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Optimal Asset Allocation with Illiquid Assets

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OPTIMAL ASSET ALLOCATION WITH ILLIQUID ASSETS

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Abstract

This thesis studies the effect of illiquidity on an investor's optimal asset allocation, where illiquidity is the restriction an asset cannot be traded for an uncertain time period. The martingale method is used to derive the optimal asset allocation in three different cases: trading in the illiquid asset is always possible, trading in the illiquid asset is not possible at some points in time and it's uncertain whether or not a trading opportunity in the illiquid asset arises. The cases are compared to measure the effect of illiquidity on the optimal asset allocation of an investor. It turns out illiquidity leads to a reduction of the allocation to the illiquid asset, especially for the less risk-averse investor. If we increase the number of time steps in the model, the investor will reduce his allocation to the illiquid asset if the sequence of no-trading points increases. This effect is even more persistent if we increase the evaluation period. However, the investor holds the same allocation if the the point in time the no-trading period arises changes.

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1 Introduction

A well-known stylized fact in financial markets is that many asset classes are illiquid. The reason for this stylized fact is often owing to the difficulty of finding appropriate counter-parties to trade with. Several asset classes require the counter-parties to have significant capital and particular knowledge about the asset classes. However, counter-parties who satisfy these capital and abilities requirements are often limited. Besides that, other asset classes are (temporarily) just not in the interest of counter-parties. This leads to infrequently trading opportunities of the illiquid asset. The most important problem hereby is that the next opportunity to trade is *uncertain*. This concept of illiquidity is studied in the paper of Ang, Papanikolaou and Westerfield (2014). In this thesis we focus on the same concept of illiquidity. Therefore, we define an illiquid asset as an asset that cannot be traded for an uncertain time period.

Obvious examples of illiquid assets are hedge funds, real estate, private equity, timber, art and infrastructure. The period between trades for hedge funds is approximately 1-2 months. In real estate markets, the common period between trades for residential housing is 4-5 years and 8-11 years for institutional real estate. This period usually varies within 3 to 10 years for private equity portfolios and timber funds. The periods between sales for infrastructure and art are even larger, normally varying from 40 to 70 years¹. Less straightforward examples are assets that seem to be liquid, but which are in fact illiquid. Whilst the very basic public equities and fixed income securities have a negligible time between trades in general, i.e. are liquid, subclasses of these assets are often illiquid. For instance corporate bonds trade approximately once a day, over-the-counter equity² once a week and municipal bonds only twice a year³. So except for very basic fixed income securities and public equities, investors have to wait for a particular period of time before they can rebalance their portfolio. Another phenomenon which sometimes arises in financial markets is a normally liquid asset which becomes temporarily illiquid. This happened for instance with Greek government bonds during the recent financial crisis. The Greek government became unable to pay back their debts and therefore investors didn't want to buy Greek government bonds at any price. During this period, investors who owned these assets were not able to rebalance their allocation in the Greek government bond market. Other examples of this phenomenon are the auction rate security market, which became illiquid in 2008 (McConell and Saretto, 2010); and the Asian emerging market crisis in 1990, where emerging market funds became illiquid during the crisis period.

¹Table 1: Holding Periods and Turnover of Various Asset Classes, Ang, Andrew, and Papanikolaou, Dimitris, and Westerfield, Mark M., April 2014, "Portfolio Choice with Illiquid Assets", p. 37.

²Over-the-counter equities have trades done directly between two parties, without any supervision of an exchange.

³Table 1: Holding Periods and Turnover of Various Asset Classes, Ang, Andrew, and Papanikolaou, Dimitris, and Westerfield, Mark M., April 2014, "Portfolio Choice with Illiquid Assets", p. 37.

Besides the fact that many asset classes are illiquid, the share of illiquid assets in the portfolio of institutional investors is also significant. The report "Global Pension Asset Study 2014" by Towers Watson shows that pension funds from the Netherlands, Australia, UK, US, Japan, Canada and Switzerland all have decreased their allocations to bonds (-12%) and cash (-5%) since 1995. The equity allocation has slightly increased (+3%), but more interestingly, the share in real estate, private equities and hedge funds etc. is increased from 5% in 1995 to 18% in 2014 (+13%).

Moreover, there is another development which makes the study to the effect of illiquidity important. As a result of the financial crisis started in 2008, the supply of long-term capital is decreasing. In order to strengthen the global economy and achieve more sustainable growth, the Organization for Economic Cooperation and Development (OECD) came up with the initiative in March 2013 to encourage institutional investors to invest in longer-term assets, such as infrastructure and energy projects. The OECD is a forum where governments from countries all over the world are able to work with each other to improve economic growth and the social well-being of people. The members of OECD try to find solutions for common economic problems and put forward recommendations for change. In this project on institutional investors and long-term investment, they focus particularly on pension funds, insurers and sovereign wealth funds, as their liabilities have a long duration.

The above two paragraphs show that pension funds invest a significant amount in illiquid assets and this amount might even increase in the future. The study to the effects of illiquidity are not only important for the pension fund itself, but also for the participants of the pension fund. In which way and in what size do the developments mentioned above affect the pensions of participants and the risks they bear?

Any transaction can be characterized by three dimensions: price, quantity and time. Along these dimensions the literature distinguishes between three concepts of illiquidity. Vayanos (1998) and Lo, Mamaysky, and Wang (2004) use the idea that illiquid assets can always become liquid as long as the investor pays sufficient costs. However, when looking at large institutional real estate projects and infrastructure projects this concept does not seem to be realistic. Counter-parties are not immediately able to trade, even at negligible prices, due to formalities like transfer of title, regulatory approval, appraisal etc. A different way to look at liquidity is to assume assets can be traded without liquidity costs, but only in limited quantities, see, e.g. Longstaff (2001). This implies a trade can always be generated if you simply wait long enough. The third idea of liquidity is that illiquid assets can only be traded at unknown, random times. Ang, Papanikolaou and Westerfield (2014) assume an illiquid asset can only be traded at randomly occurring trading opportunities, that are modeled as a Poisson process. This means illiquidity is the restriction that an asset cannot

be traded for intervals of uncertain duration. So the first concept of illiquidity imposes a restriction on the *price* of the illiquid asset, the second one imposes a restriction on the traded *quantity* of the illiquid asset and the last concept imposes a restriction on the *time* between trades of the illiquid asset. As mentioned in the beginning, the last concept of illiquidity is the conceptualization used in this thesis.

This thesis investigates the effect of illiquidity on an investor's optimal asset allocation. The main assumption in this thesis is that an illiquid asset can only be traded at an uncertain liquidity event. The time between trades has the interpretation as the searching process for finding an appropriate counter-party with whom to trade (Diamond, 1982, cited in Ang, Papanikolaou and Westerfield, 2014). We will consider three different cases and use the martingale method to characterize the optimal asset allocation of the investor in each case. We consider the following cases: the *complete* market case, the *incomplete* market case and the *combination of the complete and the incomplete* market case. In order to investigate the cases, we assume a two-time period financial market consisting of two assets, the bond and the stock. Unless stated otherwise, in all three cases the bond is assumed to be liquid, i.e. trade in the bond is always possible at deterministic points in time. In the first case we assume trading in the stock is possible at all deterministic points in time, in the second case trading in the stock is not possible at known deterministic points in time and in the last case it's not known beforehand whether or not a trading opportunity in the stock will arise at deterministic points in time. The last case is exactly the concept of illiquidity that is used in this thesis. It turns out the distracted stochastic discount factor (SDF) process, necessary to characterize the optimal asset allocation of the investor, is different in each case. Hence, different approaches in each of the cases are needed to apply the martingale method and find the optimal asset allocation of the investor. The different approaches together with their results are formulated in this thesis. Finally, we will extend the approaches to a multi-period time framework.

As mentioned above, Ang, Papanikolaou and Westerfield (2014) already studied the idea of illiquidity as the restriction assets cannot be traded for intervals of uncertain duration. This thesis has a different approach in the sense that it makes use of the martingale method instead of dynamic programming, which is the used method in the paper of Ang, Papanikolaou and Westerfield (2014). This concept of illiquidity in combination with the martingale method is a contribution to the existing literature.

In Section 2 an overview of the most relevant concepts necessary for this thesis is represented. Section 3 gives an overview of the martingale method and the SDF process turns out to be an important ingredient for this method. We will distract the SDF process in a two-time period economy. First, Section 4.1 gives a formalization of the two-period financial market. Then, the SDF process is elaborated in case of the *complete* market (Section 4.2), the *incomplete* market

(Section 4.3) and the *combination of the complete and the incomplete* market case (Section 4.4). In Section 5, the optimal asset allocation of the agent is characterized for the *complete* market (Section 5.1), *incomplete* market (Section 5.2), and the case which combines the *complete* and the *incomplete* market case to the *combination of the complete and the incomplete* market (Section 5.3). Section 6.1 gives an overview of the results for the three different cases in the two-time period setting. A sensitivity analysis of the parameters of the model is elaborated in Section 6.2, which leads to important insights in the behaviour of the investor when he is confronted with liquidity risk. Section 7 extends the model to a n -period economy, together with the corresponding results. In the last section, Section 8, conclusions and recommendations are presented.

2 Relevant concepts

This section reviews a few concepts that play a key-role in the thesis. In Section 2.1 the two most important concepts in mathematical finance are formulated for a multi-period setting: absence of arbitrage and market completeness. The concepts are respectively called the first and the second fundamental theorem of asset pricing, together defined as the fundamental theorem of asset pricing. Both concepts are defined in terms of the stochastic discount factor (SDF) process so therefore we will start with the definition of the SDF in Section 2.1. In Section 2.2 the idea of *fictitious completion* is presented, which will be used as a guidance for the techniques used in this thesis.

2.1 Fundamental theorem of asset pricing

In mathematical finance arbitrage is a self-financing trading strategy which, starting with zero initial portfolio value, creates a portfolio value at a later time that is nonnegative with probability 1 and positive with positive probability (Schumacher, 2013). The first fundamental theorem of asset pricing (FFTAP) states conditions to guarantee absence of arbitrage. The FFTAP can be defined in different ways. We consider the FFTAP in terms of stochastic discount factors (SDFs).

First of all, the definition of the stochastic discount factor (SDF) is formulated. We follow the definition introduced by Hansen and Richard (1987). The SDF is formulated under the assumption of a discrete time economy with N available assets. The price of each asset at time t is denoted by $P_{i,t}$ with $i \in 1, 2, \dots, N$.

Definition 2.1. Let M_t be a positive process adapted to the information filtration \mathcal{F}_t . This process is called a one-period stochastic discount factor if the following relations hold for all $t \in \{0, 1, \dots, T - 1\}$ and for all $i \in \{1, 2, \dots, N\}$:

$$\mathbb{E}\left[\frac{P_{i,t+1}M_{t+1}}{P_t} \mid \mathcal{F}_t\right] = 1$$

In the remainder of the thesis we will assume a discrete time economy with two available assets: the risk free bond B and the stock S . This implies the positive process M_t , adapted to the information filtration \mathcal{F}_t is called a SDF process if the following holds for all $t \in \{0, 1, \dots, T - 1\}$:

$$\mathbb{E}\left[\frac{B_{t+1}M_{t+1}}{B_t} \mid \mathcal{F}_t\right] = 1 \tag{1}$$

$$\mathbb{E}\left[\frac{S_{t+1}M_{t+1}}{S_t} \mid \mathcal{F}_t\right] = 1 \tag{2}$$

Next, going back to the general case with N available assets, we define the set of all possible solutions for the state prices by:

$$\mathcal{Y}_p = \{y_{t+1} \in \mathbb{R}^S, t \in 0, 1, 2, \dots, T-1; X_{t+1}y_{t+1} = P_t\}$$

where S is the number of possible scenarios, T the terminal time, X_{t+1} the payoff matrix at time $t+1$ of size $S \times N$ and P_t is the corresponding pricing vector at time t .

The set of all SDFs is given by:

$$\mathcal{D}_p = \{M_{t+1} = (m_{1,t+1}, \dots, m_{S,t+1})' \in \mathbb{R}^S, t \in 0, 1, 2, \dots, T-1; m_{s,t+1} = \frac{y_{s,t+1}}{\pi_{s,t+1}}, \\ s = 1, \dots, S, y_{t+1} = (y_{1,t+1}, \dots, y_{S,t+1})' \in \mathcal{Y}_p\}$$

where S is the number of possible scenarios, X_{t+1} the payoff matrix at time $t+1$ of size $S \times N$, P_t the corresponding pricing vector at time t and $\pi_{s,t+1}$ the probability of state s occurring at time $t+1$.

The first fundamental theorem of asset pricing in a multi-period setting states the following (Melenberg, Schumacher and Charlier, 2007):

Theorem 2.1. The price system P_t excludes arbitrage opportunities

$$\iff$$

The price system P_t generates a stochastic discount factor $M_{t+1} = (m_{1,t+1}, \dots, m_{S,t+1})' \in \mathcal{D}_p$, and hence the payoff pricing function $P_t = \mathbb{E}_t(M_{t+1}X_{t+1})$, satisfying $m_{s,t+1} > 0$ for all $t \in 0, 1, 2, \dots, T-1$, $s = 1, \dots, S$.

The second fundamental theorem of asset pricing (SFTAP) states conditions for market completeness. In a complete market prices are uniquely determined and any payoff as a function of variables whose value will only be known at a future time, can be replicated by a specific trading strategy (Schumacher, 2013). The SFTAP can be represented in different ways and again we consider the theorem in terms of SDFs.

The second fundamental theorem of asset pricing in a multi-period setting, given that absence of arbitrage is satisfied, states the following (Melenberg, Schumacher and Charlier, 2007):

Theorem 2.2. The market is complete with respect to the basic assets

$$\iff$$

There exists a unique, (strictly positive) stochastic discount factor $M_{t+1} \in \mathcal{D}_p$ for all $t \in 0, 1, 2, \dots, T-1$.

The fundamental theorem of asset pricing, especially in terms of SDFs, is useful for understanding the remainder of this thesis and will be referred to several times.

2.2 Fictitious-completion

In the previous section the concept of the fundamental theorem of asset pricing is elaborated. Another concept that plays a key-role in the remainder of this thesis is *fictitious-completion*. Fictitious-completion was first developed by Karatzas, Lehoczky, Shreve, and Xu (1991) in order to solve the problem of utility maximization of terminal wealth in an incomplete market.

In an incomplete market investors are not able to replicate *any* payoff, as a function of variables whose value will only be known at a future time. This implies the investor cannot always simply apply a specific trading strategy to reach the desired payoff. In case of utility maximization of terminal wealth, the payoff the investor wants to maximize is the terminal wealth in each state of the world. Incompleteness implies it will not always be possible to reach the optimal terminal wealth in each state of the world. Since there might not exist a trading strategy to reach the optimal terminal wealth, incompleteness therefore leads to a constrained problem of utility maximization of terminal wealth. Karatzas and Shreve (1998) characterize this by defining a set of constraints K and the corresponding constrained market $\mathcal{M}(K)$, where K is a subset of \mathbb{R}^N and N indicates the number of different risky assets available in the market. If $K = \mathbb{R}^N$, the corresponding market $\mathcal{M}(\mathbb{R}^N)$ is the complete market.

This implies the utility maximization problem of terminal wealth for an agent is subjected to the constraint set K . In order to solve this problem the idea is to fictitiously complete the market such that the optimal asset allocation can just be solved in the same way as is done in the complete market case, this market is denoted by $M_{\mathcal{D}}$. Karatzas and Shreve (1998) do this by adding fictitious assets in such a way that with these fictitious assets the defined market $M_{\mathcal{D}}$ becomes complete. However, we have to make sure that to obtain the optimal terminal wealth of an investor, the corresponding optimal trading strategy is defined in such a way that there is no actual position taken in the fictitious assets. The optimal position taken in a particular asset depends on the price of risk of the asset (Karatzas and Shreve, 1998). The important step is now to choose the prices of risk of the fictitious assets in such a way that the investor takes a zero position in these assets. If we have found the prices of risk such that no allocation is assigned to the fictitious assets, the optimal solution for the constrained utility maximization problem is identical to the optimal solution in the unconstrained, fictitious market $M_{\mathcal{D}}$. In this final step the problem of utility maximization of terminal wealth in an incomplete market is solved. Karatzas and Shreve (1998) have elaborated this idea in a continuous time framework.

In this thesis we look at deterministic points in time instead of a continuous time-framework as is done in Karatzas and Shreve (1998). The purpose of this concept is not to apply the described method above, but to use the idea as a guidance for our own model.

3 Optimal terminal wealth

In this section we will solve the utility maximization of the terminal wealth problem mentioned in Section 2.2. The problem of utility maximization of terminal wealth can be solved by the martingale method or by means of dynamic programming in case of continuous, deterministic functions of the parameters (Karatzas and Shreve, 1998). As mentioned in the introduction, dynamic programming is used in the paper of Ang, Papanikolaou and Westerfield (2014) to solve the problem. In this thesis we will make use of the martingale method. Solving the problem of utility maximization of terminal wealth gives us the *optimal terminal wealth* of the investor. Of course, the investor is interested in actually reaching his optimal terminal wealth. The corresponding trading strategy is therefore called the *optimal asset allocation*, which will be discussed in Section 4.

3.1 Martingale method

We start with an overview of the martingale method. The aim of this method is to characterize the optimal terminal wealth of the investor and in the next step to define the trading strategy to actually reach the optimal terminal wealth (Jeanblanc M., 2001).

Definition 3.1. A *utility function* is a concave, nondecreasing, upper semi-continuous function $U : \mathbb{R} \rightarrow [-\infty, \infty)$ satisfying (Karatzas and Shreve, 1998):

1. $\{x \in \mathbb{R}; U(x) > -\infty\}$ is a nonempty subset of $[0, \infty)$.
2. U' is continuous, positive, and strictly decreasing within $\{x \in \mathbb{R}; U(x) > -\infty\}$.

The relative risk aversion of an agent is defined as $-\frac{U''(x)}{U'(x)}$ and measures the degree of risk aversion of the agent.

Suppose we have an agent with utility function $U(\cdot)$ and an initial endowment W_0 . The main objective of the agent is to optimize his terminal wealth at time T given his initial endowment W_0 . The corresponding trading strategy to reach the target of the investor, is the optimal asset allocation. The optimization problem of the agent looks as follows:

$$\max_{W_T} \mathbb{E}[U(W_T)] \quad \text{such that} \quad \mathbb{E}[W_T M_T] = W_0 \quad (3)$$

This maximization problem can be solved using a Lagrange approach. The Lagrangian function is defined as follows:

$$\mathcal{L} = \mathbb{E}[U(W_T)] - \lambda(\mathbb{E}[W_T M_T] - W_0) \quad (4)$$

where λ is called the Lagrange multiplier. We have to maximize the Lagrangian with respect to the investor's final wealth:

$$\max_{W_T} \mathcal{L} = \max_{W_T} \mathbb{E}[U(W_T)] - \lambda(\mathbb{E}[W_T M_T] - W_0) \quad (5)$$

The optimal terminal wealth of the agent as a function of λ can now be written as:

$$W_T = [dU]^{-1}(\lambda M_T) \quad (6)$$

We can find λ by inserting W_T in the budget constraint $\mathbb{E}(W_T M_T) = W_0$. The final step is to insert λ in (6) to get the optimal terminal wealth W_T^* of the investor.

The martingale method implies that in complete markets, *any* payoff W_T can be obtained by choosing an appropriate trading strategy, conditional on the initial wealth W_0 of the investor. The martingale method presents the payoff that is desired in each state of the world for an individual investor. The corresponding optimal trading strategy of the agent to reach these payoffs is discussed in Section 4.

In the remainder of this section we assume Constant Relative Risk Aversion (CRRA) utility of the agent:

$$U(x) = \left\{ \begin{array}{ll} \frac{x^{1-\gamma}}{1-\gamma} & \text{if } \gamma > 0, \gamma \neq 1 \\ \ln(\gamma) & \text{if } \gamma = 1 \end{array} \right\}$$

The relative risk aversion parameter for a CRRA investor equals γ , i.e. γ measures the degree of risk aversion. If $\gamma = 0$, the investor is called risk-neutral since the utility is linear in x .

The optimal solution for the terminal wealth of the investor, W_T^* , in case of CRRA utility is determined as follows:

$$W_T^* = \lambda^{-\frac{1}{\gamma}} M_T^{-\frac{1}{\gamma}} = \frac{W_0}{\mathbb{E}(M_T^{1-\frac{1}{\gamma}})} M_T^{-\frac{1}{\gamma}} \quad (7)$$

Notice that an extreme risk averse investor, i.e. a high γ , results in approximately the same optimal terminal wealth for each state of the world as the investor highly dislikes variation in the payoffs.

Hence, in order to find the value of the optimal terminal wealth of the investor, we need to define the SDF process M_T . The main goal of this thesis is to investigate the effect of illiquidity in the market on the optimal asset allocation

of an agent. To make this more explicit, we will assume a two-period financial market with two available assets, a bond B and a stock S . We will consider three different cases and compare the three results to find out the impact of illiquidity on the optimal asset allocation of the investor. In the first case we assume trading is always possible in both assets at all deterministic points in time, which turns out to be the *complete* market case. In the second case we impose trading in the stock is not possible at one of the deterministic points in time and we find out this restriction leads to the *incomplete* market case. In the last case we impose the investor does not know whether or not a trading opportunity in the stock arises, which we call the *combination of the complete and the incomplete* market case. Recall again the used definition of an illiquid asset: *an illiquid asset is an asset that cannot be traded for a time period of uncertain duration*. So the last case of consideration is exactly the concept of illiquidity used in this thesis. As we will focus on this concept of illiquidity, it is assumed the stock S does *not* include a liquidity premium.

4 Stochastic Discount Factor process

In the previous chapter the martingale method is elaborated and in order to measure the effect of illiquidity explicitly, we have to characterize the SDF process in the financial market for the three different cases. First, we will give a precise formulation of the two-period financial market by means of the binomial tree in Section 4.1. Then, we will distract the SDF process for our definition of the financial market for the three different cases. In all three cases we assume the bond can be traded at all deterministic points in time. Moreover, in all three cases we assume the stock can be traded at the current time t_0 and at the terminal time $t_2 = T$. In Section 4.2 we assume trading in the stock is *always* possible at t_1 , which turns out to be the *complete* market case. In Section 4.3 we impose trading in the stock is *not* possible at t_1 and we find out this restriction leads to the *incomplete* market case. In the last section, Section 4.4, we impose the investor does not know whether or not a trading opportunity in the stock arises at t_1 , which we call the *combination of the complete and the incomplete* market case.

4.1 Formalization of the two-period financial market

In this section, we will give a detailed description of the financial market. We consider 2 time periods, the present is indicated by $t_0 = 0$ and the future times are $t_1 = 1/2$ and the terminal time $t_2 = T = 1$. To simplify notation we will remove the subscript, so we get $t = 0$, $t = 1/2$ and $t = 1$.

Suppose there are 2 assets available in our financial market, a risk free bond B and a stock S , whose initial prices at $t = 0$ are respectively given by $S_0 = 1$ and $B_0 = 1$. The bond B will grow at a fix rate $1 + r$ each time period. At time $t = 1/2$ the stock will either be in the *up* scenario or in the *down* scenario. Both scenarios have equal probability p of occurrence ($p = 1/2$)⁴. At time $t = 1$ the stock has made again an *up* move with probability $p = 1/2$ or a *down* move with probability $p = 1/2$, which implies the financial market can be in four different scenarios at time $t = 1$. Therefore, the set of all possible scenarios in this financial market equals:

$$\Omega = \{ \omega_{uu}, \omega_{ud}, \omega_{du}, \omega_{dd} \}$$

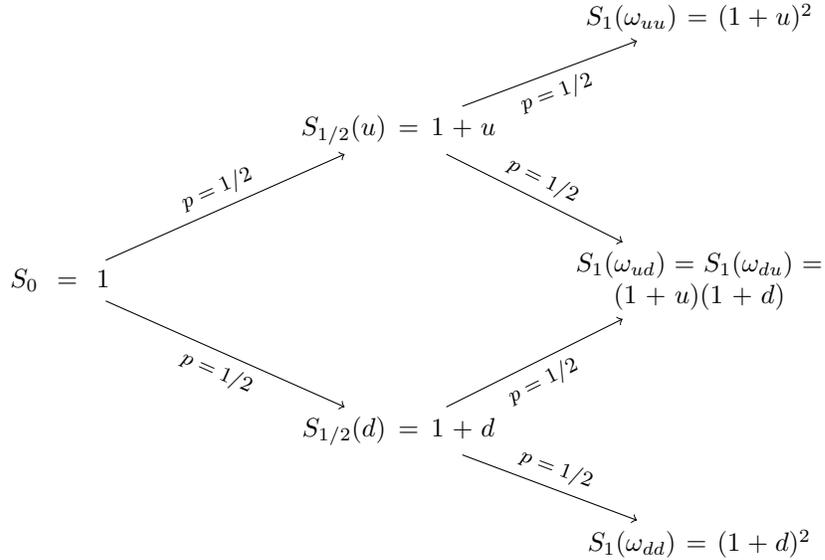
In each time period the stock price will go up by the factor $1 + u$ in the *up* scenario and down by the factor $1 + d$ in the *down* scenario. This means the moves of the stock over time are independent and identically distributed (i.i.d.) Bernoulli random variables, i.e.:

$$S_t = S_0(1 + u)^{n_{it}}(1 + d)^{i_t - n_{it}},$$

⁴The value for p is arbitrarily chosen and the value $1/2$ makes sure the results are clear for interpretation. In Section 6.2, when discussing the sensitivity analysis of the parameters of the model, we choose different values for p and show how the changes affect the results.

where i_t are the number of time periods until time t and n_{i_t} represents the number of up moves until time t .

The discrete-time model for the varying price process of the stock S_t over time can be represented in a binomial tree. The binomial tree gives an overview of the different possible paths that might be followed by the stock S . In case of deterministic points in time the binomial tree is frequently used to determine the price of an option. Moreover, we can easily distract risk-neutral probabilities and SDF processes. Distracting the SDF process will be done in the next sections. The binomial tree of the two-period financial market described above looks as follows:



Throughout this thesis the assumption is made that $u > r > d$. In Section 4.2 and Section 4.3 it becomes clear this assumption is very useful in terms of the fundamental theorem of asset pricing.

The representation of the stock price process S_t shows the process depends on the materialized market scenarios. The stock price process S_t evolves in such a way that the value at each time t , does only depend on past and current scenarios and does not depend on unobservable scenarios which will arise *after* time t . This means the price process S_t is adapted to the information filtration \mathcal{F}_t . The formal definition of the filtration \mathcal{F} is defined as follows (Campolieti and Makarov, 2014):

Definition 4.1. A filtration \mathcal{F} is a sequence of σ -algebras $(\mathcal{F}_t)_{t \in \{1, 2, \dots, T\}}$ over a set Ω such that $\mathcal{F}_s \subset \mathcal{F}_t \forall 0 \leq s \leq t \leq T$

\mathcal{F}_t is a σ -algebra if it satisfies the following conditions:

1. If $\emptyset \in \mathcal{F}_t$, then $\Omega \in \mathcal{F}_t$
2. If $A \in \mathcal{F}_t$, then $A^c \in \mathcal{F}_t$
3. If $A_i \in \mathcal{F}_t$ then $\bigcup A_i \in \mathcal{F}_t$ for $i = 1, 2, \dots$

where A is a particular subset of Ω and A^c is the complement of A .

So Definition 4.1 implies that information can only increase with time. The filtration \mathcal{F}_t at time $t = 0$, $t = 1/2$ and $t = 1$ for the market defined above, equals:

$$\begin{aligned}\mathcal{F}_0 &= \{\emptyset, \Omega\} \\ \mathcal{F}_{1/2} &= \{\emptyset, \Omega, \{\omega_{uu}, \omega_{ud}\}, \{\omega_{du}, \omega_{dd}\}\} \\ \mathcal{F}_1 = \mathcal{F} &= \{\emptyset, \Omega, \omega_{uu}, \dots, \omega_{dd}, \{\omega_{uu}, \omega_{ud}\}, \dots, \{\omega_{du}, \omega_{dd}\}, \{\omega_{uu}, \omega_{ud}, \omega_{du}\}, \dots, \\ &\quad \{\omega_{du}, \omega_{ud}, \omega_{dd}\}\}\end{aligned}$$

Now that we have formalized the price dynamics, the probability setting, the sample space and the information filtration of the financial market, we are able to distract the SDF processes for the three different cases.

4.2 The complete market case

In this section we will elaborate the distraction of the SDF process in the binomial tree by the assumption the bond and stock can always trade at $t = 0$, $t = 1/2$ and $t = 1$. Trading in the stock (bond) means that the stock (bond) can be bought or sold in unlimited quantities at $t = 0$, $t = 1/2$ and $t = 1$ at respectively well-specified prices S_0 (B_0), $S_{1/2}$ ($B_{1/2}$) and S_1 (B_1). There exists a SDF at times $t = 0$, $t = 1/2$ and $t = 1$, and we define the SDF process from the described market as follows:

$$M_0 = 1 \quad M_{1/2} = \begin{pmatrix} M_{1/2}(\omega_{uu}) \\ M_{1/2}(\omega_{ud}) \\ M_{1/2}(\omega_{du}) \\ M_{1/2}(\omega_{dd}) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \quad M_1 = \begin{pmatrix} M_1(\omega_{uu}) \\ M_1(\omega_{ud}) \\ M_1(\omega_{du}) \\ M_1(\omega_{dd}) \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}$$

From Definition 2.1, we have $M_{1/2}$ (M_1) is measurable with respect to $\mathcal{F}_{1/2}$ (\mathcal{F}_1). The formal definition of a measurable function is as follows (Campolieti and Makarov, 2014) :

Definition 4.2. The function $g : \Omega \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is \mathcal{F}_t -measurable (or just measurable), if for any Borel set, $B \in \mathcal{B}(\mathbb{R})$, it's preimage is \mathcal{F}_t -measurable, that is:

$$g^{-1}(B) = \{\omega \in \Omega : g(\omega) \in B\} \in \mathcal{F}_t$$

where Ω is the state space. The Borel σ -algebra, $\mathcal{B}(\mathbb{R})$, is the collection of all Borel sets $B \in \mathcal{B}(\mathbb{R})$ and is the smallest σ -algebra containing all intervals in \mathbb{R} . The Borel σ -algebra B satisfies the three conditions stated in Definition 4.1.

In words, Definition 4.2 means that if $M_{1/2}$ is measurable with respect to $\mathcal{F}_{1/2}$ we can calculate the values of $M_{1/2}$ if we know the events that happened in $\mathcal{F}_{1/2}$. Assume for instance $M_{1/2}(\omega_{uu}) = 1.5$ and $M_{1/2}(\omega_{ud}) = 2$ at $t = 1/2$. Take $B = 1.5$, then by Definition 4.2 $M_{1/2}^{-1}(B) = \{\omega_{uu} \in \Omega : M(\omega_{uu}) \in B\} \in \mathcal{F}_{1/2}$. This is a contradiction, as ω_{uu} is not in $\mathcal{F}_{1/2}$, and therefore $M_{1/2}(\omega_{uu})$ has to be equal to $M_{1/2}(\omega_{ud})$. So, we do only know at $t = 1/2$ whether or not we end up in the scenario set $\{\omega_{uu}, \omega_{ud}\}$ and no more information about the individual scenarios ω_{uu} and ω_{ud} is acquired. The same holds for the scenarios ω_{du} and ω_{dd} . Therefore we have:

$$M_{1/2}(\omega_{uu}) = M_{1/2}(\omega_{ud})$$

$$M_{1/2}(\omega_{du}) = M_{1/2}(\omega_{dd})$$

The bond can always be traded at $t = 0$, $t = 1/2$ and $t = 1$. This implies the pricing equations of the bond, see Definition 2.1, are defined as follows:

$$\mathbb{E}_0\{M_1 B_1\} = B_0 \tag{8}$$

$$\mathbb{E}_{1/2}\{M_1 B_1\} = B_{1/2} M_{1/2} \tag{9}$$

In this chapter we assume for all cases the stock S can always be traded at $t = 0$ and $t = 1$, therefore we have:

$$\mathbb{E}_0\{M_1 S_1\} = S_0 \tag{10}$$

Consider the implication of the zero-investment trading strategy by Hansen and Richard (1987) at time $t = 0$, which means go long 1 unit in the stock S and short S_0/B_0 in the bond. The SDF M_1 prices every traded asset, so also the payoff of this trading strategy. This implies the following equation holds:

$$\mathbb{E}_0\{(S_1 - (S_0/B_0)B_1)M_1\} = 0 \tag{11}$$

In this case, trading in the stock is always possible at $t = 1/2$. Now consider a zero-investment trading strategy at $t = 1/2$. Suppose an investor buys a $\mathcal{F}_{1/2}$ -measurable amount $a_{1/2}$ of the zero investment trading strategy going long 1 unit in the stock, with corresponding price $S_{1/2}$ and $S_{1/2}/B_{1/2}$ short in the bond

with corresponding price $B_{1/2}$. The payoff of this strategy at $t = 1$ equals:

$$a_{1/2}\{S_1 - (S_{1/2}/B_{1/2})B_1\}$$

Again, the SDF M_1 prices every traded asset, so also this zero-investment trading strategy. Hence, we have:

$$\mathbb{E}_0\{a_{1/2}\{S_1 - (S_{1/2}/B_{1/2})B_1\}M_1\} = 0 \quad (12)$$

The law of iterated expectations in terms of time series states that $\mathbb{E}_t(X) = \mathbb{E}_t(\mathbb{E}_{t+1}(X))$. Therefore we get:

$$a_{1/2}\mathbb{E}_{1/2}\{\{S_1 - (S_{1/2}/B_{1/2})B_1\}M_1\} = 0 \quad (13)$$

If we first divide out $a_{1/2}$ and substitute (9) in the above equation, we get:

$$\mathbb{E}_{1/2}\{S_1 M_1\} = S_{1/2} M_{1/2} \quad (14)$$

Equation (14) is the final pricing equation that has to be satisfied. This result is familiar, but the derivation is very useful when we extend to the case where the stock is sometimes illiquid at $t = 1/2$.

When we fill in (8), (9), (10) and (14) from the described economy, we obtain the following 6 equations:

$$\begin{aligned} 0.25((1+r)^2 m_1 + (1+r)^2 m_2 + (1+r)^2 m_3 + (1+r)^2 m_4) &= 1 \\ 0.25((1+u)^2 m_1 + (1+u)(1+d)m_2 + (1+u)(1+d)m_3 + (1+d)^2 m_4) &= 1 \\ 0.5((1+r)^2 m_1 + (1+r)^2 m_2) &= (1+r)z_1 \\ 0.5((1+r)^2 m_3 + (1+r)^2 m_4) &= (1+r)z_2 \\ 0.5((1+u)^2 m_1 + (1+u)(1+d)m_2) &= (1+u)z_1 \\ 0.5((1+u)(1+d)m_3 + (1+d)^2 m_4) &= (1+d)z_2 \end{aligned}$$

This is a system of linear equations. Solving the system gives us the following values for $M_{1/2}$ and M_1 :

$$M_{1/2} = \begin{pmatrix} \frac{2((1+r)-(1+d))}{(1+r)((1+u)-(1+d))} \\ \frac{2((1+r)-(1+d))}{(1+r)((1+u)-(1+d))} \\ \frac{(1+r)((1+u)-(1+d))}{2((1+u)-(1+r))} \\ \frac{(1+r)((1+u)-(1+d))}{2((1+u)-(1+r))} \\ \frac{2((1+u)-(1+r))}{(1+r)((1+u)-(1+d))} \end{pmatrix} \quad M_1 = (M_{1/2})^2 = \begin{pmatrix} \frac{4((1+d)-(1+r))^2}{(1+r)^2((1+u)-(1+d))^2} \\ \frac{4((1+u)-(1+r))((1+r)-(1+d))}{(1+r)^2((1+u)-(1+d))^2} \\ \frac{4((1+u)-(1+r))((1+r)-(1+d))}{(1+r)^2((1+u)-(1+d))^2} \\ \frac{4((1+u)-(1+r))^2}{(1+r)^2((1+u)-(1+d))^2} \end{pmatrix}$$

We see that $M_1(\omega_{ud}) = M_1(\omega_{du})$ and this can be clarified as follows. The value of the stock has an i.i.d. Bernoulli distribution and this means the value of the stock only depends on the number of up moves. Therefore, the stock is not only measurable with respect to \mathcal{F}_t , but also with respect to the number of up scenarios. This implies the entries of the SDF are also measurable with respect to the number of up moves, which results in the same values for $M_1(\omega_{ud})$ and $M_1(\omega_{du})$.

Another important result is that the SDFs are both uniquely determined. As a result of the assumption $u > r > d$, all entries of the SDFs are positive. In the previous section conditions on absence of arbitrage and market completeness are formulated. By Theorem 2.1 the SDFs as defined above implies that *absence of arbitrage* holds in this market. Notice that $u > r > d$ is a *necessary* condition for absence of arbitrage. Moreover, by Theorem 2.2 the market is also *complete*. This implies the conditions for the fundamental theorem of asset pricing are satisfied and for this reason the case is called the *complete market*.

4.3 The incomplete market case

Now we assume we cannot trade in the stock S at time $t = 1/2$, i.e. the stock becomes illiquid. In this case it's known beforehand for an investor trading in the stock is possible at time $t = 0$ and $t = 1$, but is not at time $t = 1/2$. So the next opportunity to trade in the stock is *certain*. As the investor is not able to trade in the stock at $t = 1/2$, (14) is removed in this case since no price is defined for the stock at time $t = 1/2$. This means we only have (8), (9) and (10) left:

$$\begin{aligned}\mathbb{E}_0\{M_1B_1\} &= B_0 \\ \mathbb{E}_0\{M_1S_1\} &= S_0 \\ \mathbb{E}_{1/2}\{M_1B_1\} &= B_{1/2}M_{1/2}\end{aligned}$$

So we have 4 equations with 6 unknowns, which implies if we solve (8), (9) and (10), we get 2 degrees of freedom in the SDF process. By doing this, we can find values for the SDF at time $t = 1$ such that $M_1(\omega_{ud})$ is not equal to $M_1(\omega_{du})$. However, it's not possible to trade the stock at $t = 1/2$ and this entails an important result in terms of the asset allocation. Recall again, trading in the stock means here the investor can sell or buy unlimited quantities of the stock at $t = 0$ and $t = 1$ at well-specified prices, respectively S_0 and S_1 . As we can only trade in the stock at time $t = 0$, irrespective of the initial allocation to the stock at $t = 0$, the optimal wealth at $t = 1$ in state ω_{ud} and ω_{du} has to be the same. In contrast to the *complete* market case, where we are able to trade the stock at $t = 1/2$ and thus take a different position in the stock, depending on

the state your in at $t = 1/2$. In this case different payments for the states ω_{ud} and ω_{du} can be found.

So we need to find a SDF which is attainable, i.e. generates the same payment in scenario ω_{ud} and ω_{du} in the *incomplete* market case, regardless of the initial allocation in the stock. Therefore, it has to be the case $M_1(\omega_{ud})$ and $M_1(\omega_{du})$ are equal. Consequently, we add the following constraint to (8), (9) and (10):

$$M_1(\omega_{ud}) = M_1(\omega_{du}) \quad (15)$$

We call this constraint the *incompleteness constraint*. The incompleteness constraint together with the three pricing equations results in 6 equations and 5 unknowns. This implies only 1 degree of freedom in the SDF process is left which leads to a significant reduction in the possible values of the SDF process. Solving the equations gives us the following SDFs at respectively $t = 1/2$ and $t = 1$:

$$M_{1/2} = \begin{pmatrix} \frac{2((1+r)^2 - (1+d)^2)}{(1+r)((1+u)^2 - (1+d)^2)} + \frac{(1+r)((1+u) - (1+d))^2}{2((1+u)^2 - (1+d)^2)} m_{2,1} \\ \frac{2((1+r)^2 - (1+d)^2)}{(1+r)((1+u)^2 - (1+d)^2)} + \frac{(1+r)((1+u) - (1+d))^2}{2((1+u)^2 - (1+d)^2)} m_{2,1} \\ \frac{2((1+u)^2 - (1+r)^2)}{(1+r)((1+u)^2 - (1+d)^2)} - \frac{(1+r)((1+u) - (1+d))^2}{2((1+u)^2 - (1+d)^2)} m_{2,1} \\ \frac{2((1+u)^2 - (1+r)^2)}{(1+r)((1+u)^2 - (1+d)^2)} - \frac{(1+r)((1+u) - (1+d))^2}{2((1+u)^2 - (1+d)^2)} m_{2,1} \end{pmatrix}$$

$$M_1 = \begin{pmatrix} \frac{4((1+r)^2 - (1+d)^2)}{(1+r)^2((1+u)^2 - (1+d)^2)} + \frac{2(1+d)((1+u) - (1+d))}{((1+u)^2 - (1+d)^2)} m_{2,1} \\ m_{2,1} \\ m_{2,1} \\ \frac{4((1+u)^2 - (1+r)^2)}{(1+r)^2((1+u)^2 - (1+d)^2)} + \frac{2(1+u)((1+u) - (1+d))}{((1+u)^2 - (1+d)^2)} m_{2,1} \end{pmatrix}$$

The free variable is chosen to be $m_{2,1}$, where the first subindex denotes the scenario your in and the second subindex denotes the time t^5 . Notice that if $m_{2,1}$ satisfies the following condition, we obtain $m_{s,t} > 0$ for $s = 1, 2, 3, 4$ and $t = 1/2, 1$:

$$0 < m_{2,1} < \frac{4((1+u)^2 - (1+r)^2)}{(1+r)^2((1+u) - (1+d))^2} \quad (16)$$

This implies by Theorem 2.1 the price system can still exclude arbitrage opportunities. The no arbitrage price of a contingent claim C with corresponding

⁵The solution would have been the same by taking another variable as the free one. However, choosing $m_{2,1}$ as the free variable leads to symmetric results, which makes the SDF process better interpretable.

payoff vector $[f(\omega_{uu}) \quad f(\omega_{ud}) \quad f(\omega_{du}) \quad f(\omega_{dd})]'$ at $t = 1$, where $f(\omega_k)$ is a particular function whose outcome depends on the materialized state ω_k , equals:

$$\{\mathbb{E}_0^{m_{2,1}}\{M_1 C_1\} : m_{2,1} \in (0, \frac{4((1+u)^2 - (1+r)^2)}{(1+r)^2((1+u) - (1+d))^2})\}$$

This implies different prices may be found when different values for $m_{2,1}$ are chosen. There are multiple SDFs at $t = 1$ and in terms of Theorem 2.2 this implies the market is incomplete. As we have seen above in terms of the contingent claim C market incompleteness causes the prices of assets are not uniquely determined anymore. Moreover, market incompleteness leads to the problem that we are not able to construct portfolio strategies that will replicate *any* payoff that can be defined at a future time, as a function of variables whose values will only be known at that time. Since in this case we have market incompleteness, we call this case the *incomplete market*.

In the remainder of this thesis we impose the preconditions on the values of the SDF process are given by (16), as if (16) is satisfied, arbitrage free prices of the assets are realized.

4.3.1 Illiquidity in the stock *and* bond

The previous two sections assume the bond can always be traded. In this section we shortly discuss the case in which both the stock *and* bond are not traded at $t = 1/2$. In this case the price of the bond is not defined at $t = 1/2$ either. Therefore, we are only left with (8) and (10), which means we have:

$$\begin{aligned}\mathbb{E}_0\{M_1 B_1\} &= B_0 \\ \mathbb{E}_0\{M_1 S_1\} &= S_0\end{aligned}$$

For the same reason as described in the case where only trading in the stock is not possible at $t = 1/2$, we add the incompleteness constraint (15):

$$M_1(\omega_{ud}) = M_1(\omega_{du})$$

Since both the prices of the stock S and bond B are not defined at $t = 1/2$, we have that the SDF at time $t = 1/2$ is not defined at all. Thus, we can only distract the SDF at time $t = 1$. This implies for the above pricing equations that we have 3 equations with 4 unknowns. Solving the equations gives us the following SDF at $t = 1$:

$$M_1 = \begin{pmatrix} \frac{4((1+r)^2 - (1+d)^2)}{(1+r)^2((1+u)^2 - (1+d)^2)} + \frac{2(1+d)((1+u) - (1+d))}{((1+u)^2 - (1+d)^2)} m_{2,1} \\ m_{2,1} \\ m_{2,1} \\ \frac{4((1+u)^2 - (1+r)^2)}{(1+r)^2((1+u)^2 - (1+d)^2)} + \frac{2(1+u)((1+u) - (1+d))}{((1+u)^2 - (1+d)^2)} m_{2,1} \end{pmatrix}$$

Again, the free variable is chosen to be $m_{2,1}$. Note that the SDF is precisely the same as the SDF at $t = 1$ in the case when only trading in the stock at $t = 1/2$ is not possible. If we take $m_{2,1} > 0$, we obtain $m_{s,1} > 0$, $s = 1, 2, 3, 4$. By Theorem 2.1 the price system can still exclude arbitrage opportunities. As in the previous section by Theorem 2.2, this market is again incomplete.

The reason we shortly touch upon the case both the stock and bond are illiquid is that in real financial markets this event is imaginable, although it might be very rare. Think for instance about the recent financial crisis, when Greek government bonds became temporarily illiquid in the bond market.

4.4 The combination of the complete and the incomplete market case

In the previous sections we assumed the stock can either be always traded or cannot be traded at all at $t = 1/2$. In this section we assume the investor does not know whether or not a trading opportunity arises in the stock at $t = 1/2$ and we will derive the pricing equations that have to hold in this situation. Again, we will derive the pricing equations in terms of the zero-investment trading strategy as introduced by Hansen and Richard (1987). Consider the following zero-investment trading strategy: buy $a_{1/2}$, $\mathcal{F}_{1/2}$ -measurable, units of the zero-investment trading strategy if possible, and do nothing otherwise. The payoff of this strategy at time $t = 1$ equals:

$$a_{1/2}I_{1/2}\{S_1 - (S_{1/2}/B_{1/2})B_1\}$$

where $I_{1/2}$ is an identity function which equals 1 in case trading in the stock is possible at time $t = 1/2$ and equals 0 otherwise. $I_{1/2}$ is a random variable at $t = 0$, which value will be known for sure at time $t = 1/2$. Since the SDF M_1 prices all traded asset, this strategy is also priced by M_1 . So for a $\mathcal{F}_{1/2}$ -measurable $a_{1/2}$, we have:

$$\mathbb{E}_0\{a_{1/2}I_{1/2}\{S_1 - (S_{1/2}/B_{1/2})B_1\}M_1\} = 0 \quad (17)$$

The law of iterated expectations in terms of time series states that $\mathbb{E}_t(X) = \mathbb{E}_t(\mathbb{E}_{t+1}(X))$. Therefore we get:

$$a_{1/2}I_{1/2}\mathbb{E}_{1/2}\{\{S_1 - (S_{1/2}/B_{1/2})B_1\}M_1\} = 0 \quad (18)$$

Notice that $I_{1/2}$ is stochastic, but it's value is known at time $t = 1/2$, therefore we can take the indicator $I_{1/2}$ out of the expectation. We can divide $a_{1/2}$ and

substitute (9) in the above equation, we get:

$$I_{1/2}\mathbb{E}_{1/2}\{S_1M_1 - S_{1/2}M_{1/2}\} = 0 \quad (19)$$

We cannot divide by $I_{1/2}$, since $I_{1/2}$ can take the value 0. By the law of total expectation we have:

$$\mathbb{E}_{1/2}\{X\} = \mathbb{E}_{1/2}\{X|I_{1/2} = 1\}\mathbb{P}\{I_{1/2} = 1\} + \mathbb{E}_{1/2}\{X|I_{1/2} = 0\}\mathbb{P}\{I_{1/2} = 0\}$$

Now take $X = I_{1/2}\{S_1M_1 - S_{1/2}M_{1/2}\}$, we get:

$$\mathbb{E}_{1/2}\{S_1M_1 - S_{1/2}M_{1/2}|I_{1/2} = 1\}\mathbb{P}\{I_{1/2} = 1\} = 0 \quad (20)$$

Assume a positive trading possibility, i.e. $\mathbb{P}\{I_{1/2} = 1\} > 0$. This implies we get:

$$\mathbb{E}_{1/2}\{S_1M_1 - S_{1/2}M_{1/2}|I_{1/2} = 1\} = 0 \quad (21)$$

The above equation, together with (8), (9) and (10), are the requirements the SDF process has to satisfy in case the investor does not know whether or not a trading opportunity will arise at $t = 1/2$. So we have:

$$\begin{aligned} \mathbb{E}_0\{M_1B_1\} &= B_0 \\ \mathbb{E}_0\{M_1S_1\} &= S_0 \\ \mathbb{E}_{1/2}\{M_1B_1\} &= B_{1/2}M_{1/2} \\ \mathbb{E}_{1/2}\{S_1M_1 - S_{1/2}M_{1/2}|I_{1/2} = 1\} &= 0 \end{aligned}$$

The above equations are a combination of the complete and the incomplete market case. Namely in case $I_{1/2}$ takes the value 0 at $t = 1/2$, we get the pricing equations formalized in Section 4.2. Whereas if $I_{1/2}$ takes the value 1 at $t = 1/2$, we get the pricing equations formalized in Section 4.1. Note, the incompleteness constraint also holds in this case if $I_{1/2} = 0$. Therefore, we call this case the *combination of the complete and the incomplete* market case.

5 Optimal asset allocation

In the previous chapter, we have formalized the SDF process for three different cases. As we have seen in Section 3.1, the resulting SDF processes are needed to define the optimal terminal wealth of the investor. In this section we will determine how we can reach the optimal terminal wealth of the investor and thus finding the *optimal asset allocation* for the three different cases. It turns out the martingale method is quite trivial in case of the complete market, but we need to go into more technical details and apply restrictions in the incomplete and the combination of the complete and the incomplete market case. In the next section, Section 6, we will discuss the results and their implications using a stylized example.

By the description of the financial market in Section 3.1, the optimal terminal wealth of the agent is defined for the four possible scenarios at the terminal date $T = 1$ as follows:

$$W_1^* = \begin{pmatrix} W_1^*(\omega_{uu}) \\ W_1^*(\omega_{ud}) \\ W_1^*(\omega_{du}) \\ W_1^*(\omega_{dd}) \end{pmatrix} = \begin{pmatrix} \frac{W_0}{\mathbb{E}(M_1^{1-\frac{1}{\gamma}})} m_1^{-\frac{1}{\gamma}} \\ \frac{W_0}{\mathbb{E}(M_1^{1-\frac{1}{\gamma}})} m_2^{-\frac{1}{\gamma}} \\ \frac{W_0}{\mathbb{E}(M_1^{1-\frac{1}{\gamma}})} m_3^{-\frac{1}{\gamma}} \\ \frac{W_0}{\mathbb{E}(M_1^{1-\frac{1}{\gamma}})} m_4^{-\frac{1}{\gamma}} \end{pmatrix}$$

5.1 The complete market case

In Section 3.1 we have formalized the martingale method and the optimal terminal wealth of the investor in the two-period economy is characterized in the introduction of this chapter. In this section we will describe how the trading strategy to reach the optimal terminal wealth of the investor can be found and the corresponding optimal asset allocation to the stock S and the bond B in the complete market case. By definition of the martingale method, we can reach the optimal terminal wealth for the agent at time $t = 1$ in the complete market if we choose the appropriate trading strategy. Define $b_{1/2,u}$ ($s_{1/2,u}$) as the number of bonds (stocks) needed in case we end up in the *up* node of the binomial tree and $b_{1/2,d}$ ($s_{1/2,d}$) if we end up in the *down* state at time $t = 1/2$. The number of bonds (stocks) needed at time $t = 0$ is defined by b_0 (s_0). The optimal asset allocation of the agent is solved backwards in time, i.e. from the end to the beginning. First we have to solve the asset allocation at time $t = 1/2$. We do this as follows:

$$(1+r)b_{1/2,u} + (1+u)s_{1/2,u} = W_1^*(\omega_{uu}) \quad (22)$$

$$(1+r)b_{1/2,u} + (1+d)s_{1/2,u} = W_1^*(\omega_{ud}) \quad (23)$$

$$(1+r)b_{1/2,d} + (1+u)s_{1/2,d} = W_1^*(\omega_{du}) \quad (24)$$

$$(1+r)b_{1/2,d} + (1+d)s_{1/2,d} = W_1^*(\omega_{dd}) \quad (25)$$

Now we can find the asset allocation at time $t = 0$, b_0 and s_0 , in the following way:

$$(1+r)b_0 + (1+u)s_0 = W_{1/2}(u) = b_{1/2,u} + s_{1/2,u} \quad (26)$$

$$(1+r)b_0 + (1+d)s_0 = W_{1/2}(d) = b_{1/2,d} + s_{1/2,d} \quad (27)$$

Note that we can always solve these equations as we have 6 unknowns and 6 independent equations. In Section 6.1 we will give the optimal asset allocation of the investor in case of a stylized example.

5.2 The incomplete market case

In the incomplete market we have more than one solution for the SDF, which implies the price of an asset is not uniquely determined anymore. Moreover, we're not able to replicate the optimal terminal wealth for the agent at time $T = 1$ for *every* choice of the free variable $m_{2,1}$. We can see this as follows. The payoffs achievable in this market at time $T = 1$ are defined as follows:

$$X_1 \begin{pmatrix} b_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} (1+r)^2 & (1+u)^2 \\ (1+r)^2 & (1+u)(1+d) \\ (1+r)^2 & (1+u)(1+d) \\ (1+r)^2 & (1+d)^2 \end{pmatrix} \begin{pmatrix} b_0 \\ s_0 \end{pmatrix}$$

As we are not able to trade in the stock at $t = 1/2$, we can only take a position in the stock at time $t = 0$. This implies the allocation in the bond cannot change at $t = 1/2$ either, as the investor is restricted to his initial total wealth W_0 . So we have to solve:

$$X_1 \begin{pmatrix} b_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} (1+r)^2 & (1+u)^2 \\ (1+r)^2 & (1+u)(1+d) \\ (1+r)^2 & (1+u)(1+d) \\ (1+r)^2 & (1+d)^2 \end{pmatrix} \begin{pmatrix} b_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} W_1(\omega_{uu}) \\ W_1(\omega_{ud}) \\ W_1(\omega_{du}) \\ W_1(\omega_{dd}) \end{pmatrix}$$

We can immediately see this system has not a solution for every choice of $m_{2,1}$. If we take for instance $m_{2,1} = 1$, then by the SDF process defined in Section 4.3, $m_{1,1} \neq 1$, $m_{4,1} \neq 1$ and $m_{1,1} \neq m_{4,1}$. As a result, $W_1(\omega_{uu}) \neq W_1(\omega_{dd}) \neq$

$W_1(\omega_{ud})$, which means we have 3 independent equations with only 2 unknowns. This system of linear equations is unsolvable.

However, we would like to find a choice for $m_{2,1}$ such that the above system *can* be solved, i.e. the agent can still reach his optimal terminal wealth by following a specific trading strategy, although the market is incomplete. We do this by introducing a new asset C . The reason for imposing a new asset is that, with certain conditions, we can make sure the system above, together with the new asset, is solvable. Moreover, imposing a new asset is in line with the approach of Karatzas and Shreve (1998) explained in Section 2.2. Recall the concept of fictitious completion, where fictitious assets are added to the constrained market $\mathcal{M}(K)$ in order to create a fictitious-complete market $M_{\hat{\mathcal{D}}}$. The utility maximization of terminal wealth problem can be solved using the fictitious market $M_{\hat{\mathcal{D}}}$. However, restrictions need to apply to make sure the agent is not actually investing in the fictitious assets. If we want to make the market of our case complete, we need to add two assets which are linearly independent from the payoffs at $t = 1$ of the two existing assets B_1 and S_1 . This means the two added assets should be such that they cannot be constructed with the assets S and B . In this way, new payoffs at time $t = 1$ can be generated, which where impossible to generate with only the assets B and S . Define the two added assets by C_1 and C_2 . In formal terms:

Definition 5.1. The vectors in the subset $V = (S, B, C_1, C_2)$ are linearly independent if the equation $a_1S + a_2B + a_3C_1 + a_4C_2 = 0$ can only be satisfied if $a_i = 0$ for all $i = 1, 2, 3, 4$.

However, as explained in Section 4.3, we want to have the same payment in scenario ω_{ud} and ω_{du} at $t = 1$ and therefore assets that payoff differently in these states are not in the interest of the investor. For this reason, we only add one fictitious assets C , which payoffs precisely the same in ω_{ud} as in ω_{du} . In the above system this means we get four equations, where two equations are exactly the same, i.e. are linear dependent by Definition 5.1. This implies we have 3 independent equations with precisely 3 unknowns. This system can be solved, so there is no need to add a second fictitious assets to actually make the market fictitious complete. This implies in formal terms we have to choose C_1 in such a way that $a_1S_1 + a_2B_1 + a_3C_1 = 0$ can only be satisfied if $a_i = 0$ for all $i = 1, 2, 3$ and $C_1(\omega_{ud}) = C_1(\omega_{du})$.

The number invested in this new asset at time $t = 0$ is defined by c_0 . Imposing the new asset implies solving the following linear system:

$$X_1 \begin{pmatrix} c_0 \\ b_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} C_1(\omega_{uu}) & (1+r)^2 & (1+u)^2 \\ C_1(\omega_{ud}) & (1+r)^2 & (1+u)(1+d) \\ C_1(\omega_{du}) & (1+r)^2 & (1+u)(1+d) \\ C_1(\omega_{dd}) & (1+r)^2 & (1+d)^2 \end{pmatrix} \begin{pmatrix} c_0 \\ b_0 \\ s_0 \end{pmatrix} = \begin{pmatrix} W_1(\omega_{uu}) \\ W_1(\omega_{ud}) \\ W_1(\omega_{du}) \\ W_1(\omega_{dd}) \end{pmatrix}$$

The added asset is *fictitious*, so we cannot actually take a position in this asset at time $t = 0$. Therefore, we have to choose $m_{2,1}$ such that $c_0 = 0$. If we have found the appropriate $m_{2,1}$, the corresponding SDF M_1 is constructed in such a way the corresponding optimal terminal wealth can still be reached by the investor in the incomplete market.

The type of options that satisfies the conditions on C are for instance binary call options. A binary call option generates a payoff which can take only 2 possible outcomes, either some fixed amount if the option expires in the money or nothing if the option expires out of the money. The advantage of the binary call options is their simplicity and usefulness when we move to the multi-period time setting. However, we could have chosen for another asset C as long as the two above mentioned conditions are satisfied. Here we add the binary option C which pays 1 in scenario ω_{uu} and pays nothing otherwise, i.e.:

$$C_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving this system gives us the following values for b_0 , s_0 and c_0 :

$$b_0 = \frac{W_1(\omega_{ud})}{(1+r)^2} + \frac{W_1(\omega_{ud}) - W_1(\omega_{dd})}{(1+r)^2(1+u)((1+d) - (1+u))} \quad (28)$$

$$s_0 = \frac{W_1(\omega_{dd}) - W_1(\omega_{ud})}{(1+d)((1+d) - (1+u))} \quad (29)$$

$$c_0 = W_1(\omega_{uu}) - W_1(\omega_{ud}) + \frac{(1+u)(W_1(\omega_{dd}) - W_1(\omega_{ud}))}{(1+d)} \quad (30)$$

As $m_{2,1}$ is the only unknown variable in the above equations, we are able to find $m_{2,1}$ such that no position is taken in the fictitious asset C , i.e. $c_0 = 0$. The corresponding SDF M_1 can be specified and substituted in (7) of Section 3.1 to find the optimal terminal wealth W_1^* of the investor in the incomplete market case. The optimal asset allocation of the investor can now be found by inserting W_1^* into (28) and (29).

5.2.1 Illiquidity in the stock *and* bond

The previous two sections assume the bond can always be traded. In this section we very shortly discuss the martingale method if both the stock *and*

bond are not traded at $t = 1/2$. If the bond is not traded, the price of the bond is not defined at $t = 1/2$. As we have seen in Section 4.3.1., the SDF at time $t = 1/2$ is not defined either and the SDF at $t = 1$ is exactly the same as for the case where only the stock is not traded at $t = 1/2$. This implies the result for the martingale method in this case is exactly the same as discussed in the previous section. For that reason, the method of adding fictitious assets, described in the previous section in order to find the optimal asset allocation of the investor, is also the same. This is not a surprising result as we only have two assets, the budget constraint of the investor implies that if no trade in the stock is possible, then the investor is also not able to trade in the bond. Therefore, the case where the bond is liquid is actually the same as the case where the bond is illiquid in this model.

5.3 The combination of the complete and the incomplete market case

In the previous two sections we have shown in which way the optimal asset allocation of an investor can be specified in the events when trading in the stock is always possible at $t = 1/2$ and when trading in the stock is not possible at $t = 1/2$. In this section we assume the investor does not know beforehand whether a trading opportunity in the stock will arise at $t = 1/2$. In order to define the optimal asset allocation in this case, we combine both the complete and the incomplete market. The set of all possible scenarios equals:

$$\Omega = \{\omega_{uu}, \omega_{ud}, \omega_{du}, \omega_{dd}\} \times \{CM, IM\}$$

Where CM indicates the case when trading in the stock is possible at $t = 1/2$, i.e. the complete market, and IM the case when trading in the stock is not possible at $t = 1/2$, i.e. the incomplete market. Thus, the set of all possible scenarios in this market equals the combination of all scenarios possible in the complete market and the scenarios in the incomplete market, which results in 8 different states of the world. Section 4.4 introduced the indicator function $I_{1/2}$, which equals 1 if the complete market materializes and equals 0 otherwise. Assume now the financial market has probability $\mathbb{P}\{I_{1/2} = 1\}$ to end up in the complete market case and probability $\mathbb{P}\{I_{1/2} = 0\} (= 1 - \mathbb{P}\{I_{1/2} = 1\})$ to end up in the incomplete market. $\mathbb{P}\{I_{1/2} = 1\}$ can be any real number on the interval $[0, 1]$ and is determined exogenously, which means $\mathbb{P}\{I_{1/2} = 1\}$, and thus $\mathbb{P}\{I_{1/2} = 0\}$, is known at $t = 0$. Moreover, as we assume no liquidity premiums in this model, we also make the assumption $I_{1/2}$ is independent from the SDF process M_T . In Section 8 we will clarify why this assumption is relatively strong and discuss its implications. To simplify notion, $I_{1/2}$ will be denoted by I in the remainder of this section.

The probability that state (ω, M) , $\omega \in \Omega$, $M \in \{CM, IM\}$ will materialize at $t = 1$ is denoted by $P(\omega, M)$. $P(\omega, M)$ is the product of the conditional probability of ω given the complete or the incomplete market is materialized

and the corresponding probability of the market that is materialized. Therefore, the marginal probability of state $\omega \in \Omega$, given the market is complete is defined by $P(\omega, CM) = \mathbb{P}[I = 1]p(\omega, CM)$, where $p(\omega, CM)$ is the conditional probability of state ω , given the market is complete. The marginal probability of state $\omega \in \Omega$, given the market is incomplete, is defined by $P(\omega, IM) = \mathbb{P}[I = 0]p(\omega, IM)$, where $p(\omega, IM)$ is the conditional probability of state ω , given the market is incomplete. Notice that the conditional probabilities are identical, as the same stock S appears in both markets.

The martingale method defines exactly the payoff an investor desires to have in each state of the world. In this market we have 8 states of the world, so for each state we have to determine the optimal terminal wealth of the investor given his initial wealth W_0 . Suppose now the investor allocates a fraction α of his initial wealth W_0 to the complete market and a fraction $(1 - \alpha)$ to the incomplete market⁶. In this case we get two budget constraint. The first budget constraint implies the expected value of the SDF times the optimal terminal wealth, if $I = 1$, i.e. the market turns out to be complete, has to be equal to αW_0 . The second budget constraint implies the expected value of the SDF times the optimal terminal wealth, if $I = 0$, i.e. the market turns out to be incomplete, has to be equal to $(1 - \alpha)W_0$.

The utility maximization of terminal wealth problem of the investor can now be formalized as follows:

$$\max_{W_T} \mathbb{E}[U(W_T)] \text{ such that } \mathbb{E}[W_T M_T I] = \alpha W_0 \text{ and } \mathbb{E}[W_T M_T (1 - I)] = (1 - \alpha)W_0 \quad (31)$$

Notice the budget constraint of the original problem described in (3) of Section 3.1 is still satisfied (by using the law of iterated expectations):

$$\begin{aligned} \mathbb{E}[W_T M_T] &= \mathbb{E}[W_T M_T | I = 1] \mathbb{P}[I = 1] + \mathbb{E}[W_T M_T | I = 0] \mathbb{P}[I = 0] \\ &= \mathbb{E}[W_T M_T I] + \mathbb{E}[W_T M_T (1 - I)] = \alpha W_0 + (1 - \alpha)W_0 \\ &= W_0 \end{aligned}$$

The Lagrange function can now be defined as follows:

$$\mathcal{L} = \mathbb{E}[U(W_T)] - \lambda_1 (\mathbb{E}[W_T M_T I] - \alpha W_0) - \lambda_2 (\mathbb{E}[W_T M_T (1 - I)] - (1 - \alpha)W_0) \quad (32)$$

⁶Notice, in this setting, allocation means the allocation to different states of the world and *not* the allocation to different assets.

We maximize \mathcal{L} with respect to W_T :

$$\begin{aligned} \max_{W_T} \mathcal{L} &= \max_{W_T} \mathbb{E}[U(W_T)] - \lambda_1(\mathbb{E}[W_T M_T I] - \alpha W_0) \\ &\quad - \lambda_2(\mathbb{E}[W_T M_T(1 - I)] - (1 - \alpha)W_0) \end{aligned} \quad (33)$$

Since we have $\mathbb{E}[W_T M_T I] = \mathbb{E}[W_T M_T | I = 1] \mathbb{P}[I = 1]$ and $\mathbb{E}[W_T M_T(1 - I)] = \mathbb{E}[W_T M_T | I = 0] \mathbb{P}[I = 0]$, we get:

$$\begin{aligned} \max_{W_T} \mathcal{L} &= \max_{W_T} \mathbb{E}[U(W_T)] - \lambda_1(\mathbb{E}[W_T M_T | I = 1] \mathbb{P}[I = 1] - \alpha W_0) \\ &\quad - \lambda_2(\mathbb{E}[W_T M_T | I = 0] \mathbb{P}[I = 0] - (1 - \alpha)W_0) \end{aligned} \quad (34)$$

Since the expectation is defined as $\mathbb{E}(X) = \sum_i p_i X_i$ and we have that $P(\omega, CM) = \mathbb{P}[I = 1]p(\omega, CM)$ and $P(\omega, IM) = \mathbb{P}[I = 0]p(\omega, IM)$, we can write (34) as follows:

$$\begin{aligned} \max_{W_T} \mathcal{L} &= \max_{W_T} \sum_{\omega \in \Omega} U(W_T(\omega)) P(\omega, M) - \lambda_1 \left[\sum_{\omega \in \Omega} W_T(\omega, CM) M_T(\omega, CM) P(\omega, CM) - \alpha W_0 \right] \\ &\quad - \lambda_2 \left[\sum_{\omega \in \Omega} W_T(\omega, IM) M_T(\omega, IM) P(\omega, IM) - (1 - \alpha)W_0 \right] \end{aligned} \quad (35)$$

By the law of iterated expectations we can write (35) as:

$$\begin{aligned} \max_{W_T} \mathcal{L} &= \\ \max_{W_T} \sum_{\omega \in \Omega} &U(W_T(\omega, CM)) P(\omega, CM) - \lambda_1 \left[\sum_{\omega \in \Omega} W_T(\omega, CM) M_T(\omega, CM) P(\omega, CM) - \alpha W_0 \right] \\ + \sum_{\omega \in \Omega} &U(W_T(\omega, IM)) P(\omega, IM) - \lambda_2 \left[\sum_{\omega \in \Omega} W_T(\omega, IM) M_T(\omega, IM) P(\omega, IM) - (1 - \alpha)W_0 \right] \end{aligned} \quad (36)$$

If we take the derivative with respect to W_T , we obtain the following:

$$W_T(\omega, CM)^{-\frac{1}{\gamma}} P(\omega, CM) - \lambda_1 M_T(\omega, CM) P(\omega, CM) = 0 \text{ for all } \{\omega \in \Omega\} \quad (37)$$

$$W_T(\omega, IM)^{-\frac{1}{\gamma}} P(\omega, IM) - \lambda_2 M_T(\omega, IM) P(\omega, IM) = 0 \text{ for all } \{\omega \in \Omega\} \quad (38)$$

The optimal terminal wealth given α can now be defined as follows:

$$W_T(\omega, CM) = \lambda_1^{-\frac{1}{\gamma}} M_T(\omega, CM)^{-\frac{1}{\gamma}} \text{ for all } \{\omega \in \Omega\} \quad (39)$$

$$W_T(\omega, IM) = \lambda_2^{-\frac{1}{\gamma}} M_T(\omega, IM)^{-\frac{1}{\gamma}} \text{ for all } \{\omega \in \Omega\} \quad (40)$$

We can find λ_1 and λ_2 by inserting (39) and (40) into the budget constraints, we have:

$$\lambda_1 = \left(\frac{\alpha W_0}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, CM)]\mathbb{P}[I = 1]} \right)^{-\frac{1}{\gamma}} \quad (41)$$

$$\lambda_2 = \left(\frac{(1-\alpha)W_0}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, IM)]\mathbb{P}[I = 0]} \right)^{-\frac{1}{\gamma}} \quad (42)$$

Substituting (41) and (42) into (39) and (40), we can define the optimal wealth given α as follows:

$$W_T(\omega, CM) = \frac{\alpha W_0}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, CM)]\mathbb{P}[I = 1]} M_T^{-\frac{1}{\gamma}}(\omega, CM) \text{ for all } \omega \in \Omega \quad (43)$$

$$W_T(\omega, IM) = \frac{(1-\alpha)W_0}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, IM)]\mathbb{P}[I = 0]} M_T^{-\frac{1}{\gamma}}(\omega, IM) \text{ for all } \omega \in \Omega \quad (44)$$

The investor wants to choose α such that he is still able to reach the optimal terminal wealth corresponding to the complete market, i.e. $W_T^*(\omega, CM)$, if the complete market materializes, but also is able to reach the optimal terminal wealth corresponding to the incomplete market, i.e. $W_T^*(\omega, IM)$, if the incomplete market materializes. If we choose α and $(1-\alpha)$ in (43) and (44) in such a way the investor can always reach his optimal terminal wealth, irrespective of the state that materializes, the optimization problem described in (31) is the same as the optimization problem described in (3) of Section 3.1. Therefore, we have constructed the following theorem:

Theorem 5.1. The utility maximization problem of terminal wealth satisfies the following:

$$W_T^* \text{ solves } \max_{W_T} \mathbb{E}[U(W_T)] \text{ such that } \mathbb{E}[W_T M_T] = W_0$$

\iff

$$W_T^* \text{ solves } \max_{W_T} \mathbb{E}[U(W_T)] \text{ such that } \mathbb{E}[W_T M_T I] = \alpha W_0 \text{ and } \mathbb{E}[W_T M_T (1-I)] = (1-\alpha)W_0$$

with $\alpha = \mathbb{E}[W_T^* M_T I] / W_0$

Proof. Denote the value for α mentioned in Theorem 5.1 by α^* . We can rewrite α^* as follows:

$$\begin{aligned} \alpha^* &= \mathbb{E}[W_T^* M_T I] / W_0 = (\mathbb{E}[W_T^* M_T I | I = 1] \mathbb{P}[I = 1] + \mathbb{E}[W_T^* M_T I | I = 0] \mathbb{P}[I = 0]) / W_0 \\ &= \mathbb{E}[W_T^* M_T | I = 1] \mathbb{P}[I = 1] / W_0 \end{aligned}$$

Notice that by assumption M_T and I are independent. This implies we get:

$$\begin{aligned} \alpha^* &= \mathbb{E}[W_T^* M_T | I = 1] \mathbb{P}[I = 1] / W_0 = \mathbb{E}[W_T^* M_T] \mathbb{P}[I = 1] / W_0 = W_0 \mathbb{P}[I = 1] / W_0 \\ &= \mathbb{P}[I = 1] \end{aligned} \quad (45)$$

If we insert α^* in (43) and (44), the terminal wealth can be defined as follows:

$$W_T^*(\omega, CM) = \frac{(\mathbb{E}[W_T^* M_T I] / W_0) W_0}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, CM)] \mathbb{P}[I = 1]} M_T^{-\frac{1}{\gamma}}(\omega, CM) \text{ for all } \omega \in \Omega \quad (46)$$

$$W_T^*(\omega, IM) = \frac{(1 - \mathbb{E}[W_T^* M_T I] / W_0) W_0}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, IM)] \mathbb{P}[I = 0]} M_T^{-\frac{1}{\gamma}}(\omega, IM) \text{ for all } \omega \in \Omega \quad (47)$$

Using $\alpha^* = \mathbb{P}[I = 1]$ and the budget constraint $\mathbb{E}[W_T^* M_T] = W_0$, we can rewrite (46) and (47) as follows:

$$W_T^*(\omega, CM) = \frac{W_0 \mathbb{P}[I = 1]}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, CM)] \mathbb{P}[I = 1]} M_T^{-\frac{1}{\gamma}}(\omega, CM) \text{ for all } \omega \in \Omega \quad (48)$$

$$W_T^*(\omega, IM) = \frac{W_0 \mathbb{P}[I = 0]}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, IM)] \mathbb{P}[I = 0]} M_T^{-\frac{1}{\gamma}}(\omega, IM) \text{ for all } \omega \in \Omega \quad (49)$$

So we get exactly the optimal terminal wealth as defined in (7) of Section 3.1:

$$W_T^*(\omega, CM) = \frac{W_0}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, CM)]} M_T^{-\frac{1}{\gamma}}(\omega, CM) \text{ for all } \omega \in \Omega \quad (50)$$

$$W_T^*(\omega, IM) = \frac{W_0}{\mathbb{E}[M_T^{1-\frac{1}{\gamma}}(\omega, IM)]} M_T^{-\frac{1}{\gamma}}(\omega, IM) \text{ for all } \omega \in \Omega \quad (51)$$

The final step is to show that α^* is indeed optimal. Suppose $\alpha > \mathbb{P}[I = 1]$,

then by (43) the terminal wealth that can be reached in the complete market will exceed the optimal terminal wealth in the complete market. However, with this α the optimal terminal wealth in the incomplete market can never be reached. If $\alpha < \mathbb{P}[I = 1]$, the argument is the precise opposite. The martingale method implies the investor wants to optimize his terminal wealth in *every* state of the world, α^* is optimal, since in this case the investor can reach the optimal terminal wealth in any scenario, irrespective of the market which materializes. \square

In order to get a better understanding about α^* intuitively, we will look at the two extreme cases, namely $\mathbb{P}[I = 1] = 1$ and $\mathbb{P}[I = 1] = 0$. Suppose first $\mathbb{P}[I = 1] = 1$, this implies α^* equals $\mathbb{P}[I = 1] = 1$. Of course, if the complete market materializes with certainty, the problem described in (31) equals the general problem of utility maximization of terminal wealth described in Section 3.1 and the optimal asset allocation can be solved as described in Section 5.1. Suppose now $\mathbb{P}[I = 1] = 0$, this implies α^* equals $\mathbb{P}[I = 1] = 0$. In this case the incomplete market materializes with certainty and the problem in (31) again equals the general problem of utility maximization of terminal wealth described in Section 3.1 and the optimal asset allocation can be solved as described in Section 5.2. If $\mathbb{P}[I = 1]$ is in between 0 and 1, the investor allocates his initial wealth accordingly to his optimal α^* .

6 Results for the two-time period economy

In this section the optimal asset allocation for the investor is presented and clarified by means of a stylized example for the three different cases; the complete market, the incomplete market and the combination of the complete and the incomplete market. A sensitivity analysis shows the impact on the behaviour of the investor when he is confronted with liquidity risk.

6.1 Descriptive results

In this section the results of the optimal asset allocation for investors are presented for the three different cases; complete market, incomplete market and the combination of the complete and the incomplete market. The time horizon is initially taken equal to one year, i.e. t_0 indicates the start of the year and $T = 1$ the end of the year. We assume in the base case the expected return on the stock equals $\mu = 5.5\%$ ⁷ and the standard deviation for the 1 year horizon is equal to $\sigma = 20\%$ ⁸. Based on the above assumption about the stock and the condition $u < r < d$, we have defined the following default values for the parameters of the model:

$$r = 0.0149, u = 0.1720, d = -0.0998, \gamma = 3, p = 0.5, \mathbb{P}\{I = 1\} = 0.6 \text{ and } W_0 = 100$$

The resulting SDF at $t = 1$ for the complete and the incomplete case, respectively M_1^{CM} and M_1^{IM} , and the corresponding optimal terminal wealth W_1 for the investor are tabulated below.

$$M_1^{CM} = \begin{pmatrix} 0.6917 \\ 0.9473 \\ 0.9473 \\ 1.2973 \end{pmatrix} \Rightarrow W_1^{CM} = \begin{pmatrix} 115.9554 \\ 104.4160 \\ 104.4160 \\ 94.0249 \end{pmatrix}$$

$$M_1^{IM} = \begin{pmatrix} 0.6814 \\ 0.9590 \\ 0.9590 \\ 1.2840 \end{pmatrix} \Rightarrow W_1^{IM} = \begin{pmatrix} 116.5302 \\ 103.9856 \\ 103.9856 \\ 94.3469 \end{pmatrix}$$

Below, the corresponding optimal trading strategies for the three cases at time $t = 0$ are tabulated.

⁷ μ is defined as the geometric expected return: $\mu = \prod_i (1 + r_i)^{p_i} - 1$, where r_i is the payoff in scenario i and p_i the probability scenario i materializes.

⁸ σ is defined as $\sigma = \sqrt{(\sum p_i (r_i - \mu)^2)}$, where r_i is the payoff in scenario i and p_i is the probability scenario i materializes.

Table 1: Optimal Asset Allocation

Case	b_0	s_0
Complete market	60.58	39.42
Incomplete market	60.62	39.38
Combination of the complete and the incomplete market	60.60	39.40

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter $\gamma = 3$ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (no trade in S at $t = 1/2$) and the combination of the complete and incomplete market (uncertain trade in S at $t = 1/2$), parameter values financial market: $r = 0.0149$; $u = 0.1720$; $d = -0.0998$ and $p = 0.5$.

The distraction of α^* in Section 5.3 implies α^* equals $\mathbb{P}\{I = 1\} = 0.6$. This means the investor allocates 60% of his initial wealth to the complete market case, i.e. trading at $t = 1/2$ in the stock is always possible. The remainder 40% of his initial wealth is allocated to the incomplete market case, i.e. trading in the stock at $t = 1/2$ is not possible. The asset allocation of the investor is independent of his initial wealth as we make use of the CRRA utility function. Therefore, the allocation to the bond and to the stock simply equals a linear combination of the optimal allocation under the complete and the incomplete case. This implies the allocation to the bond equals $(60\% \times 60.58 + 40\% \times 60.62) = 60.60$ and the allocation to the stock equals $(60\% \times 39.42 + 40\% \times 39.38) = 39.40$.

6.2 Sensitivity analysis

In this section we will clarify the results described in the previous section and a sensitivity analysis of the parameters of the model is performed. The sensitivity analysis will lead to some important implications of the behaviour of the investor when he is confronted with illiquidity risk. First of all, an overview of the behaviour of the investor in case the stock can always be traded, i.e. the complete market, is shown. Then, the behaviour of the investor in the complete market is compared with the behaviour in the incomplete market. We do this by changing the parameters of the model and compare the effects on the optimal asset allocation for both cases. Finally, we will link the results obtained from the sensitivity analysis to the combination of the complete and the incomplete market case and show the behaviour of the investor when he is confronted with *uncertainty* about the liquidity of the stock.

6.2.1 The complete market case

In the complete market case the optimal trading strategy at time $t = 1/2$ as a proportion of total wealth is precisely the same as the optimal trading strategy as a proportion of total initial wealth at time $t = 0$. This is due to the specific stochastic process of the risky stock S . At each point in time in the complete market case, the investor is confronted with the same scenario, either the next point in time the stock makes an up move with probability p or the stock makes a down move with probability $1 - p$. This implies the strategy of the investor remains the same, i.e. the proportion of his wealth at time t that is allocated to the stock, and thus to the bond, remains constant over time. This result coincides with the famous result of the Merton's Portfolio Problem in a continuous time framework, which was solved by Merton (1969). The solution for the optimal allocation to the risky assets found by Merton (1969) is as follows:

$$\pi(W, t) = \frac{\mu - r}{\sigma^2 \gamma}$$

where W is the wealth of the investor, t indicates time, μ is the expected return of the available assets, r the risk-free rate, σ^2 is the variance of the risky asset and γ is the risk-aversion parameter. So the optimal asset allocation to the risky asset is constant over time and depends only on the price of risk of the asset and the risk aversion parameter γ .

In this two-period discrete time economy, the proportions of the optimal asset allocation depends on the SDF, the risk-aversion parameter γ and the payoff matrix. The SDF itself depends again on the parameters of the model. So both the optimal trading strategy at time $t = 1/2$ and $t = 1$ depend on γ and the parameters describing the market; r , u , d and p . So therefore, as the parameters are time-independent, the proportion invested in the stock is a constant proportion of the total wealth over time.

If we change the risk aversion parameter γ , the optimal asset allocation of the investor changes. The higher γ , the less variation in the payoffs in each state of the world is desired, which implies that an asset allocation is chosen such that the optimal terminal wealth is approximately equal in all states. As the payoff of the bond does not vary between the states, the risk-averse investor will allocate a high amount of his initial wealth to the bond. If γ is relatively low, the investor is less concerned about the variation in the payoffs in different states of the world and focuses more on achieving a high payoff. Therefore, this investor will allocate more to the stock, as there is a positive probability to end up in a state where the payoff is higher for the stock than the payoff of the risk-free bond.

If we change the parameters of the financial market, i.e. change u , d , r or p , the investor will allocate more to the stock (bond) if the Sharpe ratio

of the stock increases (decreases) in comparison to the constant return r on the risk-free bond, irrespective of the risk aversion parameter, see Table 7, 8, 9 and 10 in the Appendix. The Sharpe ratio measures the risk-adjusted return on the stock and is defined by $\frac{\mu-r}{\sigma}$. The Sharpe ratio of the stock increases (decreases) for instance if we increase (decrease) u , assuming the other parameters constant. Of course, the effect of changing the Sharp ratio will be stronger visible in the asset allocation of a less risk-averse investor than for the extremely risk-averse agent for the reason described in the previous paragraph.

Finally, we increase the time horizon, i.e. increase T , which means values for the parameters of the financial market u , r and d are taken in such a way that μ and σ correspond to a longer time horizon. For instance if we take $T = 5$, we assume the expected return on the stock is $\mu = 27.5\% (= 5.50\% \times 5)$ and the standard deviation equals $\sigma = \sqrt{5} \times 20\% (= \sqrt{T} \times 20\%)$. We see in Table 11 of the Appendix that the longer the horizon, the higher the allocation in the stock S becomes. This is counter-intuitive if we interpret the optimal allocation in the risky asset found by Merton, since $\pi(W, t) = \frac{T(\mu-r)}{(\sqrt{T}\sigma)^2\gamma} = \frac{T(\mu-r)}{T\sigma^2\gamma} = \frac{\mu-r}{\sigma^2\gamma}$, which implies the asset allocation should be independent for the horizon T . However, as the Merton solution is solved in continuous time, the number of time steps increases with T and the investor can rebalance his portfolio continuously, while in our model the number of time steps does not increase with T and the investor can still only rebalance his portfolio at one point in time. Since u increases more than d if we increase the horizon T to make sure $\mu = 27.5\%$ and $\sigma = \sqrt{5} \times 20\%$, this might be the clarification the allocation to the stock increases. The goal of the sensitivity analysis is to find the behaviour of the investor when he is confronted with illiquidity, therefore we do not discuss this result further.

6.2.2 Comparison complete market and incomplete market case

The behaviour of the investor in the complete market by changing the risk-aversion parameter or the parameters of the financial market as described in the previous section is the same for the investor in the incomplete market. More interestingly is to compare the relative size of the changes in the optimal asset allocation in the complete and the incomplete market if we change certain parameters. For obvious reasons the Merton's Portfolio Problem implication is not visible in the incomplete market as there is no re-balancing opportunity at time $t = 1/2$.

The first notable observation is that the less risk-averse the investor, the higher the difference in the optimal asset allocation between the complete and the incomplete market case. In Table 2 we see that for a less risk-averse investor, e.g. $\gamma = 0.5$, the initial allocation to the bond is higher in the

incomplete market than in the complete market case. While, for a very risk averse investor, e.g. $\gamma = 30$, the differences are negligible. This can be clarified as follows. A risk-averse agent already allocates a high amount of his initial wealth to the bond in the complete market case, so the fact that he cannot trade in the stock at $t = 1/2$ does not have a high impact on his initial asset allocation in that case. Whereas a more risk-taking investor allocates a significant amount of his initial wealth to the stock in the complete market, the restriction he is not able to trade in the stock at $t = 1/2$ in the incomplete market, has a much higher impact on his initial asset allocation. The more risk-taking investor will allocate more in the bond in the incomplete market case and reduces his exposure to the stock to protect himself against the no-trading opportunity in the stock at $t = 1/2$.

Table 2: Optimal Asset Allocation

Default				
Case	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 30$
Complete market b_0	-138.71	20.64	60.58	96.08
Incomplete market b_0	-129.23	20.72	60.62	96.11
Combination market b_0	-134.92	20.67	60.60	96.09
Complete market s_0	238.71	79.36	39.42	3.92
Incomplete market s_0	229.23	79.28	39.38	3.89
Combination market s_0	234.92	79.33	39.40	3.91

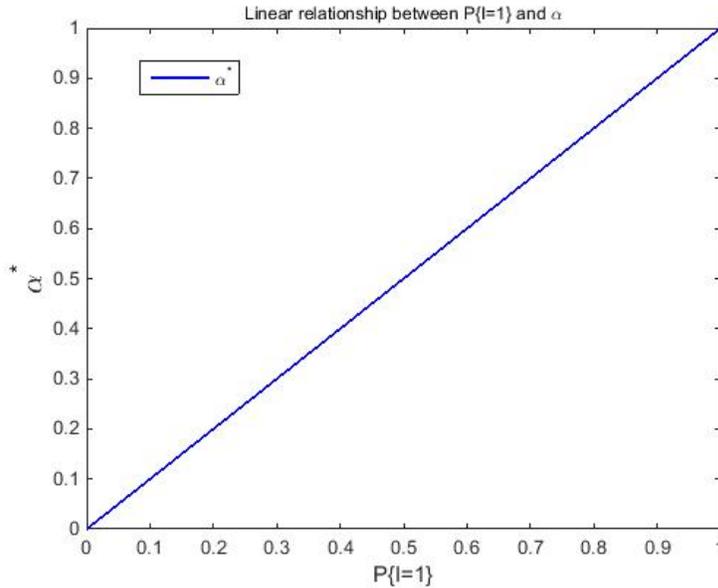
Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter γ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (no trade in S at $t = 1/2$) and the combination market (uncertainty trade in S at $t = 1/2$), parameter values financial market: $r = 0.0149$; $u = 0.1720$; $d = -0.0998$ and $p = 0.5$.

Moreover, if the Sharpe ratio of the stock increases, the above described effect becomes more visible, see Table 7 and 9 of the Appendix. As an increase in the Sharpe ratio of the stock leads to a higher allocation to the stock in the complete market, the risk of no trade in the stock at $t = 1/2$ becomes larger, irrespective of the risk aversion parameter of the investor. Therefore, the investor reduces his exposure to the stock even more than in the default case to protect himself against the scenarios no trading will be possible at $t = 1/2$. If we decrease the Sharpe ratio of the stock, the opposite effect is observed, see Table 8 and 10 of the Appendix. As the bond becomes more attractive, the investor will already allocate more to the bond in the complete market case. This means his risk is decreased in comparison to the default case if he cannot trade in the stock at $t = 1/2$. Therefore, his reduction in the exposure to the stock to protect himself against the risk of illiquidity in comparison to the complete market case, will be smaller.

Another result is that if we increase the time horizon T , which means we take values for the parameters u , r and d that correspond to a longer time horizon, the difference in the allocation to the stock in the incomplete market case compared to the complete market stays approximately the same, see Table 11 of the Appendix. In the previous section we already discussed why this result might not lead to reasonable conclusions. In order to get a better insight in the behaviour of the investor when the horizon T increases, we also have to increase the number of trading opportunities, which will be done in Section 7.5.

6.2.3 Implications for the combination of the complete and the incomplete market case

In the previous sections the effect of changing the parameters of the model in the complete and the incomplete market are discussed. Of course, all these effects will also be visible in the combination of the complete and the incomplete market as it is a combination of the distinct two cases. The complete and the incomplete market can be seen as the two extreme cases of the combination of the complete and the incomplete market, namely the cases where $\alpha^* = \mathbb{P}[I = 1] = 1$ and $\alpha^* = \mathbb{P}[I = 1] = 0$. For every $\alpha^* = \mathbb{P}[I = 1] \in (0, 1)$ the results will be in-between the complete and the incomplete market case. The graph below shows the linear relationship between α^* and $\mathbb{P}\{I = 1\}$.



This linear relationship therefore implies the optimal asset allocation of the investor will always be somewhere between the two extreme cases, depending

on the value of $\mathbb{P}\{I = 1\}$. By the assumed CRRA utility function of the investor, the allocation of the investor does not depend on his initial wealth. Therefore, the optimal asset allocation $[b_0 \ s_0]$, given α^* , can be calculated by $[b_0 \ s_0] = \mathbb{P}[I = 1] \times [b_0^{CM} \ s_0^{CM}] + \mathbb{P}[I = 0] \times [b_0^{IM} \ s_0^{IM}]$, where b_0^{CM} (s_0^{CM}) is the allocation to the bond (stock) in the complete market case and b_0^{IM} (s_0^{IM}) is the allocation to the bond (stock) in the incomplete market case.

7 The n -period economy

In the previous sections we have elaborated an economy with only two-time periods. In this section the market is extended to a n -period economy. The effect of illiquidity on the asset allocation will become more visible and realistic when we consider many time periods. First, a formalization and probability setting of the n -period economy will be presented (Section 7.1). Secondly, we will formulate the optimal asset allocation for the *complete* (Section 7.2), *incomplete* (Section 7.3) and *combination of the complete and the incomplete* (Section 7.4) market case in a n -period setting. In this section we split up the incomplete market case in two cases. Case one restricts the investors to only take a position in the stock at time t_0 and no trading opportunities arise afterwards. Case two allows the investors to trade in the stock at some predetermined points in time.

7.1 Formalization and probability setting

We consider n periods of length T/n , where T is the terminal time. The present is indicated by $t_0 = 0$ and the future times are $t_i = i \times T/n$, $i = 1, 2, \dots, n$. Again there are 2 assets available in the market, a risk free bond B and a stock S , whose initial prices at $t_0 = 0$ are respectively given by $S_0 = 1$ and $B_0 = 1$.

The bond price follows a deterministic process: $B_{t_i} = (1 + r)^i$, for all $t_i > 0$. As in the two-period economy, the varying process of the stock price S_{t_i} is represented by the binomial tree. Each time period the stock will either go up by the factor $1 + u$ or down by the factor $1 + d$. This means the stock price follows a stochastic process:

$$S_{t_i} = S_{t_{i-1}} Z_{t_{i-1}}, \text{ for all } i = 1, 2, \dots, n$$

where $Z_{t_0} = 1$ and $Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}$ is a sequence of i.i.d. random variables of the form:

$$Z_{t_i} = \left\{ \begin{array}{ll} 1 + u & \text{with probability } p \\ 1 + d & \text{with probability } 1 - p \end{array} \right\}$$

Again, it is assumed trading in the bond B is always possible at deterministic points in time at well-specified prices. Trading in the stock S is always possible at t_0 and at the terminal date T .

The set of all possible scenarios, i.e. the sample space Ω is composed by every possible path followed by the stock price in $[0, T]$. Every path is completely determined by the outcomes of the n random variables $Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}$, the scenario $\omega_k \in \Omega$ is of the form $\omega_k = (\omega_{t_1}, \omega_{t_2}, \dots, \omega_{t_n})$, where for each i , ω_{t_i} is either $1 + u$ or $1 + d$. Therefore, Ω has 2^n elements. The information filtration \mathcal{F}_{t_i} in the n -period setting is represented as follows:

$$\begin{aligned}
\mathcal{F}_{t_0} &= \{\emptyset, \Omega\} \\
\mathcal{F}_{t_1} &= \sigma(\{\{Z_{t_0} = 1 + u\}, \{Z_{t_0} = 1 + d\}\}) \\
\mathcal{F}_{t_2} &= \sigma(\mathcal{F}_{t_1} \cup \{\{Z_{t_1} = 1 + u\}, \{Z_{t_1} = 1 + d\}\}) \\
&\vdots \\
\mathcal{F}_{t_n} &= \sigma(\mathcal{F}_{t_{n-1}} \cup \{\{Z_{t_{n-1}} = 1 + u\}, \{Z_{t_{n-1}} = 1 + d\}\})
\end{aligned}$$

where $\{Z_{t_i} = 1 + u\} = \{\{\omega_{t_0}, \omega_{t_1}, \dots, \omega_{t_{n-1}}\} \in \Omega : \omega_{t_i} = 1 + u\}$,
 $\{Z_{t_i} = 1 + d\} = \{\{\omega_{t_0}, \omega_{t_1}, \dots, \omega_{t_{n-1}}\} \in \Omega : \omega_{t_i} = 1 + d\}$.

The probability to go up by the factor $1 + u$ equals p and the probability to go down by the factor $1 + d$ equals $1 - p$ for each time period i . This means we have for all $i = 1, \dots, n$:

$$\begin{aligned}
\mathbb{P}(Z_{t_i} = 1 + u) &= p \\
\mathbb{P}(Z_{t_i} = 1 + d) &= 1 - p
\end{aligned}$$

In summary, S_{t_i} is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$.

7.2 The complete market case

In this section the optimal asset allocation of an agent is determined for a n -period setting. In the complete market case the investor is able to trade in the stock at all n future points in time. Section 7.2.1 gives an overview of the necessary pricing equations in order to distract the SDF process for the complete market in a n -period economy. Section 7.2.2 shows the optimal asset allocation of the agent by means of the martingale method.

7.2.1 Stochastic Discount Factor process

The equations needed in order to distract the SDFs for the complete market is just an extension of the two-period setting and can be formulated as follows:

$$\mathbb{E}_0(M_T B_T) = B_0 \tag{52}$$

$$\mathbb{E}_0(M_T S_T) = S_0 \tag{53}$$

$$\mathbb{E}_{t_i}(M_T B_T) = B_{t_i} M_{t_i} \text{ for } i = 1, 2, \dots, n - 1 \tag{54}$$

$$\mathbb{E}_{t_i}(M_T S_T) = S_{t_i} M_{t_i} \text{ for } i = 1, 2, \dots, n - 1 \tag{55}$$

As we have seen in the two-period setting, the SDF at t_{i+1} , is just a multiplication of the SDF at t_i . Recall that this is the result of the independent and

identically distributed stock price process S_{t_i} . The total number of equations and unknowns in a n -period complete market equals $2 \times \sum_{i=0}^{n-1} 2^i$.

7.2.2 Martingale method

The martingale method in a n -period setting looks similar to the two-period setting. The Lagrangian function is defined as follows:

$$\mathcal{L} = \mathbb{E}[U(W_T)] - \lambda(\mathbb{E}[W_T M_T] - W_0)$$

where λ is called the Lagrange multiplier. We have to maximize the Lagrangian with respect to the investor's final wealth:

$$\max_{W_T} \mathcal{L} = \max_{W_T} \mathbb{E}[U(W_T)] - \lambda(\mathbb{E}[W_T M_T] - W_0)$$

Again, we assume the investors have CRRA utilities. The optimal solution for the terminal wealth of the investor, W_T , is defined as follows:

$$W_T = \lambda^{-\frac{1}{\gamma}} M_T^{-\frac{1}{\gamma}} = \frac{W_0}{\mathbb{E}(M_T^{1-\frac{1}{\gamma}})} M_T^{-\frac{1}{\gamma}} \quad (57)$$

where W_T is a vector of length 2^n

By definition of the martingale method, every payoff W_T can be reached using the appropriate trading strategy. Again, in order to find the optimal trading strategy of the investor, the problem is solved backwards in time. We start with solving the optimal trading strategy at time t_{n-1} , then at t_{n-2} and repeat the same algorithm until we reach time t_0 . The algorithm used to solve this problem is exactly the same as described in Section 5.1.

7.3 The incomplete market case

In this section the optimal asset allocation of an agent in the incomplete market is determined for a n -period economy. The incomplete market case is now split up in two cases. In the first case the investor takes an allocation in the stock time $t_0 = 0$ and is not able to trade in the stock for the future $n - 1$ time periods. In the second case, the investor takes an allocation in the stock at time $t_0 = 0$ and is allowed to trade in the stock at some predetermined points in time. Section 7.3.1 gives an overview of the necessary pricing equations in order to distract the SDF process for both cases of the incomplete market in a n -period economy. Section 7.3.2 shows the optimal asset allocation of the investor by means of the martingale method.

7.3.1 Stochastic Discount Factor process

The pricing equations necessary to distract the SDF for the first case of the incomplete market; investors can only take a position in the stocks at $t_0 = 0$ and no trading opportunities in the stock arise afterwards, are formulated as follows:

$$\mathbb{E}_0(M_T B_T) = B_0 \quad (58)$$

$$\mathbb{E}_0(M_T S_T) = S_0 \quad (59)$$

$$\mathbb{E}_{t_i}(M_T B_T) = B_{t_i} M_{t_i} \text{ for } i = 1, 2, \dots, n-1 \quad (60)$$

Notice no prices are defined for the stock on the interval $(0, T)$, so only pricing equation (59) is satisfied here. The prices for the bond are well-specified at every t_i , so we can specify the pricing equations as stated above in (58) and (60). As in the two-period economy, we have again a set of *incompleteness constraints*:

$$\begin{aligned} M_T(\omega_j | \# \text{of up moves} = i) &= M_T(\omega_k | \# \text{of up moves} = i) & (61) \\ \text{for } \{j = 1, 2, \dots, 2^n, k = 1, 2, \dots, 2^n, j > k, \\ & i = 1, 2, \dots, n\} \end{aligned}$$

The incompleteness constraints impose the scenarios with the same number of up moves (and thus the same number of down moves) need to have the same value in the SDF process. The reasoning is exactly the same as in Section 3.2. As the investors can only take an allocation in the stock at time $t_0 = 0$, irrespective of the initial allocation to stock at $t_0 = 0$, the payments in the states with the same number of up moves (and thus the same number of down moves) have to be equal to each other. In order to obtain this, it has to be the case the scenarios with the same number of up moves need to have the same value in the SDF process M_T . Condition (60) results in $\sum_{i=1}^{n-1} 2^i$ independent equations and (61) in $\sum_{i=1}^{n-1} 2^i - (n-1)$ independent equations. Thus, the total number of equations equals $2 \times \sum_{i=0}^{n-1} 2^i - (n-1)$. As the number of unknowns equals $2 \times \sum_{i=0}^{n-1} 2^i$, we get $n-1$ degrees of freedom in this case of the incomplete market.

In the second case the investor takes an allocation in the stock at time t_0 and is not allowed to trade in the stock at *some* predetermined points in time. The set of points in time the investor is not allowed to trade is denoted by l , $l \in \{1, 2, \dots, n-1\}$. The pricing equations necessary to distract the SDF process for the incomplete market in this case are as follows:

$$\mathbb{E}_0(M_T B_T) = B_0 \quad (62)$$

$$\mathbb{E}_0(M_T S_T) = S_0 \quad (63)$$

$$\mathbb{E}_{t_i}(M_T B_T) = B_{t_i} M_{t_i} \text{ for } i = 1, 2, \dots, n-1 \quad (64)$$

$$\mathbb{E}_{t_i}(M_T S_T) = S_{t_i} M_{t_i} \text{ for } \{i = 1, 2, \dots, n-1, i \notin l\} \quad (65)$$

Again, the prices for the bond are well-specified at all t_i . The prices for the stock are well-defined at t_0 , T and at the predetermined points $t_{i,i \neq T}$. If we have for instance $n = 20$ and trade in the stock is only possible at t_7 and t_{14} , the prices at t_7 and t_{14} are well-defined.

In this case we do however see that the incompleteness constraints are not valid for all j and k mentioned in (61) anymore. This can be clarified as follows. Since we are able to trade at some future points in time before the terminal date T , we do not necessarily need to have the same values for the entries of the SDF process M_T with the same number of up moves (and thus the same number of down moves). Suppose the investor is not able to trade at one predetermined point in time t_i . At t_{i-1} the investor can rebalance his portfolio and in principle he is able to take different asset allocations for each scenario that might materialize at t_{i-1} . Therefore, at t_{i+1} sample paths of the stock S which were in a different scenario at t_{i-1} , can still end up with the same number of up moves at time t_{i+1} . However, the value of the entries of the SDF process for both paths of the stock S do not have to be identical to each other, since a different asset allocation might have been taken for both paths at time t_{i-1} . To make this more clear, assume we have a 5-period economy and we are only not able to trade in the stock at t_4 , and we can always trade in the stock at the other points in time. We analyze two different paths. In the first one suppose the stock has made 2 up moves at t_3 and in the other path the stock has made only 1 up move. Suppose the investor takes a different asset allocation depending on which sample path of the stock materializes at t_3 . Then both sample paths can still end up in a scenario with the same number of up moves at terminal time T , but the payoff of the stock S differs for the two sample paths. This implies only the sample paths of the stock S that have been in exactly the same scenario at time t_3 and which have the same number of up moves at T , need to be identical to each other. In this example, we will have 46 unknowns, 30 pricing equations and only 4 incompleteness constraints. This implies we already have 12 degrees of freedom.

As became clear from the example, a significant amount of incompleteness constraints disappear and the degrees of freedom highly increase. If we take more and more time steps, the number of unknowns will grow more rapidly than the number of equations. So the total number of degrees of freedom will increase dramatically. Suppose $n = 10$ and trading is only not possible at t_9 , then the total number of degrees of freedom equals already 1013.

7.3.2 Martingale method

The martingale method is explained for the two different cases in the incomplete market. First, the case where the investor takes an allocation in the

stock at time $t_0 = 0$ and is not able to trade in the stock for the future $n - 1$ time periods is discussed. Secondly, we discuss the case where the investor cannot trade in the stock at some predetermined points in time.

As we have seen in the previous section, the more time-steps we take, the more degrees of freedom we get in the first case of the incomplete market. For every extra point in time trading is not possible, we get an extra degree of freedom. This also implies we have to increase the number of fictitious assets we add to make the system solvable. For each point in time trading is not possible, we have to add an extra fictitious asset. This means that if we take $n = 20$, we add 19 fictitious assets. As in the two-period setting, to make the system solvable we have to make sure the fictitious assets together with the stock and the bond are independent. In the two-period economy we add the binary call option that pays off 1 in scenario ω_{uu} and nothing otherwise. If we take n periods in time, we will add a set of $n - 1$ binary call options. These binary call options all pay off in different states. Denote each fictitious asset by C_i , with payoff represented by $C_{i,T}$ at terminal time T . Asset C_{i+1} pays off in the same scenario as asset C_i at time T , *and* in the scenarios that have made an *up* move one time less than the last scenario where C_i pays off. For instance, if we have a tree-period economy, we add the asset that pays off 1 in scenario ω_{uuu} and nothing otherwise and we add the asset that pays off 1 in scenario ω_{uuu} , ω_{uud} , ω_{udu} and ω_{duu} and nothing otherwise. In formal terms, we add the following set of fictitious assets:

$$C_{1,1} = \begin{pmatrix} C_{1,1}(\omega_1) \\ C_{1,1}(\omega_2) \\ \vdots \\ C_{1,1}(\omega_{2^n}) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad C_{2,1} = \begin{pmatrix} C_{2,1}(\omega_1) \\ C_{2,1}(\omega_2) \\ \vdots \\ C_{2,1}(\omega_{2^n}) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad \dots$$

$$C_{n-1,1} = \begin{pmatrix} C_{n-1,1}(\omega_1) \\ C_{n-1,1}(\omega_2) \\ \vdots \\ C_{n-1,1}(\omega_{2^n}) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

As we have seen in the previous section for the second case, we can invent different examples of cases where we get more than $n - 1$ degrees of freedom. A large number of degrees of freedom, also implies we need to add a large number of fictitious asset to make the system solvable. Recall the example with $n = 10$ time periods and the investor only cannot trade at t_9 , but is allowed to rebalance his portfolio at the other points in time. This example implies we need to add 1013 fictitious assets, which all have to be independent from each other. This gives the presumption the problem becomes computationally too large and another approach to solve this case in an efficient manner is desired. Therefore, we will only discuss the results of a 3-period economy in the next section.

7.4 The combination of the complete and the incomplete market case

In this section the optimal asset allocation of an agent in the combination of the complete and the incomplete market is determined for a n -period economy. Section 7.4.1 gives an overview of the necessary pricing equations the SDF process has to satisfy for the combination of the complete and the incomplete market in a n -period economy. Section 7.4.2 shows the optimal asset allocation of the investor by means of the martingale method.

7.4.1 Stochastic Discount Factor process

In the n -period economy we have $n - 1$ points in time where the investor is uncertain whether he will be able to trade in the stock or not. In this case we get $n - 1$ indicator functions, defined by I_{t_i} for $i = 1, 2, \dots, n - 1$. I_{t_i} equals 1 if the investor is able to trade at t_i and 0 otherwise. The pricing equations the SDFs has to satisfy in the combination of the complete and the incomplete market are the following:

$$\mathbb{E}_0\{M_T B_T\} = B_0 \tag{66}$$

$$\mathbb{E}_0\{M_T S_T\} = S_0 \quad (67)$$

$$\mathbb{E}_{t_i}\{M_T B_T\} = B_{t_i} M_{t_i} \text{ for } i = 1, 2, \dots, n-1 \quad (68)$$

$$\mathbb{E}_{t_i}\{S_T M_T - S_{t_i} M_{t_i} | I_{t_i} = 1\} = 0 \text{ for } i = 1, 2, \dots, n-1 \quad (69)$$

Again, as in the two-period case, the above pricing equations are a combination of the complete and the incomplete market case. If I_{t_i} turns out to be 1, the pricing equation $\mathbb{E}_{t_i}\{M_T S_T\} = S_{t_i} M_{t_i}$ has to be satisfied, whereas if I_{t_i} turns out to be 0 the pricing equation does not have to hold. If $I_{t_i} \neq 0$ for all $i = 1, 2, \dots, n-1$, i.e. trade is possible on the interval $(0, T)$, Section 7.3 implies the problem becomes computationally too large for sufficient number n .

7.4.2 Martingale method

In Section 5.3 we explained in which way we can combine the complete and the incomplete market in a two-period economy in such a way the investor is still able to reach his optimal terminal wealth in both markets. In this section we will intuitively explain how the method described in Section 5.3 can be extended to the n -period case, however we will not give a formal proof. The idea remains exactly the same and it becomes clear the proof for the n -period will be quite obvious if the two-period case is understood.

The market that will materialize at time T can take different forms in the n -period economy. In the two-period economy we only had either the complete or the incomplete market that could materialize. In the n -period economy, the complete market can materialize, but also one of the different forms of the incomplete market. In Section 7.3 we have distinguished the different forms in which the incomplete market can appear. At every point in time t_i there is a probability equal to $\mathbb{P}[I_{t_i} = 1]$ the investor is able to trade at t_i . The outcomes for all I_{t_i} show the form of the materialized market. Again, the investor allocates fractions of his initial wealth to the different markets that can appear in such a way he can reach his optimal terminal wealth in the market which materializes. If we have for instance the 3-period economy, with initial time t_0 and terminal time $t_3 = T$. The probability the investor is able to trade at t_1 equals $\mathbb{P}[I_{t_1} = 1]$ and equals $\mathbb{P}[I_{t_2} = 1]$ at t_2 . The market can appear in four different forms; trade is always possible, trade is only possible at t_1 , trade is only possible at t_2 or no trade is possible at all. The corresponding probabilities equal $\mathbb{P}[I_{t_1} = 1, I_{t_2} = 1]$, $\mathbb{P}[I_{t_1} = 1, I_{t_2} = 0]$, $\mathbb{P}[I_{t_1} = 0, I_{t_2} = 1]$ and $\mathbb{P}[I_{t_1} = 0, I_{t_2} = 0]$ respectively. The investor will allocate his initial wealth to the different markets according to these probabilities to obtain his optimal terminal wealth. Again, as explained in Section 7.3, the problem becomes computationally too large for sufficient number n .

7.5 Results for the three-time period economy

In this section the results of the 3-period economy are presented. We assume for the default again $T = 1$, so we have $t_0 = 0$, $t_1 = 1/3$, $t_2 = 2/3$ and $t_3 = T = 1$. As in the two-period economy, the expected return on the stock equals $\mu = 5.5\%$ and the standard deviation of the return is equal to $\sigma = 20\%$. So the only difference with the two-period economy is that we have one extra trading opportunity between t_0 and $T = 1$. At t_1 there is a probability of $\mathbb{P}[I_{1/3} = 1] = 0.6$ the investor is able to trade in the stock and the probability is again $\mathbb{P}[I_{2/3} = 1] = 0.6$ at t_2 . The other parameter values remain the same as described in Section 6.1. Therefore, the remainder of the values for the parameters of the model equal:

$$r = 0.0099, u = 0.1334, d = -0.0856, \gamma = 3, p = 0.5 \text{ and } W_0 = 100$$

In Table 3 the optimal asset allocation is shown in case of the complete market, the incomplete market and the combination of the complete and the incomplete market. The incomplete market is split up in three different cases, namely: trading in the stock is not possible at t_1 and t_2 , trading in the stock is only not possible at time t_1 and trading in the stock is only not possible at time t_2 . In the first and the third case we have to add 2 fictitious assets, whilst we only need to add 1 fictitious asset in the second case. Notice in the second case, the fictitious asset is added at time t_2 , since we can just solve the optimal trading strategy at t_2 by the backwards algorithm explained in Section 5.1. In the third case, the fictitious asset is added at time t_3 to find the optimal allocation at t_2 and the optimal trading strategy at time t_1 is again solved by the backwards algorithm explained in Section 5.1. The optimal asset allocation in the combination of the complete and the incomplete market case is a linear combination of the optimal allocation under the complete market, the incomplete market case 1, the incomplete market case 2 and the incomplete market case 3. We assume $I_{1/3}$ and $I_{2/3}$ are independent. Therefore, the investor will allocate 36% ($= \mathbb{P}[I_{1/3} = 1] \times \mathbb{P}[I_{2/3} = 1]$) of his initial wealth to the complete market case, 16% ($= \mathbb{P}[I_{1/3} = 0] \times \mathbb{P}[I_{2/3} = 0]$) to the incomplete case 1, 24% ($= \mathbb{P}[I_{1/3} = 1] \times \mathbb{P}[I_{2/3} = 0]$) to the incomplete case 2 and again 24% ($= \mathbb{P}[I_{1/3} = 0] \times \mathbb{P}[I_{2/3} = 1]$) to the incomplete case 3.

Table 3: Optimal Asset Allocation

Case	b_0	s_0
Complete market case	60.36	39.64
Incomplete market case	60.45	39.55
Incomplete market case 2	60.37	39.63
Incomplete market case 3	60.37	39.63
Combination of the complete and the incomplete market case	60.38	39.62

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter $\gamma = 3$ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (case 1: no trade in S at $t = 1/3$

and $t = 2/3$, case 2: no trade in S at $t = 1/3$, case 3: no trade in S at $t = 2/3$) and the combination of the complete and the incomplete market (uncertainty trade in S at both $t = 1/3$ and $t = 2/3$), parameter values of the model: $r=0.0099$; $u=0.1334$; $d=-0.0856$ and $p = 0.5$.

The results for the optimal asset allocation in the complete market and the the three different cases in the incomplete market do not show economically significant differences. However, we do see that the allocation to the bond in case 1 of the incomplete market, i.e. where the investor is not able to trade for 2 periods in sequence, is slightly higher than the optimal asset allocation in case 2 and 3, which are precisely the same. This implies the investor does reduce his exposure to the stock if he is not able to trade for more periods in sequence. In Table 4 the optimal asset allocation is shown for different values of the risk-aversion parameter γ . If γ is smaller, the difference between the case where the investor is not able to trade for two periods in sequence and the cases where the investor is only not able to trade once, becomes larger. This confirms the statement the investor reduces his exposure to the stock if he is not able to trade for more periods in sequence. Moreover, Table 3 and Table 4 suggest the investor is indifferent about the point in time the no-trading period arises, since the allocation in the cases where the investor is not able to trade once are approximately identical, whereas the no-trading period arises at t_1 in case 2 and at t_2 in case 3.

Table 4: Optimal Asset Allocation

Default				
Case	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 30$
Complete market b_0	-139.31	20.29	60.36	96.05
Incomplete market case 1 b_0	-126.01	20.42	60.42	96.11
Incomplete market case 2 b_0	-132.69	20.37	60.38	96.08
Incomplete market case 3 b_0	-132.69	20.37	60.38	96.08
Combination market b_0	-134.00	20.35	60.38	96.06
Complete market s_0	239.31	79.71	39.64	3.95
Incomplete market case 1 s_0	226.01	79.63	39.58	3.89
Incomplete market case 2 s_0	232.69	79.71	30.62	3.92
Incomplete market case 3 s_0	232.69	79.71	30.62	3.92
Combination market s_0	234.00	89.75	39.42	3.94

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter γ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible, the incomplete market (case 1: no trade in S at $t = 1/3$ and $t = 2/3$, case 2: no trade in S at $t = 1/3$, case 3: no trade in S at $t = 2/3$) and the combination market (uncertainty trade in S at both $t = 1/3$ and $t = 2/3$), parameter values of the model: $r=0.0099$; $u=0.1334$; $d=-0.0856$ and $p = 0.5$.

It is difficult to compare these results with the two-period case. As can be seen, the optimal terminal wealth of the investor in the complete market is not precisely the same as in the two-period case, which results from the fact we have to adjust u and d if we go from a two-period economy to a three-period economy to keep the same values for μ and σ . We now extend the horizon T with half a year and keep u , d and r precisely the same as in the two-period case. The results for the optimal asset allocation in this case are tabulated in Table 5.

Table 5: Optimal Asset Allocation

Case	b_0	s_0
Complete market case	60.58	39.42
Incomplete market case	60.75	39.25
Incomplete market case 2	60.62	39.38
Incomplete market case 3	60.62	39.38
Combination of the complete and the incomplete market case	60.63	39.37

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter γ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (case 1: no trade in S at $t = 1/3$ and $t = 2/3$, case 2: no trade in S at $t = 1/3$, case 3: no trade in S at $t = 2/3$) and the combination of the complete and the incomplete market (uncertainty trade in S at both $t = 1/3$ and $t = 2/3$), parameter values of the model: $r = 0.0149$; $u = 0.1720$; $d = -0.0998$ and $p = 0.5$.

If we compare Table 1 with Table 5, we see the allocation in the complete market is precisely the same, for the obvious reason the parameters of the model remain the same. Interestingly, the optimal asset allocation in the last two cases of the incomplete case, are both identical to the optimal asset allocation of the incomplete case in the two-period economy. The optimal asset allocation in case the investor cannot trade for two periods in sequence is again slightly higher than the cases where the investor is only not able to trade once. In Table 6 we see that for smaller values of γ , this result is even more visible. This again confirms the statement the risk-averse investor reduces his exposure to the stock when he is confronted with more periods of no-trading in sequence, whereas the point in time the no-trading period arises does not affect his optimal asset allocation.

Table 6: Optimal Asset Allocation

Default				
Case	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 30$
Complete market b_0	-138.71	20.64	60.58	96.08
Incomplete market case 1 b_0	-119.17	21.43	60.65	96.16
Incomplete market case 2 b_0	-129.25	20.72	60.62	96.11
Incomplete market case 3 b_0	-129.25	20.72	60.62	96.11
Combination market b_0	-131.04	20.80	60.61	96.11
Complete market s_0	238.71	79.36	39.42	3.92
Incomplete market case 1 s_0	219.17	78.57	39.35	3.84
Incomplete market case 2 s_0	229.25	79.28	39.38	3.89
Incomplete market case 3 s_0	229.25	79.28	39.38	3.89
Combination market s_0	231.04	79.80	39.39	3.89

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter γ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (case 1: no trade in S at $t = 1/3$ and $t = 2/3$, case 2: no trade in S at $t = 1/3$, case 3: no trade in S at $t = 2/3$) and the combination market (uncertainty trade in S at both $t = 1/3$ and $t = 2/3$), parameter values of the model: $r = 0.0149$; $u = 0.1720$; $d = -0.0998$ and $p = 0.5$.

Of course, the number of time-periods in this economy is still relatively small, but the results lead to the implication the described effect will become more and more economically significant if we extend the model with additional time-steps. In Section 7.3.2 we have explained the problem becomes computationally too large if we increase n in this model. More research is required to solve the problem for large n . Therefore, we do not proof the above statement, but we include the following conjecture.

Conjecture 7.1. Suppose we have a n -period economy, then the number of periods in sequence the investor is not able to trade *reduces* his optimal allocation to the illiquid asset, however the point in time the no-trading period arises does *not* affect his optimal asset allocation.

Moreover, we can observe from Table 3, 4, 5 and 6 that the reduction in the exposure to the stock if the investor faces illiquidity risk for two periods in sequence is in proportion larger in the case where $T = 1.5$ than the case where $T = 1$. This result might imply the investor reduces his allocation to the illiquid asset if the horizon, together with the number of trading opportunities, increases. This result is especially interesting for long-term institutional investors like pension funds, as they are particularly interested in assets that have a long duration to match their liabilities. Therefore, we will discuss the implications on this result for the institutional investors in Section 8.

8 Conclusions and recommendations

In this thesis we distinguish three different cases to identify the effect of illiquidity on the optimal asset allocation of an investor. We derive the optimal asset allocation in the three different cases in a financial market with two available assets: the bond and the stock. The bond is assumed to be always liquid, whereas the stock might be illiquid. In all the three cases we have used the martingale method to solve the asset allocation problem. It turns out the martingale method was relatively easy to apply in the complete market case, where we assume trade in the stock is always possible. More technical details were needed in the incomplete market, where we assume trade in the stock is not always possible. The essential step in order to apply the martingale method in the incomplete market case is to add fictitious assets and solve the problem with these assets in such a way no actual allocation is taken in these fictitious assets. In order to solve the combination of the complete and the incomplete market case, where we assume the investor is *uncertain* whether or not a trading opportunity arises in the stock, we stated a theorem to combine the complete market and the incomplete market case. The last case brings us exactly to the definition of illiquidity used in this thesis: the investor is uncertain whether or not a trading opportunity in the stock arises, which means the period of no-trading is of uncertain duration.

We elaborate the three different cases for a two-period economy and a three-period economy respectively, and we find interesting and useful results in the optimal asset allocation of the investor when he is confronted with illiquidity risk. We observe the investor allocates less to the stock if the stock becomes illiquid. This behaviour is even more visible for the less-risk averse investor, since this investor has a higher increased risk if he is confronted with illiquidity than the more risk-averse investor. Moreover, we observe that the number of no-trading periods in the stock reduces the exposure to the illiquid stock and implies a higher allocation to the liquid bond. It turns out the reduction to the allocation of the illiquid stock is even larger if the horizon T , together with the number of time steps, increases. Finally, we observe the investor does not necessarily care about *when* the no-trading period arises. This last observation is difficult to clarify intuitively. As the result is not immediately trivial, more research is required to get a better idea of the rationale behind this finding. Investigating the conjecture described in Section 7.5 might lead to better insight in which way time has an influence on the behaviour of the investor when he is faced with illiquidity.

Hence, the model described in this thesis identifies some interesting and useful effects on the behaviour of the investor when he is confronted with illiquidity risk. However, the financial market described in this thesis is relatively simple. Extending the model with a more realistic financial market is one possibility to improve the results of the model. Making the financial market more realistic implies for instance including more assets and allow the financial

parameters of the model to be time-dependent. However, modeling in a discrete time-framework becomes difficult if more assets are added and time-dependent parameters are included. Therefore, to realize these extensions, it might be more convenient to analyze the model in a continuous time-framework. Another reason to move to a continuous time-framework is implicitly mentioned in Section 7.2, as the problem becomes computationally too large if we take a sufficient number of time-steps n . Modeling in a continuous time-framework is easier, more efficient and it might give rise to new, creative solving techniques to solve the problem when n becomes large.

Besides the simplicity of the described financial market, we also made a strong assumption in Section 5.3, namely the independency of the indicator function $I_{1/2}$ and M_T . We are actually neglecting liquidity premiums in this model by assuming that these two random variables are independent. If we price respectively the liquid stock and the illiquid stock, which both have the same parameter values, and we assume M_T and $I_{1/2}$ are independent, then this implies the risk-neutral expectation of the payoff of the liquid stock is exactly the same as the risk-neutral expectation of the illiquid stock. The expectation under the actual probability measure is also the same for both stocks. Therefore, the liquidity risk in this model is not priced. However, this is not a realistic assumption since the investor is willing to pay a lower price for the illiquid stock than for the liquid one in order to make him indifferent about buying the liquid or the illiquid stock. The strong assumption however fits in the approach of this thesis, as we are only focusing on the *time* restriction of liquidity and assume that no liquidity premium exists. Of course, if the model will be extended, investigation on the dependency between these two random variables is highly recommended.

Moreover, this thesis only investigates the behaviour of the individual investor when he is confronted with the risk of illiquidity. As mentioned in Section 7.5, we observe the single investor allocates less to the illiquid stock when the horizon T increases. This might be counter-intuitive if we look at institutional investors. Pension funds for instance are usually interested in illiquid assets, as their liabilities are also illiquid. Since pension funds try to hedge their liabilities, it might be worthwhile for the pension funds to invest in illiquid assets. In order to investigate the behaviour of institutional investors when they are faced with illiquidity, we have to include not only the terminal wealth of the investor in the utility function, but also the liability side. In the literature different ways to include both assets and liabilities of a pension fund in the utility function are discussed. The effect of illiquidity on the optimal asset allocation when using these utility functions might be different than for the individual investor who is only interested in optimizing his final wealth, as institutional investors also have hedging incentives as a result of their liabilities.

The above two paragraphs have made clear the model can be extended and improved in several ways. Besides this, translating the model to a more

empirical approach might also lead to new insights. In this thesis we have used the time restriction to model illiquidity. From an empirical point of view, it would be interesting to combine these results with what is actually observed in the market. We could for instance calculate the size of the illiquidity premium the investor requires to make him indifferent about the liquid stock with no illiquidity premium and the illiquid stock with the illiquidity premium. We can compare this result with the illiquidity premiums observed in real market data and identify the similarities and differences for what is found by theorem and what is observed empirically.

In summary, we have touched upon a lot to improve, extend and further analyze the model. As mentioned in the introduction, the share of illiquid assets in the portfolio of institutional investors is relatively high nowadays and might even further increase in the future. Institutional investors are often struggling with the complexity of illiquid assets and aim for better models to assess the risks involved with investments in illiquid assets. On the other hand, supervisors of institutional investors face the challenge to take into account the risk character of illiquid assets in standard models for determining the solvency position in a conforming way. Therefore, both the institutional investors and the supervisor can take huge benefit from further research on this topic.

A Appendix

Table 7: Optimal Asset Allocation

Increase Sharp ratio: $\frac{\mu-r}{\sigma}$ by increasing u				
Case	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 30$
Complete market b_0	-428.72	-70.95	18.16	92.24
Incomplete market b_0	-300.43	-67.73	18.31	92.31
Combination market b_0	-377.40	-69.54	18.25	92.27
Complete market s_0	528.72	170.95	81.84	7.75
Incomplete market s_0	400.43	167.73	81.69	7.69
Combination market s_0	477.40	169.54	81.75	7.73

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter γ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (no trade in S at $t = 1/2$) and the combination market (uncertainty trade in S at $t = 1/2$), parameter values financial market: $r = 0.0149$; $u = 0.3$; $d = -0.0998$ and $p = 0.5$.

Table 8: Optimal Asset Allocation

Decrease Sharp ratio: $\frac{\mu-r}{\sigma}$ by decreasing d				
Case	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 30$
Complete market b_0	-6.41	64.56	82.30	98.23
Incomplete market b_0	-6.33	64.63	82.40	98.25
Combination market b_0	-6.38	64.59	82.30	98.24
Complete market s_0	106.41	35.44	17.70	1.77
Incomplete market s_0	106.33	35.37	17.60	1.75
Combination market s_0	106.38	35.41	17.70	1.76

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter γ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (no trade in S at $t = 1/2$) and the combination market (uncertainty trade in S at $t = 1/2$), parameter values financial market: $r = 0.0149$; $u = 0.1720$; $d = -0.12$ and $p = 0.5$.

Table 9: Optimal Asset Allocation

Increase Sharp ratio: $\frac{\mu-r}{\sigma}$ by increasing p				
Case	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 30$
Complete market b_0	-409.66	-82.53	9.13	91.02
Incomplete market b_0	-300.93	-79.40	9.08	91.08
Combination market b_0	-366.17	-81.28	9.11	91.04
Complete market s_0	509.65	182.53	90.87	8.98
Incomplete market s_0	400.93	179.40	90.92	8.92
Combination market s_0	466.17	181.28	90.89	8.96

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter γ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (no trade in S at $t = 1/2$) and the combination market (uncertainty trade in S at $t = 1/2$), parameter values financial market: $r = 0.0149$; $u = 0.1720$; $d = -0.0998$ and $p = 0.6$.

Table 10: Optimal Asset Allocation

Decrease Sharp ratio: $\frac{\mu-r}{\sigma}$ by decreasing p				
Case	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 3$	$\gamma = 30$
Complete market b_0	13.80	71.52	85.80	98.58
Incomplete market b_0	13.87	71.59	85.87	98.59
Combination market b_0	13.84	71.55	85.83	98.58
Complete market s_0	86.20	28.48	14.20	1.42
Incomplete market s_0	86.13	28.41	14.13	1.41
Combination market s_0	86.16	28.45	14.17	1.42

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter γ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (no trade in S at $t = 1/2$) and the combination market (uncertainty trade in S at $t = 1/2$), parameter values financial market: $r = 0.0149$; $u = 0.1720$; $d = -0.0998$ and $p = 0.45$.

Table 11: Optimal Asset Allocation

Horizon T				
Case	$T = 2$	$T = 3$	$T = 5$	$T = 10$
Complete market b_0	58.69	56.81	53.07	43.54
Incomplete market b_0	58.73	56.85	53.05	43.45
Combination market b_0	58.71	56.83	53.06	43.50
Complete market s_0	41.31	43.19	46.92	56.46
Incomplete market s_0	41.27	43.15	46.95	56.55
Combination market s_0	41.29	43.17	46.94	56.50

Optimal Asset Allocation (b_0 allocation to bond B and s_0 allocation to stock S) for a single investor with risk-aversion parameter $\gamma = 3$ and initial wealth $W_0 = 100$ in the complete market (trade in S always possible), the incomplete market (no trade in S at $t = 1/2$) and the combination market (uncertainty trade in S at $t = 1/2$) for different horizons, parameter values financial market:

$T = 2$: $r = 0.0296$, $u = 0.2553$, $d = -0.1158$ and $p = 0.5$

$T = 3$: $r = 0.0440$, $u = 0.3218$, $d = -0.1186$ and $p = 0.5$

$T = 5$: $r = 0.0724$, $u = 0.4296$, $d = -0.1081$ and $p = 0.5$

$T = 10$: $r = 0.1402$, $u = 0.6314$, $d = -0.0500$, and $p = 0.5$

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