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Time-Consistent Insurance Pricing

A Proof of Concept

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a Proof of Concept

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This paper investigates a new, Time-Consistent method to value insurance contracts. This new method is based on Cost of Capital pricing and is compared to the Industry standards using the Cost of Capital method.

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Introduction

This paper will give a proof of concept of a technique to price insurance contracts in a Time-Consistent way. In order to do this, the paper will first take a short look at how insurances are priced using the Cost of Capital method. Then we will look at the Industry standard for pricing contracts in contrast to a Time-Consistent pricing technique. Both methods will be discussed in detail, after which we will apply them to 2 theoretical insurance contracts, one of which is more realistic than the other. From these contracts results will be presented in order to compare both methods and to judge the relevance of introducing a new Time-Consistent method of pricing.

The aim of this proof of concept is to investigate the possible relevance of the Time-Consistent pricing technique. With this paper, I try to open up new roads for further research on the matter of Time-Consistent pricing in the hope that other researchers will continue on this path. In this introduction I would also like to thank Antoon Pelsser: without his ideas this paper would not have come into existence.

Two Paths of Pricing an Insurance

Before we start looking at practical examples and realistic scenarios it is a good idea to get a theoretic understanding of the differences between the Time-Consistent valuation method and the Industry standard. As both of these methodologies are based upon the so called Cost of Capital (CoC) valuation method, this section will start by taking a closer look at this Cost of Capital method. Finally this section will consider both the "Best Estimate" parts as well as the Cost of Capital parts for both the Industrial as well as the Time-Consistent method.

Cost of Capital and Valuation

The value of an insurance contract can (in general) be split in two parts. First of all there is what in this paper will be referred to as the "Best Estimate" value. This value is based on (conditional) expectations and is the price which reflects the average outcome of a contract. However, to price an insurance this "Best Estimate" is not enough. Within the contract there are various risks, some of which are hedge-able in the financial markets, others which are not. For the latter, an insurance company must hold a risk buffer, to ensure it meets regulatory requirements as well as to make sure it is able to continue business in stress scenarios. Clearly, the insurance company would like to calculate the costs of holding such a buffer into the value of the insurance.

The Cost of Capital method is a way to do this. It is a fairly common technique used in many countries, including EU-states as well as Switzerland[1]. The idea behind CoC is to define a risk measure, calculate an appropriate buffer and multiply this buffer with the Cost of Capital of the firm at hand. The outcome reflects the cost of holding the regulatory buffer capital for 1 year. When these costs are added to the "Best Estimate" value, one will get the total value of the insurance contract.

In this paper we assume a Cost of Capital of 6%. Furthermore we take the commonly used Value at Risk (VaR) as our risk measure. Even though Value at Risk has some clear limitations (like the risk of excessive losses behind the VaR cut-off point), the simplicity of VaR makes this risk measure perfect for a proof of concept investigation into Time-Consistent pricing. A 99,5% confidence interval will be used regarding the Value at Risk.

The "Best Estimate" Part

To evaluate the value of an insurance consider 2 types of risk drivers: changes in W1 (which we will later link to interest rate changes) and W2 (which will later on be connected to changes in the mortality table). For now W1 and W2 are simply standard Brownian motions.

The Industry method uses the following expectation to calculate the "Best Estimate" part of the value of this insurance, where IBE stands for "Industry Best Estimate":

$$IBE_{0,T} = \boldsymbol{E}[V_T(\,W1_T, W2_T\,)|W1_0\,, W2_0]$$

Here $IBE_{0,T}$ denotes the "Best Estimate" value of the insurance at time 0 till point T, and $W1_0$ and $W2_0$ denote the values of W1 and W2 at time 0. As one can see the Industry method only determines a Best Estimate at time 0.

[1] Swiss Federal Office of Private Insurance, 2004

The Time-Consistent method calculates the "Time Best Estimate" part of the value of the insurance as follows:

$$TBE_{t,T} = \boldsymbol{E}[V_T(W1_T, W2_T)|W1_t , W2_t] \quad with\; 0 < t < T$$

From this formula we see that the Time-Consistent method uses every time-step in between to get to a "Best Estimate" value. Using the Tower Property we see that both the Time-Consistent as well as the Industry method yields the same "Best Estimate" price. A 2 year example may clarify this. Say the time till maturity is 2, than working back starting at time 2, we obtain:

$$TBE_{1,2} = \boldsymbol{E}[V_2(W1_2, W2_2)|W1_1 , W2_1]$$

$$TBE_{0,2} = \boldsymbol{E}\left[TBE_{1,2}|W1_0 , W2_0\right]$$

$$TBE_{0,2} = \boldsymbol{E}[\boldsymbol{E}[V_2(W1_2, W2_2)|W1_1 , W2_1]|W1_0 , W2_0]$$

Then by the Tower Property we know that:

$$TBE_{0,2} = \boldsymbol{E}[V_2(W1_2, W2_2)|W1_0 , W2_0]$$

Which is exactly equal to the "Best Estimate " value in the Industry method. There is a difference though in the way to get there: the Industry method derives a "Best Estimate" price directly from point t=0 to a number of scenarios at time-step T. The Time-Consistent method works backward: first from T to T-1, then from T-1 to T-2 and so on till time-step 0.

The Cost of Capital Part

Let us assume that W1 is a hedge-able risk driver. This means that in the total insurance value there needs to be a buffer for the risk arising from W2. The Cost of Capital part of the value of the insurance can be written in a formula form as:

$$CoC_t = 6\% * VaR_t$$

Therefore we will look further into the Value at Risk component of the Cost of Capital.

For the Time-Consistent method we use the following formula for the Value at Risk:

$$TVaR_{t,T} = F_p^{-1}(V_T|W1_t, W2_t) - TBE_{t,T}$$

Where $F_p^{-1}(V_T|W1_t, W2_t)$ is the inverse function of the CDF from the distribution which applies to the risk drivers W1 and W2. W2 is the only risk driver for which we need to hold a buffer as we assumed W1 to be hedge-able. Hence $F_p^{-1}(V_T|W1_t, W2_t)$ equals the p-percent quantile for risk driver W2, based on random variable V_T. $VaR_{t,T}$ means we calculate the Value at Risk from t till T. Furthermore, $TBE_{t,T}$ is the Time-Consistent Best Estimate we derived before.

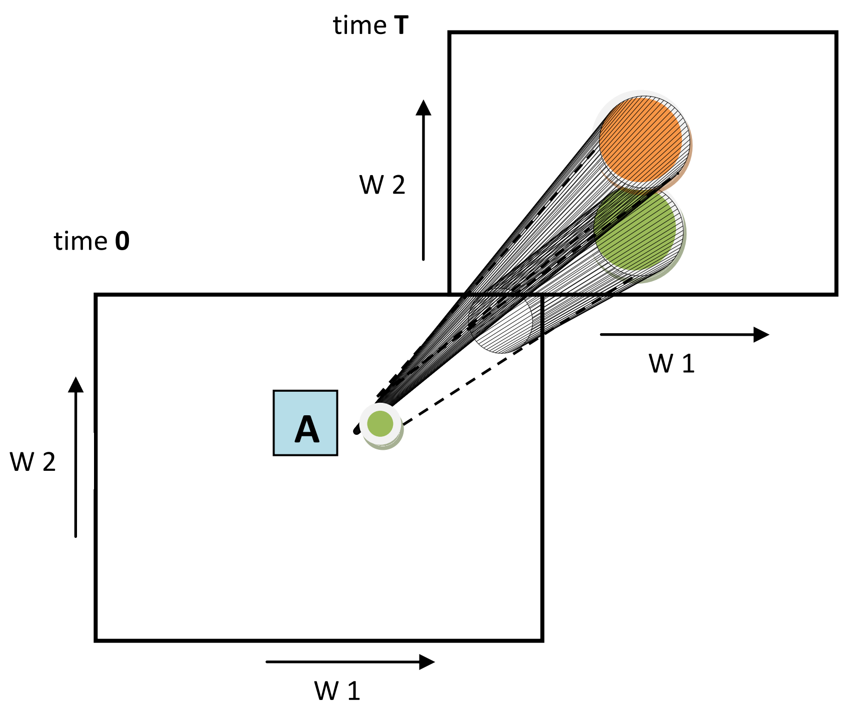
Note that this formula implies that there is a different Value at Risk for every time-step. Logically this makes sense: to determine the VaR from t to T, we use the value for W2 known at period t. This way one not only takes into account the possibilities of W2 as on the "Best Estimate" path, but also all other ones. However, in terms of calculation, this Time-Consistent method is more complicated and heavier. To acquire the Value at Risk in the Time-Consistent setting we have to start at time T and then work back in steps of 1 time period, updating our information as we progress.

In the Industry method we can write for the Value at Risk at time-step t:

$$IVaR_{0,t,T} = F_p^{-1}(V_T|W1_t, W2_t) - IBE_{0,T} \quad \text{where } W1_t = W1_0 \text{ and } W2_t = W2_0$$

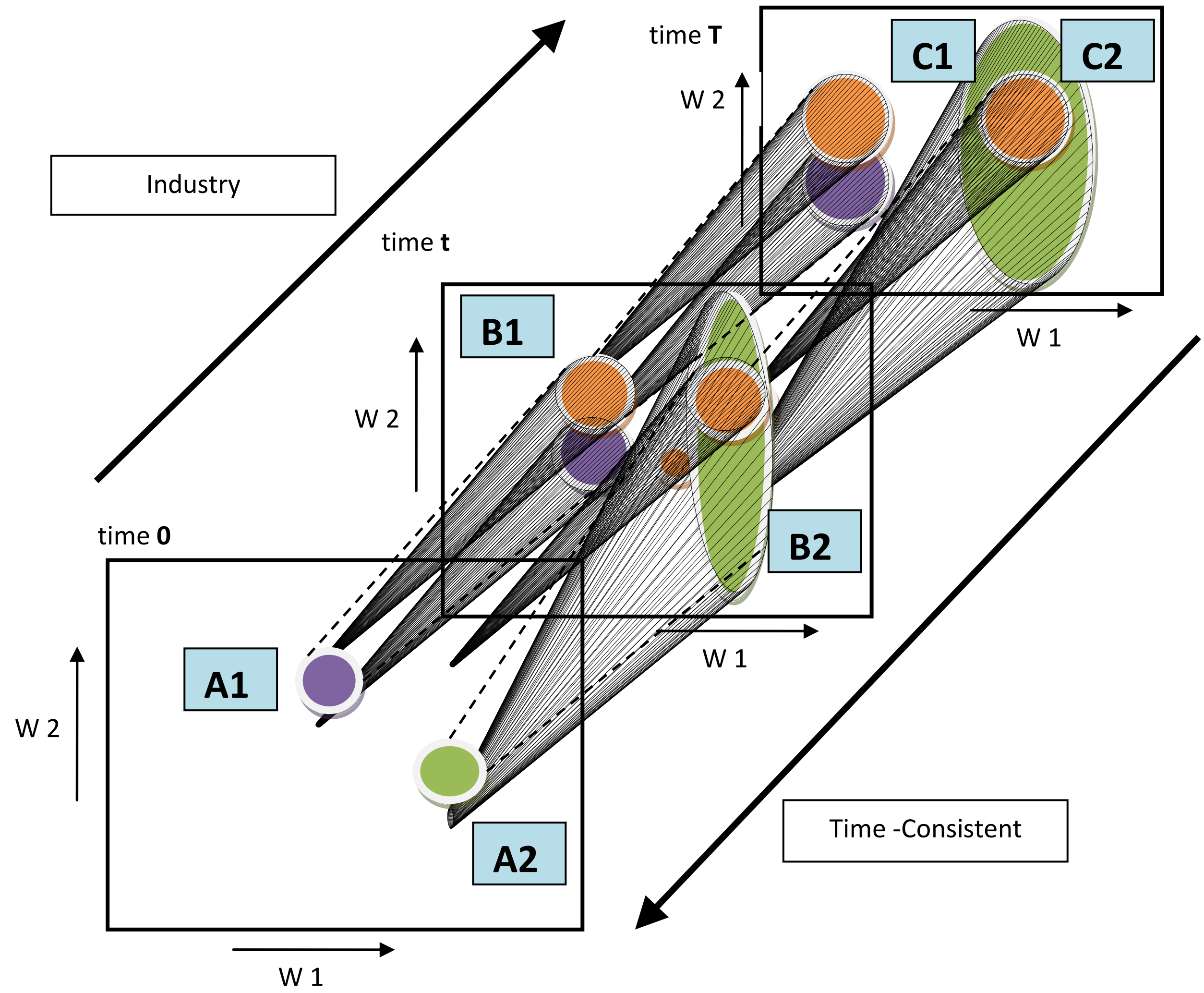
This implies all information is based on the knowledge at time 0: in other words, the Cost of Capital buffer is calculated as if we were on the "Best Estimate" path projected at time 0. This is actually counter-intuitive: the Industry-method ignores the fact that a more positive or negative outcome at time t, may change the buffer capital required from time t till time T. Again, the quantile of the function $F_p^{-1}(V_T|W1_t, W2_t)$ is only taken over W2 as this is the only non hedge able risk driver.

To better understand this difference between both methods let us try to visualize them. First consider a contract that lasts just for one year. The orange spot again illustrates the most likely "Best Estimate" outcomes. The green spot (beneath the orange one) shows these same outcomes, but then "shocked" with a Cost of Capital based on the Value at Risk.



Note that in this 1 year case the Value at Risk is the same for the Industrial method as well as for the Time-Consistent method. In both cases 1 buffer capital is calculated, namely from time 0 to time T. Both methods work with the same random variable V_T and use the information known at time 0.

The differences start to occur if we consider multiple time-step contracts. Let us consider a 2-step contract. The orange spot again shows “Best Estimate” values. The green spot illustrates the value including Cost of Capital from the Industry method and the purple spot shows this for the Time-Consistent method.



Note that the Industry-method projects forward along the Best Estimate trajectory, but that the Time-Consistent method works backward. In the Industry method we determine a Cost of Capital at point A1 for A1 to B1 and for B1 to C1 using the same underlying Best Estimate path. In the Time-Consistent method we first determine the Cost of Capital buffer from B2 to C2, using random variable V_T and $W2_t$. Using the outcome the Time-Consistent method updates the Best Estimate as well as the Cost of Capital buffer and uses these to calculate the value at A2 thereby taking into account all possible buffer capitals instead of only those along the “Best Estimate” path.

We see that the Cost of Capital component at time T for the Time-Consistent method is larger than for the Industry method: the variation around point C2 is larger than the variation around point C1. This influences the estimates for time t, thereby influencing the values at time 0. Hence, for now we conclude there is a difference in value in point A1 compared to A2.

First Steps into Practice: a Simple Insurance

In this section we will test the Time-Consistent method described above against the Industry method. The test will be based on a highly theoretical lumpsum life insurance. The first sub-section will introduce this insurance and derive a "Best Estimate" value. After this, formulas for the Cost of Capital component will be derived for both methods. Then a total insurance value will be composed. Finally there will be some (theoretical) results, in order to get a sense of magnitude regarding the differences between the Time-Consistent and the Industry methodology. For this section we assume no discounting and give no interpretation to W1 and W2.

Theoretical derivation of the Best Estimate value

Building further on the two standard Brownian motions we saw in the previous section, now consider a lumpsum life insurance of which the Best Estimate value can be determined according to the following formula:

$$V_T = e^{\alpha_1 W1_T + \alpha_2 W2_T}$$

Or in matrix notation as:

$$V_T = e^{\alpha' \boldsymbol{W}_T}$$

Furthermore, as W1 and W2 are (independent) Brownian motions:

$$W_T \sim N(0, I)$$

In which I is the identity matrix. We are however, interested in the value of the insurance at time t rather than time T (where t < T). From the Moment Generating Function we can derive directly that the "Best Estimate" value for the Industry method equals:

$$IBE_{0,T} = E\left[e^{\alpha' W_T} \middle| W_0\right] = e^{\alpha' W_0 + \frac{1}{2} T \alpha' I \alpha}$$

And that for the Time-Consistent case:

$$TBE_{t,T} = E\left[e^{\alpha' W_T} \middle| W_t\right] = e^{\alpha' W_t + \frac{1}{2}(T-t)\alpha' I \alpha}$$

For the Time-Consistent case we see that working back from T to t, and then from t to 0 that:

$$TBE_{t,T} = e^{\alpha' W_t} * e^{\frac{1}{2}(T-t)\alpha' I \alpha}$$

So after a time-step from t to T, a factorisation is made to the original value. Similar to this we have:

$$TBE_{0,t} = e^{\alpha' W_t} * e^{\frac{1}{2}(t-0)\alpha' I \alpha}$$

So making the time-step from t to T and then from t to 0 we get:

$$TBE_{0,T} = e^{\alpha' W_t} * e^{\frac{1}{2}(T-t)\alpha' I\alpha} * e^{\frac{1}{2}(t-0)\alpha' I\alpha} = e^{\alpha' W_0 + \frac{1}{2}T\alpha' I\alpha}$$

And so we see (again), that because of the Tower Property, the Best Estimate part of the price is equal in both the Time-Consistent as well as in the Industry method.

Theoretical derivation of the CoC component

Now we will derive the Cost of Capital part for valuing the insurance. Suppose that risk driver w1 is a risk that can be hedged: hence we need not hold a buffer for it. However, risk driver w2 cannot be hedged and therefore we do need some buffer capital. As both risk drivers will be split for the analysis, the notation will take a scalar form (instead of the matrix form used in the previous section).

In the previous section we saw the following formula for the (general form) of the Industry Value at Risk:

$$IVaR_{t,T} = F_p^{-1}(V_T|W1_t, W2_t) - IBE_{t,T}, \text{where } W1_t = W1_0 \text{ and } W2_t = W2_0$$

In this specific case we know that $F_p^{-1}(V_T|W1_t, W2_t)$ is lognormal distributed. Taking the quantile of $W2_t$ we get for a lognormal distribution[2]:

$$F_p^{-1}(V_T|W1_t, W2_t) = e^{\left(\mu + \sigma * \varphi^{-1}(p)\right)} \; with \; 0 < p < 1$$

Where $\varphi^{-1}(p)$ is the quantile function of the normal distribution. Since our confidence level will be 99,5% we know that $\varphi^{-1}(p) = 2{,}58$. As we are interested in the Value at Risk (only) for risk driver W2, we use the mean and standard deviation we derived earlier to obtain:

$$F_p^{-1}(V_T|W1_t, W2_t) = e^{a2*\left(W2_t + a2*2{,}58*\sqrt{T-t}\right) + a1*W1_t + a1^2*0{,}5*\sqrt{T-t}}$$

Note that for the $IBE_{t,T}$ part of the equation we use the full Best Estimate, including values of W1 as well as W2. This is because, even though the quantile is only taken over W2, the Best Estimate is influenced by both risk drivers. There is a plus in front of the second $a2$: we are interested in the quantile where the payment that has to be made to the policy holder is higher than expected, hence we take the maximum of either plus or minus a p-percentile shock. In case $a2$ is positive, there will be a plus sign. In case $a2$ is negative we will have a minus. Note, however, that this only holds in the 1 dimensional case (cases where there is only one "shocked" risk driver). Taking time-steps of 1, so T-t=1 We obtain for the Industry case:

$$IVaR_{t,T} = e^{a2*(W2_t + a2*2{,}58) + a1*W1_t + \frac{1}{2}*a1^2} - e^{a1*W1_t + a2*W2_t + \frac{1}{2}*(T-t)*a1^2 + \frac{1}{2}*(T-t)*a2^2}$$

As we condition on $W2_t = W2_0$ and only use projections along the Best Estimate path of W2 we know that $W2_t = W2_0$ and that $W1_t = W1_0$.

[2] Cassella, Berger, 2002

Hence we obtain:

$$IVaR_{t,T} = e^{a2*(w2_0+a2*2,58)+a1*W1_0+\frac{1}{2}*a1^2} - e^{a1*W1_0+a2*W2_0+\frac{1}{2}*T*a1^2+\frac{1}{2}*T*a2^2}$$

Assuming that $W2_0 = 0$ (since W2 is a Brownian Motion) we get:

$$IVaR_{t,T} = e^{a2^2*2,58+a1^2*\frac{1}{2}} - e^{\frac{1}{2}*T*a1^2+\frac{1}{2}*T*a2^2}$$

Which implies, in this theoretical framework, that the Value at Risk buffer is equal for all 1-year steps between 0 and T. In other words, $IVaR_{t,T}$ is a constant which does not depend on W1 or W2.

For the Time-Consistent case we saw the following initial formula of the general form Value at Risk:

$$TVaR_{t,T} = F_p^{-1}(V_T|W1_t, W2_t) - TBE_{t,T}$$

Following a similar derivation we obtain for quantile part of the equation:

$$F_p^{-1}(V_T|W1_t, W2_t) = e^{a2*\left(W2_t+a2*2,58*\sqrt{T-t}\right)+a1*W1_t+\frac{1}{2}*a1^2*\sqrt{T-t}}$$

Note that now the quantile depends on the real $W2_t$ (all possible outcomes) rather than those along the Best Estimate path. Furthermore we know:

$$TBE_{t,T} = e^{a1*W1_t+a2*W2_t+\frac{1}{2}*(T-t)*a1^2+\frac{1}{2}*(T-t)*a2^2}$$

$$TVaR_{t,T} = e^{a2*W2_t} * \left(e^{a2^2*2,58*\sqrt{T-t}+a1*W1_t+\frac{1}{2}*a1^2*\sqrt{T-t}} - e^{a1*W1_t+\frac{1}{2}*(T-t)*a1^2+\frac{1}{2}*(T-t)*a2^2}\right) \forall\, t, T$$

As we take time steps of T-t=1 we know:

$$TVaR_{t,T} = e^{a2*W2_t+a1*W1_t+\frac{1}{2}*a1^2} * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right) \forall\, t, T$$

Now we have both components, let us see how the total price looks using the Industry methodology as well as the Time-Consistent method.

Total Insurance price under Industry pricing

Assuming a Cost of Capital of 6% we know that the price of a 1-year contract equals for the Industry case, hence T=1 and t=0.

$$V_0 = \boldsymbol{E}[V_1|W1_1, W2_1]$$

$$V_0 = \boldsymbol{E}[V_1|W1_0, W2_0] + 6\% * IVaR_{0,1}$$

$$V_0 = IBE_{0,1} + 6\% * IVaR_{0,1}$$

$$V_0 = e^{a1*W1_0+a2*W2_0+0,5*1*a1^2+0,5*1*a2^2} + 6\% * \left(e^{a2*(w2_0+a2*2,58)+a1*W1_0+\frac{1}{2}*a1^2} - e^{a1*W1_0+a2*W2_0+\frac{1}{2}*T*a1^2+\frac{1}{2}*T*a2^2}\right)$$

When looking at a 2-year contract we set T=2. However for the Value at Risk we use T=1 and T=2 as well as t=0 and t=1 to account for 2 years of buffer:

$$V_0 = \boldsymbol{E}[V_2|W1_2, W2_2]$$

$$V_0 = \boldsymbol{E}[V_2|W1_1, W2_1] + 6\% * IVaR_{1,2}$$

$$V_0 = \boldsymbol{E}[V_2|W1_0, W2_0] + 6\% * IVaR_{1,2} + 6\% * IVaR_{0,1}$$

$$V_0 = IBE_{0,2} + 6\% * IVaR_{1,2} + 6\% * IVaR_{0,1}$$

$$V_0 = e^{a1*W1_0+a2*W2_0+0,5*1*a1^2+0,5*1*a2^2} + 6\% * \left(e^{a2*(w2_0+a2*2,58)+a1*W1_0+\frac{1}{2}*a1^2} - e^{a1*W1_0+a2*W2_0+\frac{1}{2}*T*a1^2+\frac{1}{2}*T*a2^2}\right) + 6\% * \left(e^{a2*(w2_0+a2*2,58)+a1*W1_0+\frac{1}{2}*a1^2} - e^{a1*W1_0+a2*W2_0+\frac{1}{2}*T*a1^2+\frac{1}{2}*T*a2^2}\right)$$

We see that, in this theoretical setting, the $IVaR_{1,2}$ and $IVaR_{0,1}$ are equal. However, this does not have to hold for the general case.

The examples above make it intuitive to conclude that for a T=n year contract we can derive:

$$V_0 = \boldsymbol{E}[V_n|W1_n, W2_n]$$

$$V_0 = \boldsymbol{E}[V_n|W1_{n-1}, W2_{n-1}] + 6\% * IVaR_{0,n-1,n}$$

$$V_0 = \boldsymbol{E}[V_n|W1_{n-2}, W2_{n-2}] + 6\% * IVaR_{0,n-2,n-1} + 6\% * IVaR_{0,n-1,n}$$

$$\vdots$$

$$V_0 = IBE_{0,n} + 6\% * \sum_{k=0}^{k=n-1} IVaR_{0,k,k+1}$$

$$V_0 = e^{a1*W1_0+a2*W2_0+0,5*n*a1^2+0,5*n*a2^2} + 6\% * \sum_{k=0}^{k=n-1} \left(e^{a2*(w2_0+a2*2,58)+a1*W1_0+\frac{1}{2}*a1^2} - e^{a1*W1_0+a2*W2_0+\frac{1}{2}*T*a1^2+\frac{1}{2}*T*a2^2}\right)$$

Assuming that $W1_0 = W2_0 = 0$ yields:

$$V_0 = e^{0,5*n*a1^2+0,5*n*a2^2} + 6\% * \sum_{k=0}^{k=n-1} \left(e^{a2^2*2,58+\frac{1}{2}*a1^2} - e^{\frac{1}{2}*a1^2+\frac{1}{2}*a2^2}\right)$$

Again, the Value at Risk components are the same for each step: however, this is only for this specific case. Now let us take a look at the total price of the insurance when using the Time-Consistent approach.

Total Insurance price under Time-Consistent pricing

In the Time-Consistent method, we work back from time T to obtain a value at time 0, much alike valuing an American style option with a binomial tree. Let us again start by valuing a 1-year contract, but now for the Time-Consistent setting. Again we know T=1 and t=0:

$$V_0 = \boldsymbol{E}[V_1|W1_1, W2_1]$$

$$V_0 = \boldsymbol{E}[V_1|W1_0, W2_0] + 6\% * TVaR_{0,1}$$

$$V_0 = TBE_{0,1} + 6\% * TVaR_{0,1}$$

Where we know that:

$$TBE_{0,1} = e^{a1*W1_0+a2*W2_0+0,5*1*a1^2*a2^2}$$

$$TVaR_{0,1} = e^{a2*W2_0+a1*W1_0+\frac{1}{2}*a1^2} * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)$$

So then:

$$V_0 = e^{a1*W1_0+a2*W2_0+0,5*a1^2+0,5*a2^2} + 6\% * e^{a2*W2_0+a1*W1_0+\frac{1}{2}*a1^2} * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)$$

And for this particular case we can rewrite as:

$$V_0 = e^{a2*W2_0} * \left(e^{+a1*W1_0+\frac{1}{2}*a1^2+\frac{1}{2}*a2^2} + 6\% * \left(e^{a2^2*2,58+a1*W1_0+\frac{1}{2}*a1^2} - e^{a1*W1_0+\frac{1}{2}*a1^2+\frac{1}{2}*a2^2}\right)\right)$$

For this case we find no differences with the Industry method. These start to occur as soon as we use a Value at Risk from t>0 to T. This we will see in the 2-year contract. Now, we start working from T=2 to t=1 and then from t=1 to t=0. For every time step we will develop a new Best Estimate value as well as a new Cost of Capital value.

Let us start looking at time-point 2:

$$V_2 = e^{a1*W1_T+a2*W2_T}$$

Now consider time-point 1:

$$V_1 = \boldsymbol{E}[V_2|W1_2, W2_2]$$

$$V_1 = \boldsymbol{E}[V_2|W1_1, W2_1] + 6\% * TVaR_{1,2}$$

$$V_1 = TBE_{1,2} + 6\% * TVaR_{1,2}$$

$$TBE_{1,2} = E[V_2|W1_1, W2_1] = e^{a1*W1_1+a2*W2_1+0,5*a1^2+0,5*a2^2}$$

$$TVaR_{1,2} = F_p^{-1}(V_2|W1_1, W2_1) - TBE_{1,2}$$
$$= 6\% * e^{a2*W2_1} * \left(e^{a2^2*2,58+a1*W1_1+\frac{1}{2}*a1^2} - e^{a1*W1_1+\frac{1}{2}*a1^2+\frac{1}{2}*a2^2}\right)$$

$$V_1 = e^{a1*W1_1+a2*W2_1+0,5*a1^2+0,5*a2^2} + 6\% * e^{a2*W2_1} * \left(e^{a2^2*2,58+a1*W1_1+\frac{1}{2}*a1^2} - e^{a1*W1_1+\frac{1}{2}*a1^2+\frac{1}{2}*a2^2}\right)$$

$$V_1 = e^{a2*W2_1+a1*W1_1+0,5*a1^2} * \left(e^{0,5*a2^2} + 6\% * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)\right)$$

Now consider time-point 0. Note that, instead of basing the value at time 0 on the outcomes of time 2, we do not take the expectation over V_2 but use the "updated" information we acquired in the previous step. Hence we take the expectation over V_1.

$$V_0 = \boldsymbol{E}[V_1 | W1_1, W2_1]$$

$$V_0 = \boldsymbol{E}[V_1 | W1_0, W2_0] + 6\% * TVaR_{0,1}$$

$$V_0 = TBE_{0,1} + 6\% * TVaR_{0,1}$$

$$TBE_{0,1} = E[V_1 | W1_0, W2_0] = e^{a1*W1_0+a2*W2_0+0,5*2*a1^2+0,5*a2^2} * \left[e^{0,5*a2^2} + 6\% * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)\right]$$

$$TVaR_{0,1} = F_p^{-1}(V_1 | W1_0, W2_0) - TBE_{0,1}$$

Where:

$$F_p^{-1}(V_1 | W1_0, W2_0) = e^{a2*W2_0+2,58*a2^2+a1*W1_0+0,5*2*a1^2} * \left[e^{0,5*a2^2} + 6\% * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)\right]$$

Hence we know that:

$$TVaR_{0,1} = e^{a2*W2_0+a1*W1_0+0,5*2*a1^2} * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right) * \left[e^{0,5*a2^2} + 6\% * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)\right]$$

$$TVaR_{0,1} = e^{a2*W2_0+a1*W1_0+0,5*2*a1^2} * \left[\left(e^{2,58*a2^2} - e^{\frac{1}{2}*a2^2}\right) e^{0,5*a2^2} + 6\% * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)^2\right]$$

Now we can derive, under the assumption that $W1_0 = 0$ that V_0 equals:

$$V_0 = e^{a2*W2_0+0,5*2*a1^2+0,5*a2^2} * \left[e^{0,5*a2^2} + 6\% * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)\right] + 6\% * \left[e^{\,a2*W2_0+0,5*2*a1^2} * \left[\left(e^{\,2,58*a2^2} - e^{\,\frac{1}{2}*a2^2}\right)e^{0,5*a2^2} + 6\% * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)^2\right]\right]$$

$$V_0 = e^{a2*W2_0} * \left[e^{0,5*2*a2^2+0,5*2*a1^2} + 2*6\% * e^{0,5*a2^2+0,5*2*a1^2}\left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right) + 6\%^2 * e^{0,5*2*a1^2} * \left(e^{a2^2*2,58} - e^{\frac{1}{2}*a2^2}\right)^2\right]$$

$$V_0 = e^{\,a2*W2_0} * \left(e^{0,5*a1^2+0,5*a2^2} + 6\% * \left(e^{a2^2*2,58+\frac{1}{2}*a1^2} - e^{\frac{1}{2}*a1^2+\frac{1}{2}*a2^2}\right)\right)^2$$

One could prove (using for instance Mathematical Induction) that in the n-year case we find that the total insurance value at time 0 equals:

$$V_0 = e^{\,a2*W2_0} * \prod_{k=0}^{k=n}\left(e^{0,5*a1^2+0,5*a2^2} + 6\% * \left(e^{a2^2*2,58+\frac{1}{2}*a1^2} - e^{\frac{1}{2}*a1^2+\frac{1}{2}*a2^2}\right)\right)$$

Note that the k does not appear in the product term, meaning that this term is equal for all time steps. However, this does not have to hold in the general case which we will see later on when using a more complex model.

From the formulas of the valuation in an Industry or Time-Consistent manner it is clear that there will be differences between the two values. Let us look at some results now.

Results I: Industry VS Time-Consistent using a Simple Insurance

Below a table with some results. For this table we have taken a contract of 10 years/time-steps with W1 and W2 equal to zero. Furthermore, α1 was taken equal to 0.1.

α2	BE	Industry Value	Time-Consistent Value	% Difference
-0,5	3,669	4,135	5,480	**32,51%**
-0,4	2,340	2,598	2,957	**13,83%**
-0,3	1,649	1,779	1,864	**4,81%**
-0,2	1,284	1,337	1,352	**1,13%**
-0,1	1,105	1,118	1,119	**0,11%**
0	1,051	1,051	1,051	**0,00%**
0,1	1,105	1,118	1,119	**0,11%**
0,2	1,284	1,337	1,352	**1,13%**
0,3	1,649	1,779	1,864	**4,81%**
0,4	2,340	2,598	2,957	**13,83%**
0,5	3,669	4,135	5,480	**32,51%**

Due to the square terms (next to the fact that we assume W1 and W2 equal to zero in the initial situation), we see that the differences of both methods are symmetric around zero when changing α2. Note that the difference between both methods highly depends on the chosen parameters (α2 for starters, but also α1 and the duration of the contract). The differences within these results range from big to insignificant. To be able to create results with a realistic parameterization we will now move to a more realistic model.

A small Leap: into a more Realistic Insurance

In this section we will try to make the highly theoretical framework described above more realistic. Rather than using W1 and W2, which were assumed Brownian motions, we will use more realistic risk drivers to model interest rate as well as mortality using a Vasicek model.

Vasicek model introduction

The Vasicek model was created as an interest-rate model. However, in this paper we will use it to model interest rate as well as mortality rates. Let us first take a look at the (original) interest rate model though. It is based on an instantaneous spot rate, r_t which is defined as:

$$dr_t = \gamma * (\omega - r_t)dt + \sigma dW(t)$$

Where γ is the reversion parameter and ω is the long term interest. Under some rather common assumptions the Vasicek zero-coupon rate is formulated as follows, where $Z_{T,S}$ is the S-T year zero coupon rate from the Vasicek curve at point T[3]:

$$Z_{T,S} = -\frac{1}{S-T} * \alpha_{T,S} + \frac{1}{S-T} * \beta_{T,S} * r_T \text{ , } where\ S \geq T$$

Where:

$$\alpha_{T,S} = \frac{\left(\beta_{T,S} - S + T\right) * (\gamma^2 * \omega - 0{,}5\sigma^2)}{\gamma^2} - \frac{\left(\sigma^2 {\beta_{T,S}}^2\right)}{(4 * \gamma)}$$

And:

$$\beta_{T,S} = \frac{1}{\gamma} * \left(1 - e^{-\gamma * (S-T)}\right)$$

We will start looking at the contract earlier than point T though and hence we will need estimates for the value of r_T from an earlier time, say time t. This prediction is the only source of randomness regarding the value of the insurance at time T: as soon as we arrive there, the value is based on the $\alpha_{T,S}$ and $\beta_{T,S}$ and is known. For r_T is well know that we have:[4]

$$r_T | r_t \sim N\left(\mu_{t,T}, s_{t,T}^2\right), \quad for\ S \geq T \geq t$$

Where:

$$\mu_{t,T} = r_t * e^{-\gamma * (T-t)} + \omega\left(1 - e^{-\gamma * (T-t)}\right)$$

$$s_{t,T}^2 = \frac{\sigma^2}{2 * \gamma} * \left[1 - e^{-2\gamma * (T-t)}\right]$$

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[3] Hull, 2002
[4] Hull, 2002

To take into accounting discounting of the cashflows within a contract, we need to scale $Z_{t,T}$ with a factor $(S-T)$ as for discounting we use the zero-rate multiplied with the time till the cashflow: to discount 10 years we take the exponential of minus the 10-year interest and multiply this with a factor of 10. From this it follows (by the arithmetic rules of the normal distribution) that:

$$-(S-T) * Z_{T,S}|r_t \sim N(\alpha_{T,S} - \beta_{T,S} * \mu_{t,T}, {\beta_{T,S}}^2 * s_{t,T}^2)$$

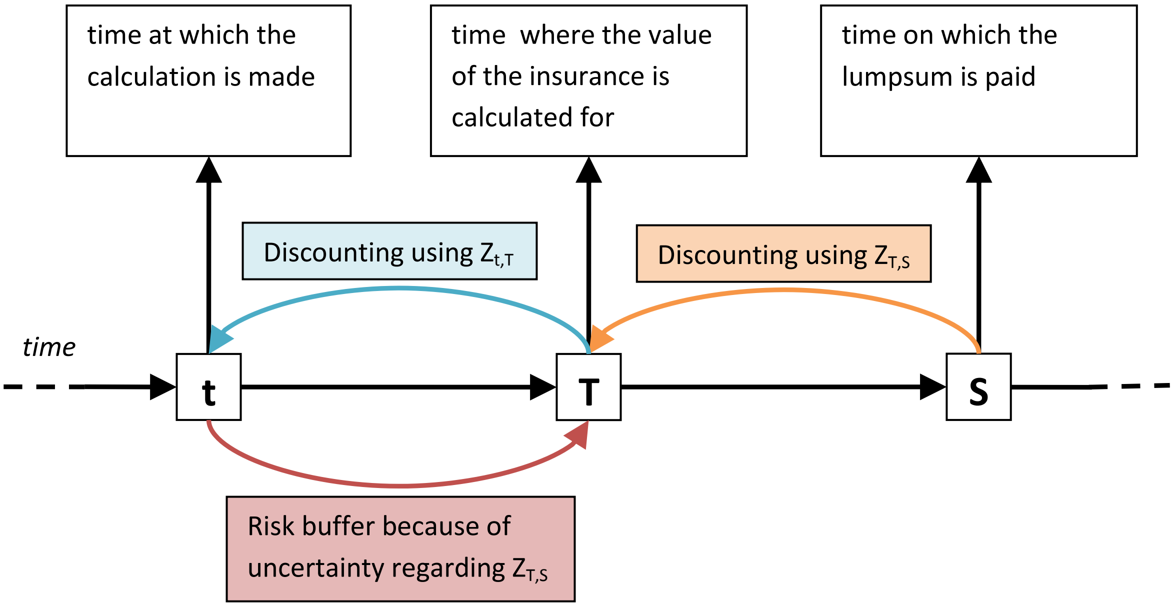
Now we will implement two of these Vasicek models into the valuation models that were discussed in the previous section.

Application to the models

In this section we will use the Vasicek model to value an insurance with risk drivers r1 and r2, where r1 represents interest rates and r2 mortality rates. For the interest rate modeling it may be intuitive to use a Vasicek model: this is what it was originally created for. And although the model is not the most complex and has its shortcomings (the biggest of which is possible negative interest rates) the Vasicek model is realistic enough to model interest rates in the proof of concept. At first sight it may seem off to use a Vasicek model to model mortality. Modeling mortality is often done by using the so called force of mortality: the instantaneous death rate for lives of a certain age. There are various approaches to model mortality, using the force of mortality: for instance with a time-homogeneous Poisson process (where the force of mortality is assumed deterministic and piecewise constant) or a Cox process (where the force of mortality is assumed stochastic). Another possibility is to use an affine stochastic model, like Vasicek, without (or with strictly negative) mean reversion. A downside of using a Vasicek model for this is that the force of mortality can actually become negative, implying that people get born at a certain age rather than die. For this proof of concept setting though, the Vasicek model will suffice.[5]

[5] Schrager, 2006

The insurance we will look at will look at will again be a lumpsum after a certain period S. However, we will value this insurance at a point T, which is between t (current time) and S (maturity). The valuation has the following time structure:



For simplicity reasons the discounting between point T and t will be ignored. As can be derived from the picture, we would have to use another interest rate curve for this, which will make the problem more complex. From the picture we see that for the value of the insurance at time t we have:

$$V_T = e^{-(S-T)* \, Z1_{T,S}+(S-T)* \, Z2_{T,S}}$$

Using a similar approach as in the previous section we will first derive Best Estimates for both methods.

Best Estimates

We first start by deriving the Industry Best Estimate. Note that $t = 0$ for the Industry method since we project forward from point 0. From the Moment Generating Function of the lognormal distribution we see that:

$$IBE_{0,T} = E[V_T | r1_0, r2_0]$$

$$IBE_{0,T} = E\left[e^{-S* \, Z1_{T,S}-(S-T)* \, Z2_{T,S}} \middle| r1_0, r2_0\right]$$

$$IBE_{0,T} = e^{\alpha1_{T,S}-\beta1_{T,S}*\mu1_{0,T}+\alpha2_{T,S}-\beta2_{T,S}*\mu2_{0,T}} * e^{\frac{1}{2}*\beta1_{T,S}{}^2*s1_{0,T}^2+\frac{1}{2}*\beta2_{T,S}{}^2*s2_{0,T}^2}$$

In order to be able to impose a similar structure onto this formula as we saw with the simple insurance contract we will rewrite the equation above in terms of $r1_t$ and $r2_t$

For the Time-Consistent method we can derive:

$$TBE_{t,T} = E[V_T|r1_t, r2_t]$$

$$TBE_{t,T} = E\left[e^{-(S-T)*\ Z1_{T,S}-(S-T)*\ Z2_{T,S}}\middle|r1_t, r2_t\right]$$

$$TBE_{t,T} = \ e^{\alpha 1_{T,S}-\beta 1_{T,S}*\mu 1_{t,T}+\alpha 2_{T,S}-\beta 2_{T,S}*\mu 2_{t,T}} * e^{\frac{1}{2}*\beta 1_{T,S}{}^2*s1_{t,T}^2+\frac{1}{2}*\beta 2_{T,S}{}^2*s2_{t,T}^2}$$

Once more one can show that, for estimating the best value at time t=0, the Industry method yields the same outcomes as the Time-Consistent method due to the Tower Property in a similar way as was done for the more theoretic insurance used before. Now we will look at the derivation of the Cost of Capital part of the insurance price.

Derivation of the Cost of Capital component

Similar to what we did earlier in order to find the Cost of Capital component we will again use the following formula to determine the Cost of Capital part of in the Industry valuation:

$$IVaR_{t,T} = F_p^{-1}(V_T|r1_t, r2_t) - IBE_{t,T}$$

Since $F_p^{-1}(V_T|r1_t, r2_t)$ has the same properties as we saw before in the more theoretical setting we can in general state that:

$$F_p^{-1}(V_T|r1_t, r2_t) = e^{\left(\mu+\sigma*\varphi^{-1}(p)\right)}\ with\ 0 < p < 1$$

Hence we have:

$$F_p^{-1}(V_T|r1_t, r2_t) = e^{\alpha 1_{T,S}-\beta 1_{T,S}*\mu 1_{t,T}+\alpha 2_{T,S}-\beta 2_{T,S}*\mu 2_{t,T}} * e^{\frac{1}{2}*\beta 1_{T,S}*s1_{t,T}+2{,}58*\ \beta 2_{T,S}*s2_{t,T}}$$

Hence we know that for the total Cost of Capital component for the Industry case we have:

$$IVaR_{t,T} = e^{\alpha 1_{T,S}-\beta 1_{T,S}*\mu 1_{t,T}+\alpha 2_{T,S}-\beta 2_{T,S}*\mu 2_{t,T}} * \left[e^{\frac{1}{2}*\beta 1_{T,S}*s1_{t,T}+2{,}58*\ \beta 2_{T,S}*s2_{t,T}} - e^{\frac{1}{2}*\beta 1_{T,S}{}^2*s1_{t,T}^2+\frac{1}{2}*\beta 2_{T,S}{}^2*s2_{t,T}^2}\right]$$

And as we use only the projections for $r1_t$ and $r2_t$ that are along the Best Estimate projected path, we know that $r1_t = r1_0$ and$r2_t = r2_0$. For this we need to introduce some new notation being:

$$\hat{\mu}_{t,T} = r_0 * e^{-\gamma*(T-t)} + \omega\left(1 - e^{-\gamma*(T-t)}\right)$$

Now we can rewrite as:

$$IVaR_{t,T} = e^{\alpha 1_{T,S}-\beta 1_{T,S}*\hat{\mu} 1_{t,T}+\alpha 2_{T,S}-\beta 2_{T,S}*\hat{\mu} 2_{t,T}} * \left[e^{\frac{1}{2}*\beta 1_{T,S}*s1_{t,T}+2{,}58*\ \beta 2_{T,S}*s2_{t,T}} - e^{\frac{1}{2}*\beta 1_{T,S}{}^2*s1_{t,T}^2+\frac{1}{2}*\beta 2_{T,S}{}^2*s2_{t,T}^2}\right]$$

Following the same method we can derive for the Time-Consistent method a Cost of Capital value as well. Note that in the Time-Consistent case we explicitly do not take only the possible outcomes along the Best Estimate path (this is the whole point of making the method Time-Consistent). Therefore we use the mean originally derived from the Vasicek model:

$$TVaR_{t,T} = e^{\alpha 1_{T,S} - \beta 1_{T,S} * \mu 1_{t,T} + \alpha 2_{T,S} - \beta 2_{T,S} * \mu 2_{t,T}} * \left[e^{\frac{1}{2} * \beta 1_{T,S} * s1_{t,T} + 2{,}58 * \ \beta 2_{t,T} * s2_{t,T}} - e^{\frac{1}{2} * \beta 1_{T,S}{}^{2} * s1_{t,T}^{2} + \frac{1}{2} * \beta 2_{t,T}{}^{2} * s2_{t,T}^{2}} \right]$$

Now let us put all components together to create the total insurance values at point T.

Total Insurance Pricing using Industry methods

The structure of pricing the insurance for in the Industry method is exactly the same as for the simple insurance case. To emphasise this, and to lighten notation a little let us define the following parts:

$$A_{t,T,S} = \alpha 1_{T,S} + \alpha 2_{T,S} + \omega 1\left(1 - e^{-\gamma 1 * (T-t)}\right) * \beta 1_{T,S} + \omega 2\left(1 - e^{-\gamma 2 * (T-t)}\right) * \beta 2_{T,S}$$

$$B_{t,T,S} = \beta_{T,S} * e^{-\gamma 1 * (T-t)}$$

$$C_{t,T,S} = \frac{1}{2} * \beta 1_{T,S}{}^{2} * s1_{t,T}^{2} + \frac{1}{2} * \beta 2_{T,S}{}^{2} * s2_{t,T}^{2}$$

$$D_{t,T,S} = \frac{1}{2} * \beta 1_{T,S} * s1_{t,T} + 2{,}58 * \ \beta 2_{T,S} * s2_{t,T}$$

With these 4 components we can create all possible insurance values in an orderly manner.

Let us again assume a Cost of Capital of 6%. Furthermore let us assume that S equals 10. We start by considering a contract where T-t equals 1. The total price of the contract at time t=0 equals:

$$V_1 = E[V_1 | r1_1, r2_1]$$

$$V_0 = \boldsymbol{E}[V_1 | r1_0, r2_0] + 6\% * IVaR_{0,1}$$

$$V_0 = \ e^{A_{0,1,10} - B1_{0,1,10} * r1_0 - B2_{0,1,10} * r2_0} * \left[e^{C_{0,1,10}} + 6\% * \left(e^{D_{0,1,10}} - e^{C_{0,1,10}} \right) \right]$$

Note that this is structure wise very similar to the formula's we saw in the simple insurance case. Now let us look at an insurance where T-t equals 2. W still assume that S is equal to 10. Now we get:

$$V_0 = E[V_2 | r1_2, r2_2]$$

$$V_0 = \boldsymbol{E}[V_2 | r1_1, r2_1] + 6\% * IVaR_{1,2}$$

$$V_0 = \boldsymbol{E}[V_2 | r1_0, r2_0] + 6\% * IVaR_{1,2} + 6\% * IVaR_{0,1}$$

$$V_0 = e^{A_{0,2,10} - B1_{0,2,10} * r1_0 - B2_{0,2,10} * r2_0} * e^{C_{0,2,10}} + 6\% * [e^{A_{1,2,10} - B1_{1,2,10} * r1_0 - B2_{1,2,10} * r2_0} * (e^{D_{1,2,10}} - e^{C_{1,2,10}})] + 6\% * [e^{A_{0,1,10} - B1_{0,1,10} * r1_0 - B2_{0,1,10} * r2_0} * (e^{D_{0,1,10}} - e^{C_{0,1,10}})]$$

Note that the subscripts of A and B change when we move from one Value at Risk buffer to another: this is because T changes, but S remains fixed through time. Also note that, as the Industry method assumes $r1_t$ lies along the "Best Estimate" path, we find the term $r1_0$.

Using the examples above it becomes easier to understand that the total value of a contract with maturity S, in which we calculate the value at point T=n, where S ≥ T can be derived as follows:

$$V_0 = \boldsymbol{E}[V_n | r1_n, r2_n]$$

$$V_0 = \boldsymbol{E}[V_n | r1_{n-1}, r2_{n-1}] + 6\% * IVaR_{n-1,n}$$

$$V_0 = \boldsymbol{E}[V_n | r1_{n-2}, r2_{n-2}] + 6\% * IVaR_{n-1,n} + 6\% * IVaR_{n-2,n-1}$$

$$\vdots$$

$$V_0 = IBE_{0,n} + 6\% * \sum_{k=0}^{k=n-1} IVaR_{k,k+1}$$

$$V_0 = e^{A_{0,n,S} - B1_{0,n,S} * r1_0 - B2_{0,n,S} * r2_0} * e^{C_{0,n,S}} + 6\% * \sum_{k=1}^{k=n} [e^{A_{k-1,k,S} - B1_{k-1,k,S} * r1_0 - B2_{k-1,k,S} * r2_0} * (e^{D_{k-1,k,S}} - e^{C_{k-1,k,S}})]$$

Note that in contrast to the more simple insurance case discussed earlier, the Cost of Capital part changes from one time-step to another through parameters alpha and beta. The rest of the structure however is exactly the same: we have a normally distributed random variable, multiplied and added with some deterministic values.

Now we have a way to calculate the Industry value of our insurance, let us take a look at the Time-Consistent value.

Time-Consistent Pricing

Let us take the same starting point as in the Industry way and start with the pricing of a contract where T-t equals 1. The maturity S is still 10 years and the Cost of Capital still 6%. We know that:

$$V_0 = \boldsymbol{E}[V_1 | r1_1, r2_1]$$

$$V_0 = \boldsymbol{E}[V_1 | r1_0, r2_0] + 6\% * TVaR_{0,1}$$

$$V_0 = TBE_{0,1} + 6\% * TVaR_{0,1}$$

Putting together all the different components (which were derived before) we obtain:

$$V_0 = e^{A_{0,1,10} - B1_{0,1,10} * r1_0 - B2_{0,1,10} * r2_0} * \left[e^{C_{0,1,10}} + 6\% * \left(e^{D_{0,1,10}} - e^{C_{0,1,10}}\right)\right]$$

One sees that this is equal to the Industry case. However, just like we saw in the more theoretic framework before, the differences start to occur when increasing the number of time-steps to between T and t to 2.

Let us start with looking at the value at time 2:

$$V_2 = e^{-(10-2)* Z1_{2,10} - (10-2)* Z2_{2,10}}$$

Once we are at this point the value is known.

Now consider the value at point 1:

$$V_1 = \boldsymbol{E}[V_2 | r1_2, r2_2]$$

$$V_1 = \boldsymbol{E}[V_2 | r1_1, r2_1] + 6\% * TVAR_{1,2}$$

$$V_1 = TBE_{1,2} + 6\% * TVAR_{1,2}$$

We see that t=1 and T=2 for these calculations. Filling in for the Time Best Estimate and the Time-Consistent Value at Risk we derived earlier we obtain:

$$V_1 = e^{A_{1,2,10} - B1_{1,2,10} * r1_1 - B2_{1,2,10} * r2_1} * \left[e^{C_{1,2,10}} + 6\% * \left(e^{D_{1,2,10}} - e^{C_{1,2,10}}\right)\right]$$

Now consider the value at point 0. To ensure time consistency we use the values obtained above instead of the expectation of V_2. Hence we use the “updated” information and take the expectation over V_1. In doing so, note that as we shift one period back we now work with t=0 and T=1:

$$V_0 = \boldsymbol{E}[V_1 | r1_1, r2_1]$$

$$V_0 = \boldsymbol{E}[V_1 | r1_0, r2_0] + 6\% * TVAR_{0,1}$$

$$V_0 = TBE_{0,1} + 6\% * TVAR_{0,1}$$

Now let us derive both parts of this equation:

$$TBE_{0,1} = e^{A_{0,2,10} - B1_{0,2,10} * r1_0 - B2_{1,2,10} * r2_0} * e^{C_{0,1,10}} * \left[e^{C_{1,2,10}} + 6\% * \left(e^{D_{1,2,10}} - e^{C_{1,2,10}}\right)\right]$$

Note that the standard deviation changed through the beta component and we get 2 different terms instead of 2 times the standard deviation within the Time-Consistent Best Estimate. Now for the Value at Risk:

$$TVaR_{0,1} = F_p^{-1}(V_1|W1_0, W2_0) - TBE_{0,1}$$

Where:

$$F_p^{-1}(V_1|W1_0, W2_0) = e^{A_{0,2,10} - B1_{0,2,10} * r1_0 - B2_{1,2,10} * r2_0} * \left[e^{D_{0,1,10}}\right] * \left[e^{C_{1,2,10}} + 6\% * \left(e^{D_{1,2,10}} - e^{C_{1,2,10}}\right)\right]$$

Hence we know that:

$$TVaR_{0,1} = e^{A_{0,2,10} - B1_{0,2,10} * r1_0 - B2_{1,2,10} * r2_0} * \left[e^{D_{0,1,10}} - e^{C_{0,1,10}}\right] * \left[e^{C_{1,2,10}} + 6\% * \left(e^{D_{1,2,10}} - e^{C_{1,2,10}}\right)\right]$$

Therefore we can write the total insurance value at time 0 as:

$$V_0 = e^{A_{0,2,10} - B1_{0,2,10} * r1_0 - B2_{1,2,10} * r2_0} * \left\{e^{C_{0,1,10}} * \left[e^{C_{1,2,10}} + 6\% * \left(e^{D_{1,2,10}} - e^{C_{1,2,10}}\right)\right] + 6\% * * \left[e^{D_{0,1,10}} - e^{C_{0,1,10}}\right] * \left[e^{C_{1,2,10}} + 6\% * \left(e^{D_{1,2,10}} - e^{C_{1,2,10}}\right)\right]\right\}$$

Note again that this structure is similar to what we saw in the simple insurance case: a normally distributed variable multiplied with a deterministic term. In contrast to the simple insurance case this term is not a constant, but it changes through time.

The formula above can be rewritten as:

$$V_0 = e^{A_{0,2,10} - B1_{0,2,10} * r1_0 - B2_{1,2,10} * r2_0} * \left[e^{C_{0,1,10}} + 6\% * \left(e^{D_{0,1,10}} - e^{C_{0,1,10}}\right)\right] * \left[e^{C_{1,2,10}} + 6\% * \left(e^{D_{1,2,10}} - e^{C_{1,2,10}}\right)\right]$$

For the more general case of a T=n-year contract it can be shown (again for instance with Mathematical Induction) that:

$$V_0 = e^{A_{0,T,S} - B1_{0,T,S} * r1_0 - B2_{0,T,S} * r2_0} * \prod_{k=1}^{k=n} \left[e^{C_{k-1,k,S}} + 6\% * \left(e^{D_{k-1,k,S}} - e^{C_{k-1,k,S}}\right)\right]$$

Now we obtained a more realistic model to draw results from let us look at some tests to determine how big the difference between both methods are.

Results II: Industry VS. Time-Consistent using Vasicek

In this section this paper will attempt to get a more realistic idea of the differences between the Time-Consistent and the Industry method. Hence it is of vital importance to choose the right input parameters for our model. Therefore this will be the starting-point of the section.

Parameterization

Choosing the right parameters is vital to create a clear view of the magnitude of the differences between both methods. For both the r1 representing interest rate as well as the r2 representing mortality this paper will fit the Vasicek model to the needed curve. Especially for the usage of the Vasicek model for mortality modelling, there are no commonly used parameters. Through this whole section we have to keep in mind that it is not about making the best fit: we should attempt to make the best fit given we use the Vasicek structure for a given maturity. Furthermore, it is more important for this proof of concept that the parameters are about the correct size, rather than that they give an accurate fit.

The Vasicek model is an Ornstein-Uhlenbeck process, which can be seen as a time-continuous variant of the time-discreet AR(1), autoregressive process[6]. We have already seen that:

$$dr_t = \gamma * (\omega - r_t)dt + \sigma dW(t)$$

This can be approximated as:

$$r_{t+1} - r_t = \gamma * (\omega - r_t)\Delta t + \sigma\sqrt{\Delta t} * \varepsilon \quad where\ \varepsilon \sim N(0,1)$$

$$r_{t+1} = \gamma\omega\Delta t + (1 - \gamma\Delta t) * r_t + \sigma\sqrt{\Delta t} * \varepsilon \quad where\ \varepsilon \sim N(0,1)$$

To find suitable parameters for the interest rates we consider the 10-year zero rates (the difference between the time-points within α and β equals 10 for each r_t), and then we know from the Vasicek model that:

$$Z_{t+1} = -\frac{1}{10} * \alpha_{10} + \frac{1}{10} * \beta_{10} * r_{t+1}$$

And:

$$Z_t = -\frac{1}{10} * \alpha_{10} + \frac{1}{10} * \beta_{10} * r_t$$

Hence we can rewrite:

$$r_t = \frac{Z_t - \frac{1}{10} * \alpha_{10}}{\frac{1}{10} * \beta_{10}}$$

[6] Greene, 1997

$$r_{t+1} = \gamma\omega\Delta t + (1-\gamma\Delta t) * \frac{Z_t - \frac{1}{10} * \alpha_{10}}{\frac{1}{10} * \beta_{10}} + \sigma\sqrt{\Delta t} * \varepsilon \qquad where\ \varepsilon \sim N(0,1)$$

So then we know that:

$$Z_{t+1} = -\frac{1}{10} * \alpha_{10} + \frac{1}{10} * \beta_{10} * \left(\gamma\omega\Delta t + (1-\gamma\Delta t)\right) * \frac{Z_{10} - \frac{1}{10} * \alpha_{10}}{\frac{1}{10} * \beta_{10}} + \sigma\sqrt{\Delta t} * \varepsilon$$

$$Z_{t+1} = -\frac{1}{10} * \alpha_{10} + \gamma\omega\Delta t + (1-\gamma\Delta t) * \left(Z_t - \frac{1}{10} * \alpha_{10}\right) + \sigma\sqrt{\Delta t} * \varepsilon$$

In case we take delta t equal to 1 we obtain:

$$Z_{t+1} = -\frac{1}{10} * \alpha_{10} + \gamma\omega + (1-\gamma) * \left(Z_t - \frac{1}{10} * \alpha_{10}\right) + \varepsilon \quad where\ \varepsilon \sim N(0,\sigma^2)$$

Which can be rewritten as:

$$Z_{t+1} = -\frac{1}{10} * \alpha_{10} + \gamma\omega - \frac{(1-\gamma)}{10} * \alpha_{10} + (1-\gamma) * Z_t + \varepsilon \ \ where\ \varepsilon \sim N(0,\sigma^2)$$

This representation is exactly the same as an AR(1) model:

$$X_{t+1} = a + b * X_t + \varepsilon \ \ where\ \varepsilon \sim N(0,\sigma^2)$$

Hence we can fit an AR(1) model in order to obtain parameters for Vasicek. This paper will do this using an OLS-regression, with the knowledge that this is not perfect. However, OLS will give unbiased estimators, and though finite sample properties are not great (and we will use a rather small sample), for the purpose of obtaining parameters in the right magnitude OLS will suffice.

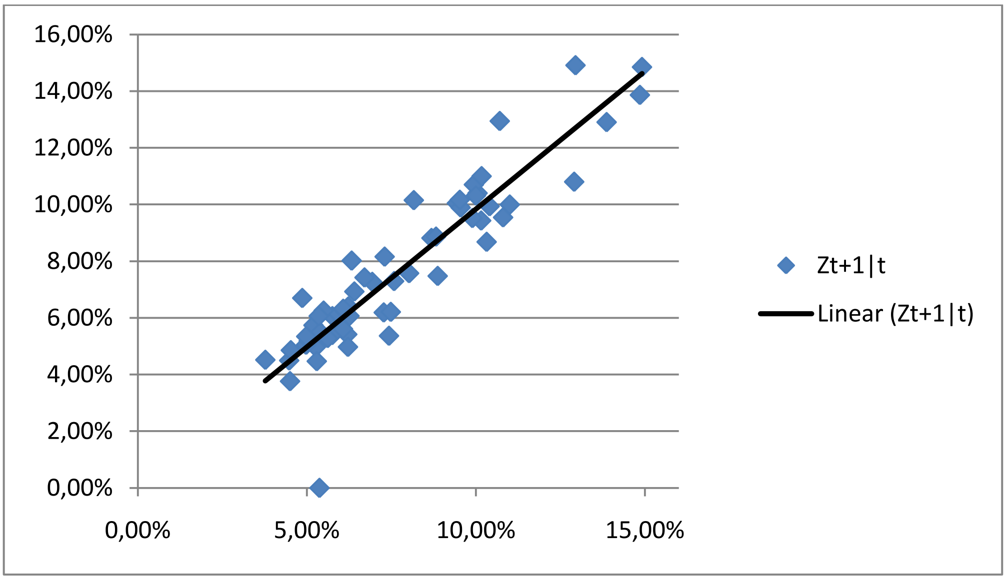
We fit the AR(1) model to 10-year Government-bond rates obtained from EuroStat. It is the average rate of the developed countries between 1949 and 2010. In doing so we obtain the following parameters for the AR(1) representation:

$$a = 0{,}007$$

$$b = 0{,}915$$

$$\sigma = 0{,}026$$

When we plot the data against a linear trendline we obtain the following picture:



Clearly the fit is not perfect: however as mentioned before, for our proof of concept this is acceptable.

From the AR(1) model it is possible to derive parameters for our Vasicek model. For the sigma terms and for the b and $\gamma 1$ it is clear that they are one to one. Note however, that a is linear and one to one as well with $\omega 1$ given that the other two parameters are known. Hence we have a (linear) system of 3 equation with 3 unknowns. Solving this system leads to:

$\omega 1 = 4{,}84\%$ and has the interpretation of the long − run interest rate

$\gamma 1 = 0{,}085$ which is the rate of reversion

$\sigma 1 = 0{,}026$ which is the volatility of the interest rate

In a similar way we can show that the 10-year mortality probabilities of a 40-year old person follow the following process:

$$Z_{t+1} = -\frac{1}{10} * \alpha_{10} + \gamma\omega + -\frac{(1-\gamma)}{10} * \alpha_{10} + (1-\gamma) * Z_t + \varepsilon \;\; where \;\; \varepsilon \sim N(0, \sigma^2)$$

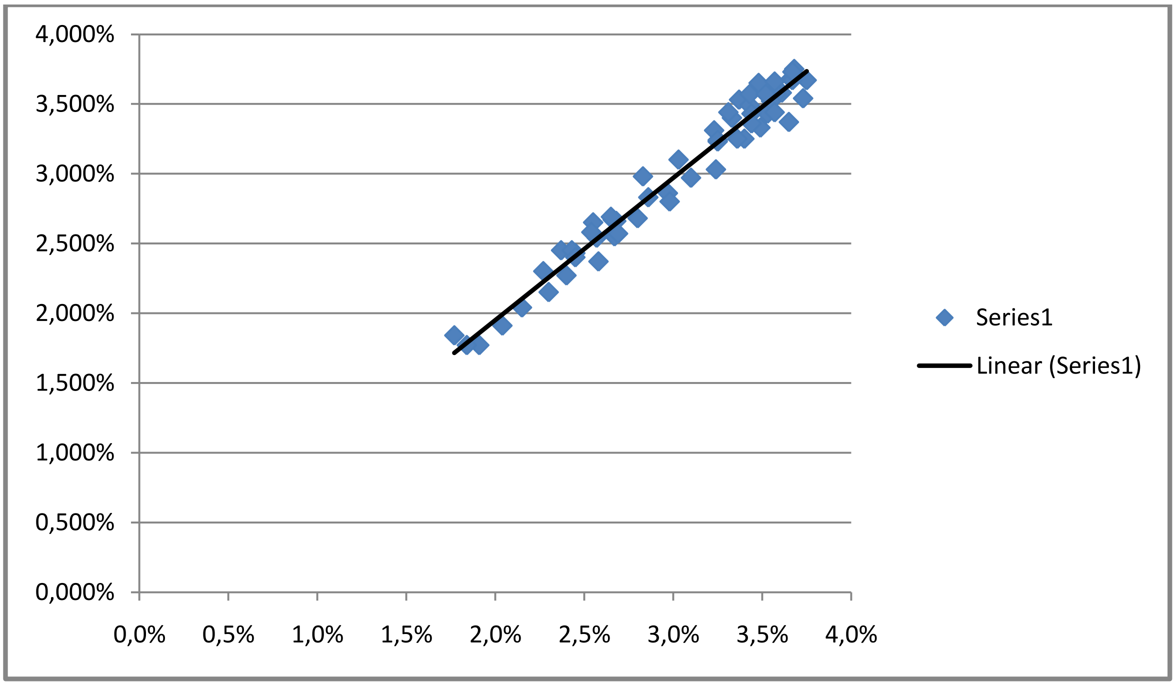
Now we fit an AR(1) model to the 10-year mortality probability of a 40-year old person. The data was taken from the CBS (Dutch bureau of Statistics) and runs from 1950 till 2009). We obtain the following parameters for the AR(1):

$a = 0{,}000$

$b = 0{,}994$

$\sigma = 0{,}0034$

When we make a similar scatter plot as before we get the following:



Using the fact that the Vasicek model and the AR(1) model are mapped one to one, we can derive the following parameters for our Vasicek mortality model as:

$$\omega 2 = 0{,}1355$$

$$\gamma 2 = 0{,}0061$$

$$\sigma 2 = 0{,}0034$$

Note that the volatility of the mortality risk is actually a factor 10 lower than the volatility of interest rate risk. This makes sense, because for the mortality factor, trend risk is a much bigger factor than the volatility within mortality. This, however is not the focus of this paper and using the parameters above we can, by using the formulas derived in the previous section derive results for the insurance value.

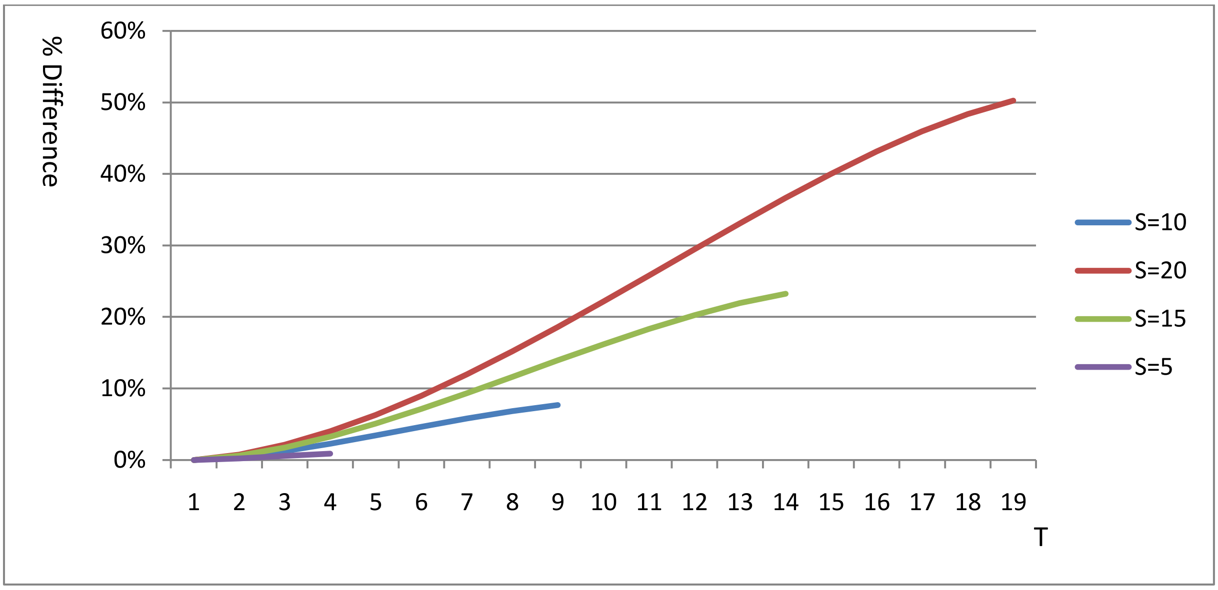
Results

The table below shows us the differences between the Industry valuation and the Time-Consistent valuation. We use an S of 10, and the table lists values for different T-points. The initial values of r were set at zero. The other parameters where used as described above.

T	BE	Industry Value	Time-Consistent Value	% Difference
1	1,994	2,012	2,012	**0,00%**
2	2,282	2,319	2,329	**0,46%**
3	2,614	2,671	2,705	**1,26%**
4	3,002	3,079	3,149	**2,28%**
5	3,456	3,552	3,675	**3,44%**
6	3,991	4,107	4,297	**4,64%**
7	4,626	4,758	5,034	**5,80%**
8	5,384	5,529	5,907	**6,83%**
9	6,292	6,446	6,939	**7,66%**

We see that, as the time T for which we value the contract increases, so does the difference between the Industry and the Time-Consistent method.

More important though is the size of the differences. For the one year contract we knew it would equal zero, but for some longer term contracts the Time-Consistent method values the insurance around 7% higher, which is definitely a significant amount. The graph below shows the percentages of difference between both methods for different values of S and of T.



We see that for every S there is zero difference when T is one (as we already saw from theory). Furthermore, there is a maximum difference for a T that is somewhere between S and time 0. Note that the S of 10 years contains the most reliable results, as we based our parameters on 10 year bond yields. Furthermore, we again see that the Time-Consistent method yields higher insurance values than the Industry method.

This is as far as this paper will go, investigating differences between Time-Consistent pricing and Industry based pricing. First, however, there will be a small conclusion as well as some ideas for further research into this matter.

One Step Beyond...

Using the examples above we saw that the Time-Consistent method of pricing an insurance definitely has potential. Even when testing with realistic parameters, the differences with the Industrial method are significant. Beyond this proof of concept though, there is a lot of room for further research into this area.

It would for instance be interesting to see how the differences behave should we choose another (more accurate) risk measure than Value at Risk, for instance Expected Shortfall. Another possibility would be to replace the Vasicek model with a model that cannot yield negative interest rates, say Cox Ingersoll and Ross. One should be careful though, that working out these models analytically, as was done for this research, may not longer be possible. In that case numerical test and programmed simulations come into play.

Even though it remains to be seen whether the Industry actually adopts the Time-Consistent method, this paper has shown that research into Time-Consistent may proof very worthwhile indeed. Therefore I will end the work on this paper, hoping it will be useful for others interested in the field of Time-Consistent valuation.

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