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Collar Design for Age-Dependent Indexation Policy

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Collar Design for Age-Dependent Indexation Policy

by
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Abstract

The traditional Dutch second pillar pension scheme is challenged by the aging of pension funds and by the recommendation of life cycle investment theory. An age-dependent indexation (ADI) policy which enables risk differentiation to age and is in line with life cycle theory has been proposed some time ago. However, this policy may be perceived as too risky, especially for the young. This paper adds a collar option for an ADI plan. The purpose of adding an indexation collar is to avoid negative indexation scenarios and to reduce the indexation volatility of an ADI plan. Collar options are structured in such a way that their initial value is zero so that they do not require upfront costs to be implemented. We make use of an asset-liability management study to evaluate a collar contingent ADI plan by comparing its indexation distribution, funding ratio distribution and certainty equivalent benefit with an ADI plan without collar and with a regular plan having a conditional indexation policy. Three conclusions are drawn from our ALM studies. First, a collar strategy successfully cuts down the indexation risk of an ADI plan. Second, in the long run, both ADI plan and collar contingent ADI plans have a lower funding ratio volatility than a traditional plan with a policy ladder. Third, an ADI plan with collar is more beneficial to the participants than an ADI plan without collar and a traditional plan with a policy ladder.

Keywords: Age-dependent Indexation policy, collar structure, option pricing, Black-Scholes, VAR model, ALM, age-dependent liability valuation, CRRA.

Contents

Chapter 1

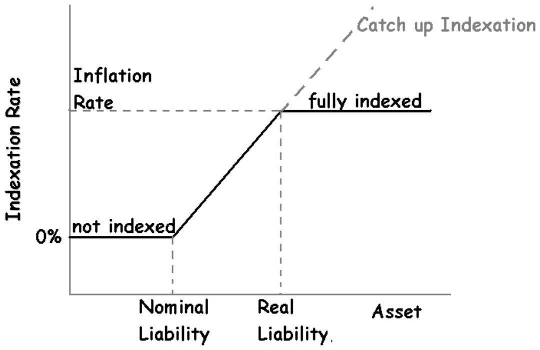
Introduction

1.1 Literature Review

At the beginning of this century, pension funds in the Netherlands were hit by a solvency crisis which was caused by a dramatic drop in the value of their assets and a jump in the value of their liabilities. This solvency crisis triggered a systematic revolution which switched the final-salary plan with unconditional indexation to an average-salary plan with a solvency-contingent indexation policy (policy ladder: see Box below). This new policy can improve the solvency control and maintain the collective risk sharing feature which is strongly supported by the participants, the sponsors and the labor unions, for it can enhance welfare and achieve cost efficiency. Cui, de Jong and Ponds (2006) show that intergenerational risk sharing is desirable within funded pension schemes.

Contingent claims have been commonly used by pension funds to manage funding risk. Sharp (1976) was the first to identify contingent claims in DB pension plans. The conditional indexation policy (policy ladder) used by the Dutch second pillar is a member of the contingent claim family. There exists an abundant literature on contingent claim design. Blake (1998) discusses pension schemes using a set of embedded option on the assets held in the pension fund. Lachance, Mitchell and Smetters (2003) investigate the buy-back option which gives DC participants a chance to switch back to a DB plan when retiring. Mens, Oerlemans and de Jong (2008) design a “zero cost collar” which provides upside and downside protection for a DC scheme. Dai (2006) designs a collar-structured pension benefit for DC schemes. The purpose of using options in the unconditional pension schemes is to guarantee a minimum level of accrual pension rights. However, many papers discover some drawbacks of an contingent claims. Ponds and Riel (2008) state that the switch from unconditional indexation policy to a ladder policy may enlarge the mismatch risk and result in individual welfare loss.

The figure below sketches the indexation ladder. When the value of the assets is larger than or equal to the real value of the pension liabilities, $A \geq L^R$, the pension fund is called fully funded. In this situation, the indexation rate equals to wage (or price) inflation rate, or in other words, the value of the accrued rights is fully indexed. When asset value drops between the real and the nominal liabilities $L^N < A < L^R$, the indexation rate is a fraction of wage inflation rate. This fraction is given by $\frac{A-L^N}{L^R-L^N}$, where $A - L^N$ is the actual indexation reserve position, and $L^R - L^N$ is the required indexation promise to the participants. When the asset value is smaller than the nominal value of the liabilities, $A \leq L^N$, no indexation would be given.



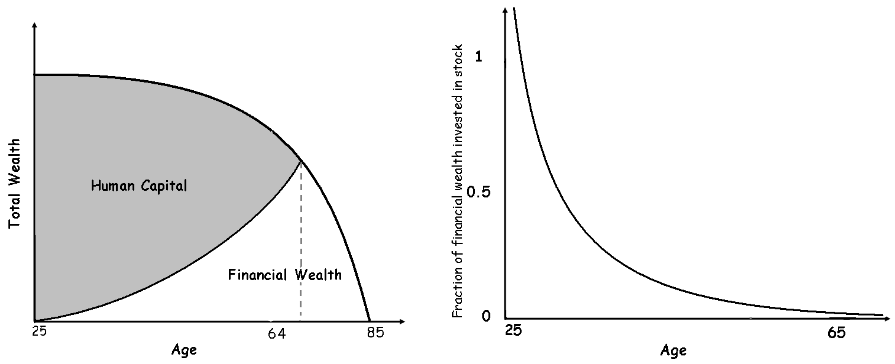
It is important to evaluate the various pension schemes. Cui, De Jong and Ponds (2005) provide a first analysis of international transfer of risks and wealth by using value-based generational accounting. De Jong (2006) evaluates pension liabilities under conditional indexation. Dai and Schumacher (2009) study the conditional indexation policy through utility functions. Asset-liability management (ALM) is a useful tool to assess the impact of policy switching, however, it is hard to impose a preference ranking to different variants based on statistical distributions. Kortleve and Ponds (2006) and Hoevenaars and Ponds (2008) design an economically fair measure named value-based ALM, which enables us to detect the welfare transfer resulting from policy switching.

The inspiration to consider age differentiation comes from life cycle investment theory. Campbell and Viceira (2002), Jagannathan and Kocherlakota (1996) all agree that people should be fully exposed to the equity market at their early career years and cut down the risk in hand when aged, such that they can optimize their saving, investment and consumption utilities over the life cycle (see Box below). Ponds (2008) brings the idea of age differentiation policy in his speech “Naar Meer Jong En Oud In Collectieve Pensioenen”(in Dutch). Policy advisors (Ponds and Riel (2007)) foresee that age-dependent risk policy might become a desirable choice for the maturing funds. The challenging question is whether it is possible to combine the implication of life-cycle theory with the proven beneficial collective pension funds. Molenaar, Munster and Ponds (2008) propose two concrete age differentiation strategies. One is called “Age-Dependent Indexation Policy”, the other is called “First DC, then DB policy”. Both policies differentiate risk exposure regarding age and maintain the advantage of collectivity and intergenerational risk sharing.

The optimal life cycle investment strategy structures the risky asset allocation over life time. The optimal investment strategy can be formulated as

$$\alpha_x = \frac{\mu - r}{\gamma\sigma^2}\frac{H_x + F_x}{F_x}$$

where
α_x=fraction of financial capital in risky assets at age x
H=human capital
F=financial capital
r=risk free rate
σ=volatility of risky assets
μ=drift of risky asset
γ=risk aversion



(a) Trajectories for human capital (H) and financial wealth (F) over the life-cycle

(b) Expectation of optimal stock weight in financial wealth (F) over the life-cycle

Total wealth consists of two parts, human capital and financial wealth. Panel (a) displays the trajectories of the two over life time. The grey area of Panel (a) represents human capital, which is decreasing over time and has none left after retirement. Financial wealth is increasing during the career time and reaches its peak at the retirement year. During the retiring period the financial wealth is diminishing till zero. Panel (b) plots the optimal investment strategy over the life-cycle. The optimal life-cycle investment strategy recommends a high risk exposure to the asset market during the early period of the life cycle, and a conservative portfolio when getting old.

1.2 Motivation

Dutch pension funds are challenged by the aging problem. In the Netherlands, pension funds have followed a uniform pension plan principle for a long time, with a single asset mix, a uniform contribution rate, and a uniform indexation rate on heterogeneous age cohorts. The increasing relative number of retirees pushes pension policies to old-member oriented settings. The asset portfolio managers have to choose a more conservative investment strategy so as to ensure the growing promised pension liabilities for the retirees. However, a conservative asset mix conflicts with the interests of young members since the low expected asset return may enlarge the contribution needed to fund their future liability, and also, a conservative mix violates the recommendation of an optimal individual life cycle investment strategy.

In recent years, some policy advisors become interested in combining the insights of life cycle theory with the guaranteed benefit of collective funding and intergenerational risk sharing. It can be shown that an age-dependent indexation policy may uphold both characteristics. The age-dependent indexation (ADI) of an x-year old participant is given by

$$ADI = \begin{cases} \frac{65-x}{40}\text{Realized Return} + \frac{x-25}{40}\text{Wage Growth Rate} & x\in[25,64] \\ \\ \text{Wage Growth Rate} & x\in[65,85] \end{cases}$$

The ADI function has two subfunctions. The upper one is for active members, and the other one is for retirees. The ADI function for active members consists of two components: one is related to the [1]realized return of asset mix, the other is related to the wage growth rate. The ADI plan captures the idea of an optimal life cycle investment strategy which is explained in the box above, for the risk exposure to the asset market is decreasing with age. Realized Return is defined as the excess return on the asset portfolio over the discount rate of the real liabilities, where the discount rate of the real liabilities is defined as the nominal interest rate minus wage (or price) growth rate. The ADI plan violates the traditional uniform pension plan principle, but is appropriate for maturing pension funds.

Figure 1.1[2] simulates three simple trajectories of the age-dependent indexation function. From steep to flat, the three straight lines represent the indexation rate of 25-year old participants, 45-year old participants, and retirees. The three straight lines all intersect point [0.02, 0.02], which means

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[1] Realized return is defined as asset return minus real liability return

[2] Suppose wage growth rate is constant $\pi = 0.02$, Realized Return$\in[-0.5, 0.5]$. The simulation consists of 100 steps. Figure 1.1 displays the simulation results.

that when the realized return is the same as the inflation rate the indexation risk is indifferent to age. It is also obtained from Figure 1.1 that the 25-year old get negative indexation once the realized return is negative, while the 45-year-old are negatively indexed only when the realized return is below -0.02. From these observations, we conclude the following feature of the ADI plan: the indexation rate of young members is more sensitive to a change of the realized return than the indexation rate of old members. Though for all active members, the indexation rate is in general, positively related to the realized return, when the realized return is above the inflation rate, the young benefit more than [3]the old, but when the realized return is below the inflation rate, young members lose more than the old.

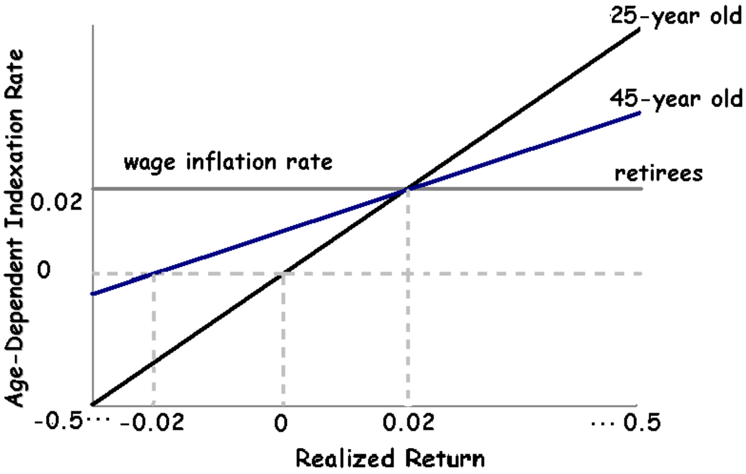

Figure 1.1: Simple Trajectories of ADI Plan

The crucial drawback of the ADI plan is the existence of negative indexation, especially during the early career years. In other words, the risk volatility may be perceived to be too high for young participants. Molenaar, Munsters and Ponds (2008) notice this problem. In chapter 3, we also confirm that members, especially the young ones, have a very large chance to get negative indexation. In recent years, pension schemes with a small amount of negative indexation have been allowed by the pension law, but if the risk volatility is too big, then such a pension variant might not be desirable. Where does the problem come from? The problem is born from the design of the ADI function for active members. The ADI function is the sum of two weighted returns. The weight of each return is an age dependent fraction. The fraction $\frac{65-x}{40}$ can be considered as the level of the risk exposure to the asset market. For instance, a 27-year-old member has to take

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[3] "The old" are still active members of the pension fund, because ADI function is mainly focusing more active pension fund contributors.

95%$(=\frac{65-27}{40})$ of the asset portfolio risk, thus the volatility of the indexation rate at 27 is very high. As he is getting older, the effect of the "Realized Return" on the indexation rate is alleviated, since $\frac{65-x}{40}$ is decreasing with age. However, the fraction $\frac{x-25}{40}$ measures the weight of the stable return, since this fraction is increasing with age, older members can thus have highly guaranteed indexation rate.

The ADI plan realizes risk differentiation towards age, maintains the advantage of intergenerational risk sharing, reduces [4]mismatch risk and is in line with the recommendation of optimal life cycle investment theory. However, as a pension variant, the ADI plan might be too risky for young members. Therefore, we need a strategy which can improve the ADI plan by reducing its risk volatility while keeping age differentiation.

1.3 Research Question

The central research question of this paper is how to get rid of negative indexation scenarios and to reduce the risk volatility of the ADI plan while keep the insights of age-differentiation. We redesign the ADI plan by embedding a contingent claim named " indexation collar".

Figure 1.2 sketches the collar structure. The collar consists of a floor and a cap. The floor is acting as a long European put option which protects from negative indexation and the cap is acting as a short European call option which cuts off the out-performing scenarios. As we aim to avoid negative indexation, the indexation floor i_{floor} is set to zero. The cap value is unknown and the determination of the indexation cap is based on the principle that the value of the collar should be zero. This will be explained in chapter 2.

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[4] Mismatch risk is defined as the degree of mismatch between asset mix and liabilities. Mismatch risk is measured by the standard deviation of funding ratio growth rate.

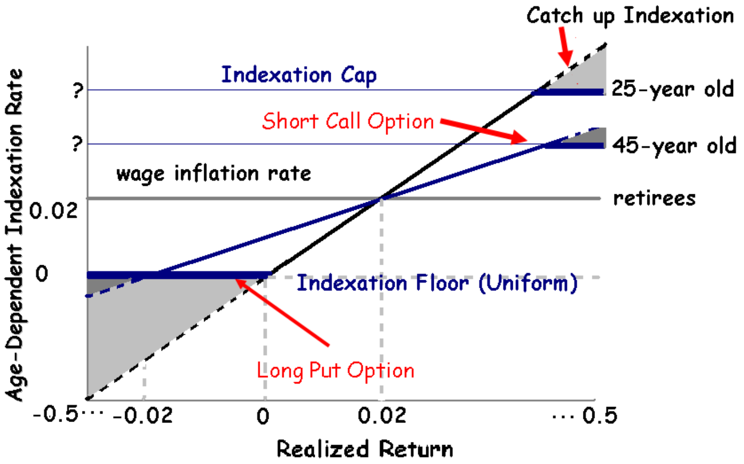

Figure 1.2: The idea of the collar contingent ADI plan

In this paper, we first price the indexation collar, then we assess the attractiveness of the collar contingent ADI plan through ALM studies. The structure of the paper in as follows: in Chapter 2, we build a single-period model in the Black-Scholes (1973) world to price the indexation collar. Two types of collar are designed, one is called "individual collar", the other is called "uniform collar". Chapter 3 evaluates collar contingent ADI plans under single-period model settings. Chapter 4 and 5 extend the study into multi periods. The indexation collars for the next 10 years are calculated in Chapter 4. Chapter 5 uses ALM study to evaluate different policy variants. Chapter 6 concludes some important findings in both single and multi-period models and also discusses some remaining challenging questions for the future research.

1.4 Policy Recommendations

Collar contingent ADI plans are attractive because of four reasons: first, they conquer the disadvantage of the ADI plan by means of avoiding negative indexation and reducing risk volatility. Second, they maintain age-differentiation which is desirable for maturing funds. Third, they are in the long run more beneficial to the participants than policy ladder. Fourth, the funding ratio volatility under collar contingent ADI plans is in the long run less volatile than under a policy ladder or under an un-collared ADI plan. Because of these advantages, we believe that the collar contingent ADI plan might become a very competitive variant in the coming future.

Chapter 2

Single Period Modeling

A policy ladder gives the indexation rate as a function of the funding ratio. While in an age-dependent indexation policy, the indexation rate is a function of the realized return and the realized inflation. In this chapter, we model the collar structure using simple settings. The model consists of one period, from the current period t to the future period $t+1$. The situation at time t is assumed to be known, while is uncertain at $t+1$. These simple settings might not well predict the real world, but enable us to work out the collar structure explicitly.

2.1 Standard Financial Settings

This section aims to explore the asset return dynamics in the Black-Scholes world with the assumption of self-financing and a fixed portfolio choice. The Black Scholes pricing model only containing one stochastic variable is not an ideal description of the real world. More often, the Ornstein-Uhlenbeck process is used to capture the dynamics of price inflation and the risk free rate, see e.g. Brennan and Xia (2002) and de Jong (2005). In a single-period model, however, we want to price the collar analytically, so the fewer the number of unknown parameters, the better. Broeders (2006) also uses the Black-Scholes framework to price conditional liabilities.

The financial dynamics following the Black-Scholes generic state-space model under the real world $\mathbf{P}$-measure is given by

$$\begin{aligned} dS &= \mu S dt + \sigma S dW_t \qquad (2.1) \\ dB &= r_n B dt \end{aligned}$$

The model consists of two state variables, the stock price S_t following a geometric Brownian motion with drift μ and volatility σ, and the risk free

Treasury Bill price B_t with fixed nominal return r_n.

When working with the standard Black-Scholes model, it is necessary to be aware of some important underlying [1]assumptions, such as a constant volatility over time, an efficient and perfectly liquid market, no dividends payoff, a constant nominal interest rate, and log-normally distributed returns.

Equation (2.1) is solvable and has the following explicit solution:

$$\begin{aligned} S_t &= S_0 \exp[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t] \\ B_t &= B_0 \exp(r_n t) \end{aligned}$$

The standard Black-Scholes model driven by Brownain motion is based on the notions of “market completeness” and “absence of arbitrage”. Both [2]equivalent martingale measure and pricing kernel measure are able to price the financial derivatives. Here, we pick the equivalent martingale measure to price the collar. Given the risk free bond B as a *numéraire*, the stock price evolution under the **Q**-measure becomes:

$$\begin{aligned} dS &= \mu S dt + \sigma S(d\widetilde{W}_t - \frac{\mu - r_n}{\sigma} dt) \\ &= r_n S dt + \sigma S d\widetilde{W}_t \end{aligned} \tag{2.2}$$

The measure transition is explained in Appendix A.1. The assets A are invested in the stock market and in the bond market only with a self-financial and constant portfolio α. The dynamics of the assets value can be captured by the following stochastic differential equation under the **Q**-measure

$$dA = r_n A dt + \sigma \alpha A d\widetilde{W}_t$$

and the explicit solution is

$$A_t = A_0 \exp[(r_n - \frac{1}{2}\alpha^2\sigma^2)t + \alpha\sigma\widetilde{W}_t] \tag{2.3}$$

Let $R_A = \frac{A_{t+1}}{A_t} - 1$ be the net return of the asset portfolio, then the log asset return r_A at time t+1 is defined as

$$r_{A,t+1} = \ln(1 + R_{A,t+1}) = \ln \frac{A_{t+1}}{A_t}$$

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[1] Schumacher (2008) The lecture note for the course “Financial Models” chapter 1,2.

[2] Equivalent martingale measure states that an asset pricing process C relative to the *numéraire* must be martingale under $\mathbf{Q}_N$, where N is the chosen *numéraire* and $\mathbf{Q}_N$ is the unique equivalent martingale measure corresponding to N.

then

$$r_{A,t+1} = r_n - \frac{1}{2}\alpha^2\sigma^2 + \alpha\sigma z \tag{2.4}$$

where $z \sim N(0,1)$.

2.2 Pension Settings

In this section, we first introduce some basic concepts such as pension plan, contribution, liabilities, investment strategy and funding ratio. Then we explain the three policy variants under investigation, namely the "policy ladder", the "ADI plan" and the "collar contingent ADI plan".

Consider a pension fund requiring mandatory participation. Each participant starts building up his or her pension savings at age 25 and retires at age 65. The survival rate before 85 is 1, and after 85 it is zero. Hence, there are 60 age cohorts under investigation. The population size of each age cohort is assumed identical. All active members earn the same annual income which remains constant over time and whose magnitude will be denoted by I.

2.2.1 Pension Fund Characteristics

Funding risks have to be borne by the current and the future participants. There are four ways to allocate the funding risk: doing nothing so that they are passing down to the future generations, adjusting the indexation rate, adjusting the contribution rate and adjusting the asset mix. Hoevenaar and Ponds (2006) state that a change of the pension characteristics may alter the funding risk allocation. In the present section we will describe the features of the fund defined above.

Pension Plan The pension plan follows an average-salary scheme with indexed liabilities. The [3]replacement rate is 80%. The pension rights are accrued every career year till retirement with an accrual rate of 2%. The pensionable income Y is measured by the total income I minus the franchise. Hence, the newly accrued pension rights in the career year t are

$$\triangle B_t = 0.02(I_t - F)$$

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[3] The replacement rate is the product of the accrual rate 2% and the number of years spent working (=40), meaning that the participants can receive the a default annuity with a payment equalling 80% of their average income. Remark: the retirement income includes the AOW

where, $\triangle B_t$ represents the newly accrued pension rights at time t, 0.02 is the accrual rate and I is the annual income. F is called the franchise which is the product of old age provision AOW_t and $\frac{10}{7}$ (i.e., $F = \frac{10}{7}AOW_t$), and $I_t - F$ is called pensionable income. In short, the newly accrued pension rights at time t equal to the accrual rate times the pensionable income.

Assume that AOW and the annual income I are both constant over time, then $\triangle B_t = \triangle B$ is also constant over time. Haneveld, Streutker and Vlerk (2007) describe the accrual pension rights calculation. Figure 2.1 gives an example illustrating the calculation of the accrual rights at time t for a 29-year old member with four-year working experience. B_t is the sum of all his newly accrued rights during his career so far. $i_{t,t-2}$ is the indexation rate at year $t-2$.

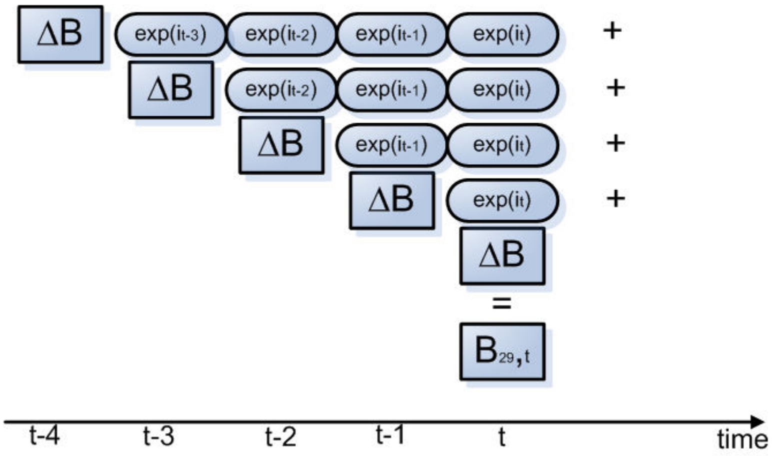

Figure 2.1: Accrued Pension Rights Calculation

Therefore, the accrual rights for any active members ($x\in[25,64]$) can be calculated by

$$B_{x,t} = \triangle B_x + \sum\nolimits_{s=25}^{x-1}\prod\nolimits_{u=t-(x-s)+1}^{t} \exp(i_u)\triangle B_s \cdot 1_{x\in[26,64]} \tag{2.5}$$

where i_t is the realized indexation rate at time t, driven by the indexation policy in use at that moment. If nominal protection is taken place, then $i_t = 0$; if real indexation is required, then $i_t = \pi_t$.

Nominal accrual pension rights with $i_t = 0$ are calculated by

$$B_{x,t}^N = \sum\nolimits_{s=25}^{x}\triangle B \tag{2.6}$$

If the indexation rate i_t always equals to some constant [4] the inflation rate π, then the fully indexed accrual rights, which are also named real

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[4] In this paper, we do not distinct price and wage inflation rate.

pension rights, can be calculated by

$$B^R_{x,t} = \triangle B \sum\nolimits_{s=25}^{x} \exp[(x-s)\pi] \tag{2.7}$$

As the inflation rate is assumed constant over time, the nominal and the real accrual pension rights only depend on [5] x and not on t, so that we obtain

$$\begin{aligned} B^N_{x,t+1} &= B^N_{x,t} \\ B^R_{x,t+1} &= B^R_{x,t} \end{aligned}$$

Contribution Rate The participants pay a fixed fraction of their pensionable income to build up their pension savings during their career years. This fraction is called the contribution rate, which in this paper is assumed to be actuarially fair. The actuarially fair contribution principle requires that the present value of contribution must be equal the present value of the total pension benefit. Therefore, the contribution level is strongly related to the pension ambition. The more benefits we want as our retirement income, the more contribution we have to pay. Solving the following equation for P, we can find the actuarially fair value of the contribution.

$$\sum\nolimits_{x=25}^{64} P \exp[-r_r(x-25)] = \sum\nolimits_{x=65}^{84} B \exp[-r_r(x-25)]$$

r_r is the real risk free rate which we assume to be fixed. B is the retirement annual income with $B = I \cdot 80\% - \text{AOW}$. P is the actuarially fair contribution value with $P = I \cdot c$ (c is the actuarially fair contribution rate). Both B and P are constant over time. If the expected real risk free rate is $r_r = 2\%$, then $c = 21.5\%$; if $r_r = 2.5\%$, the contribution rate is 18.32%.

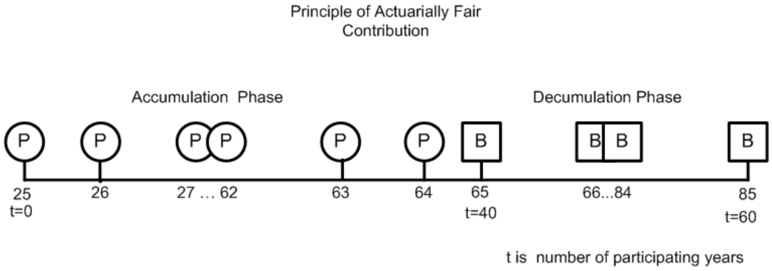

Figure 2.2: Principle of Actuarially Fair Contribution

Figure 2.2 illustrates the idea of actuarially fair contribution with respect to the risk free liabilities. The left side of Figure 2.2 (from 25 to 65) is

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[5] $x \in [25, 84]$ always refers to the age of pension participants. For example, at the current year t, a 24-year old person doesn't belong to the set of pension participants, while in the next period $t+1$, when he becomes 25 years old, this person will be a member of the pension fund.

called accumulation phase, where the participants contribute a certain fraction c of their average income I to their pension fund, and this amount of money is called the contribution P. The total contribution is accumulating till retirement. The right hand side of Figure 2.2 is called the decumulation phase, where retirees receive their annual pension benefits B till they die. When the pension funds are no longer risk free, for instance when they invest in the stock market, the participants will be confronted with funding risk. In this paper, we assume that contribution rate is fixed at the risk free actuarially fair level.

[6]The tables below summarize the model introduced so far,

Variable	**Interpretation**	**Value**
π	Inflation rate	2%
α	Asset Portfolio	50%
σ	Stock Return Volatility	18%
I	Flat Annual Income	200
AOW	First Pillar Pension Income	70
r_r	Real Return on Bond	2.5%
μ	Expected Stock Return	6%

By using the information above, we can derive

Variable	**Interpretation**	**Value**
Y	Pensionable Income	100
$\triangle B$	Newly Accrued Pension Right	2
r_n	Nominal Return on Bond	4.5%
B	Retirement Income (Excluding AOW)	90
B_x^N	Nominal Pension Right	$2(x-24)$
B_x^R	Real Pension Right	$2\sum_{s=25}^{x}\exp[(x-s)\cdot 2\%]$
c	Contribution Rate	18.32%

Liability The valuation of the liabilities is a crucial component of asset liability management. The nominal liabilities are valued by discounting the pension rights with the nominal risk free rate. In order to calculate the fair value of the liabilities, however, one should take into account the indexation policy. In a traditional DB plan, the liabilities are fully indexed. In recently years, the use of policy ladder has become wide spread. Under a policy ladder, indexation is conditional. In both policies, the indexation is independent of the age of the participants. ADI plan, however, violates this "uniform " tradition, for the age-dependent indexation policy requires risk

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[6] The values shown in the table consists both financial and pension factors. The former one is only used in single-period model, namely chapter 2 and 3. The latter one can be globally used in this paper.

differentiation towards age.

Let L_x be the accrual liability of x-year old members, then the gross liabilities of the corresponding pension fund are

$$L = \sum_{x=25}^{84} L_x$$

The nominal liability of an individual member is given by

$$L_{x,t}^{N} = \begin{cases} \sum_{s=65-x}^{84-x} [B_{x,t}^{N} \exp(-r_n s)] & x \in [25, 64] \\ \\ \sum_{s=x}^{84} B \exp[-r_n (s-x)] & x \in [65, 84] \end{cases}$$

and the real value of the accrued liabilities is given by

$$L_{x,t}^{R} = \begin{cases} \sum_{s=65-x}^{84-x} [B_{x,t}^{R} \exp(-r_n s)] & x \in [25, 64] \\ \\ \sum_{s=x}^{84} B \exp[-r_r (s-x)] & x \in [65, 84] \end{cases}$$

Therefore, the total nominal and real liabilities L_t^N, L_t^R are given as follows:

$$\begin{aligned} L_t^N &= \sum_{x=25}^{84} L_{x,t}^{N} \\ &= \sum_{x=25}^{64} L_{x,t}^{N} \cdot 1_{x \in [25,64]} + \sum_{x=65}^{84} L_{x,t}^{N} \cdot 1_{x \in [65,84]} \\ &= \sum_{x=25}^{64} \sum_{s=65-x}^{84-x} [B_{x,t}^{N} \exp(-r_n s)] + \sum_{x=65}^{84} \sum_{s=x}^{84} B \exp[-r_n (s-x)] \end{aligned}$$

$$\begin{aligned} L_t^R &= \sum_{x=25}^{84} L_{x,t}^{R} \\ &= \sum_{x=25}^{64} L_{x,t}^{R} \cdot 1_{x \in [25,64]} + \sum_{x=65}^{84} L_{x,t}^{R} \cdot 1_{x \in [65,84]} \\ &= \sum_{x=25}^{64} \sum_{s=65-x}^{84-x} [B_{x,t}^{R} \exp(-r_n s)] + \sum_{x=65}^{84} \sum_{s=x}^{84} B \exp[-r_r (s-x)] \end{aligned}$$

In a single period model, the indexation policy switches from a real DB scheme to an alternative one at time t. Therefore, the realized funds liabilities at time t is, $L_t = L_t^R$. The value of the realized liabilities of period $t+1$ is dependent on the indexation policy. In general, the value of the realized liabilities is calculated by

$$\begin{aligned} L_{t+1} &= \sum_{x=25}^{64} \sum_{s=65-x}^{84-x} B_{x,t+1} \exp[-r_n (s+1)] \\ &+ \sum_{x=65}^{84} \sum_{s=x}^{84} B \exp[-r_r (s-x)] \exp[-(r_n - i_{x,t+1})] \end{aligned} \tag{2.8}$$

We elaborate the indexation function of alternative policy variants, $i_{x,t+1}$, in next subsection.

Investment Strategy The research dealing with asset portfolio choice is fruitful. Markowitz (1952)'s mean-variance analysis opens the door of modern finance theory. Leibowitz, Kongelman and Bader (1994) use funding ratio return to approach the optimal asset allocation. Campbell and Viceira (2002) firstly connect the invertors' risk preference and portfolio choice from the utility perspective. Van Binsbergen and Brandt (2006) combine the previous two studies and apply dynamic programming approach to settle the optimal portfolio. Our paper, however, is not utility optimization targeted, therefore, our asset allocation strategy simply follows a [7] "1\n" rule. Our portfolio only consists of two asset classes, stock and bond. According to the "1\n" rule, the asset portfolio choice is 50-50.

The value of pension-fund assets is calculated by (see Cui, de Jong and Ponds (2008))

$$A_{t+1} = A_t \exp(r_{A,t+1}) + C_{t+1} - BP_{t+1} \tag{2.9}$$

where the value of A_t is known. C_{t+1}, the risk free newly received contribution (cash inflows) in the next period, is paid by the future active members. As contribution rate is assumed fixed, then $C_t = \sum_{x=25}^{64} c{\cdot}Y = 732.8$ is constant over time as well. BP_{t+1}, the value of new benefit payment (cash outflow) depends on the indexation policy towards the passive members. Nominal benefit payment is simply calculated by $BP_t = \sum_{x=65}^{84} B = 1800$.

$r_{A,t+1}$ is the asset return in the real world $\mathbf{P}$. In the real Black-Scholes model, the innovation of a self-financing asset portfolio with a fixed stock weight is given by

$$\begin{aligned} & dA = (\alpha\mu A + (1-\alpha) r_n A)dt + \alpha\sigma A dW_t \\ \Rightarrow \quad & A_t = A_0 \exp[(\alpha\mu + (1-\alpha) r_n - \frac{1}{2}\alpha^2\sigma^2)t + \alpha\sigma W_t] \end{aligned}$$

Therefore

$$r_{A,t+1} = \alpha\mu + (1-\alpha) r_n - \frac{1}{2}\alpha^2\sigma^2 + \alpha\sigma z \tag{2.10}$$

Funding ratio (FR) Funding ratio is the fraction of the market value of assets and the market value of liabilities. Funding ratio is commonly used to detect the solvency position of a pension fund. In Netherlands, the criteria of the financial solvency of the pension funds has been set by Financial Assessment Framework (FTK) since 2004 with the level of 105%. Pension funds with their funding ratios higher than this criteria are considered at

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[7] "1\n" rule states that people have the tendency to allocate equally among different available choices.

the healthy solvency position.

The nominal funding ratio, FR_n, is given by

$$FR_n = \frac{A}{L^N}$$

where A represents to the market value of the pension-fund assets. Similarly, the real funding ratio FR_r is given by

$$FR_r = \frac{A}{L^R}$$

Since the value of nominal liability is smaller than the real one if the inflation rate is positive, then we can deduce the following inequality: $FR_n \geq FR_r$. In this paper, we set the initial real funding ratio equal to 100%, thus at time t, the value of asset is exactly equal to the real value of the liabilities. The numerical summary of our pension-fund characteristics is given below:

Variable	**Value**
L_t^N	27349.7
L_t^R	33821
$FR_{r,t}$	100%
L_{t+1}^N	25837.4
L_{t+1}^R	32294.4
$A_t = FR_{r,t} \cdot L_t^R$	33821
$FR_{n,t} = \frac{A_t}{L_t^N}$	123.66%

2.2.2 Indexation Policies

We investigate and compare the performances of three indexation policies in this paper. Here, we shortly explain each of them.

Ladder Policy The ladder function is

$$i_{ladder} = \begin{cases} 0 & \text{for} \quad A < L^N \\ \beta\pi & \text{for} \quad L^N \leq A \leq L^R \\ \pi & \text{for} \quad A > L^R \end{cases}$$

where $\beta = \frac{A - L^N}{L^R - L^N}$. It is clear that, the value of conditional indexation rate depends on the solvency position.

ADI plan We have formulated the age-dependent indexation function in Chapter 1. In stead of verbal formulation, we will rewrite the function by using [8] financial notations. Let i_{ADI} be the age-dependent indexation rate, then the indexation function of an x-year old member is

$$i_{ADI_x} = \begin{cases} \frac{65-x}{40}(r_A - r_r) + \frac{x-25}{40}\pi & \text{for} \quad x \in [25, 64] \\ \\ \pi & \text{for} \quad x \in [65, 84] \end{cases}$$

Collar Contingent ADI plan An indexation collar can reduce the indexation risk and avoid the negative indexation scenarios of an ADI plan. Let i_{coADI} be the collar contingent age-dependent indexation rate, it is calculated by

$$i_{coADI_x} = \begin{cases} \min(\max(i_{ADI}, i_{floor}), i_{cap}) & \text{for} \quad x \in [25, 64] \\ \\ \pi & \text{for} \quad x \in [65, 84] \end{cases}$$

2.3 Indexation Collar in One Period Model

In this section, we are going to price the indexation collar. An indexation collar, by definition, has two sides, a floor and a cap. The indexation floor (i_{floor}) is a long European put option. The other side of the collar is called indexation cap (i_{cap}) which is a short European call option. For the purpose of avoiding the negative indexation, the indexation floor is assumed to be zero ($i_{floor} = 0$). The determination of the indexation cap is based on zero cost game which will be explained later. Two types of indexation caps are designed, one is called "Individual Indexation Cap", that mean each age cohort has one unique indexation cap so as achieve zero cost game for its own. The other sort of indexation cap is called "Uniform Indexation Cap". Members of all age cohorts, in this case, share one single indexation cap aiming to achieve the zero-cost goal jointly.

2.3.1 Individual Indexation Collar for Each Age Cohort

Before finding out the value of i_{cap}, let's review the essential purpose of adding a collar to the ADI plan. The young participants of the ADI plan, are highly exposed towards market risk, but it might be, however, too extreme for the young. The mission of a collar strategy is to reduce the indexation risk of the ADI plan while to keep age differentiation.

In this section, we aim to design an individual indexation collar of each active age cohort based on the principle of the "zero cost game". By definition, an individual collar means that each age cohort has one collar which

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[8] i_{ADI}, r_A and r_r are all defined as logarithmic returns, in order to facilitate calculation, we use the trick $r \approx R$, where r is log return and R is net price return. The proof of this assumption is given in the next section.

requires no extra cost. This individual-cohort based "zero cost game" simply indicates that the total value of beyond-indexation-cap benefit of each age group is the same as the value of its below-indexation-floor loss in the risk neutral **Q**-measure. The value of pension rights of the next time period before counting the newly accrued pension rights is $B_x \exp(i_{t+1})$, which is approximately equal to[9] $B_x(1+i_{t+1})$. The individual zero cost game under risk neutral measure **Q** is as follows

$$\begin{aligned} & E^Q[-1_{i_{ADI_{x,t+1}}<0}(B_{x,t}(1+i_{ADI_{x,t+1}}) - B_{x,t}(1+i_{floor}))] \qquad (2.11) \\ = \; & E^Q[1_{i_{ADI_{x,t+1}}>i_{cap}}(B_{x,t}(1+i_{ADI_{x,t+1}}) - B_{x,t}(1+i_{cap_{x,t+1}}))] \end{aligned}$$

where $1_{i_{ADI}}$ is an indicator function, i_{ADI_x} is the age-dependent indexation rate for all the x-year old members ($x \in [25, 64]$). The left side of Equation (2.11) is independent of i_{cap}. It determines the absolute loss from negative indexation. The right side of Equation (2.11) is a function of i_{cap}. It measures the given-up rewards resulting from the indexation cap constraint.

After simplification, Equation (2.11) becomes

$$E^Q[-1_{i_{ADI_{x,t+1}}<0}(i_{ADI_{x,t+1}})] = E^Q[1_{i_{ADI_{x,t+1}}>i_{cap_{x,t+1}}}(i_{ADI_{x,t+1}} - i_{cap_{x,t+1}})]$$

To find out the explicit solution of i_{cap} for each of the forty active age cohorts, we need a concrete expression of the age-dependent indexation function first. The ADI function contains two types of returns, namely the asset mix return and the return of the real liabilities. Let α be the fixed portfolio weight on risky asset S. The asset return r_A in the risk neutral world is given in Equation (2.4). The wage growth rate π is assumed constant in the single period model. Remark: this assumption is very strong, the evaluation results with the assumption of a constant inflation rate might be quite different from those with a flexible inflation rate. The purpose of assuming a constant inflation rate is to facilitate calculation. The return on the real liabilities equals to the real risk free rate r_r. r_r is approximately equal to the difference between the nominal rate and the wage (or price) [10] inflation rate.

$$r_r = r_n - \pi$$

---

[9] To be more precise, the total accrual pension rights before adding new right $\triangle B_{x+1}$ is $B_{x+1} = B_x \exp(i_{t+1})$, i is logarithmic indexation rate. Here we are using the trick $r \approx R$ when r is close to zero. r is logarithmic return and R is net price return. It is known that $\exp(r) = 1 + R$ and the exponential function $\exp(r)$ is defined as

$$\exp(r) = 1 + r + \frac{r^2}{2!} + \frac{r^3}{3!} \cdots$$

which is an infinite series. When r is close to zero, only the first two terms of the series dominates the final result, while the rest terms are close zero. So $\exp(r) \approx 1 + r = 1 + R$. For instance, suppose $r = 0.025$, then $\exp(r) = 1.025315$ and $R = 0.025315$. The difference between r and R is very small. The purpose of using this trick $r \approx R$ is to facilitate the calculations in conditional expectation function.

[10] Using the trick $\exp(r) \approx 1 + r$.

So the ADI function can be written as

$$\begin{aligned}
i_{ADI_x} &= \frac{65-x}{40}(r_{A,t}-r_r)+\frac{x-25}{40}\pi \\
&= \frac{65-x}{40}(r_n-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z-r_n+\pi)+\frac{x-25}{40}\pi \\
&= \frac{65-x}{40}(\pi-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z)+\frac{x-25}{40}\pi
\end{aligned}$$

Let $k=\frac{65-x}{40}$, then the ADI function is simply given by

$$i_{ADI_x}=\pi+k\cdot(-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z) \tag{2.12}$$

When $i_{ADI_x}<0$

$$\begin{aligned}
&\Rightarrow \quad \pi+k\cdot(-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z)<0 \\
&\Rightarrow \quad z<-\frac{\pi}{k\alpha\sigma}+\frac{1}{2}\alpha\sigma=d_1^x
\end{aligned}$$

When $i_{ADI_x}>i_{cap_x}$

$$\begin{aligned}
&\Rightarrow \quad \pi+k\cdot(-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z)>i_{cap_x} \\
&\Rightarrow \quad z>\frac{i_{cap_x}-\pi}{k\alpha\sigma}+\frac{1}{2}\alpha\sigma=d_2^x=\frac{i_{cap_x}}{k\alpha\sigma}+d_1^x
\end{aligned}$$

Then Equation (2.11) becomes

$$E^Q[-1_{z<d_1^x}(i_{ADI_x})]=E^Q[1_{z>d_2^x}(i_{ADI_x}-i_{cap_x})] \tag{2.13}$$

Let's split up the [11]left and the right side of the simplified Equation (2.11)

$$\begin{aligned}
\text{Left Collar}_x &= -E^Q[1_{z<d_1^x}(\pi+k\cdot(-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z))] \\
&= -\frac{1}{\sqrt{2\pi_o}}\int_{-\infty}^{d_1^x}\exp(-\frac{z^2}{2})[\pi+k\cdot(-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z)]dz \\
&= -\Phi(d_1^x)[\pi+k(-\frac{1}{2}\alpha^2\sigma^2)]+\frac{k\alpha\sigma}{\sqrt{2\pi_o}}\exp(-\frac{z^2}{2})_{-\infty}^{d_1^x} \\
&= -\Phi(d_1^x)[\pi-\frac{1}{2}k\alpha^2\sigma^2]+\frac{k\alpha\sigma}{\sqrt{2\pi_o}}\exp(-\frac{(d_1^x)^2}{2})
\end{aligned}$$

$$\begin{aligned}
\text{Right Collar}_x &= E^Q[1_{z>d_2^x}(\pi+k\cdot(-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z)-i_{cap})] \\
&= \frac{1}{\sqrt{2\pi_o}}\int_{d_2^x}^{\infty}\exp(-\frac{z^2}{2})(\pi+k\cdot(-\frac{1}{2}\alpha^2\sigma^2+\alpha\sigma z)-i_{cap_x})dz \\
&= \Phi(-d_2^x)(\pi+k(-\frac{1}{2}\alpha^2\sigma^2)-i_{cap_x})-\frac{k\alpha\sigma}{\sqrt{2\pi_o}}\exp(-\frac{z^2}{2})_{d_2^x}^{\infty} \\
&= \Phi(-d_2^x)(\pi-\frac{1}{2}k\alpha^2\sigma^2-i_{cap_x})+\frac{k\alpha\sigma}{\sqrt{2\pi_o}}\exp(-\frac{(d_2^x)^2}{2})
\end{aligned}$$

---

[11] "Left Collar" and "Right Collar" represents the left and right side of Equation (2.11)

Then Equation (2.13) becomes

$$\begin{aligned} &-\Phi(d_1^x)(\pi - \frac{1}{2}k\alpha^2\sigma^2) + \frac{k\alpha\sigma}{\sqrt{2\pi_o}}\exp(-\frac{(d_1^x)^2}{2}) \\ &\quad = \Phi(-d_2^x)(\pi - \frac{1}{2}k\alpha^2\sigma^2 - i_{cap_x}) + \frac{k\alpha\sigma}{\sqrt{2\pi_o}}\exp(-\frac{(d_2^x)^2}{2}) \end{aligned} \quad (2.14)$$

where π_o the ratio of a circle's circumference to its diameter with approximate value 3.1415927. It is hard to write down the explicit solution of i_{cap} directly, because the right collar contains three different types of functions of i_{cap}, they are: two linear functions of i_{cap}, $k = k(i_{cap})$ and $d_2 = d_2(i_{cap})$; an exponential function of i_{cap} and a probability distribution function $\Phi(i_{cap})$. However, it is still possible to arithmetize i_{cap} by running the "Solve Non-Linear Equation" (SolveNLE) package in OX. Appendix A.2 elaborates the calculation procedures.

Equation (2.14) contains three fundamental variables, namely the constant wage inflation rate π, the stock volatility σ and the portfolio weight α. If the value of the three are all known, then the indexation cap of each age cohort can also be worked out numerically. The result must be a (40×1) vector

$$i_{cap} = \begin{pmatrix} i_{cap_{25}} \\ \vdots \\ i_{cap_x} \\ \vdots \\ i_{cap_{64}} \end{pmatrix} \quad (2.15)$$

Vector (2.15) states that the indexation cap is an age dependent function. Each age cohort has its own indexation cap which could be different from other cohorts'. If the wage inflation rate is $\pi = 2\%$, the asset mix is $\alpha = 50\%$ and the volatility of the stock is $\sigma = 18\%$, then the indexation collar of each age cohort is solvable.

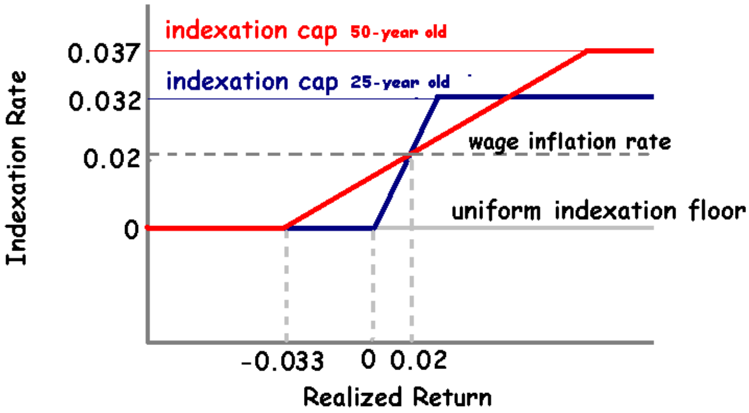

Figure 2.3: Individual indexation collar of 25 and 50-year old members

Figure 2.3 displays the collar structures of the 25-year old and the 50-year old participants. It is obtained that the indexation cap of the 25-year old participants is lower than the cap of the 50-year old participants ($3.2\% < 3.7\%$). The reasoning is as follows: a high risk taking gives investors some chance to earn a big fortune but also pushes them into a danger of having a big loss. Due to the age-differentiation structure of the ADI plan the size of the loss and the overwhelming benefit of the young participants must be bigger than the old participants'. Therefore, the indexation cap of the young have to be lower than the cap of the old so as to make sure that the given-up benefit is big enough to compensate the big loss. In short, in the individual collar model, the young participants have bigger collar effect than the old.

There are many similarities between the collar structure shown in Figure 2.3 and the policy ladder structure, however, the two are different in two ways. First, the cap of the collar and the ceiling of the ladder are different. The policy ladder limits the indexation rate between zero and the inflation rate. The conditional indexation rate of the policy ladder depends on the funding position. However, the indexation rate of the collar contingent ADI plan is restrained by a floor which is the same as policy ladder, and a cap which is in our case, higher than the inflation rate. The value of the indexation cap depends on the realized asset return as well as the inflation rate. Second, the policy ladder requires a uniform indexation rate to all the participants, while the ADI plan with indexation collar is age differentiated.

Next, we are going to do some sensitivity analysis. Given the base value of the three fundamental variables, to detect the relation between the indexation cap and each of the three fundamental variables, we play three tests. In each test, we change one of the value of the three fundamental variables while keep the other two at the base level. The three alternative cases are, an increase of the stock weight, an increase of the inflation rate and a decrease of the volatility. Remark: Equation (2.14) tells that only the product of the portfolio weight and the stock volatility $\alpha\sigma$ plays the role. Therefore, there is no surprise that α and σ may have the same effect on the indexation cap.

Case	π	σ	α
Base Case	0.02	0.18	0.5
100%	0.02	0.18	1
higher inflation	0.03	0.18	0.5
lower volatility	0.02	0.16	0.5

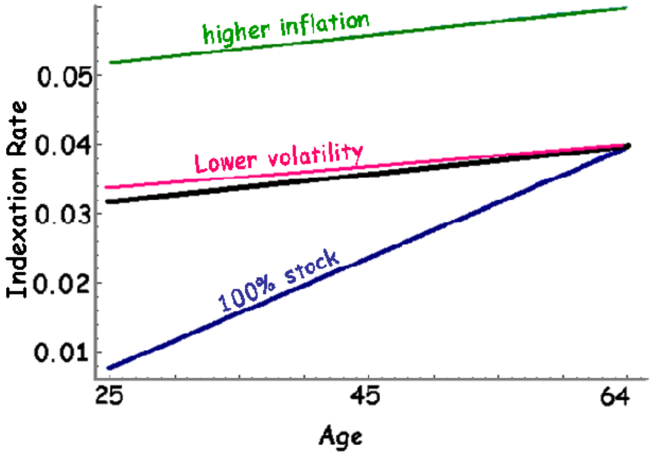

Figure 2.4: Individual Indexation Cap (Sensitivity Analysis)

Figure 2.4 plots the indexation cap of each active age cohort. y-axis is indexation rate and x-axis is age. The black curve represents the outcomes of the base case. Figure 2.3 only presents two age cohorts out of forty, while Figure 2.4 displays the complete result of Vector (2.15). The indexation cap is upward sloping and is almost linear, that means indexation cap is constantly increasing with age. The entire line is above the wage inflation rate 0.02. Therefore, the collar contingent ADI plan is more attractive compared with the policy ladder, since the participants can not only get the nominal guarantee, but can they also acquire the rights to benefit from the risky investment.

If the inflation rate goes up by 1%, then the indexation-cap curve will parallelly shift up by approximately 2% (see green curve of Figure 2.4). Because the inflation rate is assumed stable, it does not affect the volatility of the indexation rate. Therefore, we can only obtain the scale effect from a change of the inflation rate but not the slope effect which can be seen when changing the value of α or σ. According to Equation (2.14), only the product $\alpha\sigma$ impacts the value of the indexation cap, so α and σ play the same role in cap equation. If we increase the stock weight α, which is the same as increasing the stock volatility σ by the same scale, the indexation cap curve (the blue one) will become steeper and the front part will be even lower than the inflation rate. When the indexation collar is lower than the wage inflation rate, the “collar strategy” is not attractive at all, for some young age cohorts have no way to get fully indexed. If the stock volatility goes down (see purple red curve), the slope of the indexation-cap curve will be flatter. A change of either the stock weight α or the stock volatility σ does not result in the scale effect, but the two variables have the same slope effect on the indexation cap. In short, we conclude that the “collar strategy” is not always attractive to every one. It depends on the return dynamics of the market.

Assumption Relaxation

So far, we have found out the indexation collar using simple settings. Some assumptions however, might be inappropriate in the pension industry. For instance, the assumption of $i_{floor} = 0$ might be very expensive in the real world. In this subsection, we would like to relax this assumption by means of allowing the negative indexation. Another inappropriate assumption is the fixed inflation rate which is, in fact, quite volatile according to historical records. We will relax this assumption in the multi-period model (Chapter 4 and 5).

Redefine the indexation floor as the following age-dependent function

$$i_{floor_x} = -\frac{0.01}{40}(65 - x) \tag{2.16}$$

The value of the redefined indexation floor is negative and is increasing with age. Therefore, the new collar equation is given by

$$E^Q[-1_{i_{ADI_x} < i_{floor_x}}(i_{ADI_x} - i_{floor_x})] = E^Q[1_{ADI_x} > i_{cap_x}(i_{ADI_x} - i_{cap_x})]$$

Figure 2.5 displays the indexation collar structures of the 25-year old and the 50-year old participants. The 25-year old participants, according to the new indexation floor function (2.16), have a lower indexation floor than the 50-year old ($i_{floor_{25}} = -0.01 < i_{floor_{50}} = -0.00375$). Therefore, it is not a surprise to observe that the indexation cap of the young is now higher than the old’s ($i_{cap_{25}} = 0.052 > i_{cap_{50}} = 0.044$). The result is just opposite to what we conclude from Figure 2.3. The big change of the cap structure is resulting from the redefined floor function. Members now, have to borne part of their loss, hence the required compensation is less. The more negative indexation is allowed, the higher the cap value is needed. In short, Figure 2.5 indicates that allowing a certain level of negative indexation may reduce the collar effect and the mismatch risk.

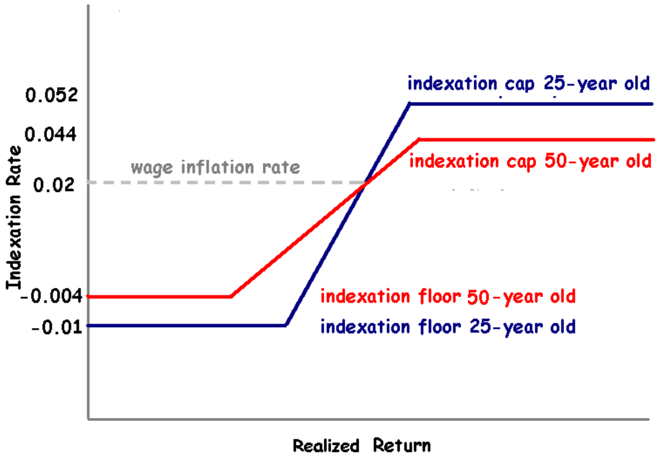

Figure 2.5: Individual Indexation Collar for 25 and 50 years

Figure 2.6 plots the sensitivity analysis result before and after using the new indexation floor function. The stronger lines represent to the indexation-cap curves when the indexation floor follows Equation (2.16). The weaker lines which have been shown up already in Figure 2.4 are the outcomes when the value of the floor is zero. Compared Figure 2.4 with Figure 2.6, we find that after changing the indexation floor function, the three fundamental variables have the same effect on the indexation cap as they do before changing the floor function. The biggest difference is that the slope of each of the four curves goes down. The base case (black line) used to be upwards sloping (see Figure 2.4), but is now decreasing with age. Though the "100% stock " curve is still increasing with age, the slope is much flatter than the corresponding one in Figure 2.4.

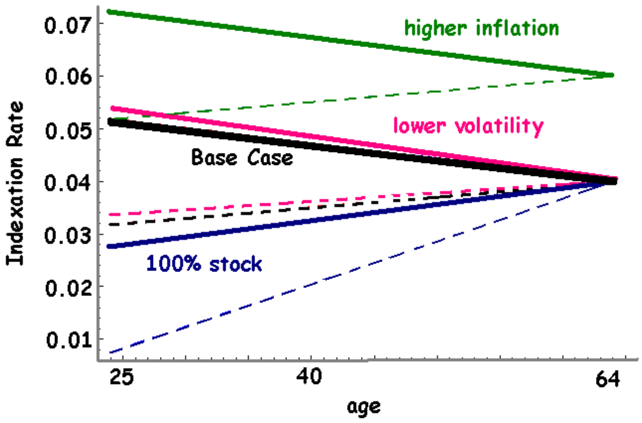

Figure 2.6: Individual Indexation Cap (Sensitivity Analysis with Negative i_{floor_x})

2.3.2 Uniform Indexation Collar for All Age Cohorts

In the previous section, we have solved the individual indexation collar. In this section, we are going to design another type of indexation collar named "uniform indexation collar". The uniform collar, by definition means that all the active participants have only one indexation collar. The mission of the uniform collar is the same as the individual one, but the value of the uniform collar is required to be jointly zero. Therefore, "zero-cost" does not necessarily to be realized by each age cohort, but the gross gain and loss has to be balanced. The indexation floor is still assumed to be zero, and the uniform indexation cap, i_{cap^*} is applied to all the active participants. The zero cost equation of the uniform indexation collar is given by

$$\begin{aligned} & E^Q[-\sum_{x=25}^{64} 1_{i_{ADI_x}<0}(B_x(1+i_{ADI_x}) - B_x(1+i_{floor}))] \qquad (2.17) \\ = \quad & E^Q[\sum_{x=25}^{64} 1_{i_{ADI_x}>i_{cap^*}}(B_{x,t}(1+i_{ADI_x}) - B_{x,t}(1+i_{cap^*}))] \end{aligned}$$

where [12] $B_{x,t} = \triangle B \sum_{s=25}^{x} \exp[(x-s)\pi]$ is the total accrued pension rights of the x-year old participants. As there is only one unknown variable i_{cap^*} in Equation (2.17), the calculation is easier than the previous case.

$$\begin{aligned} i_{ADI_x} < 0 \quad & \Rightarrow \quad z < -\frac{\pi}{\frac{65-x}{40}\alpha\sigma} + \frac{1}{2}\alpha\sigma = d_1^x \\ i_{ADI_x} > i_{cap^*} \quad & \Rightarrow \quad z > \frac{i_{cap^*} - \pi}{\frac{65-x}{40}\alpha\sigma} + \frac{1}{2}\alpha\sigma = d_2^x \end{aligned}$$

Then Equation (2.17) becomes

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[12] $B_{x,t} = \triangle B_x + \sum_{s=25}^{x-1} \prod_{u=t-(x-s)+1}^{t} exp(i_u \triangle B_S)$ where i_u is the indexation policy in use before replacing to ADI plan. Suppose previous pension rights are nominal, in other words, they are not indexed, then $B_{x,t} = B_{x,t}^R$

$$\underbrace{E^Q[-\sum\nolimits_{x=25}^{64} 1_{z<d_1^x} \sum\nolimits_{s=25}^{x} \exp[(x-s)\pi]\cdot i_{ADI_x}]}_{\sum_{x=25}^{64}\text{Left Collar}_x} =$$

$$\underbrace{E^Q[\sum\nolimits_{x=25}^{64} 1_{z>d_2^x}\cdot \sum\nolimits_{s=25}^{x} \exp[(x-s)\pi](i_{ADI_x}-i_{cap^*})]}_{\sum_{x=25}^{64}\text{Right Collar}_x}$$

where Left Collar$_x$ and Right Collar$_x$ represent to the expected loss and extra benefit of an age cohort x. The two are not necessarily equal.

$$\begin{aligned}
\text{Left Collar}_x &: \\
&= E^Q[-1_{z<d_1^x}\sum\nolimits_{s=25}^{x} \exp[(x-s)\pi](\pi + \frac{65-x}{40}(-\frac{1}{2}\alpha^2\sigma^2 + \alpha\sigma z))] \\
&= -\Phi(d_1^x)\sum\nolimits_{s=25}^{x} \exp[(x-s)\pi][\pi + \frac{65-x}{40}(-\frac{1}{2}\alpha^2\sigma^2)] \\
&+ \frac{65-x}{40}\alpha\sigma\frac{1}{\sqrt{2\pi_o}}\sum\nolimits_{s=25}^{x} \exp[(x-s)\pi]\exp(-\frac{(d_1^x)^2}{2}) \\
\text{Right Collar}_x &: \\
&= E^Q[1_{z>d_2^x}\sum\nolimits_{s=25}^{x} \exp[(x-s)\pi](\pi + \frac{65-x}{40}(-\frac{1}{2}\alpha^2\sigma^2 + \alpha\sigma z) - i_{cap^*})] \\
&= \Phi(-d_2^x)\sum\nolimits_{s=25}^{x} \exp[(x-s)\pi][\pi + \frac{65-x}{40}(-\frac{1}{2}\alpha^2\sigma^2) - i_{cap^*}] \\
&+ \frac{65-x}{40}\alpha\sigma\frac{1}{\sqrt{2\pi_o}}\sum\nolimits_{s=25}^{x} \exp[(x-s)\pi]\exp(-\frac{(d_2^x)^2}{2})
\end{aligned}$$

The equilibrium of the collar equation is 0.036 in base case. Figure 2.7 displays the uniform collar structures of the 25-year old and the 50-year old participants. Compared with Figure 2.3, we obtained that the young are better off in the uniform collar strategy and the old are worse off, because the uniform indexation cap is somewhere in the middle of the upward sloping individual-indexation-cap curve. The uniform indexation cap to the young is higher than their individual one, so they have a free lunch paid by the old. However, the uniform indexation cap to the old is lower than their individual one. Therefore, the old have to give up part of their investment benefit to pay for the free lunch.

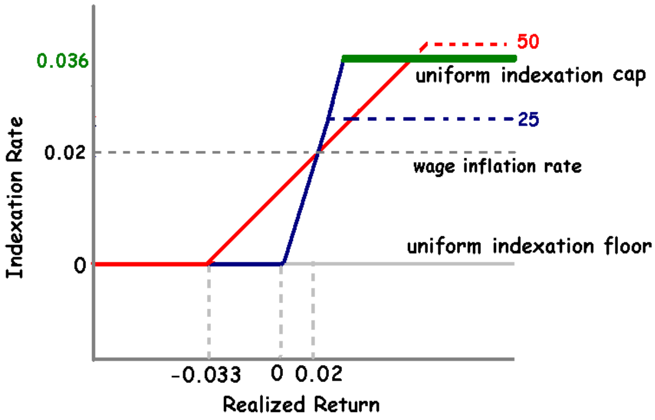

Figure 2.7: Uniform indexation collar of 25 and 50-year old members

Next, we are going to have another sensitivity analysis for the uniform collar model. The results are shown is the table below

Case	π	σ	α	Indexation Cap i_{cap^*}
Base Case	0.02	0.18	0.5	0.0361722
100% stock	0.02	0.18	1	0.0271106
higher inflation	0.03	0.18	0.5	0.0557885
lower volatility	0.02	0.16	0.5	0.0368864

The results are close to the individual caps of the middle-age participants shown in Figure 2.4. In short, the uniform indexation cap increases with the inflation rate but decreases with the stock volatility (or the asset portfolio).

Chapter 3

Evaluation of Single Period Designs

In this chapter, we make use of an asset-liability management study using the simple financial settings defined in Chapter 2 to evaluate the three policy variants. We first investigate the indexation distribution, then analyze the funding ratio distribution of the three variants.

3.1 Indexation Distribution Analysis

We simulate 1000 scenarios of the future indexation rate for each policy variant based on the simple financial settings introduced in Chapter 2. Let's shortly review the settings of the single-period model. On financial side, the asset return dynamics is captured by Equation (2.10) under **P**-distribution. The nominal rate and the inflation rate are assumed constant. The asset portfolio consists of stock and bond only, and the weight of either asset class is fixed at 50%. On pension side, we have an average-salary scheme with indexed liabilities. Besides indexation policy, the rest pension characteristics are of the homogeneity settings. The three indexation policies are under investigation. The indexation rate of the policy ladder depends on the historical funding position. The indexation rate of the three ADI plans depends on age, asset return and the inflation rate.

First, we detect the indexation rate distribution of each policy variant. Table 3.1 displays the mean and the standard deviation of the sample indexation rate of each variant. The table is separated into two, for the policy ladder is a uniform policy while the three ADI plans are age dependent. We pick eight out of forty age cohorts to find the distribution change over age. Three points can be concluded from Table 3.1. First, an age dependent indexation plan in general have a higher value of mean and standard deviations than the policy ladder. Second, the standard deviation of the three sample age-dependent indexation is decreasing with age since the risk exposure to the asset market by definition, is decreasing with age. Third, the individual collar contingent ADI plan is not very attractive to the young participants since its mean indexation rate is the lowest among all.

ind. ladder	
mean	std.
1.866	0.316

	ADI		**indi.co.ADI**		**uni.co.ADI**	
age	**mean**	**std.**	**mean**	**std.**	**mean**	**std.**
25	2.023	8.9398	1.676	1.5159	1.869	1.7109
30	2.020	7.8223	1.743	1.5608	1.887	1.7050
35	2.017	6.7049	1.804	1.5627	1.899	1.6959
40	2.014	5.5874	1.799	1.5739	1.846	1.6251
45	2.011	4.4699	1.916	1.6070	1.924	1.6159
50	2.009	3.3524	1.978	1.5725	1.951	1.5432
55	2.006	2.2350	1.951	1.4626	1.911	1.4135
60	2.003	1.1175	1.989	1.0368	1.971	1.0113

Table 3.1: Mean and Standard Deviation of Simulated Sample Indexation Rate for each variant (**in percentage**). The constant wage inflation rate is 2%. ind.ladder: Indexation Ladder policy; ADI: ADI Plan; indi.co.ADI: Individual Collar Contingent ADI plan; uni.co.ADI: Uniform Collar Contingent ADI plan

Sequently, the histogram figures are displayed in Appendix B.1. We compare the histogram of the un-collared ADI plan with the histogram of the policy ladder, the histogram of the individual collar contingent ADI plan and the histogram of the uniform collar contingent ADI plan respectively. The indexation distribution of the four policies is quite different. The sample observations of the ADI plan spread widely, while the spans of the other three variants are very small. The indexation distribution of the policy ladder has a single sharp wall and is negatively skewed. The histograms of the two collar contingent ADI plans both have two high walls, and such two-wall shaped distribution is called "San Quentin" distribution.

Table 3.2 reports several percentile values of the sample indexation rate of each variant. For the three age-dependent policies, we pick three age cohorts, namely the 25-year old, the 40-year old and the 55-year old, to represent the whole active participants. First, let's have a look at the results of the policy ladder. The 10^{th}-percentile value is 1.4% while the 1^{st} quartile is 2%, which indicates that a policy-ladder participant has more than 75%

of the chance to get [1]fully indexed and has less than 25% of the chance to be partly indexed.

Second, let's have a look at the percentile values of the ADI plan without collar. The results of the young participants are very extreme. A 25-year old member has a probability of more than 25% to receive a negative indexation rate, but also has more than 40% of the chance to get an overwhelming benefit which is much higher than the benefit from the full indexation, from the risky investment. Only a small number of sample observations have partial indexation rate (from 0 to 2%). 10% of the observations have the values of six times higher than the wage inflation rate. Though, a 45-year old participant still has a big chance to get negatively indexed, the absolute value of the loss is toned down.

The remaining part of Table 3.2 shows the percentile values of the two collar strategies. The two collar strategies have the same distribution till median and vary afterwards because of the various cap structures. Young participants have stronger collar effect than the old. For instance, in the individual collar plan, more than 40% of the 25-year old sample observations are zero (indexation floor) and more than 40% are at indexation cap level (3.19%). So the collar effect is more than $80\% (= 40\% + 40\%)$. While only 50% of the 40-year old observations are affected by the individual indexation collar. The collar effect is decreasing with age because of two reasons. First, according to the ADI function, the indexation risk is decreasing with age, so the indexation volatility is shrinking with age as well. Second, the indexation cap in the individual collar variant is increasing with age, so the collar constraint is diminishing with age. Remark, in the uniform collar strategy, the second reason does not work.

---

[1] Remark: The inflation rate of the single-period model is 2%.

Variant	Age	Percentile values						
		10^{th}	25^{th}	40^{th}	50^{th}	60^{th}	75^{th}	90^{th}
ind.ladder	all	1.4	2	2	2	2	2	2
ADI	25	-9.4	-3.7	0.0	1.8	4.4	7.7	13.3
	40	-5.3	-1.5	0.8	1.9	3.5	5.1	5.6
	55	-0.9	0.5	1.5	2.0	2.6	3.4	4.8
indi.co.ADI	25	0	0	0	2.1	3.2	3.2	3.2
	40	0	0	0.7	1.9	3.2	3.5	3.5
	55	0	0.4	1.4	2.1	2.6	3.5	3.8
uni.co.ADI	25	0	0	0	2.1	3.6	3.6	3.6
	40	0	0	0.7	1.9	3.2	3.6	3.6
	55	0	0.4	1.4	2.1	2.6	3.5	3.6

Table 3.2: Indexation Rate at Different Percentiles (**in percentage**). ind.ladder: Indexation Ladder policy; ADI: ADI Plan; indi.co.ADI: Individual Collar Contingent ADI plan; uni.co.ADI: Uniform Collar Contingent ADI plan

Figure 3.1 plots some percentiles of the sample ADI-plan indexation rate for the active participants. The two red curves represent to the 5^{th} and the 95^{th} percentile. The blue one is the median and the two green lines are the first and the third quartile. It is obtained that the red curves spread widely at early career years and are convergent to wage inflation rate. The median curve is above 2%. From Figure 3.1, we can clearly observe the existence of negative indexation and enormous volatility.

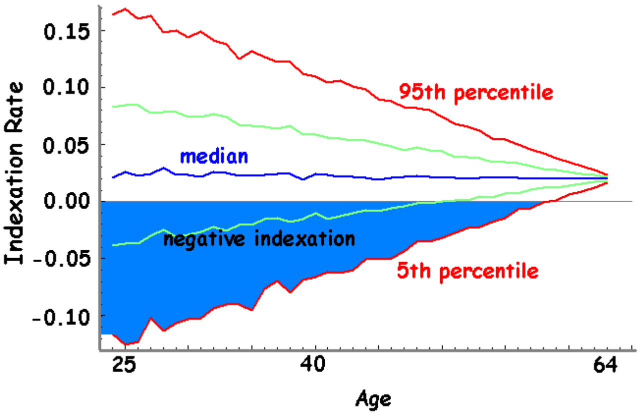

Figure 3.1: Sample Indexation Percentile of Age-Dependent Indexation Policy

Figure 3.2 compares the sample indexation rate percentiles of the ADI plan and of the policy ladder. Because the policy ladder is not age dependent, its sample indexation percentile-age curves must be horizontal. The blue area presents to the sample observations of the policy ladder between the 5^{th} and the 95^{th} percentile. The median of the sample indexation rate of the ADI plan is even bigger than the 95^{th} percentile of the policy ladder. From Figure 3.2, we can conclude that policy ladder is a very conservative variant, for all the observations are confined at a small and safe interval.

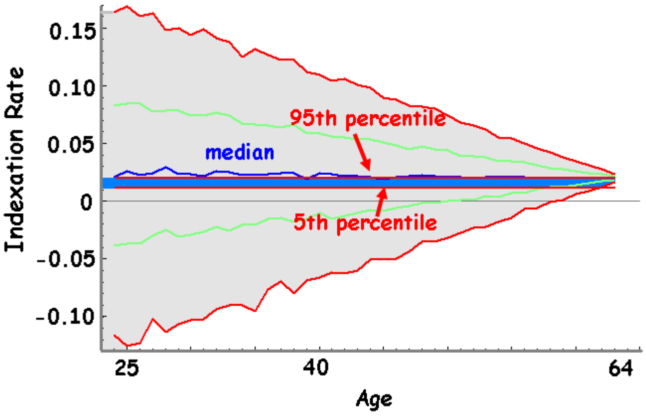

Figure 3.2: Indexation Percentile Comparison between ADI plan and Policy Ladder

Figure 3.3 and Figure 3.4 compare the sample indexation-rate percentiles of the un-collared ADI plan with the values of the individual collar and the values of the uniform collar contingent ADI plan respectively. For either collar strategy, there is a dramatic risk reduction after adding the collar. Two edges of the blue area represent to the new 5^{th} and 95^{th} percentiles after adding the collar to the ADI plan. It is obtained that for most active age cohorts (from 25 to 60) the 5^{th} percentile-age curve of both Figure 3.3 and Figure 3.4 are horizontal at the value zero. The 95^{th} percentile-age curve in Figure 3.3 is linear upward sloping since the individual indexation cap is increasing with age, but is horizontal at the value of the uniform indexation cap in Figure 3.4.

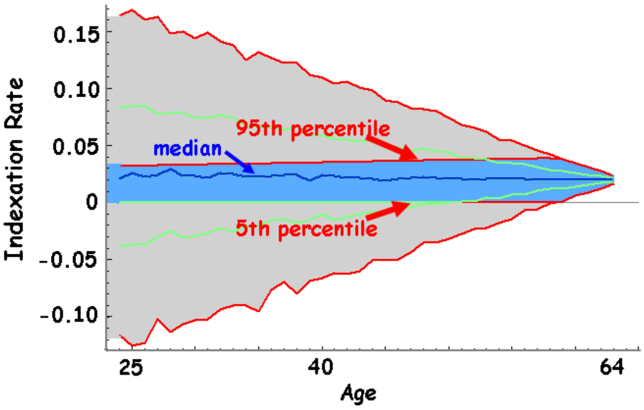

Figure 3.3: Sample Indexation Percentile Comparison between ADI plan and Individual Collar Strategy

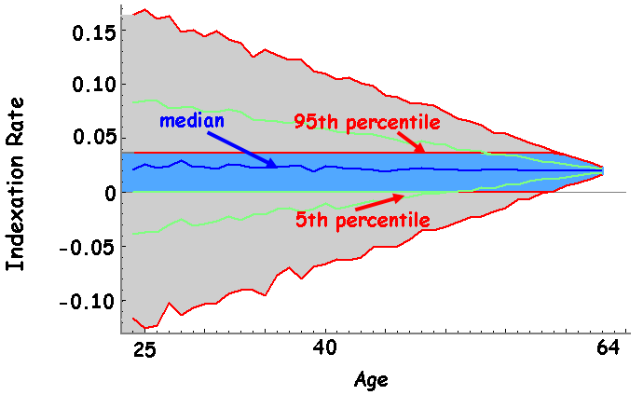

Figure 3.4: Sample Indexation Percentile Comparison between ADI plan and Uniform Collar Strategy

Compared Figure 3.2 with Figure 3.3 and with Figure 3.4, we conclude that both collar strategies can successfully control the indexation volatility of the ADI plan, but they are still riskier than the policy ladder.

3.2 Funding Ratio

Indexation policy and funding ratio are closely related. In this section, we are going to investigate the funding ratio distribution of each indexation policy. The realized funding ratio FR at time $t+1$ is given by

$$FR_{t+1} = \frac{A_{t+1}}{L_{t+1}} \tag{3.1}$$

First, we need to draw 1000 sample values of the future assets. Equation (2.9) gives the asset dynamics A_{t+1} in the real world. The function consists of two uncertain variables. One is the asset return, of which, we have already drawn 1000 sample values when simulating the indexation rates in the previous section. The other uncertain factor, the benefit payment (BP_{t+1}), is various from variant to variant. In the policy ladder, the retirees are still conditionally indexed, while in the ADI plans, the retirees are always fully indexed. Second, we simulate 1000 sample values of the realized liabilities of each variant. Equation (2.8) shows how to calculate the realized liabilities. Appendix A.3 explains the calculation procedures of the realized liabilities using different indexation functions. Appendix B.2 presents the funding ratio histograms of each variant (See Figure B.4). Table 3.3 displays the sample funding ratio mean, the standard deviation and the percentile values of each policy variant.

Variant	**mean**	**std.**	5^{th}	25^{th}	50^{th}	75^{th}	95^{th}
ind.ladder	105.7	9.587	91.9	99.1	104.2	111.4	121.8
ADI	106.1	10.11	91.4	99.2	105.3	111.9	124.2
indi.co.ADI	106.2	10.12	91.4	99.2	105.3	112.0	124.1
uni.co.ADI	106.2	10.12	91.4	99.2	105.3	112.0	124.1

Table 3.3: Funding Ratio Distribution (**in percentage**). ind.ladder: Indexation Ladder; ADI: ADI plan; indi.co.ADI: Individual collar contingent ADI plan; uni.co.ADI: Uniform collar contingent ADI plan

Before comparing the result between different indexation policies, it is important to notice that funding ratio is not a very flexible variable. Increasing the funding ratio by 1% requires an extra amount of the pension assets of around 2 billion € in [2]ABP funds.

It is obtained from Table 3.3 that each of the three age-dependent indexation policies has a higher funding ratio than the policy ladder does, and the funding ratio standard deviation of any ADI plan is around 7% higher than the value of the policy ladder.

Figure 3.5 plots the cumulative density function (CDF) of the funding ratio distribution of each policy variant. x-axis is the percentile level (from 0% to 100%), y-axis is the value of the funding ratio. There are, in principle, four curves in Figure 3.5, while three of them are almost overlapped. Therefore, we can only observe two distinct curves. The red ones represent to the sample funding-ratio CDF curves of the three ADI plans. It seems that adding collar does not cause any change on the funding ratio distribution

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[2] The total assets in ABP pension funds are around 200 billion€

of the ADI plan. The black CDF curve plots the funding ratio distribution of the policy ladder. The difference between the red and the black curves are very small because the current model only consists of one period, which is not long enough for us to obtain the funding-ratio distribution change resulting from the policy switching.

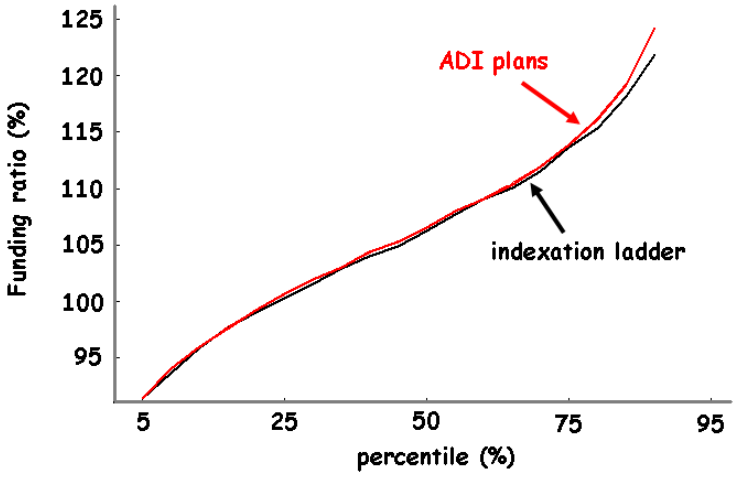

Figure 3.5: Funding Ratio Distribution

In short, we draw three conclusions from Table 3.3 and Figure 3.5. First, switching the indexation policy does not result in big impact on the funding ratio distribution in the short run. Second, the three ADI plans have higher probabilities of outperforming than the policy ladder. Third, the funding ratio of the ADI plans in the short run are more volatile than the funding ratio of the policy ladder.

The latter two conclusions contradicts the result of Molenaar, Munster and Ponds (2008). The reason for the first difference might be that we over estimate the value of the realized liabilities of the policy ladder because of either an over estimation of the value of the assets resulting from a high drift or an under estimation of the value of the real liabilities resulting from a low fixed inflation rate. The other difference might be caused by the short time horizon of the model. As single-period model can not forecast the distribution revolution in the future, a more complex model with more time periods is in demand.

Chapter 4

Multi-Period Modeling

Evaluating policy variants in a single period model has many limitations. First, a policy change results in a very small impact on the funding ratio distribution in the short run, so we cannot rank the variants through comparing the quality of the funding ratio distribution. Second, the participants especially the young care more about the value of their pension rights in the future, say ten years ahead, instead of tomorrow. Therefore, it is important to extend the single period model to the multi-period one.

Some results drawn from the single period model using simple model could be misleading. One possible way to improve the model is to enlarge the degree of freedom. In the multi-period model, the return dynamics forecasting model is switched from the [1]Black-Scholes two-state-variable model to the first order [2]Vector Auto-regression(VAR) model with multi state variables. Increasing the complexity of the model on one hand can bring a better prediction of the real world; on the other hand, however, makes the determination of indexation collar more complicated.

The plan of this chapter is as follows: we first introduce our first order VAR model; then we calculate the cap value of the indexation collar using this new return dynamics model.

---

[1] The most crucial limitation of Black-Scholes model is the use of geometric Brownian motion function to estimate price movement, as in real world stock price does not strictly follow log-normal process. Still, the risk volatility has been proven unstable over time, and the market is mostly incomplete.

[2] VAR model used in APG consists of a huge number of variables. The concrete story of Scenario generation at APG is given in Appendix A.3. The VAR model in our multi-period model is homoscedastic with constant second moment of the model. We pick nine endogenous state variables from the Core-VAR APG model.

4.1 Return Dynamics

In the multi-period model, the return dynamics are captured by the first order homoscedastic Vector Autoregression (VAR) model. The attractiveness of a VAR model is that it is able to capture the long-term dynamics out of the historical data and it can distinguish the long-run and short-run return features. The APG VAR model elaborated in Angerman, Hoevenaars, Molenaar and Steenkamp (2008) extends Campbell, Chan and Viceira (2003)'s model by adding more asset classes. The dynamic variables are separated into two groups, one group is for the core-VAR model and the other group is used derived-VAR model. In this paper, we only pick nine state variables from the APG core-VAR model as the input of our ALM studies.

4.1.1 VAR model under P-measure

VAR is an intuitive linear model describing the stochastic time series. State variables in a VAR model depend on their own lag (autoregression). Nine return variables are chosen in our VAR model (see Appendix C.1). Three of them are involved into indexation collar calculation. The three variables are: the nominal short term interest rate $r^1_{n,t}$, the price inflation π_t and the stock return $x_{s,t}$. The dynamics of these state variables is modeled in a quarterly vector autoregression system with time-varying first moment:

$$x_{t+1} = c + \Gamma x_t + \Sigma\zeta_{t+1} \qquad \zeta_{t+1} \sim N(0, I_n) \tag{4.1}$$

where x_t is a vector consisting nine state variables which are introduced in Table C.1. Appendix C.2 reports the estimation result of the VAR model. Panel (a) of Table C.2 is the value of $c = (I - \Gamma)\mu$ with μ a (9×1) vector of the historical mean. Γ is the coefficient matrix (see Table C.2 Panel (b)). Σ is the covariance matrix of the VAR residuals (see Table C.2 Panel(c)). c, Γ and Σ are estimated by the maximum likelihood measure.

4.1.2 VAR model under Q-measure

Both the equivalent martingale measure and the [3]pricing kernel measure are powerful tools for asset pricing. The former one is applied in the risk neutral **Q**-measure and the latter one is used in the real world **P**-measure. Campbell and Viceira (1999) elaborate the nominal-bond pricing measure by using stochastic discount factor. Nijman and Koijen (2005) also use the pricing kernels to value the assets.

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[3] Appendix A.3 explains asset pricing procedure by using pricing kernel measure.

Though the pricing kernel method is customary among literatures, Angerman(2004) argues that it is inefficient to price the derivatives under **P**-distribution in VAR model. When the claims are unable to be valued analytically, an alternative measure is by means of Monte Carlo simulation. However, the Monte Carlo simulation under **P**-measure is always centered around $\zeta = 0$. Thus the sum of a large number of the sample values converges slowly. In order to converge the summation of the simulation more quickly, we switch the sampling from the distribution $N(0, \Sigma\Sigma^{\top})$ to the sampling from the distribution $N(-\Sigma\lambda, \Sigma\Sigma^{\top})$. The probability of drawing x_t under the real world **P**-measure is the same as the probability of drawing $x_t - \Sigma\lambda$ under the risk neutral world **Q**-measure where $\lambda_t = \lambda_0 + \Lambda_1 x_t$ is the time-varying price of the risk (see Appendix A.14), $\Sigma\lambda_t$ is the coefficient estimate of the risk premia, the estimation result is reported in Appendix C Table C.3.

Therefore, the first order VAR model under risk neutral measure is given by

$$x_{t+1} = c + \Gamma x_t - (\Sigma\lambda_0 + \Sigma\Lambda_1 x_t) + \Sigma\zeta_{t+1} \qquad \zeta_{t+1} \sim N(0, I_n) \tag{4.2}$$

4.2 Collar Design in Multi-Period Model

In this section, we concentrate on the collar determination under the **Q** measure. In chapter 2, we designed two types of indexation collars, namely the individual and the uniform indexation collar. In the multi-period model, however, we pick the uniform one only for two reasons. First, the homogeneity characterized pension schemes are more supportive in the Netherlands. Despite the feature of partly age differentiation , the uniform collar contingent ADI plan still enjoys the characteristics of uniformity. Therefore, compared with the individual collar, the uniform one is more attractive. Second, the individual collar is less beneficial to the young members than the uniform one (see Table 3.1). The value of the individual indexation cap is increasing with age, so the young participants have a stronger collar effect than the old. For instance, there are two pension participants A and B. A is younger than B and the indexation cap for A, according to the individual collar structure, must be lower than the cap for B. Suppose the realized age-dependent indexation rate of A is the same as B $i_{ADI_A} = i_{ADI_B}$ and both are above their relevant indexation caps. In the individual collar strategy, A has to give up more benefit from the risky investment than B, though he bears more market risk. The uniform collar strategy is not financially fair neither, but is as least better than the individual one.

The inconvenience of using pricing kernel to price the collar is explained in the previous section. Therefore, we decide to price the collar in the risk neutral world. We simulate return dynamics of the next 10 years using the risk neutral VAR model (4.2). Suppose the indexation policy is switched at year t (i.e Be replaced by the ADI plan). Before time t, the liabilities are always fully indexed. The pension settings introduced in Chapter 2 are still applicable in the multi-period model. The financial settings are substituted by the VAR model.

In each of the future year $t+k$ (k is from 1 to 10), a specific uniform indexation collar has to be calculated based on the principle of "zero cost game" which states that the collar in each year requires no cost.

Let's first determine the value of the uniform cap at time $t+1$ (k=1). The collar equation under the **Q** measure can be rewritten in the following way

$$\begin{aligned} &E^Q[-\sum_{x=25}^{64} 1_{i_{ADI_{x,t+1}}<i_{floor}}(B^{ADI_x}_{x,t+1} - B^{floor}_{x,t+1})\cdot \exp(-4r^4_{n,t})] \\ &= E^Q[\sum_{x=25}^{64} 1_{i_{ADI_{x,t+1}}>i_{cap^1}}(B^{ADI_x}_{x,t+1} - B^{cap}_{x,t+1})\cdot \exp(-4r^4_{n,t})] \end{aligned} \quad (4.3)$$

where $r^4_{n,t}$ is the return of a 1-year zero coupon bond which is an [4]affine function of the state variables. The nominal term structure formula is shown in Appendix A .3, (Equation (A.16)). Equation (4.3) is just one of the possible ways to design the uniform collar. It requires the value of the floor and the cap in advance. Therefore, we need to simulate the return dynamics of the next period first before calculating the floor and the cap of the next period. An alternative way to design the uniform collar is to price the collar of the next year by using the information of the previous year. In this paper, however, we are not going to work out the latter design.

Let the indexation floor i_{floor} still be zero, them we can solve the collar equation (4.3) for the uniform indexation cap i_{cap^1} at time $t+1$.

$B_{x,t+1}$ is the total accrual pension benefit of the x-year old participants at time $t+1$. It can be calculated by

$$B_{x,t+1} = \triangle B + \triangle B\cdot \sum_{s=25}^{x-1}\prod_{u=t+1-(x-s)+1}^{t+1} \exp(i_u)\cdot 1_{x\in[26,64]}$$

since the value of the liabilities before time t is fully indexed, the indexation rate at time $u(u\leq t)$ is equal to the inflation rate at time u, in other words $i_{u_{u\leq t}} = \pi_{u\leq t}$. When $u > t$, the indexation rate at time u is driven by the indexation policy at that moment. If an ADI plan is applied then $i_{u_{u>t}} = i_{ADI_{x,u}}$ for the x-year old participants.

---

[4] Cochrane and Piazessi (2005), Nijman (2008) derive an affine term structure of interest rate for VAR model by using pricing kernel measure

Therefore, $B_{x,t+1}^{ADI_x}$ is the total accrual pension rights of the ADI-plan participants who are x-year old at the year $t+1$. Since the indexation floor i_{floor} is defined to be zero, then $B_{x,t+1}^{floor}$ means that in the year $t+1$, the value of the newly added pension rights is nominal. Similarly, $B_{x,t+1}^{cap}$ means that the value of the newly added pension rights at time $t+1$ is indexed at the rate of the uniform indexation cap i_{cap^1}.

Next, let's write down the age-dependent indexation function in the risk neutral world using the return variables involved in our VAR model.

$$\begin{aligned} i_{ADI_{x,t+1}} &= \frac{65-x}{40}(r_{A,t+1} - r_{r,t}) + \frac{x-25}{40}\pi_{t+1} \\ &= \frac{65-x}{40}(r_{n,t}^4 + \alpha x_{s,t+1} - r_{r,t}) + \frac{x-25}{40}\pi_{t+1} \\ &= \pi_{t+1} + \frac{65-x}{40}\alpha x_{s,t+1} \end{aligned}$$

or

$$i_{ADI_{x,t+1}} = \theta \cdot (c + \Gamma x_t - (\Sigma\lambda_0 + \Sigma\Lambda_1 x_t) + \Sigma\zeta_{t+1}) \tag{4.4}$$

where $\theta = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad \frac{65-x}{40}\alpha \quad 0]$. Remark: x without label stands for age; while x_t is a vector of the nine state variables in the VAR model (see Equation (4.2)).

We cannot solve the collar Equation (4.3) for i_{cap^1} analytically, so we have to choose a numerical method. The most obvious way to do that is by using the Monte Carlo simulation. The calculation consists of two procedures. First we need to determine the expectation of the right side of the collar equation, then we can try to find out the right indexation cap which equalizes the two sides of Equation (4.2). The VAR model is based on the quarterly historical data from 1973[5]II to 2003:III.

The right side of the collar equation is independent of i_{cap^1}. We draw $N = 5000$ sample scenarios of the dynamic variables of the next year using the risk neutral VAR model. The approximate value of the right side of the expectation equation (4.3) can be calculated by

$$\frac{1}{N}\sum[-\sum_{x=25}^{64} 1_{i_{ADI_{x,t+1}} < i_{floor}}(B_{x,t+1}^{ADI_x} - B_{x,t+1}^{floor}) \cdot \exp(-4r_{n,t}^4)]$$

The approximation result is 3.0922.

The left side of the collar equation is a function of i_{cap^1}. We can plug in different values of indexation cap to calculate the approximate expectation

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[5] "II" means the second season of a year

of the left side of the collar equation. We tried a sequence of numbers till we find the equilibrium one $i_{cap^1} = 8.7432\%$ which equates the expected value of the left side to 3.0922 as well.

In Chapter 2.3, we played some sensitivity analysis in the simple settings. Figure 2.4 implies that either an increase of the inflation rate or a decrease of the volatility can increase the value of the indexation cap. The former change results in a scale effect and the latter one results in a slope effect. The uniform indexation cap in the base case of the single period model is around 3.6% which is around two times larger than the fixed inflation rate 2%. However, we find that, in the multi-period model, the value of the uniform indexation cap is very large. It might be because of the following three reasons. First, a high inflation rate may result in a high cap value. The mean of our simulated sample inflation rate in year $t+1$ is 2.8659% which is higher than 2%. Second, the risk volatility of the asset return in our VAR model is 6.21% (see Table C.2 Panel(c)), which is much lower than the value we expected in the single period model in Chapter 2. A low asset-return volatility also triggers a high uniform indexation cap. Third, the inflation rate in the VAR model is not stable any more, this may also enlarge the value of the uniform indexation cap.

So far, we have worked out the indexation collar of the year $t+1$ (2004). Then we can repeat the calculation procedures to derive the indexation collar of the year $t+2$. When calculating the accrual pension benefit of the year $t+2$, it is important to notice that the indexation function at $t+1$ has to be replaced by the collar contingent ADI function, namely

$$i_{t+1} = i_{coADI_{t+1}} = \min(\max(i_{ADI_{t+1}}, i_{floor}), i_{cap^1}) \tag{4.5}$$

where $i_{floor} = 0$ and $i_{cap^1} = 8.7432\%$.

Again, by using the Monte Carlo simulation and the equilibrium searching measure, we get the right indexation cap of the year $t+2$. The solution is $i_{cap^2} = 9.1102\%$.

Recursively, to find out the indexation cap of next year k (k=1,2,...,10), we need to solve the following collar equation

$$\begin{aligned} &E^Q[-\textstyle\sum_{x=25}^{64} 1_{i_{ADI_{x,t+k}} < i_{floor}} (B_{x,t+k} - B_{x,t+k}^{floor}) \cdot \exp(-4k \cdot r_{n,t}^{4k})] \\ &= E^Q[\textstyle\sum_{x=25}^{64} 1_{i_{ADI_{x,t+k}} > i_{cap^k}} (B_{x,t+k} - B_{x,t+k}^{cap}) \cdot \exp(-4k \cdot r_{n,t}^{4k})] \end{aligned}$$

where

$$B_{x,t+k} = \triangle B + \triangle B \cdot \textstyle\sum_{s=25}^{x-1} \prod_{u=t+k-(x-s)+1}^{t+k} \exp(i_u) \cdot 1_{x \in [26,64]} \tag{4.6}$$

with

$$i_u = \begin{cases} \pi_u & \text{if} \quad u \leq t \\ i_{coADI_u} & \text{if} \quad t+1 \leq u \leq t+k-1 \\ i_{ADI_u} & \text{if} \quad u = t+k \end{cases}$$

Table 4.1 lists the values of the indexation caps of the next 10 years. The results are volatile and fluctuate around 9%.

i_{cap^1}	8.74320
i_{cap^2}	9.11020
i_{cap^3}	9.17070
i_{cap^4}	9.61700
i_{cap^5}	9.27840
i_{cap^6}	9.62870
i_{cap^7}	10.0675
i_{cap^8}	9.35289
i_{cap^9}	9.22226
$i_{cap^{10}}$	8.79717

Table 4.1: Indexation caps of next 10 years (**in percentage** %). The results are in percentage

The future cap cap^k in our model is computed in advance based on the corresponding future financial status of the funds in year $t+k$. This is only one possible design. It is also reasonable to calculate the cap on a year-by-year basis. That means future cap cap^k depends on the financial status of the previous year $t+k-1$.

Chapter 5

Evaluation of Multi-Period Designs

In this chapter, we will evaluate the three indexation policies in the multi-period model settings. The three indexation policies have been introduced in Section 2.2.2. The structure of this chapter is as follows: first, we need to generate the scenarios of the indexation rate, the value of the assets and the value of the liabilities of each indexation policy for the next 10 years starting from 2004. The sample return variables of the future years are simulated from the VAR model (4.1) in the real world **P** measure. Second, we investigate the distribution of the indexation and the funding ratio of each variant by using a classic ALM study. Third, we evaluate the certainty equivalent benefit of each variant by performing a utility-based ALM study.

5.1 Simulation

In this section, we shortly explain the procedures of the scenarios generation for an ALM study. First, we draw 5000 sample values of state variables for each of the future 10 years. Since our historical data of the VAR model is in the quarterly base, we need to simulate 40 times in total to acquire the return dynamics of the next 10 years.

Second, we shall build a sample of 5000 indexation rates of the year $t+k$ for each policy variant. Before building a sample of i_{ladder} at the year t+k, we need first, to draw a sample of 5000 values of the assets, a sample of the real and of the nominal liabilities of the year $t+k$, since the indexation ladder function is given by

$$i_{ladder_{t+k}} = \min(\max(\frac{A_{ladder_{t+k}} - L^N_{t+k}}{L^R_{t+k} - L^N_{t+k}}\pi_{t+k}, 0), \pi_{t+k}) \tag{5.1}$$

with $k = 1, ..., 10$. $A_{ladder_{t+k}}$ denotes the total amount of assets which is calculated from Equation (2.9) with the conditionally indexed Benefit Payment BP_{t+k}. The determination of A_{ladder} and i_{ladder} has to be bounded up together. The calculation procedures of L^N and L^R are illustrated in Chapter 2. The sample values of the inflation rate are simulated from the VAR model in the real world.

The age-dependent indexation function under the **P** measure is given by

$$i_{ADI_{x,t+k}} = \theta \cdot (c + \Gamma x_{t+k-1} + \Sigma \zeta_{t+k}) \quad x \in [25, 64] \tag{5.2}$$

where θ is given in Equation (4.4). The real-world return scenarios have been simulated in the first step. Based on the 5000 sample indexation-rate observations $i_{ADI_{x,t+k}}$, we can draw a sample of 5000 values of the assets of the ADI plan $A_{ADI_{t+k}}$.

Further, based on the sample values of $i_{ADI_{x,t+k}}$, we can build a sample of the indexation rate for the collar contingent ADI plan by using the formula below

$$i_{coADI_{x,t+k}} = \min(\max(i_{ADI_{x,t+k}}, 0), i_{cap^k}) \quad x \in [25, 64] \tag{5.3}$$

where i_{cap^k} is listed in Table 4.1. Given the sample indexation rate of the year $t + k$, we can build a sample of 5000 values of the assets $A_{coADI_{x,t+k}}$ for the collar contingent ADI plan.

Third, we shall build a sample of the realized liabilities. The most difficult part when calculating the realized liability is to measure the individual accrual pension rights $B_{x,t+k}$. Equation (4.6) shows the complete equation under **Q** measure. In ALM study however, Equation (4.6) is still a correct way to define the pension rights, but all the return dynamics in Equation(4.6) have to be replaced by the real world measure. The realized liability function is given in Equation (2.8). For a specific policy, the calculation procedures are presented in Appendix A.3, except that the nominal and the real discount rates are now estimated by the VAR model.

5.2 Classic ALM Study

In this section, we will show some classic ALM study results. We investigate the distribution of the indexation quality, of the pension result and of the funding ratio.

5.2.1 Indexation Quality

The value of the Indexation Quality (IQ) is calculated by

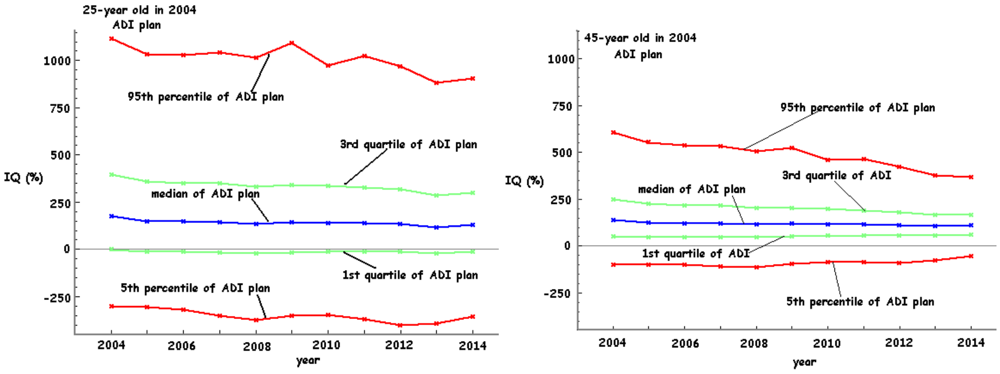
$$IQ^k = \frac{i_{t+k}}{\pi_{t+k}}$$

$IQ_{t+k} = 100\%$ means that the value of the liabilities of the year $t+k$ is fully indexed. $IQ_{t+k} = 300\%$ indicates that the indexation rate is three times as large as the inflation rate in the year $t + k$.

Figure 5.1 to 5.3 present the indexation quality distribution results of different indexation policies. Figure 5.1 is the result of the ADI plan, where panel (a) traces the movement of the indexation quality (IQ) distribution over the next 10 year for the participants who are 25-year old at the policy switching year 2004, and panel (b) is for those who are 45-year old members in 2004. The two red curves with small crossing signs plot the 5^{th} and the 95^{th} percentile of the next 10 years and the blue one is median curve. The light green curves are the 1^{st} and the 3^{rd} quartile. It is obtained from Figure 5.1 that the IQ distribution of the 25-year old sample spreads wider than the 45-year old sample and the 5th percentile-time curve of both panels is below zero. We also find that the percentile-time curves have a slight tendency to converge in the long run.

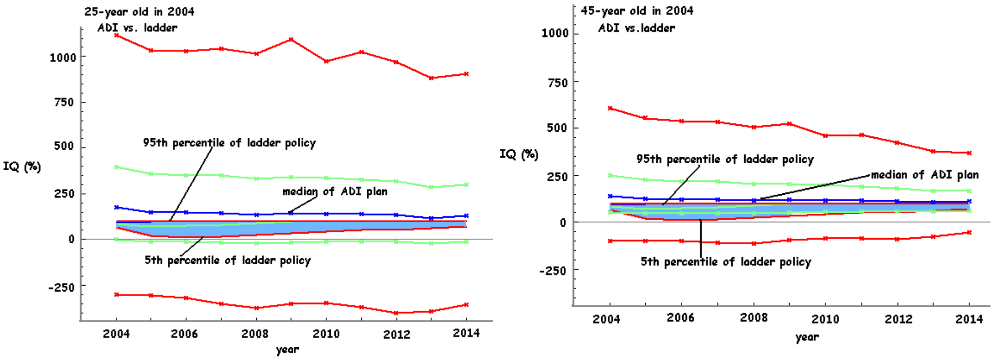
Figure 5.2 compares the (IQ) distribution of the ADI plan with the policy ladder. The narrow blue area represents the observations between the 5^{th} and the 95^{th} percentile of the sample indexation quality of the policy ladder. Since the policy ladder is not an age-dependent policy, the blue area in panel (a) is identical to the one in panel (b). Compared with the ADI plan, the policy ladder is very stable and its distribution curves is converging in the long run. The upside of the blue area (the 95^{th} percentile-time curve) is a horizontal line and is always equal to 100, for the upper bound of the conditional indexation rate, according to the policy ladder, must be equal to the inflation rate. The median curve of the ADI plan for both the 25-year old and the 45-year old members is above the 95^{th} percentile-time curve of the ladder policy because of the overwhelming upside rewards.

The comparison between the ADI plan and the collar contingent ADI plan is displayed in Figure 5.3. The grey area is referred to as the magnitude of the collar effect. The collar effect measures the size of the given-up downside risk and the amount of the deserted upside rewards from the risky investment after adding the indexation collar. There is no surprise that the collar constraint avoids the negative indexation and reduces the indexation risk. Because of the decreasing indexation risk allocation in ADI plan, the collar effect is stronger among the young participants than the old. Another finding is that though the indexation collar reduces the indexation risk of the ADI plan sharply, the collar contingent ADI plan is still risker than the policy ladder.



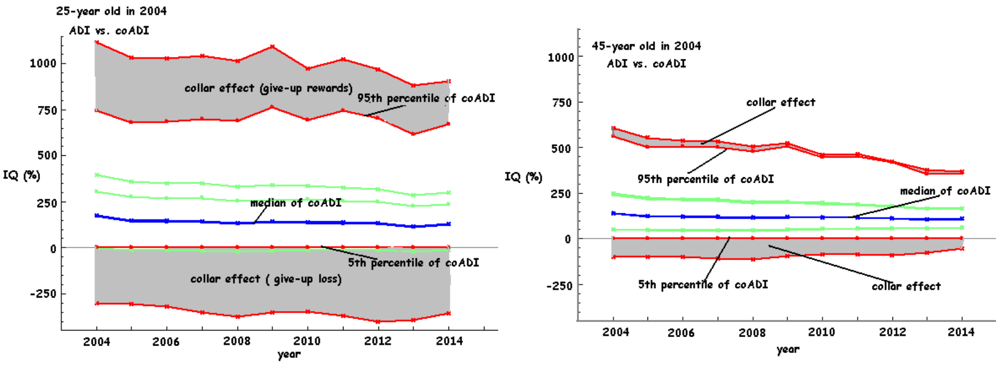
(a) (b)

Figure 5.1: Indexation Quality (**in percentage** %) of ADI plan



(a) (b)

Figure 5.2: Indexation Quality (**in percentage** %) of ladder policy vs. ADI plan



(a) (b)

Figure 5.3: Indexation Quality (**in percentage** %) of ADI plan vs. collar contingent ADI plan

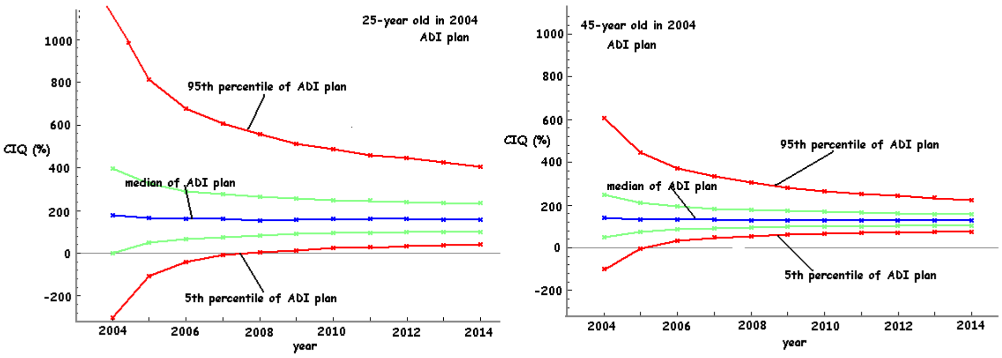
Participants prefer the cumulative indexation result to the year-by-year figure, because the current loss is likely to be smoothed by the future investment performance. Therefore, we also simulate the cumulative indexation quality in favor of the participants' interest . Cumulative Indexation Quality (CIQ) is calculated by

$$CIQ^k = \frac{\prod_{u=t+1}^{t+k}(1+i_{t+u})-1}{\prod_{u=t+1}^{t+k}(1+\pi_{t+u})-1} \tag{5.4}$$

Figure 5.4 to 5.6 show the cumulative distribution results corresponding to the previous three figures. Figure 5.4 presents the CIQ distribution curves of the ADI plan for the 25-year old and the 45-year old participants in 2004. The funnel shaped cumulative distribution curves are converging fast with a disappearing negative area. The funnel in panel (a) is wider than in panel (b), for the indexation risk with regard to the ADI plan is decreasing with age.

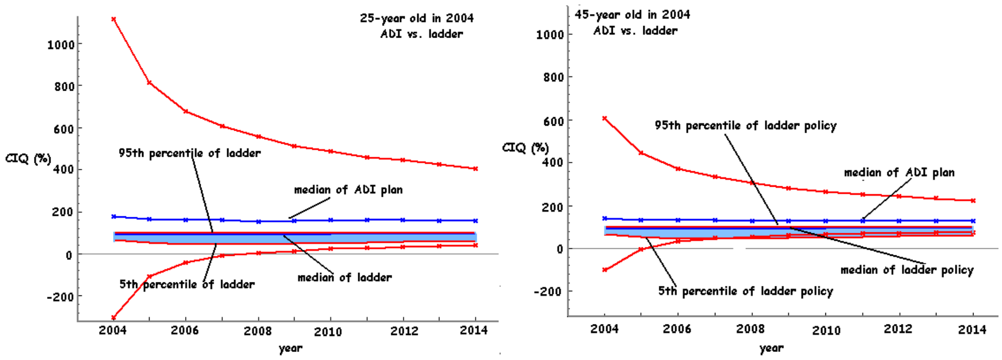
Figure 5.5 compares the (CIQ) distribution between the ADI plan and the policy ladder. The blue area again measures the span between the 5^{th} and the 95^{th} percentile-time curves of the ladder policy. Since the CIQ distribution curves of the ADI plan converge fast over time, the difference between the ADI plan and the policy ladder is getting smaller in the long run. From panel (b) of Figure 5.5 we find that the 5^{th} percentile-time curves of the two policies are almost overlapped ever since 2006. It is also obtained that the median of the sample CIQ of the ADI plan in both panels are always higher than the 95^{th} percentile of the policy ladder.

The CIQ comparison between the ADI plan and collar contingent ADI plan is displayed in Figure 5.6. The grey area is the collar effect. After adding collar, the funnel shaped distribution curves of the both selected age cohorts are getting narrower while the collar effect towards the young is always larger than towards the old.



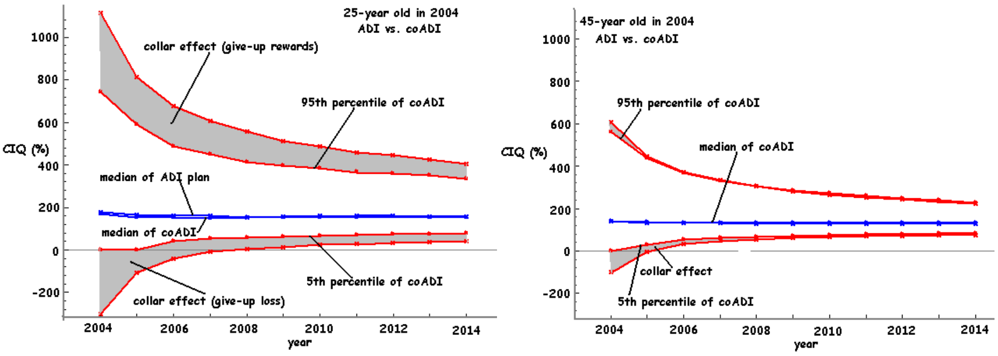
(a) (b)

Figure 5.4: Cumulative Indexation Quality (**in percentage** %) of ADI plan



(a) (b)

Figure 5.5: Cumulative Indexation Quality (**in percentage** %) of ladder policy vs. ADI plan



(a) (b)

Figure 5.6: Cumulative Indexation Quality (**in percentage** %) of ADI plan vs. collar contingent ADI plan

5.2.2 Pension Result

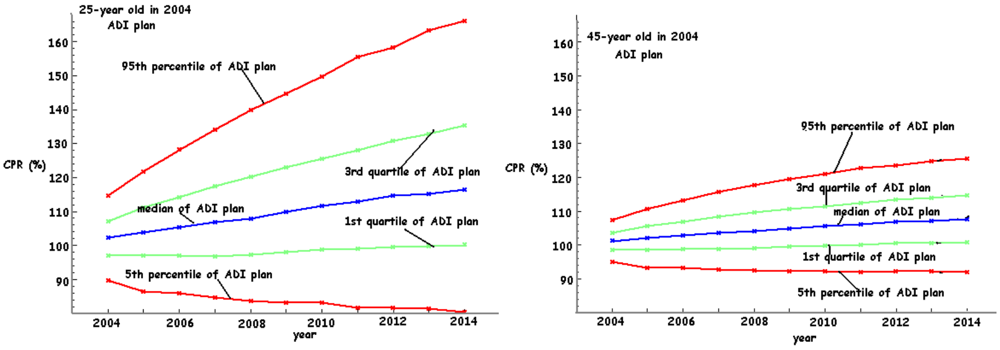
Now let's consider the cumulative pension result distribution. The cumulative pension result CPR is calculated by

$$CPR^k = \frac{\prod_{u=t+1}^{t+k}(1+i_{t+u})}{\prod_{u=t+1}^{t+k}(1+\pi_{t+u})} \tag{5.5}$$

Figure 5.7 shows the distribution result of the ADI plan for the participants who are 25-year old and 45-year old in 2004 respectively. In general, the CPR distribution curves of both cohorts are spreading over time. The spread in panel (a), however, is wider than the spread in panel (b). That means switching the indexation policy from the traditional DB scheme to the ADI plan brings more downside risk as well as upside rewards to the young participants than to the old.

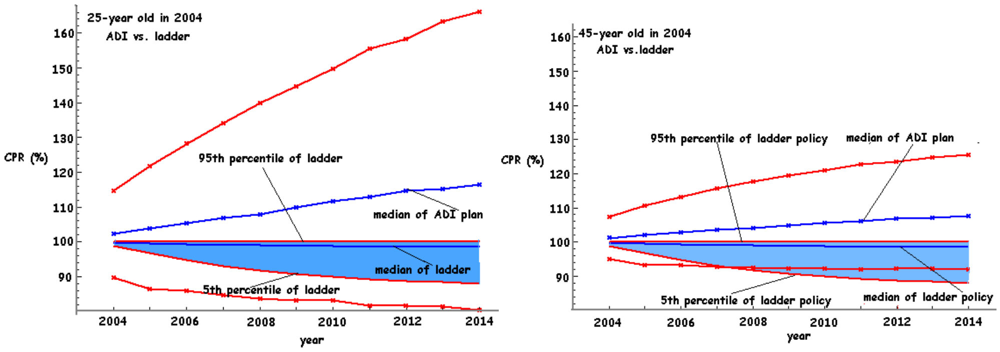
Figure 5.8 compares the CPR distribution between the ADI plan and the policy ladder (shown in blue area). It is obtained that the policy ladder does not offer the upside rewards to the participants since the 95^{th} percentile-time curve which is the upside of the blue area is always equal to 100%. The downside risk of the policy ladder is increasing over time. Compared with the policy ladder, the young participants of the ADI plan have to borne more downside risk since the 5^{th} percentile-time curve of the ADI plan for the 25-year old participants is below the policy ladder's. However, the downside risk of the ADI plan for the 45-year old participants is getting smaller than the policy ladder's after switching the policy for 3 years.

Figure 5.9 compares the CPR distribution of the ADI plans with and without collar. The grey area represents to the magnitude of the given-up risk after adding the indexation collar. The collar effect for the 45-year old participants is diminishing over time. The median curves of the ADI plans with and without collar are almost the same.



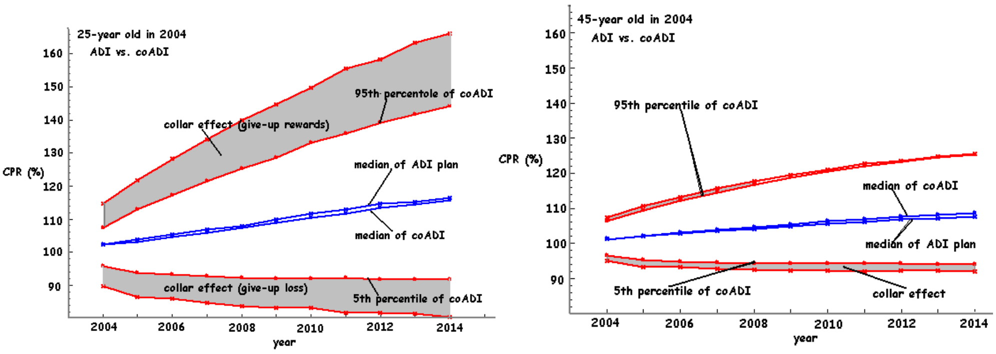
(a) (b)

Figure 5.7: Cumulative Pension Result (**in percentage** %) of ADI plan



(a) (b)

Figure 5.8: Cumulative Pension Result (**in percentage** %) of ladder policy vs. ADI plan



(a)

(b)

Figure 5.9: Cumulative Pension Result (**in percentage** %) of ADI plan vs. collar contingent ADI plan

5.2.3 Funding Ratio

Funding ratio is the fraction of the market value of the assets and the market value of the liabilities. The value of the assets is various from variant to variant. In Section 5.1, we simulated 5000 values of the assets of the next 10 years for the policy ladder. Since the policy ladder is not age dependent, the retirees also have the same conditional indexation function as the young do. The retirees of the two ADI plans however, are fully indexed. Hereby the expected assets value of the ADI plans must be lower than the value of the policy ladder. Equation (2.8) formulates the liability function. The value of the liabilities in different indexation policies is explained in Appendix A.3. The promised benefit before 2004 is fully indexed and depends on the switched policy variant after 2004. The sample nominal discount rate is drawn from the nominal term structure (see Equation (A.16)).

Figure 5.10 plots the funding ratio distribution result of the three indexation policies. The difference between the ADI plan and the collar contingent ADI plan is almost invisible. The ADI plans have a slightly lower probability of overfunding than the policy ladder, but a higher probability of low underfunding. The result is reasonable, because the ADI plans allocate more risk on the pension labilities, therefore the expected values of the liabilities of the ADI plans are higher than the value of the policy ladder. A high value of the liabilities leads to a low funding ratio. From Figure 5.10, we can not detect the funding ratio volatility change resulting from the policy switching, so we need to check the numerical result.

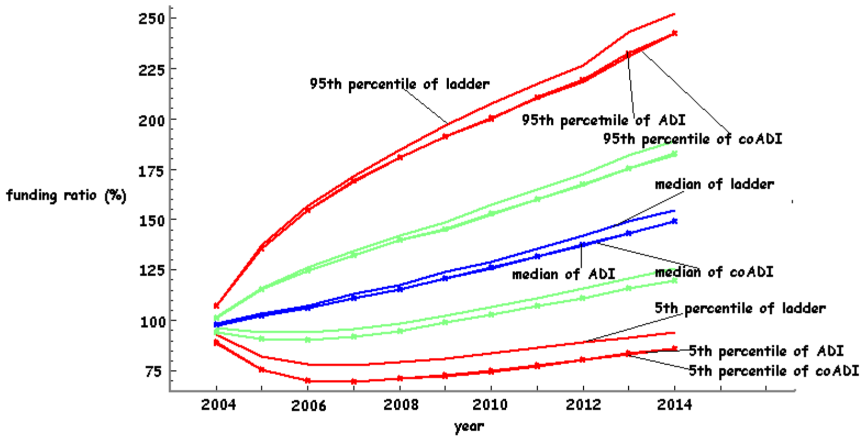


Figure 5.10: Funding ratio distribution (**in percentage** %) of three indexation policies

Table 5.1 reports the median, the mean and the standard deviation of the simulated sample funding ratio of each policy variant from the year 2004 to 2014. It is obtained that both the median and the mean are increasing over time. The mean of the sample funding ratio of each variant is always larger than its median and the difference is getting bigger over time. The growing gap between the mean and the median indicates that the funding ratio distribution is skewed, and the skewness enlarges over time. We also observe that the mean and the median of the policy ladder are always bigger than the values of the other two ADI plans. The ADI plan without collar always has a larger mean than the ADI plan with collar, while the order of the median of the two ADI plans is not consistent over time.

The funding ratio volatility of each policy is increasing over time but with a different growth rate. In the policy switching year 2004, the two ADI plans have higher values of the funding ratio volatility than policy ladder. This result is in line with our findings from the single period model (see Table 3.3). In the long run, however, the growth rate of the funding ratio volatility of each ADI plan is lower than the growth rate of the policy ladder. After switching the policy for 6 years (in 2009), the funding ratio volatility of the policy ladder firstly excesses the ADI plans and the difference is increasing over time. It is also obtained that the collar contingent ADI plan has the lowest volatility among the three ever since 2009. The result implies that the collar strategy improve the funding risk management and is more attractive in the long run.

year		**Variant**		
		ind.ladder	**ADI**	**co.ADI**
2004	median	98.586	97.662	97.651
	mean	99.234	97.800	97.800
	stdev.	4.2819	5.6036	5.5149
2005	median	103.14	102.37	102.38
	mean	105.79	103.45	103.46
	stdev.	16.605	18.068	18.045
2006	median	107.37	105.91	105.98
	mean	111.50	108.62	108.62
	stdev.	24.501	25.775	25.743
2007	median	112.92	110.97	111.02
	mean	117.22	113.89	113.87
	stdev.	29.180	30.206	30.161
2008	median	117.66	115.44	115.24
	mean	122.91	119.14	119.09
	stdev.	32.999	33.726	33.641
2009	median	124.11	120.65	120.56
	mean	128.86	124.61	124.53
	stdev.	35.939	36.258	36.149
2010	median	129.16	125.98	126.04
	mean	135.05	130.28	130.15
	stdev.	38.699	38.622	38.493
2011	median	135.45	131.80	131.40
	mean	141.50	136.22	136.04
	stdev.	41.437	40.968	40.770
2012	median	141.93	137.56	137.17
	mean	148.05	142.10	141.87
	stdev.	44.014	43.217	43.021
2013	median	148.93	143.39	142.89
	mean	155.14	148.66	148.37
	stdev.	47.309	46.149	45.905
2014	median	154.58	149.33	148.94
	mean	161.63	154.53	154.18
	stdev.	50.336	48.728	48.520

Table 5.1: Funding Ratio distribution: median, mean and standard deviation of three indexation policies (**in percentage** %). ind.ladder: Indexation Ladder; ADI: ADI plan; co.ADI: collar contingent ADI plan

The funding ratio distribution result shown in the multi-period model partially contradicts the result from single-period model and is not totally in line with Molenaar, Munster and Ponds (2008) neither. The major reason for the disagreement is the time-horizon of the model. In short, we draw the conclusion that the funding ratio volatility of ADI plans, in the short run, is higher than the volatility of the policy ladder, but because of the low growth rate, its value is getting lower than the policy ladder's in the long run. Another common conclusion can be drawn from both the single and the multi-period models is that adding collar to the ADI plan does not cause much change on the funding ratio distribution.

One issue, which is still not clear to us, is the ranking of the funding-ratio distribution of the variants. Figure 3.5 shows that ADI plans have globally higher funding ratio distribution curves than the policy ladder, while Figure 5.10 tells an opposite story. Molenaar, Munster and Ponds (2008), however, find that the probability of both low underfunding and high overfunding of the ADI plan is lower than the policy ladder. That mean the 5^{th} percentile of the ADI plan is higher than the value of the policy ladder. We temporally believe that different model settings trigger this difference.

5.3 Utility Analysis

Funding ratio is an important criterion to evaluate the solvency position, however, the participants care more about their own pension rights rather than the gross situation of the fund. Therefore, we will carry out a utility analysis to evaluate the individual pension investment performance for each policy variant. In this paper, we believe that the individual risk preference point is irrelevant to the performance of the investment. Therefore, a constant relative risk aversion (CRRA) utility function which is widely used in literatures is chosen to measure the level of the satisfaction from different pension policies. CRRA function is given by

$$u(B_t) = \frac{B_t^{1-\gamma}}{1-\gamma} \tag{5.6}$$

where B_t is the accrual pension rights at time t. γ, the risk aversion level, is assumed constant regardless of the exogenous influence. The total expected utility of the x-year old participants, generated by the accrual pension rights of the next ten years is calculated by

$$U_x = E[\sum_{k=1}^{10} \exp(-\beta k) \frac{B_{x+k,t+k}^{1-\gamma}}{1-\gamma}] \tag{5.7}$$

where $\beta = 3\%$ is an approximate discount rate, $\gamma \in [1, 10]$, and $k = 1, ..., 10$. $B_{x+k,t+k}$ measures the total accrual pension right of the participants who are x-year old at time t. The value of the pension rights depend on the implementing policy variant. We have already drawn a sample of the pension rights $B_{x+k,t+k}$ for each policy variant when calculating the realized liability, so the expectation value of Equation (5.7) is

$$\widehat{U}_x = \frac{1}{N}\sum[\sum_{k=1}^{10} \exp(-\beta k)\frac{B_{x+k,t+k}^{1-\gamma}}{1-\gamma}]$$

To facilitate the policy comparison, we translate the utility result into the Certainty Equivalent Benefit (CES), which can easily be backed out from Equation (5.7) by solving the following equation for (CES)

$$\widehat{U}_x = E[\sum_{k=1}^{10} \exp(-\beta k)\frac{CEB^{1-\gamma}}{1-\gamma}] \tag{5.8}$$

age	CEB_{ADI} $\backslash CEB_{ladder}$	$CEB_{co.ADI})$ $\backslash CEB_{ladder}$
	γ=2	
(25)	**100.74**	100.66
(35)	100.55	100.61
(45)	**99.904**	**99.993**
(54)	106.44	106.46
	γ=5	
(25)	100.07	100.07
(35)	101.05	101.30
(45)	100.36	100.50
(54)	104.76	104.79
	γ=8	
(25)	100.01	100.01
(35)	100.97	101.45
(45)	100.55	100.76
(54)	103.52	103.56

Table 5.2: Relative Certainty Equilibrium Benefit CEB of various age cohorts (**in percentage**). ladder: Indexation Ladder; ADI: ADI plan; co.ADI: collar contingent ADI plan

Table 5.2 presents the relative CEB results for four age cohorts (in percentage). We compared the CEB of the two ADI plans with the policy ladder by using three different risk aversion values. The higher the γ, the more averse to the risk. A high certainty equivalent benefit is desirable, for the participants are looking forward to receive more retirement income. The first column reports the $CEB_{ADI}\backslash CEB_{ladder}$ ratio, where CEB_{ADI} is the certainty equivalent benefit of the ADI-plan participants and CEB_{ladder} is the benefit of the policy-ladder participants. If the result is bigger than 100%, then the ADI-plan participants are better off. If the result is smaller than 100, then the policy ladder is more beneficial. The second column reports the $CEB_{co.ADI}$ over CEB_{ladder} ratio, where $CEB_{co.ADI}$ represents to the certainty equivalent benefit of the ADI-plan participants with collar constraint. For majority of the outcomes, we conclude the following inequal relation between the three.

$$CEB_{\text{ladder}} \quad \leq \quad CEB_{\text{ADI}} \quad \leq CEB_{\text{co.ADI}} \tag{5.9}$$

The two exceptions with the CEB values below 100% indicate that it is wise for the 45-year old risk lovers to choose policy ladder instead of the other two ADI plans. The other exception with value 100.74% suggests that the young risk lovers are better off if they choose un-collared ADI plan but not the collared one at the policy switching moment. Apart from the three exceptions, the ADI plan with collar is always the most beneficial one amount the three.

Chapter 6

Conclusion and Further Research

According to life cycle theory, a pension fund can optimally allocate risk to the various age groups by using an age-dependent indexation policy. Implementing such a policy is especially desirable for maturing funds. However, the basic policy variant is rather risky for the young participants. This paper aims to improve the age-dependent indexation policy by avoiding negative indexation and reducing the volatility, while keeping the advantages of age differentiation. We do this by designing the so-called indexation collar, which is some sort of contingent claim. An indexation collar requires no up-front costs to be implemented, and will prevent the occurrence of negative indexation. The collar will be evaluated in two ways; a simple and fast way, and a more complicated and accurate way. The simple evaluation method uses a Black Scholes model with a fixed inflation rate and a fixed risk free rate. These simple financial settings enable us to price the collar analytically, but the analytical formula will not very accurately described prices in the real world. The complex pricing method uses a complex multi-period model with nine state variables which dynamics are described by a first order VAR model.

In order to evaluate policy variants, one often makes use of ALM studies. Using such studies, we investigate the policy ladder, the ADI-plan, and the collar contingent ADI plan. We calculate the distributions of the indexation and the funding ratio, and we also evaluate the certainty-equivalent benefit by performing a utility-based ALM study. Both single-period and multi-period studies show that by adding a collar to an ADI plan the volatility of the indexation is reduced. Nevertheless, for young participants the collar contingent ADI plan is still riskier than a plan with a policy ladder. The volatility of the funding ratio depends on the time horizon of the model. On short time horizons, the funding ratio volatility of ADI plans is larger than the funding ration volatility of plans with a policy ladder, but on long time horizons it is the other way around. Summarizing we may say that utility-based ALM studies indicate that in general, collar contingent ADI plans are more beneficial to the participants than the other two policy variants.

In future research we plan to explore two issues. The first issue is the intergenerational transfer of welfare. Hoevenaars and Ponds (2008) state that policy changes may inadvertently reallocate welfare among the generations. The essence of the ADI plan is to reconcile intergenerational solidarity with the recommendations of life cycle investment theory. Whether this feature of the ADI plan is maintained by the collar contingent ADI plan is still an open question. To find the answer we have to evaluate the intergenerational risk transfer using value-based ALM studies. The second issue we would like to explore in the future is the influence of the choice of the utility function. In this paper we have chosen the classic CRRA utility function which has only one reference point, and which is based on the assumption that the reference framework for making a decision is independent of external influence. This assumption, however, is unrealistic since it goes against human nature. Kahneman and Tversky (1979) are famous for their prospect theory which states that people's altitude to gain differs to their altitude to loss. Holding the believe of single reference point, two well-known utility functions which also have one reference point can be used in the future research. One is based on the prospect theory. The function distinguishes the utility of loss from the utility of gain while still keeps single reference point. The other is called Epstin-Zin (1989,1991) utility function, which is designed to disentangle the elasticity of inter temporal substitution from the coefficient of the relative risk aversion.

References

[1] Angerman, H.J, 2004, Pricing derivatives in a VAR model, Internel Research Paper, APG.

[2] Angerman, H.J., Hoevenaars, R.P.M.M., Molenaar, R.D.J., Steenkamp, T.B.M., 2006, Scenario generation at APG, Internel Research Paper, APG.

[3] Binsbergen, J.H., Brandt, M.W., 2007, Optimal asset allocation in asset and liability management, working paper, Duke University.

[4] Black, F. and Scholes, M., 1973, The pricing of options and corporate liabilities, *Journal pf Prlitical Economy*, **81**, 637-659.

[5] Blake, D., 1998, Pension schemes as options on pension fund assets: implication for pension fund management, *Insurance: Mathematics and Economics*, vol. **23**, 263-286.

[6] Brennan, M.J. and Xia, Y., 2002, Dynamic asset allocation under inflation, *Journal of Finance*, **57**, 1201-1238.

[7] Broeders, D., 2006, Valuation of conditional pension laibilities and guarantees under sponsor vunlnerability, DNB working paper.

[8] Campbell, J.Y., Viceira, L.M., 2002, *Strategic asset allocation: portfolio choice for long-term investors*, Oxfors University Press.

[9] Campbell, J.Y., Chan, Y.L., Viceira, L.M., 2003, A multivariate model of strategic asset allocation, *Journal of Financial Econimics*, **67**(1), 41-80.

[10] Campbell, J.Y., Viceira, L.M., 2005, The term structure of the risk-return tradeoff, *Finanical Analysts Journal*, **61**, 34-44

[11] Cochrane, J.H. and Piazzesi, M., 2005, Bond risk premia, *American Economic Review*, vol. **95**(1), 138-160.

[12] Cui, J., De Jong,F. and Ponds, E.H.M., 2006, *Intergenerational risk sharing in collective pension schemes*, working paper Netspar, Tilburg University.

[13] Cui, J., 2008, DC pension plan defaults and individual welfare,

[14] Dai, R., 2006, Design of defined-contribution pension schemes: an option perspective, Tilburg University.

[15] Dai, R. and Schumacher, J.M., 2008, Valuation of contingent pension liabilities and implementation of conditional indexation, Tilburg University, mimeo, **8**, 321-350.

[16] Dai, R. and Schumacher, J.M., 2009, Welfare analysis of conditional indexation schemes from a two-reference-point perspective, *Journal of Pension Economics and Finance*, **8**, 321-350.

[17] De Jong, F., 2005, Valuation of pension liabilities in incomplete markets, DNB working paper.

[18] De Jong, F., 2008, Pension fund investment and the valuation of liabilities under conditional indexation, *Insurance: Mathematics and Economics*, **42**, 1-13.

[19] De Jong, J.J, Mens, H., Oerlemans, A., and Potters, J., 2008, Het nut van risicodeling bij beschikbare premieregelingen, *VBA journaal*, **1**, 25-33.

[20] Epstein, L. G. and Zin, S. E., 1989, Substitution, risk aversion, and the temporal behavior of consumption and asset returns: a theorical framework, *Econometrica*, **57**,937-969.

[21] Hoevenaar, R.P.M.M., Ponds, E.H.M., 2008, Valuation of international transfers in funded collective pension schemes, *Insurance: Mathematics and Economics*, **42**, 578-593.

[22] Hoevenaars, R.P.M.M., Molenaar, R.D.J., Schotman, P., Steemkamp, T.B.M., 2008, Strategic asset allocation with liabilities: beyond stock and bonds, *Journal of Economic Dynamics and Control*, **32**9, 2939-2970.

[23] Hoevenaars, R.P.M.M., 2008, *Strategic asset allocation and asset liability management* Maastricht University Press.

[24] Haneveld, W.K.K., Streutker, M.H. and Van der Vlerk, M.H., 2007, *ALM modeling for Dutch pension funds: Indexation and new regulatory rules*, University of Groninggen.

[25] Jagannathan, R. and Kocherlakota, N.R., 1996, Why should older people invest less in stocks than younger people? *Federal Reserve Bank of Minneapolis Quarterly Review*, Vol.**20**, no.3, 11-23.

[26] Kahneman, D. and Tversky, A., 1979, Prospect theory: an analysis of decision under risk, *Econometrica*, **47**, 263-290.

[27] Kortleve, N. and Ponds, E.H.M., 2006, *Pension deals and valued-based ALM*, Elsevier Science, Fair Value and Pension Fund Management, 2006, Editor: Kortleve, N. Nijman, T.E. and Ponds, E.H.M.

[28] Lachance, M.E., Mitchell, O.S. and Smetters, K., 2003, Guaranteeing defined contribution pensions: the option to buy back a defined benefit promise, *The Journal of Risk and Insurance*, Vol. **70**, no.1, 1-16.

[29] Leibowitz, M.L., Kogelman, S. and Bader, L.N., 1994, Funding ratio return, *Journal of Portfolio Management*, 39-47.

[30] Markowitz, H., 1952, Portfolio selection, *Journal of Finance*, **7**,77-91.

[31] Molenaar, R.D.J., Vlaar, P.J.G., 2009, *Heteroscedastic scenario generation at APG*, Internel Research Paper, APG.

[32] Molenaar, R.D.J., Munsters, R. and Ponds, E.H.M., 2008, *Towards age-differentiation in funed collective pensions*, NEA paper no.7, Netspar.

[33] Nijman, T. and Koijen, R.J., 2006, *Valuation and risk management of inflation-sentitive pension rights*, Elsevier Science, Fair Value and Pension Fund Management, 2006, Editor: Kortleve, N. Nijman, T.E. and Ponds, E.H.M.

[34] Nijman, T., 2008, *Introduction to term structure modelling* Lecture Notes for the course Asset Libaility Management, Tilburg Unicersity.

[35] Ponds, E.H.M. and Riel, B., 2007, The recent evolution of pension funds in the Netherlands: The trend to hybrid DB-DC plans and beyond, *Center for Retirement Research*, http://www.bc.edu/crr.

[36] Ponds, E.H.M. and Riel, B., 2007, ISSUES and POLICY, Sharing risk: the Netherlands' new approach to pensions, *Cambridge University Press*, **8**(1), 91-105.

[37] Ponds, E.H.M., 2008, *Naar meer jong en oud in collectieve pensioenen*, oratie ABP-Netsapr leerstoel: Economie van Collectieve Pensioencontracten, Universiteit van Tilburg.

[38] Schumacher, J.M., 2008, *Financial Models* Lecture Notes for the course Financial Models, Tilburg University.

Appendix A

This Appendix is a quantitative complement. Some missing calculations, skipped theoretical reasonings from the main body can be found here.

A.1 Equivalent Martingale Measure

The stochastic process changes can be obtained via Girsanov's theorem. Suppose $\widetilde{W}$ is a Brownian motion under risk neutral measure **Q**. The process $\widetilde{W}_t$ is defined by

$$d\widetilde{W}_t = \lambda dt + dW_t, \quad \widetilde{W}_0 = 0$$

where λ is a process adapted to W. According to absence of the arbitrage theorem, the standard Black-Scholes model is free of arbitrage if and only if there exists a $\lambda_B = \lambda_B(t, S, B)$ such that

$$\mu_{S/B} = \sigma_{S/B}\lambda$$

where

$$d(S/B) = \mu_{S/B}dt + \sigma_{S/B}dW$$

λ is the price of risk and it can be derived from "absence of arbitrage" theorem.
According to Ito's formula:

$$\begin{aligned} d\frac{S}{B} &= \frac{1}{B}dS - \frac{S}{B^2}dB \\ &= \frac{1}{B}(\mu S dt + \sigma S dW) - \frac{S}{B^2}rBdt \\ &= \frac{S}{B}[(\mu - r_n)dt + \sigma dW] \qquad \text{(A.1)} \end{aligned}$$

So (A.1) tells

$$\lambda = \frac{\mu - r_n}{\sigma}, \qquad \text{and} \quad d\widetilde{W}_t = \frac{\mu - r_n}{\sigma}dt + dW_t.$$

A.2 Solving Non-Linear Equation

Solving non-linear equation in OX follows Zero-Finding strategy.
We are going to solve Equation (2.11)

$$\begin{aligned} -\Phi(d_1)(\pi - \frac{1}{2}k\alpha^2\sigma^2) + \frac{k\alpha\sigma}{\sqrt{2\pi_o}}\exp(-\frac{d_1^2}{2}) \\ = \Phi(-d_2)(\pi - \frac{1}{2}k\alpha^2\sigma^2 - i_{cap}) + \frac{k\alpha\sigma}{\sqrt{2\pi_o}}\exp(-\frac{d_2^2}{2}) \end{aligned}$$

for i_{cap}. Let

$$x = \Phi(-d_2), \qquad y = i_{cap}, \qquad z = \exp(-\frac{d_2^2}{2})$$

x, y, z are three unknown variables of Equation (2.10), so the problem moves from solving i_{cap} to determining the value of the three newly defined variables x, y and z. Suppose

$$\begin{aligned} m &= \pi - \frac{1}{2}k\alpha^2\sigma^2, \qquad n = \frac{k\alpha\sigma}{\sqrt{2\pi}} \\ \text{then} \quad d_1 &= -\frac{m}{k\alpha\sigma}, \qquad d_2 = \frac{y-m}{k\alpha\sigma} \end{aligned}$$

Apparently, the values of m, n and d_1 are all certain, while d_2 consists of an unknown variable y. Then equation (2.11) becomes

$$-\Phi(d_1)m + n\exp(-\frac{d_1^2}{2}) = x(m-y) + nz \tag{A.2}$$

Then we are going to solve the following three non-linear equations for x, y and z

$$\begin{cases} 0 = -\Phi(-\frac{m}{k\alpha\sigma})m + n\exp(-\frac{(-\frac{m}{k\alpha\sigma})^2}{2}) - x(m-y) - nz \\ 0 = x - \Phi(-\frac{y-m}{k\alpha\sigma}) \\ 0 = z - \exp(-\frac{(\frac{y-m}{k\alpha\sigma})^2}{2}) \end{cases} \tag{A.3}$$

The next step is to solve Equation (A.3) for x, y, z, where $y(= I_{cap})$ is the final target of the investigation.

A.3 Calculating Realized Liability

In this section, we are going to complete the calculation of the realized liabilities for each variant.

Realized Liability The value of the realized liabilities of the next period is closely related to the value of the real liability L^R. Recall, the formula for calculating real liability at time t is given by

$$\begin{aligned} L_t^R &= \sum\nolimits_{x=25}^{64}\sum\nolimits_{s=65-x}^{84-x}[B_{x,t}^R \exp(-r_n s)] \\ &+ \sum\nolimits_{x=65}^{84}\sum\nolimits_{s=x}^{84} B \exp[-r_r(s-x)] \end{aligned}$$

Let's affine the long equation to

$$L_t^R = X_t + Y_t$$

where X_t is the real value of liabilities from the accumulative phase, and Y_t represents to liability of decumulation phase. Then the total value of the liabilities of next period can simply be written as

$$L_{t+1}^R = X_{t+1} + Y_{t+1} = X_t \exp(-r_n) + Y_t \exp(-r_r)$$

The complete formula for realized liabilities is

$$\begin{aligned} L_{t+1} &= \sum\nolimits_{x=25}^{64}\sum\nolimits_{s=65-x}^{84-x} B_{x,t+1} \exp[-r_n(s+1)] \\ &+ \sum\nolimits_{x=65}^{84}\sum\nolimits_{s=x}^{84} B \exp[-r_r(s-x)] \exp[-(r_n - i_{x,t+1})] \end{aligned}$$

where $B_{x,t+1} = B_{x,t+1}^R \exp(-\pi + i_{x,t+1}) + \triangle B(1 - \exp(-\pi + i_{x,t+1}))$

Policy Ladder Policy ladder requires a uniform indexation rate for all the participants, so $i_{x,t+1} = i_{t+1}$, then the value of its realized liabilitie L_{t+1}^{ladder} is given by

$$\begin{aligned} L_{t+1}^{ladder} &= X_{t+1} \exp(-\pi + i_{ladder,t+1}) \\ &+ Z(1 - \exp(-\pi + i_{ladder,t+1})) + Y_{t+1} \exp(-\pi + i_{ladder,t+1}) \\ &= (X_{t+1} + Y_{t+1} - Z) \exp(-\pi + i_{ladder,t+1}) + Z \end{aligned}$$

where $Z = \sum_{x=25}^{64}\sum_{s=65-x}^{84-x} \triangle B \exp[-r_n(s+1)]$

ADI plan In ADI plan, liabilities in accumulation phase are indexed by the age-dependent indexation rate. In decumulation phase, liabilities are fulled indexed. So the total realized liability in ADI plan is measure by

$$L_{t+1}^{ADI} = \sum\nolimits_{x=25}^{64}\sum\nolimits_{s=65-x}^{84-x} B_{x,t+1}^{ADI} \exp[-r_n(s+1)] + Y_{t+1} \tag{A.4}$$

where $B_{x,t+1}^{ADI} = (B_{x,t+1}^R - \triangle B) \exp(i_{ADI,x}) + \triangle B$

Collar contingent ADI plan Calculating the value of the realized liabilities for the collar contingent ADI plan uses the same measure as the previous one, the only difference is that the indexation function is replaced by i_{coADI}.

A.4 Scenarios Generation in APG

Angerman, Hoevenaars, Molenaar and Steenkamp (2008) elaborate the VAR model used for homoscedastic scenarios generation in APG. Later on, Molenaar and Vlaar (2009) extend to model to heteroscedastic environment. Here, we shortly summarize the VAR model in use and the term structure modeling.

VAR model at APG

This section describes the first order Vector Autoregression (VAR) model used in APG. VAR is used to capture the dynamics of economic variables. These variables are the inputs of an Asset-Liability Management(ALM) model. In most of the studies, VAR model is homoscedastic, that means the second moment of the model is assumed constant over time. However, some empirical evidences have shown that the covariances between variables are changing over time, as the result, APG has been planing to replace the homoscedastic model to heteroscedastic VAR model in recent months. VAR model at APG is an extension of Campbell and Viceira (2005) and Campbell, Chan and Viceira(2003), as it contains more parameters than the one shown in Campbell's paper. The extended VAR model enlarges the degree of freedom, so it is more complicated and more powerful.

Dynamic Variables Current first-order VAR model consists of two types of variables, namely the return variables and state variables.

State Variables s_t: State variables are the predictors of return variables. APG model contains 5 predictor variables, they are inflation rate, short term interest rate, dividend yield, excess return on stock market and credit spread.

Return Variables r_t: r_t is the log return of a certain asset. Let $r_{b,t}$ be the real return on T-bill, then the excess return of this asset is defined as $r_t - r_{b,t}$. Return variables are split into two groups: one group contains basic assets on stocks and bonds, the other group contains additional assets such as hedge funds, real estate and commodities.

Homoscedastic VAR Model Homoscedasticity in VAR model indicates an consistent correlation between dynamic variables over time. As return variables are split into two parts, the VAR model also consists of two part: one is called Core-VAR model, which contains the 5 state variables and 4 basic assets; the other is called Derived-VAR model, which contains 13 additional assets. Let Y be the gross set of dynamic variables involved in the VAR model; x is a set of 4+5=9 core variables; z is a set of 13 derived variables, then

$$y_t = \begin{pmatrix} x_t \\ z_t \end{pmatrix}.$$

Core-VAR Model The Core-VAR model is given by

$$x_{t+1} = c + \Gamma x_t + \Sigma\zeta_{t+1}, \qquad \zeta_{t+1} \sim N(0, I_n) \tag{A.5}$$

where x_{t+1} is a $(n\times1)$ vector. Γ is a block-triangular matrix. $c = (I - \Gamma)\mu$, where μ is a $(n\times1)$ vector of historical means. Σ is the covariance matrix. Under the assumption of homoscedasticity, this error term is restricted.

Derived-VAR Model The Derived-VAR model consists of 13 variables,the model can be written as

$$\begin{aligned} z_{t+1} &= \delta + D_0 x_{t+1} + H z_t + D_1 x_t + \Omega\eta_{t+1} \\ \eta_{t+1} &\sim N(0, I_m); \qquad Cov(\zeta_{t+1}, \eta_{t+1}) = 0 \end{aligned}$$

The derived variables are assumed having no dynamic effects to the core variables. D_0 and D_1 are $(m\times n)$ coefficient matrices. $H = diag\,(h_1, ..., h_m)$ is a diagonal matrix. $\Omega\eta_{t+1}$ is the shock term with zero mean and a diagonal covariance matrix Ω.

Complete Homoscedastic VAR Model The complete VAR model is the combination of Core and Derived VAR models, which can be written as

$$y_{t+1} = \Phi_0 + \Phi_1 y_t + u_{t+1} \tag{A.6}$$

where

$$\begin{aligned} \Phi_0 &= \begin{pmatrix} c \\ \delta + D_0 c \end{pmatrix}, \qquad \Phi_1 = \begin{pmatrix} \Gamma & 0 \\ D_0\Gamma + D_1 & H \end{pmatrix}, \\ u_t &= \begin{pmatrix} \Sigma\zeta_t \\ \Omega \end{pmatrix} \begin{pmatrix} \Sigma\zeta_t \\ \Omega\eta_t + D_0\Sigma\zeta_t \end{pmatrix} \end{aligned}$$

and u_{t+1} has mean zero and covariance matrix

$$\Sigma_{all} = \begin{pmatrix} \Sigma\Sigma^\top & \Sigma\Sigma^\top D_0^\top \\ D_0\Sigma\Sigma^\top & \Omega\Omega^\top + D_0\Sigma\Sigma^\top D_0^\top \end{pmatrix}$$

Term Structure

Pricing Kernel Pricing kernel is a popular measure for asset pricing. It can be treated both in a continuous-time framework and a discrete-time framework, while the latter one is more customary in pension modeling. Consider a two period economy, the additive utility over ex post consumption (C_t, C_{t+1}) is given by

$$u(C_t, C_{t+1}) = u(C_t) + \alpha E_t(u(C_{t+1})) \tag{A.7}$$

where α is time preference. The agent has the following optimization problem

$$\begin{aligned} \max \quad E(u(C_t, C_{t+1})) &= u(C_t) + \alpha E_t(u(C_{t+1})) \qquad \text{(A.8)} \\ \text{s.t.} \qquad\qquad C_t &= L_t - \beta P_t, \\ C_{t+1} &= L_{t+1} + \beta P_{t+1} \end{aligned}$$

where L_t is labor income at time t. P_t is the price of a stock at time t and the stock price at next period P_{t+1} is a random variable. β is the units of stock holding at the current moment. First order condition respect to β can help to solve the optimization problem.

$$\begin{aligned} P_t u'(C_t) &= E_t[\alpha u'(C_{t+1})P_{t+1}], \text{ so} \\ \Longrightarrow P_t &= E_t[\frac{\alpha u'(C_{t+1})}{u'(C_t)} P_{t+1}] \end{aligned}$$

Let $M_{t+1} = \frac{\alpha u'(C_{t+1})}{u'(C_t)}$, then

$$P_t = E_t[M_{t+1}P_{t+1}] \tag{A.9}$$

Equation (A.9) is the standard two-period pricing kernel model. M is called pricing kernel or stochastic discount factor. A crucial restriction of the pricing kernel model is the absence of arbitrage. The model is also applicable in the multi-period economy. Suppose the price of a n-period zero coupon bond is $P_t^{(n)}$, then the pricing kernel is given by

$$P_t^{(n)} = E_t[M_{t+1}P_{t+1}^{(n-1)}] \tag{A.10}$$

Assume that asset price and pricing kernel are both joint log-normal, so the pricing kernel model is

$$p_t^{(n)} = E_t[m_{t+1} + p_{t+1}^{(n-1)}] + \frac{1}{2} Var_t[m_{t+1} + p_{t+1}^{(n-1)}] \tag{A.11}$$

where $P_t \equiv ln[p_t]$, $M_{t+1} \equiv ln[m_{t+1}]$.

The yield of the n-period zero coupon bond is given by

$$-r_{n,t}^{n} = p_t^{(n)}/n$$

When n is 1, the 1-period zero coupon bond is risk free, so

$$\begin{aligned} p_{t+1}^{(0)} &= 0, \qquad \text{and} \\ -r_{n,t}^{1} &= p_t^{(1)} \\ &= E_t[m_{t+1}] + \frac{1}{2}Var_t[m_{t+1}] \end{aligned}$$

Cochrane and Piazzessi (2005) relates the log pricing kernel to the state variables

$$-m_{t+1} = \delta_0 + \delta_1^\top x_t + \frac{1}{2}\lambda_t^\top \lambda_t + \lambda_t^\top \zeta_{t+1} \tag{A.12}$$

where $r_{n,t}^1 = \delta_0 + \delta_1^\top x_t$ is one-period risk-free rate (short rate) and $\lambda_t = \lambda_0 + \Lambda_1 x_t$.

Nominal term structure The pricing kernel model implies that the log prices of n-period nominal zero coupon bond is an affine function of state variable x_t

$$p_t^{(n)} = -A_n - B_n^\top x_t \tag{A.13}$$

where A_n and B_n can be determined in the following steps

$$\begin{aligned} -A_n - B_n^\top x_t &= E_t[m_{t+1}] + E_t[p_{t+1}^{(n-1)}] + \frac{1}{2}Var_t[m_{t+1}] \\ &\quad + \frac{1}{2}Var_t[p_{t+1}^{(n-1)}] + Cov_t[m_{t+1}, p_{t+1}^{(n-1)}] \\ &= -\delta_0 - \delta_1^\top x_t + E_t[-A_{n-1} - B_{n-1}^\top x_{t+1}] \\ &\quad + \frac{1}{2}Var_t[-A_{n-1} - B_{n-1}^\top x_{t+1}] \\ &\quad + Cov_t[m_{t+1}, -A_{n-1} - B_{n-1}^\top x_{t+1}] \end{aligned}$$

Plug in the core-VAR model Equation (A.5) into the right side of the equation above, the we can get

$$\begin{aligned} -A_n - B_n^\top x_t &= -\delta_0 - \delta_1^\top x_t - A_{n-1} - B_{n-1}^\top (c + \Gamma x_t) \\ &\quad + \frac{1}{2}B_{n-1}^\top \Sigma\Sigma^\top B_{n-1} + B_{n-1}^\top \Sigma \lambda_t \end{aligned}$$

with $\lambda_t = \lambda_0 + \Lambda_1 x_t$

Formulas for A_n and B_n can be derived through coefficients matching,

$$A_n = \delta_0 + A_{n-1} + B_{n-1}^\top c - \frac{1}{2}B_{n-1}^\top \Sigma\Sigma^\top B_{n-1} - B_{n-1}^\top \Sigma \lambda_0 \tag{A.14}$$

$$B_n^\top = \delta_1^\top + B_{n-1}^\top \Gamma - B_{n-1}^\top \Sigma \Lambda_1 \tag{A.15}$$

Equation (A.14) and (A.15) guarantee the assumption of free of arbitrage of the objective model.
In general,

$$r_{n,t}^{n} = \frac{A_n}{n} + \frac{B_n^{\top}}{n} x_t \tag{A.16}$$

formulates all the nominal term structure of the interest rate.

Appendix B

This appendix shows some missing figures from the main body.

B.1 Histogram comparison between ADI plan and other three variants

This section shows some histograms of sample indexation rate of different policy variants. Figure B.1 compares the indexation distribution of the ADI plan and the policy ladder. We simulate 1000 scenarios. The x-axis of Figure B.1 is the value of the indexation rate and y-axis is the number of observations from the 1000 scenarios. ADI plan histogram is shown in red collar and policy ladder is in black. Since the ADI plan is age differentiated, we pick four active age cohorts out of forty to detect the indexation distribution. While policy ladder is a uniform indexation plan, that mean the histograms of policy ladder in the four panels of Figure B.1 are actually the same. It is obtained from Figure B.1 that histogram of ADI plan spreads widely, while the result of the policy ladder is limited at a very small indexation range. That mean the risk volatility of the ADI plan is many times larger than the policy ladder.

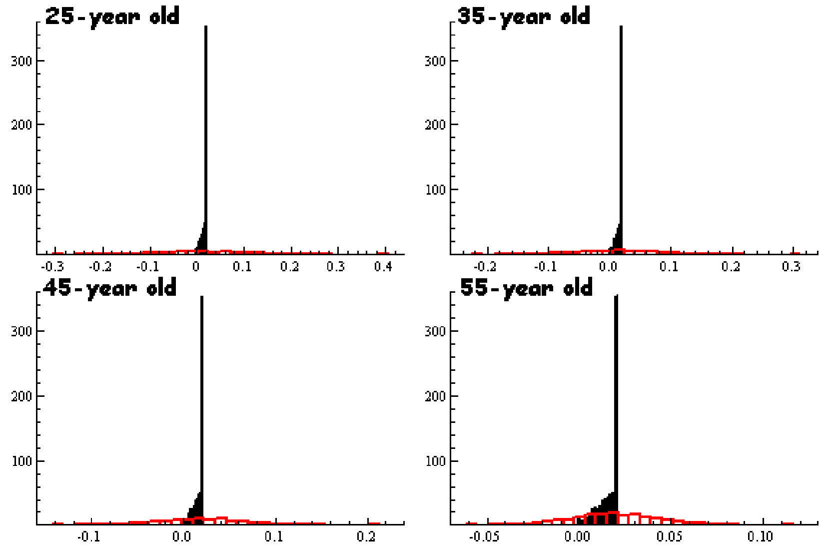


Figure B.1: Histogram comparison between ADI plan (red) and policy ladder (black)

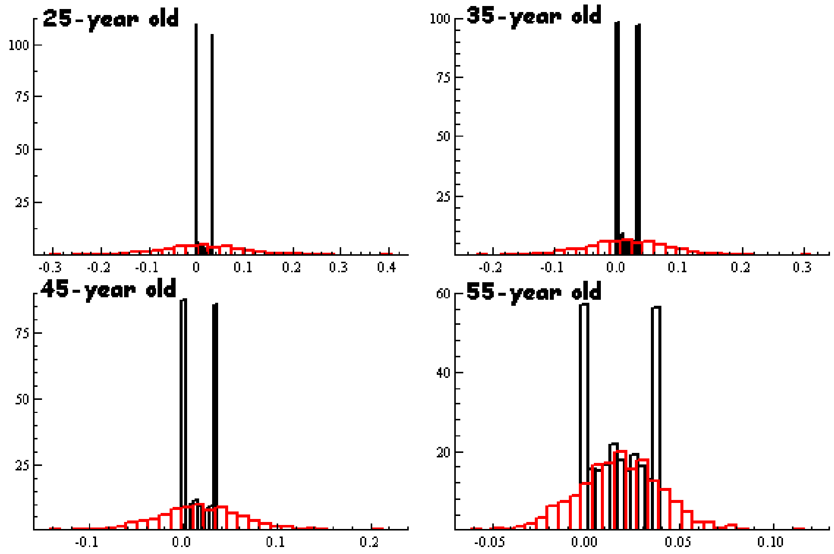


Figure B.2: Histogram comparison between ADI plan (red) and individual collar contingent ADI plan (black)

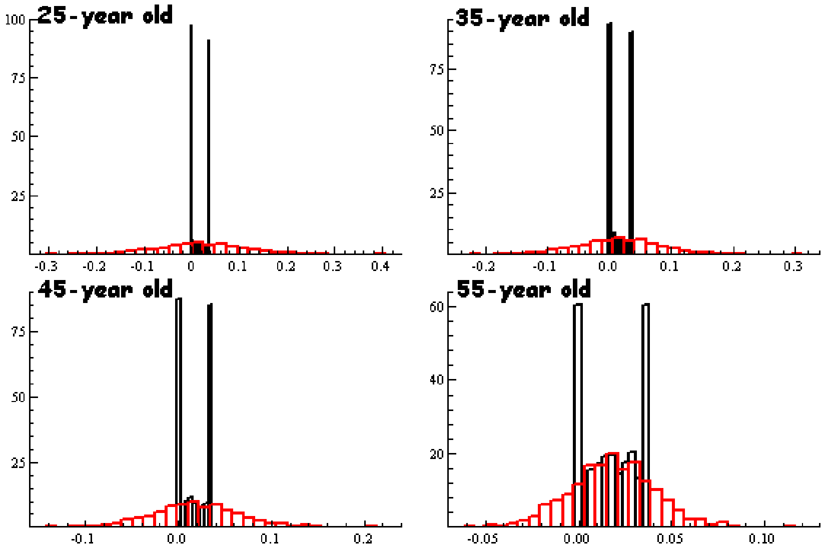
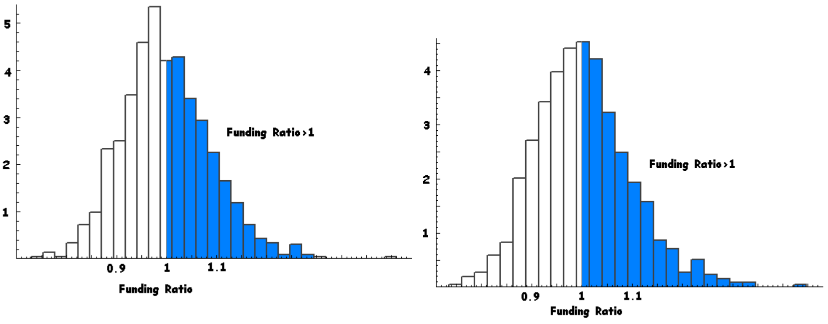


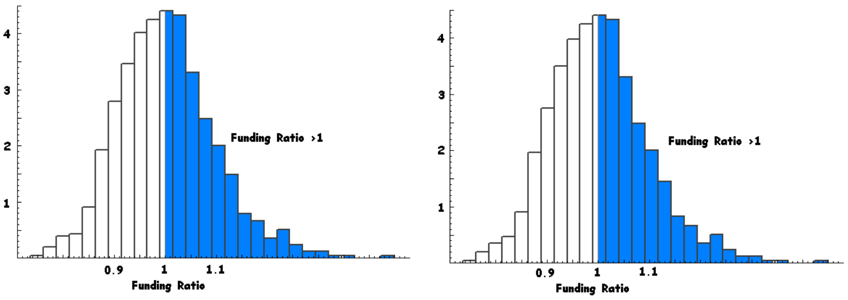
Figure B.3: Histogram comparison between ADI plan (red) and uniform indexation collar contingent ADI plan (black)

Figure B.2 compares the histograms of the ADI plan with the individual collar contingent ADI plan. The collar contingent indexation rate is a “ San Quentin” distribution as its histograms have two high walls. The two walls are the location of indexation floor and cap respectively. It is obtained form Figure B.2 that, the indexation risk has been well controlled and all the observations are on the positive side. It is also obtained that the two walls is getting lower from the young cohort to the old. That means the collar effect is diminishing with age. Figure B.3 compares the histograms of the ADI plan and the uniform collar contingent ADI plan. The histograms are very similar to the previous one.

B.2 Histogram of Funding Ratio



(a) Policy Ladder (b) ADI plan



(c) ind. coADI (d) uni. coADI

Figure B.4: Histogram of Funding Ratio. ADI: Age-dependent indexation; ind. coADI: Individual collar contingent ADI plan; uni. coADI: Uniform collar contingent ADI plan.

Figure B.4 has four panels representing the funding ratio histograms of the four policy variants. The blue area represents the observations with the funding ratio bigger than one. It is obtained from Figure B.4 that, the [1]mode value of panel (a) is smaller than 1, while the mode value of the other panels is 1. Without numerical evidence, we are not able to conclude which plan is better.

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[1] Mode is the value that has the the largest number of observations from sample (http://en.wikipedia.org/wiki/Mode)

Appendix C

C.1 Variables used in VAR model

r_n^1	3 months Euribor
$r_r^(40)$	Dutch 10-year real zero coupon yield
π	Dutch price inflation
r_n^{40}	German 10-year zero coupon
x_{re}^{liq}	Listed US Real Estate (NAREIT)
cs	Credit Spread (BAA vs AAA)
x_{conv}	Convertibles (UBS Global Convertibles)
x_s	Developed Equity (MSCI world)
dy	Dividend -price ratio

SCI is short for Morgan Stanley Capital International. MSCI world is a stock market index which consists of 1500 global stocks from emerging markets. Resource: Wikipedia, MSCI World.

Table C.1: Variables used in VAR model

C.2 VAR estimation result

This table reports the VAR estimation result. Period (1973:II-2003:III). Panel (a) is value $c(=(I-\Gamma)\cdot\mu)$. Panel (b) is VAR estimates Γ and Panel (c)is cross-correlations of the VAR residuals Σ.

(a)	r_n^1	r_r^{40}	π	r_n^{40}	x_{re}^{liq}	cs	x_{conv}	x_s	dy
c	-0.0016	0.0005	-0.0025	0.0034	0.3181	-0.0014	-0.0765	0.4623	-0.0039
(b)Γ	$r_{n,t}^1$	$r_{r,t}^{40}$	π_t	$r_{n,t}^{40}$	$x_{re,t}^{liq}$	cs_t	$x_{conv,t}$	$x_{s,t}$	dy_t
$r_{n,t+1}^1$	0.9085	0.0066	-0.0420	0.1366	0.0014	-0.1511	-0.0105	0.0001	-0.0360
$r_{r,t+1}^{40}$	0.0094	0.9521	-0.0092	0.0104	0.0003	-0.0356	-0.0016	0.0007	0.0034
π_{t+1}	0.1682	-0.7786	0.6445	0.3513	-0.0003	0.1036	-0.0072	0.0006	-0.1081
$r_{n,t+1}^{40}$	0.0759	-0.008	-0.0113	0.8514	-0.0006	-0.0636	-0.0050	0.0020	0.0425
$x_{re,t+1}^{liq}$	-4.4037	11.3435	1.5408	-6.2533	-0.0783	3.7078	0.3591	0.0527	7.7147
cs_{t+1}	0.0221	-0.0316	0.0018	0.0333	-0.0003	0.8182	-0.0031	-0.0009	-0.0434
$x_{conv,t+1}$	-2.3431	0.9086	-1.1964	4.3111	0.0096	4.1755	0.0239	-0.0798	-0.7573
$x_{s,t+1}$	-3.1025	2.7528	-2.1088	-4.3923	-0.1338	2.0563	0.4080	0.0161	9.9273
dy_{t+1}	0.0265	-0.0899	0.0137	0.0618	0.0010	-0.0560	-0.0040	0.0002	0.9050
(c)Σ	r_n^1	r_r^{40}	π	r_n^{40}	x_{re}^{liq}	cs	x_{conv}	x_s	dy
r_n^1	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
r_r^{40}	0.0002	0.0004	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
π	0.0005	-0.0002	0.0030	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
r_n^{40}	0.0002	0.0007	0.0002	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000
x_{re}^{liq}	-0.0101	-0.0178	-0.0017	-0.0176	0.0750	0.0000	0.0000	0.0000	0.0000
cs	0.0001	-0.0000	0.0000	0.0000	-0.0004	0.0007	0.0000	0.0000	0.0000
x_{conv}	-0.0095	-0.0190	-0.0068	-0.0177	0.0125	0.0003	0.0250	0.0000	0.0000
x_s	-0.0106	0.0105	-0.0030	-0.0178	0.0427	-0.0111	-0.0039	0.0621	0.0000
dy	0.0002	-0.0000	0.0000	0.0002	-0.0003	0.0000	0.0000	-0.0006	0.0002

Table C.2: Summary Statistics and VAR Estimation Results

C.3 Term structure estimation result

This table reports coefficient estimates of the risk premia (i.e. $\Sigma\lambda_0$ and $\Sigma\Lambda_1$). The first row is the constant of risk premium ($\Sigma\lambda_0$), the rest rows are exposures to the state variables $\Sigma\Lambda_1$.

$\Sigma\lambda_t$	r_n^1	r_r^{40}	π	r_n^{40}	x_{re}^{liq}	cs	x_{conv}	x_s	dy
	-0.0176	0.0003	0.0007	0.0024	0.3213	-0.0256	-0.0757	0.4655	0.0566
r_n^1	0.2520	-1.7357	-0.0247	0.4073	-0.0029	-0.7142	0.01311	-0.0034	-0.5418
r_r^{40}	0.0347	-0.0371	-0.0274	-0.0009	0.0002	-0.0403	-0.0014	0.0007	0.0047
π	0.0000	0.0000	0.0000	-0.0000	-0.0000	0.0000	0.0000	0.0000	0.0000
r_n^{40}	0.1168	-0.0756	-0.0101	-0.1568	-0.0009	-0.0866	-0.0046	0.0019	0.0188
x_{re}^{liq}	-4.4037	11.3435	1.5408	-6.2533	-0.0783	3.7078	0.3591	0.0527	7.7147
cs	-0.0674	-0.2960	0.0263	0.5897	0.0113	0.0290	0.0376	-0.0132	-0.5581
x_{conv}	-2.3431	0.9086	-1.1964	4.3111	0.0096	4.1755	0.0239	-0.0798	-0.7573
x_s	-3.1025	2.7528	-2.1088	-4.3923	-0.1338	2.0563	0.4080	0.0161	9.9273
dy	-0.4201	4.3416	0.0133	-1.4859	-0.0105	1.1758	-0.0597	0.0163	1.5610

Table C.3: Estimations Results of Term Structure Model

Terminology

2.1

S	Stock price
B	Bond price
μ	Drift of stock price
σ	Volatility of stock price
r_n	Nominal return
W_t	Brownain motion under **P**-measure
$\widetilde{W}_t$	Brownian motion under **Q**-measure
A	Assets value
α	Asset portfolio
r_A	Asset portfolio return

2.2

$\triangle B$	newly accrued pension rights
I	Flat annual total income
Y	Pensionable income
F	Franchise
B_x	Total accrual pension rights of x-year old members
B_x^N	Individual nominal pension rights
B_x^R	Individual real pension rights
π	Wage growth rate
i	Indexation rate
r_r	Real interest rate
P	Actuarially fair annual individual contribution
B	Annual retirement income
c	Actuarially fair contribution rate
L_x	Individual liability for x-year old members
L^N	Nominal liability of the fund
L^R	Real liability of the fund
C	Newly received contribution from the active members
BP	Benefit payments to the retirees

FR_n	Nominal funding ratio
FR_r	Real funding ratio
i_{ladder}	Indexation rate under policy ladder
i_{ADI}	Age-dependent indexation rate
i_{coADI}	Collar contingent age-dependent indexation rate
i_{floor}	Indexation floor
i_{cap}	Indexation cap

4.1

x_t	State variables of VAR model
Γ	Coefficient matrix of VAR model
μ	Historical mean vector of state variables
Σ	Covariance matrix of VAR residuals
ζ	Standard normal distribution

5.1

IQ^k	Indexation quality
CIQ^k	Cumulative indexation quality
CPR^k	Cumulative pension result
CEB	Certainty equivalent benefit
γ	Risk aversion
β	Constant discount rate