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Optimal Investment for Retirement with Intergenerational Benchmarking

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Abstract

Countries have demonstrated a tendency to switch their second pillar to defined contribution plans, increasing the market-sensitivity of pensions. This can lead to large differences across generations within a pension fund. In this paper we investigate how each cohort should invest if the goal is to prevent the existence of unlucky generations, without reducing overall pension levels. We find a closed-form solution for the optimal life-cycle investment that differs substantially from the classical CRRA investment and bears a strong resemblance to the $100-$age rule of thumb. We show that this effectively reduces differences between consecutive cohorts.

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1 Introduction

The Future Pensions Act (2023) adds the Netherlands to the growing list of countries that have moved their second pillar away from a defined benefit (DB) plan. This transition to defined contribution (DC), which has already been undergone in the private sectors of major economic forces such as the USA, UK and Canada some time ago, transfers the risk-bearing toward the participants of the plan. One of the principle changes entails, as the name of the plan suggests, the cessation of guarantees of the participant's benefits. To mitigate the consequences of the absence of guarantees in this new plan, pension funds' responsibilities are broadened to also include the prevention of so-called fortunate and unfortunate generations, due to fluctuating market returns. This has led to an increase in academic literature on optimal investment under uncertainty, most of them variations on the well-known optimal investment problem introduced by Merton (1969) and Samuelson (1969).

This problem has been widely investigated, with many additions to further improve the real-world applicability, such as stochastic volatility (Heston, 1993), stochastic interest rates (Munk & Sørensen, 2004), stochastic labor supply (Benzoni et al., 2007) and mean-reverting market returns (Wachter, 2002). Others have aimed to improve the specification of behavioral characteristics, through an adjustment of the utility function(von Neumann & Morgenstern, 2004). The earlier works usually consist of concave functions that have wealth (or consumption) as a variable (Hakansson, 1970; Merton, 1969, 1971), whereas the literature afterwards incorporates often wealth relative to some reference or benchmark, rather than as an absolute amount on its own, leaning onto ideas discussed by e.g. Duesenberry (1949), Kahneman and Tversky (1979), and Ryder and Heal (1973). In this paper we employ a form of *habit formation*, which depicts a benchmark that is of a dynamic nature.

The idea of habit formation has gained quite some popularity, ever since it was introduced to model consumption in an economy by Constantinides (1990) and Sundaresan (1989), who apply the habit formulation of Ryder and Heal (1973), and by Abel (1990) who utilizes a discreet framework. By adopting entirely different behavioral models, they also demonstrate the diversity of the formulation of habit formulation. Constantinides (1990) assumes a consumer that evaluates the *difference* between their current consumption and their habit level, whereas Abel (1990) considers the *ratio*. Models that consider differences with some habit level, so-called *additive* models, have shown to be solvable in closed form (Detemple & Zapatero, 1991, 1992; Munk, 2008; Polkovnichenko, 2007; Schroder & Skiadas, 2002) and empirical support has been found in food expenditures (Dynan, 2000). However, additive models suffer from "addictive" behavior (Carroll et al., 2000; Chapman, 1998), meaning consumption will always exceed the habit level. This is not a desirable property in the defined contribution framework of this paper, as there are no "hard" guarantees anymore.

This paper therefore considers a *multiplicative* habit model, meaning we consider the *ratio* of wealth and the habit level, rather than the difference. This has the great advantage that, although individuals aim to exceed their current habit level, it is not a strict necessity. The notion of multiplicative habit formation was introduced by Abel (1990) and further discussed by Galí (1994). Furthermore, they discuss the distinction between internal and external habit models, where habits are endogenously or exogenously formed, respectively. Internal habits have often been invoked to explain individual consumption and saving levels (Bambi & Gozzi, 2020; Carroll, 2000; Carroll et al., 2000; Díaz et al., 2003; Fuhrer, 2000; Gomes & Michaelides, 2003). Furthermore Bovenberg et al. (2020) solve the optimal consumption-investment problem for an individual exhibiting internal multiplicative habit formation, using a linear approximation suggested by Schroder and Skiadas (2002). Although these models are very appropriate to explain consumption and savings on an individual level, they overlook the intergenerational or cross-individual effects. This is problematic when we aim to reduce differences between generations, which is the setting of this paper. To be able to capture the desired intergenerational setting, we will therefore focus our attention on external multiplicative habit formation.

The start of the rise of the external habit literature can be dated back to Abel (1990), who added the term *catching up with the Joneses*[1] to the standard vocabulary in the external habit literature. This term expresses the behavioral phenomenon of individuals comparing their wealth to an aggregate value, or to others in general. This is closely related to the relative income model from Duesenberry (1949). This social comparison has been used in the context of tax policies (Alonso-Carrera et al., 2005), in explaining risk premia (Campbell & Cochrane, 1999; Xiouros & Zapatero, 2010) and in clearing up the growth-savings paradox (Harbaugh, 1996). In this paper, we implement external multiplicative habit formation such that there is a continuous effects of comparisons of retirement wealth between successive cohorts. External habit formations holds the advantage that it can account for cross-generational comparisons, which are implicitly assumed when we assume the possibility of (un)fortunate generations in a pension landscape. Furthermore, multiplicative habit formation ensures the presence of generational comparisons, but prevents the undesirable property that this becomes a hard lower bound (Carroll et al., 2000; Chapman, 1998). Empirical evidence for this social comparison in pension setting was found by Montizaan and Vendrik (2014).

The literature on optimal investment in DC pensions plans has made quite some advances as well, in line with a steadily increase in participation in DC plans in developed countries that has been explored by several authors among who Kruse (1995) and Zelinsky (2004), hinting at the effect of tax reforms and a changing business landscape at the time. Moreover, Larsen and Munk (2023) demonstrated the large positive effect of a mandatory pension plan on welfare of households. Bodie et al. (1988) noted that DB plans hold the advantage over DC in terms of guarantees, whereas DC plans allow for more flexibility in investment strategies. They state that interesting future research might be to combine the best of both worlds: a DC plan with some (low-level) guarantee included. Our formulation of external habit formulation addresses this issue by introducing an intergenerational benchmark, which can be interpreted as a *soft* lower bound[2] for each generation. This distinguished our work from other papers in this context, that either model a minimum guarantee (Deelstra et al., 2003; Grossman & Zhou, 1993), assume a defined benefit pension plan (Frankfurter & Hill, 1981; Tepper, 1974) or consider different benchmarks, without comparisons between generations (Dong & Zheng, 2019; Vigna & Haberman, 2001).

Important to mention is that this paper is concerned with cohorts that invest purely for themselves, without any form of risk-sharing. In other words, we are in *autarky*. In this way this paper separates itself from recent works such as Anthropelos et al. (2024) who consider a linear risk-sharing rule to guarantee a certain retirement benefit. Furthermore, the advantage of our model

[1]"Catching up" is used rather than "keeping up", to emphasize the dependence on lagged values.

[2]As mentioned before, we explicitly do not make this a hard lower bound, to avoid the addictive property (Carroll et al., 2000; Chapman, 1998) and to keep investment flexibility (Chen et al., 2021).

specification is that it can be used *in combination with* methods that look at optimal risk sharing, examples of which are Gollier (2008) and Cui et al. (2011). Moreover, this paper is only looking at investment during the accumulation phase, at the end of which (at retirement) a lump-sum is accumulated by a member of a cohort. Our cohort considers the wealth at retirement as the value of interest, bearing a strong resemblance to a terminal wealth maximization problem. In our setting, cohorts are aiming to reduce the difference between their wealth at retirement and that of preceding cohorts, where no assumptions are made about their post-retirement pension decisions. More specifically, our specification bears a strong resemblance to a "catching up with the Joneses" utility, as introduced by Abel (1990). However, the benchmark represents a comparison to a directly preceding cohort, rather than aggregate consumption, to fit the intergenerational pension setting. Because every cohort compares their wealth to that of the preceding cohort, this results in a continuous "domino" effect of social comparisons.

In this paper, we answer how optimal investment for retirement changes when we aim to prevent large difference between consecutive cohorts, while pension levels remain the same. This is achieved by obtaining a closed-form solution to the terminal wealth problem with a utility function of the "catching up with the Joneses" form (Abel, 1990). This paper is structured as follows. Chapter 2 introduces the model of the economy, pension capital and motivates the choice of utility function. In chapter 3, we solve the optimal control problem, calibrate the model parameters and illustrate the solution graphically. Chapter 4 and 5 are dedicated to demonstrating the effectiveness of our newly found investment strategy in reducing differences between cohorts and in absorbing shocks. Chapter 6 then shows that our strategy is robust, in terms of welfare loss, to misspecification of the utility function. Result will then be concluded.

2 Model

2.1 Economy

First, we define the financial market. We consider a fixed terminal time $\mathcal{T}$, such that we can invest during the horizon $[0, \mathcal{T}]$. We have a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{0\leq t\leq \mathcal{T}}, \mathbb{P})$ and the randomness comes from a Brownian motion, denoted by $\{W_t\}_{0\leq t\leq \mathcal{T}}$. In this paper, the market consist of two assets: a risky asset (or stock), denoted by $\{S_t\}_{0\leq t\leq \mathcal{T}}$ and a risk-free asset (or bond), denoted by $\{B_t\}_{0\leq t\leq \mathcal{T}}$. These two assets satisfy the following processes respectively:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_0 = 1, \tag{1}$$
$$dB_t = r B_t dt, \qquad B_0 = 1. \tag{2}$$

Here, r denotes the risk-free rate and μ and σ denote the risky asset's expected return and volatility. They are assumed to be scalar processes. As usual, the Wiener process, $\{W_t\}_{0\leq t\leq \mathcal{T}}$, obeys a normal distribution, i.e., $W_t \sim \mathcal{N}(0, t)$. Furthermore, All of these processes are $\{\mathscr{F}_t\}_{0\leq t\leq \mathcal{T}}-$ measurable. Hence, our risky asset follows a geometric Brownian motion, while the bond is completely predicable.

We define κ to be the *price of risk*, i.e., $\kappa = \frac{\mu - r}{\sigma}$, which is again a constant. Moreover, there are no transaction costs in our model and investments can be continuously adjusted. In short, we have a complete and arbitrage-free "Black-Scholes" economy (Black & Scholes, 1973).

2.2 Cohorts

In our setting we aim to find an optimal investment strategy for the participants of a pension plan. To this end, we focus on the *accumulation* phase of the participants, i.e., the period that starts when the participants begin their working career and ends with retirement. We assume that this period is of the same duration for every participant. In particular, we consider the phase to be of length τ, which is a parameter than can be determined afterwards. Furthermore, We divide the participants into *cohorts*. A cohort consists of all participants whose accumulation phase start at the same time. By assumption, this immediately means that these participants have the same retirement date. As a notational convention, we denote the cohort that accumulates during the interval $[T-\tau, T]$ by "Cohort T"

We define $C_{t,T}$ to be the value of the investment portfolio at some intermediate point $t \in [T-\tau, T]$ for cohort T. In this paper, we assume that each cohort only invests for itself, i.e., neither returns nor risks are to be shared. In other words, we assume *autarkic* cohorts. As each cohort faces exactly the same maximization problem, we can, without loss of generality, focus on cohort T specifically. Now, this cohort can, during the accumulation phase, either invest in the risky or risk-free asset. The fraction of total wealth[3] invested in the risky asset at time t by this cohort is denoted by $f(t-T+\tau)$. The amount invested in the risky asset and the risk-free asset are then defined as $f(t-T+\tau)C_{t,T}$ and $(1-f(t-T+\tau))C_{t,T}$ respectively.

The random variable $C_{t,T}$ is a process following the stochastic differential equation (SDE) given by

$$dC_{t,T} = f(t-T+\tau)C_{t,T}\frac{dS_t}{S_t} + (1-f(t-T+\tau))C_{t,T}\frac{dB_t}{B_t} \tag{3}$$

In words, the only factor within our influence that determines changes in our portfolio value is the amount we decide to invest in the risky asset. With the definitions given in (1) and (2) the SDE can be rewritten slightly:

$$dC_{t,T} = (r + (\mu - r)f(t-T+\tau))C_{t,T}\, dt + \sigma f(t-T+\tau)C_{t,T}\, dW_t \tag{4}$$

To approach the problem in terms of returns rather than portfolio value, we can take the natural logarithm of $C_{t,T}$, i.e, $c_{t,T} := \ln(C_{t,T})$. By Itô's lemma, this admits the following SDE

$$\begin{aligned} dc_{t,T} = &\left(r + (\mu - r)f(t-T+\tau) - \frac{1}{2}\sigma^2 f(t-T+\tau)^2\right)\, dt \\ &+ \sigma f(t-T+\tau)\, dW_t. \end{aligned} \tag{5}$$

This SDE can also be written in integral representation, which clarifies the interpretation. To this end, let us consider the returns at retirement, i.e., $c_{T,T}$. To simplify notation, let us denote this simply by c_T.

$$\begin{aligned} c_T = &\ln(A_0) + \int_{T-\tau}^{T} r + (\mu - r)f(t-T+\tau) - \frac{1}{2}\sigma^2 f(t-T+\tau)^2\, dt \\ &+ \int_{T-\tau}^{T} \sigma f(t-T+\tau)\, dW_t. \end{aligned} \tag{6}$$

[3]Total wealth is defined by adding human capital to financial wealth.

Here, A_0 denotes the present value at time $T - \tau$ of all contributions for cohort T. From this definition, it becomes clear that this can be interpreted as the cumulative returns, consisting of contributions and returns on these contributions by investments during the accumulation phase. The wealth accumulated at retirement can be obtained by taking the exponential of c_T. Finally, it worth mentioning that wealth at retirement is then a log-normally distributed random variable.

2.3 Utility Function

To determine the optimal fraction invested in the risky asset at each point in time, we need to define the manner in which cohorts evaluate their pension capital C_T at retirement date T. The utility function exhibiting constant relative risk aversion (CRRA) only takes into account the accumulated wealth of the cohort itself, meaning that this will never result in the desired *intergenerational benchmarking* between cohorts. Therefore, we adjust the utility function such that a cohort also benchmarks their pension against the pension of preceding cohorts.

This adjustment could be defined in a multitude of ways. Our choice of utility functions, which we introduce in this section, is closely related to the utility function formulated by Abel (1990). Their utility function, dubbed catching up with the Joneses (CUJ), is based on the notion that appreciation of wealth is comparison-driven. Abel (1990) introduces a preference parameter, defined by lagged aggregate consumption, against which their consumer evaluates their consumption. In a similar fashion, with our choice of utility function, we incorporate that our cohort looks at its own accumulated wealth and evaluates the ratio with the accumulated wealth of the preceding cohort, following closely the utility specification of Alonso-Carrera et al. (2005), Abel (1990) and Alvarez-Cuadrado et al. (2004). If this ratio is larger than one, i.e., if the cohort's accumulated wealth exceeds that of the preceding cohort, the cohort is content with their pension. This positive difference thus gives a "boost" to their perceived utility. On the other hand, however, we incorporate that a lowering in wealth with respect to the preceding cohort greatly diminishes the level of contentment. We can therefore interpret the accumulated wealth of the preceding cohort as the benchmark of the current cohort. In other words, cohorts now measure their wealth relative to an exogenously given habit level, defined by the previous generation.

With this in mind, we introduce the following term that denotes our benchmark (habit)

$$\Xi_T = \lim_{\Delta t \downarrow 0} \left(\frac{C_T}{C_{T-\Delta t}} \right)^{\frac{1}{\Delta t}}. \tag{7}$$

This term defines the ratio of the wealth at retirement of two immediately successive cohorts, corrected for time differences, Δt. Note that this term is strictly larger (smaller) than one if our wealth at retirement exceeds (is exceeded by) that of the directly preceding cohort. Our utility function has the following form:

$$U(C_T, \Xi_T) = \frac{(C_T \Xi_T^{\alpha})^{1-\gamma_I}}{1-\gamma_I}. \tag{8}$$

Notice how, with this utility function, a cohort not only considers its wealth at retirement in absolute terms, but also compares this to the wealth of the preceding cohort. In this way, we incorporate the fact that people tend to feel worse about their gains when they know others have had better returns. In other words, the cohorts in our model are "jealous". Realize that, as every cohort in our model compares itself with the immediately preceding cohort, and because there are a multitude of cohorts in the long run, that we get a continuous effect of cohorts that benchmark their accumulated wealth against the preceding cohort. In this sense, this resembles external habit formation (Abel, 1990; Galí, 1994; Xiouros & Zapatero, 2010) very closely.

Observe that we have added a new parameter to the utility function, α, which we restrict to be positive. We shall henceforth refer to this parameter as the "intergenerational factor", as it defines the importance that the cohort attaches to the ratio of pensions. Furthermore, note that we have added a subscript to the risk aversion parameter, γ_I. This is meant to emphasize that this value does not necessarily align with the value in the usual CRRA utility function, which we shall call γ_P.

Let us analyze the utility function, considering C_T and Ξ_T to be two separate (albeit highly correlated) processes, which are both strictly positive. Notice that our function satisfies the standard property of decreasing marginal utility with respect to both processes, implying concavity in C_T and Ξ_T. Furthermore, note that the relative risk aversion with respect to C_T is γ_I, as usual. Moreover, we have that the "benchmark elasticity" of marginal utility is constant, given by

$$\Xi_T \frac{\frac{\partial^2 U(C_T,\Xi_T)}{\partial \Xi_T \partial C_T}}{\frac{\partial U(C_T,\Xi_T)}{\partial C_T}} = \alpha(1-\gamma_I). \tag{9}$$

The effect of the external habit is determined by the sign of α (evidently) and the size of γ_I. Under our restriction that $\alpha > 0$, we have that the benchmark elasticity of marginal utility is negative for $\gamma_I > 1$. This would indicate that, under this reasonable assumption, an increase in the intergenerational benchmark results in a lower marginal utility of own pension capital. Or, in other words, a decrease in the pension capital of the preceding cohort results in a lower utility gain from an increase in own pension capital. Conversely, if the terminal wealth of the preceding cohort increases, we experience a stronger utility gain from an increase in own wealth, as we want to "catch up with the Joneses". Furthermore, we can observe that the elasticity depends on an interaction between α and γ_I. This suggests that these values depend on each other and are not to be estimated separately. In section 3.3, we introduce a way to quantify the relationship between the two parameters.

Realize that, we can write our function in terms of cumulative returns at retirement, rather than wealth at retirement. We introduce the logarithm of the benchmark process, ξ_T, defined as:

$$\xi_T := \ln(\Xi_T) = \lim_{\Delta t \downarrow 0} \frac{c_T - c_{T-\Delta t}}{\Delta t}. \tag{10}$$

This new process can be interpreted as the difference in cumulative returns between immediately successive cohorts. We can therefore define our utility function in terms of cumulative returns:

$$U(c_T, \xi_T) = \frac{e^{(1-\gamma_I)(c_T + \alpha \xi_T)}}{1-\gamma_I}. \tag{11}$$

Notice that these two functions are identical, both in mathematical and interpretation terms. Therefore, we shall make use of the latter representation as it facilitates a more efficient and succinct solution approach.

3 Optimal Investment

In this chapter we first show the step-by-step solution to this problem. After this, we graphically and analytically compare our solution to the solution found when the CRRA utility function is assumed.

3.1 Problem Definition

As stated before, we focus on an autarkic solution for cohort T. This means that we aim to find an optimal investment strategy for this cohort, without the possibility of wealth transfers between cohorts. It can be shown that our solution is a deterministic function, even if we consider a larger set of (possibly stochastic) investment strategies[4] So, without altering the result, we restrict our investment policy to be deterministic, in line with most pension practice[5], and uniform for all cohorts that participate in the plan of the pension fund.

Considering the random nature of c_T and ξ_T, we obtain the formulation of our optimal control problem:

$$\max_{f()} \mathbb{E}\left[\frac{e^{(1-\gamma_I)(c_T+\alpha\xi_T)}}{1-\gamma_I}\right], \tag{12}$$

where c_T is a normally distributed variable, given by (6). As the random variables c_T and ξ_T are determined by the investment policy $f(t-T+\tau)$, we can directly evaluate (12). We consider the definition of (10) applied to the cumulative returns as defined by (6).

$$\begin{aligned}
&\lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\bigg\{\int_{T-\tau}^{T}(\mu-r)f(t-T+\tau)-\frac{1}{2}\sigma^2 f(t-T+\tau)^2\,dt\\
&-\int_{T-\tau-\Delta t}^{T-\Delta t}(\mu-r)f(t-T+\tau+\Delta t)-\frac{1}{2}\sigma^2 f(t-T+\tau+\Delta t)^2\,dt\\
&+\int_{T-\tau}^{T}\sigma f(t-T+\tau)\,dW_t-\int_{T-\tau-\Delta t}^{T-\Delta t}\sigma f(t-T+\tau+\Delta t)\,dW_t\bigg\}.
\end{aligned}\tag{13}$$

Notice that the deterministic part is simply a time derivative that we can directly evaluate using the Leibniz integral rule. We then have that this term becomes

$$\begin{aligned}
&(\mu-r)\left(f(\tau)-f(0)+\int_{T-\tau}^{T}\tfrac{\partial f(t-T+\tau)}{\partial T}\,dt\right)\\
&-\frac{1}{2}\sigma^2\left(f(\tau)^2-f(0)^2+\int_{T-\tau}^{T}2f(t-T+\tau)\tfrac{\partial f(t-T+\tau)}{\partial T}\,dt\right).
\end{aligned}\tag{14}$$

Note that $\frac{\partial f(t-T+\tau)}{\partial T}$ is well-defined because we are only considering deterministic investment strategies. The stochastic part, however, must be evaluated with more caution, as the Leibniz

[4] Proof available upon request.

[5] Asset mixes of target date funds in the US, UK and Australia are predetermined as a function of age (Basu et al., 2011; Chan et al., 2017; Mitchell & Utkus, 2022; Sass, 2014; Surz & Israelsen, 2008).

rule is not applicable. Consequently, let us rewrite the stochastic part as the limit:

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}\Bigg\{ \int_{T-\tau}^{T-\Delta t} \sigma(f(t-T+\tau) - f(t-T+\tau+\Delta t))\, dW_t \\ + \int_{T-\Delta t}^{T} \sigma f(t-T+\tau)\, dW_t - \int_{T-\tau-\Delta t}^{T-\tau} \sigma f(t-T+\tau+\Delta t)\, dW_t \Bigg\}. \tag{15}$$

Notice that we have split the integral into three parts, in order to emphasize that there are two intervals that are not shared and one common interval of accumulation. We consider three limits separately. The first limit we need to evaluate is defined by the following expression and is immediately solvable.

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}\Bigg\{ \int_{T-\tau}^{T-\Delta t} \sigma(f(t-T+\tau) - f(t-T+\tau+\Delta t))\, dW_t \Bigg\} \tag{16}$$

$$= \int_{T-\tau}^{T} \sigma \tfrac{\partial f(t-T+\tau)}{\partial T}\, dW_t \tag{17}$$

The second limit is given by:

$$\lim_{\Delta t \downarrow 0} \int_{T-\Delta t}^{T} \frac{\sigma f(t-T+\tau)}{\Delta t}\, dW_t. \tag{18}$$

Notice that for any $\lim_{t\uparrow T} f(t-T+\tau) = f(\tau) \neq 0$ this expression diverges to a random variable with infinite variance. For the utility function to be well-defined, we need that this is finite. Therefore, it is necessary to assume that $f(\tau) = 0$. In a similar fashion we impose $\lim_{t\downarrow T-\tau} f(t-T+\tau) = f(0) = 0$.

So, the stochastic part remains finite if and only if we assume $f(0) = f(\tau) = 0$. This also reduces the complexity of the deterministic part. In particular, note that we have now the following deterministic term.

$$\int_{T-\tau}^{T} \sigma \tfrac{\partial f(t-T+\tau)}{\partial T}(\kappa - \sigma f(t-T+\tau))\, dt.$$

Note that, by definition, we have that $\frac{\partial f(t-T+\tau)}{\partial T} = -\frac{\partial f(t-T+\tau)}{\partial t}$. Therefore, the deterministic term above reduces to

$$\int_{T-\tau}^{T} \sigma \tfrac{\partial f(t-T+\tau)}{\partial t}(\sigma f(t-T+\tau) - \kappa)\, dt = 0. \tag{19}$$

This means that, with the boundary conditions $f(0) = f(\tau) = 0$, ξ_T only depends on the random parts between $T-\tau$ and T. This is not unexpected, as the deterministic returns should be the same in case of multiple cohorts following the same investment strategies. The process ξ_T is thus defined as follows.

$$\xi_T = \int_{T-\tau}^{T} \sigma \tfrac{\partial f(t-T+\tau)}{\partial T}\, dW_t = \int_{T-\tau}^{T} -\sigma \tfrac{\partial f(t-T+\tau)}{\partial t}\, dW_t. \tag{20}$$

Notice that ξ_T now has a Gaussian distribution with mean 0 and variance equal to $\int_{T-\tau}^{T} \sigma^2 \left(\frac{\partial f(t-T+\tau)}{\partial t}\right)^2 dt$, by Îto's isometry. As both the cumulative returns and the log-benchmark are now normally distributed, we can directly evaluate (12).

Let us now fix $T = \tau$ to lighten the notation. This does not influence our problem (or solution), as we are considering a uniform solution for autarkic cohorts. Furthermore, we shall henceforth denote the time-derivative $\frac{\partial f(t)}{\partial t}$ by $\dot{f}(t)$. Then, applying (20) and (6) to our maximization problem (12), our problem becomes (together with the boundary conditions):

$$\max_{f()} \mathbb{E}\left[\frac{A_0^{1-\gamma_I}}{1-\gamma_I} e^{(1-\gamma_I)\left(\int_0^\tau r+(\mu-r)f(t)-\frac{1}{2}\sigma^2 f(t)^2\, dt+\sigma \int_0^\tau f(t)-\alpha \dot{f}(t)\, dW_t\right)}\right]. \tag{21}$$

3.2 Solution

Notice that we can explicitly evaluate (21) for deterministic $f(t)$, providing us with the following deterministic optimal control problem.

$$\begin{aligned} &\max_{f()} \frac{A_0^{1-\gamma_I}}{1-\gamma_I} e^{(1-\gamma_I)\int_0^\tau \mathscr{L}(t,f,\dot{f})\, dt} \qquad (22)\\ &\text{s.t. } f(0) = f(\tau) = 0 \\ &\mathscr{L}(t,f,\dot{f}) = r + (\mu - r)f(t) - \frac{1}{2}\gamma_I \sigma^2 f(t)^2 + \frac{1}{2}(1-\gamma_I)\sigma^2(\alpha^2 \dot{f}(t)^2 - 2\alpha f(t)\dot{f}(t)). \end{aligned}$$

Notice that this is a calculus of variation problem, where we have to make a choice for $f(t)$. Therefore, we need to consider the effect of a perturbation of the control function on the functional $\mathscr{L}(t, f, \dot{f})$, which is defined by the objective function in (22). Because we have a functional of time, f and its time derivative, we can immediately apply the Euler-Lagrange equation. Note that a stationary point of $\mathscr{L}(t, f, \dot{f})$ is immediately also a stationary point of the entire expression. Therefore, by the Euler-Lagrange equation, the function f is found by solving the following equation:

$$\frac{\partial \mathscr{L}(t,f,\dot{f})}{\partial f} - \frac{d}{dt}\frac{\partial \mathscr{L}(t,f,\dot{f})}{\partial \dot{f}} = 0. \tag{23}$$

This leads to the optimality condition

$$(\mu - r) - \gamma_I \sigma^2 f(t) - (1-\gamma_I)\sigma^2\alpha^2 \ddot{f}(t) = 0, \tag{24}$$

together with $f(0) = f(\tau) = 0$. This then is a ordinary differential equation (ODE) which we can directly solve. Our optimal investment is then given by:

$$f^*(t) = \frac{\mu - r}{\gamma_I \sigma^2}\frac{(e^{qt}-1)(e^{q(\tau-t)}-1)}{e^{q\tau}+1}, \tag{25}$$

where q is a new risk parameter, defined as $q = \sqrt{\frac{\gamma_I}{(\gamma_I - 1)\alpha^2}}$. Remark that this parameter is a decreasing function of both γ_I and α.

3.3 Calibrating risk parameters

This section compares our $f^*(t)$, which we shall henceforth refer to as the CUJ investment strategy, to the widely-known Merton (1969) fraction, which we denote by $f^P(t)$ and is defined as $f^P(t) = \frac{\mu-r}{\gamma_P\sigma^2}$, that is a result of the power utility function. To be able to do this fairly, we need to determine the relation between risk parameters of both, i.e. γ_I and γ_P. This relation is of great importance, as this determines the height of the *expected* pension wealth in both cases. We assume that cohorts do not favor wealth reductions in comparison to the current situation, even if this reduces the differences between cohorts. Therefore, this paper assumes that, for every value of γ_P, we determine γ_I such that the expected wealth at retirement remains the same. In other words, a change in α will need to be compensated with a change in γ_I.

Mathematically, we are determining γ_I such that $\mathbb{E}[e^{c_T}|f^P(t)] = \mathbb{E}[e^{c_T}|f^*(t)]$. The left expectation leads to the following term

$$\mathbb{E}\left[A_0 e^{\int_0^\tau r+(\mu-r)f^P(t)-\frac{1}{2}\sigma^2 f^P(t)^2\,dt+\int_0^\tau \sigma f^P(t)\,dW_t}\right] = A_0 e^{\tau\left(r+\frac{(\mu-r)^2}{\gamma_P\sigma^2}\right)}. \tag{26}$$

The other side of the equation equals

$$\mathbb{E}\left[A_0 e^{\int_0^\tau r+(\mu-r)f^*(t)-\frac{1}{2}\sigma^2 f^*(t)^2\,dt+\int_0^\tau \sigma f^*(t)\,dW_t}\right] \tag{27}$$

$$= A_0 e^{\tau\left(r+\frac{(\mu-r)^2}{\gamma_I\sigma^2}\right)+\frac{(\mu-r)^2}{\gamma_I\sigma^2}\left(\frac{4}{qe^{q\tau}+q}-\frac{2}{q}\right)} \tag{28}$$

It can be shown that these equal each other if the risk parameters satisfy the following equation.

$$\frac{\gamma_P}{\gamma_I} = \frac{\tau q(e^{q\tau}+1)}{(e^{q\tau}+1)(\tau q-2)+4}. \tag{29}$$

Note that this means that γ_I is never bigger than γ_P, which makes intuitive sense since $\alpha > 0$ needs to be "compensated" with a lower risk-aversion. To give an intuition on the size of the difference, we have that $\gamma_I \approx 2.99$ when $\alpha = 10$ and $\gamma_P = 5$.

3.3.1 Limit behavior for the intergenerational factor

From the diversity of methods to capture risk aversion from participants proposed in behavior economics literature (Alserda et al., 2019; Chetty, 2006; Clark & Strauss, 2008; Schooley & Worden, 1996; Thomas, 2016; van Rooij et al., 2007; Wik et al., 2004), it has become clear that estimating risk parameters is quite a burden for pension funds. To alleviate this burden, we shall derive the optimal investment strategy when $\alpha \to \infty$, such that we only need the estimate for γ_P.

In order to properly evaluate the limit, let us write γ_I in terms of q. We then get

$$\gamma_I = \frac{q^2\alpha^2}{q^2\alpha^2-1}. \tag{30}$$

Note that, by the original definition, $q > 0$. Then, in turn, we are solving

$$\frac{\gamma_P}{\frac{q^2\alpha^2}{q^2\alpha^2-1}} = \frac{\tau q(e^{q\tau}+1)}{(e^{q\tau}+1)(\tau q-2)+4}. \tag{31}$$

This can be written in terms of α

$$\alpha^2 = \frac{\gamma_P(e^{q\tau}+1)(q\tau-2)+4\gamma_P}{q^2\left(q\tau(e^{q\tau}+1)(\gamma_P-1)-2\gamma_P(e^{q\tau}-1)\right)} =: h(q). \tag{32}$$

To find the behavior of q as α grows to ∞, we need to take a closer look at $h(q)$.

Note that we only have to focus on the positive domain of the function ($q > 0$). On this domain, the numerator is strictly greater than zero. This means that $h(q) \to \infty$ is equivalent to finding the finite value of q for which the denominator goes to zero[6]. To facilitate a straightforward analysis, let us focus on $q\tau$ rather than q itself. Therefore, we need to find finite $q\tau > 0$ such that

$$q\tau(e^{q\tau}+1)(\gamma_P-1) - 2\gamma_P(e^{q\tau}-1) \to 0. \tag{33}$$

This expression goes to zero if

$$\frac{q\tau}{\tanh\left(\frac{q\tau}{2}\right)} \to \frac{2\gamma_P}{\gamma_P-1} \tag{34}$$

This can be solved numerically. The table below shows the values for q, for a variety of parameter values of γ_P and τ.

	$q\tau$	q		
		$\tau = 35$	$\tau = 40$	$\tau = 45$
$\gamma_P = 2$	3.830	0.109	0.096	0.085
$\gamma_P = 5$	1.776	0.051	0.044	0.039
$\gamma_P = 10$	1.168	0.033	0.029	0.026

Table 1: Values for q when $\alpha \to \infty$.

3.4 Illustration

We first analyze the optimal investment strategy in terms of total wealth, after which we also show it in terms of financial wealth, i.e., the life-cycle investment. We will assume, henceforth, that every cohort has an accumulation phase of 40 years, i.e., $\tau = 40$. Figures 1 and 2 then show the investment strategy for a cohort that accumulates during $[0, 40]$.

The first principal change with respect to the CRRA fraction, is that in the beginning and end op the accumulation phase there is a de-risking process with respect to total wealth. This decrease in exposure is exactly during the periods that we consider non-shared by consecutive cohorts. This means that during the periods in which not all of the cohorts that we consider are accumulating (and thus investing), we reduce the exposure as these periods contribute most to the difference in accumulated wealth at retirement. The larger the value of α, the larger the periods are that we consider non-shared, and the lower the exposure during these periods. On the other hand, however, we have that during the shared periods, i.e., the period between the two non-shared periods, we expose ourselves more to the market. The goal of this is to compensate for the lower exposure in the non-shared periods, such that we still obtain the same *expected* wealth at retirement. Meanwhile, this period cannot lead to large differences, as successive cohorts are all investing during that period. A larger value of α leads to a shorter shared period with a higher exposure, leading to a higher correlation between the pension capitals of consecutive cohorts.

[6] Note that if $q \to \infty$ that the denominator goes to ∞ as well, meaning that the entire expression remains finite.

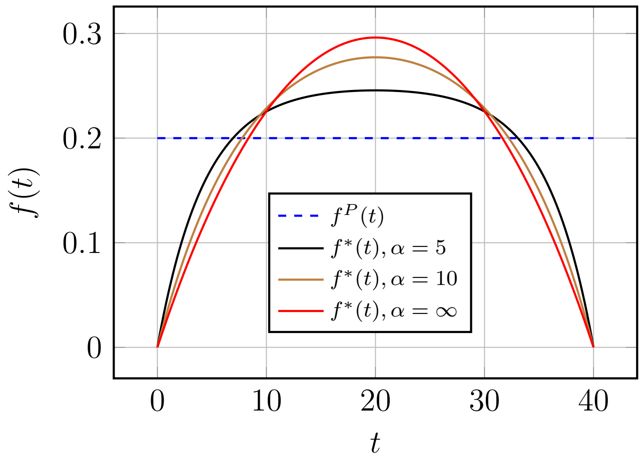


Figure 1: Optimal "CUJ" and "CRRA" fraction of total wealth invested in risky portfolio for different values of "intergenerational factor" α. We assume $\kappa = 0.2$ and $\gamma_P = 5$.

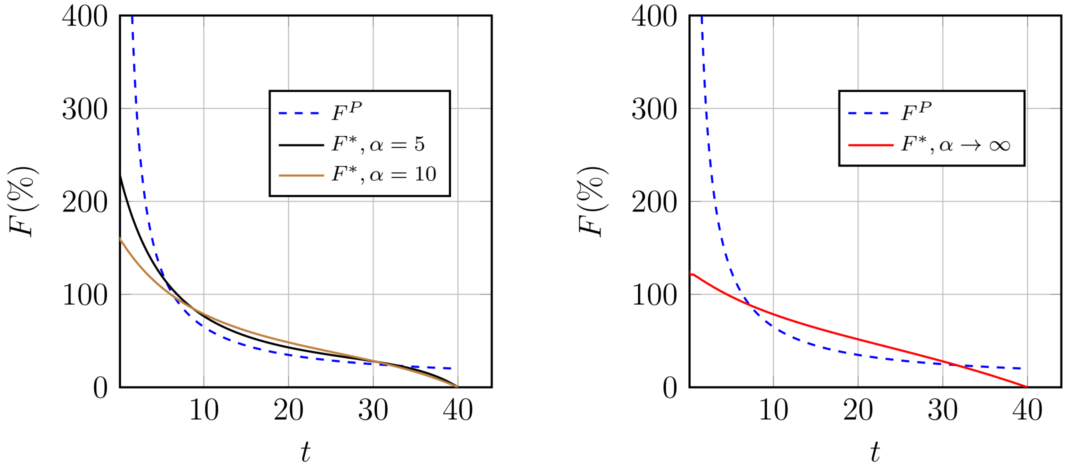


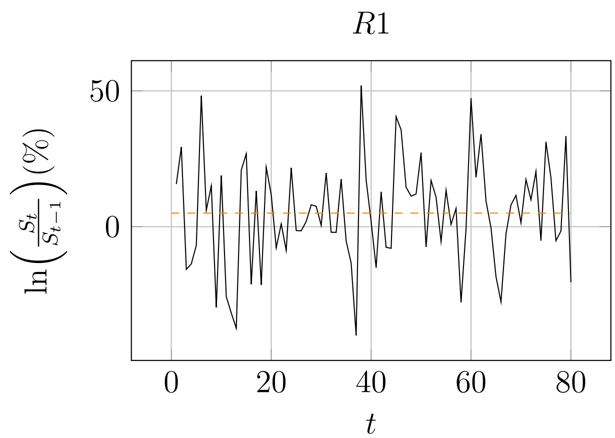
Figure 2: Optimal CUJ and CRRA fraction of financial wealth invested in the risky asset for $\gamma_P = 5$ and different values of α.

The main finding that figure 2 shows us, next to figure 1, is that F^* bears a strong resemblance to the life-cycle based on the well-known rule of thumb that tells us to invest $(100 - \text{age})\%$ in the risky asset. The resemblance is due to the quasi-linearity of our life-cycle and because of the finite leverage, as opposed to the CRRA fraction which results in infinite leverage as $t \downarrow 0$. This similarity becomes more pronounced for larger values of α. Furthermore, it worth mentioning that if α goes to zero in the limit, f^* will align with the CRRA investment strategy.

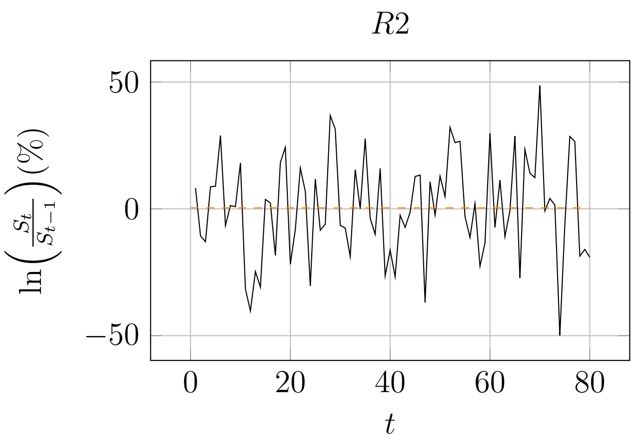
4 Scenario Projections

In this chapter we investigate the wealth accumulated at retirement of different cohorts, where we compare the situation in which the CRRA life-cycle is followed and the situation in which our CUJ investments are assumed.

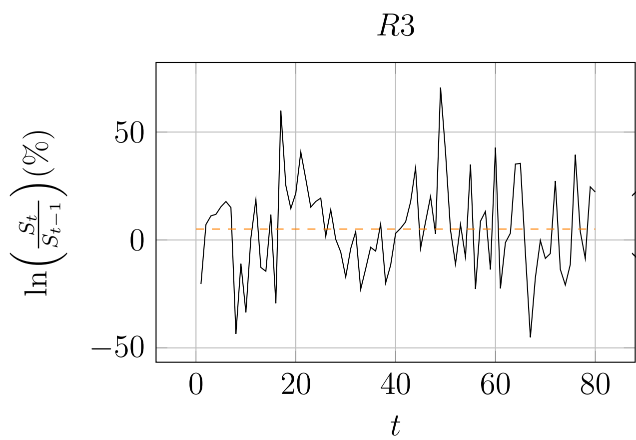
We simulate a risky asset, S_t that follows daily movements for a duration of eighty years, according to (1) with $\mu = 0.06$ and $\sigma = 0.2$. The bond is assumed to pay a steady risk-free rate of 0.02. For conciseness, the development of the asset is assumed to happen during the interval $[0, 80]$. We assume forty cohorts, each one accumulating for 40 years ($\tau = 40$) such that the pensions are accumulated from $t = 41$ to $t = 80$. During these eighty years, we have two types of cohorts. There are forty cohorts that invest according to the CRRA fraction, and forty that follow the CUJ life-cycle, where investments are adjusted on a daily basis. To properly evaluate and demonstrate the effectiveness of the CUJ life-cycle, we take into consideration four distinct paths that the risky asset follows. We refer to these four situations by R_1, R_2, R_3 and R_4.



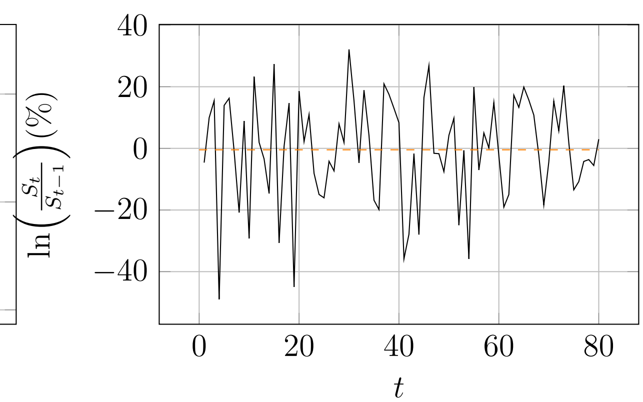
(a) $\hat{\mu} \approx 5$ en $\hat{\sigma} \approx 19.66$.



(b) $\hat{\mu} \approx 0.39$ en $\hat{\sigma} \approx 19.94$.



(c) $\hat{\mu} \approx 5.05$ en $\hat{\sigma} \approx 21.7$.



(d) $\hat{\mu} \approx -0.47$ en $\hat{\sigma} \approx 17.28$.

Figure 3: Four scenario's annual returns risky asset. Parameters are set at $\mu = 0.06, \sigma = 0.2$. Numbers are displayed in percentages.

We can observe that in the first of these situations, R_1, we are dealing with an asset that is relatively stable, with generally positive returns. Furthermore, the returns show, on average, an increase over the years. The second path shows, on the other hand, a stock that barely increases over eighty years. R_3 bears a strong resemblance to R_1, the only exception being the large decline after all the increases, whereas R_1 does not experience this decline. In the last situation, we are dealing with an asset that performs quite poorly. We now show the average annualized realized returns that the forty cohorts experience, in each of the scenario's.

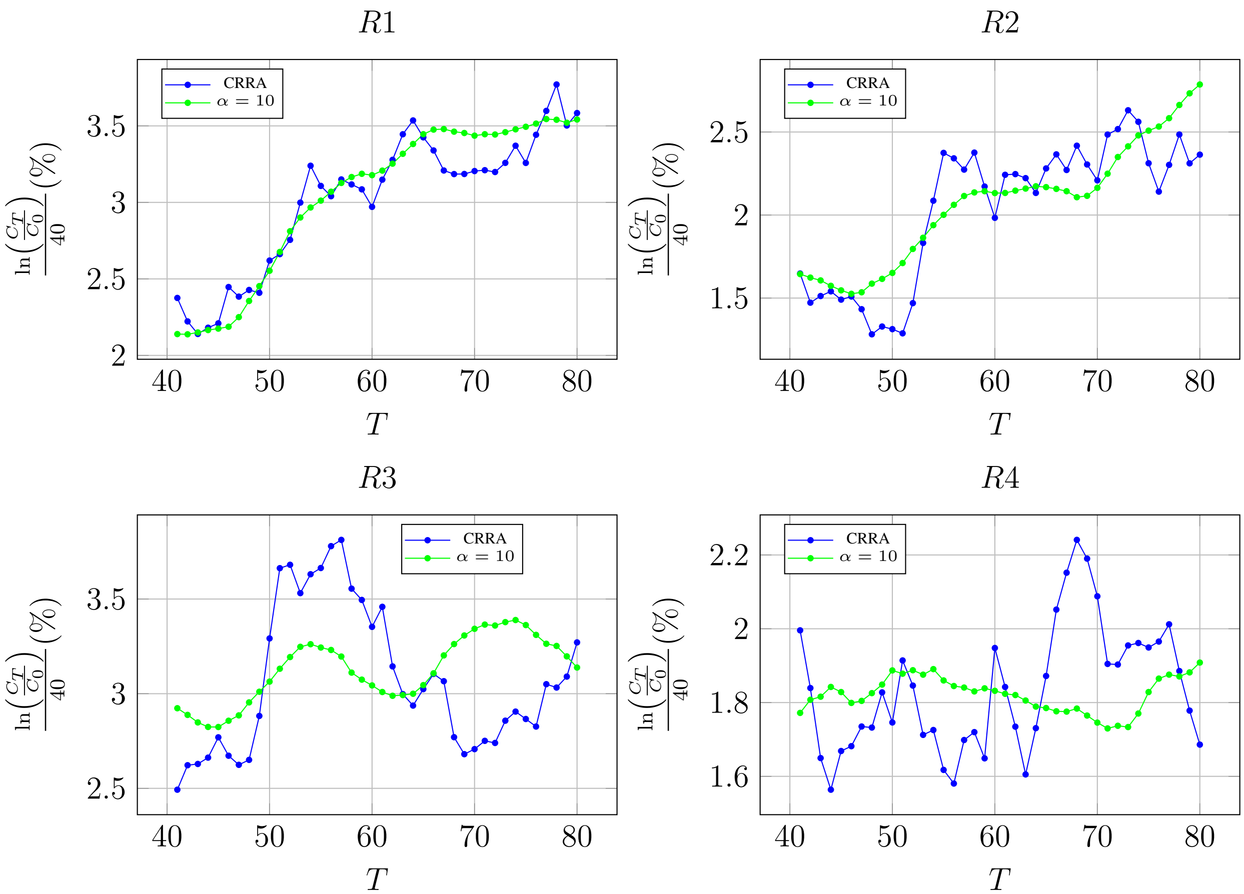


Figure 4: Simulation four scenario's pensions. Parameters are set at $r = 0.02, \mu = 0.06, \sigma = 0.2$. Displayed are the average annual returns for all cohorts, in percentages.

In figure 4 we compare the average annual returns of forty consecutive cohorts that either invest according to the CRRA strategy (blue) or the CUJ strategy (green) , that this paper proposes. We will refer to these forty cohorts by CRRA cohorts and CUJ cohorts, respectively. Every dot in figure 4 represents the average annual returns that a cohort that retires at time T has experienced. As the goal of our proposed investment strategy is to reduce the differences among (immediately) successive cohorts, while keeping the expected wealth the same, we expect the resulting lines between the dots to be relatively flat, compared to the CRRA cohorts.

First of all, we see that the differences between immediately successive cohorts have been tremendously reduced: the largest immediate decline in annual returns for the CUJ cohorts, which can be found in R_3, between the cohorts that retire at $T = 57$ and $T = 58$, is just below to 0.085%, which amounts to approximately 3.40% in forty years time. On the other hand, for the CRRA cohorts, the largest decline amounts to almost 12.6% over forty years. Furthermore, considering the entire set of forty cohorts, the CUJ investment strategy results in a much lower volatility of returns. The green line displays a much *smoother* development than the blue line, that shows tremendous volatility. This also means that over longer periods, i.e., between cohorts that differ more than one year, there are fewer declines in returns. Moreover, increases and decreases happen more gradually, rather than sudden. In short, the CUJ investment strategy this paper proposes is an effective tool to reduce differences between cohorts and thus reduce the probability of so-called unfortunate and fortunate generations. This is all achieved, without compromising on the level of returns (or pensions) and without any form of risk or wealth sharing.

5 Impulse-Response

In this chapter we assess the ability of the CUJ strategy to absorb shocks in the value of the asset. This is an important aspect of the strategy, as shocks may lead to differences in returns and thus create unfortunate generations.

To be able to illustrate the effect of our investment strategy as clearly as possible, we consider the simplified situation in which the asset follows a deterministic path, with a constant growth. This means that, in the absence of a shock, the returns (and pensions) of the cohorts are exactly equal. Therefore, the shock will be the only factor that determines differences. A shock can either be a sudden increase or decrease in the value of the asset, indicating either a lucky or unlucky generation. As decreases are mirrored results of increases, we will only consider the latter.

More specifically, we consider two distinct scenario's, where in both we again consider a stock development of eighty years and forty annual cohorts. In the first scenario we have an asset that undergoes a shock of 50%, that occurs around $t = 20$. In the second scenario the there is again a shock with a magnitude of 50%, after 55 years. This means that in the first scenario the earlier generations are fortunate, and in the second scenario the later generations. Graphs 5 and 6 depict the logarithm of the stock and the shock absorbing mechanism that is the result of the investment strategies. Again, as in figure 4, each point in graph 6 represents the average annual returns for a cohort that retires at T.

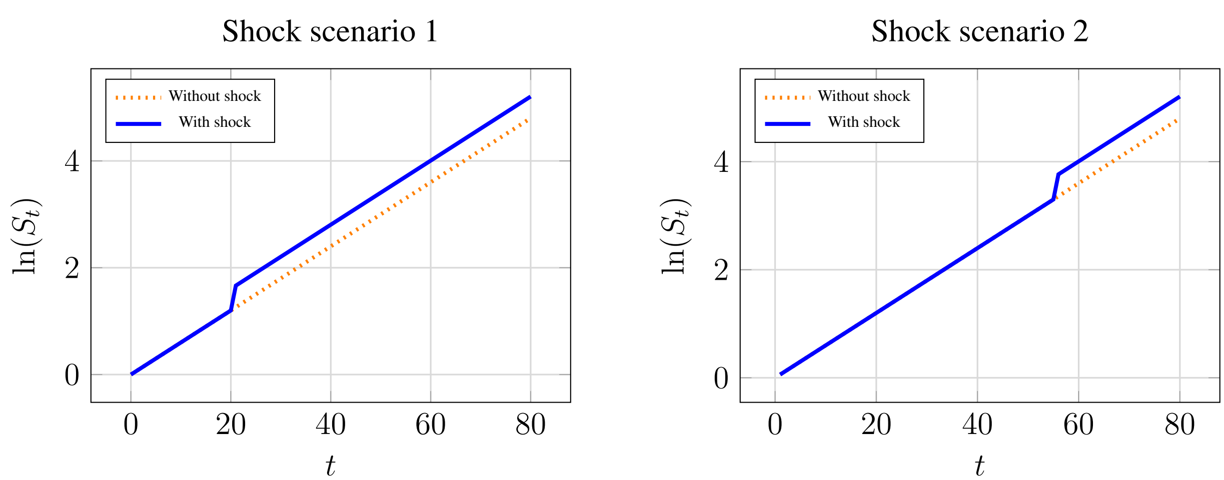


Figure 5: Asset values in scenario 1 and scenario 2.

We observe the resulting shift in returns for cohorts that either follow the CRRA investment strategy or the CUJ strategy. Notice that a shock in the asset value corresponds one-to-one with a shock in the returns of the cohorts. This is due to the constant exposure with respect to total wealth.

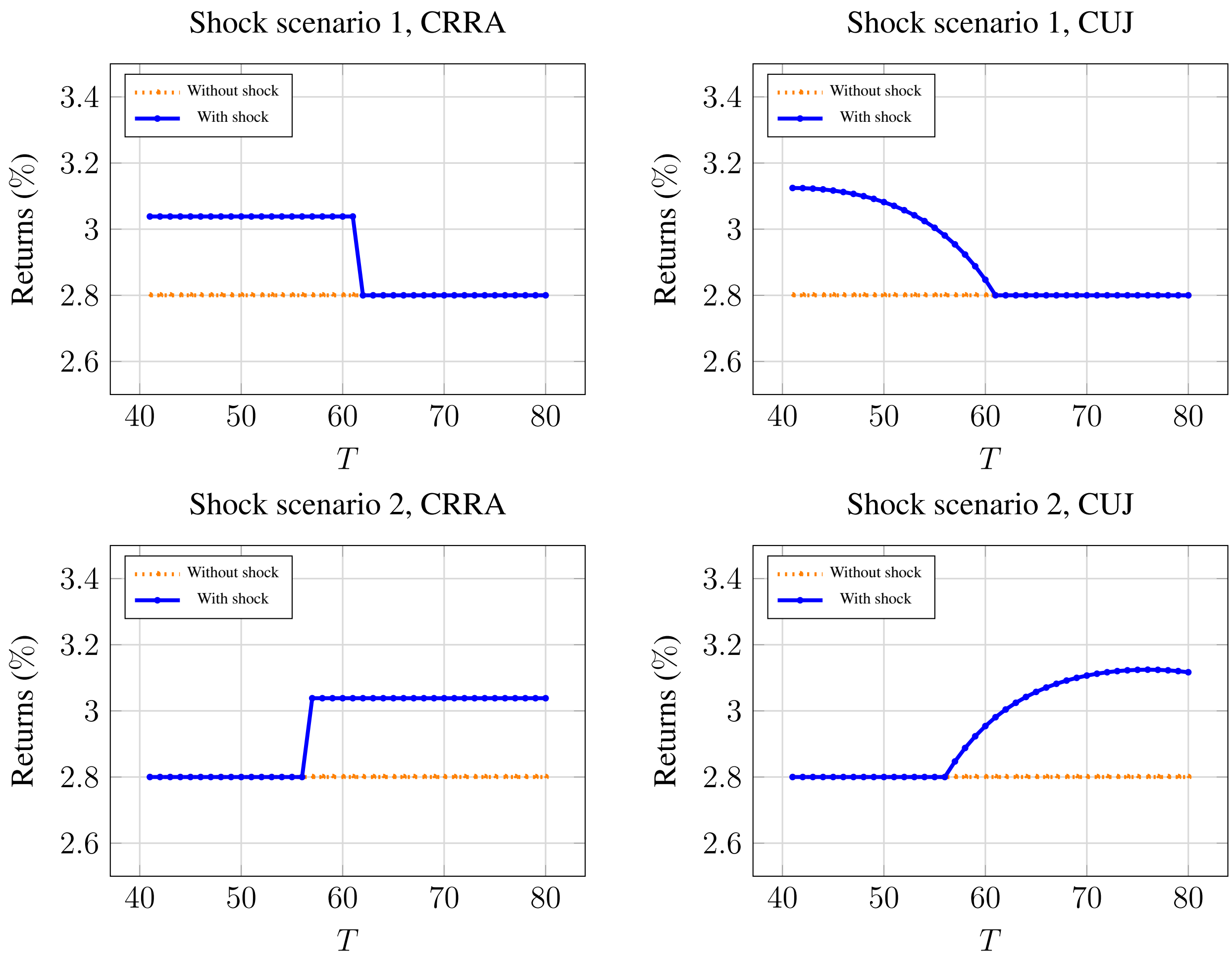


Figure 6: Effect of shock on average annual returns of the forty cohorts in scenarios 1 and 2, for both CRRA and CUJ model.

On the other hand, however, we see that the CUJ investment strategy is much more apt in *spreading* the shock over multiple cohorts. This means the decline or increase happens gradually rather than all at once. This prevents large differences in a short amount of time and, thus, the probability of an unlucky cohort. In short, the CUJ investment strategy is much more apt in absorbing shocks and results, as before, in *smoother* increases or decreases in wealth.

6 Welfare impact misspecification

In this chapter, we discuss the impact on welfare of misspecification of the utility function. We shall express this in terms of differences in CEQ returns per annum.

6.1 CRRA utility

In this first part we illustrate the welfare losses participants of a pension plan experience whenever the wrong utility function is assumed. In particular, we look at the CUJ utility function assumed in this paper and the CRRA utility function.

There are two possible ways welfare loss could happen. The first occurs when the investor evaluates according to our CUJ utility function but invests the Merton (1969) fraction. Note that this fraction does not consider the pensions of any previous cohorts nor does it impose any restriction on the values at $t = 0$ or $t = \tau$. This absence of boundary conditions leads to a process ξ_T with infinite variance. This also means that there is an infinite amount of welfare loss for a CUJ participant in that case. On the other hand, the CUJ investment strategy, $f^*(t)$, leads to a finite welfare loss if the CRRA utility is assumed. We shall express this loss in terms of difference in annual certainty equivalent (CEQ) returns. In mathematical terms, we are solving for some annual return g_P, such that

$$\mathbb{E}\left[\frac{C_T[f^P]^{1-\gamma_P}}{1-\gamma_P}\right] = \frac{(A_0 e^{g_P \tau})^{1-\gamma_P}}{1-\gamma_P}, \tag{35}$$

where $C_T[f^P]$ denotes the pension capital at retirement when we follow invest strategy f^P. Similarly, we denote $C_T[f^*]$ and determine g_I such that

$$\mathbb{E}\left[\frac{C_T[f^*]^{1-\gamma_P}}{1-\gamma_P}\right] = \frac{(A_0 e^{g_I \tau})^{1-\gamma_P}}{1-\gamma_P}. \tag{36}$$

The first is the well-known Merton CEQ:

$$g_P = r + \frac{\kappa^2}{2\gamma_P}. \tag{37}$$

For g_I we find:

$$g_I = r + \frac{\kappa^2}{\gamma_I}\frac{e^{q\tau}(q\tau-2)+q\tau+2}{qe^{q\tau}+q} - \frac{\kappa^2\gamma_P}{2\gamma_I^2}\frac{4q\tau e^{q\tau}+q\tau+e^{2q\tau}(q\tau-3)+3}{q\left(e^{q\tau}+1\right)^2}. \tag{38}$$

In order to get a better idea of the difference of the two, let us look at some differences for specific parameter values. We show the difference between the two CEQ's graphically, as a function of α and include the CEQ loss for a variety of parameter values in table 2.

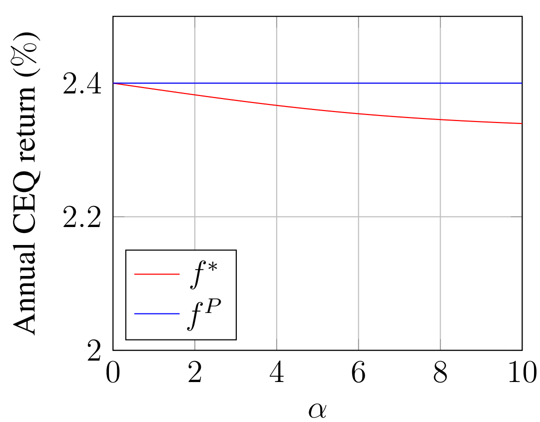


Figure 7: CEQ as a function of α, for $\gamma_P = 5$ and $\kappa = 0.2$.

By construction, any investment strategy that is not the Merton (1969) fraction, is suboptimal for the participant that evaluates its wealth according to CRRA utility. Therefore, $f^*(t)$ necessarily realizes a lower CEQ per annum than $f^P(t)$. However, we see here that this differences is almost negligible in size. Even high levels of α, which we have seen can have a large effect on reducing the differences in pensions, lead to a loss in CEQ of a mere 0.06% per year, meaning a cumulative difference of 2.4% over forty years.

Welfare loss	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 2$	$\alpha = 5$	$\alpha = 10$	$\alpha \to \infty$
$\gamma_P = 2$	0.009	0.017	0.034	0.077	0.121	0.167
$\gamma_P = 5$	0.004	0.009	0.018	0.040	0.060	0.075
$\gamma_P = 10$	0.002	0.005	0.009	0.021	0.032	0.038

Table 2: Welfare loss in terms of annual CEQ returns in percentages (%), for $\kappa = 0.2$.

From table 2 we can conclude that, unsurprisingly, the CEQ loss increases with the level of α and decreases whenever γ_P increases. However, the welfare loss remains limited, even for higher levels of α and lower levels of γ_P. This all suggests that the CUJ investment strategy is robust to utility misspecification, making it a prudent choice whenever we aim to prevent welfare losses.

6.2 Intergenerational factor

In this subsection, we consider the welfare loss in terms of CEQ per annum whenever there is a misspecification of the value of our parameter α. We shall assume that we invest according to $\hat{\alpha}$ with corresponding $\hat{\gamma}_I$ and $\hat{q}$, but evaluate according to some different α with corresponding γ_I. Therefore, consider our investment strategy f^*. We then calculate y such that

$$\mathbb{E}\left[\frac{(C_T[f^*]\Xi_T^{\alpha}[f^*])^{1-\gamma_I}}{1-\gamma_I}\right] = \frac{(A_0 e^{y\tau})^{1-\gamma_I}}{1-\gamma_I}. \tag{39}$$

This leads to the term

$$y = r + \frac{\kappa^2(\hat{q}\tau + e^{\hat{q}\tau}(\hat{q}\tau - 2) + 2)}{\hat{\gamma}_I \hat{q}\tau(e^{\hat{q}\tau} + 1)} - \frac{\kappa^2 \gamma_I \hat{q}(e^{2\hat{q}\tau} - 2\hat{q}\tau e^{\hat{q}\tau} - 1)}{2\hat{\gamma}_I^2 q^2 \tau\left(e^{\hat{q}\tau} + 1\right)^2} \tag{40}$$

$$+ \frac{\kappa^2 \gamma_I (e^{2\hat{q}\tau}(3 - \hat{q}\tau) - 4\hat{q}\tau e^{\hat{q}\tau} - \hat{q}\tau - 3)}{2\hat{\gamma}_I^2 \hat{q}\tau\left(e^{\hat{q}\tau} + 1\right)^2} \tag{41}$$

To illustrate the impact on annual CEQ for specific values of α, we include table 3.
We can deduce from table 3 that it is much more costly to invest according to a preference parameter value that is lower than the "actual" value, than the other way around. For example, for $\gamma_P = 5$, an investment according to $\hat{\alpha} = 0.5$ together with an evaluation based on $\alpha = 10$ leads to a total CEQ return loss of $40 * 0.775\% = 31\%$ over a span of forty years. On the other hand, $\hat{\alpha} = 100$ together with $\alpha = 0.5$ leads to a limited loss of 2.64% over an accumulation phase of forty years. In short, it is a much more robust strategy to overestimate the preference level α on which the investment strategy is based, than to underestimate it.

	$\hat{\alpha} = 0.5$	$\hat{\alpha} = 1$	$\hat{\alpha} = 2$	$\hat{\alpha} = 5$	$\hat{\alpha} = 10$	$\hat{\alpha} \to \infty$
$\gamma_P = 2$						
$\alpha = 0.5$	-	0.004	0.018	0.059	0.102	0.147
$\alpha = 1$	0.008	-	0.008	0.045	0.086	0.130
$\alpha = 2$	0.068	0.015	-	0.023	0.059	0.100
$\alpha = 5$	0.480	0.184	0.05	-	0.015	0.046
$\alpha = 10$	$1,385$	0.597	0.220	0.026	-	0.013
$\alpha \to \infty$	3.517	1.608	0.670	0.143	0.024	-
$\gamma_P = 5$						
$\alpha = 0.5$	-	0.002	0.010	0.031	0.051	0.065
$\alpha = 1$	0.004	-	0.003	0.023	0.042	0.056
$\alpha = 2$	0.035	0.008	-	0.012	0.029	0.042
$\alpha = 5$	0.260	0.101	0.027	-	0.007	0.017
$\alpha = 10$	0.775	0.335	0.121	0.013	-	0.005
$\alpha \to \infty$	2.095	0.949	0.381	0.069	0.009	-
$\gamma_P = 10$						
$\alpha = 0.5$	-	0.001	0.005	0.016	0.029	0.033
$\alpha = 1$	0.002	-	0.002	0.012	0.022	0.029
$\alpha = 2$	0.019	0.004	-	0.006	0.015	0.021
$\alpha = 5$	0.140	0.054	0.014	-	0.003	0.008
$\alpha = 10$	0.420	0.181	0.06	0.007	-	0.002
$\alpha \to \infty$	1.135	0.511	0.203	0.036	0.005	-

Table 3: Welfare loss in terms of annual CEQ returns in percentages (%) for different values of γ_P. Losses are expressed in differences relative to the CEQ of the correctly specified case, i.e. $\alpha = \hat{\alpha}$.

7 Conclusion

Over the last few decades, more and more countries have seen their second pillar transition to a DC scheme. As this results in pensions that are highly sensitive to market returns, this can create large differences between generations. To prevent the undesirable situation in which there are so-called unlucky and lucky generations, pension funds aim to find means to mitigate this risk. Therefore, we have investigated in this paper whether we can already prevent these unlucky generations, by changing only the investment strategy during the accumulation phase. More specifically, we have considered generations in *autarky*, i.e. where every generation only invests for themselves, without any wealth transfers between generations. We have included the presence of unlucky generations (or cohorts) in our model by assuming a "catching up with the Joneses" utility function. This means that we have adapted the utility function such that a cohort not only looks at their own accumulated wealth at retirement, but also compares this to the wealth of the preceding cohort at retirement, bearing a strong resemblance to external habit formation models.

We have solved this model in closed form and find that the solution differs substantially from the optimal investment that is found for standard CRRA utility. First, we observe a much lower exposure to the market at the start and end of the accumulation phase. This is also intuitively plausible, as these are the intervals during which consecutive cohorts are not all accumulating.

These are then the periods that could, in the absence of such de-risking, create large differences in wealth. Then, in the large interval in which we consider that all these consecutive cohorts are investing (the "shared" interval), we increase the level of exposure to compensate for the lower exposure in the "non-shared" intervals. During this shared interval, all these consecutive cohorts are investing in the same market, meaning this cannot result in large differences between them. Furthermore, if we consider the investment strategy in terms of financial wealth, rather than total wealth, we observe a strong similarity to the rule of thumb that tells us we need to invest $(100 - \text{age})\%$ in the risky asset.

To demonstrate the effect of the new investment strategy on pensions, we include a few scenario trajectories and conduct an impulse response analysis. These simulations show that differences between closely following cohorts are indeed strongly reduced. Whereas the classical CRRA strategy can lead to a volatile movement of pensions over generations, the "catching up with the Joneses" strategy smooths these pensions out. We observe that this smoothing effect happens, while on average the level of pension capital is not reduced. This smoothing effect in general grows with the level of the intergenerational factor. We also see that, in a pension context, shocks in the asset value are absorbed in a much more desirable way. Rather than creating an all-or-nothing scenario where the shock is felt all at once, the CUJ investment strategy spreads the shock over multiple years, meaning the differences in pension capitals between consecutive cohorts are reduced. In total, the simulations show that the CUJ investment strategy effectively reduces the probability of unlucky generations.

Finally, we also look at the robustness of our utility function to misspecification. First, we consider the error where the wrong utility function is assumed for the investment strategy. In particular, we have analyzed the case where the CRRA utility or the CUJ utility is assumed, while the other is the "true" way of evaluation. The case where the CRRA fraction is invested, but our CUJ utility is used for evaluation, we have an infinite welfare loss, due to the infinite variance of the intergenerational benchmark. Conversely, we see that the CUJ investment strategy results in a very limited welfare loss, in case the cohort evaluates according to the CRRA utility function, even for high levels of the intergenerational factor. Furthermore, we see that, in case of the CUJ investment strategy, an overestimation of the intergenerational factor is much less costly than an underestimation. So, if we want to avoid large welfare losses due to misspecification, the most robust option is to choose the CUJ investment strategy with a high intergenerational factor.

To conclude, this paper proposes a new life-cycle investment strategy for retirement that prevents unlucky generations by reducing the differences in wealth at retirement between consecutive cohorts, without lowering pension levels relative to the CRRA strategy. The advantage of this method is that the smoothing of pensions is already achieved, without any risk-sharing or wealth transfers. This means that our strategy can be used next to such measures. We propose that further research is done into the design of a solidary scheme to further reduce differences between generations, given that our investment strategy is already followed.

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