

# Discussion of paper Boon-Werker

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# Outline

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## Summary of paper

- Investigate Variable Annuity for optimal consumption problem
- Use CRRA utility and Black-Scholes-Vašíček model setup
- Analyse optimal solution via “lockbox” approach
- Quantify welfare losses due to sub-optimal VA strategy

# Optimal Consumption Problem

- General problem formulation:

$$\begin{aligned} \max_{\{C_t\}} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T U(t, C_t) dt \right] \\ \text{s.t. } \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T \frac{C_t}{N_t} dt \right] = \frac{W_0}{N_0} \end{aligned}$$

- Maximise  $\mathbb{P}$ -expected utility of consumption
- Under a  $\mathbb{Q}$  budget-constraint (w.r.t. numéraire  $N_t$ )
- $\mathbb{Q}$  is unique for complete and arbitrage-free economy (given  $N_t$ )
- Note: general formulation!

# Optimal Consumption Problem 2

- Use  $\mathbb{P}$ -expectations everywhere:

$$\begin{aligned} \max_{\{C_t\}} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T U(t, C_t) dt \right] \\ \text{s.t. } \mathbb{E}^{\mathbb{P}} \left[ \int_0^T C_t \frac{Q_t}{N_t} dt \right] = W_0 \frac{Q_0}{N_0} \end{aligned}$$

- The process  $Q_t$  is the Radon-Nikodym derivative  $d\mathbb{Q}/d\mathbb{P}$ .
- Note:  $Q_t/N_t$  is the *unique* stochastic discount factor for the economy (irrespective of choice for numéraire  $N_t$ )

# Optimal Consumption Problem 3

- Use stochastic discount factor  $M_t := Q_t/N_t$ :

$$\begin{aligned} \max_{\{C_t\}} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T U(t, C_t) dt \right] \\ \text{s.t. } \mathbb{E}^{\mathbb{P}} \left[ \int_0^T C_t M_t dt \right] = W_0 M_0 \end{aligned}$$

- $M_t$  is the unique stochastic discount factor for the economy
- Note: this is the generalised version of the Boon-Werker setup

# Optimal Consumption Solution

- Build Lagrange function  $\mathcal{L}(\{C_t\}, \lambda)$ :

$$\mathcal{L}(\{C_t\}, \lambda) := \mathbb{E}^{\mathbb{P}} \left[ \int_0^T U(t, C_t) dt \right] - \lambda \left( \mathbb{E}^{\mathbb{P}} \left[ \int_0^T C_t M_t dt \right] - W_0 M_0 \right)$$

- Simplify as:

$$\mathcal{L}(\{C_t\}, \lambda) := \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \left( U(t, C_t) - \lambda C_t M_t \right) dt \right] + \lambda W_0 M_0$$

## Optimal Consumption Solution 2

- Consider impact of small change in consumption plan  $\{C_t + h_t\}$ :

$$\mathcal{L}(\{C_t + h_t\}, \lambda) - \mathcal{L}(\{C_t\}, \lambda) \approx \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \left( U_C(t, C_t) - \lambda M_t \right) h_t dt \right]$$

- For *optimal* consumption path  $C_t^*$ , value function  $\mathcal{L}()$  is not affected by *any* perturbation  $\{h_t\}$ , hence:  $U_C(t, C_t^*) - \lambda M_t \equiv 0 \quad \forall t$
- Express solution as:

$$C_t^* = I(t, \lambda M_t) \quad \forall t,$$

where  $I(t, \cdot)$  is the inverse function of  $U_C(t, \cdot)$

## Optimal Consumption Solution 3

- Consider optimal solution:

$$C_t^* = I(t, \lambda M_t) \quad \forall t$$

- Note: (almost) explicit solution to general formulation of problem!
- Utility function  $U(t, C_t)$  reflected in  $I(t, \cdot)$
- Economy reflected in stochastic discount factor  $M_t$
- Optimal consumption  $C_t^*$  at time  $t$  is determined by  $I(t, \cdot)$  and  $\lambda M_t$
- Lagrange parameter  $\lambda$  “scales” whole consumption path
- Solve budget equation, which is non-linear for  $\lambda$

## Optimal Consumption for CRRA utility

- CRRA:  $U(t, C) = e^{-\beta t} C^{1-\gamma}/(1-\gamma)$ , with  $U_C(t, C) = e^{-\beta t} C^{-\gamma}$
- Solving  $e^{-\beta t} C^{-\gamma} = \lambda M$  gives:  $C_t^* = e^{-(\beta/\gamma)t} (\lambda M_t)^{-(1/\gamma)}$
- For CRRA we can solve budget constraint explicitly for  $\lambda$ :

$$\lambda^{-(1/\gamma)} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T e^{-(\beta/\gamma)t} M_t^{-(1/\gamma)} M_t dt \right] = W_0 M_0$$

- Hence, CRRA allows for fully explicit solution for  $C_t^*$ :

$$C_t^* = e^{-(\beta/\gamma)t} \frac{W_0 M_0}{\mathbb{E}^{\mathbb{P}} \left[ \int_0^T e^{-(\beta/\gamma)t} M_t^{1-(1/\gamma)} dt \right]} M_t^{-(1/\gamma)}$$

- Note: product structure is *very specific* for CRRA!

## Two-Stage Lockbox Formulation

- Authors discuss “Lockbox Separation” of Sharpe (2007)
- Separation into series of static strategies that finances consumption at each date
- What is added value of lockbox separation?
- Follows immediately from general solution:

$$C_t^* = I(t, \lambda M_t) \quad \forall t$$

- Lockbox feels like shoehorning general result into CRRA product-form
- Delete from the paper! But, perhaps I completely missed the point. . .

# Variable Annuity

- Variable Annuity (VA) product has a reference investment portfolio that finances a consumption stream
- I like this! VA has qualitatively the right structure
- We don't know "true" utility-function and "true" model for economy
- But we do know it is definitely not CRRA and BS-Vašíček
- I don't like that you shoehorn VA into CRRA-BSV setup
- Optimal solution in CRRA-BSV is "more accurately computing the wrong answer"
- I don't care about welfare loss relative to CRRA-BSV setup. . .

## Variable Annuity – Robustness

- VA has qualitatively the right structure
- Extend with time-dependent (life-cycle) investment strategy and AIR
- For which combinations of  $U()$  and  $M_t$  would (extended) VA strategy be optimal, given general solution:

$$C_t^{\text{VA}} = I(t, \lambda M_t) \quad \forall t$$

- Can we then frame this into a “robustness” perspective?
- When is VA “approximately the right answer”?
- Could we always find a  $U()$  and  $M_t$  combination that is “close” to a specific benchmark model? Then, VA would be a “robust” strategy!

## Robust VA – Sketch

- For constant MPR's  $\kappa$  the stoch.disc.factor has a log-normal form

$$dM = -rM dt + M \kappa' \cdot dW$$

- For generalised power-util:  $I(t, X) = h(t)X^{-1/\gamma(t)}$  and

$$C_t^{\text{VA}} = h(t)(\lambda M_t)^{-1/\gamma(t)}$$

- Hence,  $C_t^{\text{VA}}$  still has log-normal form, which can be replicated by (extended) VA reference portfolio
- Would this setup be “sufficiently close” to any benchmark model?