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Optimizing an Investment Portfolio under Solvency II

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Abstract

Per 1 January 2016 the new supervision framework Solvency II will be implemented for insurance companies. In this paper we want to investigate the optimal investment strategy for an insurance company under the new Solvency II framework. We can formulate this as an optimization problem where the expected surplus is maximized subject to several constraints: a budget constraint and various ”risk constraints” meaning that the Solvency Capital Requirements (SCR) the company needs to hold for an asset portfolio cannot exceed a certain fraction of the initial surplus.

Contents

1 Introduction

Per January 1, 2016 the new supervision framework Solvency II will be implemented for insurance companies. Solvency II was proposed as a successor to the original Solvency I directive introduced in 1973 which already provided more realistic minimum capital requirements for insurance firms. The main goals of Solvency II is to enable harmonization of the European insurance market regarding rules and regulations. Similar to the banking regulations Basel II, Solvency II is composed of three pillars: financial requirements, governance/supervision and reporting/disclosure. This paper will study the solvency capital requirements (SCR) framework which is part of the first pillar.

We can formulate the introduction of the SCR as an optimization problem where the expected surplus at time $T = 1$ is maximized subject to several constraints: a budget constraint and ”risk constraints” implying that the SCR the company needs to hold for an asset portfolio cannot exceed 0.8 times the initial surplus (i.e. this is equivalent to a 125% solvency position at time 0).

The academic contribution of this thesis is a linear programming ”simplification” to the optimization problem with Solvency II constraints under multiple stress scenarios resulting in a stochastic LP. We thereby link principles of the Operations Research field to Finance and Actuarial Science. Previous literature has mostly dealt with either the portfolio optimization problem with value-at-risk constraints in general (Cho, 2008; Kaura, 2005) or used a more complex model but using the Solvency II constraints (Braun et al., 2013; Gatzert and Martin, 2012). Therefore, the stochastic linear programming approach would be a new contribution to the discussion on Solvency II. The obvious practical contribution of this study is to offer a method for insurers to optimize their investment portfolio in regard to the new Solvency II regulations. Depending on the obtained expected surplus of their investment portfolio contributions may need to be adjusted to cover the ”costs” induced by the new constraints.

In this paper we want to investigate the optimal investment strategy for an insurance company under the new Solvency II framework. In section 2, we will introduce the Solvency Capital Requirement framework, its sub-modules and how it is applied in an insurance company. In section 3, the SCR framework is transformed into a stochastic linear optimization problem maximizing the expected surplus of an insurance company subject to constraints imposed by the SCR. Section 4 will extend the methodology with the KNW-capital market model which provides us with a large set of market returns, i.e. term structures (bond returns) and stock returns (and thus, scenarios) for the stochastic linear program. Next, the methodology is applied using current data for (i) two bonds, (ii) all asset types and using the *haalbaarheidstoets* scenario set returns (based on the KNW model) for (iii) all asset types with multiple states. In section 6 the results obtained by the linear program will be discussed. Finally, section 7 will conclude the paper.

2 SCR Framework

The Solvency Capital Requirement (SCR) framework determines how much capital is required to ensure that insurance firms are able to meet their obligations over the next 12 months with a probability of at least 99.5% (Value-at-Risk). Insurance companies have the choice of using a standard model provided by the Solvency framework or adopting an internal model that has to be tested and approved by the regulator. Even in the case of adopting an internal model, applying the standard model parallel to the internal model might give a benchmark on what regulations require an individual insurer to take into account in their day-to-day business. In the UK, for instance the Financial Services Authority requires all companies to determine their SCRs using the standard formula for the first two years under Solvency II. Due to its relevance to any insurance firm we therefore further examine the standard approach of solvency calculations.

The standard model of the SCR can be described as a bottom-up process consisting of multiple different risk modules which are then combined in the basic SCR (BSCR) as depicted in Figure 1. Rather than including all modules of the SCR, in this paper we will concentrate on equity, interest rate and property risk of the market risk module, which already include the most prominent forms of financial instruments of an insurance company, i.e. stocks, bonds, real estate and other alternative assets, such as hedge funds, private equity and commodities.

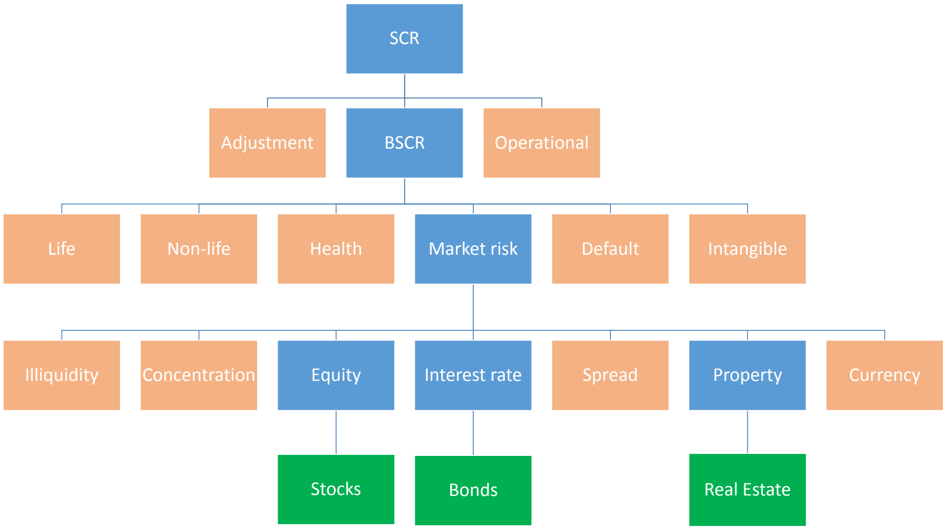

Figure 1: Overall structure of the SCR (see EIOPA (2014a))

Both the German Federal Financial Supervisory Authority (Bundesanstalt für Finanzdienstleistungsaufsicht (BaFin), 2011) and Gatzert and Martin (2012) have identified that market risk is the largest risk driver for life and health insurers and second largest in the property-casualty sector of the German insurance industry. Within the market risk module interest rate (and also spread risk) require the largest solvency capital for life and health insurers, whereas for property-casualty insurers only equity risk is a larger risk driver. Therefore, combining these three modules already captures most of the risk that insurance firms are exposed to. Adding the property risk to the interest rate and equity risk module enables us to include all the major alternative investments, as the equity risk module already includes private equity, commodities and hedge funds, via so-called type 2 equity.

The capital requirements are then calculated based on predefined shock scenarios and the usage of the basic own funds (BOF) where BOFs are market value of assets minus the best estimate of liabilities. The SCR covers the variation of the market value over a one year period.

2.1 Interest Rate Risk

The European Insurance and Occupational Pensions Authority (EIOPA, 2014b) specifies the interest rate risk sub-module under the following assumptions:

- *"Only interest rate risk that arises from changes in the level of the basic risk free interest rates is captured.*
- *Volatility and changes in the shape of the yield curve are not covered explicitly in the interest risk sub-module.*
- *The undertaking is not exposed to material inflation or deflation risk."*

The objective of this sub-module is to capture interest rate risk concerning asset and liability positions exposed to this kind of risk. The stressed upward and downward term structures are calculated using prescribed changes to the basic risk-free interest rates for given maturities. Furthermore, it is assumed that when interest rates are lower then the absolute shocks are lower as well which also holds for the opposite case. However, risks of inflation or deflation are not included in the interest risk module, therefore inflation/deflation risk should be considered separately.

The reason for the regulator to not include volatility and changes in the shape of the yield curve is due to the fact that these factors are only relevant when the insurer's asset portfolio and/or obligations are actually exposed to changes in interest rate volatility which is the case when liabilities explicitly contain embedded options and guarantees. On the asset side of the insurer's balance sheet in order to be exposed to interest rate volatility they specifically would need to hold derivatives hedging against interest rate fluctuations. Interest rate volatility is therefore not covered explicitly in the interest rate risk sub-module but can be described implicitly through positions in swaptions or other derivatives.

Four representative indicators for the term structure of interest rates were used in order to construct the interest rate shocks in the standard formula, namely EUR government zero coupon term structures (1997 to 2009), GBP government zero coupon term structures (1979 to 2009), and both Euro and GBP LIBOR/swap rates (1997 to 2009) with maturities ranging from 3 months to 30 years. The shock coefficients were determined through a Principal Component Analysis (PCA) for each of the four indicators with their respective maturities. Subsequently, the mean of the PCA results are taken for each maturity in order to obtain single shock coefficients.

The reasoning behind the choice of these four series used for the setup of the interest risk sub-module lies in their representation of the deepest and most liquid markets for fixed income in the European area. Using the mean of all four series ensures individual uncertainties of the series are balanced out and are less affected by factors that influence single bond markets. The EIOPA (2014b) also notes that the aforementioned series are based on EUR and GPB interest rates that reflect the European economic development over the past 30 years but may evolve differently compared to the past. As an example they give that the scenario similar to Japan's deflation in the 1990's might be possible in Europe.

The regulator also defines that the interest rate risk sub-module can be applied to captives of insurers and the underlying assumptions explaining its use; however, this paper will not further discuss captives.

The EIOPA has determined upward and downward shift regarding the interest rates as follows. The shocked term structures are calculated by multiplying the current interest rates by $(1+s^{\text{up}})$ and $(1+s^{\text{down}})$ given in the Solvency II guidelines (see Table 4), e.g. the shocked 3-year interest rate $r_1(3)$ in the upward shock scenario is given by

$$r_1(3) = r_0(3) \cdot (1 + 0.64).$$

Nevertheless, the absolute increase of interest rates in the upward shock scenario at any maturity is required to be at least one percentage point. Additionally, if the initial interest rate for a given maturity is negative the increase or decrease of the interest rate is calculated as the product of the relative change and the absolute value of the initial interest rate. An initially negative shocked 3-year interest rate $r_1(3)$ in the upward shock scenario is given by

$$r_1(3) = r_0(3) + |r_0(3)| \cdot 0.64$$

and under the downward shock scenario by

$$r_1(3) = r_0(3) - |r_0(3)| \cdot 0.56.$$

2.2 Equity Risk

The EIOPA (2014b) specifies the equity risk sub-module under the following assumptions:

- *"Assets and liabilities subject to equity risk are only exposed to a fall in the level of equity prices and not to a rise in those prices.*
- *The value of equity investments cannot fall below zero.*
- *For the split between type 1 and type 2 equities it is assumed that type 2 equities consist of more risky equities than the equities covered in the type 1 category. For this reason, the stress factor for type 2 equities is higher than for type 1 equities.*
- *The undertaking holds a type 1 equity portfolio that is well diversified with respect to geography (developed market countries), stock size (large, mid, small, micro cap), sectors and investment style (growth, value, income etc.).*

- *With regard to the type 2 equity portfolio, the undertaking owns a private equity portfolio of mainly large private equity companies, a commodity portfolio of liquid commodities, a hedge funds portfolio of medium and large size hedge funds trading on a transparent basis and a portfolio of equities in emerging markets. All these portfolios are assumed to be well diversified with respect to geography, stock size, composition, investment and financing style among other factors."*

The equity risk sub-module incorporates changes in the level of equity prices but only considers the downward equity stress scenario, unlike in the interest rate risk sub-module. The reasoning behind this lies in the fact that only a downward movement of equity prices is detrimental to the insurer's balance sheet whereas both up- and downward movements of the interest rate can reduce the size of the balance sheet depending on the positions taken on the asset side and how liabilities are valued. By holding investments in equities or equity derivatives and by embedding equity-linked options and guarantees to their liability portfolio insurers are affected by changes in equity volatility as well; nevertheless, the equity risk sub-module does not capture volatility explicitly.

Computing the equity risk capital charge can be done by applying either of two methods, the "standard" approach or the "duration based" approach. The standard approach is specified using the 99.5% VaR level for two different types of equities and includes a symmetric adjustment mechanism which allows the equity shock to move within a band of 10% on either side of the standard equity stress. The duration-based approach on the other hand is applied when the insurer considered provides occupational retirement provisions or retirement benefits and the position in equity is linked to the average duration of liabilities and exceeds 12 years. However, as this paper is only considering the investment optimization for a short horizon of 1 year we will apply the standard approach concerning the equity risk sub-module.

As previously stated, equities are distinguished between two types where Type I equities are equities listed in regulated markets in member countries of the EEA or the OECD and Type 2 equities include equities in stock exchanges not in the aforementioned countries as well as non-listed and private equities, hedge funds, commodities and other alternative investments. The EIOPA (2014b) specifies type 1 equity stress scenarios using data from the MSCI World Developed Price Equity Index from 1979 to the end of 2009, including data from recent and past crises while for the type 2 equity stress scenarios indices representing the private equity, commodities, hedge funds and emerging markets, namely LPX50 Total Return, S&P GSCI Total Return Index, HFRX Global Hedge Fund Index and MSCI Emerging Markets BRIC. They further observe that the distribution of equity returns appears to be normally distributed in the long run but at shorter horizons reveals non-normal features, i.e. fat tails which is also incorporated into the computation of equity risk capital charges.

The symmetric adjustment mechanism is implemented in order to adjust the initial 99.5% VaR and include the mean reverting behavior of equity prices assumed by the EIOPA. The prescribed falls in the value of equities in each category are given by Table 1.

Table 1: Prescribed drop in value of equity investments given by the EIOPA

Equity shock	
Type 1	**Type 2**
46.5%	56.5%

2.3 Property Risk

The EIOPA (2014b) specifies the property risk sub-module under the following assumptions:

- *"The risk-profile of any of the undertaking's exposures to property located in third countries is not materially different from the risk profile of European property markets.*
- *The distributions of property returns are characterized by long left-fat tails and excess kurtosis (signifying disparity from normal distribution)."*

Property shocks are defined as the direct effect in the case when there is a fall in real estate benchmarks and there are positions in property investments embedded in the insurer's balance. Again, volatility is not explicitly but implicitly included in the specification of property shocks. The EIOPA has used data on total return indices from UK retrieved from the Investment Property Databank (IPD) for the calibration of the property risk module. The IPD indices are generated using survey data from various property related institutions and are representative for the European market due to their common usage among commercial property indices. Even though there are indices for most European markets and even outside of the Euro area, most of them lack time series from further in the past, resulting in the choice of the UK market with data available from 1987 (data used for the calibration ranges until 2008). We should note that the total return index is not based on actual sales but on appraised market values which may incorporate previous valuated prices as appraisers tend to take into account former prices when valuating property related products, referred to as the backward-looking behavior of appraisers. Furthermore, it is assumed that the risk profile of the UK property market is very similar to the ones in other European and non-European markets as well as among other different property classes. We therefore have no different characterizations within the property risk sub-module and a property shock is simply defined as an instantaneous decrease of 25% on the value of investments in immovable property.

3 Stochastic Linear Program (LP)

Let an insurance firm hold an asset portfolio to cover its liabilities consisting of bonds, equity and property. Then its surplus at time $T = 1$ can be formulated as

$$x'V_1 - l_1,$$

assets less liabilities. If we now consider different states of the world then the insurer's objective is to maximize their surplus subject to a regular budget constraint and risk constraints, i.e. in each possible state of the world their surplus has to be at least 0 in order to stay solvent. This portfolio optimization problem can be translated into a stochastic linear program, giving us a framework which is able to deal with additional variation and uncertainty in a linear program. Figure 2 depicts the scenario tree of the portfolio optimization of an insurance company.

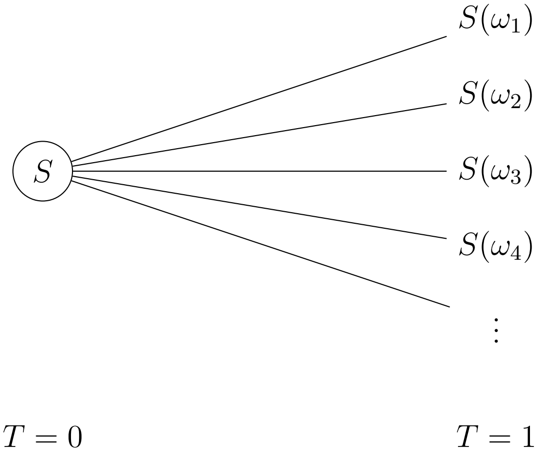

Figure 2: Investment portfolio optimization scenario tree

We therefore obtain the following maximization problem

$$\begin{aligned} \max_x \quad & \mathbb{E}[x'V_1(\omega) - l_1(\omega)] & \\ \text{s.t.} \quad & S(\omega_i) = x'V_1(\omega_i) - l_1(\omega_i) \geq 0 & \text{(risk constraints)} \\ & x'V_0 \leq A & \text{(budget constraint)} \end{aligned}$$

where

- $x = (x_1\ x_2\ \ldots\ x_K)'$ is the distribution of funds,
- $V_t = (V_{t1}\ V_{t2}\ \ldots\ V_{tK})'$ are the assets' prices at time t,
- l_t are the total liabilities at time t, and
- $\omega = (\omega_1\ \omega_2\ \ldots\ \omega_N)'$ correspond to different states of the world or "shock scenarios".

Insurance companies normally hold assets such that their surplus ratio lies at least in the range of 1.05 to 1.20. Since liabilities in the objective function are just a constant (maximizing the asset side of the surplus also maximizes the surplus regardless of the liabilities) and moving them to the right hand side in the risk constraints gives us

$$\begin{aligned} \max_x \quad & \mathbb{E}[x'V_1(\omega)] \\ \text{s.t.} \quad & x'V_1(\omega_i) \geq l_1(\omega_i) \\ & x'V_0 \leq A. \end{aligned}$$

After formulating the linear program with the appropriate parameters we use the `lpSolve` package (Berkelaar et al., 2015) implemented in `R` to solve the linear program. The linear programming solver is based on the commonly used revised simplex method which originates from Dantzig's simplex method but uses a matrix-oriented approach instead of the tableau form.

3.1 Assets

For the portfolio optimization the set of assets an insurance firm can choose from has to be determined. Staying within the Solvency II framework different smaller portfolios which represent the stock market and the bond market will be considered instead of single individual assets. We therefore assume that an insurance company diversifies its assets within the different asset classes. Hence, each asset class can be described by a prevalent benchmark index or returns will be provided by a market model.

In the Solvency II market risk framework, we will consider European government bonds, equity and investments in immovable property and their respective risk modules. The prices of bonds at time $T = 0$ are determined by using the zero rates and transforming them into discount factors, or equivalently zero prices

$$\delta_0(t) = (1 + r_t)^{-t}$$

where r_t is the spot rate of interest on a zero coupon bond with a time to maturity of t. Using zero coupon bonds proves to be quite practical since we can combine zero coupon bonds of different maturities in order to replicate the cash flows of a regular coupon paying bond. Stocks and property prices are simply given by 1 in the budget constraint whereas bond prices are determined by the current zero rates.

For the expected value of bonds we use forward rates to transform the current term structure. The implied forward rates are defined as

$$f(j, k) = \left(\frac{(1 + y_k)^k}{(1 + y_j)^j}\right)^{\frac{1}{k-j}} - 1$$

which is an agreed-upon rate between two future dates j and k, where $k \geq j$, at which one can borrow or lend on date j for a loan maturing on date k (Sundaresan, 2009). For $T = 1$ in the LP we have that $j = 1$. The forward rates are transformed into discount rates as well:

$$\begin{aligned}\delta_1(t) &= (1 + f(1, t))^{-(t-1)} \\ &= \frac{\delta_0(t)}{\delta_0(1)}\end{aligned}$$

Concerning the risk constraint, shock scenarios have to be taken into account in the asset value calculation. For this purpose, the shock scenarios defined by the EIOPA as stated above are applied to a given term structure. Figure 3 illustrates the implied forward rate curve as of end of September 2016 between its upward and downward stressed states. Note, that next to a slight increase in the upward rates slope there is a 0.01 upward shift as specified in the Solvency II guideline.

Using the resulting shocked interest rates we can determine discount factors for the asset valuation in the risk constraint. Note that, for equity and property risk modules the EIOPA only considers downward shifts in contrast to the interest risk sub-module.

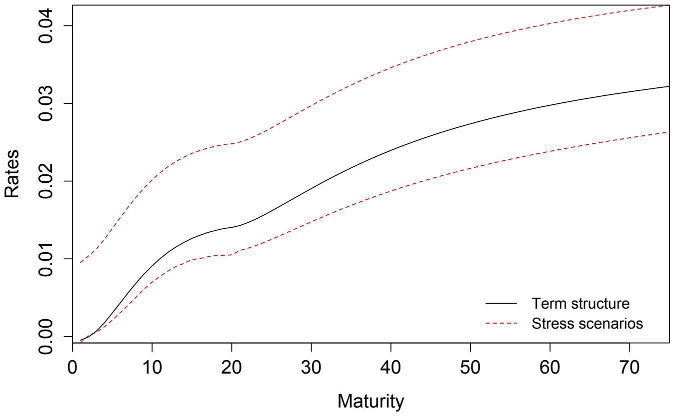

Figure 3: Implied forward rate curve and upward/downward stress scenarios

3.2 Liabilities

For simplicity, regarding the liability side of the balance sheet suppose the insurer faces a geometric decay function describing the payments, i.e. at time t the insurance firm has to pay out

$$c(t) = c_0 \, t \cdot 0.95^t$$

where c_0 is the initial amount of the insurance policy. We take $c_0 = 1$ as assets will be scaled to liabilities through the budget constraint, therefore the value can be chosen arbitrarily, as long as $c_0 > 0$. Let the insurance policies of the firm have a maturity of 75 years, then at time $T = 1$ the insurance firm has to make sure that it is able to pay the payments for 74 years in the future. We therefore multiply the payments with the (shocked) discount factors and accumulate them to get the (shocked) liabilities. For $s \in \{0, 1\}$

$$\begin{aligned} l_s(\omega_i) &= \sum_{j=1}^{75} c(j) \cdot \delta_s(j, \omega_i) \\ &= \sum_{j=1}^{75} j \, 0.95^j \cdot \delta_s(j, \omega_i) \end{aligned}$$

Figure 4 shows an example of the present value of future cash flows, i.e. the insurance company's liabilities valued at time $T = 0$ discounted with the term structure as of 30/09/2015 retrieved from the EIOPA website.

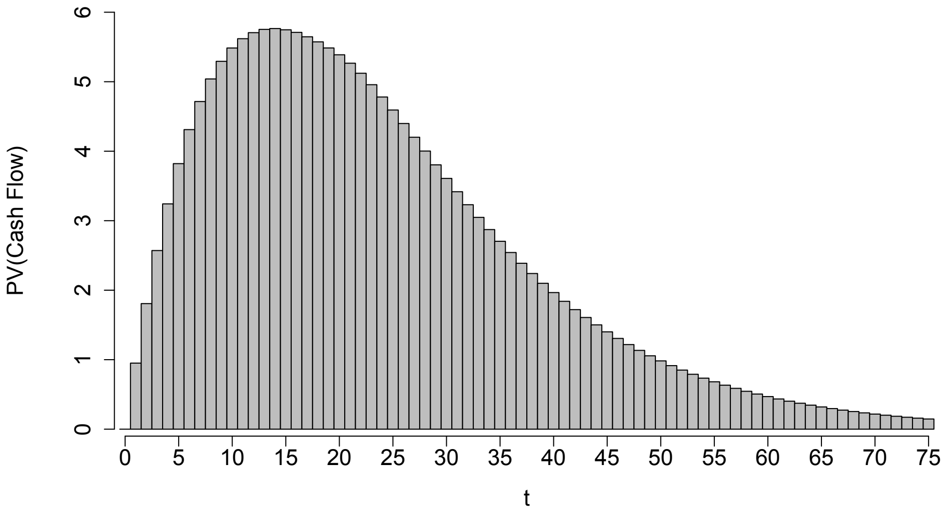

Figure 4: Liabilities of an insurance firm discounted with 30/09/2015 term structure

3.3 Scenario generation

We can distinguish between four main groups of scenarios that form the risk constraints of the stochastic LP. The market risk module of the SCR framework provides us with stress scenarios for given current prices of fixed income, equity and property investments, namely a sudden fall in the value of equity and investments in immovable property and both an increase and decrease of the term structure of interest rates. In order to obtain a more robust result in the portfolio optimization we further consider a model that provides us with a larger scenario set, which will be introduced in the following section.

4 KNW-capital market model

The capital market model by Koijen, Nijman & Werker (KNW) (2010) is based on the investment behavior of a life-cycle investor subject to borrowing and short-sale constraints. Even though with the KNW-model the so called *haalbarheidstoets* scenario sets were developed for pension fund stress tests we can use these for insurance companies as well, as the theory of life-cycle investment behavior is also applicable to life insurers.

The authors distinguish between four age periods in the life cycle of households. The first being in the age of 25 to 35 where the individual has a large stock of nontradable human capital but due to borrowing constraints is not able to use future labor income to increase their present consumption. Therefore, all income is mostly consumed and participation in financial markets is highly restricted. The next period between age 35 and 45, the investor is now able to invest their accumulated income (now, less limited by borrowing constraints) in equity markets essentially. Koijen et al. (2010) argue that as human capital can be treated as a nontradable position in inflation-linked bonds and in order to reduce their equity allocation to invest in bonds cannot borrow using human capital as collateral. The resulting high opportunity costs which cannot be offset by any realistic bond risk premium lead to the choice of equity. During the next stage between age 45 and 55, the investor's human capital has declined significantly and they shift their investment to bonds. Additionally, it is assumed that they optimally increase their investments in long-term bonds during times of high bond risk premia. In the last stage during age 55-65 human capital is mostly depleted. Consequently, the individual shifts their equity and bond positions further to cash of up to 20% until retirement. In times of high bond risk premia the investor will reduce their cash position first before reducing their equity position as the opportunity costs that arise due to this shift are smaller for cash than for equity, though it should be noted that this will only happen when bond risk premia are sufficiently high.

Koijen et al. (2010) assume that the asset menu of the life-cycle investor basically consists of stocks, (long-term nominal) bonds and cash (nominal money market account). Muns (2015) summarizes the key determinants of pension risk as inflation $\mathrm{d}\Pi/\Pi$, the stock return $\mathrm{d}S/S$, the bond portfolio return $\mathrm{d}P^B(\tau)/P^B(\tau)$ with constant maturity τ, the term structure

$y(\tau)$ and their dynamics as follows

$$\begin{aligned}
&\text{State parameters} & \mathrm{d}X_t &= -KX_t\mathrm{d}t + \mathrm{d}\tilde{Z}_t && (1)\\
&\text{Instantaneous expected inflation} & \pi_t &= \delta_{0\pi} + \delta_{1\pi}'X_t && (2)\\
&\text{Price index process} & \frac{\mathrm{d}\Pi}{\Pi} &= \pi_t\mathrm{d}t + \sigma_{\Pi}'\mathrm{d}Z_t && (3)\\
&\text{Instantaneous nominal interest rate} & R_t^0 &= \delta_{0R} + \delta_{1R}X_t && (4)\\
&\text{Stock return process} & \frac{\mathrm{d}S}{S} &= (R_t^0 + \eta_S)\mathrm{d}t + \sigma_s'\mathrm{d}Z_t && (5)\\
&\text{Bond return process} & \frac{\mathrm{d}P_t^B(\tau)}{P_t^B(\tau)} &= \left(R_t^0 + B(\tau)'\tilde{\Lambda}_t\right)\mathrm{d}t + B(\tau)'\mathrm{d}\tilde{Z}_t && (6)\\
&\text{Prices of risk} & \Lambda_t &= \Lambda_0 + \Lambda_1 X_t && (7)\\
&\text{Stochastic discount factor} & \frac{\mathrm{d}\phi_t}{\phi} &= -R_t^0\mathrm{d}t - \Lambda'\mathrm{d}Z_t && (8)
\end{aligned}$$

where $\mathrm{d}Z \sim N(0_{(k+2)\times 1}, I_{(k+2)\times(k+2)})$ and

$$\begin{array}{lll}
K \in \mathbb{R}^{k\times k} & \sigma_{\Pi}, \sigma_S, \Lambda_0 \in \mathbb{R}^{k+2} & \mathrm{d}\tilde{Z}_t = [I_{k\times k}\ 0_{k\times 2}]\,\mathrm{d}Z_t\\
\delta_{1\pi}, \delta_{1R} \in \mathbb{R}^k & \Lambda_1 \in \mathbb{R}^{(k+2)\times k} & \tilde{\Lambda}_t = [I_{k\times k}\ 0_{k\times 2}]\,\Lambda_t.
\end{array}$$

Furthermore, Koijen et al. (2010) specify that $\sigma_{\Pi(k+2)} = 0$ and K to be a lower triangular matrix such that X and Z are identifiable.

The complete model is then summarized as a multivariate Ornstein-Uhlenbeck process (Muns, 2015):

$$\mathrm{d}\Upsilon_t = (\Theta_0 - \Theta_1\Upsilon_t)\mathrm{d}t + \sigma_{\Upsilon}\mathrm{d}Z_t \tag{9}$$

where

$$\Upsilon_t = \begin{bmatrix} X_t \\ \log\Pi_t \\ \log S_t \\ \log P^B(\tau) \end{bmatrix} \qquad \sigma_Y = \begin{bmatrix} I_{k\times k} & 0_{k\times 3} \\ \multicolumn{2}{c}{\sigma_{\Pi}'} \\ \multicolumn{2}{c}{\sigma_S'} \\ B(\tau)' & 0_{1\times 3} \end{bmatrix}$$

$$\Theta_0 = \begin{bmatrix} 0_{k\times 1} \\ \delta_{0\pi} - \frac{1}{2}\sigma_\Pi'\sigma_\Pi \\ \delta_{0R} + \eta_S - \frac{1}{2}\sigma_S'\sigma_S \\ \delta_{0R} + B(\tau)'\tilde{\Lambda}_0 - \frac{1}{2}B(\tau)'B(\tau) \end{bmatrix} \qquad \Theta_1 = \begin{bmatrix} -K & 0_{k\times 3} \\ \delta_{1\pi}' & 0_{1\times 3} \\ \delta_{1R}' & 0_{1\times 3} \\ \delta_{1R}' + B(\tau)'\tilde{\Lambda}_1 & 0_{1\times 3} \end{bmatrix}$$

$$\tilde{\Lambda}_0 = [I_{k\times k} \quad 0_{k\times 2}]\,\Lambda_0 \qquad \tilde{\Lambda}_1 = [I_{k\times k} \quad 0_{k\times 2}]\,\Lambda_1.$$

In order to estimate the model (9) is discretized into a VAR(1)-model

$$Y_t = \gamma + \Gamma Y_{t-h} + \varepsilon_t \qquad \varepsilon \sim N(0,V)$$

where

$$Y_t = \begin{bmatrix} X_t \\ \Delta \log \Pi_t \\ \Delta \log S_t \\ \Delta \log P^B(\tau) \end{bmatrix} \quad \text{and} \quad \begin{aligned} \gamma_t &= UFU^{-1}\Theta_0 \\ \Gamma &= U\exp(Dh)U^{-1} \\ V &= UWU' \\ \Theta_1 &= UDU^{-1} \end{aligned}$$

are obtained from eigenvalue decomposition. The model is estimated using data ranging back to 1970 on yields, stock returns and inflation and by maximum likelihood applying Kalman filtering. For inflation the Western German consumer price index (1970-1990) and the Harmonized Index of Consumer Prices of the euro area were used while for yields of six different maturities (three-month, one-year, two-year, three-year, five-year and ten-year), German money market rates, interbank rates, zero-coupon rates constructed from swap rates by *De Nederlandsche Bank* (DNB) and zero-coupon yields based on government bonds were used. Stock returns are based on the MSCI index in euros or *Deutsche Mark* before 1999 hedged against the US dollar. [1]

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[1] For further details on the exact details on the derivation of the KNW model, see the original paper of Koijen et al. (2010) or Draper (2014) and Muns (2015)

5 Application

In the following the methodology is applied to several numerical examples with increasing complexity.

5.1 Two assets

As a simple example, consider only two zero-coupon bonds with 5 and 30 years maturity respectively. The scenario tree in this example merely consists of two different states, the upward (ω^{I+}) and the downward shift (ω^{I-}) of the term structure.

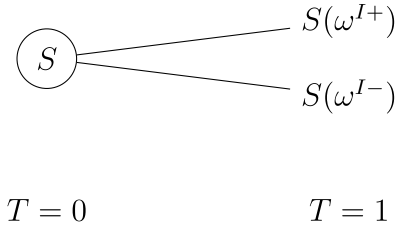

Figure 5: Investment portfolio optimization scenario tree with only two bonds

In this case, we consider the term structure from 30/09/2015, i.e. the annual zero-coupon spot rates of the Euro area denoted as the basic risk-free rate retrieved from the EIOPA website. The rates are given in Table 2, with r_s being the rates retrieved from the data.

Table 2: Zero coupon bond rates, 5 and 30 year maturity

	5-year	30-year
r_s	0.248%	1.949%
$f(1,s)$	0.329%	2.202%
$f(1,s)^{up}$	1.329%	3.202%
$f(1,s)^{dn}$	0.165%	1.457%

With a 1-year zero rate of -0.076% we obtain the forward rates, $f(1,k)$, i.e. the zero rates at time $T = 1$.

$$f(1,5) = \left(\frac{1.00245^5}{0.99924}\right)^{\frac{1}{4}} - 1 = 0.00329$$
$$f(1,30) = \left(\frac{1.01949^{30}}{0.99924}\right)^{\frac{1}{29}} - 1 = 0.02020$$

These forward rates are then shocked using the interest rate risk module of the SCR standard model, resulting in $f(1,k)^{up}$ and $f(1,k)^{dn}$. Note, that at time $T = 1$ bonds with maturity t are now bonds with maturity $t-1$, therefore we use the shock factors of maturity 4 and 29 respectively. Furthermore, the upward shock is required to be at least 1%.

$$\begin{aligned}
f(1,5)^{up} &= f(1,5) + 0.01 = 0.01329 \\
f(1,30)^{up} &= f(1,30) + 0.01 = 0.03020 \\
f(1,5)^{dn} &= (1 + s^{dn}(4))\, f(1,5) = 0.00165 \\
f(1,30)^{dn} &= (1 + s^{dn}(29))\, f(1,30) = 0.01457
\end{aligned}$$

where $s^{dn}(29) = s^{dn}(20)\frac{dn(90)-dn(20)}{90-20}(29-20) = -27.843\%$ was obtained from linear interpolation. All the obtained rates are transformed into discount factors.

$$\begin{aligned}
\delta_0(5) &= (1.00248)^{-5} = 0.98769 & \delta_0(30) &= (1.01949)^{-30} = 0.56042 \\
\delta_1(5) &= (1.00329)^{-4} = 0.98694 & \delta_1(30) &= (1.02020)^{-29} = 0.55999 \\
\delta_1^{up}(5) &= (1.01329)^{-4} = 0.94855 & \delta_1^{up}(30) &= (1.03020)^{-29} = 0.42202 \\
\delta_1^{dn}(5) &= (1.00165)^{-4} = 0.99344 & \delta_1^{dn}(30) &= (1.01457)^{-29} = 0.65734
\end{aligned}$$

While for the asset side the discount factors are used as the zero prices, for the liabilities side they are combined with the aforementioned payment scheme (geometric decay) with $c_0 = 1$.

After having calculated the discount factors of the bonds and the liabilities in the shock scenarios, the optimization problem is then given by

$$\begin{aligned}
\max \quad & 0.98694\, x_5 \; + 0.55999\, x_{30} \\
\text{s.t.} \quad & 0.94855\, x_5 \; + 0.42202\, x_{30} \geq 152.62818 \\
& 0.99344\, x_5 \; + 0.65734\, x_{30} \geq 214.13706 \\
& 0.98769\, x_5 \; + 0.56042\, x_{30} \leq 198.83041
\end{aligned}$$

where we have chosen $A = 1.05\, l_0$ for the budget, i.e. the minimum surplus level allowed. As one can see in Figure 6, both shock constraints and the budget constraint form a feasible set. Solving the maximization problem gives us the intersection point between budget and downward shock constraint, $(x_5,\ x_{30}) = (115.59,\ 151.06)$ with a maximum of 198.68. We have that $(x_5,\ x_{30})\, V_0 = (114.17,\ 84.66)$ which gives us the amount invested in each bond. Given that the expected value of the liabilities at time $T = 1$ is 188.27 we obtain an expected surplus of $\mathbb{E}[S] = 10.41$, i.e. the initial surplus increased from 5% to 5.529%.

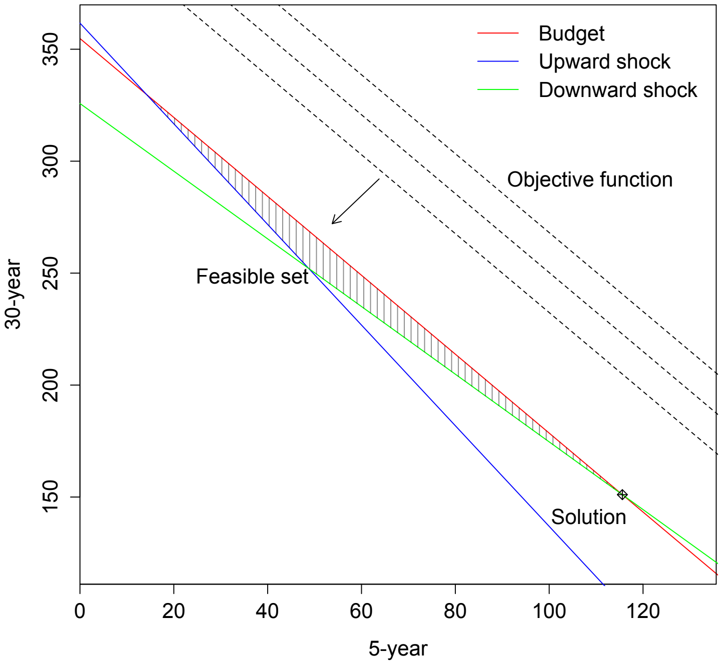

Figure 6: Graphical representation of the LP.

5.2 Adding equity and property assets

In this application of the LP methodology, next to the zero-coupon bonds with maturity 5 and 30 years as before, we will include equity of both type 1 and type 2 and property for the asset side. Including these three asset types extends the scenario tree by three additional states at time $T = 1$ depicted in Figure 7.

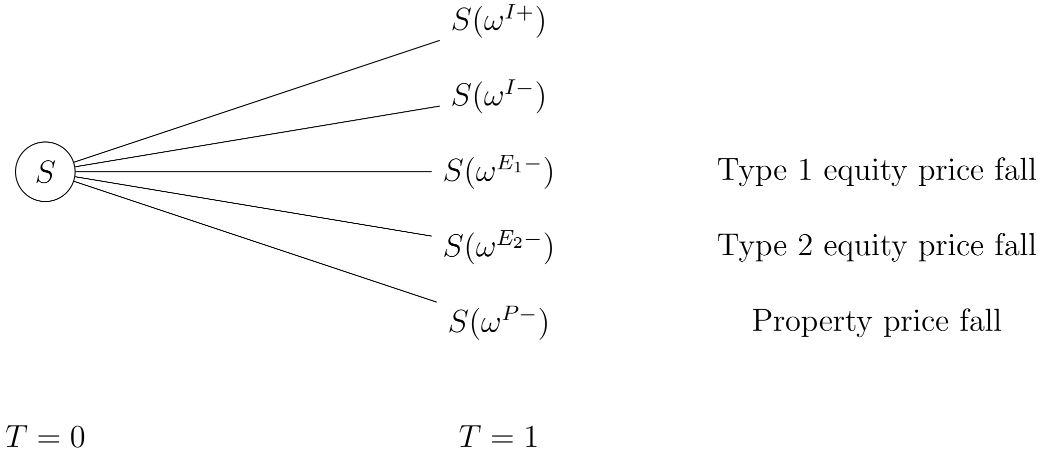

Figure 7: Scenario tree including equity and property value stress scenarios

For the purpose of this example we again consider the term structure from 30/09/2015 retrieved from the EIOPA website. Valuation of the bond prices 1-year ahead is done by transforming the spot rates to forward rates as explained previously. The bond prices are the same as in the previous example. For the equity and property valuation we are using historical data of the respective indices up until September 2015 and fit an ARIMA(p, d, q) to the data. These indices are the same that were used for the calibration of the solvency stress factors. The data series of the six indices were retrieved from *Datastream*. The appropriate parameters of the ARIMA model are determined by applying a stepwise algorithm developed by Hyndman and Khandakar (2007) by means of the `auto.arima` function in the `forecast` package implemented in `R` [2]. Using the ARIMA models we can forecast the price 12-months ahead and obtain a 1-year expected return for each of the indices [3]. The resulting expected returns are summarized in Table 3 which in turn are used to obtain the expected value of equity or property.

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[2] See the appendix for further notes on the `auto.arima` function

[3] See Figure 15 in the appendix for the plots of the historical data and forecasts

Table 3: Expected returns obtained from the ARIMA forecasts on indices

Index	**Expected Return**
MSCI World Developed Price Equity	1.706%
LPX50 Total Return	-5.940%
S&P GSCI Total Return	10.628%
HFRX Global Hedge Fund	-1.564%
MSCI Emerging Markets BRIC	3.262%
IPD Total Return UK	6.317%

Consequently, the return for type 1 equity equals 1.706% (MSCI World Developed Price Equity), type 2 equity equals 1.596% (averaged over the four type 2 indices) and property equals 6.317% (IPD Total Return UK).

Liabilities (at $T = 0$ and shocked at $T = 1$) are calculated using the expected cash flows explained above discounted by the respective term structures. The budget is restricted to 1.05 times the liabilities at time $T = 0$ as in the previous example. Allowing the budget constraint to be less restrictive simplifies the LP problem in terms of the asset allocation, therefore the lower bound was chosen.

The resulting linear program consists of 5 decision variables and six constraints, the two interest rate, two equity and one property shock scenario(s) and a budget constraint. With $x = (x_5\ x_{30}\ x_{e_1}\ x_{e_2}\ x_p)'$ the LP is given by

$$\begin{aligned}
\max_x \quad & x'\ (0.98694 \quad 0.55999 \quad 1.01706 \quad 1.01596 \quad 1.06317)' \\
\text{s.t.} \quad & \begin{pmatrix} 0.94855 & 0.42202 & 1.01706 & 1.01596 & 1.06317 \\ 0.99344 & 0.65734 & 1.01706 & 1.01596 & 1.06317 \\ 0.98694 & 0.55999 & 0.54414 & 1.01596 & 1.06317 \\ 0.98694 & 0.55999 & 1.01706 & 0.44194 & 1.06317 \\ 0.98694 & 0.55999 & 1.01706 & 1.01596 & 0.79738 \end{pmatrix} \begin{pmatrix} x_5 \\ x_{30} \\ x_{e_1} \\ x_{e_2} \\ x_p \end{pmatrix} \geq \begin{pmatrix} 152.62818 \\ 214.13706 \\ 188.26838 \\ 188.26838 \\ 188.26838 \end{pmatrix} \\
& x'\ (0.98769 \quad 0.56042 \quad 1 \quad 1 \quad 1\ )' \leq \quad 198.83041
\end{aligned}$$

Solving this linear program yields an optimal value of 203.30 with an asset mix of $\hat{x} =$

$(0,\ 150.45,\ 31.78,\ 26.18,\ 56.55)'$, which corresponds to investing €0, €84.31, €31.78, €26.18 and €56.55 into the respective assets (obtained by $\hat{x}'V_0$). Figure 8 shows the optimal asset allocation of an insurance company using current data and the ARIMA model to forecast expected returns.

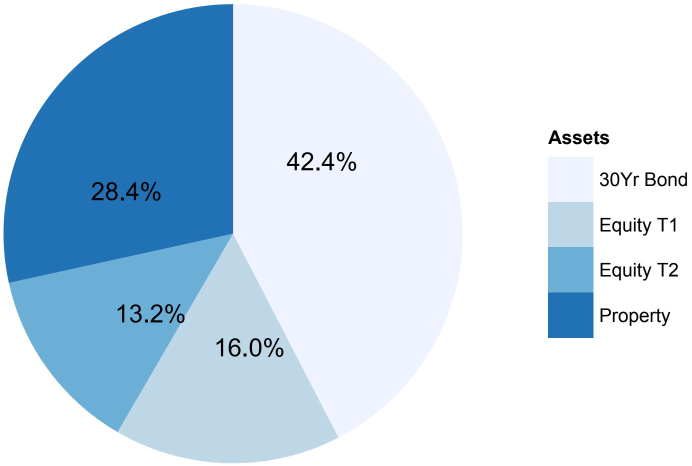

Figure 8: Optimal asset allocation projected to Sept. 2016

5.3 *Haalbaarheidstoets* - Feasibility Tests

Instead of one single term structure we will now consider multiple term structures generated for the *haalbaarheidstoets*, or feasibility tests provided by DNB. In theses tests the risk free rates, ${}_iR_t^m$, for scenario i ($i = 1, \ldots, 2000$) in the projection year t ($t = 0, ..., 60$) for period m ($m = 1, \ldots, 75$) are given by:

$${}_iR_t^m = \exp\left[a^m + b^m(1) \cdot {}_iX_1(t) + b^m(2) \cdot {}_iX_2(t)\right] - 1$$

where, a and b are (nominal or real) interest rate parameters and $X1$ and $X2$ are state variables. All the parameters a, b, X_1 and X_2 are provided in the *haalbaarheidstoets* scenario set and were initially estimated by using the KNW-model applied with data underlying the annual statements for the financial year. For the LP we are only interested in the term structure at $T = 0$ and the projection year $T = 1$. Furthermore, the Solvency II risk module specifies that interest rate stresses are applied to the basic risk-free rates only, thus the nominal interest rate parameters are used. We therefore obtain the current term structure ($T = 0$) and 2000 term structures for different states of the world in the projection year $T = 1$.

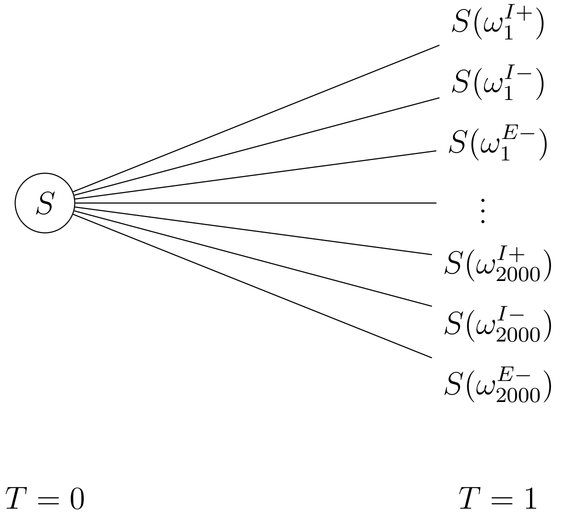

Figure 9: Scenario tree *haalbaarheidstoets*

The scenario tree of the stochastic LP is now extended to 6000 different states, for each of the 2000 *haalbaarheidstoets* scenarios we obtain three shock scenarios through the SCR framework, namely the up- and downward shifts in the term structure and the negative shock in stock prices (See Figure 9).

Figure 10 illustrates the current term structure and 20 of the 2000 projected term structures (the remaining terms structures lie in the range). Furthermore, equity returns are also given by the *haalbaarheidstoets* framework. Unfortunately, the KNW model and thus the *haalbaarheidstoets* scenario set do not provide us with any real estate related asset returns, therefore the property risk module could not be included in the following stochastic LP.

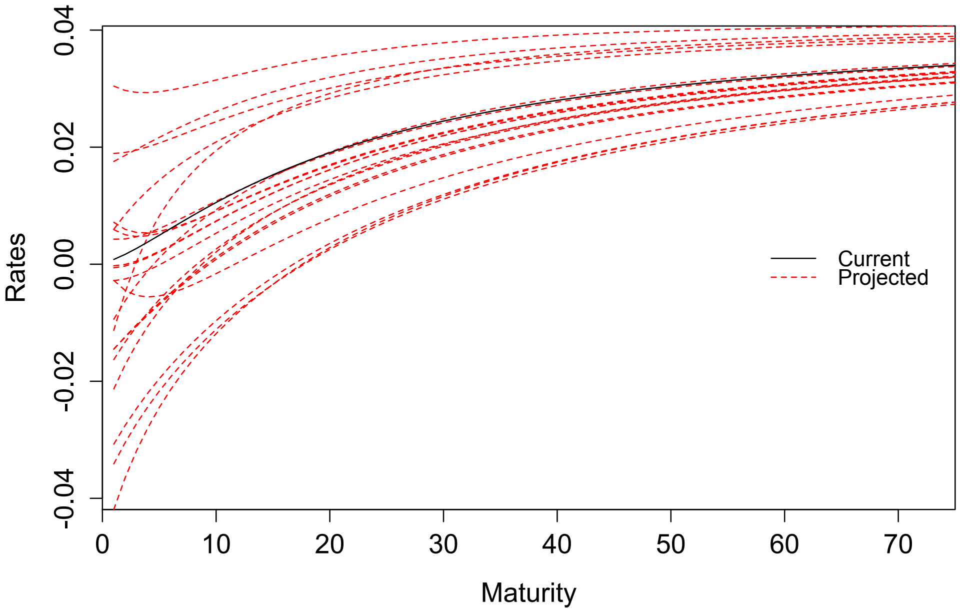

Figure 10: Current term structure and 20 of the 2000 different projected term structures generated for the *haalbaarheidstoets*

The projected term structures and equity prices are again stressed using the Solvency II risk module framework in the same manner as before, liabilities are calculated similarly as well. The only difference is that the term structures can be used without having to transform the current term structure to forward rates. In this application of the methodology the insurance firm can invest in zero coupon bonds with 1 to 30 years maturity and equity, while on the liabilities side the whole range of the term structure, i.e. 75 years, is used. [4] The

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[4] See Figure16 in the appendix for the distributions of a selection of interest rates

resulting stochastic LP the insurance firm faces can be summarized as follows

$$\begin{aligned} \max_x \quad & x' \, \mathbb{E} \begin{pmatrix} V_1 \\ V_e \end{pmatrix} \\ \text{s.t.} \quad & \begin{pmatrix} V_1^{up} & V_e \\ V_1^{dn} & V_e \\ V_1^0 & 0.535 \, V_e \end{pmatrix} x \geq \begin{pmatrix} L_1^{up} \\ L_1^{dn} \\ L_1^0 \end{pmatrix} \\ & x' \begin{pmatrix} V_0 \\ 1 \end{pmatrix} \leq s \, l_0 \end{aligned}$$

where the allocation x and the expected asset prices are defined as

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_{30} \\ x_e \end{pmatrix}, \qquad \mathbb{E} \begin{pmatrix} V_1 \\ V_e \end{pmatrix} = \mathbb{E} \begin{pmatrix} 1 \\ \delta_1(1) \\ \vdots \\ \delta_1(29) \\ 1 + r_e \end{pmatrix}, \qquad \text{where} \qquad \begin{array}{rcl} \mathbb{E}\left(\delta_1(s)\right) & = & \frac{1}{2000} \sum_{i=1}^{2000} {}_i\delta_1(s) \\ \mathbb{E}\left(r_e\right) & = & \frac{1}{2000} \sum_{i=1}^{2000} {}_i r_e \end{array}$$

i.e. in the objective function the expected asset prices are averaged over the 2000 scenarios. In the risk constraints for different states of the term structure, $\omega \in \{\text{up, dn, } 0\}$ (upward, downward and unchanged) bonds and stock prices are given by

$$V_1^\omega = \begin{pmatrix} 1 & {}_1\delta_1^\omega(1) & \dots & {}_1\delta_1^\omega(29) \\ \vdots & \vdots & \dots & \vdots \\ 1 & {}_{2000}\delta_1^\omega(1) & \dots & {}_{2000}\delta_1^\omega(29) \end{pmatrix}, \qquad V_e = \begin{pmatrix} 1 + {}_1r_e \\ \vdots \\ 1 + {}_{2000}r_e \end{pmatrix}.$$

Bonds of maturity m now have maturity $m-1$ at $T = 1$, e.g. bonds with 30 years maturity have 29 years remaining until maturity, while 1-year maturity bonds have matured and are not discounted which is expressed by the 1's in the first column of the matrices V_1^ω. Liabilities in the risk constraints are computed by

$$L_1^\omega = \begin{pmatrix} {}_1l_1^\omega \\ \vdots \\ {}_{2000}l_1^\omega \end{pmatrix}, \qquad \text{where} \qquad {}_il_1^\omega = \sum_{j=2}^{75} j \, 0.95^j \, {}_i\delta_1^\omega(j-1).$$

Lastly, in the budget constraint we have

$$V_0 = \begin{pmatrix} \delta_0(1) \\ \vdots \\ \delta_0(30) \end{pmatrix} \qquad \text{and} \qquad s\, l_0 = s \sum_{j=1}^{75} j\, 0.95^j \delta_0(j)$$

where s is the surplus ratio. Solving the LP for $s = 1.05$ gives us the solution in Figure 11(a). The graphs in Figure 11 illustrate the optimal asset allocations obtained from solving the linear program for different values of s in comparison to the insurance firm's liabilities. Note, that the left vertical axis corresponds to the discounted liabilities while the right vertical axis corresponds to the asset allocation. While bonds are linked to their maturity on the horizontal axis, stocks were simply plotted in the middle of the graph for illustrative purposes and are independent of the maturity. [5]

The first and clearest observation one can make from these results is that the "wealthier" the insurance company is, the higher its allocation into equity optimizes its surplus. This is quite sensible as when the insurance firm has matched its liabilities by investing in bonds the leftover assets can be spend in equity in order to profit from extra returns.

Taking a look at the asset distribution among bonds, there seems to be more diversification in the medium-term to lower end of long-term bonds next to the major position of around 50% in 30 Year bonds when the surplus ratio is low. With increasing surplus these two main positions seem to converge to a middle ground. Compared to Figure 11(a) in Figure 11(b) the position in 30 Year bonds has decreased to 7.5% while the remaining asset allocation has shifted with a small proportion to 14 and 15 year bonds and with a position of over 60% to 23 Year bonds. Increasing the surplus ration by 0.05 further, apart from an increased position into equity, shifts the bond position solely to 23 and 24 year bonds with almost 60% in the former and 23% in the latter (See Figure 11(c). Finally, figure 11(d) reveals that if the insurance firm has an even higher surplus ratio of 1.20 the optimal allocation is further shifted to bonds with a longer maturity with 35% invested in 25 Year bonds and 42% in 26 Year bonds whereas the rest of its budget is spend on equity.

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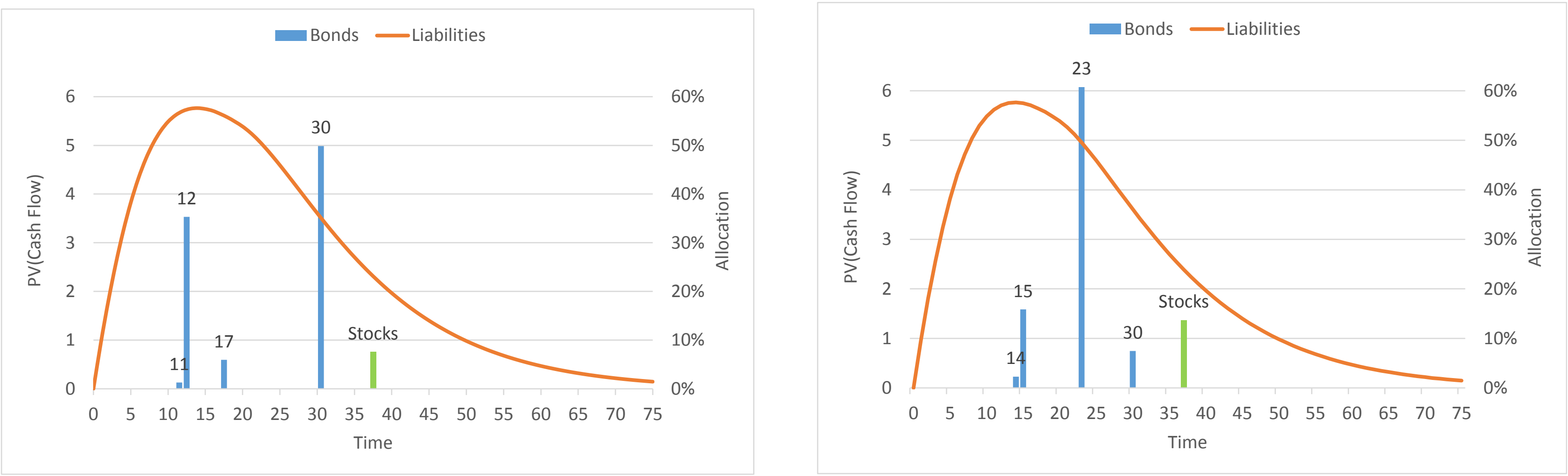
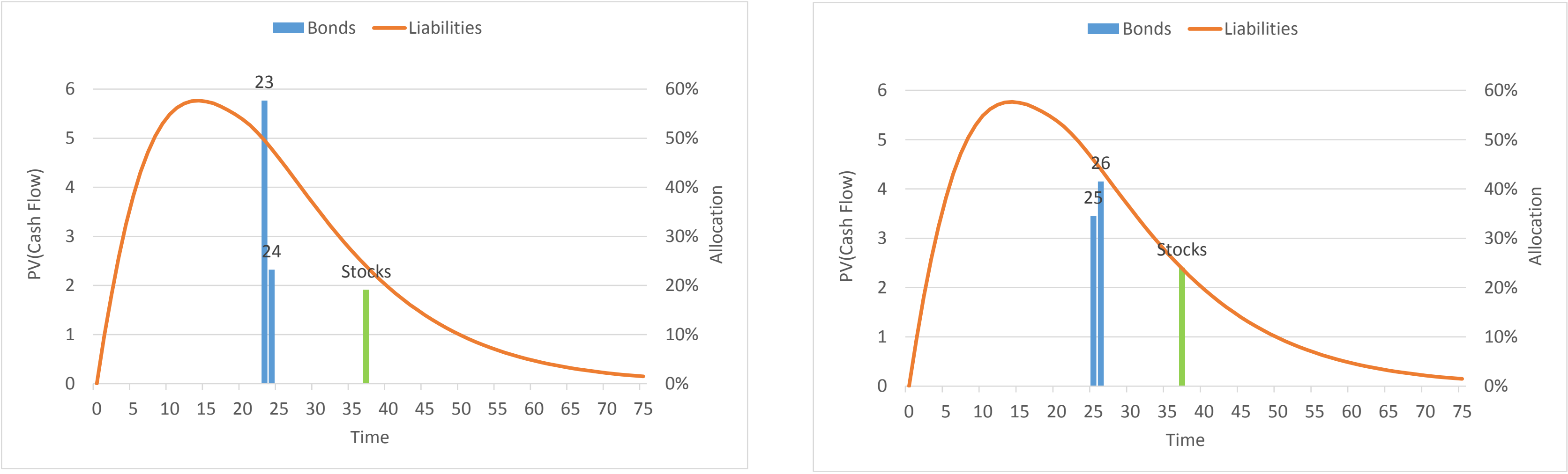
[5] See Table 6 in the appendix for the numerical values



(a) Surplus ratio 1.05

(b) Surplus ratio 1.10



(c) Surplus ratio 1.15

(d) Surplus ratio 1.20

Figure 11: Asset allocation vs. liabilities at T=0

These four cases already show that given the scenario set the solution to the portfolio optimization highly depend on how large the insurance company's budget and thus its initial surplus is. We therefore computed all the optimal asset allocations for surplus ratios ranging from the lowest ratio, 1.007, that provided us a feasible solution to 1.50.

Figure 12 represents the optimal asset allocations for surplus ratios between 1.007 and 1.08. The equity position growth pattern is clearer in this graph. The allocation into equity increases linearly with the surplus ratio, more specifically approximately 2.38% per additional 1% of surplus. Bonds with 30 Years maturity start off with a position of 70% and decrease almost linearly with an increasing surplus. In the interval 1.007 to 1.017 apart from the position in 30 Year bonds the optimal allocation includes positions in 5, 6 and 7 Year maturity bonds, while the allocation into the latter is rather small, between 20 and 50 basis points. Distribution in 5 year bonds first increases, decreases and increases again while the position in 6 year bonds do the opposite, together they amount to about 30%. In the surplus ratio interval 1.017 to 1.047 the allocation into bonds has a more regular pattern, gradually shifting the optimal allocation in bonds from 6 years to 12 years maturity, while increasing equity and

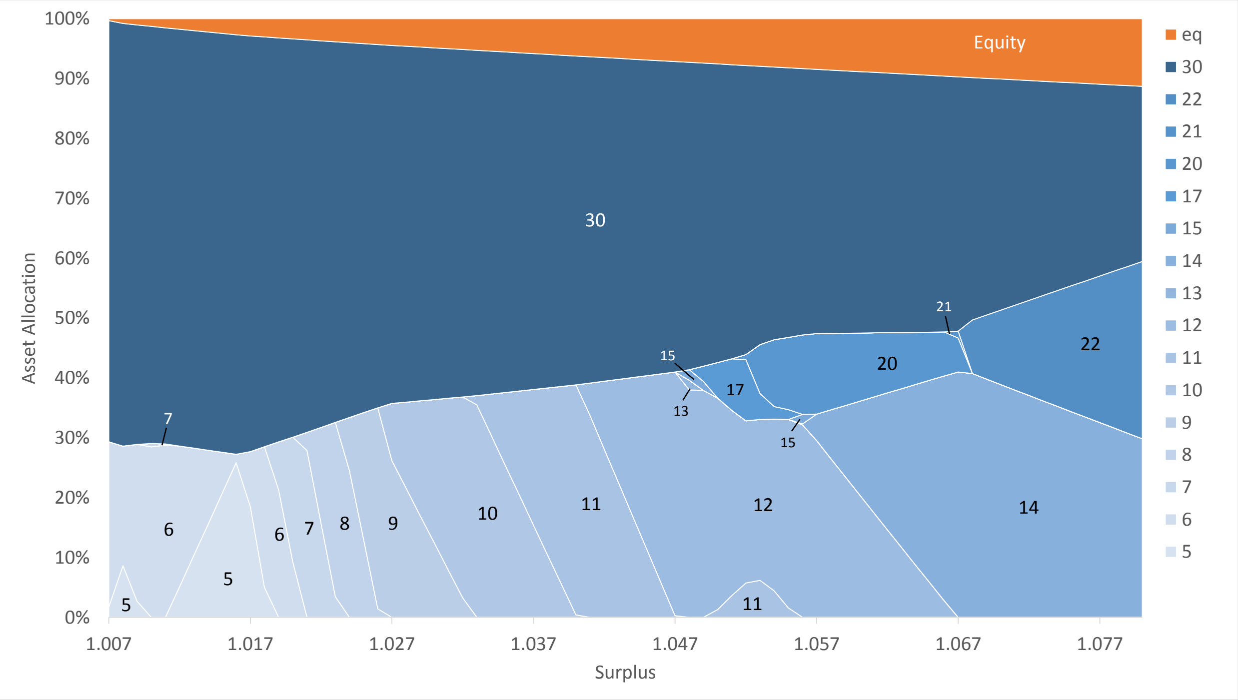


Figure 12: Asset allocations dependent on surplus ratio, 1.007 to 1.08

decreasing 30 year bond positions linearly. Unlike the previous interval increasing the surplus further changes the bond allocation to a more irregular pattern. While the bond position is still shifted to the next higher maturity the portfolio optimization suggests investing in bonds further away as well, a higher initial surplus allows the insurance firm to diversify its bond portfolio among different maturities.

Figure 13 illustrates the optimal asset allocations for initial surplus ratios of 1.08 to 1.15. Again, the equity position increases linearly here, while the allocation into 30 year bonds decreases almost linearly from close to 30% to 0 when the surplus just exceeds 1.10. While the allocation in lower maturity bonds shifts from 14 to 17 years it decreases the higher the initial surplus. On the contrary, the bond position in 22 year bonds increases until there is a shift to 23 and 24 year bonds. Above an initial surplus of 1.11, bonds with 22 years maturity are becoming superior regarding the portfolio optimization again. Starting from a ratio of 1.12 the regular pattern recommences, the exposure to 22 year bonds shifts to 23 year bonds and so forth.

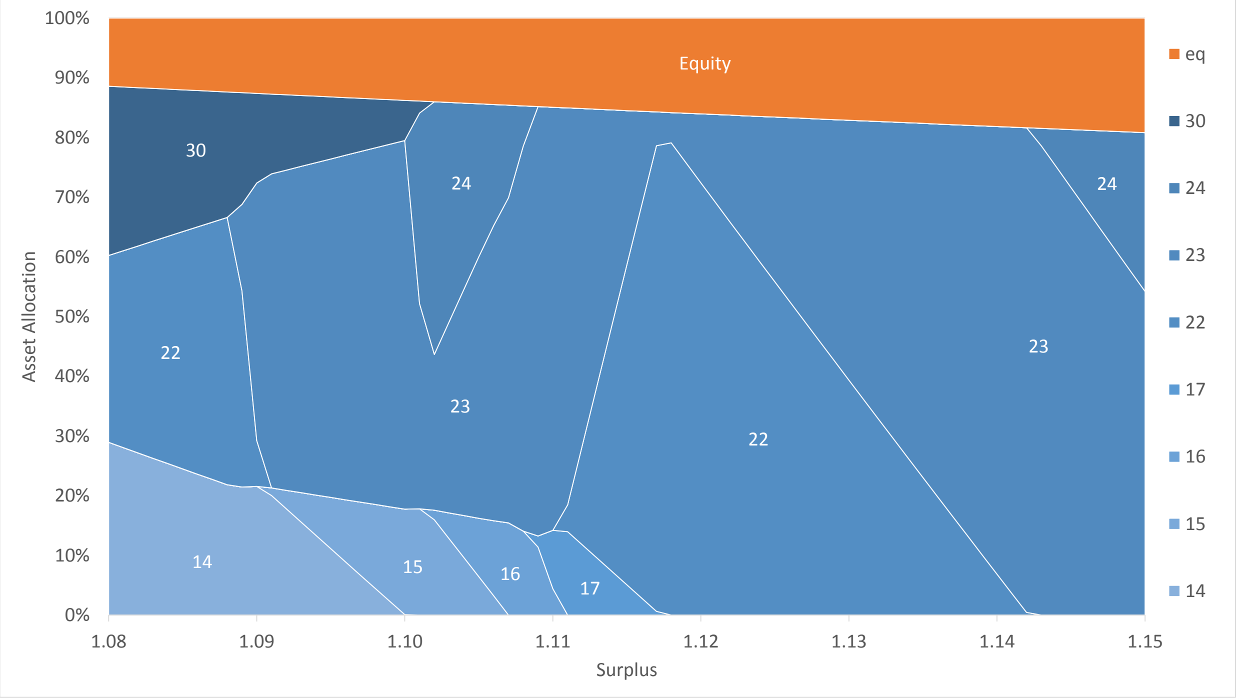


Figure 13: Asset allocations dependent on surplus ratio, 1.08 to 1.15

Taking a look at Figure 11(c), which plots the optimal asset allocations with initial surplus ranging from 1.15 to 1.50, the pattern continues until the optimal investment solely consists of 30 year bonds and equity, at an initial surplus of 1.278. Equity continues to grow linearly with increasing budget. At the upper end of the range the asset allocation is split among stocks and 30 Year bonds equally.

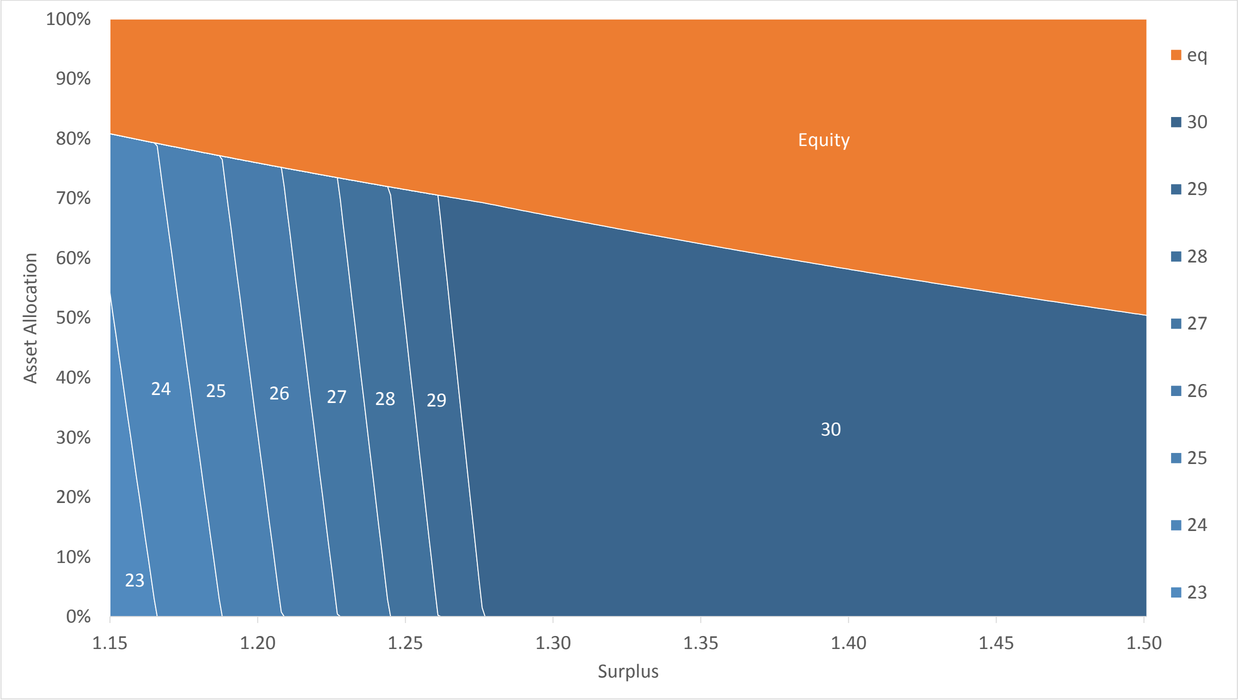


Figure 14: Asset allocations dependent on surplus ratio, 1.15 to 1.50

6 Discussion and Conclusion

We have developed an approach to investigate the optimal investment strategy for insurance companies under the new supervision framework, Solvency II. A stochastic linear program was formulated maximizing the expected surplus subject to risk constraints stemming from the SCR framework that have to be fulfilled in scenarios generated by the KNW-model and a budget constraint. The methodology was then applied to numerical examples with increasing complexity.

The first example only included two types of bonds and was mainly done in order to illustrate the methodology in a simplistic approach. Using solely two assets enabled us to illustrate the portfolio optimization with a graphical representation. The linear program specifies risk and budget constraints that form a feasible set of solutions. By adjusting the utility function, represented by the insurance firm's surplus we can find the intersection between the feasible set and the objective function which yields us with the solution of the maximization problem. With only two bonds and a budget constraint of 5% surplus the LP solver concludes that investing 114.17 and 84.66 in the respective bonds earns the highest surplus in the next year under the Solvency II regulations.

In the second example we further included equity and property related assets to the surplus optimization. In order to obtain the expected value of the additional assets we had to forecast their returns using historical data. Fitting an ARIMA(p, d, q) to the data resulted in rather low returns for stocks and type 2 equity in comparison to the fairly high returns in real estate. The result of the LP therefore does not surprise us with a large allocation in property, but also in both types of equity. While adding more types of assets might seem to be sensible as insurers have the possibility to invest in various securities other than stocks and bonds, the result in this example seems to be biased due to the forecasting method of the asset returns and recent performance of these assets. Furthermore, regulations require insurers to have a higher position in fixed income securities, even though other assets might provide higher potential returns they increase the insurer's exposure to risk. As this example is still quite limited due to the lack of an adequate set of scenarios we further expanded the stochastic LP with DNB's *haalbaarheidstoets* scenario set obtained from the KNW model.

With the implementation of the KNW term structures and stock returns we obtain a more robust result regarding the optimal investment decision of an insurer under the new Solvency II regulations. The optimal asset allocation that maximizes the insurers surplus at time $T = 1$ shows strong variation depending on the insurers initial surplus. The first observation we made was that the higher the initial surplus of the insurance firm, the larger the position in stocks will maximize the future surplus. In general, there seems to be a pattern regarding the allocation to bonds with increasing budget as well. The wealthier the insurer is the higher its exposure to longer maturity bonds will optimize its expected surplus. If we take a closer look at the intervals, especially at the lower end of the surplus, there are less clearer patterns. Starting from the lowest surplus that provides a feasible solution to slightly above 1.10 the optimal bond portfolio mainly consist of 30-year bonds combined with lower maturity bonds ranging from 5 up to 24 years. In the range between 1.05 and 1.11 a more diversified bond portfolio maximizes the insurers surplus with small positions in medium to long-term bonds. In the lower end of the surplus there seems to be a strong need to invest in 30 year bonds to match the long-term liabilities that have long duration of 75 years while the rest is then invested in shorter term bonds and equity. Furthermore, bonds with 30 years maturity are the least affected by the solvency stress scenarios, which is evident from the shock scenario factors provided by the EIOPA (Table 4). They offer a safer investment especially when the budget is tight. Reaching an initial surplus of above 25% reduces the optimal investment portfolio entirely to 30 year bonds and equity. At this point the optimal position in stocks is quite high and continues to grow with the surplus. One reason why this is the case might be due to the fact that we have not included the correlation parameters between market risk sub-modules as provided in the standard model of the SCR framework. If we introduce correlation to the portfolio optimization we would expect a smaller exposure to stocks in general as they will be also affected by changes in interest rates and become less attractive when interest rates are low.

The methodology of this paper has certain limitations and improvements that can be made. First of all, as already mentioned, the standard model of the SCR framework includes correlations between sub-modules of the markets risk module. The original standard model determines the changes in BOFs due to shocks in market risk drivers which are the capital requirements Mkt_i for that particular risk. These capital requirements are then combined to an overall capital requirement for market risk, SCR_{mkt} via the use of their correlations

$$SCR_{mkt} = \sqrt{\sum_{i,j} CorrMkt_{i,j} \cdot Mkt_i \cdot Mkt_j}.$$

Excluding the correlation between certain products implies that the risk drivers are independent. Looking at recent developments, while during times of stable markets, risk factors seem to be independent, in times of crises there are signs that they are strongly correlated. In 2008 stock indices dropped sharply while interest rates plummeted to historic lows as well. Other market risk drivers, such as property prices had similar developments and decreased similarly (CEIOPS, 2010). These joint movements have also been observed in crises prior to the recent ones. While in an older draft of the Solvency II SCR standard formula the model was calibrated such that equity/property and interest rate risk are uncorrelated the CEIOPS (now replaced by the EIOPA) suggested a correlation of 0.5 when insurers are exposed to falls in the interest rate. Furthermore, there are several other sub-modules in the market risk module that are included in the standard model which are correlated, too. Adding these correlation parameters would require to reformulate the portfolio optimization and steer away from the linear programming approach.

Another improvement to the methodology would be to use a more sophisticated and realistic model to simulate an insurers liabilities in comparison to the simple geometric decay function. Depending on the liability structure and its distribution, the optimal asset allocation will be most likely shifted to different weights.

Even though the SCR framework is only based on the 1 year 99.5% Value at Risk, we could extend the time horizon of the portfolio optimization several time steps further, using the KNW-model term structures and stock returns which are also provided by the *haalbaarheidstoets* scenarios (up to 60 years).

While the KNW-model provides us with a large set of possible outcomes of the bond and the stock market, we lack simulated data for alternative assets such as private equity, hedge funds or real estate. Due to the current low interest rate environment several financial service providers such as Towers Watson (2015), KPMG (2015) or JP Morgan Asset Management (2014) conclude that alternative investment are becoming more and more relevant. They potentially offer a higher level of diversification and enabling higher returns. Nevertheless, these upsides naturally come with their downsides, as higher returns cannot be realized without taking more risk. To which extent these risks are affecting the asset's profitability is still being discussed. Therefore, including alternative investments to the portfolio optimization under Solvency II would be an interesting addition to the study.

7 Appendix

7.1 Interest rate shock factors

Table 4: Stressed interest rates according to the EIOPA

Maturity t (years)	**relative change** $s^{up}(t)$	**relative change** $s^{down}(t)$
1 or shorter	70%	-75%
2	70%	-65%
3	64%	-56%
4	59%	-50%
5	55%	-46%
6	52%	-42%
7	49%	-39%
8	47%	-36%
9	44%	-33%
10	42%	-31%
11	39%	-30%
12	37%	-29%
13	35%	-28%
14	34%	-28%
15	33%	-27%
16	31%	-28%
17	30%	-28%
18	29%	-28%
19	27%	-29%
20	26%	-29%
90 or longer	20%	-20%

7.2 auto.arima function

The auto.arima function implemented in the forecast package in R applies a step-wise algorithm proposed by Hyndman and Khandakar (2007) to find the appropriate specifications of an ARIMA$(p, d, q)(P, D, Q)_m$ model after the orders of integration d and D were determined by successive KPSS unit-root tests. Initially the algorithm starts with fitting four models for seasonal or non-seasonal data.

Table 5: Starting models of auto.arima

non-seasonal	seasonal
$(2, d, 2)$	$(2, d, 2)(1, D, 1)$
$(0, d, 0)$	$(0, d, 0)(0, D, 0)$
$(1, d, 0)$	$(1, d, 0)(1, D, 0)$
$(0, d, 1)$	$(0, d, 1)(0, D, 1)$

If $d + D \leq 1$ then the models are fitted including a constant term which is excluded otherwise. From these models the one with the smallest Akaike Information Criterion (AIC) value is chosen. The next step of the algorithm examines 13 alternative models increasing and decreasing one of p, q, P and Q, both p and q, both P and Q by 1 and including or excluding a constant depending on the initial model. If the modified model has a lower AIC value the new model is kept and the procedure is reiterated. The algorithm stops when no model is found with a lower AIC. In order to enable convergence and avoid issues with near unit-roots constraints are additionally imposed, i.e. parameters have upper bounds, namely p, $q \leq 5$ and P, $Q \leq 2$, close to non-invertible or non-causal models are rejected (by computing the roots of the model polynomials and rejecting models with roots smaller than 1.001 in absolute values). Furthermore, in case the model estimation procedure returns any errors the model is rejected as well. Figure 15 shows the forecasts on indices used to determine the expected return for the equity and property risk sub-module calculations.

7.3 Equity and property indices

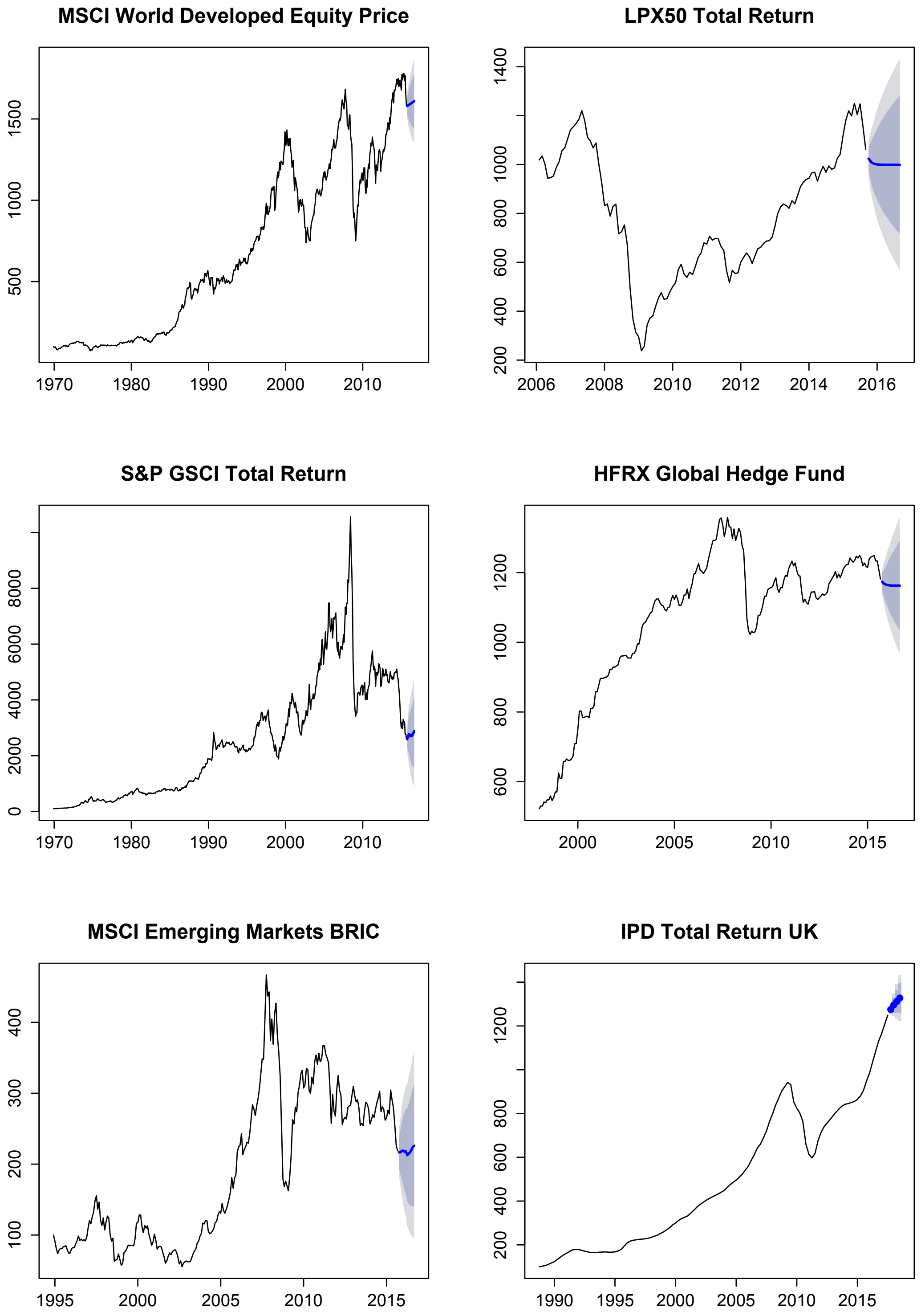


Figure 15: Historical data and ARIMA 1-year forecasts of equity and property indices

7.4 *Haalbaarheidstoets* rates

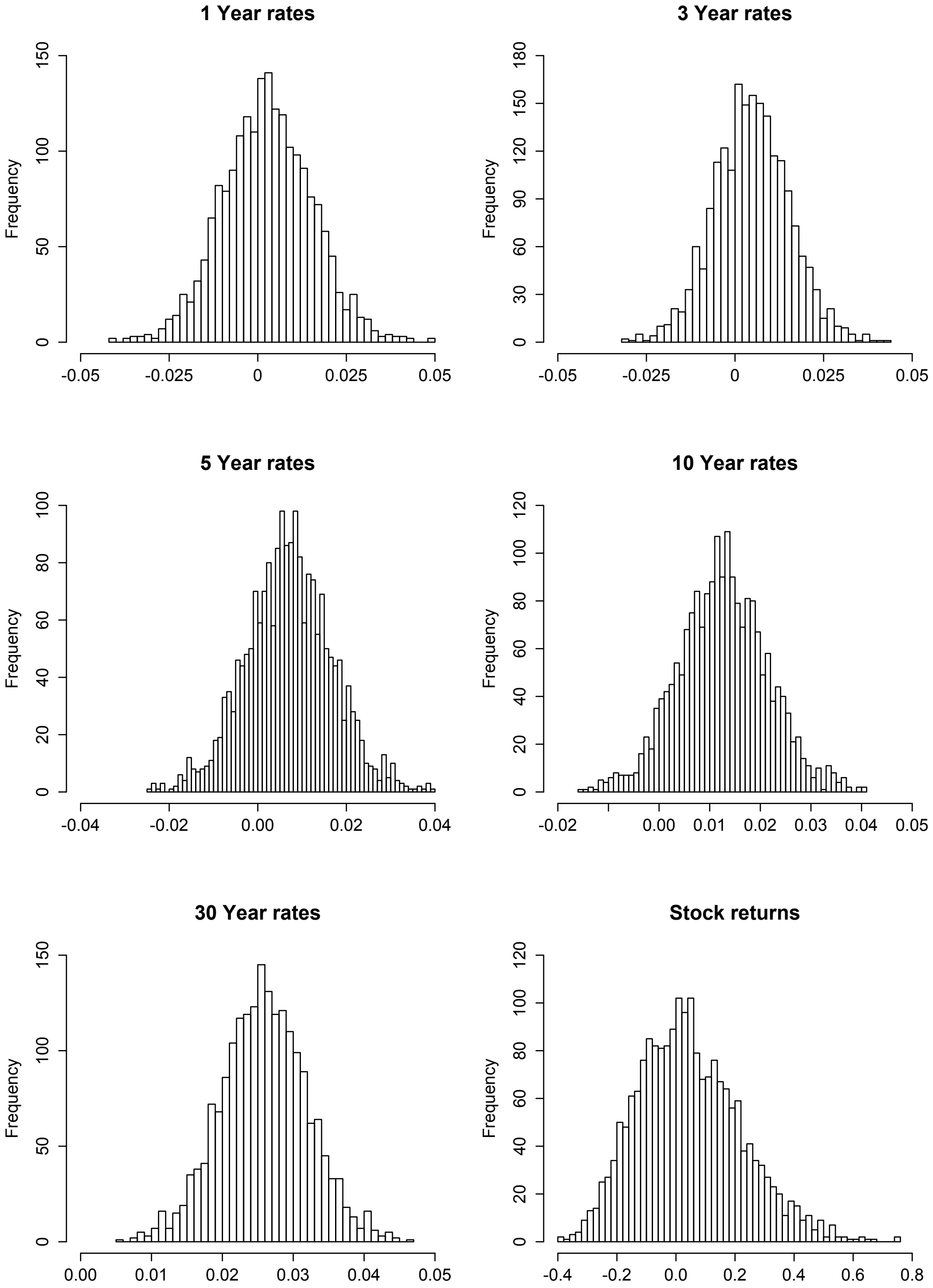


Figure 16: Histograms of scenario set interest rates and stock returns, $T = 1$

7.5 *Haalbaarheidstoets* LP solutions

Table 6: Asset allocations for different levels of surplus

Assets	Surplus			
	1.05	1.10	1.15	1.20
11 Yr	2.54	-	-	-
12 Yr	70.03	-	-	-
14 Yr	-	4.79	-	-
15 Yr	-	33.47	-	-
17 Yr	12.09	-	-	-
23 Yr	-	132.56	131.58	-
24 Yr	-	-	53.18	-
25 Yr	-	-	-	82.68
26 Yr	-	-	-	99.70
30 Yr	105.88	16.57	-	-
Stocks	13.82	26.20	38.24	50.13
Total	204.36	213.59	223.01	232.51

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