# Quantifying ambiguity bounds through hypothetical statistical testing

Anne G. Balter & Antoon Pelsser

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# Outline

- Literature
- Model
  - Application I: Robust Investment
  - Deterministic
  - Rectangularity
  - Stochastic
- Divergences
- Application II: Stochastic Example
- Conclusion
- References

# Preview

- Robustness
- Set of alternative models
- Optimisation
- Research on quantification of uncertainty
  - Characteristics of indistinguishable set
  - Link with Type I and II error, test horizon
  - All deviations $|\lambda(t,\omega)| \leq \frac{\Phi^{-1}(1-\beta)-\Phi^{-1}(\alpha)}{\sqrt{T}}$

# Literature Uncertainty

- Risk versus uncertainty
- The Ellsberg paradox (Ellsberg, 1961)
- Bayesian prior, posterior (Thomas Bayes, 1701 1761)
- Multiple prior model from Gilboa and Schmeidler (1989)
- Extension of the standard multiple prior approach Garlappi et al. (2007)

# Literature Uncertainty

- Robustness: Hansen, Sargent and Tallarini (1999); Hansen, Sargent and Turmuhambetova (2006); Hansen, Sargent and Wang (2002); Hansen and Sargent (2008); Hansen and Sargent (2015)
- $\phi$-Divergence: Ben-Tal, Den Hertog, De Waegenaere, Melenberg and Rennen (2013)
- Model Confidence Set: Hansen, Lunde and Nason (2011)
- Confidence Interval for Parameters

# Literature Uncertainty

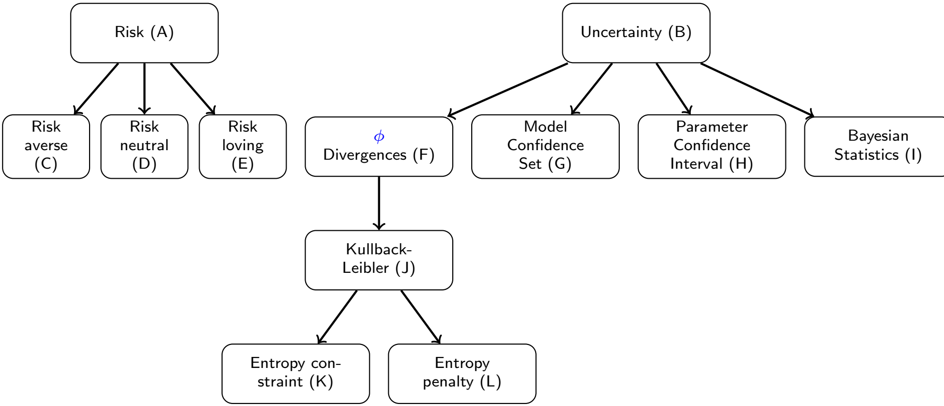



Figure: Overview of Uncertainty Sets

# Robustness

- Gilboa and Schmeidler (1989) maxmin:

$$\max_{\pi} \min_{\mathbb{L}\in\mathcal{L}} \mathbb{E}^{\mathbb{L}}[U(X)]. \tag{1}$$

- Maccheroni et al. (2006) variational preferences

$$\max_{\pi} \min_{\mathbb{L}} \mathbb{E}^{\mathbb{L}}[U(X)] + c(\mathbb{L}|\mathbb{P}). \tag{2}$$

- Hansen and Sargent (2001) relative entropy: $c(\mathbb{L}|\mathbb{P}) = \theta D_{KL}(\mathbb{L}|\mathbb{P})$
- Klibanoff et al. (2005) smooth ambiguity model:

$$\max_{\pi} \mathbb{E}^{P^{\mathbb{L}}} \left[\phi\left(\mathbb{E}^{\mathbb{L}}\left[U(X)\right]\right)\right] \tag{3}$$

- Bayesian: support of prior

# Motivation

- Goal: we would like to identify and characterise the set of alternative models *ex ante*
- Independent from objective problem
- Consider a large class of alternatives
- Statistically indistinguishable models:
  - Based on Type I and II error and test horizon
- Outline
  - Model
  - Stochastic example
  - Deterministic

# Model

- SDEs of form

$$dX = \mu(t,\omega)\,dt + \sigma(t,\omega)\,dW(t)$$

- Possible alternative models $dW_t + \lambda(t,\omega)\,dt$
- Ex ante ($t = 0$)
- Those $\lambda(t,\omega)$ indistinguishable from $\lambda = 0$
- $dW_t + \lambda(t,\omega)dt$ not only adjusting the mean of the probability distribution
- Stochastic example

# Robust Investment

- Merton (1969): Agent generates utility from terminal wealth

$$\max_{\pi} \min_{\mathbb{L} \in \mathcal{L}} \mathbb{E}^{\mathbb{L}}[u(X(T))]$$

- Allocate $\pi$ to risky asset $S$ with

$$dS_t = \mu S_t dt + \sigma S_t \left(dW_t + \lambda dt\right)$$

- Rest on bank account $B$ with riskfree rate $r$
- Equity premium puzzle
- Robust optimal investment strategy subject to $\lambda^2 \leq k^2$

$$\pi^* = \max\left(\frac{\mu - r - \sigma k}{\gamma \sigma^2}, 0\right)$$

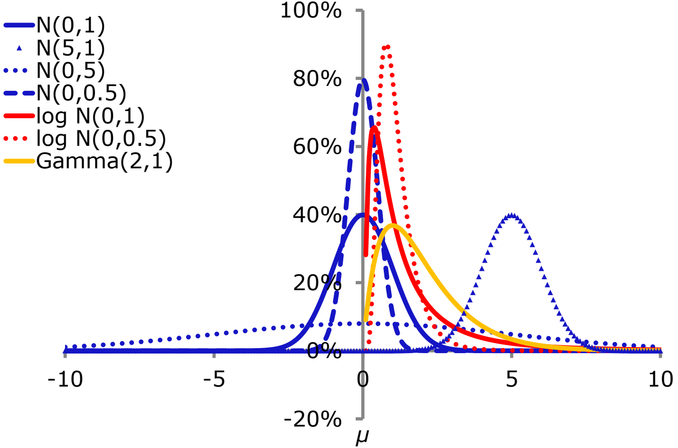
# Strategies

Table: Robust Optimal Investments

| $\gamma$ | $\pi^M$ | $\pi^C_{T=100}$ | $\pi^C_{T=200}$ |
|---|---|---|---|
| 0.5 | 469% | 159% | 250% |
| 1 | 234% | 79.4% | 125% |
| 3 | 78.1% | 26.5% | 41.6% |
| 5 | 46.9% | 15.9% | 25.0% |
| 10 | 23.4% | 7.94% | 12.5% |

For $k = \frac{2.48}{\sqrt{T}} = 0.248, 0.175$ and $\mu - r = 6\%$.

# Alternatives



# Stochastic Example

- Mean repelling process: $\lambda(t,\omega) = a\tanh(aW(t))$
- Alternative fatter tails
- Let

$$L(\mathcal{T}) = \frac{1}{2}\left(e^{-\frac{1}{2}a^2\mathcal{T}+aW(\mathcal{T})} + e^{-\frac{1}{2}a^2\mathcal{T}-aW(\mathcal{T})}\right)$$

- Under $\mathbb{L}$[1] a mixture of $N(a\mathcal{T},\mathcal{T})$ and $N(-a\mathcal{T},\mathcal{T})$, together not normal, mean $0$ and variance $\mathcal{T}+(a\mathcal{T})^2$
- Under $\mathbb{P}$ always $W(\mathcal{T}) \sim N(0,\mathcal{T})$

[1] $\mathbb{P}$ and $\mathbb{L}$ stand for baseline and alternative.

# Model (2)

- Test $H_0 : \mathbb{P} \text{ versus } H_A : \mathbb{L}$
- Hence, $L(\mathcal{T})$ equals the likelihood ratio test statistic
- Radon-Nikodym derivative (Girsanov)

$$L(\mathcal{T}) = \exp\left\{-\frac{1}{2}\int_0^{\mathcal{T}} \lambda(t,\omega)^2 dt + \int_0^{\mathcal{T}} \lambda(t,\omega) dW^{\mathbb{P}}(t,\omega)\right\}$$

- Value $L(\mathcal{T},\omega)$ determined by realisation $\omega$
- Test if model $\mathbb{P}$ should be rejected in favour of model $\mathbb{L}$
- Two simple hypotheses, Neyman-Pearson Lemma most powerful test is likelihood ratio test

# Model (3)

- Type I error: incorrectly rejecting model $\mathbb{P}$

$$\mathbb{P}[L(\mathcal{T}) \geq \gamma] = \alpha$$

- Type II error: incorrectly rejecting model $\mathbb{L}$

$$\mathbb{L}[L(\mathcal{T}) < \gamma] = \beta$$

- Power: probability of accepting model $\mathbb{L}$ when model $\mathbb{L}$ is the true model

$$\begin{aligned} \mathbb{L}[L(\mathcal{T}) \geq \gamma] &= 1 - \beta \\ &= \mathbb{E}^{\mathbb{P}}\left[L(\mathcal{T})\mathbb{1}(L(\mathcal{T}) \geq \gamma)\right] \end{aligned}$$

# Deterministic

- Radon-Nikodym, without $\omega$, log normal
- E.g. $\alpha = 0.05$, then $\Phi^{-1}(\alpha) = -1.64$
- If we take $\beta = 0.20$ then power is $0.80$ and we have $\Phi^{-1}(0.80) = 0.84$
- Hence, the class of all indistinguishable models is then given by all models that satisfy

$$\left(\int_0^{\mathcal{T}} \lambda(t)^2 \, dt\right)^{\frac{1}{2}} \leq \Phi^{-1}(1-\beta) - \Phi^{-1}(\alpha) = 0.84 - (-1.64) = 2.48$$

# Test horizon

$\mathcal{T}$

- Future moment in time at which test would *hypothetically* be performed
- Extra amount of data that one would take into consideration
- Time period during which model would remain the same; no structural break

# Rectangularity

- Rectangularity $\Leftrightarrow$ time-consistency $\Leftrightarrow$ m-stability $\Leftrightarrow$ BSDEs
- Power $\Leftrightarrow$ CVaR $\Leftrightarrow$ Coherent risk measure
- For time-consistent coherent risk measures (Barrieu and El Karoui (2007))
  - $|\lambda(t,\omega)| \leq k$
- Optimal solution at any time-point $t$ does not depend on history between $[0,t]$
- Optimal policy devised at time $0$ for $t > 0$ is still valid at time $t$ given information $\mathcal{F}_t$
- Intersect classes

# Optimisation

- Recall

$$L(\mathcal{T}) = \exp\left\{-\frac{1}{2}\int_0^{\mathcal{T}} \lambda(t,\omega)^2 dt + \int_0^{\mathcal{T}} \lambda(t,\omega) dW^{\mathbb{P}}(t,\omega)\right\}$$

- Optimisation problem

$$\max_{\gamma,|\lambda(t,\omega)|\leq k} \mathbb{E}\left[L(\mathcal{T})\mathbb{1}(L(\mathcal{T}) \geq \gamma)\right] \quad \text{(MP)}$$
$$\text{s.t. } \mathbb{E}\left[\mathbb{1}(L(\mathcal{T}) \geq \gamma)\right] = \alpha$$
$$dL = \lambda(t,\omega) L dW, L_0 = 1$$

- Maximum for $|\lambda(t,\omega)| \equiv k \Rightarrow$ log Normal

**Theorem (Rectangular and Coherent Sets of Indistinguishable Models)**

*Consider a baseline model* $dX(t) = \mu(t,\omega)dt + \sigma(t,\omega)dW(t)$. *The set of all models with* $dW(t) + \lambda(t,\omega)dt$ *and* $|\lambda(t,\omega)| \leq k$ *is rectangular and coherent, where*

$$k = \frac{\Phi^{-1}(1-\beta) - \Phi^{-1}(\alpha)}{\sqrt{\mathcal{T}}} \tag{4}$$

*forms an indistinguishable set for a Type I error of* $\alpha$, *a Type II error of* $\beta$ *and a test horizon* $\mathcal{T}$.

# Bounds on Divergences

- $\phi$-Divergences/$f$-Divergences (non-symmetric distance measures)
- Robust results in optimisation problems
- Continuous $\phi$-divergence

$$D_\phi(L(\mathcal{T},\omega)) = \mathbb{E}^{\mathbb{L}}\left[\phi\left(\frac{1}{L(\mathcal{T})}\right)\right] = \mathbb{E}^{\mathbb{P}}\left[L(\mathcal{T})\phi\left(\frac{1}{L(\mathcal{T})}\right)\right]$$

- For each measure $\phi(\cdot)$ given and convex
- Size of the uncertainty quantified by $c$

$$D_\phi(L(\mathcal{T},\omega)) \leq c$$

- The divergences are expressed in terms of the ratio $x = \frac{1}{L(\mathcal{T},\omega)}$ under the measure $\mathbb{L}$

# Bounds on Divergences

| Divergence | $\phi(x)$ | $c \Leftrightarrow \|\lambda(t,\omega)\| \equiv k$ | $k\sqrt{\mathcal{T}} = 2.48$ |
|---|---|---|---|
| Kullback-Leibler | $x \ln x - x + 1$ | $\frac{1}{2}k^2\mathcal{T}$ | 3.08 |
| Burg entropy | $-\ln x + x - 1$ | $\frac{1}{2}k^2\mathcal{T}$ | 3.08 |
| J-divergence | $(x-1)\ln x$ | $k^2\mathcal{T}$ | 6.15 |
| $\chi^2$-divergence | $\frac{1}{x}(x-1)^2$ | $e^{k^2\mathcal{T}} - 1$ | 467.90 |
| Modified $\chi^2$-divergence | $(x-1)^2$ | $e^{k^2\mathcal{T}} - 1$ | 467.90 |
| Hellinger distance | $(\sqrt{x}-1)^2$ | $2 - 2e^{-\frac{1}{8}k^2\mathcal{T}}$ | 1.07 |
| Variation distance | $\|x-1\|$ | $4N(\frac{1}{2}k\sqrt{\mathcal{T}}) - 2$ | 1.57 |
| $\chi$-divergence of order $\vartheta > 1$ | $\|x-1\|^\vartheta$ | *numerical* | *Table 2* |
| Cressie-Read $\vartheta \neq 0, 1$ | $\frac{1-\vartheta+\vartheta x - x^\vartheta}{\vartheta(1-\vartheta)}$ | *expr* | *Table 2* |

| Divergence $\setminus\vartheta$ | 1.5 | 2.0 | 2.5 | 3.0 |
|---|---|---|---|---|
| $\chi$-divergence of order $\vartheta$ | 10.40 | 467.90 | $1.02 \times 10^6$ | $1.03 \times 10^8$ |
| Cressie-Read | 12.05 | 233.95 | $2.72 \times 10^4$ | $1.72 \times 10^7$ |

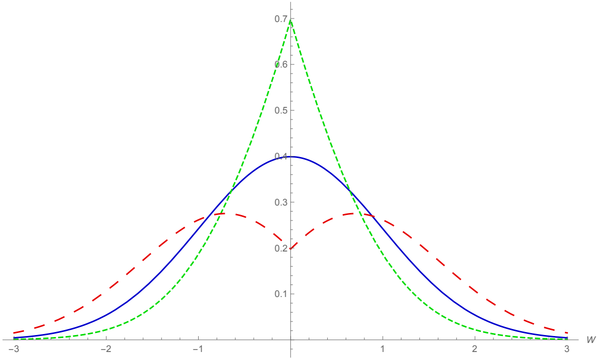
# Stochastic example

- Baseline model: Brownian motion
- Stochastic deviation $\lambda(t, \omega) = a \cdot \mathrm{sgn}\,(W(t, \omega))$
- $a > 0$
  - Mean repelling process
  - Increase the variance of $X(T)$ under model $\mathbb{L}$
- $a < 0$
  - Mean reverting process
  - Decrease the variance of $X(T)$ under model $\mathbb{L}$

# Stochastic example

- Under $\mathbb{P}$
  - $dX(t) = dW(t)$
  - Model $X$ normal distributed
- Under $\mathbb{L}$
  - $dX(t) = a \cdot \mathrm{sgn}(X(t))dt + dW(t)$
  - Model $X$ unknown distribution and variance changed
- Distribution
  - $\ln L(T) \sim^{\mathbb{P}} N\left(-\frac{1}{2}a^2T, a^2T\right)$
  - $\ln L(T) \sim^{\mathbb{L}} N\left(\frac{1}{2}a^2T, a^2T\right)$
- Explicit bound $|a| \leq \frac{\Phi^{-1}(1-\beta)-\Phi^{-1}(\alpha)}{\sqrt{T}}$
- Does satisfy rectangular and coherence axioms

# Stochastic example



# Maenhout

| | 1891-1994 | 1947.2-1996.3 |
|---|---|---|
| $\mu - r$ | 6.258% | 7.852% |
| $r$ | 1.955% | 0.785% |
| $\sigma$ | 0.18534 | 0.15218 |
| $\theta$ | 14 | 237 |
| $\gamma$ | 7 | 10 |
| $k$ | 0.2251 | 0.4951 |
| $(\alpha, 1-\beta, \mathcal{T})$ | (0.05,0.80,122) | (0.05,0.80,25) |
| $(\alpha, 1-\beta, \mathcal{T})$ | (0.05,0.73,100) | (0.05,0.9995,100) |

# Conclusion

- Quantify uncertainty
- Most powerful test
- Ex ante
- For given size and power
- Stochastic deviation from the drift

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