$\int_0^{T} e^{-\delta s} \cdot S(s) \cdot \frac{\partial u(C_s^*, 0)}{\partial C} \frac{\partial C_s^*}{\partial R} \, \mathrm{d}s = e^{-\delta R^*} \cdot S(R^*) \cdot d(R^*). \tag{13}$$

The equation above states that at the optimal retirement age R^*, the expected disutility from labour at R^* is equal to the marginal expected life-time utility from optimal consumption with respect to R.

For a more intuitive understanding, a stylized example is given. Assume the agent has a known lifespan with death at time T, a constant income level $Y_t = Y$, and time discounting factor $\delta = r = 0$. Furthermore, assume that the optimal consumption path is constant[1] $C_t^* = C^*$ given any retirement age R. With this the first order condition (13) simplifies to:

$$T \cdot \frac{\partial u(C^*)}{\partial C} \frac{\partial C^*}{\partial R} = d(R^*) \tag{14}$$

For this agent, the life-time accrued wealth is:

$$W_{life}(R) = Y \cdot R + Z(T - sra) \tag{15}$$

Now as utility is increasing in consumption, we want to consume all wealth:

$$C^* = W_{life}(R)/T \quad \Rightarrow \quad \frac{\partial C^*}{\partial R} = Y/T, \tag{16}$$

which implies that at the optimal retirement age:

$$\frac{\partial u(C^*)}{\partial C} Y = d(R^*). \tag{17}$$

Thus at the optimal retirement age, disutility from labour is equal to the marginal utility from consumption multiplied by income. This shows that the income level Y is a core factor in the retirement age trade-off, as it determines the rate at which more consumption becomes available while remaining employed.

3.5 Continuous Markov Decision Problem

To find the optimal life-cycle $\{C^P, R\}_t^* \in \Theta_{L_t, W_t, t}$, the problem can be rewritten as a continuous Markov decision problem. This can then be tackled with dynamic programming. In dynamic programming it is assumed that the optimal response in the future is known for each possible state. The agent then determines the optimal current decision by trading-off the gain from the transition to the next state, with the optimal utility achieved in this next state. To ensure that optimal behaviour in the future is known, the process is executed backwards in time. In continuous-time problems the transition is infinitesimally small with a continuously changing state space.

Consider the possible decisions of an agent at some moment t. If the agent is currently employed, he may choose his current consumption level $C_t \in [0, \infty)$. Infinite consumption is not allowed, as this causes an infinite reduction in wealth. Any other non-negative consumption level is possible in a continuous time problem[2]. Further, the agent may choose to retire right now $L_t = 1$, or he could choose to retire at some later moment $L_t = 0$. A retired agent can solely choose their consumption level, where $L_t = 1$ is enforced. As such we have $L_t \equiv \mathbb{1}_{\{t \geq R\}}$.

[1] In this stylized example, a constant consumption level is indeed optimal, as long as it would not result in negative wealth at any time.

[2] If the agent has exactly zero wealth $W_t = 0$, consumption is limited by the current income level, as otherwise wealth would become negative.

If at moment t the agent is already retired: $R \leq t$, then leisure $L_{t+x} = 1$ for all $x \in [0, T-t]$. As such the search space is identical, and the same optimal value can be reached at time t, regardless of how long the agent has already been retired:

$$V_t\left(\{C^P, t-x\}_t^*\right) = V_t\left(\{C^P, t\}_t^*\right) \quad \forall x \in [0, t]. \tag{18}$$

To simplify notation, let $F^R(t, W_t) = V_t\left(\{C^P, t\}_t^*\right)$ be the optimal achievable value for a retired person at time t with wealth W_t, and let $F(t, W_t) = V_t\left(\{C^P, R\}_t^*\right)$ be the optimal achievable value for a still employed person. Then equation (6) can be written in dynamic programming form as:

$$F^R(t, W_t) = \max_C \left\{ \int_t^{t+h} e^{-\delta s} S(s) u_s(C, 1)\,\mathrm{d}s + F^R(t+h, W_{t+h}), \right\} \tag{19}$$

$$F(t, W_t) = \max_{C,L} \begin{cases} \int_t^{t+h} e^{-\delta s} S(s) u_s(C, 0)\,\mathrm{d}s + F(t+h, W_{t+h}) & \text{if } L = 0, \\ F^R(t, W_t) & \text{if } L = 1. \end{cases} \tag{20}$$

Here h is an infinitesimally small unit of time. Equation (20) gives $F(t, W_t) \geq F^R(t, W_t)$, as a not yet retired person may choose to retire, and thus can always achieve at least $F^R(t, W_t)$. As there is no bequest desire, we set $F^R(T, W_T) = F(T, W_T) = 0$ for the maximum possible age.

Let's first consider the optimal decision if the agent is already retired. By the mean value theorem for integrals, for any time moment $t \in [0, T)$ we have:

$$F^R(t, W_t) = \max_{C_t} \left\{ h \cdot e^{-\delta t} S(t) u_t(C_t, 1) + F^R(t+h, W_{t+h}) \right\}. \tag{21}$$

Now consider two identical agents, but with two different wealth levels $\hat{W}_t > W_t$ at the same time moment t. Let C_t^* be the optimal consumption level for the agent with wealth W_t, and let W_{t+h}^* be the corresponding optimal remaining wealth. If we define $k > 0$ as the amount of additional consumption which can be realized from $\hat{W}_t$, such that we obtain remaining wealth W_{t+h}^*. Then if utility is strictly increasing in consumption, the following relation holds:

$$F^R(t, \hat{W}_t) \geq h \cdot e^{-\delta t} S(t) u_t(C_t^* + k, 1) + F^R(t+h, W_{t+h}^*) > F^R(t, W_t) \tag{22}$$

This proofs that optimal life-time utility is strictly increasing in wealth. As resulting wealth W_{t+h} is strictly decreasing in consumption (5), optimal utility $F^R(t+h, W_{t+h})$ is also strictly decreasing in consumption.

With utility $u_t(C_t, 1)$ strictly increasing in consumption, the optimization problem in (21) is concave with a singular local and global maximum. Thus a simple hill-climbing search is guaranteed to find the globally optimal consumption level for a retired agent at time t, with wealth W_t.

Now consider a not yet retired agent, with remaining optimal life-time utility $F(t, W_t)$. If it is optimal to remain employed, then with the same logic as in (22), it holds that $F(t, \hat{W}_t) > F(t, W_t)$. If it is optimal to retire, then $F(t, W_t) = F^R(t, W_t)$, and thus $F(t, W_t)$ is again increasing in wealth. This means that the optimal corresponding consumption level is again guaranteed to be found by a simple hill-climbing search. Whichever is higher, $F^R(t, W_t)$ or maximal utility while remaining employed, determines whether the agent should retire.

As the optimal decision $\{C_t^*, L_t^*\}$ can be found in all possible states $\{L_t, W_t, t\}$ of an agent, the ideal life-cycle is determined by tracing these optimal decisions from the starting point $\{L_{t_0}, W_{t_0}, t_0\}$. Furthermore given that the optimization problems for consumption have a unique maximum, the optimal life-cycle is also unique[3].

3.6 Dynamic Programming Solution

The above explains the optimal life-cycle problem in terms of a continuous Markov decision problem and continuous dynamic programming strategy. To find a solution for this problem, we will discretize the continuous variables wealth W_t and time t. This yields a set of discrete grid values over wealth and time, where at each grid point the optimal behaviour will be computed explicitly. Whenever values between grid points are required, they will be approximated using linear interpolation to the grid. In line with the discretization, the agent is only allowed to change their decision (consumption rate and employment status) at the time grid points. The idea with such a discretization, is that the resulting optima tend to the true continuous solution for sufficiently fine time and wealth grids. The pseudocode for this discretized dynamic programming solver is given in Algorithm 1. For reproducibility, the code used in this work will be provided by emailing the author[4]. The remainder of this section will describe the approach in some more detail.

For the discretization, the planning horizon $t \in [0, T]$ is split into N different points yielding $N - 1$ equally sized intervals. The wealth domain $W_t \in \mathbb{R}_{\geq 0}$ is bounded from above by W_{max}, and split into $M - 1$ equally sized intervals. The choice of a maximum wealth

[3] Technically the optimal life-cycle path may not be unique. This happens if the utility while remaining employed is identical to utility while retired, and stay identical in the resulting optimal states if we remain employed. In this case there would be a non-trivial interval within which the optimal life-cycle is indifferent to retirement.

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level, implies the additional constraint $W_t \leq W_{max}$. This constraint is not desired, thus W_{max} should be chosen sufficiently high, such that it will never be active in the resulting optimal wealth paths.

At each of the $M \times N$ different grid points we need to store both the optimal behaviour and their corresponding outcomes. The time corresponding to each of the M different levels are determined by the vector $t^{grid} \in \mathbb{R}^M$. Similarly each of the different wealth points are given by $W^{grid} \in \mathbb{R}^N$. As described in Section 3.5, the problem is split into two components, the retired agent and the not yet retired agent. For each agent the optimal behaviour will be stored separately.

The matrices $\mathbf{C}, \mathbf{C}^R \in \mathbb{R}^{M\times(N-1)}$ will store the optimal consumption levels for the employed and the retired agent respectively. $\mathbf{F}, \mathbf{F}^R \in \mathbb{R}^{M\times N}$ will store the maximal achievable utility as in equation (19) and (20). Lastly $\mathbf{L} \in \{0,1\}^{M\times(N-1)}$ stores the optimal leisure level for the employed agent. If $\mathbf{L}_{j,i} = 1$, then an employed agent with wealth W_j^{grid} at time t_i^{grid} should retire. Note that the consumption and retire grids are of dimension $M \times (N-1)$, here the last time moment $t_N^{grid} = T$ is omitted, as there is no optimal decision to be made at the maximum possible age.

Now consider the optimization problem for a given grid point $\{j, i\}$. For two time points $t_1 = t_i^{grid}$ and $t_2 = t_{i+1}^{grid}$, the utility from constant consumption for the interval is:

$$val(C, L) = \frac{e^{-\delta t_1} - e^{-\delta t_2}}{\delta} u_{t_1}(C, L) S(t_1). \tag{23}$$

Given this constant consumption and leisure level during the interval, with starting wealth $W = W_j^{grid}$ the resulting wealth at the next time period is:

$$W_{next}(C, L) = \frac{C - Y_{t_1}(1-L) - \mathbb{1}_{\{t_1 \geq sra\}} Z}{r} \left(1 - e^{r(t_2 - t_1)}\right) + W e^{r(t_2-t_1)}. \tag{24}$$

The life-time utility at this resulting wealth level would be given by $F(t_2, W_{next}(C, 0))$ for the employer, or $F^R(t_2, W_{next}(C, 1))$ for the retired. These are however only computed at the grid levels, as such some interpolation is required. Let k be the index such that $W_k^{grid} \leq W_{next}(C, L) < W_{k+1}^{grid}$, then the interpolation is given by:

$$val_F(C, 0) = (1-\theta)\mathbf{F}_{k,i+1} + \theta \mathbf{F}_{k+1,i+1} \approx F(t_2, W_{next}(C, 0)), \tag{25}$$

$$val_F(C, 1) = (1-\theta)\mathbf{F}^R_{k,i+1} + \theta \mathbf{F}^R_{k+1,i+1} \approx F^R(t_2, W_{next}(C, 1)), \tag{26}$$

$$\text{where } \theta = \frac{W_{next}(C, L) - W_k^{grid}}{W_{k+1}^{grid} - W_k^{grid}}. \tag{27}$$

Now the discretized version of the optimal decision problem in (19) and (20), can be written as:

$$\mathbf{F}^R_{j,i} = \max_C \{val(C,1) + val_F(C,1)\}, \tag{28}$$

$$\mathbf{F}_{j,i} = \max_{C,L} \{val(C,L) + val_F(C,L)\}. \tag{29}$$

These maximization problems are solved with a simple hill-climbing algorithm. The corresponding optimal behaviour grid points are stored as:

$$\mathbf{C}^R_{j,i} = \arg\max_C \{val(C,1) + val_F(C,1)\}, \tag{30}$$

$$\{\mathbf{C}_{j,i}, \mathbf{L}_{j,i}\} = \arg\max_{C,L} \{val(C,L) + val_F(C,L)\}. \tag{31}$$

With known 0 remaining life-time utility at the terminal age T, this process can be executed backwards in time to find the full optimal decision grid. Algorithm 1 does exactly this.

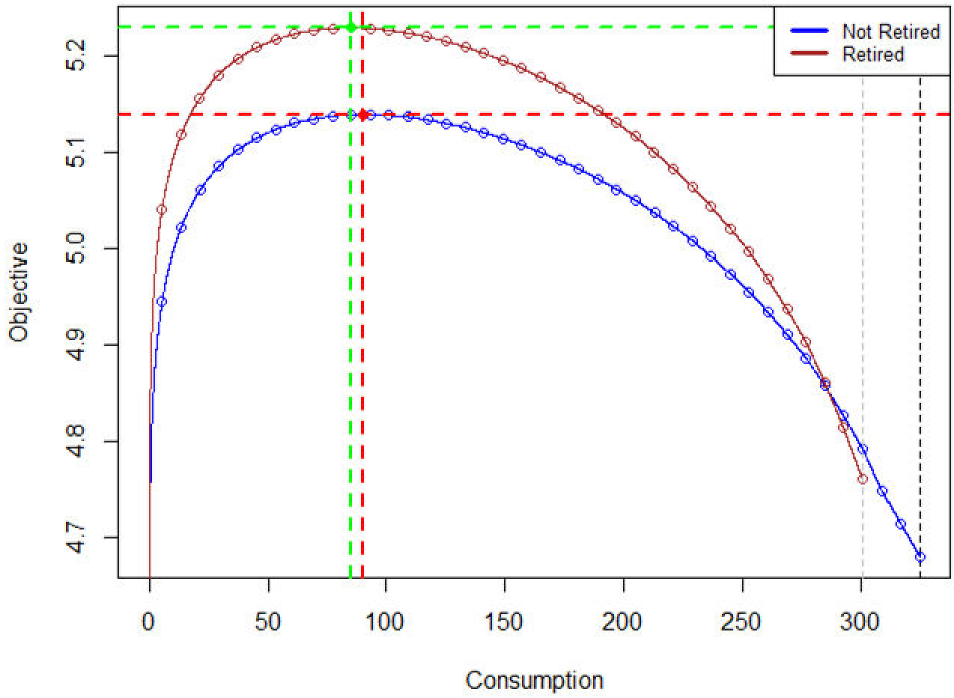
For illustration, Figure 3 shows an example of the optimization trade-off made at each grid point. The dots on the dotted lines denote the consumption levels for which $W_{next}(.)$ falls on a wealth grid level, meaning no interpolation in $\mathbf{F}$, $\mathbf{F}^R$ is required. When consumption is low, gains from consuming more in the current interval, outweigh the utility loss from lower future wealth. Essentially the optimal consumption level is reached when first-order-condition is satisfied:

$$\frac{\partial val(C,L)}{\partial C} = -\frac{\partial val_F(C,L)}{\partial C}. \tag{32}$$

Though as $val_F(C,L)$ is based on an interpolation, it is piecewise linear and its partial derivative is discontinuous, meaning the two partial derivatives may not intersect. This is why a hill-climbing algorithm is used to find the maximum. In the example in Figure 3, the maximum for the *Retired* line is higher than the maximum for the *Not Retired* line, meaning that the agent should retire.

Now the optimal decisions are known for the grid of possible states of an agent, the optimal life-cycle is given by following the optimal decisions from a starting point $\{W_0, L_0, 0\}$. As wealth after any particular interval will likely not fall on a wealth grid level, the optimal consumption and leisure levels are approximated by interpolating from the nearest grid levels. Additionally the optimal leisure level is rounded such that $L \in \{0,1\}$. When the first state is reached for which $L = 1$ is optimal, the decision grid $\mathbf{C}^R$ is used instead of $\mathbf{C}$, and all subsequent leisure level points are forced to 1. This guarantees the agent stays retired, in line with equation (19). The transition moment t_k, for which $L_{k+1} - L_k = 1$, is the optimal retirement age R^*.

Figure 3: Visualization for the optimization problem at any particular grid point. In this example retiring is preferred.



4 Results

In this section we describe the results obtained from implementing the model in Section 3. For this a benchmark individual is considered, and their optimal life-cycle is analyzed. The conditions for this benchmark case will then be altered, as to gain insights in the optimal retirement age response.

4.1 Benchmark Case

Consider an agent who starts at $age_0 = 25$ in the year 2018 and has a maximum theoretical age of 110. The planning horizon for this agent spans from $t_0 = 0$ to $T = 85 = 110 - age_0$. We assume the agent has a constant relative risk aversion level $\eta = 2$, which is similar to values typically observed in the literature. The agent has a rate of time preference equal to the market return at $\delta = r = 0.02$. He has a final net wage level $Y_{final} = 24\,000$, where wage growth is as described in Appendix B. The statutory retirement age is 67 ($sra = 42 = 67 - age_0$), after which a public pension is received at a rate of $Z = 12\,000$. The survival and disutility functions are based on the average mortality characteristics of men and women. Lastly the agent is employed at the start and has zero initial wealth.

As there is no external data from which the trade-off between utility from leisure and consumption can be elicited, we instead require that some particular retirement age is considered optimal. For this it is assumed that the benchmark individual optimally retires at the sra. From calibration, we find that for a value of $\beta = 1.58 \cdot 10^{-2}$ the disutility curve is scaled such that $R^* = sra$.

Unless noted otherwise the simulations from this section are generated using four time points per year $t^{grid} = \{0, 0.25, ..., T\}$. The maximum wealth level considered in the grid is $W_{max} = 1\,000\,000$, where $W^{grid} = \{0, 2\,000, ..., 1\,000\,000\}$ splits the wealth range into 500 intervals. In the various simulations the optimal wealth paths typically reaches a maximal wealth level of around 250 000, and thus the implied constraint $W_t \leq W_{max}$ will not be active. Though this constraint could technically still affect optimal decision computations while not active, we find that its affect is negligible as long as $W_t \leq W_{max}$ is not active in the considered life-cycle.

4.2 The Optimal Life-cycle

In Figure 4a the optimal life-cycle and corresponding cash flows are displayed for the benchmark individual. When impatience is equal to the rate of return $\delta = r$, an agent with known time of death and without leverage restrictions will have a constant optimal consumption level $C_t^* = C^*$. For the benchmark individual in Figure 4a it does hold that $\delta = r$, however their consumption path is not constant. This is due to the leverage limitation and the uncertain lifespan.

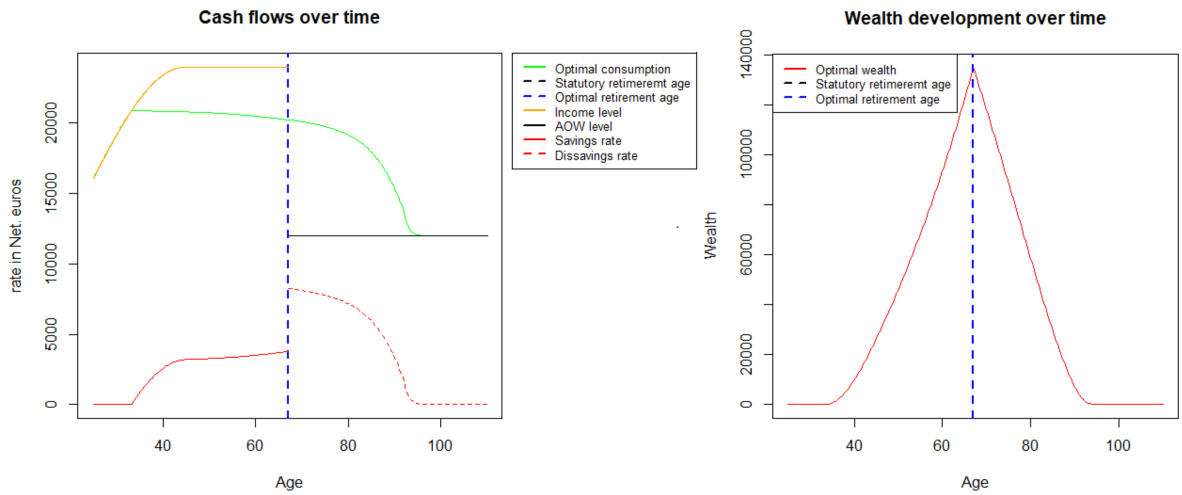
At higher ages the probability of being alive drops quickly. This lowers the expected utility from a certain amount of consumption, effectively increasing the subjective discounting rate δ with age, making old-age utility more expensive. As a result consumption is relatively concentrated towards earlier life, which also reduces the expected wealth at death. This wealth represents a loss of utility as a result of uncertain lifespan. The agent however chooses to accept this loss, as reducing this loss further would require them to have less savings later in life, allowing less old-age consumption. As such it could be referred to as the optimal excess wealth at death. In a collective pension fund this wealth would be shared among still living participants in some manner, allowing higher life-time utility.

At more advanced ages still, the consumption path approaches the public pension (AOW) income rate $C_t^* = Z$. The probability of reaching such an age is low, and almost all financial wealth will have been consumed as a result. While alive however, AOW income is certain. Furthermore with the zero leverage constraint future AOW income cannot be discounted and consumed earlier in life. So to keep excess wealth at death low, all AOW income is immediately consumed.

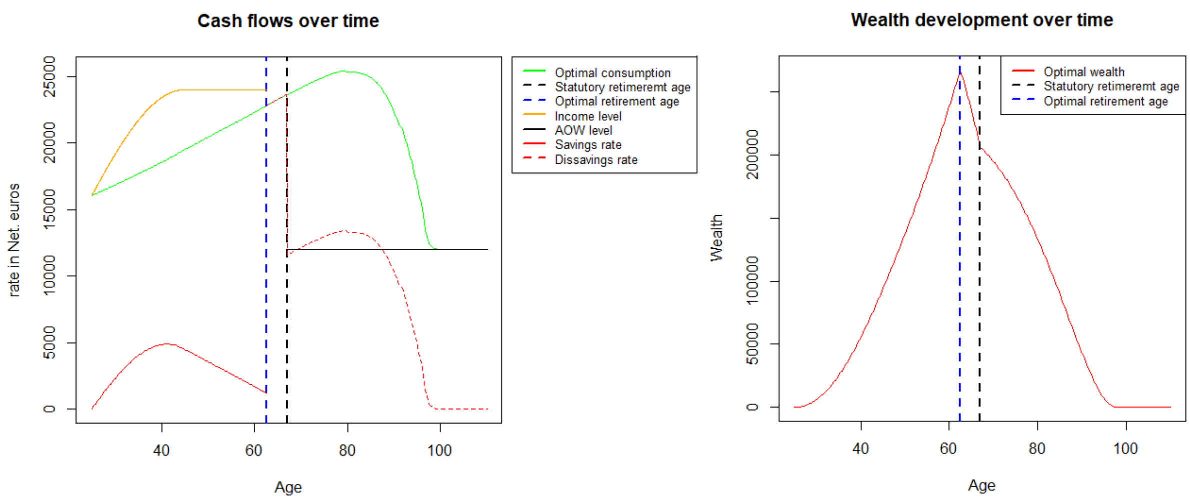
Early in life the leverage constraint also has an effect. Here the agent would ideally increase their consumption level further. It however is limited by the still growing income level, and without leverage or initial wealth they cannot consume more than their income. This causes the agent to live "paycheck-to-paycheck" early in life while income levels are still lower. At a certain point however the agent has to start financing his retirement period, such that they can mitigate disutility from labour at old-age. When this point is reached, saving additional income for retirement for future utility, is preferred over immediate utility from consumption.

In order to fund retirement the agent starts saving wealth at 32 years old. By calibration of the scale parameter β of disutility, the agent optimally retires at 67 years. The total expected amount of premiums payed over the life-cycle is 98 000, for which they will receive an expected 137 000 of benefit payments after retirement. The highest wealth level is reached at retirement at 135 000, with an expected excess wealth at death of 22 000.

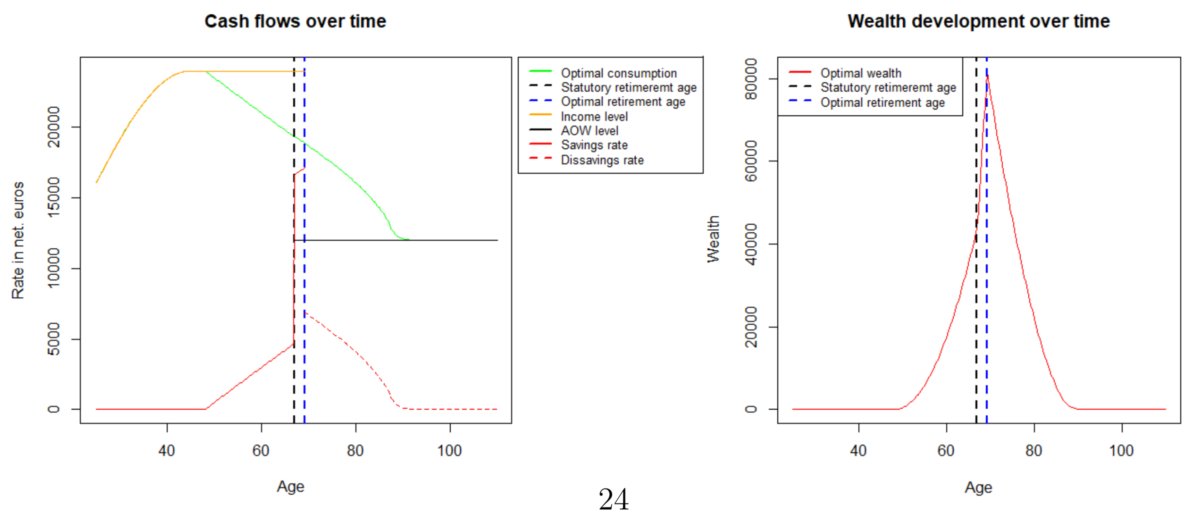
Figure 4: (a) The optimal life-cycle for the base person. (b) The optimal life-cycle with higher returns from the market $r = 0.04$. (c) The optimal life-cycle with lower returns $r = 0$.



(a)



(b)



(c)

4.3 Effect of Market Performance

Now consider the same individual with the same disulity parameter β and impatience $\delta = 0.02$, but the return on investment is doubled $r = 0.04$, see Figure 4b. Here $r > \delta$, making future consumption relatively more favourable, causing optimal consumption to increase with age. With the higher return on investment the agent starts saving immediately, and he also saves less when nearing retirement. Notably with more favourable savings, the agent not only consumes at a higher rate during retirement, but also retires 4.5 years earlier. Choosing to substitute some wealth for more leisure. Due to the additive utility from consumption and leisure being additive (3), the consumption pattern does not show a discontinuous jump or kink at the retirement moment R^*.

Before reaching the statutory retirement age this consumption path is fully financed from the savings-account. The agent chooses to temporarily increase the payed benefits to fund consumption and bridge the income gap before the AOW age. In the Dutch pension context the agent chooses to use an “AOW overbrugging” (AOW bridging). As saving is more profitable the agent consumes more later in life, but also saves a larger amount of total income. Here the total expected premiums payed are 118 000, compared to 98 000 in the base case. For this a much larger expected amount of benefits are also received 355 000, compared to 137 000 before. Correspondingly a larger amount of wealth is kept at higher ages, causing a larger expected wealth at death of 59 000, compared to 22 000.

In Figure 4c a similar picture is given without a return on investment $r = 0$. This perhaps corresponds more to the recent economic climate for the Dutch pension industry, where pension liabilities are near zero discounted. A core difference being however that this is only a recent development, whereas the agent in Figure 4c is guaranteed not to receive a return on savings for the entire life-cycle, and thus can prepare accordingly. In this case immediate consumption is more favourable, reducing saving rates. Savings are also pushed to later life periods, extending the full wage consumption period. The agent only starts saving at 49 years old. Furthermore retirement is postponed with 2.25 years, retiring beyond the AOW age. In the period between the AOW age and the retirement moment, AOW income is still received and used in a “negative” AOW bridge to quickly increase savings. The amount saved for this $r = 0$ case is also decreased. The agent pays an expected 81 000 in pension premiums, for which he will receive an expected 70 000 in benefits. The effective return on savings is less than one as a result of mortality.

The model displays clearly that in a climate with low return on investment retirement becomes more expensive, and as a result one should retire later and consume less during retirement. In the Dutch pension context however, there typically is a minimum savings rate as long as someone is employed. Furthermore a common tendency is to increase saving rates to retain a similar consumption level post retirement. One could conceive of a plethora of complexities which may induce this behaviour in practice. However the model indicates that it should not be self-evident that such behaviour is desirable.

4.4 Retirement Age Sensitivities

The relation between the return on investment r and the rate of time preference δ, has a large impact on the optimal consumption path, and thereby implicitly on the optimal (dis)-saving pattern. Changes in other parameters tend to just perturb the general cash-flow behaviour seen in Figure 4a. We will now consider how changes in the various parameters affect the optimal retirement age R^*.

Table 1: This table display optimal retirement age R^* sensitivity, when different parameters are adjusted with respect to the benchmark case. The benchmark individual optimally retires at age 67. Both R^* and corresponding C^{P*} are redetermined in each case.

Factor	Symbol	Change	Retirement age	Change
Disutility of Labour	β	+20%	64.75	-2.25 years
Impatience	δ	+20%	67.5	+0.5 years
Return on Investment	r	+20%	66	-1 years
Income Level	Y_t	+20%	65.5	-1.5 years
Public Pension Level	Z	+20%	66	-1 years
Statutory Retirement Age	sra	+4 years	67.75	+0.75 years
Life Expectancy		+2 years[5]	**70**	**+3 years**
CRRA for Consumption	η	+20%	**39.75**	**-27.75 years**

Table 1 displays the optimal retirement age response R^* for changes in various model parameters. We find that the sign (increase or decrease) of these effects are largely as expected. The response to a changing life expectancy, and especially to a change in relative risk aversion, are larger than one might expect. The causes for this will be considered separately in Sections 4.5 and 4.6 respectively.

The height of disutility from labour β seems to have a large effect size, where increasing disutility inevitably causes earlier retirement. Increasing impatience δ, will cause the agent to push utility gains towards a short-term horizon. This causes them to always have a larger desire for consumption, but also gives a similar larger desire for leisure.

[5]Increasing the life expectancy by 2 years is done by changing the starting year of the planning horizon from 2018 to 2035. An individual who is 25 years old in 2035 has a 2 years longer life expectancy than someone who is 25 in 2018, according to the AG2018 data.

Larger consumption will delay retirement, but a higher leisure desire postpones retirement. As Table 1 indicates, even with an agent with perfect foresight, the consumption effect dominates as the more impatient agent retires later. The return on investment has a similar but opposite effect, as it makes future consumption less expensive with respect to current wealth. It furthermore reduces the present value of old-age labour income. Both of these effects increase the retirement age as r increases. Out of δ and r, the return on investment has a slightly larger effect size. Increasing both $\delta = r = 2.4$, the agent retires slightly earlier (0.25 years).

With respect to the public pension system, we find that the effect of increasing the statutory retirement age by four years, has a similar effect size as decreasing pension payments by 20%. The actual effect seems relatively small, where increasing the sra by 4 years, only causes 0.75 years of postponement. Increasing such exogenous old-age benefits reduces funding constraints for post-retirement consumption, decreasing the retirement age. As the actual dutch pension system is PAYG tax funded, increasing the benefits should increase the burden of taxation on the younger population. This reduces net income levels, which according to Table 1 should offset the effect and increase the retirement age. As income from employment Y_t causes the bulk of life-time wealth gains, increasing it causes life-time consumption levels to increase. It also affects the cost of retirement, as earlier retirement implies a larger cost in missed income. Again however, according to Table 1 the consumption effect dominates, as higher income levels cause earlier retirement.

4.5 Retirement Age with Changing Life Expectancy

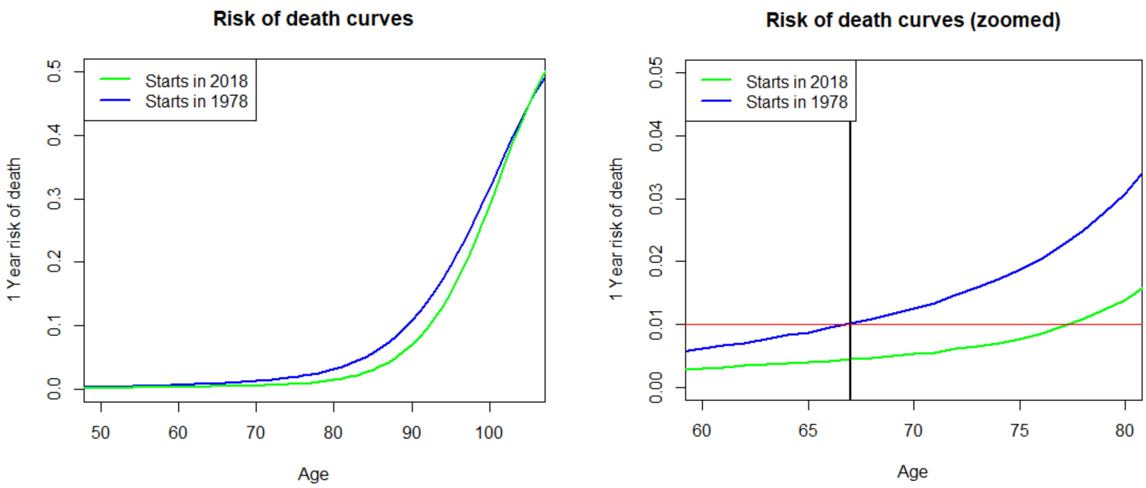
The last effect is that of life expectancy. Here for the benchmark individual, we have used total population mortality characteristics. In 2018 this agent has a life expectancy at $age_0 = 25$ of 88.8 years. The alternative agent is $age_0 = 25$ years old in 2035, and has a life expectancy at this point of 90.8. Increasing life expectancy increases the retirement age to finance consumption for an extended old-age period. Furthermore as disutility (2) is linked to mortality, a higher life expectancy also reduces the benefit of leisure, again increasing the retirement age.

Table 1 indicates that an increased lifespan does indeed increase the optimal retirement age. Counter-intuitively however, a lifespan increase of 2 years increases the ideal retirement level by 3 years. With some investigation, we find that this is caused by the altered disutility curve (2) corresponding to the changed mortality pattern.

In Section 3.2 we modelled disutility of labour as a calibrated scalar multiple of the probability of dying in the next year. This is done under the intuition that less healthy people are more likely to die, where moreover less healthy people also receive a higher disutility of labour. These one-year death curves are displayed in Figure 5. Here two examples are given, one agent who starts their planning horizon in 1978, and one who starts in 2018.

Figure 5a shows that mortality has been reduced in general. At advanced ages however, mortality improvements have been lower. This results in a compression of morbidity, where a larger fraction of the population reaches old-age, but then pass away in a more concentrated range of ages. The net result of this compression, is that mortality and by extension disutility of labour, increased more at the relevant ages than overall life expectancy. Specifically Figure 5b shows that the 2018 agent only reaches the same disutility of labour at 77 years old, as the 1978 agent did in at 67 years old. The life expectancy op the 2018 agent however was 88.8 years, whereas the 1978 agent has a life expectancy of 82.9 years. Thus in this example a 6 year increase in life expectancy has resulted in a 10 year delay before the same disutility level is reached. This compression of morbidity with its delayed disutility, combined with the longer life-expectancy, causes the more than unity retirement age response shown in Table 1.

Figure 5: One-year risk of death curves for someone with $age_0 = 25$ in 1978 and someone with $age_0 = 25$ in 2018. Figure 5b, shows that the agent who starts in 2018 has the same risk of death at 77 years, as the other agent did at 67 years.



(a) (b)

This more than unity increase in the ideal retirement age is not matched by observed behaviour. Here Kalwij et al. (2016) showed that, while (healthy) life expectancy has increased, old-age labour participation rates have barely increased over the generations. Bloom et al. (2014) argues that this may be explained by income growth between generations. In particular, if we consider Table 1, then someone with a 2 year higher life expectancy may retire at the same age if their life-time real wages are approximately 40% higher. Assuming the sra is unchanged between generations. Alternatively the disutility of labour may not be proportional to short-term risk of death, such as assumed in both our model and in Bloom et al. (2014). Changing the disutility function (2) will yield potentially much different sensitivity responses in Table 1. Without direct evidence on people's preferences, it is however hard to justify that an alternative disutility specification is superior.

4.6 Marginal Utility of Consumption

Finally, we must address the extreme response when changing the constant relative risk aversion coefficient for consumption η in Table 1. When increasing this by just 20% to $\eta = 2.4$, the agent will suddenly retire a sizeable 27.25 years earlier. To find what disutility of labour would be required for this new agent to retire at the sra, we repeat the calibration of β and find a value of $\beta^{(2.4)} = 3.02 \cdot 10^{-04}$. For the benchmark individual we instead had a value $\beta^{(2)} = 1.58 \cdot 10^{-02}$. This would imply that the amount of disutility from labour needs to be reduced by $\approx 98\%$ to compensate. As risk aversion levels in the literature typically range from 1 to 3, and retirement ages do not vary as extremely, this implies a very large range for β.

Though perhaps unexpected, this extreme behaviour is not unaccountable for. In equation (13) from Section 3.4 we find the first order condition for the optimal retirement age. This states that the expected disutility from labour at R^* must be equal to the expected gain of lifetime utility from increased consumption as a result of postponing retirement. For the stylized example with known age of death and $r = \delta = 0$, we have found that this simplifies to (17):

$$\frac{\partial u(C^*)}{\partial C} Y = d(R^*). \tag{17}$$

Thus for R^* to be optimal, the disutility at this moment should approximately be equal to the marginal utility of consumption at the optimal constant consumption level, times the income level. Now with an optimal retirement age of $R^* = 67$, then:

$$d(67) = \beta \cdot q_{67}^{2018-age_0+67} \tag{33}$$

Furthermore for power utility as in (3), we have:

$$\frac{\partial u(C^*)}{\partial C} = C^{*-\eta}. \tag{34}$$

As such, using the first order condition from the stylized model, we may try to roughly predict β as:

$$\beta \approx \frac{C^{*-\eta} Y}{q_{R^*}^{2018-age_0+R^*}}. \tag{35}$$

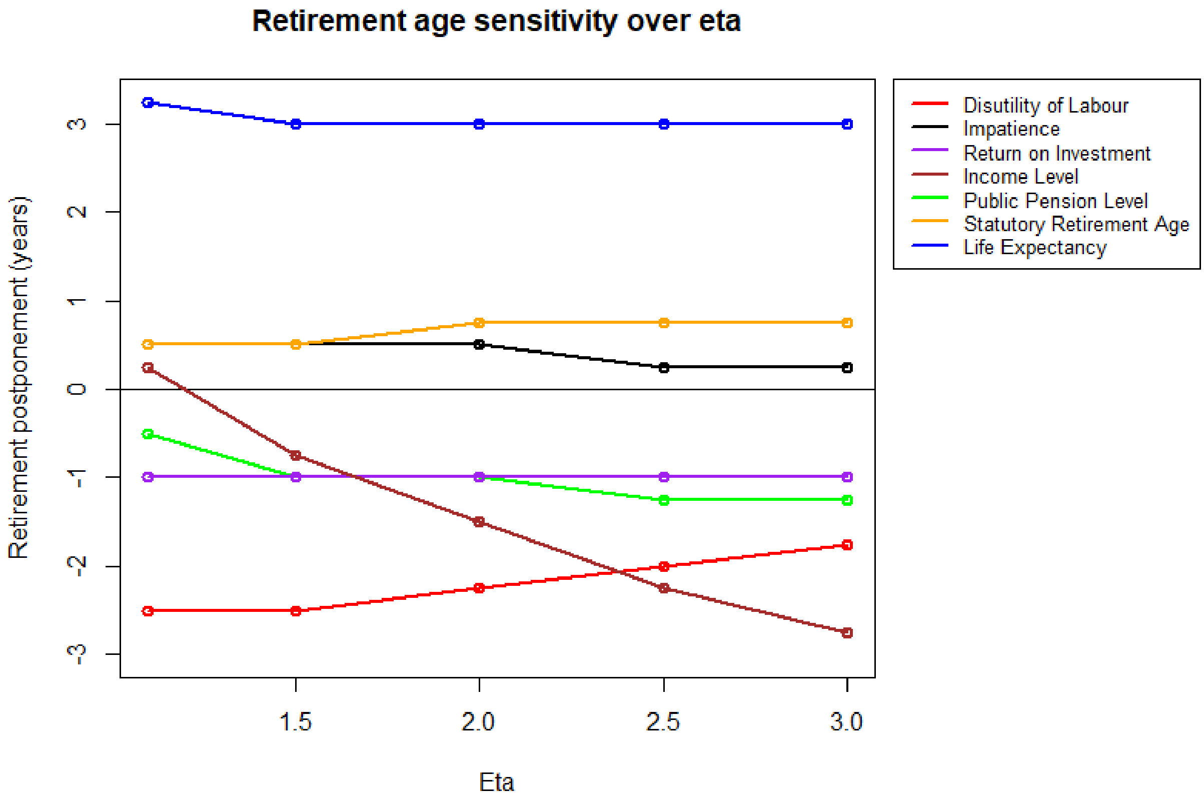
Now with a mean expected consumption level for the benchmark case approximated by $C \approx 20\,000$, income at the retirement age of $Y = 24\,000$, and $q_{67}^{2018-age_0+67} = 3.734 \cdot 10^{-3}$. We find an expected level of $\beta^{(2)} = 1.61 \cdot 10^{-02}$. This approximation is only off by 1.9%, when compared to the calibrated level in the full model. Similarly, for $\eta = 2.4$ equation (34) predicts a level of $\beta^{(2.4)} = 3.06 \cdot 10^{-04}$, which is off by 1.3% when compared to the calibrated level for the full model.

Given the accuracy of this approximation, we find that the marginal utility of consumption is crucial in determining the disutility of labour. As $20\,000^{-2}$ is approximately 50 times larger than $20\,000^{-2.4}$, it is expected that the corresponding disutility from labour is also around 50 times larger. Scaling down all financial quantities by 1 000, we find that $20^{-2}/20^{-2.4} \approx 3$ which means that the problem is affected by scale, unlike the classic Merton model. A purely multiplicative function such as Cobb-Douglas utility used in Appendix C, may retain scale invariance but will cause a discontinuous jump in consumption at retirement.

As changing η slightly seems to induce an extreme change in R^*, we consider whether different levels of η, also causes the retirement age sensitivity to change for other parameters. We essentially repeat the results of Table 1 for different η, to check if our results are reasonably stable despite the lack of scale invariance. For this we consider five different values of $\eta = \{1.1, 1.5, 2, 2.5, 3\}$. At each level the original benchmark individual is considered, however the disutility from labour coefficient β, is recalibrated for each η such that the benchmark individual retires optimally at $R^* = sra = 67$. The results for this are displayed in Figure 6.

As shown in Figure 6 the sensitivities do change, however there appear to be no extreme effects as a result of no scale invariance. The sensitivity of R^* with respect to the income level changes most with η. Here for $\eta = 3$, an increase in wages by 20% causes 3.75 years earlier retirement. Whereas with $\eta = 1.1$, the same increase yields almost no change in retirement. When η is low, the marginal utility from consumption remains higher. As a result there is a lower incentive to smooth consumption over time. This is what the interpretation of η as the *inverse of intertemporal elasticity of substitution* entails. Furthermore with a higher marginal utility from consumption, when additional income becomes available, more of it will be put towards consumption instead of leisure. As a result we see that the income effect in Figure 6 becomes smaller as η becomes smaller. The other parameter sensitivities are less affected by changing η.

Figure 6: The sensitivity of the optimal retirement age for different levels of η. Here each line plots the retirement age, resulting from the corresponding increase shown in Table 1. In particular when $\eta = 2$, the same values are obtained as in the last column of Table 1. For each of the different levels of η, the value β is recalibrated, such that $R^* = 67$ is optimal in the base case.



5 Conclusions

In this work we have considered the voluntary optimal retirement problem with perfect foresight, where we simultaneously find the corresponding optimal consumption cycle. Here we have focused on an individually funded pension in a Dutch type retirement system, with a publicly financed old-age pension. The public system forms an exogenous influence on the retirement age decision. Instead of an occupational pension with mandatory participation, the agent is allowed considerable flexibility in their (dis)saving decision. This allows for a comparison of the resulting optimal (dis)saving paths and the mandatory participation system.

We aimed to produce a model where the optimal solution could be implemented by a typical citizen. Thus for practical feasibility we did not allow unretirement or financial leverage. The latter means that the agent cannot fund current consumption by discounting future income streams. To our knowledge, this is the first quantitative retirement age model which includes both these real-world restrictions. In such voluntary retirement models the agent is tasked to maximize life-time utility by making a trade-off between utility from consumption, and the disutility of employment. The disutility of employment is increasing with age, and is based on the short-term mortality at a given age as proxy for health.

To obtain a solution, the life-cycle optimization problem is reformulated as a continuous Markov decision problem. This reformulated problem is tackled with dynamic programming, where we show that the globally optimal decision in any particular state can be guaranteed as the outcome of a simple hill-climbing algorithm. As the optimal decision can be found in any particular state, the continuous dynamic program will yield the globally optimal life-cycle. The implemented life-cycle solver is a discretized approximation of this continuous dynamic programming strategy. Apart from not needing a financial leverage and unretirement assumption, the dynamic programming strategy also puts few restrictions on the functional forms used in the model. This allows the use of real-world mortality and wage growth.

To produce our results, we consider a benchmark individual with a rate of time preference equal to the market return, and see how the retirement age optimally changes as a response to changes in this benchmark. In the classical Merton model an agent with time preference equal to market return chooses to consume at a constant rate. This is also approximately correct in our retirement model with additive utility from leisure. The benchmark individual however does not consume at a constant rate, as uncertain survival makes consumption at advanced ages less favourable. The absence of leverage furthermore causes the optimizing agent to live paycheck-to-paycheck, both early in life when income is low, and late in life if he has outlived his retirement savings. As the optimizing agent chooses to start saving only later in life, this indicates that the mandatory minimum saving rate in the Dutch second pillar, may not be ideal. When return on investment is higher than the rate of time preference, we find that consumption increases with age, and that the agent both saves earlier and retires earlier.

The return on investment and impatience determines the general (dis)saving and consumption pattern. Other parameters simply perturb the optimal life-cycle, but do influence the ideal retirement age. In particular for the benchmark agent, increasing the statutory retirement age by 4 years, increases the optimal retirement age by only 0.75 years. This is a similar effect as decreasing public pension benefits by 20%. Doubling the return on investment from 2% to 4% causes earlier retirement by 4.5 years. With zero return on investment however retirement is postponed by 2.25 years. Here retirement occurs beyond the statutory retirement age, and the public pension income is used to quickly increase retirement savings. Increasing life-time wages by 20% causes an earlier retirement of 1.5 years. This is however affected by the marginal utility of consumption, and ranges approximately from -0.5 to $+3$ years of earlier retirement for typical relative risk aversion levels $\eta \in [1, 3]$.

Life expectancy changes seem to have a large effect on the ideal retirement age. An agent who is born later, and as a result has a 2 year longer life expectancy than the benchmark case, will choose to retire 3 years later, ceteris paribus. The analysis shows that this is caused by a compression of morbidity, where a larger fraction of the population reaches old-age, and later pass away in a more concentrated range of ages. As a result the point at which the same short-term mortality is reached, increases more quickly than the life-expectancy. With the model assumption to link health with short-term mortality, and to link disutility of labour with health, the ideal retirement age also increases more quickly than life-expectancy.

Lastly, we find that ceteris paribus changes to the coefficient of relative risk aversion for consumption causes extreme changes in the optimal retirement age. This is caused by a lack of invariance to scale, which is present in the classic Merton model, but is absent here due to the additive disutility from labour. Both this and related work on optimal retirement do however use such additive utility models, as alternatives cause distinct jumps in consumption at retirement and are more complex to interpret. The absence of invariance of scale withholds sensible comparison between individuals with different risk aversion levels η, however the effect of other parameters given specific risk aversion levels can still be determined. Here the income effect is most affected by different risk aversion levels. The effect on the retirement age with changes in other parameters remain relatively stable.

We have produced a relatively flexible and practical model and solution approach for the voluntary retirement age problem. All such models however require a trade-off of consumption and leisure, for which there appears to be no recommended approach in the literature. As a result these models assume that some observed retirement age is optimal, and as a result should not be used to recommend any particular retirement age, but instead only be used to determine differential effects as the situation changes.

For future work we recommend that this trade-off should be nailed down first. For the leisure component, an extensive survey may help where individuals are asked how much wealth they are willing to trade for earlier retirement or days off. Here one will have to find out how this desire for leisure changes for different ages. Once such a trade-off target is known for the considered agent, we may use these models to recommend a particular retirement age and savings pattern. Moreover if the aim is to predict how the ideal retirement age changes over the generations, a model for the disutility of labour will have to be produced, which can predict future disutility. A simple link to short-term mortality is likely insufficient, though no better alternative is present in the related work. Lastly an extension including habit formation in utility from consumption and the inclusion of annuities may allow these models to better represent observed behaviour.

Appendices

A Survival and Disutility Functions

Figure 7: The survival function for the average person, starting at $age_0 = 25$ in 2018.

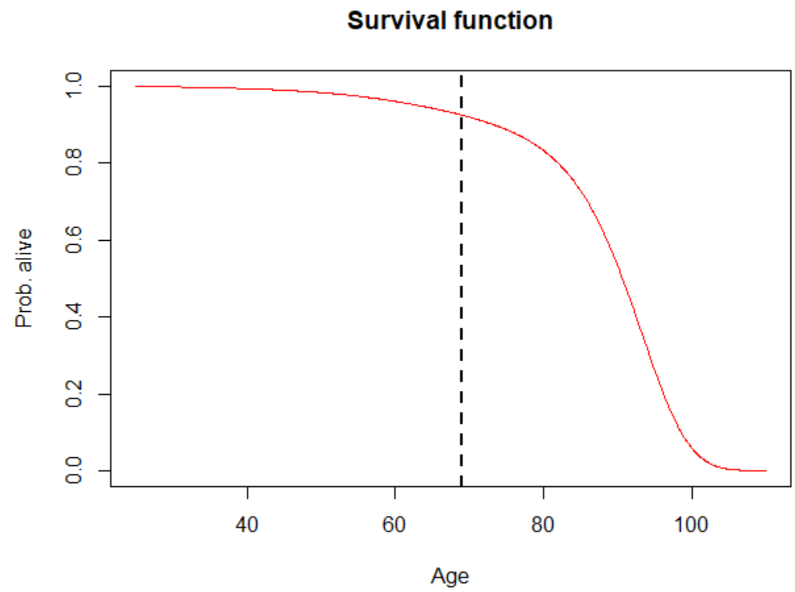
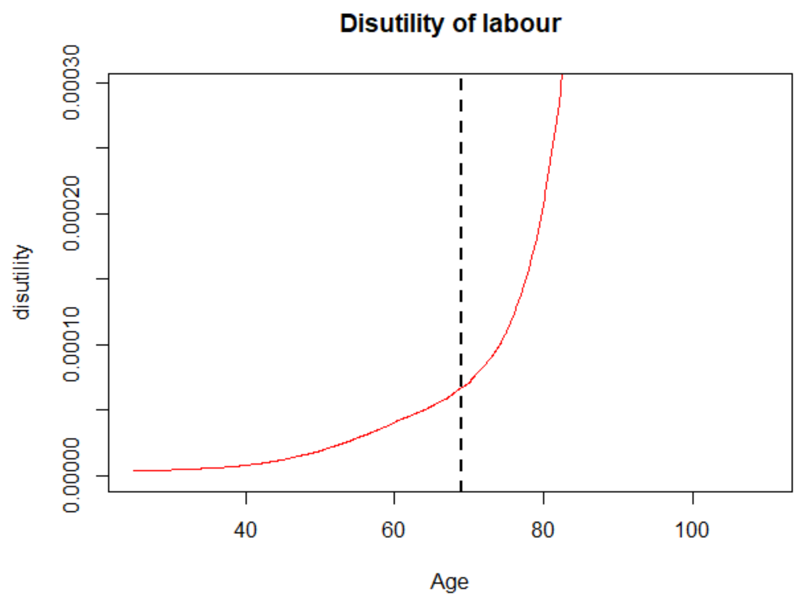


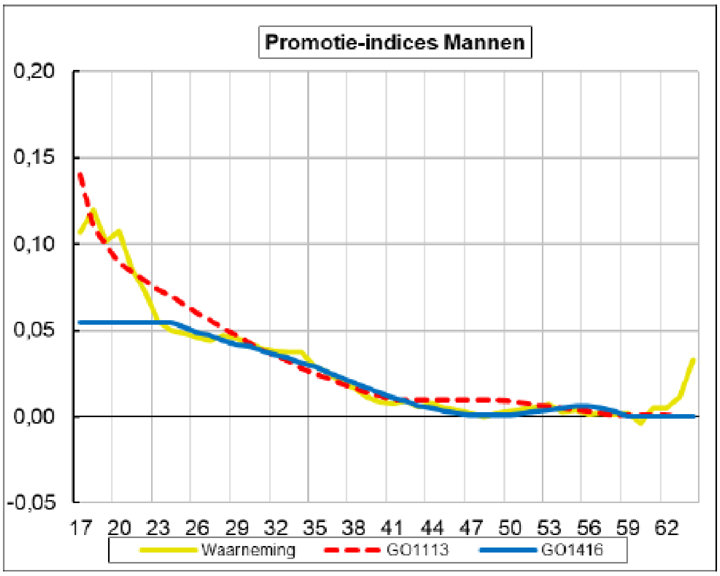
Figure 8: The disutility function for the average person, starting at $age_0 = 25$ in 2018. Here only the relative height with respect to zero is relevant, as disutility is scaled by a calibration parameter β.



B The Assumed Wage Structure

The wage path is determined by the promotion-indices from the ABP foundation-research of 2014-2016. See Figure 9. Here GO1416 is the most recent estimate of the promotion-indices. Based on this pattern, we make the stylized assumption that the wage growth rate is 5.5% before the age of 25, it linearly decreases from 5.5% to 0% between the ages of 25 and 45, and wages are constant after 45. Note that these growth levels consider gross wages, whereas our model requires net wages. As such we instead use 4% wage growth instead of 5.5%, in order to compensate for the progressive tax system.

Figure 9: Promotion-indices for males according to the ABP foundation-research. The blue line gives the most recent estimate.

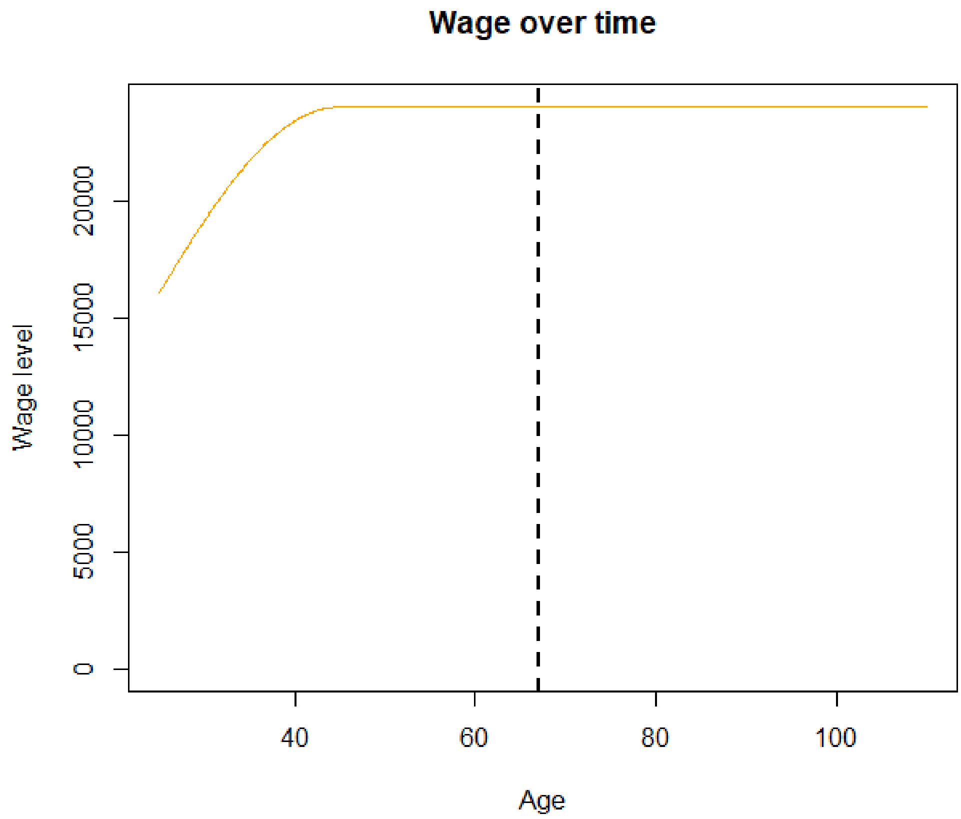


Let Y_t denote the wage level at time t, and let Y_{final} be the final wealth level. Let $S = 25 - age_0$ be the time at which income growth starts to decrease, and let $E = 45 - age_0$ be the time at which wage growth has stopped. Then we approximate wage growth according to the following ordinary differential equation:

$$dY_t = g(t) \cdot Y_t dt, \tag{36}$$

$$\text{where } g(t) = \begin{cases} 0.04, & \text{if } t \leq S \\ 0.04 \cdot \left(1 - \frac{t-S}{E-S}\right), & \text{if } S < t \leq E \\ 0, & \text{if } E < t \end{cases} \tag{37}$$

Figure 10: Income from employment for different ages, with a final net income level of 24,000. The dashed line denotes the statutory retirement age.



With as boundary condition $Y_E^G = Y_{final}^G$, the solution to (36) is the following:

$$Y_t = \begin{cases} Y_{final} \cdot \exp\left\{0.05S\left(1 + \frac{S}{2(E-S)}\right) - 0.05E\left(1 + \frac{2S-E}{2(E-S)}\right) + 0.05(t-S)\right\}, & \text{if } t \leq S \\ Y_{final} \cdot \exp\left\{0.05t\left(1 + \frac{2S-t}{2(E-S)}\right) - 0.05E\left(1 + \frac{2S-E}{2(E-S)}\right)\right\}, & \text{if } S < t \leq E \\ Y_{final}, & \text{if } E < t \end{cases} \tag{38}$$

Figure 10 shows the resulting wage path over time for an individual starting at $age_0 = 25$ and a final income $Y_{final} = 24,000$.

C Merton Consumption Model with Labour Participation

As stated in Section 1.1, our initial research considered extending the classical Merton model, Merton (1969), with a labour participation choice. This is inspired by the work of Bodie et al. (1992). They however exclusively considered the implications for optimal investment. In this appendix we display our work on the implications for optimal consumption (saving) and labour participation instead.

Here we do not directly consider a discrete retirement choice. Instead an individual is flexible in his choice of consumption and the amount of labour during the life-cycle. That is, we consider an optimal control problem for consumption and leisure in continuous-time. The classic continuous time consumption/investment problems have first been analyzed in Merton (1969), Merton (1975), and Samuelson (1975). Here it is found that an analytical solution exists, if we assume a Black-Scholes type world and the individual has preferences which fall in the Hyperbolic Absolute Risk Aversion (HARA) family.

The focus in Bodie et al. (1992), is to describe the optimal investment portfolio when there is labour flexibility. Like the related work on retirement planning in Section 1.1, we abstract from the investment problem, and use purely risk-free investment. Like Bodie et al. (1992) we will consider Cobb-Douglas utility, due to its similarity to power utility, which is a member of the HARA family. Denote by C_t the consumption level at time t, and by L_t the amount of leisure at time t, then:

$$\text{Cobb-Douglas utility:} \quad u(C_t, L_t) = \frac{C_t^{\gamma} L_t^{\beta}}{\gamma} \tag{39}$$

In the particular case where the coefficient on leisure $\beta = 0$, we obtain the power utility function typically associated with the Merton model. Note that we require β and γ to be of the same sign, as otherwise increasing leisure would reduce utility. Furthermore imposing the condition $\beta + \gamma \leq 1$ ensures that we cannot realize increasing marginal utility with scale. Increasing marginal utility with scale implies it may be optimal to save wealth, in order to consume it in infinitesimally short impulses. Such a discontinuous consumption pattern is intractable mathematically.

Without further ado, consider an individual with perfect foresight, who starts his employment career at time 0 and lives to a known date T. There is no risk in this world, with an instantaneous return on wealth of r. The wage paid for labour at time t is denoted by Y_t, and is determined by some known function of time $Y_t = y(t)$. In the particular case where $y(t) = Y_0 e^{gt}$, we obtain the same deterministic wage pattern as used in Bodie et al. (1992).

Now let h_t be the amount of work performed at time t, and let leisure L_t be such that $L_t + h_t = 1$. Assume that wages are paid out linearly based on the amount worked, then the optimizing agent is subject to the following dynamic budget equation:

$$dW_t = (rW_t - C_t - Y_t L_t)dt. \tag{40}$$

Note that W_t is the total wealth for that individual at time t. Here $W_t = F_t + H(t)$, where F_t is his financial wealth and $H(t)$ is the value of his remaining human capital at time t. As remaining human capital is included in the budget equation, taking leisure time reduces wealth at a rate equal to income at that moment. This is why (40) contains the term $-Y_t L_t$. Human capital is the present value of future non-capital gains, assuming the agent works full-time during his entire life, that is:

$$H(t) = \int_t^T Y_s e^{-r(s-t)} \, \mathrm{d}s. \tag{41}$$

Like in the standard Merton model, and related work that quantitatively optimizes the retirement age, the agent aims to maximize his life-time utility from consumption and leisure. This life-time utility function is assumed to be strongly separable in time, where utility in each future moment is discounted with some rate δ:

$$\int_0^T e^{-\delta t} u(C_t, L_t) \, \mathrm{d}t. \tag{42}$$

Let $J(W_t, Y_t, t)$ be defined as the optimal attainable value of life-time utility (42) at time t. Then $J(W_t, Y_t, t)$ is called the implicit value function for this problem:

$$J(W_t, Y_t, t) \equiv \max_{\{C,L\} \in \Theta} \int_t^T e^{-\delta s} u(C_s, L_s) \, \mathrm{d}s. \tag{43}$$

Here Θ is the set of feasible consumption and leisure paths. For now however, we do not consider boundary conditions on C_t and L_t, focusing instead on a regular interior maximum. Equation (43) can be written in dynamic programming form as:

$$J(W_t, Y_t, t) = \max_{\{C,L\}} \left\{ \int_t^{t+h} e^{-\delta s} u(C_s, L_s) \, \mathrm{d}s + J(W_{t+h}, Y_{t+h}, t+h) \right\}. \tag{44}$$

Now applying Taylor's theorem for multivariate functions and the mean value theorem for integrals to (44), the Hamilton–Jacobi–Bellman equation for optimality is found. Note that no second-order derivatives are prevalent, as there is no stochastic process in this model:

$$0 = \max_{\{C_t, L_t\}} \left\{ e^{-\delta t} u(C_t, L_t) + J_t + J_Y y'(t) + J_W [rW_t - C_t - Y_t L_t] \right\}. \tag{45}$$

Here the subscript x in J_x, denotes the partial derivative of $J(.)$ to x. This should not be confused with W_t, Y_t, C_t, L_t which simply denote the value of their respective variables at time t. For a regular interior maximum no Karush-Kuhn-Tucker or Lagrange conditions need to be satisfied, therefore in the optimum of (45) we have the first order conditions:

$$u_C(C_t^*, L_t^*)e^{-\delta t} - J_W = 0, \tag{46}$$
$$u_L(C_t^*, L_t^*)e^{-\delta t} - Y_t J_W = 0. \tag{47}$$

Here by dividing (47) by (46), we find $u_L/u_C = Y_t$, which indicates that the marginal gains on leisure relative to consumption is determined by the wage level at that moment. Now introducing Cobb-Douglas utility 39, the first order conditions (46) and (47) become:

$$C_t^{*\gamma-1} L_t^{*\beta} e^{-\delta t} - J_W = 0, \tag{48}$$
$$\frac{\beta}{\gamma} C_t^{*\gamma} L_t^{*\beta-1} e^{-\delta t} - Y_t J_W = 0, \tag{49}$$

Substituting (49) into (48), we find the following proportionality condition:

$$C_t^* = \left(\frac{\gamma}{\beta} Y_t\right) L_t^*. \tag{50}$$

Assuming at a specific moment t, we know a certain total-consumption level is optimal, then (50) denotes how the total-consumption should be split among consumption and leisure. Note that γ/β is always positive, due to requirement that they are of the same sign. As such for a given wage level and a given total-consumption rate, increasing $|\gamma|$ will increase consumption and reduce leisure, and increasing $|\beta|$ will do the reverse.

Using the proportionality condition (50) and (49), we can write the optimal leisure and consumption levels in terms of partial derivatives of the implicit value function $J(W_t, Y_t, t)$:

$$0 = \frac{\beta}{\gamma}\left(\left(\frac{\gamma}{\beta} Y_t\right) L_t^*\right)^{\gamma} L_t^{*\beta-1} e^{-\delta t} - Y_t J_W, \tag{51}$$
$$\Rightarrow L_t^* = \left[J_W e^{\delta t}\right]^{\frac{1}{\gamma+\beta-1}} \left(\frac{\gamma}{\beta} Y_t\right)^{\frac{1-\gamma}{\gamma+\beta-1}}, \tag{52}$$
$$\text{proportionality:} \quad \Rightarrow C_t^* = \left[J_W e^{\delta t}\right]^{\frac{1}{\gamma+\beta-1}} \left(\frac{\gamma}{\beta} Y_t\right)^{\frac{\beta}{\gamma+\beta-1}}. \tag{53}$$

Substituting this result into the Hamilton-Jacobi-Bellman equation (45), we obtain the fundamental partial differential equation which $J(W_t, Y_t, t)$ needs to satisfy:

$$
\begin{aligned}
0 =& e^{-\delta t}\frac{1}{\gamma}\left\{\left[J_W e^{\delta t}\right]^{\frac{1}{\gamma+\beta-1}}\left(\frac{\gamma}{\beta}Y_t\right)^{\frac{\beta}{\gamma+\beta-1}}\right\}^{\gamma}\left\{\left[J_W e^{\delta t}\right]^{\frac{1}{\gamma+\beta-1}}\left(\frac{\gamma}{\beta}Y_t\right)^{\frac{1-\gamma}{\gamma+\beta-1}}\right\}^{\beta} \\
&+ J_W\left\{rW - \left[J_W e^{\delta t}\right]^{\frac{1}{\gamma+\beta-1}}\left(\frac{\gamma}{\beta}Y_t\right)^{\frac{\beta}{\gamma+\beta-1}} - Y_t\left[J_W e^{\delta t}\right]^{\frac{1}{\gamma+\beta-1}}\left(\frac{\gamma}{\beta}Y_t\right)^{\frac{1-\gamma}{\gamma+\beta-1}}\right\} \\
&+ J_t + J_Y y'(t). \qquad (54)
\end{aligned}
$$

With some arithmetic one finds that this simplifies to:

$$
0 = J_W^{\frac{\gamma+\beta}{\gamma+\beta-1}}\exp\left(\frac{\delta t}{\gamma+\beta-1}\right)\xi(Y_t) + J_W r W_t + J_t + J_Y y'(t), \qquad (55)
$$

$$
\text{where } \xi(Y_t) = \frac{1-\gamma}{\gamma}\left[1 - \left(\frac{\gamma}{\beta}Y_t\right)^{\frac{\beta}{\gamma+\beta-1}}\right]. \qquad (56)
$$

Note that if there is an indifference to leisure $\beta = 0$, and there is zero wage income $Y_t = 0$, then (55) is effectively identical to a risk-less version of the classic consumption problem of Merton (1969) with power utility. As is described in the classic work, a trial-solution is required which simplifies the partial differential equation (55), and makes it solvable. Finding these trial-solutions is however not a trivial effort, and their existence depends on the exact functional forms used in the model. For example that the utility function belongs to the HARA family in the classic case. To solve the partial differential equation (55), consider the following trial solution, and its partial derivatives:

$$
\begin{aligned}
\bar{J}(W_t, Q_t, t) &= \frac{b(t)}{\gamma+\beta}e^{-\delta t}W_t^{\gamma+\beta}, && (57)\\
\bar{J}_W &= b(t)e^{-\delta t}W_t^{\gamma+\beta-1}, && (58)\\
\bar{J}_Q &= 0, && (59)\\
\bar{J}_t &= \frac{b'(t)}{\gamma+\beta}e^{-\delta t}W_t^{\gamma+\beta} - \frac{\delta b(t)}{\gamma+\beta}e^{-\delta t}W_t^{\gamma+\beta}. && (60)
\end{aligned}
$$

Substituting these partial derivatives into (55), we find that the dependence on wealth drops out, reducing it to an ordinary differential equation in $b(t)$:

$$
\begin{aligned}
0 =& \left(\frac{b(t)}{\gamma+\beta}e^{-\delta t}W_t^{\gamma+\beta}\right)^{\frac{\gamma+\beta}{\gamma+\beta-1}}\exp\left(\frac{\delta t}{\gamma+\beta-1}\right)\xi(Y_t) + \frac{b(t)}{\gamma+\beta}e^{-\delta t}W_t^{\gamma+\beta+1}r \\
&+ \frac{b'(t)}{\gamma+\beta}e^{-\delta t}W_t^{\gamma+\beta} - \frac{\delta b(t)}{\gamma+\beta}e^{-\delta t}W_t^{\gamma+\beta} + 0, \qquad (61)\\
\Rightarrow& \; b'(t) = \left[\delta - r(\gamma+\beta)\right]b(t) - \xi(Y_t)(\gamma+\beta)b(t)^{\frac{\gamma+\beta}{\gamma+\beta-1}}. \qquad (62)
\end{aligned}
$$

As wage Y_t is a known function of time, equation (62) is an ordinary differential equation in time. If furthermore there is no wage change over time, then $Y_t = Y$ for all t, and $\xi(Y) = \xi$ is constant. In this case (62) has a known analytical solution for $b(t)$:

$$b(t) = \left(\frac{(\beta+\gamma)\xi - \exp\left\{(t-c_1)\left[r + \frac{\delta - r}{1-\gamma-\beta}\right]\right\}}{\delta - r(\beta+\gamma)} \right)^{1-\gamma-\beta}, \tag{63}$$

where c_1 is a constant to be determined by a boundary condition. Assume the gain from terminal wealth is: $\epsilon^{1-\gamma-\beta}/(\gamma+\beta) \cdot e^{-\delta T} W_T^{\gamma+\beta}$, then (57) yields the boundary condition $b(T) = \epsilon^{1-\gamma-\beta}$, and the full solution is obtained:

$$c_1 = T - \frac{\ln[(\beta+\gamma)\xi - \epsilon(\delta + r(\beta+\gamma))]}{\left[r + \frac{\delta - r}{1-\gamma-\beta}\right]}, \tag{64}$$

$$\Rightarrow b(t) = \left(\frac{1 + [v\epsilon - 1]\exp\left\{(t-T)\left[r + \frac{\delta - r}{1-\gamma-\beta}\right]\right\}}{v} \right)^{1-\gamma-\beta}, \tag{65}$$

$$\text{where } v = \frac{\delta - r(\beta+\gamma)}{(\beta+\gamma)\xi}. \tag{66}$$

For an infinitesimally small bequest desire $\epsilon > 0$, this boundary condition is a proxy for no utility from terminal wealth. Substituting (65) into (58) an expression for J_W is obtained, and consequently we find the optimal implicit leisure and consumption paths:

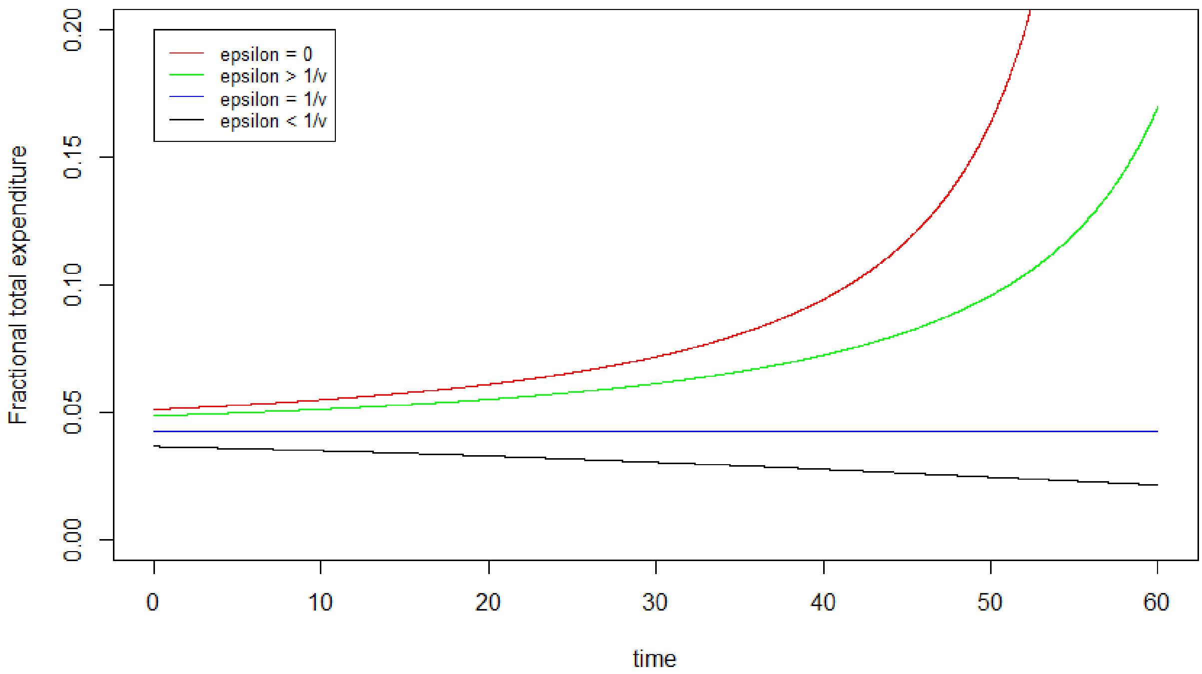
$$L_t^* = \frac{v}{1 + [v\epsilon - 1]\exp\left\{(t-T)\left[r + \frac{\delta - r}{\gamma+\beta-1}\right]\right\}} \left(\frac{\gamma}{\beta} Y\right)^{\frac{1-\gamma}{\gamma+\beta-1}} W_t, \tag{67}$$

$$C_t^* = \frac{v}{1 + [v\epsilon - 1]\exp\left\{(t-T)\left[r + \frac{\delta - r}{\gamma+\beta-1}\right]\right\}} \left(\frac{\gamma}{\beta} Y\right)^{\frac{\beta}{\gamma+\beta-1}} W_t. \tag{68}$$

Note that we have used a positive utility from terminal wealth determined by ϵ. In the agent's goal function (42) however, there is no utility from terminal wealth $\epsilon = 0$. This will cause an impulse of infinite relative consumption and leisure at time $t = T$, forcing W_T to zero. In general the value of ϵ may have a significant impact on the optimal consumption and leisure behaviour.

Consider the fraction of total wealth expenditure on consumption and leisure $(C_t + L_t Y)/W_t$. In Figure 11 the effect for different values of ϵ is displayed. While ϵ only determines utility from terminal wealth, it indirectly affects the amount of total consumption during the entire life. In particular if $\epsilon < 1/v$ fractional total-consumption is

Figure 11: The relative expenditure of total wealth, $(C_t + L_tY)/W_t$, plotted over an individual's life for different levels of utility from excess wealth at the end of life.



increasing over time.

With a terminal utility determined by $\epsilon = 1/v$, the time dynamics of the relative expenditure of wealth cancel out. In the classical Merton model, it is then generally considered that optimal consumption is constant if the $\delta = r$. In this model with labour participation we can similarly find a condition such that total-expenditure is constant:

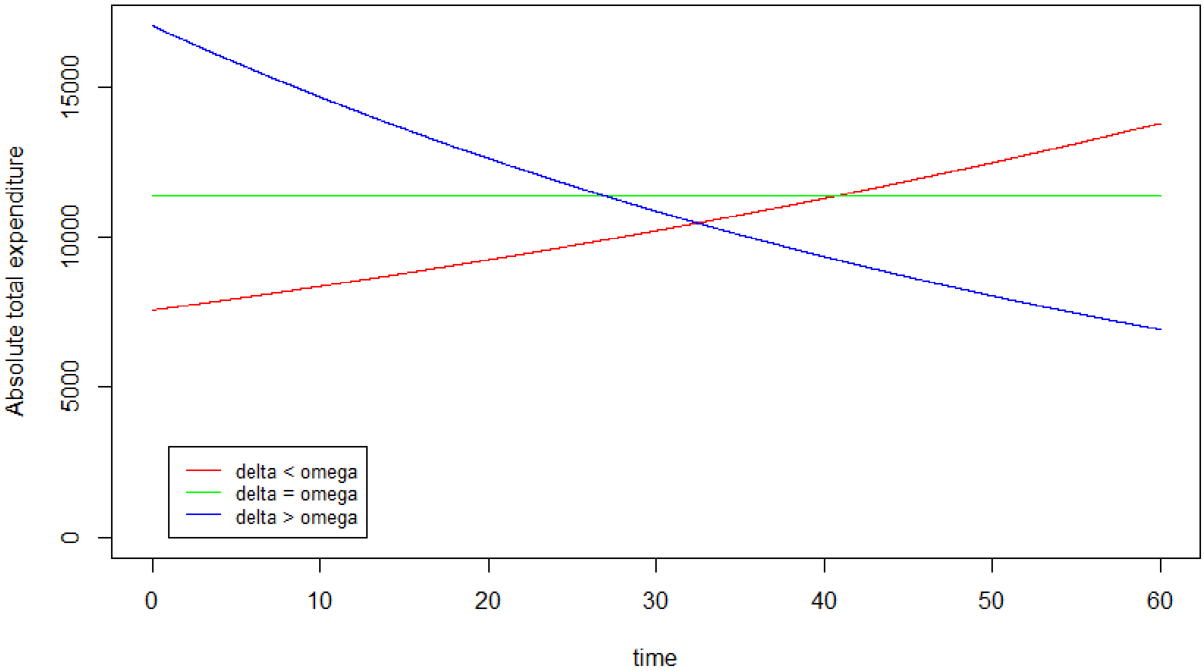
$$dW_t = (rW_t - (C_t^* + YL_t^*))dt = 0, \tag{69}$$

$$\Rightarrow r = v\left(\left(\frac{\gamma}{\beta}Y\right)^{\frac{\beta}{\gamma+\beta-1}} + Y\left(\frac{\gamma}{\beta}Y\right)^{\frac{1-\gamma}{\gamma+\beta-1}}\right), \tag{70}$$

$$\Rightarrow \delta = \omega \equiv r(\gamma+\beta) + r(1-\gamma)\left[\left(\frac{\gamma}{\beta}Y\right)^{\frac{\beta}{1-\gamma-\beta}} - 1\right]. \tag{71}$$

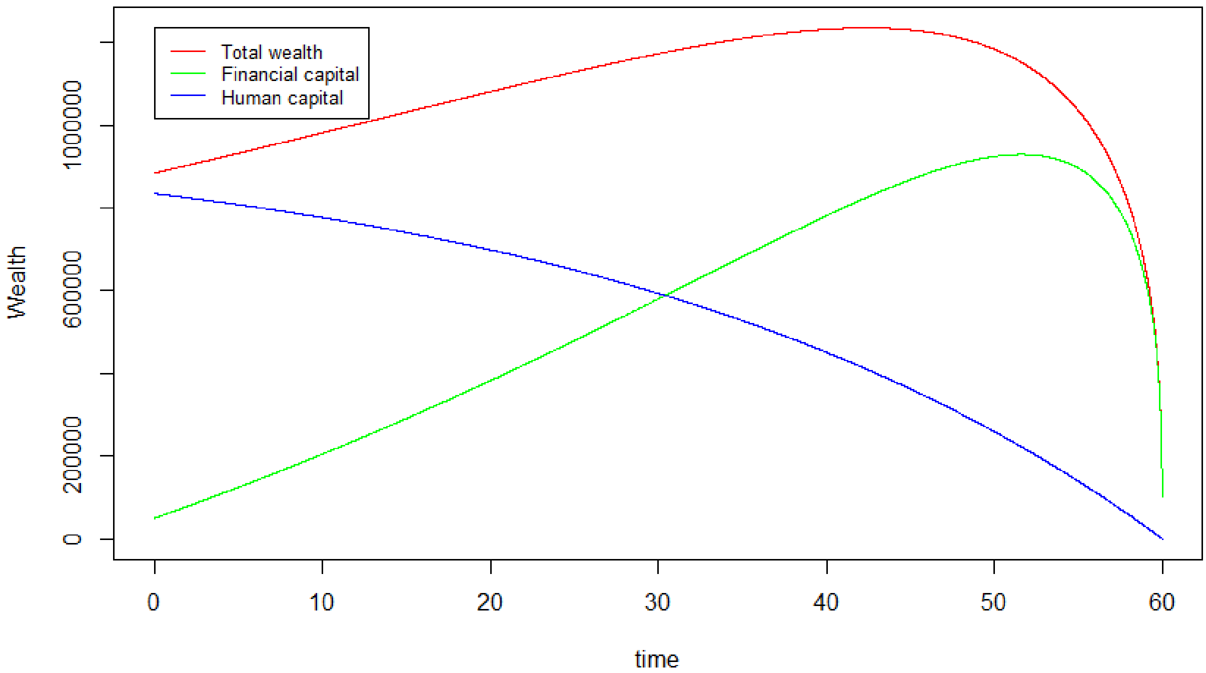
Define ω as the value for δ such that (71) holds. Figure 12 displays the effect of different subjective discounting rates when $\epsilon = 1/v$. As shown, for a relatively patient agent with $\delta < \omega$ it is preferred to consume less early in life, which causes total wealth to accrue, and in turn both increases consumption later in life as well as terminal wealth. As utility from terminal wealth is $J(W_T, Y, T) = \epsilon^{1-\gamma-\beta}/(\gamma+\beta) \cdot e^{-\gamma T} W_T^{\gamma+\beta}$, a lower discounting

Figure 12: The absolute expenditure of total wealth, $C_t + L_t Y$, plotted over an individual's life for different subjective time-discounting factors δ and $\epsilon = 1/v$.



factor δ also causes a larger gain for a given level of W_T.

Figure 13: The development of total wealth and its components financial and human capital.

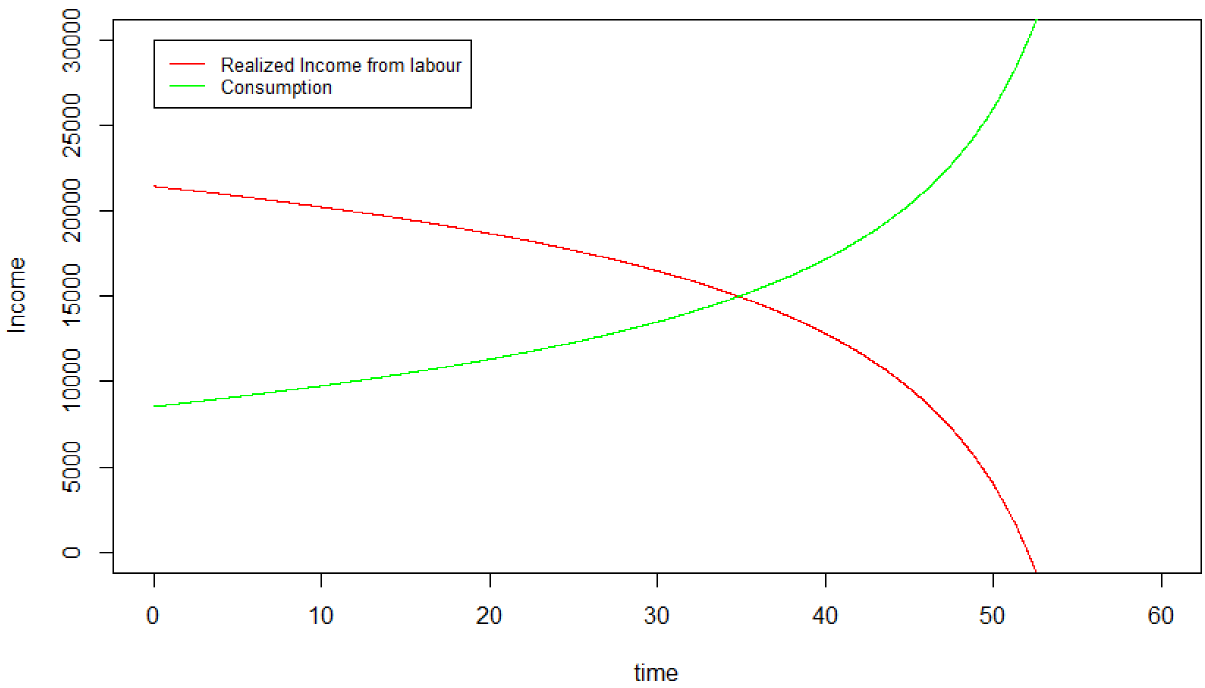


For a numerical example, consider an individual who starts his working career at time $t_0 = 0$ and lives to $T = 60$. This individual earns an instantaneous income rate of $Y = 30,000$ when he works full-time, and there is no change in wage level $Y_t = Y$. He starts with an initial financial wealth $F_0 = 50,000$, which is invested with a rate of return $r = 0.03$. His personal preference on consumption $\gamma = 0.1$, and on leisure $\beta = 0.1$. Further he is impatient compared to the market, with a discounting factor $\delta = 0.05$, and does not wish to leave a bequest $\epsilon = 0$.

In Figure 13 the development of total wealth and its components financial and human capital are portrayed. Remember that human capital (41) does not consider the chosen amount of leisure, but instead assumes full employment. Total wealth is affected by the chosen amount of leisure, as it is consumed by not realizing full employment income.

Figure 14 displays the realized income from labour $Y(1 - L_t)$ and the consumption level C_t. For the first 35 years there is excess income, which is invested into financial capital, after 35 years consumption dominates which is financed by the bank account. After 52 years of employment, L_t is larger than 1 and the individual has stopped working. As we consider an interior point solution, there is no boundary constraint which states that $L_t \leq 1$. This effectively means that time can be bought with money.

Figure 14: Income from realized labour $Y(1 - L_t)$ compared to consumption C_t.



D Pseudocode Dynamic Programming Solver

Algorithm 1 Find optimal decisions

$t^{grid} \leftarrow$ Set of discretized time points
$N \leftarrow$ Length of t_{grid}
$W^{grid} \leftarrow$ Set of discretized wealth levels
$M \leftarrow$ Length of W_{grid}

procedure FINDOPTIMALGRIDLEVELS()
    $\mathbf{F}, \mathbf{F}^R \leftarrow$Matrix(init=0, $\mathbb{R}^{M\times N}$)
    $\mathbf{C}, \mathbf{C}^R \leftarrow$Matrix(init=0, $\mathbb{R}^{M\times (N-1)}$)
    $\mathbf{L} \leftarrow$Matrix(init=0, $\{0,1\}^{M\times (N-1)}$)
    for i in $N-1, N-2, ..., 1$ **do**
        for j in $1, 2, ..., M$ **do**
            $W \leftarrow W_j^{grid}$ ▷ Wealth and time points for considered state
            $t_1 \leftarrow t_i^{grid}$
            $t_2 \leftarrow t_{i+1}^{grid}$
            ▷ Assuming agent is retired
            $C_{max} \leftarrow$ Maximal consumption such that $W_{next}(C_{max}, L=1) = 0$
            $C_{min} \leftarrow$ Maximal consumption such that $W_{next}(C_{min}, L=1) = W_{max}$
            $C_{min} \leftarrow \max\{C_{min}, 0\}$
            $opt^R \leftarrow$ HillClimb(DPequation($C=x, W=W, i=i, L=1$), $x \in [C_{min}, C_{max}]$)

            ▷ Assuming agent remains employed
            $C_{max} \leftarrow$ Maximal consumption such that $W_{next}(C_{max}, L=0) = 0$
            $C_{min} \leftarrow$ Maximal consumption such that $W_{next}(C_{min}, L=0) = W_{max}$
            $C_{min} \leftarrow \max\{C_{min}, 0\}$
            $opt \leftarrow$ HillClimb(DPequation($C=x, W=W, i=i, L=0$), $x \in [C_{min}, C_{max}]$)

            $\mathbf{C}_{j,i}^R \leftarrow opt^R$.arg ▷ Store optimal decisions:
            $\mathbf{F}_{j,i}^R \leftarrow opt^R$.max
            if opt^R.max $\geq$ opt.max **then**
                $\mathbf{C}_{j,i} \leftarrow opt^R$.arg
                $\mathbf{F}_{j,i} \leftarrow opt^R$.max
                $\mathbf{L}_{j,i} \leftarrow 1$
            else
                $\mathbf{C}_{j,i} \leftarrow opt$.arg
                $\mathbf{F}_{j,i} \leftarrow opt$.max
                $\mathbf{L}_{j,i} \leftarrow 0$
    return $(\mathbf{F}, \mathbf{F}^R, \mathbf{C}, \mathbf{C}^R, \mathbf{L})$

procedure DPEQUATION(C, W, i, L)
    $t_1 \leftarrow t_i^{grid}$
    $t_2 \leftarrow t_{i+1}^{grid}$
    $W_{next} \leftarrow W_{next}(C, L)$
    $val \leftarrow u_t(C, L) \cdot S(t_1) \cdot (e^{-\delta t_1} - e^{-\delta t_2})/\delta$
    if $L = 1$ **then**
        $val_F \leftarrow$ Interpolated value of $\mathbf{F}_{i+1}^R$ for W_{next}
    else
        $val_F \leftarrow$ Interpolated value of $\mathbf{F}_{i+1}$ for W_{next}
    return $val + val_F$

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