An Empirical Investigation of Affine Term Structure Model Uncertainty

Jing LI

Tiburg University

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Motivation

- Practitioners prefer simple models for tractability reason.
- Model uncertainty might be significant, if using simple (nominal) models.
  - nominal model (*NOM*): the model you use.
  - model uncertainty: all uncertainty that causes your model to fail to capture the true DGP.
- Affine term structure models of different estimation approaches and factors choices.

Research Question: How does model uncertainty affect asset pricing using Affine Term Structure Models (ATSM).

Literature

- Hansen and Sargent [2007] v.s. Schneider and Schweizer [2015]
  - possibly complicated nominal models v.s. potentially simple ones.
  - parametric model uncertainty v.s. model misspecification uncertainty.
- Glasserman and Xu [2014] v.s. Perez-Cruz [2008]
  - Divergence calculation by definition v.s. by empirical approach
- Adrian et al. [2013] easy-to-implement estimation for ATSM
  - focus on the different structures of factor models

This Paper

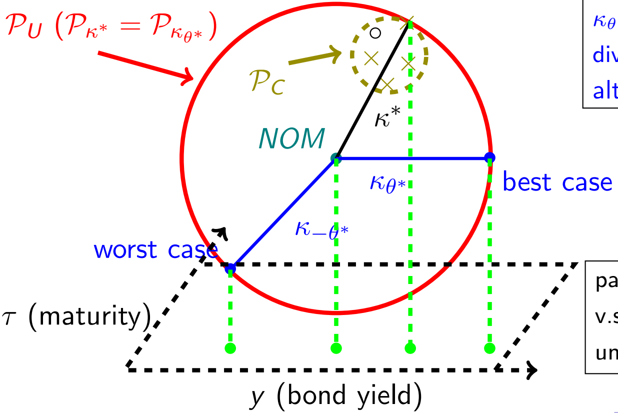
- Analyzes the **uncertainty impacts** of expected yield curves.
    - uncertainty impacts: the outcomes of the best case and the worse case.
- Compares the impacts of **chosen** nominal ATSM.
- Evaluates the impacts by **two empirical approaches** based on different divergence calculations.

Contents

Model Uncertainty Illustration I

- Based on data, **select a model confidence set** ($\mathcal{P}_C$) from a big collection of models with various structures and variables, deemed as a set of empirically indistinguishable models that **captures the true DGP** [Hansen et al., 2011].
- **Quantify the model uncertainty** of the *NOM* by κ^*, the maximal KL divergence from the $\mathcal{P}_C$, and use it to **construct the uncertainty set $\mathcal{P}_U$**.
- Consider the impacts as **the best outcome and the worst outcome**, which are obtained from the alternative models in $\mathcal{P}_U$ **by a change of measure** from the *NOM* [Glasserman and Xu, 2014] .

Model Uncertainty Illustration II



○ : the true DGP.
κ^* : the maximum divergence from $\mathcal{P}_C$.
κ_{θ^*}: equals to κ^*; the divergence from the alternative model.

parameter uncertainty v.s. model misspecification uncertainty.

Application: Affine Term Structure Models

- The bond yield y with maturity of τ at time t is given by

$$y_t^{(\tau)} = A_\tau + B_\tau' X_{t,j} + u_t^{(\tau)}, \quad (1)$$

  where parameters A_τ and B_τ are estimated by Adrian et al. [2013], and $X_{t,j}$ is the pricing-factor vector of model j following a VAR(1) process.
- A model j is defined as a condtional probability $p_j\left(y|X\right)$.
- We are interested in the expected pricing outcomes from $\mathcal{P}_U$
  - $\mathbb{E}_{\boldsymbol{j}}(\boldsymbol{y})$ under $p_j\left(y\right)$;
  - the best case $\mathbf{sup}\,\mathbb{E}_{\boldsymbol{j}}(\boldsymbol{y})$ under $p_{j,sup}\left(y\right)$
  - the worst case $\mathbf{inf}\,\mathbb{E}_{\boldsymbol{j}}(\boldsymbol{y})$ under $p_{j,inf}\left(y\right)$

Evaluation of the Impacts I

Consider the optimization problem

$$\sup_{m \in \mathcal{P}\kappa} \mathbb{E}_{NOM}\left(c \cdot m \cdot y\right), \tag{2}$$

where $\boldsymbol{m}$ is the Radon-Nikodym derivative equal to $\frac{\boldsymbol{p_j(y)}}{\boldsymbol{p_{NOM}(y)}}$. $c = 1$ is for the best case, and $c = -1$ is for the worst case.

- Quantify model uncertainty by **KL divergence**, defined by $\mathbb{E}\left(m \log m\right)$.
- $m_j \in \mathcal{P}\kappa$ implies $\mathbb{E}\left(\boldsymbol{m} \log \boldsymbol{m}\right) \leq \boldsymbol{\kappa}$.

Evaluation of the Impacts II

- Form the dual optimization problem,

$$\inf_{\theta>0} \sup_{m} \mathbb{E}_{NOM} \left[c \cdot m \cdot y - \frac{1}{\theta} \left(m \log m - \kappa \right) \right],$$

The optimal solution is

$$m_\theta^* = \frac{\exp\left(c\theta \cdot y\right)}{\mathbb{E}_{NOM} \left\{ \exp\left[c\theta \cdot y \right] \right\}}, \qquad \theta > 0, \tag{3}$$

By change of measure, the **probability measure** of the best case or the worst case is

$$p_{\theta,NOM}(y) = m_\theta^* \cdot p_{NOM}(y). \tag{4}$$

Approach 1-Evaluation of the Impacts

- The impacts are evaluated by

$$\mathbb{E}_j(y) = \mathbb{E}_{NOM}\left(m^*_{\theta_1} \cdot y\right). \tag{5}$$

- **Measure the KL divergence** by

$$\kappa_{\theta_1} = \mathbb{E}\left(m^*_{\theta_1} \log m^*_{\theta_1}\right), \tag{6}$$

and **calibrate** $\boldsymbol{\theta_1}$ such that $\kappa_{\theta_1} = \kappa^*$, giving θ_1^*.

Approach 2-Evaluation of the Impacts I

- Given $p_{NOM}(y) \sim \mathcal{N}(\mu_y, \Sigma_y)$, then

$$p_{\theta,NOM}(y) \sim \mathcal{N}(\mu_y + c\theta_2\Sigma_y, \Sigma_y)$$

  by $p_{\theta,NOM}(y) = m^*_{\theta_2} \cdot p_{NOM}(y)$.

- The impacts are evaluated by

$$\mathbb{E}_j(y) = \mathbb{E}_{NOM}(y + c\theta_2\Sigma_y) \quad (7)$$

Approach 2-Evaluation of the Impacts II

- KL divergence is **empirically calculated** by the kNN approach [Perez-Cruz, 2008]

$$\kappa_{\theta_2} = \frac{d}{N_2} \sum_{n_2=1}^{N_2} \log \frac{r_k\left(\widetilde{y}_{n_2}\right)}{s_{k+1}\left(\widetilde{y}_{n_2}\right)} + \log \frac{N_1}{N_2 - 1}, \quad (8)$$

  using i.i.d samples $\widetilde{y} = \{\tilde{y}_{n_2}\}_{n_2=1}^{N_2}$ and $\widehat{y} = \{\hat{y}_{n_1}\}_{n_1=1}^{N_1}$ drawn from $\boldsymbol{p_{\theta,NOM}(y)}$ and $\boldsymbol{p_{NOM}(y)}$ respectively.

  - $r_k\left(\widetilde{y}_{n_2}\right)$: the Euclidean distance of the k-th nearest-neighbour of $\widetilde{y}_{n_2}$ in $\widehat{y}$;
  - $s_{k+1}\left(\widetilde{y}_{n_2}\right)$ the Euclidean distance of the $(k+1)$-th nearest-neighbour of $\tilde{y}_{n_2}$ in $\widetilde{y}$.

- $\boldsymbol{\theta_2^*}$ is **calibrated** such that $\kappa_{\theta_2} = \kappa^*$

Data Description I

Panel A : The Estimation Errors of the NSS Approach

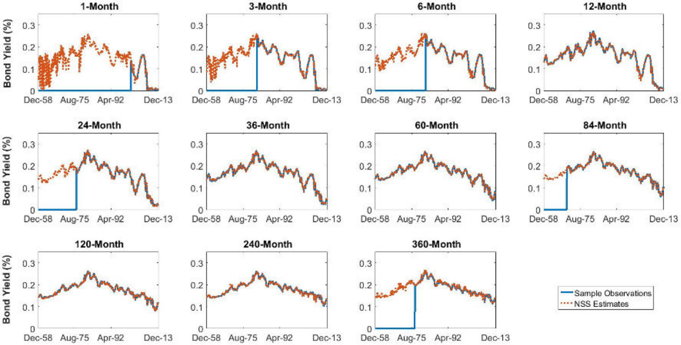
Maturity (month)	|mean| (%)	std (%)	obs
1	0.0020	0.0066	150
3	0.00002	0.0035	384
6	0.0012	0.0033	384
12	0.0005	0.0027	661
24	0.00002	0.0022	451
36	0.0010	0.0020	661
60	0.0004	0.0015	661
84	0.0004	0.0015	534
120	0.0007	0.0017	661
240	0.0005	0.0027	661
360	0.0013	0.0034	443

Panel B :Data descriptions of NSS estimates of bond yields.

Maturity (month)	mean (%)	std (%)	minimum (%)	maximum (%)
1	0.1371	0.0634	0.0004	0.2574
3	0.1437	0.0606	0.0022	0.2586
6	0.1486	0.0596	0.0016	0.2638
12	0.1540	0.0584	0.0033	0.2679
24	0.1605	0.0541	0.0146	0.2710
36	0.1648	0.0497	0.0257	0.2702
60	0.1699	0.0430	0.0443	0.2650
84	0.1728	0.0389	0.0600	0.2617
120	0.1752	0.0353	0.0771	0.2607
240	0.1793	0.0319	0.1032	0.2600
360	0.1828	0.0324	0.1123	0.2656

- Original data of bond yields are **unbalanced.**
- Apply **Nelson-Siegel-Svensson** (NSS) approach to get data balanced.
- Compare the estimates with the original data. **NSS performs well.**

Data Description II



Use NSS estimates as the data input for the following studies.

Model Confidence Set I

- The Great Candidate Set

$$\mathcal{M}_0 = \{[1], [6], [9], [24], [60], [84], [120], [1, 6], [6, 9], [9, 12], [12, 36], [12, 60], [36, 84], [60, 120], [1, 6, 9], [6, 9, 12], [12, 24, 60], [36, 60, 84], [12, 60, 120], [1, 6, 9, 12], [6, 9, 12, 24], [24, 36, 60, 84], [6, 9, 12, 24, 36], [6, 24, 36, 60, 84], [1, 6, 9, 12, 24], [6, 12, 24, 36, 60, 120], [9, 12, 36, 60, 84, 120], [1, 6, 9, 12, 24, 36, 60], [12, 24, 36, 60, 84, 120], [6, 9, 12, 24, 36, 60, 84, 120]\}.$$

- A collection of ATSM based on **factor models**.
- Factors are bond yields with τ-*month* maturities as the numbers indicate.
- The selection procedure first **ranks** them according to a **loss function**, and then **tests** whether the worst has to be eliminated.
- Set significant level $\alpha = 5\%$.

Model Confidence Set II

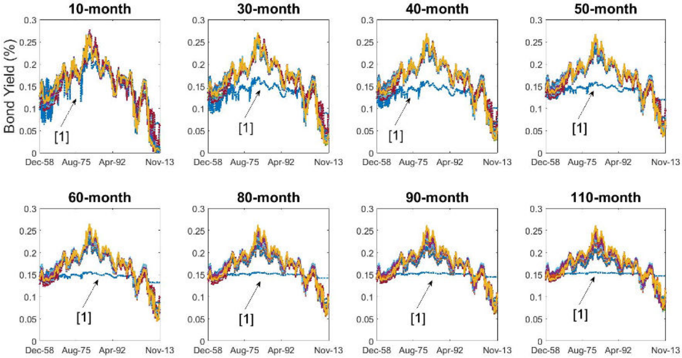
- For instance, pricing the 30-**month bond** gives the ranked set

$$\begin{aligned}\mathcal{M}_r = \{&[1], [120], [84], [6], [1,6], [60], [9], [6,9], [1,6,9], [60,120], [9,12], [24],\\ &[6,9,12], [1,6,9,12], [12,60], [36,84], [36,60,84], [12,36], [12,60,120],\\ &[24,36,60,84], [12,24,60], [6,9,12,24], [1,6,9,12,24],\\ &[9,12,36,60,84,120], [12,24,36,60,84,120], [6,24,36,60,84],\\ &[\mathbf{6,9,12,24,36}], [\mathbf{6,12,24,36,60,120}],\\ &[\mathbf{1,6,9,12,24,36,60}], [\mathbf{6,9,12,24,36,60,84,120}]\}.\end{aligned}$$

The p-value for each r round of test collected correspondingly in the set

$$\begin{aligned}p_r\text{-val}(\mathcal{M}_r) = \{&0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%,\\ &0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%, 0.00\%,\\ &0.01\%, 0.01\%, 0.01\%, 0.01\%, 0.01\%, 0.01\%, 0.01\%, 0.01\%,\\ &1.01\%, 1.22\%, 1.80\%, \mathbf{5.68}\%, \mathbf{6.24}\%, \mathbf{56.54}\%, \mathbf{100}\%\}.\end{aligned}$$

A General Picture of Pricing Results from models in $\mathcal{M}_0$



Model Uncertainty Investigation

- Investigate the bond pricing with maturities of **10-month, 30-month, 40-month, 50-month, 60-month, 80-month, 90-month and 110-month**.
- X_{NOM}s for comparison

$$NOM_1 = [1]; \quad NOM_2 = [120]; \quad NOM_3 = [12, 36];$$
$$NOM_4 = [12, 60]; \quad NOM_5 = [6, 9, 12]; \quad NOM_6 = [12, 60, 120];$$
$$NOM_7 = [1, 6, 9, 12, 24].$$

- **The uncertainty impacts** will form **"uncertainty bands" across maturity**.

κ^* obtained by the kNN approach

Table: The KL divergence κ^*

X_{NOM} \ τ	10	30	60	110
NOM_1	0.0467	0.1854	0.1506	0.1696
NOM_3	0.0008	0.0007	0.0009	0.0093
NOM_5	—	0.0006	0.0021	0.0144
NOM_6	0.0007	0.0005	0.0006	0.0001*
NOM_7	—	0.0004	0.0007	0.0019

- NOM_5 and NOM_7 are in the MCS when pricing 10-month bond, cases **not** considered in this paper.

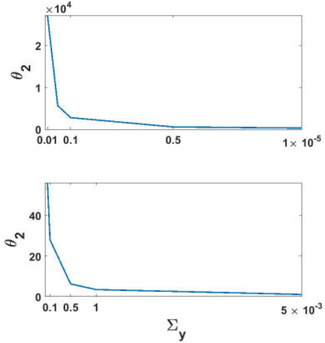
θ^* obtained by Approach 1 and Approach 2

Table: The θ^* for the best cases, based on calibrations by Approach 1 and Approach 2.

θ^*	X_{NOM} \ τ	10	30	60	110
θ_1^*	NOM_1	6.2800	12.1700	12.1680	14.1700
	NOM_3	0.6970	0.7130	0.9620	3.7700
	NOM_5	—	0.6850	1.5250	4.7000
	NOM_6	0.6344	0.6835	0.8250	0.2813
	NOM_7	—	0.5219	0.8500	1.7000
θ_2^*	NOM_1	20.9330	14.8469	12.4250	14.22
	NOM_3	919.00	683.00	108.18	35.1200
	NOM_5	—	111.60	24.00	24.10
	NOM_6	1300.00	747.00	5006.00	3650.00
	NOM_7	—	12060.00	66.00	18.00

A conjecture: θ_2 decreases in Σ_y

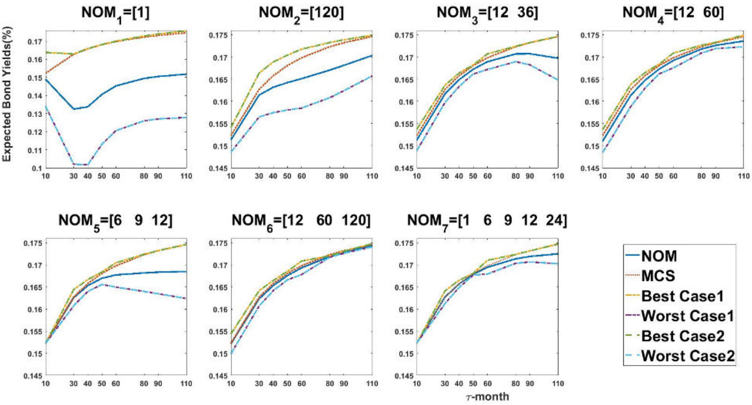
Recall Approach 2



Σ_y	θ_2
1×10^{-7}	2.73×10^{-4}
5×10^{-7}	5.65×10^{-3}
1×10^{-6}	2.78×10^{-3}
5×10^{-6}	548
1×10^{-5}	275
5×10^{-5}	56.2
1×10^{-4}	28
5×10^{-4}	6.3
1×10^{-3}	3.5
5×10^{-3}	1.115

Figure: θ_2 and Σ_y for a same $\kappa^* = 0.0021$ in pricing 60-month bond using NOM_5. The table on the right lists the values plotted in the figure.

Uncertainty Bands of Yield Curves



Summary & Conclusions

- *Evaluation of the impacts by two empirical approaches.* **Impacts** by both approaches are **the same**. But $\boldsymbol{\theta}^*$ are **not**, conjectured resulting from θ_2 decreasing in Σ_y.
- *Comparison of the impacts of chosen nominal ATSM.* All uncertainty bands **satisfy** to include the outcomes from $\mathcal{P}_C$.
- Financial interpretations
  - The largest impacts are in the case using a single-factor model with a short rate.
  - The minimal impacts are in the case using three-factor model with short, medium and long rates.

Topics of Future Research

- Extend by considering more advanced ATSM.
- Study the empirical model uncertainty impacts on institutional investments.
- Develop advanced ATSM incorporating model uncertainty.

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Thank You !

Table: The expected sample variances $\widehat{\Sigma}_y$

X_{NOM} \ τ	10	30	60	110
NOM_1	0.0007	0.0021	0.0020	0.0017
NOM_3	$2.64 \cdot 10^{-6}$	$2.87 \cdot 10^{-6}$	$1.62 \cdot 10^{-5}$	$1.40 \cdot 10^{-4}$
NOM_5	—	$1.68 \cdot 10^{-5}$	$1.16 \cdot 10^{-4}$	$2.54 \cdot 10^{-4}$
NOM_6	$1.69 \cdot 10^{-6}$	$2.43 \cdot 10^{-6}$	$3.21 \cdot 10^{-7}$	$1.03 \cdot 10^{-7}$
NOM_7	—	$1.19 \cdot 10^{-7}$	$2.33 \cdot 10^{-5}$	$1.27 \cdot 10^{-4}$

θ^*