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Existence of Optimal Consumption Strategies in Markets with Longevity Risk

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Abstract

Survival bonds are financial instruments with a payoff that depends on human mortality rates. In markets that contain such bonds, agents optimizing expected utility of consumption and terminal wealth can mitigate their longevity risk. To examine how this influences optimal portfolio strategies and consumption patterns, we define a model in which the death of the agent is represented by a single jump process with Cox-Ingersoll-Ross intensity. This implies that our stochastic mortality rate is guaranteed to be nonnegative, in contrast to many other models in the literature. We derive explicit conditions for existence of an optimal solution in terms of model parameters by analyzing certain inhomogeneous Riccati equations. We find that constraints must be imposed on the market price of longevity risk to have a well-posed problem and we derive the optimal strategies when such constraints are satisfied.

Keywords: Optimal consumption, portfolio selection, longevity risk, CIR process, Laplace transform

1 INTRODUCTION

This paper investigates the optimal consumption and asset allocation of an investor in a market which contains financial assets and contracts that are sensitive to longevity risk. These contracts, with a payoff that depends on the realised mortality rate in a large population, can be thought of as insurance products that can help to mitigate the effects of changes in survival probabilities during the lifetime of the agent. There is at the moment no liquid market for such products but by introducing them in an asset allocation optimization problem we formulate a consistent and arbitrage-free way of analyzing the influence of the market price of longevity risk on investment behavior. The precise value of such a market

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price of risk may be difficult to estimate in practice, but it would not be realistic to put it at zero, especially for the analysis of retirement provisions.

The classical formulation of the optimal consumption and investment problems that we wish to consider here goes back to Merton (1969, 1971). In that setup, markets are complete and there is a fixed time period for investment and consumption. These results have been extended in many directions, by changing preferences as in Musiela and Zariphopoulou (2010) and Kraft et al. (2011), or by specifying different assumptions on asset price dynamics. Some authors, including Kraft (2005) and Chacko and Viceira (2005), have considered stochastic volatility processes for equity prices and solved the optimization problems for that case. Others have introduced more realistic fixed income markets by introducing stochastic interest rates. Many papers use Gaussian models for the short rate but Deelstra et al. (2000) and Kraft (2005, 2009) derive the optimal strategy for an agent maximizing power utility from terminal wealth by investing in a market with a short-rate following a Cox-Ingersoll-Ross (CIR) process, which will thus remain positive at all times.

If the investment horizon is uncertain, the optimal strategies need to be adjusted. We can distinguish between problems where the end of the investment period is chosen by the agent himself, such as the optimal stopping problem for flexible retirement in Dybvig and Liu (2010), and problems where the end of the investment period cannot be chosen by the agent. Models in which the death of the agent is included in the model form an obvious example of the latter category. In that case agents face a trade-off between obtaining an amount of utility now with certainty versus an amount of utility in the future which is uncertain due to both financial risk and mortality risk. Yaari (1965) seems to be the first to consider such consumption and investment problems in which the lifetime of the agent is stochastic. Under the assumption that the probability of death at a given age is known and constant in time, he solves the optimal investment problem in continuous time with uncertain lifetime by dynamic programming methods. The solution shows that one can interpret the deterministic mortality rate in terms of adjusted discount rates. Hakansson (1969) obtained similar results in a discrete time. Pliska and Ye (2007), building on previous work by Richard (1975), introduced life insurance as an extra asset for investment. They derive closed form solutions for the investment in stocks, bonds and life insurance products under the assumption that mortality rates are time-varying but deterministic and known a priori. Huang and Milevsky (2008) extended these results to HARA utility functions and include a stochastic income process. In Charupat and Milevsky (2002) and Milevsky and Young (2007) annuities instead of life insurance contracts are added to the asset mix.

All these models use time-varying but deterministic mortality rates. One of the first authors to study optimal consumption strategies when mortality rates are stochastic is Menoncin (2008). He solves the Hamilton-Jacobi-Bellman equation associated with the optimization problem of an investor who is exposed to a stochastic mortality rate while allowing for a general specification for the asset price dynamics; a longevity bond is available in the economy to mitigate the effects of this risk. Blanchet-Scalliet et al. (2008) allow the conditional distribution function of an agent's remaining time-horizon to be stochastic and correlated to asset returns[1]; to get closed form expressions, the rates which determine the random time at which the investor must liquidate the asset portfolio are allowed to become negative with positive probability. Huang et al. (2012) compare the optimal consumption strategy for an agent with a deterministic versus a stochastic mortality rate in a market where only

[1] See Maurer (2011) for another example of markets where mortality rates are correlated with asset returns.

the money market account is available for investment. They show that, compared to the deterministic case, agents increase or decrease their initial consumption in the stochastic case, depending on whether their constant risk aversion coefficient makes them more or less risk averse than an investor with a logarithmic utility function. In very recent work Shen and Wei (2014) solve an optimal consumption and asset allocation problem in a general diffusion framework under the premise that an exponential integrability condition is satisfied. Guambe and Kufakunesu (2015) generalize these results to Lévy processes.

As an extension of these results we provide a detailed analysis of conditions that guarantee existence of an optimal consumption strategy in a market where rates follow CIR dynamics. We will thus model mortality by a stochastic process which will almost surely stay positive at all times. Affine mortality rates lead to analytically tractable survival probabilities which remain smaller than one, as was noted earlier in work by Biffis (2005) and Schrager (2006); see also Dahl (2004). We include both financial assets and survival bonds in the asset mix and allow for a stochastic market price of mortality risk. The optimal consumption and investment strategies are derived in semi-closed form and we show that the hedging demand in our model is bounded. The impact of stochastic mortality on initial consumption is characterized in terms of the risk aversion coefficient.

The existence of a solution to the problem studied in the present paper depends on the asymptotic behaviour of the utility function for large values of wealth and the existence of moments of the state-price density. It was noted in Korn and Kraft (2004) and Kraft (2009) that not all moments of the state-price density in a model based on a CIR short rate may be well-defined. Therefore, the model studied in the present paper is different from the case considered in Shen and Wei (2014). We provide explicit conditions ensuring that the optimal investment and consumption problem is well-defined and a unique solution exists. By assuming that the market prices of risk are proportional to the square root of the short rate and the mortality rate (with a time-varying proportionality coefficient) we can derive such conditions by analyzing the existence and uniqueness of bounded solutions for certain inhomogeneous Riccati equations.

The remainder of this paper is organized as follows. Section 2 introduces the model for the economy. In Section 3 the investment problem is formulated and the main result of the paper is presented. Section 4 provides a characterization of the Laplace tranform of a Cox-Ingersoll-Ross process which is needed to prove the main result. The proof of the main result is given in Section 5. The economic implications of the model are studied in Section 6. Section 7 concludes.

2 MODEL FOR THE ECONOMY

In this Section we will construct a complete financial market in which asset returns, interest rates and mortality rates are uncertain. Let $(\Omega, \mathcal{G}, P)$ be a probability space on which a three-dimensional standard Brownian motion $W(t) = (W_1(t), W_2(t), W_3(t))'$ and an exponentially distributed random variable Θ are defined. We study the optimal investment and consumption strategy of an investor during the timespan $[0, T]$, for some $0 < T < \infty$, and we allow for the possibility that the investor does not survive until T. Let $\mathcal{F}(t)$ be the P-augmentation of the filtration generated by $W(t)$. The exponential random variable Θ is taken to be independent of $\mathcal{F}(T)$.

To model the time of death of the investor we introduce the nonnegative, $\mathcal{G}$-measurable random time

$$\tau \;=\; \inf\Big\{\, t \geq 0 \,:\, \int_0^t \lambda(u)\,du \;\geq\; \Theta \,\Big\}\;, \tag{2.1}$$

in which the so-called mortality rate $\lambda(t)$ follows a Cox-Ingersoll-Ross process, that is,

$$\lambda(t) \;=\; \lambda_0 \;+\; \int_0^t (\mu_2(u) - \kappa_2\,\lambda(u))\,du + \int_0^t \xi_2\sqrt{\lambda(u)}dW_2(u)\;. \tag{2.2}$$

The coefficients λ_0, κ_2, ξ_2 and μ_2 are assumed to be bounded, strictly positive, and μ_2 is continuously differentiable. The extended Feller condition[2]

$$2\,\mu_2(t) \;\geq\; \xi_2^2\,, \qquad \text{for all } t \in [0,T]\,, \tag{2.3}$$

ensures that the mortality rate is strictly positive almost surely.

Since $\int_0^t \lambda(u)\,du$ is $\mathcal{F}(t)$-measurable and Θ is independent of $\mathcal{F}(T)$, the survival probability of the agent satisfies

$$\overline{F}(t) \;:=\; P\,[\tau > t \mid \mathcal{F}(t)] \;=\; P\left[\int_0^t \lambda(u)\,du < \Theta \,\middle|\, \mathcal{F}(t)\right] \;=\; \exp\Big(-\int_0^t \lambda(u)\,du\Big)\;. \tag{2.4}$$

To finance his/her consumption and long-term wealth objectives the agent invests in a number of assets: a stock, a zero-coupon bond, a longevity bond and the money-market account.

The value $\beta(\cdot)$ of the money market account, based on the continuously compounded stochastic short rate $r(\cdot)$, satisfies

$$\beta(t) \;=\; 1 \;+\; \int_0^t \beta(u)r(u)du\;.$$

The short rate is assumed to follow a Cox-Ingersoll-Ross process, i.e.

$$r(t) \;=\; r_0 \;+\; \int_0^t (\mu_1(u) - \kappa_1\,r(u))\,du \;+\; \int_0^t \xi_1\sqrt{r(u)}dW_1(u)\;, \tag{2.5}$$

in which the coefficients r_0, κ_1, ξ_1 and μ_1 are bounded, strictly positive and μ_1 is continuously differentiable. By assuming that

$$2\,\mu_1(t) \;\geq\; \xi_1^2\,, \qquad \text{for } t \in [0,T]\,, \tag{2.6}$$

we enforce that the short rate almost surely remains strictly positive.

Given $S(0) > 0$, the stock price $S(\cdot)$ is assumed to follow the dynamics

$$S(t) \;\;=\;\; S(0) + \int_0^t \mu_S(u)S(u)du + \int_0^t \sigma_S(u)\,S(u)dW(u)\;,$$

in which

$$\mu_S(t) = r(u) + \sigma_S(t)\theta(t)'\;, \qquad \sigma_S(t) = (-\xi_3\rho\,\sqrt{r(t)},\,0,\,-\xi_3\overline{\rho})\;,$$

[2]See Theorem 4.2 in Maghsoodi (1996).

and

$$\theta(t) \quad = \quad \left(-\psi_1(t)\,\xi_1^{-1}\sqrt{r(t)},\ -\psi_2(t)\,\xi_2^{-1}\sqrt{\lambda(t)},\ -\psi_3(t)\,\xi_3^{-1}\right) ,$$

while ξ_i, for $i \in \{1,2,3\}$, are constants; $\rho \in]-1,1[$ is the instantaneous correlation between asset returns and interest rate risk; and $\overline{\rho}^2 = 1-\rho^2$. The market price of interest rate risk $\theta_1(t)$ is thus assumed to be proportional to the square root of the short rate (with a time-varying proportionality coefficient) and the market price of mortality risk $\theta_2(t)$ is taken to be proportional to the square root of the mortality rate. The deterministic scaling functions ψ_i are required to be continuous and bounded on the time horizon $[0,T]$.

A zero-coupon bond which pays one unit of currency at its expiration date T_1 is available for trading in the economy. We assume[3] that the expiration date of the bond lies beyond the investment horizon of the agent, i.e. $T_1 > T$. [4] The price process $P(t,T_1)$ of the zero-coupon bond at an earlier time $t \leq T_1$ satisfies, see Filipović (2009, Prop. 5.1),

$$P(t,T_1) \quad = \quad P(0,T_1) + \int_0^t P(u,T_1)\mu_P(u)du + \int_0^t P(u,T_1)\sigma_P(u,T_1)dW(u) ,$$

in which

$$\mu_P(t) = r(u) + \sigma_P(t)\theta(t)' , \qquad \sigma_P(t) = (-\xi_1 B_1(t,T_1)\sqrt{r(t)},\, 0,\, 0) ,$$

and the bond duration B_1 satisfies the differential equation

$$\begin{cases} -\partial_t B_1(t,T_1) + (\kappa_1 - \psi_1(t))\,B_1(t,T_1) + \frac{1}{2}\xi_1^2\,B_1(t,T_1)^2 \ = \ 1\,, \quad 0 \leq t \leq T_1\,, \\ B_1(T_1,T_1) \ = \ 0\,. \end{cases} \tag{2.7}$$

The economy that we wish to study permits agents to trade longevity risk via survival bonds.[5] Survival bonds are financial securities paying, at their expiration date T_1, an amount proportional to the expected fraction $\overline{F}(T_1) = \exp(-\int_0^{T_1}\lambda(u)du)$ of survivors at time T_1 among individuals in a (large) population with a common mortality rate process λ. Such bonds can be thought of as insurance products; they can be used by investors to hedge uncertainty about future survival probabilities. The price process $F(\cdot,T_1)$ of the survival bond satisfies

$$F(t,T_1) \quad = \quad F(0,T_1) + \int_0^t F(u,T_1)\mu_F(u)\,du + \int_0^t F(u,T_1)\sigma_F(u)dW(u) , \tag{2.8}$$

[3] This assumption is required to prevent that the volatility of the bond tends to zero near T which would lead to a singularity in the optimal asset allocation. For a discussion of the case $T = T_1$ the reader is referred to Bielecki et al. (2005).

[4] Instead of a long-term bond we could alternatively introduce a 'rolling bond' as a tradeable asset in the economy. This corresponds to a self-financing strategy which continuously invests in a bond with fixed expiry $T_2 > 0$. Using results in Rutkowski (1999) it can be shown that the price process $P^R(\cdot,T_2)$ of a rolling bond satisfies

$$P^R(t,T_2) \ = \ P^R(0,T_2) + \int_0^t P^R(u,T_2)\,r(u)\,du - \int_0^t \xi_1\,B_1(u,u+T_2)\,P^R(u,T_2)\,\sqrt{r(u)}\,dW_1(u) ,$$

in which B_1 is the solution to Eq. (2.7).

[5] Although currently there does not exist a liquid market for this type of securities, a number of mortality-linked instruments with notionals exceeding 10 billion EUR have traded recently. For a discussion and overview of trading in longevity instruments the reader is reffered to Blake et al. (2013).

where

$$\mu_F(t) = r(u) + \sigma_F(t)\theta(t)' \ , \qquad \sigma_F(t) = (-\xi_1 B_1(t,T_1)\sqrt{r(t)},\ -\xi_2 B_2(t,T_1)\sqrt{\lambda(t)},\ 0) \ .$$

The function $B_1(\cdot, T_1)$ solves Eq. (2.7), and $B_2(\cdot, T_1)$ is the solution to

$$\begin{cases} -\partial_t B_2(t,T_1) + (\kappa_2 - \psi_2(t))\, B_2(t,T_1) + \frac{1}{2}\xi_2^2\, B_2(t,T_1)^2 \ = \ 1 \ , \quad 0 \le t \le T_1 \ , \\ B_2(T_1,T_1) \ = \ 0 \, . \end{cases} \tag{2.9}$$

We will show later on, see Remark 5.2, that this indeed implies that $F(\cdot, T_1)$ is an asset price process which replicates the survival fraction at time T_1, i.e. that $F(T_1, T_1) = \overline{F}(T_1)$.

The investor continuously adjusts his/her portfolio of assets in order to achieve a consumption and terminal wealth profile that is optimal in a sense that will be defined shortly. The vector $\pi(t) = (\pi_1(t), \pi_2(t), \pi_3(t))$ denotes the amount invested, at time $t \in [0, T \wedge \tau]$, in zero-coupon bonds with maturity T_1, survival bonds maturing at T_1 and stocks respectively. The remaining wealth $\pi_0(t)$ is invested in the money-market account. This portfolio represents the savings of an agent who starts with initial wealth $x > 0$ at time $t = 0$. The nonnegative $\mathcal{F}(t)$-progressively measurable process $c(\cdot)$, satisfying $\int_0^T c(u)du < \infty$ a.s., represents the instantaneous consumption of the agent.

The wealth process $X^{x,c,\pi}(\cdot)$, corresponding to an asset allocation $\pi(\cdot)$, a nonnegative consumption process $c(\cdot)$ and initial endowment $x > 0$, is given by

$$\begin{aligned} X^{x,c,\pi}(t) \ &= \ x - \int_0^t c(u)du \ + \ \int_0^t \pi_0(u)\frac{d\beta(u)}{\beta(u)} \ + \ \int_0^t \pi_1(u)\frac{dP(u,T_1)}{P(u,T_1)} \\ &\quad + \int_0^t \pi_2(u)\frac{dF(u,T_1)}{F(u,T_1)} \ + \ \int_0^t \pi_3(u)\frac{dS(u)}{S(u)} \, . \end{aligned}$$

Expressed in terms of units in the money market account the wealth process satisfies

$$\frac{X^{x,c,\pi}(t)}{\beta(t)} \ = \ x \ - \ \int_0^t \frac{c(u)}{\beta(u)}\, du \ + \ \int_0^t \frac{\pi(u)}{\beta(u)} \mathscr{S}(u,T_1) \Big(dW(u) + \theta(u)'\, du \Big) \ , \tag{2.10}$$

where

$$\mathscr{S}(t,T_1) \ = \ (\sigma_P(t), \sigma_F(t), \sigma_S(t))' \ . \tag{2.11}$$

Remark 2.1. *Due to the extended Feller conditions (2.6) and (2.3) the volatility matrix (2.11) is nonsingular almost surely for $t \le T$.*

In order to exclude asset allocations leading to infinite wealth in finite time we restrict our attention to the subset of $\mathcal{F}(t)$-progressively measurable asset allocations $\pi(\cdot)$ satisfying

$$\int_0^T \big|\pi_0(t) + \pi_1(t) + \pi_2(t) + \pi_3(t)\big|\, r(t)\, dt \ < \ \infty \quad a.s. \ , \tag{2.12}$$

$$\int_0^T \big|\big|\pi(t)\mathscr{S}(t,T_1)\big|\big|^2\, dt \ < \ \infty \quad a.s. \ , \tag{2.13}$$

and

$$\int_0^T |\pi(t)\mathscr{S}(t,T_1)\theta(t)'|\, dt \ < \ \infty \quad a.s. \ , \tag{2.14}$$

An asset allocation for which conditions (2.12), (2.14) and (2.13) are met will be called a portfolio process.

In the subsequent Sections, we will formulate the investment problem and characterize its solution.

3 THE OPTIMAL INVESTMENT AND CONSUMPTION PROBLEM

In Section 2 we introduced a financial market in which the short rate and the mortality rate are both stochastic. In this Section we will define an optimal investment problem for an investor whose remaining lifetime is uncertain.

Definition 3.1 (Main problem)**.** *Let $x > 0$ be an amount of initial capital and let $T \in]0, T_1[$. We define the value function of the portfolio and consumption plan maximizing utility from consumption and terminal wealth on an uncertain time horizon as*

$$V(x) \;:=\; \sup_{(c,\pi)\in\mathscr{A}} \mathbb{E}\left[\int_0^{T\wedge\tau} \mathscr{U}_1(t, c(t))dt + \mathscr{U}_2(X^{x,c,\pi}(T))1_{\{\tau>T\}}\right] , \qquad (3.1)$$

in which, for certain $p \in (-\infty, 1) \setminus \{0\}$ and $\Upsilon \in [0, \infty)$,

$$\mathscr{U}_1(t, x) \;=\; \frac{x^p}{p} \qquad \text{and} \qquad \mathscr{U}_2(x) \;=\; \Upsilon\,\frac{x^p}{p} , \qquad (3.2)$$

or when $p = 0$,

$$\mathscr{U}_1(t, x) \;=\; \log x \qquad \text{and} \qquad \mathscr{U}_2(x) \;=\; \Upsilon \log x , \qquad (3.3)$$

and where

$$\mathscr{A} \;:=\; \big\{\, (c, \pi) \text{ is } \mathcal{F}(t)\text{-progressively measurable, satisfies (2.12) - (2.14)} , \; c \geq 0,\; X^{x,c,\pi} \geq 0 \,\big\} ,$$

denotes the set of admissible strategies. □

The above definition of admissible strategies ensures that $\int_0^T c(u)\,du < \infty$. Indeed, since wealth is required to be nonnegative on the one hand, and almost surely finite on the other hand, see the definition of a portfolio process in Eq. (2.12)-(2.14), it follows that the amount of wealth available for consumption is almost surely finite. Moreover, since for every admissible (c, π) the discounted gains are bounded from below, i.e.

$$P\left[\frac{1}{\beta(t)}\left(X^{x,c,\pi}(t) + \int_0^t c(u)du\right) \;\geq\; 0\right] \;=\; 1 ,$$

the definition of admissibility excludes doubling strategies on a finite time interval, see Karatzas and Shreve (1998, Def. 1.2.4).

The set $\mathscr{A}$ of admissible strategies is not empty and contains a strategy which yields more than negative infinite utility, so $V(x) > -\infty$ for all $x > 0$. Indeed, consider the strategy which allocates all initial wealth $x > 0$ to the money-market account, and continuously consumes wealth, at a rate of $\frac{1}{2}x/T$ per unit time. This strategy is admissible since $\mathscr{U}_1(t, \frac{1}{2}x/T) > -\infty$ for all $t \in [0, T]$ and $\mathscr{U}_2(X^{x,c,\pi}(T)) \geq \mathscr{U}_2(\frac{1}{2}x) > -\infty$.

Using a dual approach, see for example Cox and Huang (1989) and Karatzas and Shreve (1998), we will solve the problem from Definition 3.1 and prove the following result.

Theorem 3.2 (Main result)**.** *Assume that the scaling functions for the market price of risk* ψ_i, $i = 1, 2$, *are analytic on* $[0, T]$. *Then the problem from Definition 3.1 has a solution which is unique up to almost-everywhere equivalence when* $p \leq 0$. *When* $p \in]0, 1[$ *this is still the case if the parameters of the short rate and mortality rate processes satisfy*

$$\min_{s\in[0,T]} \left(\kappa_i + \tfrac{p}{1-p}\psi_i(s)\right) > 0 \tag{3.4}$$

for $i = 1, 2$ *and*

$$p\,[\ (\kappa_1 - \psi_{1,min})^2 \ + \ \tfrac{(\psi_1^2)_{max} - (\psi_{1,min})^2}{1-p}\ ] \ < \ \kappa_1^2 - 2p\xi_1^2\ , \tag{3.5}$$

$$p\,[\ (\kappa_2 - \psi_{2,min})^2 \ + \ \tfrac{(\psi_2^2)_{max} - (\psi_{2,min})^2}{1-p}\ ] \ < \ \kappa_2^2 + 2\xi_2^2\ , \tag{3.6}$$

where

$$\psi_{i,min} = \min_{s\in[0,T]} \psi_i(s), \qquad (\psi_i^2)_{max} = \max_{s\in[0,T]} \psi_i(s)^2.$$

The optimal wealth process $X(\cdot)$ *satisfies*

$$\frac{d\left(X(t) + \int_0^t c(u)du\right)}{X(t)} \ = \ \eta_0(t)\frac{d\beta(t)}{\beta(t)} + \eta_1(t)\,\frac{dP(t,T_1)}{P(t,T_1)} + \eta_2(t)\,\frac{dF(t,T_1)}{F(t,T_1)} + \eta_3(t)\frac{dS(t)}{S(t)}\ ,$$

in which the observable[6] *quantities* $\eta_i(\cdot)$, $i = 0, \ldots, 3$, *defined in Eqs. (5.35) – (5.38), represent the fraction of wealth invested in each of the asset classes. Moreover, the optimal strategy comprises consumption at the rate* $c(t) = X(t)/n(t)$ *at each time* t *during the life* $[0, \tau)$ *of the agent, where* $n(\cdot)$, *defined in Eq. (5.22), is an observable process.*

The expression for the optimal asset allocation in Theorem 3.2 is explicit up to the solution of a Riccati equation. In the special case where the parameters of the short rate and the mortality rate process are (piecewise) constant, these Riccati equations admit a closed-form solution which is given in Lemma 4.4.

Notice that Theorem 3.2 is not a special case of the results in Shen and Sherris (2014) and Shen and Wei (2014) or the extension of these results in Guambe and Kufakunesu (2015). The analysis in these papers relies on an exponential integrability condition which is not satisfied in general for our model, see also Korn and Kraft (2004, Prop. 3.2).

4 THE LAPLACE TRANSFORM OF A CIR PROCESS

To prove our main theorem, we first derive properties of the Laplace transform of an integrated Cox-Ingersoll-Ross process which is scaled by a deterministic function $h(t)$. These results will facilitate the computation of semi-closed form expressions for the optimal consumption and investment strategy and they enable us to obtain explicit conditions under which the investment problem from Section 3 has a unique solution.

We start by establishing, in Lemma 4.1, that under appropriate conditions, the Laplace transform of an (integrated) Cox-Ingersoll-Ross process on a finite time-interval has an

[6] A variable is said to be 'observable' if its value at any time t in $[0, T]$ is completely determined by known prices of tradeable assets (current and historical) at that time.

affine representation. A well-known application of this result is the pricing of bonds in a CIR short rate model, which corresponds to the choice $h(t) \equiv 1$ in Lemma 4.1. We are however interested in the case where $h(t)$ is time-varying and possibly negative. Moreover, we will not constrain the mean-reversion speed to be nonnegative. We thus extend results in Pitman and Yor (1982), Kraft (2003), Wong and Heyde (2006) and Gnoatto and Grasselli (2013) to the case of time-varying parameters and bounded, but possibly negative, mean-reversion speed.

Lemma 4.1. *Let h, κ, μ and ξ be bounded and continuous functions $[0,T] \to \mathbb{R}$ satisfying $\mu(t) \geq 0$ and $\xi(t) > 0$ for $0 \leq t \leq T$. Suppose that $r(\cdot)$ follows a Cox-Ingersoll-Ross process*

$$r(t) \;=\; r_0 \;+\; \int_0^t (\mu(s) - \kappa(s)r(s))\, ds \;+\; \int_0^t \xi(s)\sqrt{r(s)}\, d\bar{W}(s)\,, \tag{4.1}$$

in which $\bar{W}$ is standard Brownian motion and r_0 is a strictly positive constant. For $\alpha \in \mathbb{R}$ and $0 \leq t \leq T$, consider the Laplace transform

$$g(t,T,x) \;=\; \mathbb{E}\left[\exp\left\{-\alpha r(T) - \int_t^T h(s)r(s)\, ds\right\} \,\middle|\, r(t) = x\right]. \tag{4.2}$$

If the Riccati equation

$$\begin{cases} \partial_t B(t,T) \;=\; \mathcal{L}(t, B(t,T))\,, & 0 \leq t \leq T\,, \\ B(T,T) \;=\; \alpha\,, \end{cases} \tag{4.3}$$

in which

$$\mathcal{L}(t,x) \;=\; \kappa(t)x + \frac{1}{2}\xi^2(t)x^2 - h(t)\,, \tag{4.4}$$

as well as the differential equation

$$\begin{cases} \partial_t A(t,T) \;=\; -\mu(t)B(t,T)\,, & 0 \leq t \leq T\,, \\ A(T,T) \;=\; 0\,, \end{cases} \tag{4.5}$$

have bounded solutions $A(\cdot,T)$ and $B(\cdot,T)$, then the Laplace transform $g(t,T,r)$ has an affine representation

$$g(t,T,r) \;=\; e^{-A(t,T)-B(t,T)r}\,. \tag{4.6}$$

The previous lemma provides sufficient conditions for finiteness of the Laplace transform, but it is shown in Korn and Kraft (2004, Prop. 3.2) that in the special case where $h(t) = \bar{h}$ for some $\bar{h} < 0$, there exist constants μ and ξ, for any given constant mean reversion speed κ, such that the Laplace transform in Lemma 4.1 is infinite. Conversely, given a set of constant CIR parameters there is a number $\bar{h} < 0$ such that the Laplace transform in Lemma 4.1 with $h(t) = \bar{h}$ is infinite.

Before proving Lemma 4.1, we need the following result which, for the case of nonnegative mean-reversion speed, is due to Shirakawa (2002).

Lemma 4.2. *Let $\phi : [0,T] \to \mathbb{R}$ be a continuous and bounded function and let the process $r(\cdot)$ be defined as in Eq. (4.1). Assume that μ, κ and ξ are continuous and bounded functions and that $\xi(t) > 0$ and $\mu(t) \geq 0$ for $0 \leq t \leq T$. The stochastic exponential*

$$Z(t) \;=\; \exp\left\{\int_0^t \phi(s)\sqrt{r(s)}d\bar{W}(s) - \frac{1}{2}\int_0^t \phi^2(s)r(s)\, ds\right\}\,, \tag{4.7}$$

is a martingale for $0 \le t \le T$*.*

Proof:

Since κ is bounded we have that $\kappa(t) > -m$ for some $m > 0$. Define $\widetilde{r}(t) = e^{-mt} r(t)$. It follows from Itô's lemma that

$$\begin{aligned} d\widetilde{r}(t) &= -me^{-mt} r(t)dt + e^{-mt} dr(t) \\ &= \left[\widetilde{\mu}(t) - \widetilde{\kappa}(t) r(t)\right] dt + \widetilde{\xi}(t)\sqrt{r(t)} d\bar{W}(t) , \end{aligned}$$

where $\widetilde{\mu}(t) = \mu(t)\, e^{-mt}$, $\widetilde{\kappa}(t) = (\kappa(t) + m)\, e^{-mt} > 0$ and $\widetilde{\xi}(t) = \xi(t)\, e^{-mt}$ are all bounded and continuous on $[0, T]$. Hence $\widetilde{r}(t)$ is a CIR process with strictly positive mean-reversion speed. Observe that

$$dZ(t)/Z(t) = \phi(t)\sqrt{r(t)} d\bar{W}(t) = \widetilde{\phi}(t)\sqrt{\widetilde{r}(t)} d\bar{W}(t) ,$$

in which $\widetilde{\phi}(t) = \phi(t) e^{\frac{1}{2} mt}$ is a bounded function on $[0, T]$. It thus remains to prove the result for $\kappa > 0$. A proof for this case can be found in Shirakawa (2002, Thm. 3.2). □

Using Lemma 4.2 we can now proceed to establish the affine representation of the Laplace transform (4.2) and prove Lemma 4.1.

Proof of Lemma 4.1:

By assumption of the lemma there exist bounded functions $A(\cdot, T)$ and $B(\cdot, T)$ satisfying the Riccati equations (4.3) and (4.5) on the interval $[0, T]$. Define

$$f(t, x) = e^{-\int_0^t h(s) r(s) ds - A(t,T) - B(t,T) x} ,$$

and observe that

$$f(T, r(T)) = e^{-\alpha r(T) - \int_0^T h(s) r(s) ds} .$$

If the process $f(t, r(t))$ is a martingale, i.e.

$$\mathbb{E}\left[f(T, r(T)) \mid \mathcal{F}_t \right] = f(t, r(t)) ,$$

then it follows that

$$\mathbb{E}\left[e^{-\alpha r(T) - \int_0^T h(s) r(s) ds} \,\middle|\, \mathcal{F}_t \right] = e^{-\int_0^t h(s) r(s) ds - A(t,T) - B(t,T) r(t)} ,$$

or, equivalently,

$$g(t, T, r(t)) = e^{-A(t,T) - B(t,T) r(t)} .$$

We conclude that, if $f(t, r(t))$ is a martingale, then the Laplace transform has the affine representation as stated in the lemma.

Applying Itô's lemma to $f_t := f(t, r(t))$ yields

$$df_t = \left[-h(t) r(t) - \partial_t A(t, T) - \partial_t B(t, T) r(t)\right] f_t\, dt - B(t, T) f_t dr(t) + \frac{1}{2} B(t, T)^2 f_t d\langle r, r\rangle_t .$$

Substituting the dynamics of $r(\cdot)$ we find

$$\begin{aligned} \frac{df_t}{f_t} &= \left[-h(t) - \partial_t B(t, T) + \kappa(t) B(t, T) + \frac{1}{2}\xi^2(t) B(t, T)^2 \right] r(t)\, dt \\ &\quad - \left[\partial_t A(t, T) + \mu(t) B(t, T) \right] dt - B(t, T)\xi(t)\sqrt{r(t)}\, d\bar{W}(t) . \end{aligned}$$

Hence, if $B(\cdot,T)$ solves the Riccati equation (4.3) and $A(\cdot,T)$ satisfies (4.5) then $f_t = f(t,r(t))$ is a martingale provided that the local martingale solving

$$\frac{df_t}{f_t} \;=\; -B(t,T)\xi(t)\sqrt{r(t)}\,d\bar{W}(t)\;, \tag{4.8}$$

is a true martingale. This follows from Lemma 4.2 and the assumption that that $B(\cdot,T)$ and $\xi(t)$ are bounded.

□

In order to apply Lemma 4.1 we need conditions ensuring existence of a bounded solution to the inhomogeneous Riccati equation (4.3). In Lemmata 4.5 and 4.6 we will provide such conditions for the case where $\xi(t) = \bar{\xi}$ for some $\bar{\xi} \in]0,\infty[$. First we need the following comparison result which will be used to extend a local solution of a Riccati equation to a solution on the interval $[0,T]$.

Lemma 4.3. *Let $G\colon [0,\infty[\times\mathbb{R} \to \mathbb{R}$ be a function which is locally Lipschitz in its second argument. Consider, for $t \in [T',T]$, the differential equation*

$$\partial_t y(t) \;=\; G(t,y(t))\;,\quad y(T)=0\;,$$

and let $x(t)$ and $z(t)$ be bounded functions satisfying the differential inequalities

$$\partial_t x(t) \;\le\; G(t,x(t))\;,\quad x(T)=0\;,$$

and

$$\partial_t z(t) \;\ge\; G(t,z(t))\;,\quad z(T)=0\;.$$

Then $z(t) \le y(t) \le x(t)$ on $[T',T]$.

Proof: See Walter (1998, p. 96). □

The following Lemma provides an explicit solution to the Riccati equations (4.3)–(4.5) for the case where the functions h and κ are constant. More general results appear in Kraft (2003) and Maghsoodi (1996).

Lemma 4.4. *Suppose that $\xi(t) = \bar{\xi}$, $\kappa(t) = \bar{\kappa}$, $\mu(t) = \bar{\mu}$ and $h(t) = \bar{h}$ for some $\bar{\xi},\bar{\kappa},\bar{\mu},\bar{h} \in \mathbb{R}$ such that*

$$-\,2\,\bar{\xi}^2\bar{h} \;<\; \bar{\kappa}^2\;, \tag{4.9}$$

and

$$\bar{\kappa} + a \;>\; 0\;, \tag{4.10}$$

where $a = \sqrt{\bar{\kappa}^2 + 2\bar{h}\bar{\xi}^2}$. Then the (real-valued) solution of the Riccati equations (4.3)–(4.5) is given by

$$A(t,T) \;=\; -\frac{\bar{\mu}}{\bar{\xi}^2}\left[(\bar{\kappa}-a)\,(T-t) - 2\log\left(\frac{1-\nu e^{-a(T-t)}}{1-\nu}\right)\right]\;, \tag{4.11}$$

$$B(t,T) \;=\; 2\bar{h}\left[\bar{\kappa} + a\coth\left(\frac{1}{2}\,a(T-t)\right)\right]^{-1}\;, \tag{4.12}$$

and $\nu \;=\; (\bar{\kappa}-a)\,/\,(\bar{\kappa}+a)$. The functions $A(\cdot,T)$ and $B(\cdot,T)$ are bounded and are continuously differentiable with bounded first-order derivatives.

Proof: It is straightforward to verify that A and B solve Eq. (4.3) and (4.5). Observe that condition (4.9) ensures that A and B are real-valued. Since $x \mapsto \coth(x)$ is decreasing and bounded from below by 1 on $[0,T]$, it follows that B is monotone and bounded on $[0,T]$. Condition (4.10) implies that $\nu < 1$ so A is also bounded on $[0,T]$. Since $A(\cdot,T)$ and $B(\cdot,T)$ satisfy (by definition) Eq. (4.3) and (4.5), it follows that their first-order derivatives are continuous and bounded. □

Let $h_{min} = \min_{0\le s\le T} h(s)$, $h_{max} = \max_{0\le s\le T} h(s)$, and $\kappa_{min} = \min_{0\le s\le T} \kappa(s)$. Consider the family of Riccati equations

$$\begin{cases} \partial_t \varphi_L(t,u) \;=\; \kappa_{min}\varphi_L(t,u) + \frac{1}{2}\bar{\xi}^2\varphi_L(t,u)^2 - \min(0,\, h_{min}) \;, & t\in[0,u]\,, \\ \varphi_L(u,u) = 0 \end{cases} \tag{4.13}$$

and the family

$$\begin{cases} \partial_t \varphi_U(t,u) \;=\; \kappa_{min}\varphi_U(t,u) + \frac{1}{2}\bar{\xi}^2\varphi_U(t,u)^2 - \max(0,\, h_{max}) \;, & t\in[0,u]\,, \\ \varphi_U(u,u) \;=\; 0 \;, \end{cases} \tag{4.14}$$

both indexed by $u \in [0,T]$. The next Lemma establishes a lower bound and an upper bound for solutions of the Riccati equation (4.3) for the case where $\alpha = 0$ and $\xi(t) = \bar{\xi}$ for some $\bar{\xi} \in]0,\infty[$. Note that we explicitly allow for the possibility that the source term $h(t)$ of the Riccati equation (4.3) changes sign.

Lemma 4.5. *Suppose that h is bounded and either equals zero or changes sign finitely many times, and that*

$$h_{min} > 0 \qquad \text{or} \qquad \kappa_{min} > \sqrt{-2\bar{\xi}^2 h_{min}}\;. \tag{4.15}$$

If on the interval $(T',T]$ the function $B(\cdot,T)$ is a solution to the Riccati equation (4.3) with $\alpha = 0$ and $\xi(t) = \bar{\xi}$ for some $\bar{\xi} \in]0,\infty[$, then

$$\varphi_L(t,T) \;\le\; B(t,T) \;\le\; \varphi_U(t,T)\;,$$

for $t \in (T',T]$.

Proof:

Condition (4.15) implies that Eqs. (4.9) and (4.10) are satisfied for the values $\bar{h} = h_{min}$ and $\bar{\kappa} = \kappa_{min}$. Indeed, $\bar{h} > 0$ implies (4.9) is trivially satisfied while $\bar{\kappa} + a = \bar{\kappa} + \sqrt{\bar{\kappa}^2 + 2\bar{h}\bar{\xi}^2} > \bar{\kappa} + |\bar{\kappa}| \ge 0$. If $\bar{h} \le 0$ then $\bar{\kappa} + a = \bar{\kappa} + \sqrt{\bar{\kappa}^2 + 2\bar{h}\bar{\xi}^2} \ge \bar{\kappa} > 0$ so (4.10) is also satisfied and (4.9) is satisfied by construction. It thus follows from Lemma 4.4 that the solutions $\varphi_L(t,u)$ and $\varphi_U(t,u)$ to the Riccati equations (4.13) and (4.14) exist and are bounded on $[0,u]$ for all $0 \le u \le T$.

If $B(\cdot,T)$ changes sign at t, then $\partial_t B(t,T) = -h(t)$. Hence on intervals where $h(t)$ does not change sign, $B(\cdot,T)$ can change sign only once. Since, by assumption, the function $h(t)$ changes sign only finitely many times, it follows that $B(\cdot,T)$ changes sign only finitely many times. Therefore we may partition the interval $(T',T]$ into a finite number of subintervals

$$(t_0,t_1],\; (t_1,t_2],\; \ldots,\; (t_{N-1},t_N]\;,$$

with $t_0 = T'$, $t_N = T$ and such that $B(t_i,T) = 0$ while the function $B(\cdot,T)$ has constant sign on each of these subintervals.

Take $1 \leq i \leq N$ and consider the interval $(t_{i-1}, t_i]$. Assume that $B(\cdot, T)$ is negative on $(t_{i-1}, t_i]$. By Eq. (4.12) we have that $\varphi_L(t, t_i)$ is also negative on that interval, so

$$\partial_t \varphi_L(t, t_i) \;\geq\; \mathcal{L}(t, \varphi_L(t, t_i)) \;, \tag{4.16}$$

with $\mathcal{L}$ as defined in Eq. (4.4), while

$$\partial_t B(t, T) \;=\; \mathcal{L}(t, B(t, T)) \;, \tag{4.17}$$

for $t \in (t_{i-1}, t_i]$ and $B(t, T) = \varphi_L(t, t_i)$ at the endpoint $t = t_i$. Hence by Lemma 4.3 the solution $B(t, T)$ is bounded from below by $\varphi_L(t, t_i)$ on $(t_{i-1}, t_i]$. Note that $\mathcal{L}$ is locally Lipschitz in its second argument. We also see that $\varphi_L(t, T)$ is decreasing in T by (4.12), so it follows that

$$\varphi_L(t, T) \;\leq\; \varphi_L(t, t_i) \;\leq\; B(t, T) \;\leq\; 0 \;,$$

for $t \in (t_{i-1}, t_i]$.

Now consider the case where $B(\cdot, T)$ is positive on $(t_{i-1}, t_i]$. We have that $\varphi_U(t, t_i) \geq 0$ by (4.12) so

$$\partial_t \varphi_U(t, t_i) \;\leq\; \mathcal{L}(t, \varphi_U(t, t_i)) \;, \tag{4.18}$$

for $t \in (t_{i-1}, t_i]$. Hence by Lemma 4.3 the solution $B(t, T)$ is bounded from above by $\varphi_U(t, t_i)$ on $(t_{i-1}, t_i]$. Since $\varphi_U(t, T)$ is increasing in T, see Lemma 4.4, it follows that

$$0 \;\leq\; B(t, T) \;\leq\; \varphi_U(t, t_i) \;\leq\; \varphi_U(t, T) \;,$$

for $t \in (t_{i-1}, t_i]$.

Since $\varphi_L(t, T) \leq 0$ and $\varphi_U(t, T) \geq 0$ on all intervals $(t_{i-1}, t_i]$ we see that $\varphi_L(t, T) \leq B(t, T) \leq \varphi_U(t, T)$ on all these intervals, which proves the result as stated. □

The following Lemma establishes conditions under which the Laplace transform Eq. (4.2) has a finite-valued solution.

Lemma 4.6. *Suppose that $\xi(t) = \bar{\xi}$ for some $\bar{\xi} \in]0, \infty[$. If the conditions of Lemma 4.5 hold, then the (unique) solution $B(t, T)$ to the Riccati equation*

$$\begin{cases} \partial_t B(t, T) = \mathcal{L}(t, B(t, T)) \;, & 0 \leq t \leq T \;, \\ B(T, T) = 0 \;, \end{cases} \tag{4.19}$$

in which $\mathcal{L}$ is defined as in Eq. (4.4), exists and is bounded on $[0, T]$. Moreover $B(t, T)$ is continuously differentiable and $\partial_t B(t, T)$ is bounded.

Proof:

If we can show that a solution $B(t, T)$ exists and is bounded for all $t \in [0, T]$, then we immediately have that $\partial_t B(t, T)$ is bounded for all such t due to Eq. (4.19).

The right-hand side of the Riccati equation (4.19) is continuous (since h and κ are continuous) and locally Lipschitz continuous in the second argument with a Lipschitz constant which does not depend on time. From Picard-Lindelöf, see Teschl (2012, Thm. 2.2), we know that there exists a unique solution $B(\cdot, T)$ to the Riccati differential equation on the interval $(T', T]$ for some $T' < T$.

Suppose that $(T', T]$ is the largest interval on which a solution to (4.3) exists. We will argue that

$$\lim_{s \downarrow T'} B(s, T) \ , \tag{4.20}$$

exists. From Lemma 4.5 we know that B is bounded on $(T', T]$, therefore both $a = \limsup_{s\downarrow T'} B(s,T)$ and $b = \liminf_{s\downarrow T'} B(s,T)$ exist and are finite. Suppose that $a > b$. There exist sequences $a_n \downarrow T'$ and $b_n \downarrow T'$ with $B(a_n, T) \to a$ and $B(b_n, T) \to b$. There also exists an $\bar{L} \geq 0$ such that $|\mathcal{L}(t, B(t,T))| = |\kappa(t)B(t,T) + \frac{1}{2}\bar{\xi}^2 B(t,T)^2 - h(t)| \leq \bar{L}$ for all $t \in]T', T]$ because of Lemma 4.5 and the continuity of φ_L, φ_U, κ and h. But then $|B(b_n, T) - B(a_n, T)| \leq \bar{L}|b_n - a_n|$ which gives a contradiction for $n \to \infty$. Hence $a = b$ and the limit in (4.20) exists. We can thus apply Picard-Lindelöf again and extend the interval of existence beyond $(T', T]$. This contradicts the assumption that $(T', T]$ is the largest interval of existence.

□

5 THE OPTIMAL CONSUMPTION AND INVESTMENT STRATEGY

In this Section we will prove Theorem 3.2. First, to show that that the market introduced in Section 2 does not admit arbitrage, we will construct an equivalent measure $\widetilde{P}$ such that all assets, expressed in units of the money-market account, are $\widetilde{P}$-martingales.

Lemma 5.1. *The stochastic exponential*

$$Z_0(t) \ = \ \exp\left\{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t ||\theta(u)||^2 du\right\} \tag{5.1}$$

defines a Radon-Nikodym derivative and the associated measure

$$\widetilde{P}(A) \ = \ \mathbb{E}_P\left[Z_0(T)1_A\right] \ , \quad \text{for all } A \in \mathcal{F}(T) \ , \tag{5.2}$$

is a martingale measure for the economy, i.e. β-discounted prices of all tradeables are martingales under this measure.

Proof: Using Lemma 4.2 together with the boundedness of the functions $t \mapsto \xi_i B_j(t, T_1)$, $i, j = 1, 2$, and $t \mapsto \psi_i(t)/\xi_i$, $i = 1, 2$, it can be shown that β-discounted prices of tradeables are $\widetilde{P}$-martingales. A detailed proof is given in Appendix A. □

From Lemma 5.1 we conclude that the economy does not admit arbitrage; and it follows from Remark 2.1 that the economy introduced in Section 2 is complete so that the martingale measure defined in Lemma 5.1 is unique, see Karatzas and Shreve (1998, Thm. 1.4.2, Thm. 1.6.6. and Prop. 1.7.4).

Remark 5.2. *From a standard result in arbitrage theory*[7] *the price of a survival bond, which pays the amount $\overline{F}(T_1)$ at time T_1, is given by*

$$F(t, T_1) \ = \ \beta(t)\, \mathbb{E}_{\widetilde{P}}\left[\beta(T_1)^{-1}\, \overline{F}(T_1) \,\middle|\, \mathcal{F}(t)\right] \ . \tag{5.3}$$

[7] See for example Karatzas and Shreve (1998, Prop. 2.2.3).

The survival distribution $\overline{F}(\cdot)$ is bounded, hence we may apply the Feynman-Kac theorem[8] *to express the price of the survival bond in terms of the solution to the Riccati differential equation (2.9). We conclude that the price process (2.8) of the survival bond coincides with the replication cost of its payoff.* □

Applying Itô's lemma to the product of the state price density $H(t) = Z_0(t)/\beta(t)$ and the wealth process $X^{x,c,\pi}(\cdot)$ yields

$$H(t)X^{x,c,\pi}(t) + \int_0^t H(u)c(u)du \;=\; x + \int_0^t H(u)[\mathscr{S}(u,T_1)\pi(u) - X^{x,c,\pi}(u)\theta(u)]' dW(u)\,, \tag{5.4}$$

for $0 \le t \le T$. The left-hand side of this expression represents the total (discounted) wealth of the investor, including the wealth that has been consumed, expressed in terms of the money-market account. The definition of admissible strategies requires the wealth process and consumption to be nonnegative. Consequently every admissible strategy (c, π) satisfies the budget constraint

$$\mathbb{E}\left[H(T)X^{x,c,\pi}(T) + \int_0^T H(u)c(u)du\right] \;\le\; x\,. \tag{5.5}$$

Indeed, since the left-hand side of Eq. (5.4) is nonnegative if $(c, \pi) \in \mathscr{A}$, while the right-hand side is a local martingale, it follows that the right-hand side of Eq. (5.4) is a supermartingale. Taking expectations in Eq. (5.4) thus leads to the budget constraint (5.5); this inequality plays an essential role in the verification of the candidate optimal solution in the proof of Lemma 5.4.

The stopping time τ, which models the time of death of the investor, is not adapted to the filtration $\mathcal{F}(t)$ since it depends on both the mortality rate and the exponentially distributed random variable Θ. Using iterated conditioning the following alternative representation of the value function can be established.

Lemma 5.3. *The value function equals*

$$V(x) \;=\; \sup_{(c,\pi)\in\mathscr{A}} \mathbb{E}\left[\int_0^T U_1(t, c(t))dt + U_2(X^{x,c,\pi}(T))\right] \tag{5.6}$$

in which $U_1(t,x) = \overline{F}(t)\mathscr{U}_1(t,x)$ and $U_2(x) = \overline{F}(T)\mathscr{U}_2(x)$.

Proof: See Appendix A. □

The following lemma, the proof of which is adapted from Karatzas and Shreve (1998), provides conditions ensuring that the problem from Definition 3.1 has a unique solution. Let $I_1\colon [0,T]\times(0,\infty] \to [0,\infty)$ and $I_2\colon (0,\infty] \to [0,\infty)$, defined by

$$I_1(t,y) \;=\; \overline{F}(t)^{\frac{1}{1-p}}\, y^{\frac{1}{p-1}} \qquad \text{and} \qquad I_2(y) \;=\; \Psi\, \overline{F}(T)^{\frac{1}{1-p}}\, y^{\frac{1}{p-1}}\,, \tag{5.7}$$

with $\Psi = \Upsilon^{\frac{1}{1-p}}$, be the inverses of the marginal utility functions $c \mapsto \partial_c U_1(t,c)$ and $x \mapsto \partial_x U_2(x)$ respectively.

[8] See Karatzas and Shreve (1991, Thm. 7.6).

Lemma 5.4. *Let $x > 0$ be an amount of initial wealth. There exists an optimal strategy $(c, \pi) \in \mathscr{A}$ for Problem 3.1 if there is a $v(x) \in [0, \infty[$ such that*

$$w(0, v(x)) \;=\; x\;, \tag{5.8}$$

where

$$w(t,y) \;=\; \frac{1}{H(t)}\mathbb{E}\left[\int_t^T H(s)I_1(s, yH(s))ds + H(T)I_2(yH(T))\middle|\,\mathcal{F}(t)\right]\;.$$

The optimal consumption strategy $c(\cdot)$ and optimal wealth process $X(\cdot)$ are given by

$$c(t) \;=\; I_1(t, v(x)H(t))\;, \tag{5.9}$$

and

$$X(t) \;=\; w(t, v(x))\;. \tag{5.10}$$

If $V(x) < \infty$ then the optimal strategy is unique up to almost-everywhere equivalence.

Proof:

First we will construct an investment strategy $\pi(\cdot)$ such that the wealth process $X^{x,c,\pi}(\cdot)$, with consumption $c(\cdot)$ as in Eq. (5.9), replicates the (nonnegative) candidate optimal wealth process Eq. (5.10) and, in particular, leads to the terminal wealth

$$\zeta \;=\; w(T, v(x)) \;=\; I_2(v(x)H(T))\;. \tag{5.11}$$

Let $M(t) \;=\; H(t)X(t) + \int_0^t H(u)\,c(u)\,du$, and note that

$$M(t) \;=\; \mathbb{E}\left[\int_0^T H(t)I_1(t, v(x)H(t))dt + H(T)I_2(v(x)H(T))\middle|\,\mathcal{F}(t)\right]\;.$$

Since $\mathbb{E}|M(t)| = w(0, v(x)) = x < \infty$ we conclude that $M(\cdot)$ is a Doob martingale. The martingale representation theorem[9] guarantees the existence of an adapted process $\gamma(\cdot)$ satisfying $\int_0^T ||\gamma(u)||^2 du < \infty$ and

$$M(t) \;=\; x + \int_0^t \gamma(u)dW(u)\;.$$

The reciprocal of the process $Z_0(\cdot)$ defined in (5.1) satisfies

$$Z_0^{-1}(t)\theta(t)d\widetilde{W}(t) \;=\; Z_0^{-1}(t)\theta(t)(dW(t) + \theta'(t)dt) \;=:\; d(Z_0^{-1}(t))\;.$$

Applying Itô's lemma we find

$$\begin{aligned}
d\left(\frac{X(t)}{\beta(t)}\right) \;&=\; Z_0^{-1}(t)d(H(t)X(t)) + H(t)X(t)d(Z_0^{-1}(t)) + d\left\langle H(t)X(t), Z_0^{-1}(t)\right\rangle_t \\
&=\; Z_0^{-1}(t)\Big(\gamma(t)d\widetilde{W}(t) - H(t)c(t)dt\Big) + H(t)X(t)\theta(t)Z_0^{-1}(t)d\widetilde{W}(t) \\
&=\; -\frac{1}{\beta(t)}c(t)dt \\
&\quad + \frac{1}{\beta(t)}\frac{1}{H(t)}\left[\gamma(t) + \Big(M(t) - \int_0^t H(u)c(u)du\Big)\theta(t)\right]d\widetilde{W}(t)\,.
\end{aligned} \tag{5.12}$$

[9]See Karatzas and Shreve (1991, pp. 182–184).

Comparing this expression to Eq. (2.10) we find that if we choose the asset allocation

$$\pi(t) \;=\; \frac{1}{H(t)}\Big[\gamma(t) + \Big(M(t) - \int_0^t H(u)c(u)du\Big)\theta(t)\Big]\,\mathscr{S}(t,T_1)^{-1}\;, \tag{5.13}$$

then the corresponding wealth process $X^{x,c,\pi}(\cdot)$ satisfies $X^{x,c,\pi}(T) = \zeta$. It remains to show that the allocation (5.13) defines a portfolio process, and that the pair (c,π) is admissible. This can be proven using similar techniques as in Karatzas and Shreve (1998, Thm 6.3).

Next we will establish optimality and uniqueness of the optimal strategy. Due to Lemma 5.3 we may rewrite the problem from Definition 3.1 as

$$V(x) \;=\; \sup_{(c,\pi)\in\mathscr{A}} \mathbb{E}\left[\int_0^T U_1(t,c(t))dt + U_2(X^{x,c,\pi}(T))\right]\,, \tag{5.14}$$

in which $U_1(t,x) \;=\; \overline{F}(t)\frac{x^p}{p}$ and $U_2(x) \;=\; \Upsilon\,\overline{F}(T)\frac{x^p}{p}$ are concave functions of x.

Let (c',π') be an admissible pair. Due to the concavity of $U_1(t,\cdot)$ we have

$$U_1(t,c'(t)) - U_1(t,c(t)) \;\le\; (c'(t)-c(t))\,\partial_c U_1(t,c(t)) \;=\; (c'(t)-c(t))\,v(x)H(t)\,, \tag{5.15}$$

almost surely for every $0 \le t \le T$. Similarly, due to the concavity of $U_2(\cdot)$,

$$U_2(X^{x,c',\pi'}(T)) - U_2(\zeta) \;\le\; (X^{x,c',\pi'}(T)-\zeta)\,\partial_x U_2(\zeta) \;=\; (X^{x,c',\pi'}(T)-\zeta)\,v(x)H(T)\,, \tag{5.16}$$

almost surely. Integrating the inequalities (5.15) and (5.16) we find

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T U_1(t,c'(t))dt + U_2(X^{x,c',\pi'}(T))\right] - \mathbb{E}\left[\int_0^T U_1(t,c(t))dt + U_2(\zeta)\right]\\
\le\;\; & v(x)\,\mathbb{E}\left[\int_0^T (c'(t)-c(t))H(t)dt + (X^{x,c',\pi'}(T)-\zeta)H(T)\right]\\
\le\;\; & 0\,,
\end{aligned}$$

by the budget constraint (5.5). This proves optimality. Uniqueness (up to almost-everywhere equivalence) follows from Eq. (5.15) and Eq. (5.16) provided that $V(x) < \infty$.

□

Remark 5.5. *If $w(0,1) < \infty$ then condition (5.8) in Lemma 5.4 is satisfied for any initial level of wealth $x > 0$. Indeed, since $w(0,y) \;=\; y^{\frac{1}{p-1}}\,w(0,1)$, it follows that if $w(0,1) < \infty$ then the function $v\colon\,]0,\infty[\to [0,\infty[$ defined by*

$$v(x) = x^{p-1}w(0,1)^{1-p} \tag{5.17}$$

is the inverse of $w(0,\cdot)$.

We will need the following auxiliary result, which gives the multiplicative Doob-Meyer decomposition[10] of $H(s)I_1(s,H(s))$, for $0 \le s \le T$, and $H(T)I_2(H(T))$.

[10] See Jamshidian (2007, Prop. 4.2).

Lemma 5.6. *Assume $w(0,1) < \infty$. The optimal consumption and optimal terminal wealth satisfy*

$$H(s)I_1(s, H(s)) \;=\; \Lambda(s)m(s)\,, \tag{5.18}$$

for $0 \le s \le T$, and

$$H(T)I_2(H(T)) \;=\; \Psi\Lambda(T)m(T)\,, \tag{5.19}$$

in which

$$\Lambda(t) \;=\; \exp\left\{\frac{p}{1-p}\int_0^t \theta(u)dW(u) - \frac{1}{2}\left(\frac{p}{1-p}\right)^2\int_0^t ||\theta(u)||^2 du\right\} \tag{5.20}$$

is a local martingale and

$$m(t) \;=\; \exp\left\{-\frac{1}{1-p}\int_0^t \lambda(u)du + \frac{p}{1-p}\int_0^t r(u)du + \frac{p}{2(1-p)^2}\int_0^t ||\theta(u)||^2 du\right\}\,, \tag{5.21}$$

is a process of finite first-order variation.

Proof: The identities (5.18) and (5.19) follow from straightforward computation. □

We proceed to derive expressions for the optimal consumption strategy and optimal terminal wealth which are explicit up to the evaluation of a conditional expectation. Later we will show that this conditional expectation has a representation as a Laplace transform that can be evaluated using Lemma 4.1.

Proposition 5.7. *Let*

$$n(t) \;=\; \int_t^T L(t,u)du \;+\; \Psi\, L(t,T)\,, \tag{5.22}$$

in which

$$L(t,s) \;=\; \frac{1}{\Lambda(t)m(t)}\mathbb{E}\left[\Lambda(s)m(s)|\mathcal{F}(t)\right]\,. \tag{5.23}$$

If $w(0,1) < \infty$ then the optimal consumption and wealth process for Problem 3.1 are given by

$$c(t) \;=\; \frac{1}{n(t)}X(t)\,, \qquad X(t) \;=\; \frac{x}{H(t)}\frac{n(t)}{n(0)}\Lambda(t)m(t)\,. \tag{5.24}$$

The value function is finite and satisfies

$$V(x) \;=\; \begin{cases} \dfrac{1}{p}\,n(0)^{1-p}\,x^p\,, & \text{if } p \neq 0\,,\\[2ex] -M + n(0)\log\left(\dfrac{x}{n(0)}\right)\,, & \text{if } p = 0\,, \end{cases} \tag{5.25}$$

in which M is a (finite) constant.

Proof: See Appendix A. □

In the special case where the short rate is deterministic and $\lambda \equiv 0$ we find that formula (5.22) simplifies to $n(t) \;=\; {}^1\!/_{m(t)}\int_t^T m(u)du \;+\; {}^{m(T)}\!/_{m(t)}$. We thus see that Proposition 5.7 in this case coincides with the solution of Merton's classical portfolio optimization problem.

The following result provides conditions on the parameters of the short rate and the mortality rate process that imply $w(0,1) < \infty$; thus ensuring, by Remark 5.5, the existence of a solution for the problem from Definition 3.1.

Proposition 5.8. *Assume that the functions ψ_i, $i = 1, 2$, are analytic on $[0, T]$. The problem from Definition 3.1 has a solution which is unique up to almost-everywhere equivalence when $p \le 0$. When $p \in]0, 1[$ this is still the case if the parameters of the short rate and mortality rate processes satisfy*

$$\min_{s\in[0,T]} \left(\kappa_i + \tfrac{p}{1-p}\psi_i(s)\right) > 0 \ , \tag{5.26}$$

for $i = 1, 2$ and

$$p\,[\ (\kappa_1 - \psi_{1,min})^2 \ + \ \tfrac{(\psi_1^2)_{max} - (\psi_{1,min})^2}{1-p}\ ] \ < \ \kappa_1^2 - 2p\xi_1^2 \ , \tag{5.27}$$

$$p\,[\ (\kappa_2 - \psi_{2,min})^2 \ + \ \tfrac{(\psi_2^2)_{max} - (\psi_{2,min})^2}{1-p}\ ] \ < \ \kappa_2^2 + 2\xi_2^2 \ , \tag{5.28}$$

where

$$\psi_{i,min} = \min_{s\in[0,T]} \psi_i(s), \qquad (\psi_i^2)_{max} = \max_{s\in[0,T]} \psi_i(s)^2.$$

Furthermore, for given $T > 0$, $L(t, T)$ has an affine representation which is almost surely continuous in t and bounded on the domain $0 \le t \le T$.

Proof:

By Lemma 4.2 the stochastic exponential $\Lambda(\cdot)$ in (5.20) is a P-martingale on $[0, T]$ since we required the functions $\psi_i(t)$, $i \in \{1, 2, 3\}$, to be continuous and bounded on $[0, T]$. Hence we may define the measure $\hat{P}$ by

$$\frac{d\hat{P}}{dP} \ = \ \Lambda(T) \ . \tag{5.29}$$

From Girsanov's Theorem[11] we find that the short rate

$$r(t) \ = \ r_0 + \int_0^t (\mu_1(u) - \hat{\kappa}_1(u) r(u))du \ + \ \int_0^t \xi_1 \sqrt{r(u)}\ d\hat{W}_1(u)\,,$$

in which $\hat{W}_1(t) = W_1(t) - \frac{p}{1-p}\int_0^t \theta_1'(s)ds$, $\hat{\kappa}_1(t) = \kappa_1 + \frac{p}{1-p}\,\psi_1(t)$, and the mortality rate

$$\lambda(t) \ = \ \lambda_0 + \int_0^t (\mu_2(u) - \hat{\kappa}_2(u)\lambda(u))du \ + \ \int_0^t \xi_2 \sqrt{\lambda(u)}\ d\hat{W}_2(u)\,,$$

in which $\hat{W}_2(t) = W_2(t) - \frac{p}{1-p}\int_0^t \theta_2'(s)ds$, $\hat{\kappa}_2(t) = \kappa_2 + \frac{p}{1-p}\,\psi_2(t)$, remain independent under $\hat{P}$. Using Bayes' rule we can rewrite Eq. (5.23) as

$$L(t, T) \ = \ \frac{1}{\Lambda(t)m(t)}\mathbb{E}_P\left[\Lambda(T)m(T)|\mathcal{F}(t)\right] \ = \ \frac{1}{m(t)}\mathbb{E}_{\hat{P}}\left[m(T)|\mathcal{F}(t)\right] \ .$$

Hence we find, using Eq. (5.21),

$$\begin{aligned} L(t, T) \ &= \ \mathbb{E}_{\hat{P}}\left[\exp\left\{-\int_t^T h_1(u)r(u)du\right\}\bigg| \, r(t)\right] \\ &\quad \times \mathbb{E}_{\hat{P}}\left[\exp\left\{-\int_t^T h_2(u)\lambda(u)du\right\}\bigg| \, \lambda(t)\right] \ \exp\left\{-\int_t^T h_3(u)\,du\right\} \ , \end{aligned} \tag{5.30}$$

[11] See Karatzas and Shreve (1991, Thm. 5.1).

where

$$h_1(t) \;=\; -\frac{1}{1-p}\left(p+\frac{1}{2}\frac{p}{1-p}\left(\frac{\psi_1(t)}{\xi_1}\right)^2\right)\,, \quad h_2(t) \;=\; -\frac{1}{1-p}\left(-1+\frac{1}{2}\frac{p}{1-p}\left(\frac{\psi_2(t)}{\xi_2}\right)^2\right)\,,$$

and

$$h_3(t) \;=\; -\frac{1}{2}\frac{p}{(1-p)^2}\left(\frac{\psi_3(t)}{\xi_3}\right)^2\,.$$

We will now use Lemma 4.6 to prove that the conditions of Lemma 4.1 are satisfied for the Laplace transforms in Eq. (5.30); it then follows that $L(t,T)$, $0 \le t \le T$, has an affine representation. Since ψ_i, $i=1,2$, are analytic on an open set containing $[0,T]$,[12] it follows from the identity theorem for real analytic functions [13] that, for every $\alpha \in \mathbb{R}$, either $\psi_i(t)=\alpha$ on $[0,T]$ or the zero set $\{t : \psi_i(t)-\alpha=0\}$ does not contain an accumulation point. Consequently, the functions h_i are either identically zero on $[0,T]$ or change sign only a finite number of times. Moreover, the functions h_i are continuous and hence bounded on $[0,T]$. The last factor in (5.30) which involves the function h_3 is therefore always finite. To apply Lemma 4.6 to the first two factors in Eq. (5.30), we need to verify that condition (4.15) is satisfied. For $p \le 0$ we see that $h_i(t) \ge 0$ for $i=1,2$ so the condition is always satisfied in that case. We therefore only need to consider $p \in\,]0,1[$. For $i=1$ we need to check that

$$\begin{aligned}\min_{s\in[0,T]}(\kappa_1+\tfrac{p}{1-p}\psi_1(s))^2 \;&>\; -2\xi_1^2\min_{s\in[0,T]}\tfrac{-p}{1-p}(1+\tfrac{1}{2(1-p)}(\psi_1(s)/\xi_1)^2)\\ &=\; \frac{2p\xi_1^2}{1-p} \;+\; \frac{p}{(1-p)^2}\max_{s\in[0,T]}\psi_1(s)^2.\end{aligned}$$

But since $p>0$ and $\kappa_1+\frac{p}{1-p}\psi_1(s)>0$ for all $s\in[0,T]$, the minimum of the expression on the lefthand side must be attained at the point where ψ_1 is minimal. We then multiply both sides of the inequality with $1-p$ to get

$$\kappa_1^2(1-p)+2p\kappa_1\psi_{1,min}+\tfrac{p^2}{(1-p)}\psi_{1,min}^2 \;>\; 2p\xi_1^2 \;+\; \tfrac{p}{(1-p)}(\psi_1^2)_{max}$$

which gives

$$\begin{aligned}\kappa_1^2-2p\xi_1^2 \;&>\; p(\kappa_1-\psi_{1,min})^2-p\psi_{1,min}^2-\tfrac{p^2}{(1-p)}\psi_{1,min}^2 \;+\; \tfrac{p}{(1-p)}(\psi_1^2)_{max}\\ &=\; p\,[\;(\kappa_1-\psi_{1,min})^2 \;+\; \tfrac{(\psi_1^2)_{max}-(\psi_{1,min})^2}{(1-p)}\;].\end{aligned}$$

The case $i=2$ can be treated in a similar fashion.

We can thus apply Lemma 4.1 to the Laplace transforms in Eq. (5.30) and we find that

$$L(t,T) \;=\; e^{-\overline{A}_1(t,T)-\overline{A}_2(t,T)-\overline{B}_1(t,T)r(t)-\overline{B}_2(t,T)\lambda(t)-\overline{A}_3(t,T)}\,, \tag{5.31}$$

in which $\overline{A}_3(t,T)=\int_t^T h_3(u)\,du$, while $\overline{A}_i(\cdot,T)$ and $\overline{B}_i(\cdot,T)$, $i=1,2$, are the solutions to

$$\partial_t\overline{B}_i(t,T) \;=\; \hat{\kappa}_i(t)\overline{B}_i(t,T)+\frac{1}{2}\xi_i^2\overline{B}_i(t,T)^2-h_i(t)\,, \qquad \overline{B}_i(T,T)\;=\;0\,, \tag{5.32}$$

$$\partial_t\overline{A}_i(t,T) \;=\; -\mu_i(t)\,\overline{B}_i(t,T)\,, \qquad \overline{A}_i(T,T)\;=\;0\,, \tag{5.33}$$

[12] A function being analytic on a closed set implies that it is analytic on an open covering of that set, see Rudin (1976, Thm. 8.4).

[13] See for example Rudin (1976, Thm. 8.5).

and these functions are bounded and continuously differentiable by Lemma 4.6.

Finally, to establish existence of a solution to Problem 3.1 we have to verify that the conditions of Lemma 5.4 are satisfied. From Remark 5.5 and the identity $w(0,1) = n(0)$ it follows that this is equivalent to proving that $n(0) < \infty$, which follows from the almost sure boundedness of $L(t,T)$ on $[0,T]$. □

The next Proposition characterizes the optimal hedging strategy. Given any nonnegative consumption process, there exists a portfolio process which replicates the optimal wealth, see Eq. (5.12) in the proof of Lemma 5.4. Moreover, the choice of the portfolio process only affects the diffusion term of the numéraire rebased wealth process while the drift term depends on the consumption process. By comparing the dynamics of the optimal wealth process to the dynamics of the tradeables in the (complete) market we can determine the trading strategy which finances the optimal consumption plan and leads to the optimal distribution of terminal wealth at time T.

Proposition 5.9. *Under the conditions of Lemma 4.5 the dynamics of the optimal wealth and consumption process satisfy*

$$\frac{d\left(X(t) + \int_0^t c(u)du\right)}{X(t)} = \eta_0(t)\frac{d\beta(t)}{\beta(t)} + \eta_1(t)\frac{dP(t,T_1)}{P(t,T_1)} + \eta_2(t)\frac{dF(t,T_1)}{F(t,T_1)} + \eta_3(t)\frac{dS(t)}{S(t)}\ , \tag{5.34}$$

where,

$$\eta_0(t) = 1 - \sum_{i=1}^{3} \eta_i(t) \tag{5.35}$$

$$\eta_1(t) = \frac{1}{B_1(t,T_1)}\left(\frac{1}{1-p}\left(\frac{\psi_1(t)}{\xi_1^2} - \frac{\rho\,\psi_3(t)}{\xi_1\xi_3\overline{\rho}}\right) + \frac{\Xi_1(t)}{n(t)}\right) - \eta_2(t)\ , \tag{5.36}$$

$$\eta_2(t) = \frac{1}{B_2(t,T_1)}\left(\frac{1}{1-p}\,\frac{\psi_2(t)}{\xi_2^2} + \frac{\Xi_2(t)}{n(t)}\right)\ , \tag{5.37}$$

$$\eta_3(t) = \frac{1}{1-p}\,\frac{\psi_3(t)}{\overline{\rho}\,\xi_3^2}\ , \tag{5.38}$$

in which,

$$\Xi_i(t) = \Psi\,\overline{B}_i(t,T)L(t,T) + \int_t^T \overline{B}_i(t,u)L(t,u)\,du\ , \tag{5.39}$$

and $\overline{B}_i$ are the solutions of (5.32) and L is given by Eq. (5.31).

Proof: See Appendix A. □

We conclude from Proposition 5.9 that the optimal strategy invests a fraction $\eta_1(t)$ of wealth in bonds, a fraction $\eta_2(t)$ in survival bonds, a fraction $\eta_3(t)$ in stocks and the remaining wealth $\eta_0(t)$ is invested in the money-market account. The portfolio weights depend on the short rate $r(\cdot)$ and the mortality rate $\lambda(\cdot)$. These rates are observable due to their affine structure; their values can be inferred from the price of the zero-coupon bond $P(\cdot,T_1)$ and the price of the survival bond $F(\cdot,T_1)$ respectively.

The demand for survival bonds in Eq. (5.37) has a deterministic component which earns the risk premium and a stochastic component that represents the hedging demand. The stochastic part reduces to a deterministic term only in the special case where $\overline{B}_2$ is constant. Due to the boundary condition in Eq. (5.32) this happens only if $\overline{B}_2 \equiv 0$. Hence we conclude that there will not be a market price of risk function in this model which makes the demand for longevity derivatives vanish. This means that there is no 'least favourable market completion' in the sense of Karatzas et al. (1991).

Proposition 5.10. *The hedging demand (as a fraction of wealth) for all assets in the economy is bounded.*

Proof: Fix $i \in \{1,2\}$ and let $\varphi_L(\cdot,u)$ and $\varphi_U(\cdot,u)$ be the family, indexed by $u \in [0,T]$, of lower and upper bounds for $\overline{B}_i(\cdot,u)$ as constructed in Lemma 4.5. Since $\varphi_L(t,u)$ is decreasing in u, it follows that

$$\begin{aligned}
\varphi_L(t,T) - \frac{\Xi_i(t)}{n(t)} &= \frac{1}{n(t)}\Big(\varphi_L(t,T)n(t) - \Xi_i(t)\Big) \\
&= \frac{1}{n(t)}\int_t^T \Big(\varphi_L(t,T) - \overline{B}_i(t,u)\Big)L(t,u)du + \Psi L(t,T)\left(\varphi_L(t,T) - \overline{B}_i(t,T)\right) \\
&\leq \frac{1}{n(t)}\int_t^T \Big(\varphi_L(t,T) - \varphi_L(t,u)\Big)L(t,u)du \\
&\leq 0\ .
\end{aligned}$$

Similarly, since $\varphi_U(t,u)$ is increasing in u,

$$\begin{aligned}
\varphi_U(t,T) - \frac{\Xi_i(t)}{n(t)} &\geq \frac{1}{n(t)}\int_t^T \Big(\varphi_U(t,T) - \overline{B}_i(t,u)\Big)L(t,u)du \\
&\geq \frac{1}{n(t)}\int_t^T \Big(\varphi_U(t,T) - \varphi_U(t,u)\Big)L(t,u)du \\
&\geq 0\ .
\end{aligned}$$

Hence

$$\varphi_L(t,T) \ \leq\ \frac{\Xi_i(t)}{n(t)} \ \leq\ \varphi_U(t,T)\ .$$

□

6 ECONOMIC ANALYSIS OF THE MODEL

Huang et al. (2012) establish a relation between the parameter p of the power utility functions defined in Eqs. (3.2) and (3.3), and the optimal initial consumption $c(0)$. In Theorem 6.1 we show that a similar relation holds in our model setup.

Theorem 6.1. *Consider two models: the first model has a deterministic mortality rate $\lambda^{det}(\cdot)$, while in the second model the mortality rate $\lambda^{stoch}(\cdot)$ is stochastic. Assume that $\psi_2^{stoch}(t) \geq 0$,[14] whereas $\psi_2^{det}(t) = 0$.[15] If the survival probabilities are the same in both models, that is, if for all $s \in [0,T]$*

$$\mathbb{E}_P\left[e^{-\int_0^s \lambda^{stoch}(u)du}\right] \ =\ e^{-\int_0^s \lambda^{det}(u)du}\ , \tag{6.1}$$

[14]This corresponds to the case where the drift of the survival bond carries a positive risk premium.

[15]If the mortality rate is deterministic, then there is no longevity risk, therefore we have $\psi_2^{det}(t) = 0$.

then for $0 \leq p < 1$,

$$c^{det}(0) \;\geq\; c^{stoch}(0) \qquad \textit{and} \qquad V^{det}(x) \;\leq\; V^{stoch}(x) \;,$$

while for $p \leq 0$,

$$c^{det}(0) \;\leq\; c^{stoch}(0) \qquad \textit{and} \qquad V^{det}(x) \;\geq\; V^{stoch}(x) \;.$$

We thus find that a stochastic mortality rate, and a nonnegative market price of longevity risk, lead to a difference in the consumption rate and the value function. The sign of the change depends on the coefficient of relative risk aversion. Note that in Huang et al. (2012), where it is assumed that longevity risk is not tradeable, the value function in the stochastic mortality model always exceeds the value function in the deterministic model, regardless of the coefficient of relative risk aversion. To prove Theorem 6.1 we need the following Lemma.

Lemma 6.2. *Assume that* $\psi_2(s) \geq 0$ *for all* $s \in [0, T]$.
If $0 \leq p < 1$, *then, for all* $t \in [0, T]$,

$$\mathbb{E}_{\hat{P}}\left[e^{-\int_t^T \lambda(u)du} \,\middle|\, \mathcal{F}(t) \right] \;\geq\; \mathbb{E}_P\left[e^{-\int_t^T \lambda(u)du} \,\middle|\, \mathcal{F}(t) \right] \;, \tag{6.2}$$

whereas if $p \leq 0$, *then, for all* $t \in [0, T]$,

$$\mathbb{E}_{\hat{P}}\left[e^{-\int_t^T \lambda(u)du} \,\middle|\, \mathcal{F}(t) \right] \;\leq\; \mathbb{E}_P\left[e^{-\int_t^T \lambda(u)du} \,\middle|\, \mathcal{F}(t) \right] \;. \tag{6.3}$$

Proof: See Appendix A. □

Lemma 6.2 relates the survival probability under P to its value under $\hat{P}$ and shows that their relation depends qualitatively on the risk aversion coefficient. We can now prove Theorem 6.1.

Proof of Theorem 6.1:

Fix $s \in]0, T]$ and recall that, by Eq. (5.30),

$$L^{stoch}(0,s) \;=\; M\,\mathbb{E}_{\hat{P}}\Big[\exp\left\{-\frac{1}{1-p}\int_0^s \lambda^{stoch}(u)\,du\right\}\exp\left\{\frac{1}{2}\frac{p}{(1-p)^2}\int_0^s ||\theta_2(u)||^2\,du\right\}\Big],$$

in which

$$M \;=\; \mathbb{E}_{\hat{P}}\left[\exp\left\{-\int_0^s h_1(u)r(u)du\right\}\right]\;\exp\left\{-\int_0^s h_3(u)\,du\right\} \;\geq\; 0\,.$$

For $0 < p < 1$ we find

$$L^{stoch}(0,s) \;\geq\; M\,\mathbb{E}_{\hat{P}}\left[\phi\left(e^{-\int_0^s \lambda^{stoch}(u)\,du}\right)\right]\;,$$

in which $\phi(u) = u^{\frac{1}{1-p}}$ is an increasing, strictly convex function. Hence, by Jensen's inequality

$$\mathbb{E}_{\hat{P}}\left[\phi\left(e^{-\int_0^s \lambda^{stoch}(u)\,du}\right)\right] \;\geq\; \phi\left(\mathbb{E}_{\hat{P}}\left[e^{-\int_0^s \lambda^{stoch}(u)\,du}\right]\right)\;. \tag{6.4}$$

Using Lemma 6.2 and the fact that ϕ is an increasing function we conclude that

$$\phi\left(\mathbb{E}_{\hat{P}}\left[e^{-\int_0^s \lambda^{stoch}(u)\,du}\right]\right) \;\geq\; \phi\left(\mathbb{E}_P\left[e^{-\int_0^s \lambda^{stoch}(u)\,du}\right]\right)\,,$$

in which the P-expectation represents the survival probability.

On the other hand, for the deterministic model we find, using the fact that $\psi_2^{det}(t) = 0$,

$$L^{det}(0,s) \;=\; M\,\phi\left(e^{-\int_0^s \lambda^{det}(u)\,du}\right)\,,$$

Since, by assumption,

$$\mathbb{E}_P\left[e^{-\int_0^s \lambda^{stoch}(u)\,du}\right] \;=\; e^{-\int_0^s \lambda^{det}(u)\,du}\,,$$

we conclude that $L^{stoch}(0,s) \geq L^{det}(0,s)$ for $0 < p < 1$ and for all $s \in]0,T]$. Consequently,

$$n^{stoch}(0) \;=\; \int_0^T L^{stoch}(0,s)\,ds + \Psi\,L^{stoch}(0,T) \;\geq\; \int_0^T L^{det}(0,s)\,ds + \Psi\,L^{det}(0,T) \;=\; n^{det}(0)\,,$$

and the result for $0 < p < 1$ follows from Eqs. (5.24) and (5.25).

Note that for $p = 0$ the inequalities above hold as equalities, hence $L^{stoch}(0,s) = L^{det}(0,s)$ for $p = 0$ and $s \in]0,T]$. If $p < 0$ then ϕ is a concave, increasing function. In this case all inequalities above are reversed and we find that $L^{stoch}(0,s) \leq L^{det}(0,s)$ for all $s \in]0,T]$. □

7 CONCLUSION

In this paper we have formulated an optimal consumption and investment problem with positive stochastic mortality rates and time-varying market prices of risk. We have derived conditions which guarantee that there is no convex set of admissable investment strategies, or even a trivial strategy which only invests in the bank account, which give rise to infinite expected utility of consumption or terminal wealth. We can characterize the optimal consumption and investment strategy explicitly and show that the hedging demand for all assets in the economy remains bounded.

In our setup we have completed the market by the introduction of an asset which allows the investor to earn a risk premium for longevity risk. If such an asset would not be available, the market becomes incomplete in the sense that no mortality-dependent terminal wealth and consumption profiles can be generated by the investor. A common approach for handling incompleteness in optimal investment problems is to search for the 'least favourable' market completion, that is, the market price of risk under which it is optimal not to invest in a non-tradeable asset such as the survival bond, see Karatzas et al. (1991). In our market setting no specification of the market price of longevity risk can make the survival bond redundant, as can easily be seen from our optimal portfolio equations. This means that an investor will always benefit from the extension of investment opportunities that is provided by longevity-based derivatives.

Our results show that for investors who are only moderately risk averse[16], restrictions must be imposed on the risk premium for longevity risk to get a well-posed problem. These

[16] in the sense that our risk aversion parameter p is larger than zero

restrictions depend on the level of risk aversion, on the mean reversion speed and on the volatility of the mortality rate process. Some care must therefore be taken if one wants to use utility indifference pricing methods to establish an asset pricing theory for survival bonds, longevity swaps and other mortality-dependent financial products.

A PROOFS

Proof of Lemma 5.1:

From Lemma 4.2 it follows that, for $i \in \{1,2,3\}$ and $0 \le t \le T$,

$$\mathbb{E}\left[\exp\left\{-\int_0^t \theta_i(u)dW_i(u) - \frac{1}{2}\int_0^t ||\theta_i(u)||^2 du\right\}\right] \;=\; 1 \;.$$

Since W_1, W_2 and W_3 are independent we conclude that $Z_0(\cdot)$ is a martingale[17], and Eq. (5.2) defines by Girsanov's theorem, see Karatzas and Shreve (1991, Thm 5.1), a measure equivalent to P. Moreover, $\widetilde{W}(t) = W(t) + \int_0^t \theta(u)du$ is a $\widetilde{P}$-Brownian motion.

From Itô's lemma we obtain that the product of the state price density $H(t) = Z_0(t)/\beta(t)$ and the stock price satisfies

$$H(t)S(t) \;=\; S(0) + \int_0^t \left[\sigma_S(u) - \theta(u)\right] H(u)S(u)dW(u) \;. \tag{A.1}$$

The product of the bond price and the state price density satisfies

$$H(t)P(t,T_1) \;=\; P(0,T_1) + \int_0^t \left[\sigma_P(u) - \theta(u)\right] H(u)P(u,T_1)dW(u) \;. \tag{A.2}$$

Similarly, for the survival bond we find

$$H(t)F(t,T_1) \;=\; F(0,T_1) + \int_0^t \left[\sigma_F(u) - \theta(u)\right] H(u)F(u,T_1)dW(u) \;. \tag{A.3}$$

Observe that the processes in Eq. (A.1), (A.2) and (A.3) are local martingales and can be rewritten as stochastic exponentials of the form (4.7). Since the functions $t \mapsto \xi_i B_j(t,T_1)$, $i,j = 1,2$, and $t \mapsto \psi_i(t)/\xi_i$, $i = 1,2$, are bounded and using the independence of $W_i(\cdot)$, $i = 1,2,3$, under P, we conclude from Lemma 4.2 that the product of any tradeable with the state price density is a P-martingale. Since $Z_0(t)$ is a Radon-Nikodym derivative by Girsanov's theorem, it follows that β-discounted prices of tradeables are $\widetilde{P}$-martingales. □

Proof of Lemma 5.3:

Using the tower rule and the fact that $\mathscr{U}_2(X^{x,c,\pi}(T))$ is $\mathcal{F}(T)$-measurable we obtain

$$\begin{aligned}\mathbb{E}\left[\mathscr{U}_2(X^{x,c,\pi}(T))1_{\tau>T}\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathscr{U}_2(X^{x,c,\pi}(T))1_{\tau>T}|\mathcal{F}(T)\right]\right] \\ &= \mathbb{E}\left[\mathscr{U}_2(X^{x,c,\pi}(T))\mathbb{E}\left[1_{\tau>T}|\mathcal{F}(T)\right]\right] \\ &= \mathbb{E}\left[\mathscr{U}_2(X^{x,c,\pi}(T))\overline{F}(T)\right] \;.\end{aligned}$$

[17] It is shown in Kraft (2003, p. 59) that Novikov's condition, i.e. $\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T ||\theta(u)||^2\, du\right\}\right] \;<\; \infty$ a.s., is not satisfied in general for our model setup.

Using Fubini's theorem (the integrand is nonnegative) and the fact that $\mathscr{U}_1(t, c(t))$ is $\mathcal{F}(t)$-progressively measurable, we can apply the same argument to obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^T \mathscr{U}_1(t,c(t))\, 1_{\tau>t}\, dt\right] &= \mathbb{E}\left[\int_0^T \mathbb{E}\left[\mathscr{U}_1(t,c(t))\, 1_{\tau>t} \mid \mathcal{F}(t)\right] dt\right] \\ &= \mathbb{E}\left[\int_0^T \overline{F}(t)\mathscr{U}_1(t,c(t))dt\right]\end{aligned}$$

Consequently,

$$\begin{aligned}V(x) &= \sup_{(c,\pi)\in\mathscr{A}} \mathbb{E}\left[\int_0^{T\wedge\tau} \mathscr{U}_1(t,c(t))dt + \mathscr{U}_2(X^{x,c,\pi}(T))1_{\tau>T}\right] \\ &= \sup_{(c,\pi)\in\mathscr{A}} \mathbb{E}\left[\int_0^{T} \mathscr{U}_1(t,c(t))\, 1_{\tau>t}\, dt \;+\; \mathscr{U}_2(X^{x,c,\pi}(T))1_{\tau>T}\right] \\ &= \sup_{(c,\pi)\in\mathscr{A}} \mathbb{E}\left[\int_0^{T} \overline{F}(t)\mathscr{U}_1(t,c(t))dt + \overline{F}(T)\mathscr{U}_2(X^{x,c,\pi}(T))\right].\end{aligned}$$

□

Lemma A.1. *The utility functions $U_1(t,\cdot)$ and $U_2(\cdot)$, defined in Lemma 5.3, satisfy the relations*

$$U_1(s, I_1(s,y)) = \begin{cases} \dfrac{1}{p}\, y\, I_1(s,y)\ , & p \neq 0\ , \\ m(s)\log(m(s)/y)\ , & p = 0\ , \end{cases} \tag{A.4}$$

for $0 \le s \le T$ and for all $y > 0$, and

$$U_2(I_2(y)) = \begin{cases} \dfrac{1}{p}\, y\, I_2(y)\ , & p \neq 0\ , \\ \Psi\, m(T)\, \log(\Psi\, m(T)/y)\ , & p = 0\ , \end{cases} \tag{A.5}$$

for all $y > 0$.

Proof:

If $p \neq 0$ then for power utility it holds that $p\,U_1(t,x) = x\,\partial_x U_1(t,x)$. Take $x = I_1(t,y)$ then $p\,U_1(t, I_1(t,y)) = I_1(t,y)\,\partial_x U_1(t, I_1(t,y)) = y\, I_1(t,y)$. The other identities follow from similar computations. □

Proof of Lemma 5.7:

From the proof of Lemma 5.4 we know that the optimal consumption strategy and the optimal terminal wealth are given by Eq. (5.9) and Eq. (5.11) respectively. Observe that the integrand of the optimal wealth process (5.10) is positive; hence we may apply the conditional Fubini theorem together with Lemma 5.6 to obtain

$$\begin{aligned}X(t) &= \frac{1}{H(t)}\mathbb{E}\left[\int_t^T H(u)I_1(u, v(x)H(u))du + H(T)I_2(v(x)H(T))\,\middle|\, \mathcal{F}(t)\right] \\ &= \frac{1}{H(t)}v(x)^{\frac{1}{1-p}}\,\Lambda(t)m(t)\Big(\int_t^T L(t,u)dt + \Psi\, L(t,T)\Big) = \frac{x}{H(t)}\frac{n(t)}{n(0)}\Lambda(t)m(t)\,.\end{aligned}$$

In the last step we used Eq. (5.17) and the fact that $w(0,1) = n(0)$. For the optimal consumption process Eq. (5.9) we find

$$c(t) \;=\; I_1(t, v(x)H(t)) \;=\; \frac{x}{w(0,1)} I_1(t, H(t)) \;=\; \frac{x}{w(0,1)} \frac{\Lambda(t)m(t)}{H(t)} \;=\; \frac{X(t)}{n(t)} \,.$$

By substitution of the optimal wealth (5.11) and optimal consumption strategy (5.9) into the value function Eq. (5.14) we obtain

$$V(x) \;=\; v(x)^{\frac{p}{p-1}}\, \mathbb{E}\left[\int_0^T U_1(t, I_1(t, H(t)))\, dt + U_2(I_2(H(T)))\right] \,.$$

From Lemma A.1 we have for $p \neq 0$ that $\mathbb{E}\left[U_1(t, I_1(t, H(t)))\right] = \Psi^{-1}\, \mathbb{E}\left[U_2(I_2(H(t)))\right] = p^{-1}L(0,t)$. Consequently,

$$V(x) \;=\; \frac{1}{p}\, v(x)^{\frac{p}{p-1}}\, n(0) \;=\; \frac{1}{p}\left(\frac{x}{w(0,1)}\right)^p n(0) \;=\; \frac{1}{p}\, n(0)^{1-p}\, x^p \;<\; \infty.$$

The value function for the case $p = 0$ is given by

$$V(x) = -M + n(0)\log\left(\frac{x}{n(0)}\right)$$

in which

$$M \;=\; \mathbb{E}\left[\int_0^T \overline{F}(t)\log\left(\frac{H(t)}{\overline{F}(t)}\right) dt + \Psi\, \overline{F}(T)\, \log\left(\frac{H(T)}{\Psi \overline{F}(T)}\right)\right] \,,$$

is a constant and

$$\log\left(\frac{H(t)}{\overline{F}(t)}\right) \;=\; -\int_0^t r(u)\, du + \int_0^t \lambda(u)\, du - \int_0^t \theta(u) dW(u) - \frac{1}{2}\int_0^t ||\theta(u)||^2\, du \,.$$

If, for some random variable Y, the Laplace transform $\mathbb{E}\left[e^{-\bar{h}Y}\right]$ is finite for any $\bar{h} < 0$ then Y has finite moments of any order. Take $Y = \int_0^t r(u)\, du$ and $\frac{\kappa_1^2}{-2\xi_1^2} < \bar{h} < 0$, then the conditions of Lemma 4.6 are satisfied so that the Laplace transform is finite by Lemma 4.1. The other terms can be bounded in expectation by similar arguments. It also follows that the stochastic integral is a martingale and vanishes in expectation. Thus $M < \infty$. □

Lemma A.2 (Leibniz' rule for Itô integrals)**.** *For all $0 < T < \infty$ define the process $\widetilde{L}(\cdot, T)$ by*

$$\widetilde{L}(t,T) \;=\; \widetilde{L}(0,T) \;+\; \int_0^t \widetilde{\mu}(u,T)\, du + \sum_{i=1}^{2}\int_0^t \widetilde{\xi}_i(u,T) dW_i(u) \tag{A.6}$$

where $W(t) = (\, W_1(t), W_2(t)\,)$ is a Brownian motion, $\widetilde{\mu}$ is an adapted function $[0,T] \times [0,T] \times \Omega \to R$ and $\widetilde{\xi}$ is an adapted function $[0,T] \times [0,T] \times \Omega \to R^2$.

If for $i = 1, 2$,

$$\int_0^t \int_0^t |\widetilde{\mu}(u,s)| ds\, du \;<\; \infty \quad a.s. \text{ for all } t \in [0,T] \;, \tag{A.7}$$

$$\int_0^t \left\{\widetilde{\xi}_i(u,s) 1_{u \le s}\right\}^2 du \;<\; \infty \;\; a.s. \text{ for all } t \in [0,T] \text{ and } s \in [0,T]\,, \tag{A.8}$$

$$\int_0^t \left\{ \int_0^T \widetilde{\xi}_i(u,s) 1_{u\le s}\, ds \right\}^2 du \;<\; \infty \quad \textit{a.s. for all } t\in[0,T]\,, \tag{A.9}$$

and

$$t \mapsto \int_0^T \left\{ \int_0^t \widetilde{\xi}_i(u,s) 1_{u\le s}\, dW_i(u) \right\}^2 ds \tag{A.10}$$

is almost surely continuous then, for fixed T,

$$d\left(\int_t^T \widetilde{L}(t,s) ds \right) \;=\; \left\{ -\widetilde{L}(t,t) + \int_t^T \widetilde{\mu}(t,s) ds \right\} dt + \sum_{i=1}^2 \left\{ \int_t^T \widetilde{\xi}_i(t,s) ds \right\} dW_i(t)\,.$$

Proof:

A proof can be found in Munk and Sørensen (2000, Appendix A). The conditions of the Lemma are taken from Heath et al. (1992) and are required to justify the interchange of a stochastic integral and a Lebesgue integral. □

Proof of Lemma 5.9:

Applying Itô's lemma to the optimal wealth process in Eq. (5.24) gives

$$\begin{aligned} \frac{X(t)}{\beta(t)} &= \frac{x}{n(0)} \frac{\Lambda(t)}{Z_0(t)} m(t) n(t) \\ &= x + \int_0^t \alpha_0(u)\, du + \int_0^t \frac{x\, m(u) n(u)}{n(0)} d\left(\frac{\Lambda(u)}{Z_0(u)}\right) + \int_0^t \frac{x\, m(u)\Lambda(u)}{n(0) Z_0(u)}\, dn(u) \\ &= x + \int_0^t \alpha_1(u)\, du + \int_0^t \frac{1}{1-p} \frac{X(u)}{\beta(u)} \theta(u) d\widetilde{W}(u) + \int_0^t \frac{1}{n(u)} \frac{X(u)}{\beta(u)}\, dn(u)\,, \end{aligned} \tag{A.11}$$

for some processes $\alpha_0(\cdot)$ and $\alpha_1(\cdot)$. Notice that the last stochastic integral on the right-hand side has a diffusion component and a component of finite variation. To determine the dynamics of the process $n(t) = \int_t^T L(t,u) du + \Psi L(t,T)$ we apply Lemma A.2. First, we verify that the conditions of this lemma are satisfied. From Eq. (5.31) and Itô's lemma we obtain

$$\begin{aligned} L(t,T) \;&=\; L(0,T) - \int_0^t \Big(r(u)\partial_t \overline{B}_1(u,T) + \lambda(u)\partial_t \overline{B}_2(u,T) + \textstyle\sum_{i=1}^3 \partial_t \overline{A}_i(u,T) \Big) L(u,T)\, du \\ &\quad - \int_0^t \overline{B}_1(u,T) L(u,T)\, dr(u) - \int_0^t \overline{B}_2(u,T) L(u,T)\, d\lambda(u) \\ &\quad + \frac{1}{2} \int_0^t \overline{B}_1(u,T)^2 d\left\langle r,r \right\rangle_u + \frac{1}{2} \int_0^t \overline{B}_2(u,T)^2 d\left\langle \lambda,\lambda \right\rangle_u \,. \end{aligned} \tag{A.12}$$

Observe that $\overline{B}_i(\cdot,s)$ is bounded, uniformly in s, for $i=1,2$ by Lemma 4.5. Consequently the same holds for $\overline{A}_i(\cdot,s)$, for $i=1,2$, and hence for $L(\cdot,s)$. Moreover, the paths $r(\cdot)$ are almost surely continuous, hence finite on $[0,T]$. It follows that, for all $t\in[0,T]$ and $s\in[0,T]$,

$$\int_0^t \left\{ L(u,s) \overline{B}_1(u,s) \xi_1 \sqrt{r(u)} \right\}^2 du \;<\; \infty, \qquad \text{a.s.,} \tag{A.13}$$

and, for all $t \in [0,T]$,

$$\int_0^t \left\{ \int_0^T L(u,s)\overline{B}_1(u,s)\xi_1\sqrt{r(u)}\,ds \right\}^2 du \;<\; \infty, \qquad \text{a.s.} \tag{A.14}$$

Similarly, from the continuity of $\lambda(\cdot)$ we have, for all $t \in [0,T]$ and $s \in [0,T]$,

$$\int_0^t \left\{L(u,s)\overline{B}_2(u,s)\xi_2\sqrt{\lambda(u)}\right\}^2 du \;<\; \infty, \qquad \text{a.s.}, \tag{A.15}$$

and, for all $t \in [0,T]$,

$$\int_0^t \left\{ \int_0^T L(u,s)\overline{B}_2(u,s)\xi_2\sqrt{\lambda(u)}\,ds \right\}^2 du \;<\; \infty, \qquad \text{a.s.} \tag{A.16}$$

Hence conditions (A.8) and (A.9) are met. It follows from Eqs. (5.32)–(5.33), Lemma 4.5 and Eqs. (4.11)–(4.12) the derivatives $\partial_t\overline{A}_i(\cdot,s)$, $\partial_t\overline{B}_i(\cdot,s)$, $i=1,2$ are bounded uniformly in $s \in [0,T]$. Therefore, the drift term in Eq. (A.12) satisfies condition (A.7). Thus we can apply Lemma A.2 and conclude that, for some process $\alpha_2(\cdot)$,

$$d\left(\int_t^T L(t,u)du\right) = \alpha_2(t)\,dt \,-\int_t^T \overline{B}_1(t,u)L(t,u)du\,dr(t) - \int_t^T \overline{B}_2(t,u)L(t,u)du\,d\lambda(t)\,. \tag{A.17}$$

For $\Xi_1(\cdot)$ and $\Xi_2(\cdot)$ as in Eq. (5.39), we obtain

$$dn(t) \;=\; d\left(\int_t^T L(t,u)du + \Psi L(t,T)\right) \;=\; \alpha_3(t)\,dt \;-\; \Xi_1(t)\,dr(t) \;-\; \Xi_2(t)\,d\lambda(t)\,, \tag{A.18}$$

for a certain process $\alpha_3(\cdot)$. Substituting (A.18) into (A.11) we find, for certain processes $\alpha_4(\cdot)$ and $\alpha_5(\cdot)$,

$$\begin{aligned}
\frac{X(t)}{\beta(t)} \;&=\; x \;+\; \int_0^t \alpha_4(u)\,du \\
&\quad + \int_0^t \frac{X(u)}{\beta(u)}\left\{\frac{1}{1-p}\left(-\frac{\psi_1(u)}{\xi_1}\sqrt{r(u)}d\widetilde{W}_1(u) - \frac{\psi_2(u)}{\xi_2}\sqrt{\lambda(u)}d\widetilde{W}_2(u) - \frac{\psi_3(u)}{\xi_3}d\widetilde{W}_3(u)\right)\right. \\
&\quad \left. - \frac{1}{n(u)}\Big(\Xi_1(u)dr(u) + \Xi_2(u)d\lambda(u)\Big)\right\} \\
&=\; x \;+\; \int_0^t \alpha_5(u)\,du + \int_0^t \frac{X(u)}{\beta(u)}\left(\frac{1}{1-p}\frac{-\psi_1(u)}{\xi_1^2} - \frac{\Xi_1(u)}{n(u)}\right)\xi_1\sqrt{r(u)}\,d\widetilde{W}_1(u) \\
&\quad + \int_0^t \frac{X(u)}{\beta(u)}\left(\frac{1}{1-p}\frac{-\psi_2(u)}{\xi_2^2} - \frac{\Xi_2(u)}{n(u)}\right)\xi_2\sqrt{\lambda(u)}\,d\widetilde{W}_2(u) \\
&\quad - \int_0^t \frac{X(u)}{\beta(u)}\left(\frac{1}{1-p}\,\frac{\psi_3(u)}{\xi_3^2\,\overline{\rho}}\right)\xi_3\,\overline{\rho}\,d\widetilde{W}_3(u)\;.
\end{aligned} \tag{A.19}$$

On the other hand, if $\eta(\cdot)$ is the fraction of wealth invested in each of the asset classes, then from Eq. (2.10) we have

$$\frac{X(t)}{\beta(t)} \;=\; x - \int_0^t \frac{c(u)}{\beta(u)}\,du \;+\int_0^t \frac{X(u)}{\beta(u)}\eta(u)\,\mathscr{S}(u,T_1)\,d\widetilde{W}(u)\,. \tag{A.20}$$

The optimal portfolio weights are obtained when we use martingale representation to equate the diffusion terms in Eq. (A.19) and Eq. (A.20). □

Proof of Lemma 6.2:

From Lemma 3.4 we know that the survival probability, which is calculated as an expectation under P, satisfies

$$\mathbb{E}_P\left[e^{-\int_t^T \lambda(u)du} \,\middle|\, \mathcal{F}(t)\right] \;=\; e^{-\widetilde{A}_2(t,T)-\widetilde{B}_2(t,T)\,\lambda(t)} \;,$$

where $\widetilde{B}_2(\cdot,T)$ solves

$$\begin{cases} \partial_t \widetilde{B}_2(t,T) \;=\; \kappa_2 \widetilde{B}_2(t,T) + \frac{1}{2}\xi_2^2 \widetilde{B}_2(t,T)^2 - 1 \;, \\ \widetilde{B}_2(T,T) = 0 \,, \end{cases}$$

and $\widetilde{A}_2(t,T) \;=\; \int_t^T \mu_2(u)\widetilde{B}_2(u,T)\,du$.

Similarly, under $\hat{P}$ we have

$$\mathbb{E}_{\hat{P}}\left[e^{-\int_t^T \lambda(u)du} \,\middle|\, \mathcal{F}(t)\right] \;=\; e^{-\widehat{A}_2(t,T)-\widehat{B}_2(t,T)\,\lambda(t)} \;,$$

where $\widehat{B}_2(\cdot,T)$ solves

$$\begin{cases} \partial_t \widehat{B}_2(t,T) \;=\; \hat{\kappa}_2(t) \widehat{B}_2(t,T) + \frac{1}{2}\xi_2^2 \widehat{B}_2(t,T)^2 - 1 \;, \\ \widehat{B}_2(T,T) = 0 \,, \end{cases}$$

and $\widehat{A}_2(t,T) \;=\; \int_t^T \mu_2(u)\widehat{B}_2(u,T)\,du$.

Further note that $\widehat{B}_2(t,T) \geq 0$ and $\widetilde{B}_2(t,T) \geq 0$ for all $t \in [0,T]$ due to the negative source term and the terminal condition.

Consider first the case where $0 \leq p < 1$. Since $\psi_2(t) \geq 0$ by assumption, we have

$$\hat{\kappa}_2(t) \;=\; \kappa_2 + \frac{p}{1-p}\,\psi_2(t) \;\geq\; \kappa_2 \;,$$

for all t. By Lemma 4.3 we thus obtain that $0 \leq \widehat{B}_2(t,T) \leq \widetilde{B}_2(t,T)$. This implies that $e^{-\widehat{B}_2(t,T)z} \geq e^{-\widetilde{B}_2(t,T)z}$ for all $z \geq 0$. Moreover,

$$\widehat{A}(t,T) \;=\; \int_t^T \mu_2(u)\widehat{B}(u,T)\,du \;\;\leq\;\; \int_t^T \mu_2(u)\widetilde{B}(u,T)\,du \;=\; \widetilde{A}(t,T) \;,$$

hence

$$e^{-\widehat{A}_2(t,T)-\widehat{B}_2(t,T)z} \;\geq\; e^{-\widetilde{A}_2(t,T)-\widetilde{B}_2(t,T)z} \;,$$

for all $z \geq 0$.

Inequality (6.3) is obtained by similar reasoning. Note that if $p \leq 0$ then $\hat{\kappa}_2(t) \leq \kappa_2$ for all t, hence the direction of the inequality is reversed. □

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