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Correlations in Economic Capital Models for Pension Fund Pooling

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Correlations in Economic Capital Models for Pension Fund Pooling

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Contents

1 Introduction

Supervisory authorities and internal risk management require pension funds to hold economic capital (risk capital) as a buffer against adverse economic shocks to assets and liabilities. Pension companies have to deal with several kinds of risks and for each of these risks they should hold some amount of their capital. In this respect, economic capital can be used as a 'common currency' for risk. By combining assets and liabilities of different (national) pension funds, a reduction in economic capital can be achieved through risk diversification, as the movements of different funds' assets and liabilities are likely to be less than perfectly correlated. Since the assets of pension funds are usually well diversified, the major potential gains from diversification across funds lie in the liabilities, which are usually concentrated on interest rate, inflation, and longevity risk particular to a country and population respectively.

Do not put ALL your eggs in ONE basket!

This is a well known country lore from farmers. If you drop the basket then you will lose all the eggs. Therefore, it is better if you use several baskets to carry all the eggs. This simple saying constitutes the basics of risk diversification in financial markets. Diversification arises because not all risks materialize in the same period. For example an insurance company insuring cars and ships would not expect claims from accidents on cars and ships to be interlinked. Similarly, a major motor accident (insurance risk) would not necessarily coincide with turbulence in the financial markets (financial risk). As a result, since it is unlikely that different types of events occur at same time, the company may not need to hold capital for all events going wrong at the same time.

Pension companies aim to measure and manage their risks. It is essential to develop a methodology for required economic capital. While determining this economic capital, diversification benefits play an essential role. Failure to take into account diversification would lead to higher economic capital than necessary and hence would be costly.

There are several ways to quantify diversification benefits. For example, one can employ either economic capital models based on Value-at-Risk or Expected Shortfall. However, the results from such models are highly sensitive to assumptions. The characteristics of risks should be assessed prudently since diversification benefits are driven by the risk model in use.

The dependence structure between risks vary widely. In this study only linear dependence will be taken into account with certain (normal) distribution assumptions. For this reason, to obtain reliable results, the construction of stable and economically sound correlation matrices is crucial. Furthermore, supervisory authorities demand the use of relatively conservative correlations in risk models, representing long-term views and carrying economic meaning.

This study reveals the economic relevance of dependence/correlations for a pension fund. By defining economic capital as a 'common currency' for risk, the framework has the potential to overcome with inconsistent measurement of risks (Kuritzkes et al., 2003). The main goal of the thesis is to answer the question of 'How to produce stable and sensible correlation matrices'.

Construction of correlation matrices is not always an easy job. This study examines how to infer bivariate correlations. Mixture of consistent and inconsistent data, expert opinions and including regularities might lead to a matrix which is not positive semidefinite and so not a correlation matrix. Then, one should seek for the nearest valid positive semidefinite correlation matrix. When computing the nearest correlation matrices one might obtain a matrix which requires higher economic capital than needed or the derived matrix might require smaller economic capital to hold. A methodology to construct the "nearest" correlation matrix is studied by giving an ability to decide on the conservativeness of the resultant matrix.

The most dangerous case for risk management is extreme events such as Black Monday (1987), Internet Bubble (2001) and Spanish Flu (1918). Extreme events that will happen in the future did not happen yet in the past. The only possibility we have is the use of scientific evidence on dependencies, based on semi-worse case events in the past and expert opinion and to get an agreement between industry partners and the regulators (Solvency II Groupe Consultatif Working Group, 2005). Therefore, expert opinion is very important. Also in case of lack of data, one needs to consult for an expert opinion. The problem is how to get an expert opinion and how to combine with the data driven correlations in a consistent way.

In summary, the questions of the thesis can be listed as follows:

- What is the economic relevance of dependence/correlations for a pension fund?
- How to produce stable and sensible correlation matrices?
  - How to infer bivariate correlations?
  - How to obtain and make use of an expert opinion?
  - How to construct the "nearest" multivariate correlation matrix and deal with conservativeness?
- How to deal with problems about fat tails, tail correlations, dependence structures and their pitfalls?

In the financial literature, most of the studies are devoted to the cases in insurance and banking sectors. The studies on pensions are not that much. For this thesis, the main benefits are taken from 3 papers: *Diversification–Technical Paper* by Solvency II Groupe Consultatif Working Group (2005), *Risk Measurement, Risk Management and Capital Adequacy in Financial Conglomerates* by Kuritzkes et al. (2003) and *A Framework for Incorporating Diversification in the Solvency Assessment of Insurers* by CRO Forum (2005). In addition, the paper *Decomposing Portfolio Value-at-Risk: A General Analysis* by Hallerbach (1999) is consulted for technical issues.

The structure of paper is as follows. Chapter 2 reveals the relation between diversification and risk by introducing risk classes. Chapter 3 proposes the methodology to determine a correlation matrix and some related definitions. In Chapter 4 an application is studied. Finally, Chapter 5 concludes the thesis and presents some suggestions for further research.

2 Risk & Diversification

2.1 Pension Pooling

Pension fund pooling is getting more important in the financial sector both within a country and between countries. The benefit of pooling can be listed as:

1. **Economies of scale:** by transferring asset and liability management into one hand.
2. **Risk offset:** netting of exposures to different risk classes (interest rate, currency; longevity) across individual funds by pooling positions in one vehicle. Positive and negative positions subject to the same risks cancel each other out. Hence, a natural hedge is realised.
3. **Risk diversification:** less than perfect correlations of assets and liabilities respectively reduce volatility, both within and across risk classes. The gains are strongly dependent on correlation assumptions.

There is an incentive for small pension funds to merge, so as to increase their pool in order to obtain the benefits mentioned above. Merging with other companies and investing on a multinational level helps pension companies to hold less economic capital.

Risk diversification is the core of pension fund pooling. Even a small percentage of diversification benefits can return pension companies with a large amount of profits. diversification benefitswill be discussed in more detail later on.

2.2 Economic Capital

Economic Capital (risk capital) can be defined as the methods or practices that allow pension funds to consistently assess risk and attribute capital to cover the economic effects of risk-taking activities (Basel Committee on Banking Supervision, 2009). Regulators require financial institutions to hold economic capital as a buffer capital in order to prevent for insolvency. Holding economic capital at a higher level of confidence reduces the probability of default, and thereby raises the credit rating of a pension company. Lower economic capital for a given degree of risk taking will make a pension company less solvent, but more profitable (Kuritzkes et al., 2003). Furthermore, the risk capital held by the company is an indicator for rating agencies as a measure of the company's capacity to bear risks. Therefore, almost all internationally active banks set their economic capital solvency standard at a level they perceive to be required to preserve a specific external rating (eg AA) (Basel Committee on Banking Supervision, 2009).

On the other hand, economic capital is also a business tool developed and used by individual institutions for internal risk management purposes. It is often parameterised as an amount of capital that a pension fund needs to absorb unexpected losses over a certain time horizon at a given confidence level (Basel Committee on Banking Supervision, 2009). In that sense, economic capital can be seen as a common measure across all risks and businesses. It provides pension funds with a common currency for measuring, monitoring, and controlling different risk types.

2.3 Levels of Diversification

This thesis is based on a bottom up approach. In order to get the total capital needed at the highest level of a group, we start with the calculation at the lowest level the *sub-risks*, which composes a definable risk type such as interest rate risk, currency risk etc. The issue is then how to combine these sub-risks to obtain the capital at various levels of a pension fund. The combination will cause a reduction of the total risk. This reduction is called *diversification benefits*.

A financial company might have different kinds of business units such as insurance, banking, pension etc. Similar to risk combination, the mixture of various business units again would cause diversification benefits. Furthermore, a global company that has multinational branches can also derive benefit from diversification. For example, a reinsurer might encounter a big loss in a country because of a big disaster like an earthquake, but there is a very low probability to encounter another big catastrophe in another country at the same time.

The diversification effect can be calculated at several levels:

1. **Stand-alone risks, within a risk type:** In this level sub-risk types are combined into one classified risk type. The first level aggregates the stand-alone risks within a single risk factor. For example, sub-risks of currency risk which might be influenced by several currencies.

2. **Between risk types, within a business line:** The result is the economic capital for the stand-alone business. It contains the diversification between risk types and within a single business line. For example, the pooling of a pension company in one

country by combining the risks that the company encounters such as market risk, actuarial risk and operational risk.

3. **Between business lines, within a country:** This level aggregates risk across different business line. It is encountered at financial conglomerates which are a combination of diverse businesses operating under a common ownership structure such as banking, insurance, pension etc (Kuritzkes et al., 2003). In this study, this level will be skipped because we only consider pension companies.

4. **Between geographies:** Pooling several countries into one pool will result in further diversification because of adding risk parts, combining risk types over a larger range than in first level depending on geographic and economic situations.

Kuritzkes et al. (2003) show that diversification benefits are usually greatest within a single risk factor, decrease at the level of a business line, and are smallest across business lines. They also clarify that the capital management problem for a conglomerate is to determine both within and across businesses how much capital is needed to support the level of risk taking. There is a trade-off such that lower capital for a given degree of risk taking will make an institution less solvent, but more profitable, and vice-versa.

The reasons for diversification benefits are (CEA, 2007, Kuritzkes et al., 2003):

- *Law of Large Numbers:* In the first level, for diversification within risk types, the law of large numbers plays an important role since as the number of risky assets increases it is more predictable to determine the necessary risk capital.

- *Opposite risks:* If two risks move in opposite directions then it will cause a natural hedge effect between them and so decreases the risk of the total portfolio. This is also called *netting effect*.

- *Risks that are not perfectly dependent:* The total capital linked to the combination of sub-risks will be equal or lower than the sum of the capitals for each sub-risk. This property depends on the choice of related risk measure, which is going to be explained on next section. Unconnected (independent) risks reduce the total risk of a portfolio.

In general, the diversification benefits increase with the number of positions, decreases with greater concentration, and decreases with greater correlation. Diversification is a major rationale for financial institutions to internationalize portfolios (Kuritzkes et al., 2003).

Figure 1 shows that diversification levels in which a bottom up approach is followed. Starting with the capitals for each sub-risk within an entity, we want to derive the capitals needed at the higher levels. It is important to notice that the bottom up calculation always has to start at the lowest level. In the figure, *level 0* represents the diversification benefits between sub-risks, within a risk type. EC1, EC2, EC3, and EC4 indicates the stand-alone economic capital that has to be separated for 4 sub-risks and a distribution is obtained for one risk type. Similarly, at *level 1* these risk types are combined and a general distribution within a business line is obtained. Since only pension companies are considered, between business lines are skipped and lastly, *level 2* shows the diversification benefits between countries. For any country, suchlike distribution can be attained.

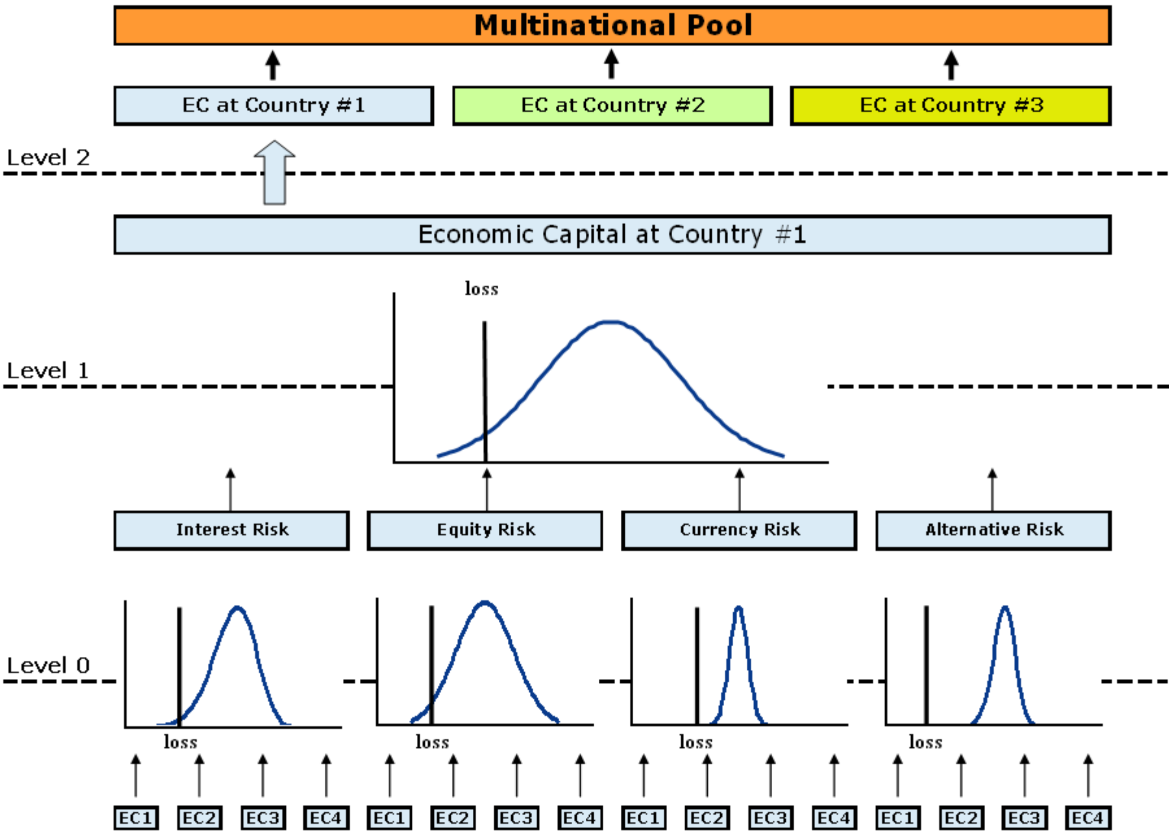


Figure 1: Levels of diversification

One point needs to be clarified about bottom-up approach, which is shown in Figure 2. When determining the diversification benefits at the upper levels, the correlation between all of the included sub-risks must be taken into account. At every level of diversification, the relation between any components must not be ignored.

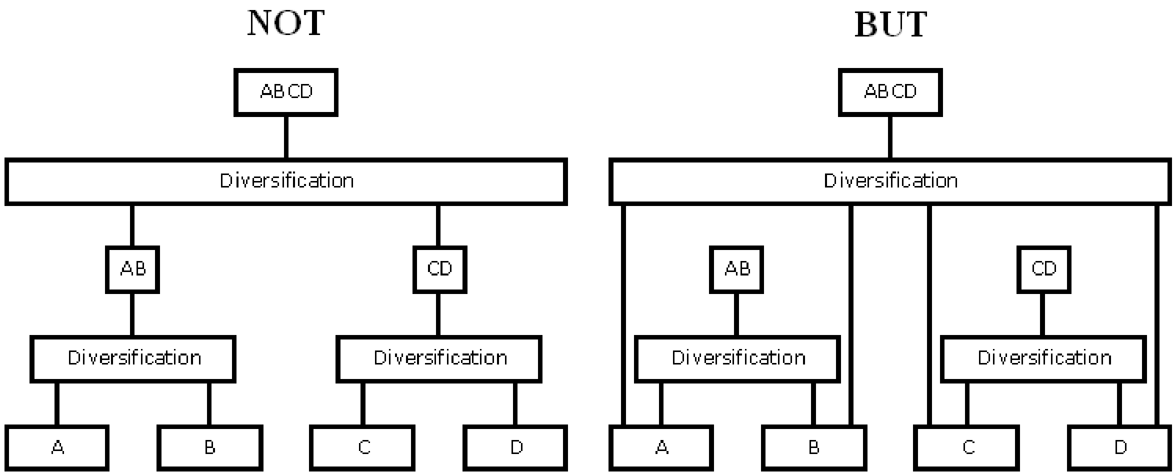


Figure 2: Bottom up approach (Solvency II Groupe Consultatif Working Group, 2005)

Solvency II warns that "failure to take into account diversification - fully and in all forms - would lead to higher capital requirements than necessary with adverse consequences for policyholders and the competitiveness of the European insurance industry" (CEA, 2007).

2.4 Risk Measures

A *risk measure* is described as a mapping from a set of random variables to the real numbers. This set of random variables stands for risk. The common notation for a risk measure related with a random variable X is $\rho(X)$.

Determination of economic capital depends on choosing a proper risk measure. Dowd and Blake (2006) mention that it is difficult to give a good assessment of financial risk except the cases, in which we specify what a measure of financial risk actually means. They elucidate by this example, "the notion of temperature is difficult to conceptualize without a clear notion of a thermometer, which tells us how temperature should be measured." Hence, in order to clarify the notion of risk itself, Artzner et al. (1999) propose several axioms, which are monotonicity, subadditivity, positive homogeneity and translation invariance and defines a *coherent risk measure* as a measure which satisfies these four properties:

- **(Axiom M)** Monotonicity: If $X \leq Y$ then $\rho(X) \leq \rho(Y)$.
- **(Axiom S)** Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
- **(Axiom PH)** Positive Homogeneity: If $a \in \mathbb{R}_+$ then $\rho(\lambda X) = \lambda\rho(X)$.
- **(Axiom T)** Translation Invariance: If $a \in \mathbb{R}$ then $\rho(a + X) = a + \rho(X)$.

Axiom M intuitively indicates that if a risk Y is always bigger than another risk X, then the risk measures should be similarly ordered. *Axiom S* states that if two portfolios are combined, the risk is not greater than the sum of the risks associated with each portfolio. So, coherence ensures a risk measure that accounts for diversification benefits. *Axiom PH* obligates that portfolio size should linearly influence risk, and finally *Axiom T* postulates that adding a constant loss to a portfolio raises the necessary risk measure by the same amount.

Coherent risk measures seem to be accepted because of their helpful characterization of risk measures under fairly general conditions. However, there is a huge discussion about the axioms of coherence in the financial literature. Particularly, the Axiom S and Axiom PH are widely criticised. Value-at-Risk (see section 2.4.2), generally fails the Axiom S, due to its disregard for the extreme tails of distribution. Value-at-Risk is subadditive for elliptical distributions. Hardy (2006) says in spite of an attempt to reject Value-at-Risk in favour of coherent measures, it is useful and well understood, that in a few situations is not sufficiently important to reject this risk measure.

Dhaene et al. (2003) give the following example about subadditivity. In earthquake risk insurance, it is better to insure two independent risks rather than two positively dependent risks, like two buildings in the same neighbourhood. For insuring both buildings, the premium should be more than twice the premium for insuring only a single building because these buildings are highly dependent. So, $\rho(X+Y) \geq \rho(X) + \rho(Y)$ might be hold. This illustration points out that in some circumstances it may be valuable to dis-aggregate risks.

Axiom PH is the most disputed one. Artzner et al. (1999) argue that, if a portfolio is so large that adjustment of the risk measure would seem necessary, it is actually less valuable in the market, so that adjustment takes place through X. This leads to *liquidity risk*, which is the risk that the market cannot easily absorb the sell-off of large asset positions (Eberlein et al., 2007). Therefore, Follmer and Schied (2002) propose to relieve the Axiom S and Axiom PH by introducing the property of *convexity*. A monotonic and translation invariant risk measure ρ is called convex if it satisfies the following property:

- Convexity: $\rho(aX + (1-a)Y) \leq a\rho(X) + (1-a)\rho(Y)$ where $a \in [0,1]$.

The set of coherent risk measures can be defined as the class of convex risk measures that satisfy the positive homogeneity property. Since the set of convex risk measures is larger than the class of coherent risk measures, it is sometimes referred as the set of *weakly coherent risk measures* (Dhaene et al., 2008).

Basel Committee on Banking Supervision (2009) declares the characteristics of risk measures as "An ideal risk measure should be intuitive, stable, easy to compute, easy to understand, coherent and interpretable in economic terms."

Next, some of the commonly used risk measures are presented.

2.4.1 Standard Deviation Principle

The first use of risk measures in actuarial science and insurance was the development of premium principles. The risk measures such as expected value principle and standard deviation principle were applied to a loss distribution in order to determine a proper premium to charge for the risk. As a consequence a premium calculation principle can be directly interpreted as a risk measure. The Standard Deviation Principle is briefly explained as follows:

$$\rho(X) = E[X] + \kappa\sigma[X], \qquad \kappa \geq 0.$$

The *safety loading*, $\kappa\sigma[X]$ is risk-sensitive, as it is a proportion of the standard deviation.

This principle is mostly used by reinsurance pricing and also related to Markowitz portfolio theory. Standard Deviation Principle is not a coherent risk measure since it does not satisfy monotonicity axiom. Standard deviation principle is probably the most commonly used approach in property and casualty insurance (Buhlmann, 1970). It is linear due to a proportional change in the claims experience, and this is most likely the reason for its popularity.

2.4.2 Value-at-Risk

$$\rho(X) = \mathrm{VaR}_p(X) = F_X^{-1}(p), \qquad p \in (0,1),$$

where F_X is the cumulative probability distribution of X. $\mathrm{VaR}_p(X)$ is easily understood as the amount of capital that, when added to the risk X, limits the probability of default to $1-p$. As Jorion (2001) points out, "VaR summarizes the worst loss over a target horizon with a given level of confidence." (p.22)

J.P. Morgan introduced VaR at the early 1990s and since then it has become more and more popular methodology for the measurement and reporting of risk. The Market Risk Amendment of the Basel Accord, represented in 1995, permitted the use of VaR to set regulatory capital for market risk.

VaR is subadditive for elliptical distributions such as Gaussian distribution. It is not subadditive for non-elliptical distributions. So, it may not satisfy subadditivity condition always (lack of coherence) and as a consequence VaR might discourage diversification.

2.4.3 Expected Shortfall

$$\rho(X) = \mathrm{ES}_p[X] = \int_p^1 F_X^{-1}(q)dq, \qquad p \in (0,1).$$

Expected Shortfall is also called Conditional Tail Expectation (CTE). Although Expected Shortfall is a coherent risk measure it has a difficult interpretation and does not provide a clear link to a bank's aimed target rating (Basel Committee on Banking Supervision, 2009).

VaR assesses the worst case loss, where the worst case is defined as the event with a $1-p$ probability. It does not take into consideration what the loss will be if that $1-p$ worst case event actually realized. On the other hand, Expected Shortfall addresses these problems by measuring the loss in tails. In other words, Expected Shortfall is the expected loss given that the loss falls in the worst $1-p$ part of the loss distribution.

2.5 Risk Classes

Pension companies are exposed to various risks. In order to decide better pooling strategies, one should understand the basic salient features of risk factors separately. The risks can be divided into five specific types: systematic (market), actuarial, operational, liquidity and credit risks.

2.5.1 Systematic (Market) Risk

A pension company's illustrative portfolio categorisation is shown in Figure 3. All asset classes are exposed to different kinds of risks. However, the risk management has to evaluate this portfolio with respect to risk classes not asset categories. In Figure 4, you see the risk classes by the related asset classes.

Systematic risk is the risk of asset and liability value changes related with systematic factors (Santomero and Babbel, 1997). It is the risk of loss as a result of movements in the level or volatility of market prices (Jorion, 2005). Systematic risk cannot be diversified completely but can be hedged. For this reason, systematic risk is also called *undiversifiable risk*.

In order to achieve a hedge strategy and reduce the dependence on the volatilities of systematic risk, pension firms analyse and track them individually. Jorion (2005) categorises the four main risk factors of systematic risk as follows:

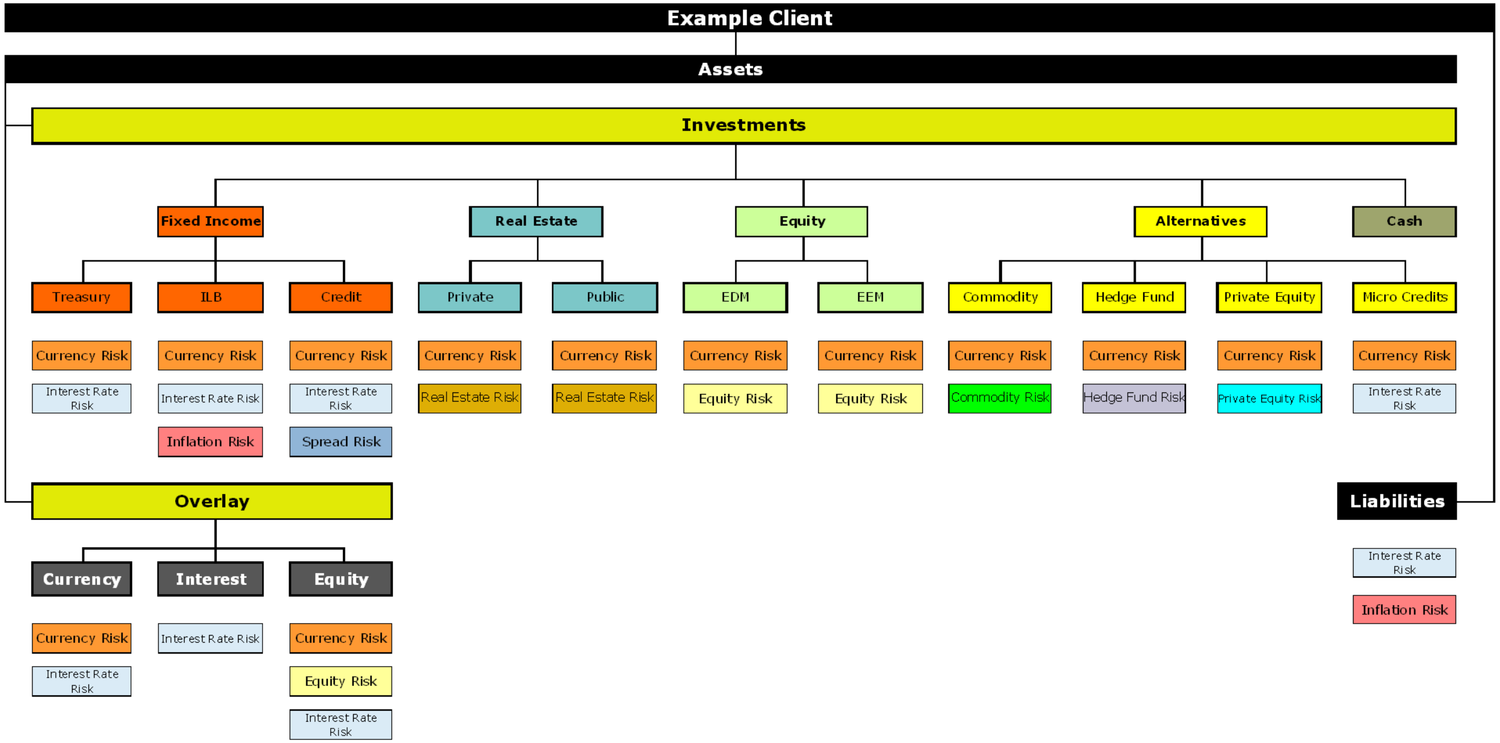


Figure 3: Market risk classes by asset class

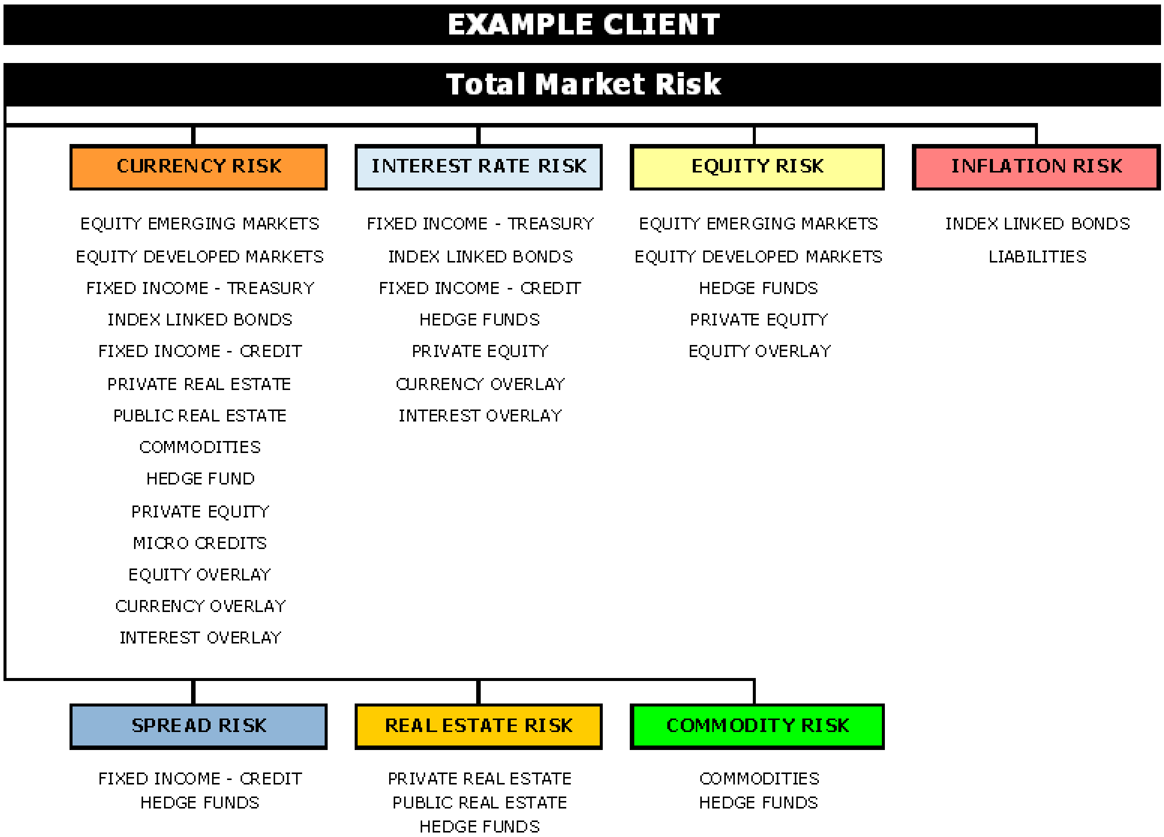


Figure 4: Risk classes

- **Equity Risk** is the risk that derives from potential movements in the value of stock prices.
- **Interest Rate (Fixed-Income) Risk** is the risk derives from potential movements in the level and volatility of bond yields.
- **Currency Risk** is the risk that derives from potential movements in the value of foreign currencies.
- **Commodity Risk** is the risk that derives from potential movements in the value of commodity contracts, which include agricultural products, metals, and energy products.

In addition, the following two risk factors can be analysed separately:

- **Real Estate Risk** is the risk that derives from potential movements in the value of real estate prices.
- **Inflation Risk** is the risk that the value of the money obtained in the future will be worth less when it is obtained.
- **Spread Risk** is risk due to exposure to some spread. It often appears with a long-short position or with derivatives.

Value-at-Risk is widely used in order to measure systematic risk.

2.5.2 Actuarial Risk

Actuarial risk is the risk arising from participants life uncertainty. Actuarial risk may derive both because of *longevity risk* and *risk of short life*. The latter is the risk that a participant lives shorter than expected. In this instance, more benefits payments may have to be made to surviving relatives. Longevity risk is the risk that a participant lives longer than expected and causes pension fund to pay more benefits. Hence, actuarial risk affects liabilities much.

2.5.3 Operational Risk

Basel Committee on Banking Supervision (2009) defines *operational risk* as the risk of loss associated with human or system failures, as well as fraud, natural disaster and litigation. Operational risk is not a pure economic risk. It is more idiosyncratic to companies and rather then risk management departments, IT services cares for services and human resources cares for human-derived risks. Therefore it is difficult to find data and estimate operational risk. It represents losses from all types of activity where a company encounters, such as:

- Business
- Legal/Compliance
- Fraud
- Administration
- Staff
- Physical Assets
- Systems
- Tax

2.5.4 Liquidity Risk

Liquidity risk occurs in case of lack of funding. In that case, a given security or asset cannot be traded quickly enough in the market to prevent a loss. Jorion (2005) divides liquidity risk into two types:

- **Asset liquidity risk:** Arises when a transaction cannot be conducted at prevailing market prices due to the size of the position relative to normal trading lots.

- **Funding liquidity risk:** refers to the inability to meet payment obligations.

Asset liquidity risk generally falls under the market risk management function. Liquidity risk can be lessen by diversification of assets.

2.5.5 Credit Risk

Credit risk is the risk that arises when a portfolio value changes due to shifts in the probability that an obligor (or counterparty) may fail to submit cash flows (principal and interest) as previously contracted (Basel Committee on Banking Supervision, 2009). Credit risk is mostly important for banks. The history of financial institutions has also shown that the biggest banking failures were due to credit risk.

Credit risk is diversifiable. However, it is difficult to eliminate totally (Santomero and Babbel, 1997). There are some credit rating agencies such as Moody's, which surveys and assigns credit ratings for issuers of certain types of debt obligations.

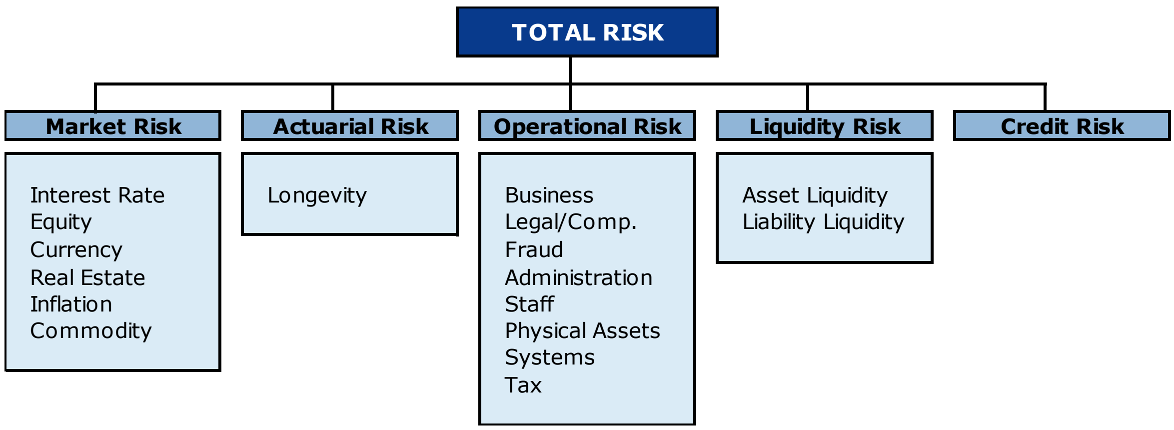


Figure 5: Risk types

3 Methodology for Determining Diversification Benefits by Correlation Matrices

3.1 Linear (Pearson) Correlation

The pooling of risks across portfolios, business lines, organisations achieves diversification. The extent of the diversification benefits base on the degree of dependence between the pooled risks. This dependence structure can be defined by many ways. The most commonly used method ordinary is linear (Pearson) correlation. In probability theory and statistics, correlation indicates the strength and direction of a linear relationship between two random variables.

The *correlation coefficient* $\rho_{X,Y}$ between two random variables X and Y with expected values μ_X and μ_Y and standard deviations ρ_X and ρ_Y is defined as:

$$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\rho_X \rho_Y} = \frac{E((X-\mu_X)(Y-\mu_Y))}{\rho_X \rho_Y} \in [-1,1], \qquad (1)$$

where E is the expected value operator and cov means covariance. A widely used alternative notation is:

$$\mathrm{corr}(X,Y) = \rho_{X,Y}.$$

The correlation is defined only if both of the standard deviations are finite and both of them are nonzero. It is a corollary of the CauchySchwarz inequality that the correlation cannot exceed 1 in absolute value.

The correlation is 1 in the case of an increasing linear relationship, -1 in the case of a decreasing linear relationship, and some value in between in all other cases, indicating the degree of linear dependence between the variables. The closer the coefficient is to either -1 or 1, the stronger the correlation between the variables.

If the variables are independent then the correlation is 0, but the converse is not true because the correlation coefficient detects only linear dependencies between two variables.

Solvency II Groupe Consultatif Working Group (2005) states that it is difficult to use theoretically accepted methods, which includes the use of the combination of the risk measure Tailvar and Copula-functions since they will only work "perfect" in case good information about tail is available. Therefore a practical method is adopted by using correlation matrices. The main advantage of using copula-method is that they can be used to accurately combine other distributions than the Normal Family and that they can recognise dependencies that change in the tail of the distribution. On the other hand, the usage of copulas is so difficult and generally there is limited data available to estimate the copula function in the tail.

The correlation matrix of N random variables R_1,...,R_N is an $N \times N$ symmetric positive semidefinite matrix A with $\rho_{ii} = 1$.

$$A = \begin{array}{c} \\ R_1 \\ R_2 \\ \vdots \\ R_N \end{array} \begin{array}{c} \begin{array}{cccc} R_1 & R_2 & \cdots & R_N \end{array} \\ \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & 1 \end{bmatrix} \end{array}$$

It has the following properties:

- symmetric
- 1s on the diagonal
- eigenvalues nonnegative
- off-diagonal elements between -1 and 1.

The term *correlation matrix* comes from statistics, describing a matrix in which the (i, j) entry points to the correlation between two random variables R_i and R_j. Hence correlation matrices have ones on the diagonal and are symmetric. The reason is that the correlation between a random variable and itself is always 1 and the correlation between R_i and R_j is the same as between R_j and R_i.

3.2 Caveats

It is important to model the dependence between risk factors before revealing the diversification benefits. In such cases one has to model multivariate distributions. Multivariate probabilistic modelling is much more complex than univariate modelling. A couple of warnings caveats should be mentioned about our model assumptions. In this thesis, correlation coefficients are used in order to detect the dependencies. Although correlations are very popular because of their straightforward calculations, there are some fallacies of correlations:

1. In order to calculate the correlation coefficient of two variables (see (1)), the variances of both variables must be finite or the linear correlation is not defined. This is not ideal for a dependence measure and causes problems in case of heavy-tailed distributions such as large claims in insurance (Embrechts et al., 2002).

2. Correlation coefficient only gives a scalar summary of linear dependence. All other sort of dependencies are ignored. It may happen that Y is a nonlinear function of X, so is completely determined by X, and still the correlation coefficient between X and Y is zero. For example, let X be standard normal, and assume $Y = X^2$; then $\text{cov}(X, Y) = E[X^3] - E[X]E[X^2] = 0$.

3. Correlation coefficient does not tell us anything about the degree of dependence in the tail of the underlying distribution. Distributions with same correlation coefficients may show totally different dependence in the tails. In addition, Embrechts et al. (2002) emphasise that "same correlation may have qualitatively very different dependence structures and, ideally, we should consider the whole dependence structure which seems appropriate for the risks we wish to model."

4. Independence of two random variables implies they are uncorrelated with a correlation coefficient of zero. However, zero correlation does not in general imply independence.

The correlation coefficient entirely defines the dependence structure only in very special cases, for example when the cumulative distribution functions are the multivariate normal distributions. Therefore, to overcome the shortcomings of correlation coefficients, in this study, normality is assumed for all kind of distributions.

On the other side, the drawbacks of normality should be considered. Although this distribution assumption is widely used in theoretical studies, most of the time, it does not reflect the reality. In addition, Gaussian (Normal) copula model has a limited ability to capture the impact of extreme events, due to its thin tails. The multivariate copula can be a better alternative. In order to conceive dependencies on tails, a further study is required related to fat tailed distributed risks. In this thesis only linear dependency is utilised and any other kinds of dependencies are not taken into account.

3.3 Correlation Matrix Constitution Methodology

3.3.1 Correlation Assessments

In Figure 6, a representative example is shown as a risk categorization of a pension company. Liquidity and credit risks are not included. Here, we can observe the risk factors of economic capital.

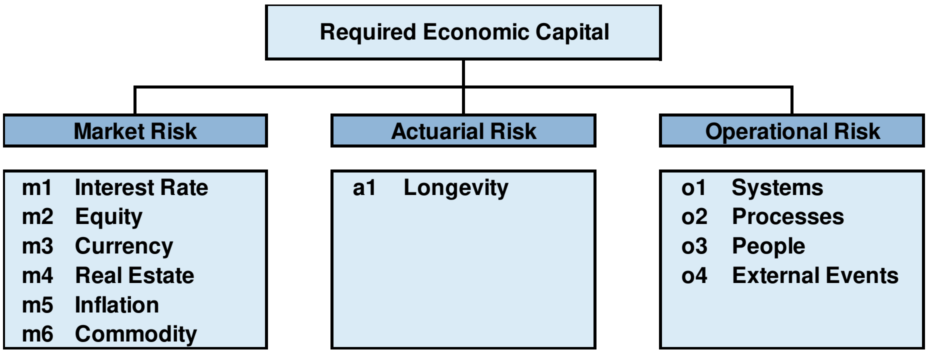


Figure 6: Required Economic Capital

In this figure, m_i, a_1 and o_i indicates to three risk types: market, actuarial and operational risk. The first step is to determine the bivariate dependencies/correlations. For one country, the relation between risks by using correlation ρ_i is represented in the following correlation matrix:

	Country 1										
	Market Risk						Actuarial	Operational Risk			
	m1	m2	m3	m4	m5	m6	a1	o1	o2	o3	o4
m1	1										
m2	ρ1	1									
m3	ρ2	ρ11	1								
m4	ρ3	ρ12	ρ20	1							
m5	ρ4	ρ13	ρ21	ρ28	1						
m6	ρ5	ρ14	ρ22	ρ29	ρ35	1					
a1	ρ6	ρ15	ρ23	ρ30	ρ36	ρ41	1				
o1	ρ7	ρ16	ρ24	ρ31	ρ37	ρ42	ρ46	1			
o2	ρ8	ρ17	ρ25	ρ32	ρ38	ρ43	ρ47	ρ50	1		
o3	ρ9	ρ18	ρ26	ρ33	ρ39	ρ44	ρ48	ρ51	ρ53	1	
o4	ρ10	ρ19	ρ27	ρ34	ρ40	ρ45	ρ49	ρ52	ρ54	ρ55	1

Figure 7: Correlation matrix in one country

Some correlations can be obtained from historical data. However, for several risk types there might be lack of data such as real estate since there is not enough historical data available or even there might not be any data at all such as in the case of operational risks. In this situation, one needs to consult for expert opinion and qualitatively assign the necessary correlations. In Figure 8 the orange boxes denote the qualitatively assigned correlations.

	Country 1										
	Market Risk						Actuarial	Operational Risk			
	m1	m2	m3	m4	m5	m6	a1	o1	o2	o3	o4
m1	1										
m2	ρ1	1									
m3	ρ2	ρ11	1								
m4	ρ3	ρ12	ρ20	1							
m5	ρ4	ρ13	ρ21	ρ28	1						
m6	ρ5	ρ14	ρ22	ρ29	ρ35	1					
a1	ρ6	ρ15	ρ23	ρ30	ρ36	ρ41	1				
o1	ρ7	ρ16	ρ24	ρ31	ρ37	ρ42	ρ46	1			
o2	ρ8	ρ17	ρ25	ρ32	ρ38	ρ43	ρ47	ρ50	1		
o3	ρ9	ρ18	ρ26	ρ33	ρ39	ρ44	ρ48	ρ51	ρ53	1	
o4	ρ10	ρ19	ρ27	ρ34	ρ40	ρ45	ρ49	ρ52	ρ54	ρ55	1

Figure 8: Correlation matrix in one country where orange indicates there is no data

Naturally, the anticipated correlations are not regarded as precise results. The expert opinion is either numerical or verbal. In the latter case, the views taken by words can be transformed into numbers as follows:

Table 1: Expert opinion translation

Expert Opinion	ρ_i
Independent	0
Some correlation	0.25
Significant correlation	0.50
High correlation	0.75
Full correlation	1

The next step is to determine the correlation coefficients between countries same as before and then a big correlation matrix is obtained as illustrated for two countries in Figure

9 where the diversification levels can be observed. Here the correlation matrices for each country are assumed to be same. Also, note that the diagonal elements of between countries sub-matrix are not necessarily 1. For these 11 sub-risks, 121 correlations are required. Similarly, more countries or risk types can be added into construction. Pension companies usually, encounter with huge correlation matrices.

3.3.2 Nearest Correlation Matrix

So far the positive semi-definiteness (PSD) property of correlation matrices have not been taken into account. After combining with expert opinions, the generated matrix that comes out might not be correlation matrix any longer due to its failure to satisfy PSD. Observed correlation matrices must satisfy many constraints. For example, if A is highly correlated to B, and B is highly correlated to C, then A and C must also be highly correlated. With the rounding method while receiving expert opinions, it is possible that one might end up violating PSD because of ignoring this triple relation.

In case of such a situation one need to find the nearest correlation matrix without changing or slightly changing the blue correlations in Figure 8 since they are valid data-driven correlations. For example, in the paper of Kuritzkes et al. (2003), to calculate the diversification benefits between business units, they refer to correlations from different sources. For banking correlations, they use Dimakos and Aas (2002) and for insurance correlations, they use KSW. However, the resulting correlation matrix does not satisfy PSD.

There is not a big literature about finding the nearest correlation matrices. Rebonato and Jackel (1999) have suggested a simple algorithm but it was not that efficient. Later on, by the help of faster processors in computer science, new methods are proposed.

			Country 1											Country 2										
			Market Risk						Act.	Operational Risk				Market Risk						Act.	Operational Risk			
			m1	m2	m3	m4	m5	m6	a1	o1	o2	o3	o4	m1	m2	m3	m4	m5	m6	a1	o1	o2	o3	o4
Country 1	Market Risk	m1	1																					
		m2	ρ1	1																				
		m3	ρ2	ρ11	1																			
		m4	ρ3	ρ12	ρ20	1																		
		m5	ρ4	ρ13	ρ21	ρ28	1																	
		m6	ρ5	ρ14	ρ22	ρ29	ρ35	1																
	A.	a1	ρ6	ρ15	ρ23	ρ30	ρ36	ρ41	1															
	Op. Risk	o1	ρ7	ρ16	ρ24	ρ31	ρ37	ρ42	ρ46	1														
		o2	ρ8	ρ17	ρ25	ρ32	ρ38	ρ43	ρ47	ρ50	1													
		o3	ρ9	ρ18	ρ26	ρ33	ρ39	ρ44	ρ48	ρ51	ρ53	1												
		o4	ρ10	ρ19	ρ27	ρ34	ρ40	ρ45	ρ49	ρ52	ρ54	ρ55	1											
Country 2	Market Risk	m1	ρ56											1										
		m2	ρ57	ρ67										ρ122	1									
		m3	ρ58	ρ68	ρ77									ρ123	ρ132	1								
		m4	ρ59	ρ69	ρ78	ρ86								ρ124	ρ133	ρ141	1							
		m5	ρ60	ρ70	ρ79	ρ87	ρ94							ρ125	ρ134	ρ142	ρ149	1						
		m6	ρ61	ρ71	ρ80	ρ88	ρ95	ρ101						ρ126	ρ135	ρ143	ρ150	ρ156	1					
	A.	a1	ρ62	ρ72	ρ81	ρ89	ρ96	ρ102	ρ107					ρ127	ρ136	ρ144	ρ151	ρ157	ρ162	1				
	Op. Risk	o1	ρ63	ρ73	ρ82	ρ90	ρ97	ρ103	ρ108	ρ112				ρ128	ρ137	ρ145	ρ152	ρ158	ρ163	ρ167	1			
		o2	ρ64	ρ74	ρ83	ρ91	ρ98	ρ104	ρ109	ρ113	ρ116			ρ129	ρ138	ρ146	ρ153	ρ159	ρ164	ρ168	ρ171	1		
		o3	ρ65	ρ75	ρ84	ρ92	ρ99	ρ105	ρ110	ρ114	ρ117	ρ119		ρ130	ρ139	ρ147	ρ154	ρ160	ρ165	ρ169	ρ172	ρ174	1	
		o4	ρ66	ρ76	ρ85	ρ93	ρ100	ρ106	ρ111	ρ115	ρ118	ρ120	ρ121	ρ131	ρ140	ρ148	ρ155	ρ161	ρ166	ρ170	ρ173	ρ175	ρ176	1

Level 0 : between sub-risks, within a risk type (for Market Risk)
Level 1 : between risk types, within a country (for Country 1)
Level 2 : between countries

Figure 9: Full correlation matrix for 2 countries

Given a symmetric matrix $G \in S^n$, computing its nearest correlation matrix, a problem from finance, is studied by Higham (2002) and is given by

$$\begin{aligned} \min \ & \frac{1}{2}||G - X||^2 \\ \text{s.t. } & X_{ii} = 1, \quad i = 1, \dots, n \\ & X \in S^n_+, \end{aligned} \tag{2}$$

where S^n and S^n_+ are respectively the space of $n \times n$ symmetric matrices and the cone of positive semi-definite matrices in S^n; and $|| \cdot ||$ is the Frobenius norm. In the paper of Qi and Sun (2006) a method, which uses "The Newton Algorithm" is proposed to solve this optimisation problem and obtain the nearest correlation matrix. They explain the reason of using the Newton Method as, "The success of Newton's method for solving the convex best interpolation problem motivates us to study Newton's method for matrix nearness problem." The algorithm uses an iterative approach.

In a recent paper of Gao and Sun (2009), this optimisation problem is improved. New two constraints are added and a more general case than (2) is found. By the help of first constraint, the individual may fix not only the diagonal elements of the matrix X to 1 but also off-diagonal elements to any given number. The second constraint allows to set upper and lower bounds for the intended correlations in X. Therefore, the following optimisation problem can be derived for our purpose:

$$\min \frac{1}{2}||G - X||^2 \tag{3}$$

$$\text{s.t. } X_{ij} = G_{ij}, \qquad G_{ij} : \text{fixed correlations in } G \tag{4}$$

$$X_{ij} \geq (G_{ij} - k_{ij}) \qquad G_{ij} : \text{conservative correlations in } G \tag{5}$$

$$X \in S_+^n \qquad S_+^n : \text{the cone of PSD matrices}$$

The constraint (4) preconditions fixed correlations from given matrix G, including the diagonal elements equals to 1. It helps during the process of finding the nearest correlation matrix, to keep the correlations that are data driven or unwanted to be changed as fixed. Next, the constraint (5) allows to set lower bounds for some correlations in X_{ij} where k_{ij} is a positive number for each X_{ij} reflecting conservativeness allowance. This feature prevents to end up dramatic reductions in correlations in order to keep the conservativeness at least at the same level.

So, in Figure 8, the blue correlations can now be fixed and the nearest PSD correlation matrix is obtained. The authors Gao and Sun (2009) also provide an algorithm[1] in order to solve the optimisation problem (3). The results are convincing where the lower bounds are not exceeded, the correlations that are preconditioned as fixed are slightly changed and the nearest correlation matrix is achieved by satisfying PSD as well.

[1]The MATLAB code is available from `http://www.math.nus.edu.sg/~matsundf/CaliMat1.m`

3.4 Diversification Benefits and Allocation

For pension funds, the pooling of risks across portfolios and countries acquires diversification. The success of the diversification benefits depend on the degree of dependence between the pooled risks. Aggregate solvency capital should reflect the diversification benefits. In order to measure the diversification benefits, two risk measures, which are Expected Shortfall and Value-at-Risk, are going to be mentioned in details.

3.4.1 Value-at-Risk Calculations

Value-at-Risk (VaR) is a widely used risk measure in order to determine the economic capital required. The basic formula for VaR is:

$$\mathrm{VaR}_i = v_i \times \alpha \times \sigma_i \tag{6}$$

where v_i is the market value of the i^{th} asset, σ_i is the volatility of that asset, and α represents the desired level of confidence. This structure uses the market value of the position denominated in local currency, and as a result the standard deviation parameter is a dimensionless volatility.

Examining the formula for the variance of the portfolio returns is essential because it reveals how the correlations of the returns of the assets in the portfolio affect volatility.

The variance of the portfolio returns, R_i, is represented as follows (Jorion, 2001, chap. 7):

$$\begin{aligned} \sigma_P^2 &= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \rho_{i,j} \sigma_i \sigma_j \\ &= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \rho_{i,j} \sigma_i \sigma_j \qquad (7) \\ &= \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \sigma_{i,j} \qquad (8) \end{aligned}$$

where:

$$\begin{aligned} \sigma_P^2 &= \text{the variance of the portfolio returns} \\ w_i &= \text{the portfolio weight invested in position } i \\ \sigma_i &= \text{the standard deviation of the return in position } i \\ \sigma_{i,j} &= \text{cov}(R_i, R_j) \\ \rho_{i,j} &= \text{the correlation between the returns of asset } i \text{ and asset } j \end{aligned}$$

The variance of rate of return σ_P^2 also has a matrix representation:

$$\sigma_P^2 = [w_1 \cdots w_N] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \cdots & \sigma_{NN} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}.$$

In case the covariance matrix is denoted as Σ, the variance of the portfolio rate of return can be written more precisely as $\sigma_P^2 = w'\Sigma w$.

Next, (7) leads us to the standard deviation, σ_P, which is:

$$\sigma_P = \sqrt{\sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j<i}^{N} w_i w_j \rho_{i,j}\sigma_i\sigma_j}$$

and the total VaR of the portfolio becomes:

$$\begin{aligned}\text{VaR}_P &= v_P \times \alpha \times \sigma_P \\ &= v_P \times \alpha \times \sqrt{\sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j<i}^{N} w_i w_j \rho_{i,j}\sigma_i\sigma_j} \\ &= \alpha\sqrt{V^T\Sigma V} \end{aligned} \tag{9}$$

where V is the vector of N current market values of each assets and $\sum$ is their covariance matrix. For example, in case of there are two assets:

$$\begin{aligned}\text{VaR}_P &= v_P \times \alpha \times \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2} \\ &= \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2 + 2\rho_{1,2}\text{VaR}_1\text{VaR}_2}\end{aligned}$$

VaR for uncorrelated assets i.e. when $\rho_{1,2} = 0$ is:

$$\text{VaR for uncorrelated positions: } \text{VaR}_P = \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2}$$

VaR for perfectly correlated assets i.e. when $\rho_{1,2} = 1$ is:

$$\text{Undiversified VaR: } \text{VaR}_P = \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2 + 2\text{VaR}_1\text{VaR}_2} = \text{VaR}_1 + \text{VaR}_2$$

In general, undiversified VaR is the sum of all the VaRs of the individual positions in the portfolio when none of those positions are short positions. In this situation, instead of investing in a single asset, the usage of two uncorrelated assets achieves diversification benefits:

$$\text{Diversification Benefits } = (\text{VaR}_1 + \text{VaR}_2) - \sqrt{\text{VaR}_1^2 + \text{VaR}_2^2}$$

Under certain assumptions, the portfolio standard deviation of returns for a portfolio with more than two assets has a very precise formula (Kaplan, 2008). The assumptions are:

- The portfolio is equally weighted.
- All the individual positions have the same standard deviation of returns.
- The correlations between each pair of returns are the same.

Then the formula is:

$$\sigma_P = \sigma\sqrt{\frac{1}{N} + \left(1 - \frac{1}{N}\right)\rho}$$

where:

N = the number of positions

σ = the standard deviation that is equal for all N positions

ρ = the correlation between the returns of each pair of positions

So, as the number of assets increases, the volatility of the total portfolio decreases.

Incremental Value-at-Risk (IVaR): *What is the change in portfolio VaR due to the inclusion of a particular position?* Incremental VaR is the answer for that question. IVaR is the change in VaR from the addition of a new position in a portfolio (Kaplan, 2008).

The calculation is straightforward; VaR is determined with, and without, the position of interest and the difference is the Incremental VaR.

For an existing position i; $\quad \mathrm{IVaR}_i = \mathrm{VaR}_P - \mathrm{VaR}_{P-i}$

For a new or potential position i; $\quad \mathrm{IVaR}_i = \mathrm{VaR}_{P+i} - \mathrm{VaR}_P$

Marginal Value-at-Risk (MVaR): *Where should the next investment dollar be spent?* Marginal VaR is the answer for that question. MVaR applies to a particular position in a portfolio, and it is the per dollar change in a portfolio VaR that occurs from an additional investment in that position (Kaplan, 2008). It is the partial derivative with respect to the component weight:

$$\mathrm{MVaR}_i = \frac{\partial \mathrm{VaR}_P}{\partial v_i}$$

Differentiating (8) with respect to w_i (Jorion, 2001, chap. 7):

$$\frac{\partial \sigma_P^2}{\partial w_i} = 2w_i\sigma_i^2 + w\sum_{j=1,j\neq i}^{N} w_j\sigma_{ij} = 2\mathrm{cov}\left(R_i, w_iR_i + \sum_{j\neq i}^{N} w_jR_j\right) = 2\mathrm{cov}(R_i, R_P)$$

Since $\partial\sigma_P^2 = 2\sigma_P\partial\sigma$, we have

$$\frac{\partial \sigma_P}{\partial w_i} = \frac{\mathrm{cov}(R_i, R_P)}{\sigma_P}$$

Hence,

$$\mathrm{MVaR}_i = \frac{\partial \mathrm{VaR}_P}{\partial v_i} = \frac{\partial \alpha\sigma_P v_P}{\partial w_i v_P} = \alpha\frac{\partial\sigma_P}{\partial w_i} = \alpha\frac{\mathrm{cov}(R_i, R_P)}{\sigma_P} = \frac{\alpha\sigma_{iP}}{\sigma_P}$$

Note that $\sigma_P^2 = w'\Sigma w$ and $\sigma_{ij} = \Sigma w$. Thus, the marginal VaR for the i^{th} component corresponds to the i^{th} component of the following vector:

$$\mathrm{MVaR}_i = \alpha\frac{\Sigma w}{\sqrt{w'\Sigma w}} \tag{10}$$

In practice, the Marginal VaR of all positions are typically calculated at one time by using (10).[2]

Component Value-at-Risk (CVaR): Component VaR can be used as an Economic Capital allocation method. It provides the proportion of the portfolio VaR that can be attributed to each of the components of the portfolio. CVaR has at least three desirable properties:

1. If the components partition the portfolio (i.e. are disjoint and exhaustive), then the CVaRs should sum to the (diversified) portfolio VaR.
2. If the components were to be deleted from the portfolio, the CVaR should tell us, at least approximately, how the portfolio VaR will change.
3. CVaR will be negative for components which act to hedge the remainder of the portfolio.

Component VaR is calculated by using the Marginal VaR to allocate the portfolio VaR across the various sub components of that portfolio:

$$\mathrm{CVaR}_i = \mathrm{MVaR}_i \times v_i = v_i \frac{\partial \mathrm{VaR}_P}{\partial v_i} \tag{11}$$

By using CVaR, the total VaR of the portfolio can be expressed as:

$$\mathrm{VaR}_P = \sum_{i=1}^{N} \mathrm{CVaR}_i$$

It is quite tempting to use Component VaR as a proxy for allocation of risk capital across positions, desks, or countries since they will sum to the enterprise VaR.

[2]Some practitioners define Marginal VaR as Incremental VaR and vice versa.

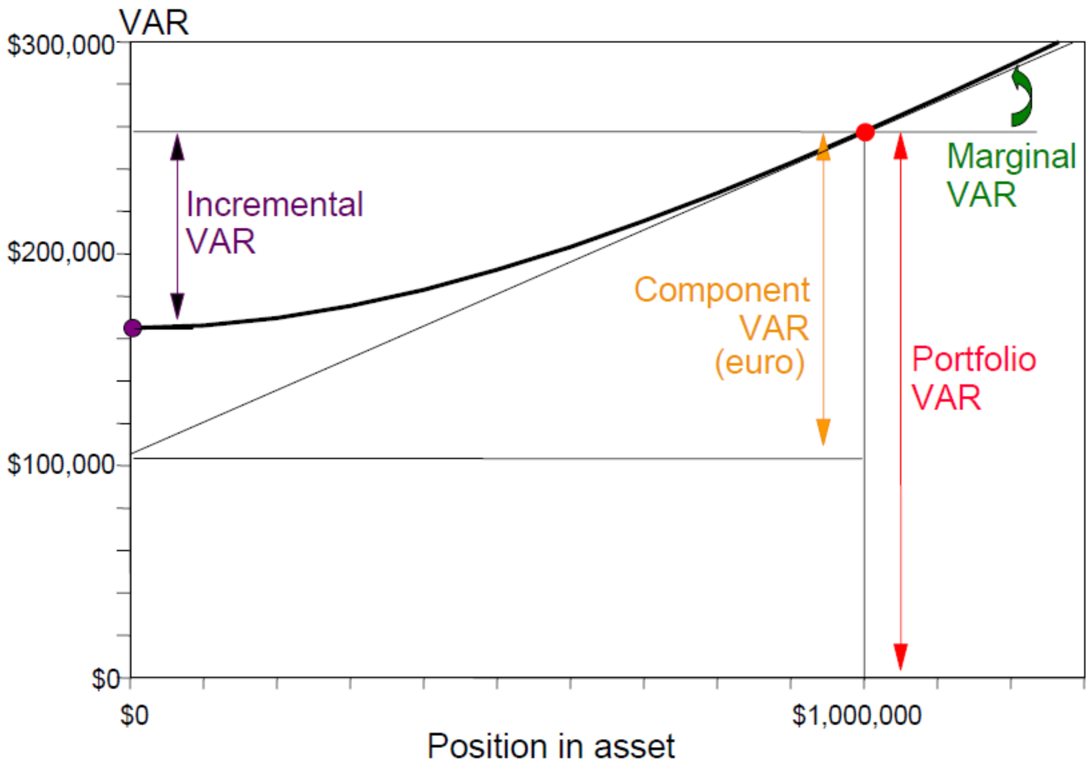


Figure 10: VaR Decomposition (Jorion, 2001, chap. 7)

In conclusion, Incremental VaR measures the impact of adding or deleting a position, Marginal VaR measures the impact of small changes in a position, and Component VaR allocates the portfolio VaR to the positions or desk, or countries based on Marginal VaR (See Figure 10). All three measures provide a wealth of information to risk managers, but need to be used properly with a clear understanding of their interpretation.

3.4.2 Expected Shortfall Calculations

In comparison to VaR, Expected Shortfall is more sensitive to the shape of the loss distribution in the tail of the distribution. For a given probability level p, Expected Shortfall, $\mathrm{ES}_p[X]$ of a random variable X is defined by [3]

$$\mathrm{ES}_p[X] = E[X|X > Q_p[X]], \qquad 0 < p < 1, \tag{12}$$

where Q_p stands for the quantile function (Dhaene et al., 2008):

$$Q_p[X] = \inf\{x|F_X(x) \geq p\}, \qquad 0 < p < 1. \tag{13}$$

Dhaene et al. (2008) mention ES of a normal random variable as follows;

Assume that $X \sim N(\mu, \sigma^2)$ with $\sigma^2 > 0$. Then the ES's of X:

$$ES_p[X] = \mu + \sigma \frac{\Phi'(\Phi^{-1}(p))}{1-p}, \qquad 0 < p < 1, \tag{14}$$

where Φ stands for the cumulative distribution function (cdf) and Φ' the related pdf of the standard normally distributed random variable $Z \sim N(0,1)$. Furthermore, Φ^{-1} is the quantile function of the standard normal cdf.

ES of a portfolio is easy to compute if the returns are assumed to be normally distributed. This study is materialized under the assumption of normality. In addition, $\mu = 0$ can be assumed without loss of generality and then the Expected Shortfall of a position v_i

[3] Expectations of random variables are assumed to exist when required.

becomes:

$$\mathrm{ES}_p[X] = v_i \times \frac{\Phi'(\Phi^{-1}(p))}{1-p} \times \sigma_i \tag{15}$$

So, when the profit-loss distribution is normal, both VaR and ES give almost the same information. They are both scalar multiples of the standard deviation.

Tasche (1999) presents a capital allocation principle where the capital allocated to each risk unit can be stated in terms of its contribution to the ES of the aggregate risk. Panjer (2002) introduces a closed-form expression for this allocation principle in the multivariate normal case. Landsman and Valdez (2002) generalise Panjer's result to the class of multivariate elliptical distributions.[4] Afterwards, Dhaene et al. (2008) simplify their result into more elegant way, which is shown in the following statement:

In case $\bar{X} \sim N_n(\bar{\mu}, \bar{\Sigma})$ with $\bar{\Sigma}$ positive definite we have that $S \sim N_1(\mu_S, \sigma_S^2)$ with $\sigma_S^2 > 0$. Then $E[X_i|S > Q_p[S]]$ is given by using (14) as:

$$E[X_i|S > Q_p[S]] = \mu_i + \frac{\sigma_{i,S}}{\sigma_S} \times \Phi' \frac{(\Phi^{-1}(p))}{1-p}, \qquad 0 < p < 1 \tag{16}$$

where:

$$\sigma_S^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_{jk}, \qquad \text{and} \qquad \sigma_{k,S} = \sum_{j=1}^{n} \sigma_{kj}.$$

Likewise Component VaR, (17) can be used in order to allocate the economic capital for position with v_i as follows:

$$E[X_i|S > Q_p[S]] = v_i \times \frac{\sigma_{i,S}}{\sigma_S} \times \Phi' \frac{(\Phi^{-1}(p))}{1-p} \tag{17}$$

[4] The chronology is taken from Dhaene et al. (2008)

3.4.3 Value-at-Risk vs Expected Shortfall

Value-at-Risk is the loss level that will not be exceeded with a specified probability and Expected Shortfall is the expected loss given that the loss is greater than the VaR level. Two portfolios with the same VaR can have very different expected shortfalls because of the distribution of their rate of returns as shown in Figure 11.

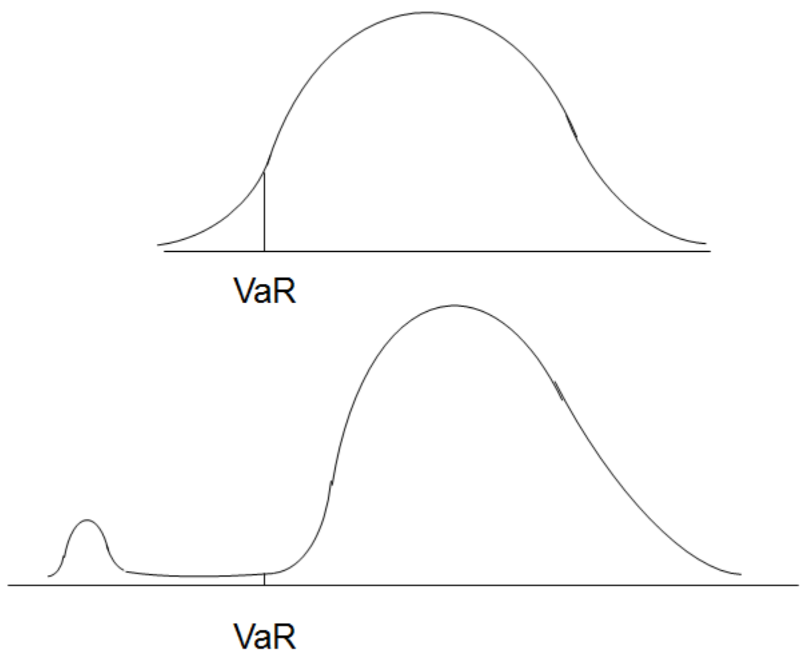


Figure 11: Distribution with the same VaR but different ES (Hull, 2006, chap. 7)

Although VaR is the most commonly used risk measure, ES has an additional property that it is a coherent risk measure. Therefore, ES guarantees that the portfolio diversification is always positive. In comparison to VaR, the main drawback of ES is that it is more difficult to understand and it is complicated to compute when normality is not valid. The reasons why VaR is so popular can be listed as follows:

- Regulators base the capital they requires banks to keep on VaR.
- It is easy to understand and the methodology is based on well-known techniques.

- VaR calculation is fast with the covariance-variance method since it is not simulation-based but analytical.

4 Numerical Example

In this section, a numerical example is studied in order to clarify the methodology discussed so far. There is an international pension company which has a pension fund pooling consisting of 3 countries; Netherlands, United Kingdom and Germany. The portfolio weights are 50%, 25% and 25% respectively. As it is shown in Figure 12, only 3 risk categories are considered:

1. **Market (Systematic) Risk** is constituted from 4 sub-risks, which are interest rate, equity, currency and real estate. The market weights are close to average Dutch Pension fund allocation.

2. **Actuarial Risk** consists of one category, which is longevity risk.

3. **Operational Risk** has 3 sub-risks, which are systems, people and external events.

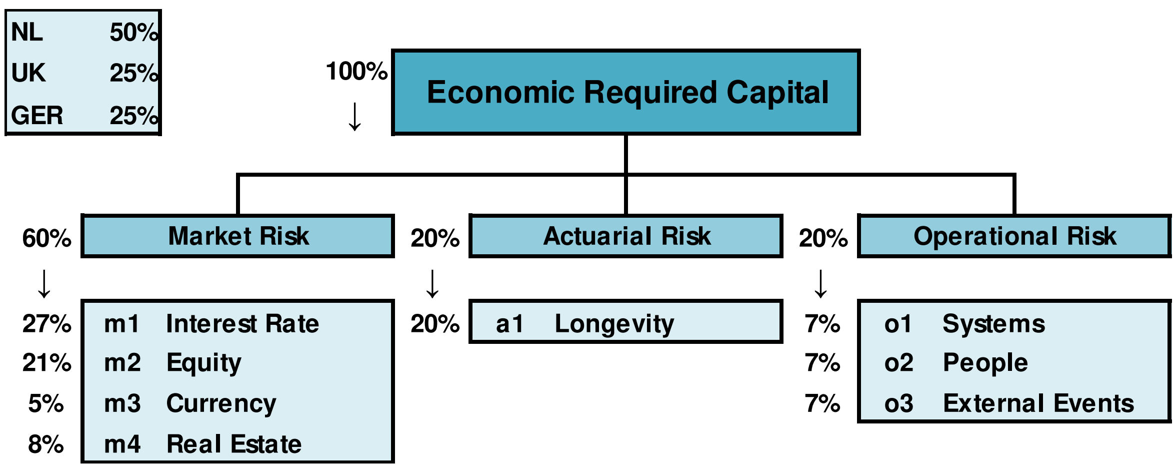


Figure 12: Risk Decomposition

Before starting to numerical example, it is important to emphasize that for the distribution of all risk types normality is assumed. Figure 13 denotes the weights of pension fund pooling, which are the exposed amount to the corresponding risk types. All countries are identical in terms of risk exposure.

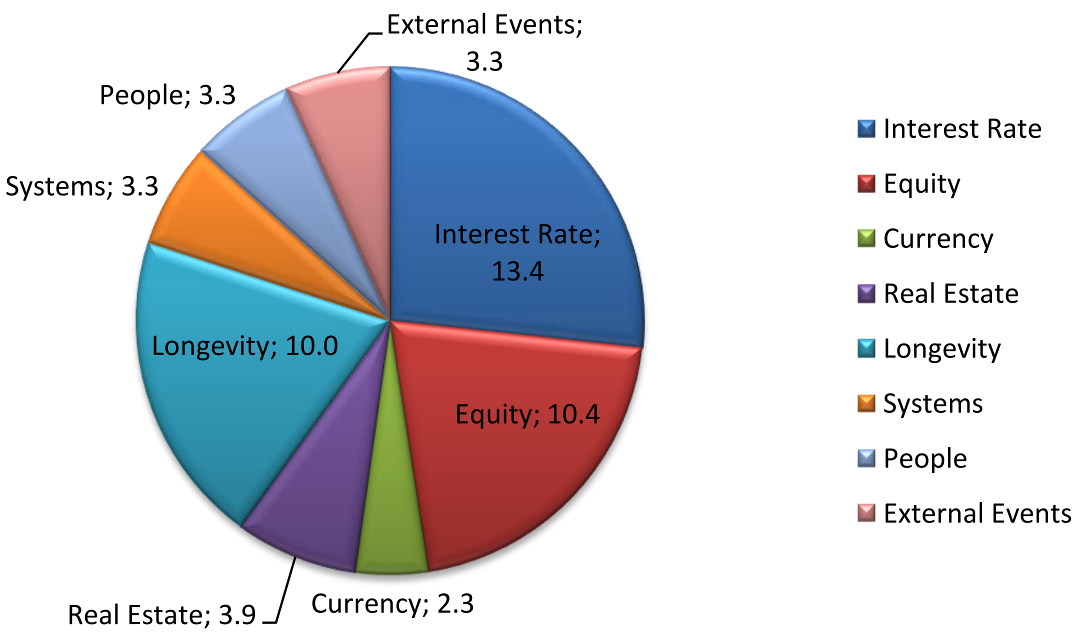


Figure 13: Risk Exposures

In this example, we are seeking for the necessary economic capital for that pension fund. As it is mentioned earlier, the first task is to determine the correlation matrix. Plenty of data are available for market and actuarial risk. However, for the operational risk there is no data available in order to generate the correlations matrix. Therefore, as an expert opinion, APG senior risk managers are consulted to complete the correlation matrix and provide the corresponding volatilities of operational risk factors. Figure 19 shows the constituted matrix.

The next step is to ensure whether the given matrix is a mathematically sound correlation matrix or not. The most overlooked requirement is the positive semi-definiteness of the correlation matrix. The failure to leave PSD out of account could end up with inaccurate results. The given matrix has negative eigenvalues. So, it does not satisfy the PSD property and therefore, it is not a sound correlation matrix. Then one may use the optimisation problem mentioned in (3) to find the nearest correlation matrix. For the constraint (4), all the diagonal components are set as 1 and all of the data-driven correlations (market and actuarial risks) are fixed to the given correlations. It is not expected to see big changes in these correlations. For the constraint (5), $k_{ij} = 0.15$ is assumed for all correlations, which are consulted by an expert, as a secure lower bound. In other words, k_{ij} guarantees those components will not be too smaller such that the difference is not bigger then 0.15.

By using the algorithm proposed in the appendix, the nearest correlation matrix is found as shown in Figure 20. During the execution of the algorithm, first of all the fixed components are detected, which are the correlations between market and actuarial risk. These are presumed as fixed because no expert opinion is involved so far. However, in case there is a computational error, the algorithm tests whether it is appropriate to use those correlations as fixed and if they are not then suggests the nearest correlations to them. Next, the algorithm detects lower bounds for the rest of the correlation matrix and gives the result with respect to assigned parameters.

Figure 21 exhibits the absolute difference between the given matrix and the nearest correlation matrix. The results are:

- There are minor changes in the fixed correlations, which are between market and actuarial risks.
- The substantial changes are observed in operational risk related correlations.

- The biggest changes are between operational-to-operational risks.
- None of the correlations exceed the lower bounds.

After having an economically accurate correlation matrix, now we can move on to determine the necessary economic capital and observe the diversification benefits. Both of the risk measures Value-at-Risk and Expected Shortfall are examined.

4.0.4 VaR as a Risk Measure

Firstly, the stand-alone economic capital is calculated by using the formula given in (6). v_i is the amount of the position to corresponding risk factor. $\alpha = \Phi^{-1}(0.95) = 1.64$ is the confidence level and σ_i is the daily volatility. VaR_i indicates the stand-alone risk capital for each risk factor i as shown in Table 2:

Table 2: *Level 0* Diversification Benefits

α= 1.64			v_i	σ_i	VaR_i	w_i	Σw	$w'\Sigma w$	MVaR_i	CVaR_i	EC	Sum	D. Benefit
NL	m1	Interest Rate	133.50	0.03	7.38	45%	0.001		0.03	3.42			
	m2	Equity	104.40	0.20	34.34	35%	0.018		0.32	32.95			
	m3	Currency	23.10	0.10	3.80	8%	0.002		0.03	0.68			
	m4	Real Estate	39.00	0.20	12.83	13%	0.013	0.009	0.24	9.21	46.26	58.35	21%
	a1	Longevity	100.00	0.06	9.87	100%	0.004	0.004	0.10	9.87	9.87	9.87	0%
	o1	Systems	33.33	0.05	2.74	33%	0.002		0.07	2.25			
	o2	People	33.33	0.05	2.74	33%	0.002		0.07	2.25			
	o3	External Events	33.33	0.05	2.74	33%	0.002	0.002	0.08	2.63	7.13	8.22	13%
UK	m1	Interest Rate	66.75	0.03	3.69	45%	0.002		0.03	1.88			
	m2	Equity	52.20	0.20	17.17	35%	0.018		0.32	16.56			
	m3	Currency	11.55	0.10	1.90	8%	0.002		0.03	0.32			
	m4	Real Estate	19.50	0.20	6.41	13%	0.014	0.009	0.24	4.76	23.52	29.18	19%
	a1	Longevity	50.00	0.06	4.93	100%	0.004	0.004	0.10	4.93	4.93	4.93	0%
	o1	Systems	16.67	0.05	1.37	33%	0.002		0.07	1.08			
	o2	People	16.67	0.05	1.37	33%	0.002		0.07	1.08			
	o3	External Events	16.67	0.05	1.37	33%	0.002	0.002	0.08	1.34	3.51	4.11	15%
GER	m1	Interest Rate	66.75	0.03	3.69	45%	0.002		0.03	1.93			
	m2	Equity	52.20	0.20	17.17	35%	0.018		0.31	16.39			
	m3	Currency	11.55	0.10	1.90	8%	0.002		0.03	0.37			
	m4	Real Estate	19.50	0.20	6.41	13%	0.013	0.009	0.23	4.43	23.12	29.18	21%
	a1	Longevity	50.00	0.06	4.93	100%	0.004	0.004	0.10	4.93	4.93	4.93	0%
	o1	Systems	16.67	0.05	1.37	33%	0.002		0.06	1.07			
	o2	People	16.67	0.05	1.37	33%	0.002		0.06	1.07			
	o3	External Events	16.67	0.05	1.37	33%	0.002	0.002	0.08	1.31	3.45	4.11	16%

The economic capital within a risk type is determined by (9). However, then the question is how to allocate EC to risk factors. By using (10), Marginal VaR is calculated for each risk type and Component VaR is given by (11). At the last column of Table 2, the *level 0* diversification benefits are shown.

Similarly, the diversification benefits with-in a country (*level 1*) can be computed as denoted in Table 3:

Table 3: *Level 1* Diversification Benefits

α= 1.64			v_i	σ_i	VaR$_i$	w_i	Σw	$w'\Sigma w$	MVaR$_i$	CVaR$_i$	EC	Sum	D. Benefit
NL	m1	Interest Rate	133.50	0.03	7.38	27%	0.001		0.02	3.30			
	m2	Equity	104.40	0.20	34.34	21%	0.011		0.31	31.93			
	m3	Currency	23.10	0.10	3.80	5%	0.001		0.03	0.68			
	m4	Real Estate	39.00	0.20	12.83	8%	0.008		0.23	8.85			
	a1	Longevity	100.00	0.06	9.87	20%	0.001		0.02	2.05			
	o1	Systems	33.33	0.05	2.74	7%	0.000		0.01	0.33			
	o2	People	33.33	0.05	2.74	7%	0.000		0.01	0.38			
	o3	External Events	33.33	0.05	2.74	7%	0.001	0.004	0.04	1.30	48.82	76.45	36%
UK	m1	Interest Rate	66.75	0.03	3.69	27%	0.001		0.03	1.82			
	m2	Equity	52.20	0.20	17.17	21%	0.011		0.31	16.05			
	m3	Currency	11.55	0.10	1.90	5%	0.001		0.03	0.33			
	m4	Real Estate	19.50	0.20	6.41	8%	0.009		0.24	4.59			
	a1	Longevity	50.00	0.06	4.93	20%	0.001		0.02	1.01			
	o1	Systems	16.67	0.05	1.37	7%	0.000		0.01	0.22			
	o2	People	16.67	0.05	1.37	7%	0.001		0.01	0.25			
	o3	External Events	16.67	0.05	1.37	7%	0.001	0.004	0.03	0.51	24.77	38.22	35%
GER	m1	Interest Rate	66.75	0.03	3.69	27%	0.001		0.03	1.89			
	m2	Equity	52.20	0.20	17.17	21%	0.011		0.30	15.81			
	m3	Currency	11.55	0.10	1.90	5%	0.001		0.03	0.37			
	m4	Real Estate	19.50	0.20	6.41	8%	0.008		0.22	4.29			
	a1	Longevity	50.00	0.06	4.93	20%	0.001		0.02	1.03			
	o1	Systems	16.67	0.05	1.37	7%	0.000		0.01	0.18			
	o2	People	16.67	0.05	1.37	7%	0.000		0.01	0.20			
	o3	External Events	16.67	0.05	1.37	7%	0.001	0.003	0.03	0.45	24.23	38.22	37%

Finally, the total diversification benefits are found as shown in Table 4. There are 39% diversification benefits in that example. Figure 14 indicates the diversification benefits of each level. VaR (Full Pooling) is the necessary economic capital needed and the VaR (undiversified) is the economic capital in case of no diversification. The difference is the company's total diversification benefits.

Table 4: *Level 2* Diversification Benefits

α= 1.64			v_i	σ_i	VaR_i	w_i	Σw	$w'\Sigma w$	$MVaR_i$	$CVaR_i$	EC	Sum	D. Benefit
NL	m1	Interest Rate	133.50	0.03	7.38	13%	0.001		0.03	3.43			
	m2	Equity	104.40	0.20	34.34	10%	0.010		0.30	31.37			
	m3	Currency	23.10	0.10	3.80	2%	0.001		0.03	0.60			
	m4	Real Estate	39.00	0.20	12.83	4%	0.007		0.22	8.49			
	a1	Longevity	100.00	0.06	9.87	10%	0.000		0.01	1.44			
	o1	Systems	33.33	0.05	2.74	3%	0.000		0.01	0.41			
	o2	People	33.33	0.05	2.74	3%	0.000		0.01	0.44			
	o3	External Events	33.33	0.05	2.74	3%	0.002		0.04	1.45			
UK	m1	Interest Rate	66.75	0.03	3.69	7%	0.001		0.02	1.56			
	m2	Equity	52.20	0.20	17.17	5%	0.010		0.29	15.37			
	m3	Currency	11.55	0.10	1.90	1%	0.001		0.02	0.20			
	m4	Real Estate	19.50	0.20	6.41	2%	0.007		0.22	4.23			
	a1	Longevity	50.00	0.06	4.93	5%	0.000		0.01	0.48			
	o1	Systems	16.67	0.05	1.37	2%	0.000		0.01	0.16			
	o2	People	16.67	0.05	1.37	2%	0.000		0.01	0.16			
	o3	External Events	16.67	0.05	1.37	2%	0.001		0.03	0.48			
GER	m1	Interest Rate	66.75	0.03	3.69	7%	0.001		0.03	1.77			
	m2	Equity	52.20	0.20	17.17	5%	0.010		0.29	15.29			
	m3	Currency	11.55	0.10	1.90	1%	0.001		0.03	0.29			
	m4	Real Estate	19.50	0.20	6.41	2%	0.007		0.21	4.12			
	a1	Longevity	50.00	0.06	4.93	5%	0.000		0.01	0.53			
	o1	Systems	16.67	0.05	1.37	2%	0.000		0.01	0.16			
	o2	People	16.67	0.05	1.37	2%	0.000		0.01	0.17			
	o3	External Events	16.67	0.05	1.37	2%	0.001	0.003	0.03	0.45	93.07	152.89	39%

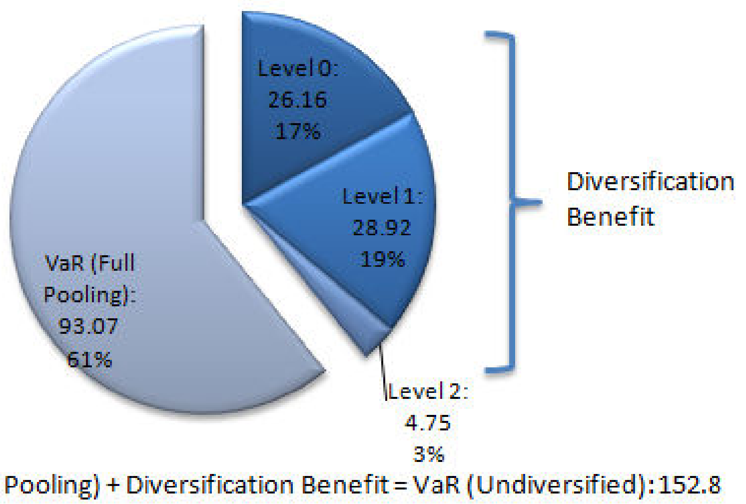


Figure 14: Diversification Benefits

The biggest diversification benefits are especially at the first two levels, 17% and 19%, respectively. The smallest diversification benefits derive from level 2 by 3%. Figure 15

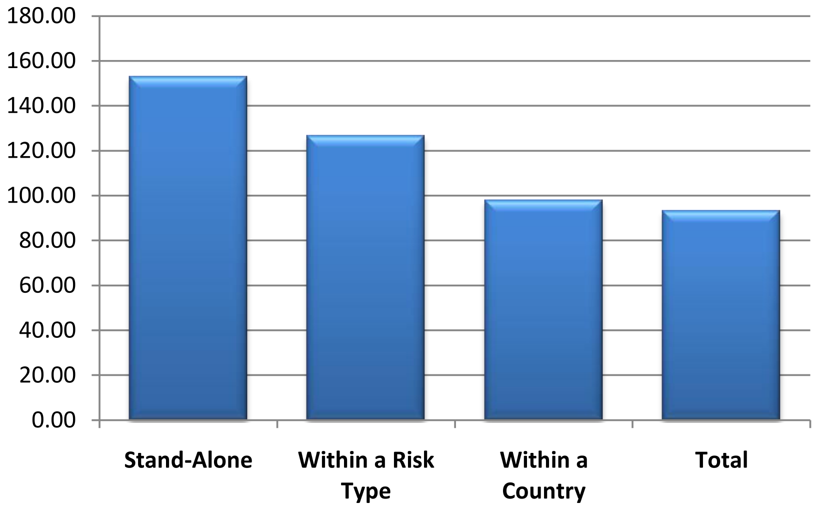
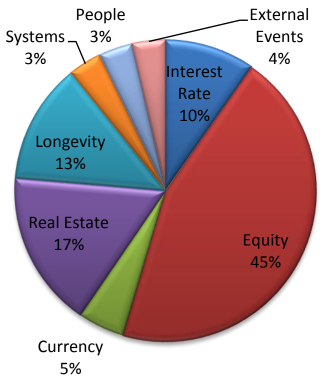
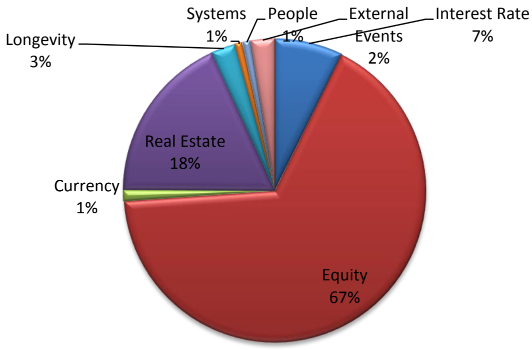


Figure 15: Levels of Diversification

compares the economic capital required at each level. Figure 16a reflects the stand-alone economic capitals before diversification and 16b is the economic capitals required for each sub-risk after diversification. If we compare two figures, there is a noticeable reduction in longevity. This is because longevity's low correlation to other risk factors.



(a) Pre-Diversified Risk Capital, EC = 152.8



(b) Diversified Risk Capital, EC = 93

Figure 16: Economic Capital Comparison

4.0.5 ES as a Risk Measure

The same methodology is used also for Expected Shortfall. Table 5 shows the level 0 diversification benefits:

Table 5: *Level 0* Diversification Benefits

c= 2.06			v_i	σ_i	$ES_p[X_i]$	$\sigma_{k,S}$	$E[X_i \mid S > Q_p[S]]$	σ_S	EC	Sum	D. Benefit
NL	m1	Interest Rate	133.50	0.03	9.25	0.004	3.20				
	m2	Equity	104.40	0.20	43.07	0.066	36.98				
	m3	Currency	23.10	0.10	4.76	0.014	1.69				
	m4	Real Estate	39.00	0.20	16.09	0.065	13.49	0.39	58.02	73.18	21%
	a1	Longevity	100.00	0.06	12.38	0.00	12.38	0.06	12.38	12.38	0%
	o1	Systems	33.33	0.05	3.44	0.005	2.82				
	o2	People	33.33	0.05	3.44	0.005	2.82				
	o3	External Events	33.33	0.05	3.44	0.006	3.30	0.13	8.94	10.31	13%
UK	m1	Interest Rate	66.75	0.03	4.63	0.005	1.93				
	m2	Equity	52.20	0.20	21.53	0.069	18.98				
	m3	Currency	11.55	0.10	2.38	0.012	0.75				
	m4	Real Estate	19.50	0.20	8.04	0.065	6.75	0.39	29.49	36.59	19%
	a1	Longevity	50.00	0.06	6.19	0.00	6.19	0.06	6.19	6.19	0%
	o1	Systems	16.67	0.05	1.72	0.005	1.36				
	o2	People	16.67	0.05	1.72	0.005	1.36				
	o3	External Events	16.67	0.05	1.72	0.006	1.68	0.13	4.40	5.16	15%
GER	m1	Interest Rate	66.75	0.03	4.63	0.006	1.98				
	m2	Equity	52.20	0.20	21.53	0.065	18.14				
	m3	Currency	11.55	0.10	2.38	0.014	0.87				
	m4	Real Estate	19.50	0.20	8.04	0.063	6.62	0.38	28.99	36.59	21%
	a1	Longevity	50.00	0.06	6.19	0.00	6.19	0.06	6.19	6.19	0%
	o1	Systems	16.67	0.05	1.72	0.005	1.34				
	o2	People	16.67	0.05	1.72	0.005	1.34				
	o3	External Events	16.67	0.05	1.72	0.006	1.64	0.13	4.33	5.16	16%

Equation (15) is used to derive $ES_p[X_i]$, which is the stand-alone economic capital for each sub-risks. The number $c = \frac{\Phi'(\Phi^{-1}(0.95))}{1-0.95} = 2.06$ is constant. Since $c = 2.06 > \alpha = 1.64$, ES gives higher results than VaR method. $E[X_i|S > Q_p[S]]$ reflects the allocated amount for the corresponding sub-risk (see (17)). The economic capital required is found by using correlation matrix and the last column of Table 5 reveals the *level 0* diversification benefits.

Similarly, the *level 1* and *level 2* diversification benefits are computed as shown in Tables 6 and 7.

Table 6: *Level 1* Diversification Benefits

c= 2.06			v_i	σ_i	$ES_p[X_i]$	$\sigma_{k,S}$	$E[X_i \mid S > Q_p[S]]$	σ_S	EC	Sum	D. Benefit
NL	m1	Interest Rate	133.50	0.03	9.25	0.005	3.05				
	m2	Equity	104.40	0.20	43.07	0.070	35.23				
	m3	Currency	23.10	0.10	4.76	0.015	1.63				
	m4	Real Estate	39.00	0.20	16.09	0.066	12.48				
	a1	Longevity	100.00	0.06	12.38	0.004	1.87				
	o1	Systems	33.33	0.05	3.44	0.005	0.86				
	o2	People	33.33	0.05	3.44	0.006	0.91				
	o3	External Events	33.33	0.05	3.44	0.013	2.01	0.43	61.22	95.87	36%
UK	m1	Interest Rate	66.75	0.03	4.63	0.01	1.84				
	m2	Equity	52.20	0.20	21.53	0.07	18.02				
	m3	Currency	11.55	0.10	2.38	0.01	0.74				
	m4	Real Estate	19.50	0.20	8.04	0.07	6.28				
	a1	Longevity	50.00	0.06	6.19	0.00	0.93				
	o1	Systems	16.67	0.05	1.72	0.01	0.47				
	o2	People	16.67	0.05	1.72	0.01	0.50				
	o3	External Events	16.67	0.05	1.72	0.01	0.89	0.43	31.06	47.93	35%
GER	m1	Interest Rate	66.75	0.03	4.63	0.01	1.92				
	m2	Equity	52.20	0.20	21.53	0.07	16.99				
	m3	Currency	11.55	0.10	2.38	0.02	0.85				
	m4	Real Estate	19.50	0.20	8.04	0.07	6.21				
	a1	Longevity	50.00	0.06	6.19	0.00	0.95				
	o1	Systems	16.67	0.05	1.72	0.01	0.43				
	o2	People	16.67	0.05	1.72	0.01	0.45				
	o3	External Events	16.67	0.05	1.72	0.01	0.85	0.42	30.39	47.93	37%

Likewise, again 39% diversification benefits are obtained. This is not surprising since under normality assumption, the only difference between ES and VaR is the confidence levels. The proportion $c/\alpha = 1.64/2.06 = 1.25$ is same for all stand-alones and the necessary economic capitals at each levels of diversification between VaR and ES.

Figure 17 represents the levels of diversification with respect to expected shortfall.

Table 7: *Level 2* Diversification Benefits

c= 2.06			v_i	σ_i	$ES_p[X_i]$	$\sigma_{k,S}$	$E[X_i \mid S > Q_p[S]]$	σ_S	EC	Sum	D. Benefit
NL	m1	Interest Rate	133.50	0.03	9.25	0.01	3.39				
	m2	Equity	104.40	0.20	43.07	0.19	33.99				
	m3	Currency	23.10	0.10	4.76	0.04	1.49				
	m4	Real Estate	39.00	0.20	16.09	0.17	11.23				
	a1	Longevity	100.00	0.06	12.38	0.01	1.10				
	o1	Systems	33.33	0.05	3.44	0.02	0.90				
	o2	People	33.33	0.05	3.44	0.02	0.92				
	o3	External Events	33.33	0.05	3.44	0.04	2.14				
UK	m1	Interest Rate	66.75	0.03	4.63	0.01	1.63				
	m2	Equity	52.20	0.20	21.53	0.20	18.04				
	m3	Currency	11.55	0.10	2.38	0.03	0.65				
	m4	Real Estate	19.50	0.20	8.04	0.17	5.78				
	a1	Longevity	50.00	0.06	6.19	0.01	0.49				
	o1	Systems	16.67	0.05	1.72	0.01	0.38				
	o2	People	16.67	0.05	1.72	0.01	0.39				
	o3	External Events	16.67	0.05	1.72	0.03	0.83				
GER	m1	Interest Rate	66.75	0.03	4.63	0.02	1.80				
	m2	Equity	52.20	0.20	21.53	0.19	17.19				
	m3	Currency	11.55	0.10	2.38	0.04	0.74				
	m4	Real Estate	19.50	0.20	8.04	0.17	5.66				
	a1	Longevity	50.00	0.06	6.19	0.01	0.52				
	o1	Systems	16.67	0.05	1.72	0.01	0.39				
	o2	People	16.67	0.05	1.72	0.01	0.39				
	o3	External Events	16.67	0.05	1.72	0.03	0.82	1.20	116.71	191.73	39%

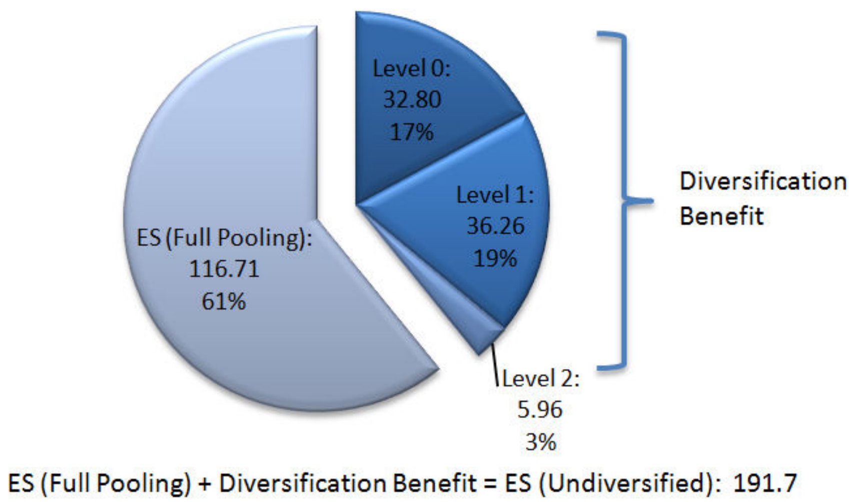
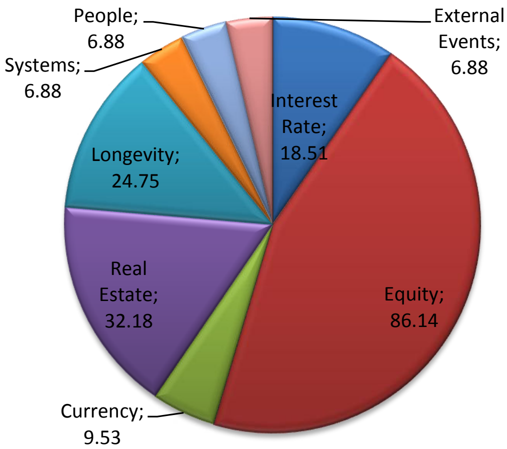
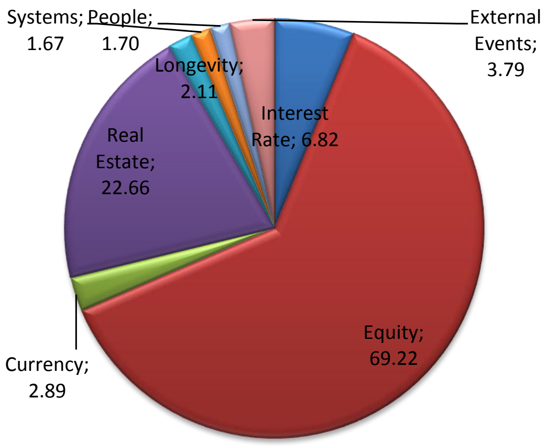


Figure 17: Diversification Benefits

Figure 18 compares the pre-diversified and diversified risk capitals. In comparison to Figure



(a) Pre-Diversified Risk Capital, EC = 191.7



(b) Diversified Risk Capital, EC = 110.8

Figure 18: Economic Capital Comparison

16b, the second noticeable difference between VaR and ES is observed at the allocation of risks. Although in percentage the diversification benefits are same with VaR, EC allocated to each sub-risk is different than VaR.

Finally, a sensitivity analysis is performed in order to observe the changes in the required economic capital. Table 8 demonstrates the results. The variable k stands for the conservativeness level and important for determining the nearest correlation matrix. For $k = 0$, the highest EC is obtained. For the sake of conservativeness, expert-consulted correlations are not allowed to reduce. As expected, for larger values of k the economic capital decreases because of more freedom in correlations and after some value, k makes no sense since the nearest correlations don't get that much smaller values.

The choice of confidence level has the biggest impact on Economic Capital rather than k. Under normality, $EC_{0.99}/EC_{0.95} \approx \Phi^{-1}(0.99)/\Phi^{-1}(0.95)$ for both of the risk measures. An-

Table 8: Economic Capital with respect to different parameters

k	p =0.95		p =0.99	
	VaR	ES	VaR	ES
0.00	95.7	120.0	135.4	155.1
0.05	94.3	118.3	133.4	152.8
0.10	93.5	117.2	132.2	151.4
0.15	93.1	116.7	131.6	150.8
0.20	92.8	116.4	131.3	150.4
0.25	92.7	116.3	131.2	150.3

other result of normality is that both of the risk measures are just multiples of the standard deviation. Expected Shortfall is always higher than Value-at-Risk this is because ES shows the average loss when the loss exceeds the VaR level (Yamai and Yoshiba, 2005):

$$\mathrm{ES}_p(X) = E[X|X \geq \mathrm{VaR}_p(X)].$$

			NL 50%									UK 25%									GER 25%								
			Market Risk				Act.	Operational Risk			Market Risk				Act.	Operational Risk			Market Risk				Act.	Operational Risk		
			m1	m2	m3	m4	a1	o1	o2	o3	m1	m2	m3	m4	a1	o1	o2	o3	m1	m2	m3	m4	a1	o1	o2	o3
NL 50%	Market R.	m1	1	0.35	0	0.15	0	0	0	0.2	0.8	0.3	-0.1	0.2	0	0	0	0.2	0.9	0.4	0	0.25	0	0	0	0.2
		m2	0.35	1	0.1	0.55	0	0	0	0.2	0.3	0.85	0	0.5	0	0	0	0.2	0.4	0.9	0.1	0.5	0	0	0	0.2
		m3	0	0.1	1	0.1	0	0	0	0.2	-0.1	0	0.7	0.1	0	0	0	0.2	0	0.1	1	0	0	0	0	0.2
		m4	0.15	0.55	0.1	1	0	0	0	0.2	0.2	0.5	0.1	0.65	0	0	0	0.2	0.25	0.5	0	0.65	0	0	0	0.2
	A.	a1	0	0	0	0	1	0	0.1	0	0	0	0	0	0.3	0	0	0	0	0	0	0	0.4	0	0	0
	Op. R.	o1	0	0	0	0	0	1	0.2	0.9	0	0	0	0	0	0.7	0.1	0.6	0	0	0	0	0	0.8	0.15	0.7
		o2	0	0	0	0	0.1	0.2	1	0.9	0	0	0	0	0	0.1	0.7	0.6	0	0	0	0	0	0.15	0.8	0.7
		o3	0.2	0.2	0.2	0.2	0	0.9	0.9	1	0.2	0.8	0.2	0.2	0	0.6	0.6	0.4	0.2	0.2	0.2	0.2	0	0.7	0.7	0.5
UK 25%	Market R.	m1	0.8	0.3	-0.1	0.2	0	0	0	0.2	1	0.4	0.1	0.2	0	0	0	0.2	0.4	0.3	-0.1	0.25	0	0	0	0.2
		m2	0.3	0.85	0	0.5	0	0	0	0.8	0.4	1	0.1	0.6	0	0	0	0.2	0.3	0.8	0	0.6	0	0	0	0.2
		m3	-0.1	0	0.7	0.1	0	0	0	0.2	0.1	0.1	1	0	0	0	0	0.2	-0.1	0	0.7	0.1	0	0	0	0.2
		m4	0.2	0.5	0.1	0.65	0	0	0	0.2	0.2	0.6	0	1	0	0	0	0.2	0.25	0.6	0.1	0.6	0	0	0	0.2
	A.	a1	0	0	0	0	0.3	0	0	0	0	0	0	0	1	0	0.1	0	0	0	0	0	0.2	0	0	0
	Op. R.	o1	0	0	0	0	0	0.7	0.1	0.6	0	0	0	0	0	1	0.1	0.9	0	0	0	0	0	0.5	0	0.6
		o2	0	0	0	0	0	0.1	0.7	0.6	0	0	0	0	0.1	0.1	1	0.9	0	0	0	0	0	0	0.5	0.6
		o3	0.2	0.2	0.2	0.2	0	0.6	0.6	0.4	0.2	0.2	0.2	0.2	0	0.9	0.9	1	0.2	0.4	0.2	0.2	0	0.6	0.6	0.3
GER 25%	Market R.	m1	0.9	0.4	0	0.25	0	0	0	0.2	0.4	0.3	-0.1	0.25	0	0	0	0.2	1	0.4	0.1	0.2	0	0	0	0.2
		m2	0.4	0.9	0.1	0.5	0	0	0	0.2	0.3	0.8	0	0.6	0	0	0	0.4	0.4	1	0.1	0.5	0	0	0	0.2
		m3	0	0.1	1	0	0	0	0	0.2	-0.1	0	0.7	0.1	0	0	0	0.2	0.1	0.1	1	0.1	0	0	0	0.2
		m4	0.25	0.5	0	0.65	0	0	0	0.2	0.25	0.6	0.1	0.6	0	0	0	0.2	0.2	0.5	0.1	1	0	0	0	0.2
	A.	a1	0	0	0	0	0.4	0	0	0	0	0	0	0	0.2	0	0	0	0	0	0	0	1	0	0.1	0
	Op. R.	o1	0	0	0	0	0	0.8	0.15	0.7	0	0	0	0	0	0.5	0	0.6	0	0	0	0	0	1	0.2	0.8
		o2	0	0	0	0	0	0.15	0.8	0.7	0	0	0	0	0	0	0.5	0.6	0	0	0	0	0.1	0.2	1	0.8
		o3	0.2	0.2	0.2	0.2	0	0.7	0.7	0.5	0.2	0.2	0.2	0.2	0	0.6	0.6	0.3	0.2	0.2	0.2	0.2	0	0.8	0.8	1

Figure 19: Correlation Matrix where Operational Risk is derived from expert opinion

			NL 50%								UK 25%								GER 25%							
			Market Risk				Act.	Operational Risk			Market Risk				Act.	Operational Risk			Market Risk				Act.	Operational Risk		
			m1	m2	m3	m4	a1	o1	o2	o3	m1	m2	m3	m4	a1	o1	o2	o3	m1	m2	m3	m4	a1	o1	o2	o3
NL 50%	Market R.	m1	1	0.35	-0.02	0.16	-0	0	0	0.15	0.78	0.31	-0.09	0.2	-0	0.02	0.02	0.19	0.87	0.39	0.02	0.24	0	0.01	0.01	0.19
		m2	0.35	1	0.1	0.55	-0	0	0	0.36	0.3	0.85	0	0.5	0	-0	-0	0.17	0.4	0.9	0.1	0.5	0	-0.01	-0.01	0.16
		m3	-0.02	0.1	1	0.09	-0	0.03	0.03	0.11	-0.09	-0.01	0.7	0.1	0	0.03	0.03	0.18	0.02	0.1	0.97	0.01	0	0.03	0.03	0.19
		m4	0.16	0.55	0.09	1	0	-0.01	-0.01	0.18	0.2	0.5	0.1	0.65	-0	0.02	0.02	0.17	0.24	0.5	0.01	0.65	-0	0.01	0.01	0.19
	A.	a1	-0	-0	-0	0	1	-0.01	0.09	0.02	-0	0	-0	-0	0.3	0	0	-0	0	0	0	-0	0.4	-0	0	0
	Op. R.	o1	0	0	0.03	-0.01	-0.01	1	0.38	0.75	0.02	0.23	0.03	0.01	-0	0.75	0.16	0.52	0.02	-0.02	0.01	0	-0	0.86	0.22	0.61
		o2	0	0	0.03	-0.01	0.09	0.38	1	0.75	0.02	0.23	0.04	0.01	0	0.16	0.75	0.52	0.02	-0.02	0.01	0	0	0.23	0.85	0.61
		o3	0.15	0.36	0.11	0.18	0.02	0.75	0.75	1	0.21	0.65	0.17	0.25	0	0.45	0.45	0.59	0.16	0.32	0.1	0.25	0	0.55	0.55	0.67
UK 25%	Market R.	m1	0.78	0.3	-0.09	0.2	-0	0.02	0.02	0.21	1	0.4	0.1	0.2	-0	0.02	0.02	0.14	0.42	0.3	-0.11	0.26	0	0.01	0.01	0.17
		m2	0.31	0.85	-0.01	0.5	0	0.23	0.23	0.65	0.4	1	0.1	0.6	-0	0.06	0.06	0.22	0.3	0.8	0.01	0.6	-0	0.09	0.1	0.22
		m3	-0.09	0	0.7	0.1	-0	0.03	0.04	0.17	0.1	0.1	1	0	0	0.03	0.03	0.14	-0.1	-0	0.7	0.1	0	0.02	0.02	0.15
		m4	0.2	0.5	0.1	0.65	-0	0.01	0.01	0.25	0.2	0.6	0	1	0	0.01	0.01	0.17	0.25	0.6	0.1	0.6	0	0	0	0.17
	A.	a1	-0	0	0	-0	0.3	-0	0	0	-0	-0	0	0	1	-0.01	0.09	0.01	0	-0	-0	0	0.2	-0	-0.01	0.01
	Op. R.	o1	0.02	-0	0.03	0.02	0	0.75	0.16	0.45	0.02	0.06	0.03	0.01	-0.01	1	0.27	0.75	0.02	0.05	0.02	0.02	-0	0.61	0.1	0.45
		o2	0.02	-0	0.03	0.02	0	0.16	0.75	0.45	0.02	0.06	0.03	0.01	0.09	0.27	1	0.75	0.02	0.05	0.02	0.01	-0.01	0.1	0.61	0.45
		o3	0.19	0.17	0.18	0.17	-0	0.52	0.52	0.59	0.14	0.22	0.14	0.17	0.01	0.75	0.75	1	0.16	0.29	0.16	0.16	0.01	0.45	0.45	0.57
GER 25%	Market R.	m1	0.87	0.4	0.02	0.24	0	0.02	0.02	0.16	0.42	0.3	-0.1	0.25	0	0.02	0.02	0.16	1	0.4	0.08	0.21	0	0.01	0.01	0.18
		m2	0.39	0.9	0.1	0.5	0	-0.02	-0.02	0.32	0.3	0.8	-0	0.6	-0	0.05	0.05	0.29	0.4	1	0.1	0.5	-0	0.02	0.02	0.14
		m3	0.02	0.1	0.97	0.01	0	0.01	0.01	0.1	-0.11	0.01	0.7	0.1	-0	0.02	0.02	0.16	0.08	0.1	1	0.09	-0	0.01	0.01	0.18
		m4	0.24	0.5	0.01	0.65	-0	0	0	0.25	0.26	0.6	0.1	0.6	0	0.02	0.01	0.16	0.21	0.5	0.09	1	0	0.01	0.01	0.17
	A.	a1	0	0	0	-0	0.4	-0	0	0	0	-0	0	0	0.2	-0	-0.01	0.01	0	-0	-0	0	1	-0	0.1	0
	Op. R.	o1	0.01	-0.01	0.03	0.01	-0	0.86	0.23	0.55	0.01	0.09	0.02	0	-0	0.61	0.1	0.45	0.01	0.02	0.01	0.01	-0	1	0.27	0.7
		o2	0.01	-0.01	0.03	0.01	0	0.22	0.85	0.55	0.01	0.1	0.02	0	-0.01	0.1	0.61	0.45	0.01	0.02	0.01	0.01	0.1	0.27	1	0.7
		o3	0.19	0.16	0.19	0.19	0	0.61	0.61	0.67	0.17	0.22	0.15	0.17	0.01	0.45	0.45	0.57	0.18	0.14	0.18	0.17	0	0.7	0.7	1

Figure 20: Nearest Correlation Matrix

			NL 50%								UK 25%								GER 25%							
			Market Risk				Act.	Operational Risk			Market Risk				Act.	Operational Risk			Market Risk				Act.	Operational Risk		
			m1	m2	m3	m4	a1	o1	o2	o3	m1	m2	m3	m4	a1	o1	o2	o3	m1	m2	m3	m4	a1	o1	o2	o3
NL 50%	Market R.	m1	0	0	0.02	0.01	0	0	0	0.05	0.02	0.01	0.01	0	0	0.02	0.02	0.01	0.03	0.01	0.02	0.01	0	0.01	0.01	0.01
		m2	0	0	0	0	0	0	0	0.16	0	0	0	0	0	0	0	0.03	0	0	0	0	0	0.01	0.01	0.04
		m3	0.02	0	0	0.01	0	0.03	0.03	0.09	0.01	0.01	0	0	0	0.03	0.03	0.02	0.02	0	0.03	0.01	0	0.03	0.03	0.01
		m4	0.01	0	0.01	0	0	0.01	0.01	0.02	0	0	0	0	0	0.02	0.02	0.03	0.01	0	0.01	0	0	0.01	0.01	0.01
	A.	a1	0	0	0	0	0	0.01	0.01	0.02	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Op. R.	o1	0	0	0.03	0.01	0.01	0	0.18	0.15	0.02	0.23	0.03	0.01	0	0.05	0.06	0.08	0.02	0.02	0.01	0	0	0.06	0.07	0.09
		o2	0	0	0.03	0.01	0.01	0.18	0	0.15	0.02	0.23	0.04	0.01	0	0.06	0.05	0.08	0.02	0.02	0.01	0	0	0.08	0.05	0.09
		o3	0.05	0.16	0.09	0.02	0.02	0.15	0.15	0	0.01	0.15	0.03	0.05	0	0.15	0.15	0.19	0.04	0.12	0.1	0.05	0	0.15	0.15	0.17
UK 25%	Market R.	m1	0.02	0	0.01	0	0	0.02	0.02	0.01	0	0	0	0	0	0.02	0.02	0.06	0.02	0	0.01	0.01	0	0.01	0.01	0.03
		m2	0.01	0	0.01	0	0	0.23	0.23	0.15	0	0	0	0	0	0.06	0.06	0.02	0	0	0.01	0	0	0.09	0.1	0.02
		m3	0.01	0	0	0	0	0.03	0.04	0.03	0	0	0	0	0	0.03	0.03	0.06	0	0	0	0	0	0.02	0.02	0.05
		m4	0	0	0	0	0	0.01	0.01	0.05	0	0	0	0	0	0.01	0.01	0.03	0	0	0	0	0	0	0	0.03
	A.	a1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0.01	0.01	0	0	0	0	0	0	0.01	0.01
	Op. R.	o1	0.02	0	0.03	0.02	0	0.05	0.06	0.15	0.02	0.06	0.03	0.01	0.01	0	0.17	0.15	0.02	0.05	0.02	0.02	0	0.11	0.1	0.15
		o2	0.02	0	0.03	0.02	0	0.06	0.05	0.15	0.02	0.06	0.03	0.01	0.01	0.17	0	0.15	0.02	0.05	0.02	0.01	0.01	0.1	0.11	0.15
		o3	0.01	0.03	0.02	0.03	0	0.08	0.08	0.19	0.06	0.02	0.06	0.03	0.01	0.15	0.15	0	0.04	0.11	0.04	0.04	0.01	0.15	0.15	0.27
GER 25%	Market R.	m1	0.03	0	0.02	0.01	0	0.02	0.02	0.04	0.02	0	0	0	0	0.02	0.02	0.04	0	0	0.02	0.01	0	0.01	0.01	0.02
		m2	0.01	0	0	0	0	0.02	0.02	0.12	0	0	0	0	0	0.05	0.05	0.11	0	0	0	0	0	0.02	0.02	0.06
		m3	0.02	0	0.03	0.01	0	0.01	0.01	0.1	0.01	0.01	0	0	0	0.02	0.02	0.04	0.02	0	0	0.01	0	0.01	0.01	0.02
		m4	0.01	0	0.01	0	0	0	0	0.05	0.01	0	0	0	0	0.02	0.01	0.04	0.01	0	0.01	0	0	0.01	0.01	0.03
	A.	a1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.01	0.01	0	0	0	0	0	0	0	0
	Op. R.	o1	0.01	0.01	0.03	0.01	0	0.06	0.08	0.15	0.01	0.09	0.02	0	0	0.11	0.1	0.15	0.01	0.02	0.01	0.01	0	0	0.07	0.1
		o2	0.01	0.01	0.03	0.01	0	0.07	0.05	0.15	0.01	0.1	0.02	0	0.01	0.1	0.11	0.15	0.01	0.02	0.01	0.01	0	0.07	0	0.1
		o3	0.01	0.04	0.01	0.01	0	0.09	0.09	0.17	0.03	0.02	0.05	0.03	0.01	0.15	0.15	0.27	0.02	0.06	0.02	0.03	0	0.1	0.1	0

Figure 21: Absolute changes between matrices

5 Conclusion

This thesis considers the problem of determining appropriate solvency capital requirements to be set by pension funds. It has been shown that the diversification benefits play an important role during that process. Levels of diversification benefits are discussed. The risk classes are briefly mentioned.

Some of the problems about determining economic capital is examined in the frame of normality. Classification of the underlying risks of a pension fund pooling is mentioned and the dependence structure among these risk classes are observed by using linear correlations. In case of lack of data, expert opinion gains a big importance. For a given correlation matrix, an iterative method has been proposed to find the nearest correlation matrix by allowing the individual to keep a conservative behaviour. This is a practical relevance to risk management studies by showing a simple path.

Next, it has been demonstrated how to measure the associated risk by using two simple methods, Value-at-Risk and Expected Shortfall. These risk measures are compared to each other by emphasizing the allocation problem of the total economic capital.

Risk measures, diversification benefits, finding nearest correlation matrix are very crucial and still improving issues. For the future researches, this thesis can be expanded into a more general frameworks rather than normality. Because of some drawbacks of correlations, copula models or other models can be preferred. In addition, a criteria might be proposed about deciding an expert opinion. Different risk measures might be used.

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Appendix

Matlab Code:

```
% Based on the algorithm at
% http://www.math.nus.edu.sg/~matsundf/CaliMat1.m

tau   = .0e-8;                    % parameters
OPTIONS.tau0 = tau;
k = 0.15;                         % conservativeness level
G = xlsread('matrix.xls');        % reads data
% first row of G reflects the fixed components as 1 and others as 0
T = G(1,:)';                      % fixed risk indicator
s = sum(T);                       % number of fixed risk factors
G(1,:)=[];                        % given matrix
[n m]= size(G);

% assigns the fixed components
j=0;u=0;
for i=1:n
    if T(i)==1
        j=j+1;
        K(j)=i;
    else
        u=u+1;
        V(u,:)=[1 i i];
    end
end
for t=1:j^2
    w2(t) = K(ceil(t/j));
    w3(t) = K(mod(t-1,j)+1);
    w1(t) = G(w2(t), w3(t));
end

% checks whether the preconditioned-fixed components are feasible
wx = reshape(w1,s,s);
if min(eig(wx)) < 0 % means wx is not a PSD matrix
    wy = CaliMat1(wx, ones(s,1),[1:1:s]', [1:1:s]', -2,1,1,2,1,1);
    w1 = reshape(wy,1,s^2);
end
```

```
w = [w1' w2' w3']; U = [w; V];
w = sortrows(U,2);
b = w(:,1);
I_b = w(:,2);
J_b = w(:,3);

% assigns the lower bounds
r=1;
for p = 1:n
    for q = 1:n
        x(r,:) = [G(p,q) p q];
        if x(r,:) == w(1,:)
            w(1,:)=[];
            if p==n && q==n
                x(r,:)=[];
            end
        else
            r = r+1;
        end
    end
end

l = x(:,1)-k;
I_l = x(:,2);
J_l = x(:,3);

% the nearest correlation matrix is calculated wrto given constraints
X = CaliMat1(G,b,I_b,J_b,l,I_l,J_l,2,1,1,OPTIONS);
xlswrite('matrix.xls',X,'nearest');     %writes data
```