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Valuation Analysis of Funded Notional Defined Contributions Pensions Exposed to Stochastic Demographics

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Abstract

We carry out a market consistent valuation of funded Notional Defined Contributions (NDC) pension plans implemented on a national scale or a scale wide enough to make it sensitive to stochastic demographics. We report the *fair* values of expected returns on participating in such programs and compare it with the benchmark returns on individually saving and investing in the financial markets. Our results show that NDC benefits have higher short run exposure to market risks; however, the very high growth rate of NDC benefits renders its long run exposure rate, making it a safer investment with a lower expected return. We repeat the analysis for different scenarios with several values of fertility parameters and fund conditions. In each case, we characterise the exposure to risk and expected returns of the system. Our framework features an affine term structure (static and dynamic), and we price three risks (real rate, inflation and equity) out of eight different stochastic shocks driving the model.

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1 Introduction

The recent years have witnessed major changes in the policies governing public and private pension programs. Pay-as-you-go (PAYG) public schemes are no more sustainable on the long run due to critical long-term fiscal challenges spurring from population ageing. On the other hand, second and third pillars of the pension system which rely more on individuals' accounts and funded assets have been shown to carry significant solvency risk, especially in times of economic distress. Uncertainty stemming from these two types of risk imposes pressure on regulators for pension reform, in order to create long term sustainable and transparent systems.

One of the alternative that emerged in the mid-1990s to provide partial solutions to the above concerns, and has already caught a lot of attention since then is the notional defined contribution (NDC) system, which has been introduced in six countries: Sweden (1994), Italy (1995), Latvia (1996), the Kyrgyz Republic (1997), Poland (1999), and Mongolia (2000). It features individual accounts, but they grow on a pay-as-you-go basis, see Williamson and Williams (2003). We concern ourselves with a special type of NDC plans, namely, funded NDC. It falls at the intersection of NDC and funded systems, by counting on PAYG structure along with investment in the financial market. For political reasons, in order to ensure a smooth transition, the non-funded/ funded NDC model is being introduced as packages that compliment present social coverage plans. This delegates the decision making to the prospective participant; she can choose from the different options available based on her preferences and the prices of retirement plans traded in the market.

However, since funded NDC plans are not risk free assets, and their model follows non-linear processes, determining the fair market price becomes complicated. Moreover, the changes, imposed by the new framework of Solvency II in how insurance and pension firms are supposed to valuate and manage risks, make valuation even more technically demanding. Traditional actuarial approach is being replaced by fair valuation framework that links pension funds stability with the state of financial markets. Since a pension shares the same features of a long term financial asset, despite its more complicated exposure to risk, it should be valued as such. Therefore, the main goal of any valuation approach, to meet Solvency II requirements, is to reach an objective analysis of the solvency state of a pension fund and the risk involved in its ability to meet promised retirement benefits.

This paper is motivated by the lack of literature on market consistent valuation of funded NDC pension programs, that are implemented on a national scale or a scale wide enough to expose them to changes in population demographics. We address this concern by simulating the non-linear NDC model evolution, whereas an analytical solution is not attainable. We start by determining the market present time value (hence,expected returns) of the retirement package within a *No Arbitrage* framework for several scenarios of population projections and present time fund balance. Then, we characterise the individual participant exposure to risk and the role played by population parameters and fund balance. We, eventually, use the results to draw a conclusion on the economic

value of subscribing in a funded NDC plan from the perspective of a young worker. Throughout we compare funded NDC plans with the alternative of holding an individual savings account which invests in risky or fixed income assets. Still, one of the important points raised by Kortleve et al. (2006) is that the prices we report here only constitute the upper bound on the market value. From the point of view of a beneficiary, she expects an additional premium for the uncertainty associated with lack of transparency about the fund conditions.

The paper is structured as follows: Section 2 and 3 introduce the conceptual and technical frameworks, and section 4 details our model implementation. Section 5 analyses the model main results besides valuation which is presented in section 6. Section 7 concludes. A reader who may only be interested in the findings, could skip directly to sections 6 and 7.

2   Conceptual Framework

2.1   Notional Defined Contributions

A notional account is a virtual account in which individual contributions are deposited and virtual returns are added to the account until the contributor retires. Returns are calculated according to a notional rate; this rate takes into account factors like GDP growth, average earnings, income, and the stability prospect of the fund. Upon retirement, participants receive a pension based on their individual accumulated notional accounts. The annuity of the pension paid depends on the specific mortality rate and life expectancy of every cohort in order to ensure stability against longevity risks,Vidal-Meliá et al. (2006).

This type of systems makes an appealing option to policy makers mainly for three reasons. First, and formost, it links benefits with contributions which is a key goal for many reformers who are concerned about labor force incentives. Second, since it is financed on a mixed basis of PAYG and investment returns, it does not involve transition costs; hence, current generation of workers does not have to pay twice: for current retirees and their own retirements. Finally, it could be implemented in a way that compliments an existing structure, which means that it has a lower political cost during a transition period.

Nonetheless, implementation of the NDC model has faced strong criticism, which revolves around two major points. First, NDC plans are designed to tie benefits to contributions like all defined contribution schemes, and therefore they are generally less redistributive compared with defined benefit plans. Less redistribution inflicts a larger degree of inequality among retirees. The second point is about transparency of NDC systems; the model is more transparent, like the funded account model, from the point that, at any instance, beneficiaries can inquire about their exact account balance; this is not possible with the defined benefit alternatives. Although how much is credited to an individual account is indeed tractable, the benefit expected upon retirement is, however, less certain compared with a defined benefit system. This imposes a sort of political pressure, especially

under regimes that are more concerned with a wider social coverage, and more inclusive insurance net for all workers.

2.2 Fair Value/Market Consistent Valuation

From the perspective of a participant, valuation answers one main question: what is the economic value of a certain financial contract? So as to say, what is the value of the contract in the same currency used to decide on every day consumption and saving decisions? It is in this sense that market consistent valuation leads to optimal investment allocations, and therefore, higher welfare gains. Market-consistent valuation requires that the valuation of any liability is to be determined by the price for which it would be traded in a market transaction between two willing and knowledgeable counter-parties. This price reflects the fair value of the liability, and encodes both growth and risk potentials. Technically, a mark-to-market approach should be used for assets and liabilities whenever the prices can be observed in a liquid market. Otherwise, a mark-to-model approach is used, where the valuations models incorporate all relevant market information, see Pelsser and Vlaar (2008), and Kortleve et al. (2006).

Mark to Model Or Mark to Market Pricing?

Traditional actuarial and accounting uses terms like mark to model or mark to market to define the approach to pricing an asset; however, in modern asset pricing theory, they complement each other. Theoretically, pricing an asset builds on one simple axiom, namely, price equals discounted payoff. Two popular approaches are derived from this basic idea; Cochrane (2005) calls them *absolute* and *relative* pricing. In *absolute* pricing, assets are priced based on their exposure to fundamental sources of macroeconomic risk, such as consumption risk. On the other hand, for *relative* pricing, an asset value is determined given the prices of some other assets, where market indices and yield returns are common examples. The prices of the assets we condition on are exogenously fed to the model without justifying how they have been first determined.

Relative pricing, which we implement here, represents a blend of the traditional mark to model and mark to market approaches. Although pension payoffs are not liquid in the market and can only be expected based on their model, present valuation can be deduced based on liquid traded assets. The theory behind this can be summarised in the following two equations,

$$P_t = \mathbb{E}_t(M_{t+1}X_{t+1})$$
$$M_{t+1} = \mathbb{F}(liquid\ factors, parameters)$$

where M is the stochastic discount factor, and X is the pay off. So, price depends on the expected growth in X and M, and the covariance between the two which determines the exposure to what is called 'factor risk'. Higher exposure leads to higher discounting, or, in the language of

expected returns, higher risk premium. The important caveat here is that only factor risks earn a premium; any source of volatility that is not represented by a liquid traded asset is not priced. This is of direct importance to our current application where demographic risk, for example, is not priced since there is no liquid asset than can capture it.

The one principle behind the implementation of this approach is the Law of One Price, which follows from the assumption of efficient markets/law of No Arbitrage. This assumption, which means that we expect the same thing to sell for the same price, is deduced from two basic premises, namely, participants are rational and risk averse. No Arbitrage/Law of one price are enough conditions to ensure the existence of a positive discount factor, which is the cornerstone of our valuation framework.

3   Valuation Framework

Market consistent valuation requires contingent pricing of assets of uncertain future values which are exposed to market risks. Within an affine term structure framework, the stochastic discount factor is endogenously constructed from the assets traded in a complete market of short and long term bonds and risky stocks. We calibrate the model with stylised values in the literature; therefore, we pre-specify either risk premia or prices of risk in order to identify all our model parameters. This pricing framework (stochastic discount factor, risk prices, bond prices) is motivated by the work of Cox et al. (1985), Duffie and Kan (1996), Dai and Singleton (2000), Dai and Singleton (2002), and Duffee (2002), which we briefly discuss in what follows.

3.1   Affine Term Structure Models

3.1.1   State Factors

Affine term structure models make assumptions on the functional form of the yield curve modelling, which makes tractable pricing formulas possible. The assumptions are on the functional forms of the state vector, where the model is affine if the state variables follow an affine time homogeneous Markov process, and the short-rate is affine, see Piazzesi (2010) for overview of recent literature.

Most affine models are versions of the following basic early models:

1. Vasicek-type models Vasicek (1977): state factors are Gaussian.

2. Cox-Ingersoll-Ross (CIR)-type models Cox et al. (1985): state vector consists of independent (uncorrelated) square-root processes.

3. Mixture models Duffie and Kan (1996): x consists of possibly correlated affine processes.

Vasicek-type models are characterised by a constant volatility, unlike volatilities of CIR-type models which depend on the square root of the state variable. Mixture models are multi-factor models that are built from the two basic building blocks where volatility may depend on a number of state variables. In affine models, state factors X_t follow a first-order autoregressive $AR(1)$ process which corresponds to an affine diffusion in continuous time,

$$X_{t+1} = (1 - \phi_x)\mu_x + \phi_x X_t + \Sigma_{x,t}\epsilon_{x,t+1} \tag{1}$$

where,

$$\Sigma_{x,t} = \Sigma \qquad (Constant/Homoscedastic/Gaussian) \tag{2}$$

or,

$$\Sigma_{x,t} = \Sigma\sqrt{S(t)} \qquad \text{with } [S(t)]_{ii} = \alpha_i + \beta_i' X_t \qquad (State Dependent/Hetroscedastic) \tag{3}$$

This $AR(1)$ features mean reversion where X is pulled towards its unconditional mean μ_x with a speed controlled by ϕ_x. The horizon over which a factor contributes to bonds prices is determined by its persistence level. Factors with the highest persistence contrubute the most to bonds with long term maturity. ϵ_x disturbs X from reverting to its mean. This shocks is normally distributed with volatility determined by Σ_x. X in (2) is allowed to take negative values with positive probability. On the other hand, CIR-type model in (3) with a square root process introduce conditional heteroskedasticity by allowing the volatility to depend on the state of X. However, not all state variables have to contribute to volatility and additional conditions are needed to ensure that volatility stays positive. In case of a single factor, the Feller condition $\phi_x\mu_x > \frac{1}{2}\sigma_x^2$ makes sure that zero is never reached. This, intuitively, ensures that the ϕ_x is strong enough to pull X away from the zero boundary, and rules out unit roots. Duffie and Kan (1996) provide generalisation for the Feller condition in multi factor cases, and Dai and Singleton (2000) specify necessary conditions for all possible multi-factor models sub-families $A_m(X_n)$ where m is the number of factors contributing to the state volatilities, and n is the total number of factors.

3.1.2 Bond Prices

In an affine term structure model, the next step after assuming factors that are affine in mean and volatility is to let the short rate r_t be affine in the factors,

$$r_t = \delta_0 + \boldsymbol{\delta_1' X_t} \tag{4}$$

where δ_0 is a constant and δ_1 is N-vector of constants, where N is the number of state factors. Due

to the direct link between the pricing kernel and the short rate (one period yield), SDF becomes affine in the state factors as well. It is this feature in addition to the multiplicative recursive nature of SDF pricing that makes a closed form analytical solution to long term bonds possible,

$$-p_t^{(1)} = y_t^{(1)} = -logE_t(M_{t+1}) = E_t(-m_{t+1}) - \frac{1}{2}Var_t(m_{t+1}) \tag{5}$$

$$P_t^{(n)} = E_t(M_{t+1}P_{t+1}^{(n-1)}) \tag{6}$$

using induction,

$$-p_t^{(n)} = ny_t^{(n)} = A_n + \boldsymbol{B_n X_t} \tag{7}$$

$$\tag{8}$$

where A_n and $\boldsymbol{B_n}$ are functions of bond maturity n but not of time t and $\boldsymbol{B_n}$ is $1 \times N$-vector. Solving a set of Ordinary Differential Equations gives the solution to the exact form of the coefficients, where the level of complexity of the solution depends of course on the number of factors involved.

3.1.3 Risk Prices

The model is completed by specifying risk prices. Depending on the form of this specification and how it relates the state factors, affine models can be categorised in sub-classes : (Completely vs Essentially Affine Models), and (Static vs Dynamic Term Structure Models).

Generally speaking, excess returns on bond prices of different maturities are determined by both their exposure to risk and price of risk; in the case of a an affine term structure model, this takes the form,

$$r_e^{(n)} = -\boldsymbol{B_{(n)}\Sigma}\sqrt{\boldsymbol{S_t}}\boldsymbol{\Lambda_t} \tag{9}$$

$\boldsymbol{B_{(n)}}$ encodes information of factors contributions to long run exposure to risk. $\boldsymbol{\phi_x}$ enters the equation for $\boldsymbol{B_{(n)}}$, which means that persistence of factors and cross correlation between them play a role in the compensation for risk at multiple time horizons.

Completely Affine Models
Let λ_1 be n-vector, then the price of risk vector $\boldsymbol{\Lambda_t}$ is given by,

$$\boldsymbol{\Lambda_t} = \boldsymbol{\lambda_1'}\sqrt{\boldsymbol{S_t}} \tag{10}$$

where $\boldsymbol{\lambda_1}$ is an n-vector of constants. Vasicek (1977), and Cox et al. (1985) models are nested in this specification. In case of the former model where factor volatilities are constants, risk prices are constant, $\boldsymbol{\Lambda_t = \Lambda}$. This leads to the class of static term structure models where excess returns are constant over time. Although this class provides poor estimated fit to actual term structure dynamics, we use it as a benchmark specification for our valuation exercise. This helps us highlight the role of other dynamics in the pension model.

On the other hand, applying this risk price specification to a mixture model $A_m(X_n)$ of the type in Dai and Singleton (2000) leads to a dynamic term structure model with excess returns solely dependent on factors volatilities. The main reason why this model is popular is that the vector $\sqrt{S_t}\Lambda_t$ becomes affine in X_t; this leads to closed form solutions to different properties of the yield curve, and hence the name "completely affine". We do not implement this model in the current paper.

Essentially Affine Models

Duffee (2002), with the purpose of introducing higher model flexibility to capture term structure dynamics, characterises risk prices as,

$$\boldsymbol{\Lambda_t = \lambda_1'}\sqrt{\boldsymbol{S_t}} + \boldsymbol{\lambda_2'}\sqrt{\boldsymbol{S_t^-}}\boldsymbol{X_t} \tag{11}$$

where where $\boldsymbol{\lambda_2}$ is an $n \times n$-matrix of constants, and $\boldsymbol{S_t^-}$ has zeros in its first m diagonal elements and $(\boldsymbol{\alpha_i + \beta_{ii} X_t})^{-1}$ in the remaining $n - m$ elements. $\boldsymbol{\Lambda_t'\Lambda_t}$ is not affine in X_t, yet, this does not affect bonds prices; hence, the name "essentially affine". The main added advantage in this model is that the tight connection between risk prices and factors volatilities in the previous model is broken; prices are allowed to evolve independently as a function of the state factors. Nonetheless, this model still imposes some limitations on term structure dynamics; the number of factors that contribute to risk prices is limited by the number of factors in state dependent volatilities. Therefore, the higher flexibility gained by mixture models to capture the historical conditional moments of the yield curve limits the ability of the model to forecast expected excess returns for long term bonds. Dai and Singleton (2002) provide extensive discussion of this topic. We implement a three factors variant of this type of models and compare valuation results with the results obtained with the static version explained above.

3.2 Implications For Long Run Valuation

Characterisation of the long term implications of a certain model on present time valuation of stochastic future cash flows is an essential tool to understand the impact of perturbation of certain parameters on current valuation of long run risk, and how exposure to risk changes over time. In order to see how different assumptions about the long run are reflected in expected returns, we

decompose expected returns into sum of risk-free component and a long-term risk premium. This premium is further broken into a product of a measure of long run exposure to risk and the price of long-run risk. The result is a structure equivalent to the risk-return trade off for one period returns (e.g. $r_e^{(1)} = -\beta\lambda$, where λ is the risk price of a certain risk factor, and β reflects the exposure of asset returns to this risk). We are unable to implement that last step due to model non-linearity. Nonetheless, in what follows, we give a brief presentation of the work of Hansen et al. (2008), Hansen and Scheinkman (2009), Borovička et al. (2011), Lettau and Wachter (2011), Hansen (2012), and Borovicka and Hansen (2013) in order to develop an intuition about the model outcomes below.

As in Hansen et al. (2008), Hansen and Scheinkman (2009), assume a cash flows process D_t that grows stochastically over time and is linked to factors that evolve according to stationary Markov processes x_t,

$$D_t = D_t^* f(x_t) \tag{12}$$

$$D_t^* = exp(\zeta t + \sum_{j=1}^{t} \mathbf{\Gamma_j \epsilon_j}) \tag{13}$$

with the state factor evolution,

$$\mathbf{x_{t+1} = (1 - \phi_x)\bar{x} + \phi_x x_t + \Sigma_x \epsilon_{t+1}} \tag{14}$$

and the process for log SDF,

$$m_{t+1} = \mu_m + \mathbf{\phi_m x_t + \Lambda\epsilon_{t+1}} \tag{15}$$

where D_t^* is modelled as a the exponential of a random walk to capture the growth part of the process, and $f(x_t)$ represents local/transient state dependency. $f(x_t)$ can be linear as in the state vector, see Hansen et al. (2008), or non-linear as in Hansen and Scheinkman (2009). Multiplicative decomposition of time series is well documented in the literature and has many applications in analysing non-stationary series; see Hansen and Scheinkman (2009) for necessary conditions for existence of this decomposition. Λ represents exposure to long run risk in a growing cash flow, and Λ are the one period risk prices. A key point to realise is that eventually the price of cash flows decline as the horizon of valuation increases; this is mainly a result of the multiplicative and recursive nature of the pricing factor. The rate of decay in present valuation depends on the expected growth in the cash process relative to the discount rate. This gives birth to the concepts of long term growth and decay rates; the former reflects the rate by which a future payoff increases as the time horizon increases, while the latter is the rate by which the compounded pricing kernel decays as the time horizon of the payoff increases.

In a linear log normal case, where $f(x) = exp(\omega x + \kappa)$, Hansen et al. (2008) give closed form formulas for the asymptotic rate of decay v and the asymptotic rate of growth η of the valuation

process, reflecting the two competing forces of long run risk value,

$$-v = \mu_m + \zeta + \frac{|\bar{\omega}\Sigma_x + \Lambda + \Gamma|^2}{2} \tag{16}$$

$$\eta = \zeta + \frac{1}{2}\Gamma'\Gamma \tag{17}$$

$$\bar{\omega} = \phi_m(1 - \phi_x)^{-1} \tag{18}$$

Hence, asymptotic rate of return is given by,

$$\eta + v = \kappa + \Lambda^*\Gamma \tag{19}$$

where,

$$\Lambda^* \equiv -\Lambda - \phi_m(1 - \phi_x)^{-1}\Sigma_x \tag{20}$$

$$\kappa \equiv -\mu_m - \frac{\Lambda'^*\Lambda^*}{2} \tag{21}$$

Λ^* is the price of long run risk, and Γ is the exposure to this risk. Equation (18) is the long run risk-return trade off counterpart of the familiar one period term $(\lambda \times \beta)$. κ reflects the long run return on assets not exposed to long run risk. Equation(19) shows that long run risk price depends on persistence in the pricing kernel relative to persistence in factors. Highly persistent factors magnify the effect of exposure to state factors risk and leads to higher discounting of future cash flows. Although the analytical results above are for the special case of log linear state dependency $f(x_t)$, the same intuition holds for the non-linear cases which is investigated in the later work of Hansen and Scheinkman (2009) and Hansen (2012); they extract the asymptotic rates by solving for the principal eigen-values driving the cash flows or discounting process.

4   Model of NDC with Stochastic Demographics

4.1   Static Two-Factor Model

Following Campbell et al. (2001), we use an explicit reduced form model of a nominal affine term structure. This offers the simplest way to understand the system in both real and nominal terms. The model is set in discrete time; it has a state vector X of two variables: real short rate, and expected inflation. Variables are log normally distributed with constant variances. The model dynamics are driven by 8 uncorrelated shocks to (real rate, expected inflation, inflation, equity return, economic growth, household income, fertility, and mortality). The market prices three risks (real rate, inflation, and equity returns).

State vector X takes the form,

$$X_t = \begin{pmatrix} r_t^R \\ \mu_{\pi,t} \end{pmatrix}$$

$$X_{t+1} = (1 - \phi_x)\bar{X} + \phi_x X_t + \Sigma_x \epsilon_{x,t+1}$$

where,

$$\bar{X} = \begin{pmatrix} \bar{r} \\ \bar{\mu}_\pi \end{pmatrix}, \quad \phi_x = \begin{pmatrix} \phi_r & 0 \\ 0 & \phi_{\mu_\pi} \end{pmatrix}, \quad \Sigma_x = \begin{pmatrix} \sigma_r & 0 \\ \beta_{\mu_\pi,r}\sigma_r & \sigma_{\mu_\pi} \end{pmatrix}, \quad \epsilon_{x,t+1} = \begin{pmatrix} \epsilon_{r,t+1} \\ \epsilon_{\mu_\pi,t+1} \end{pmatrix}$$

$\bar{X}$ is the unconditional mean and ϕ_x is the persistence of factors. Σ_x is the exposure of factors to factor risks.

Real short rate:

Real short rate is determined at the *end* of the previous period and constitutes the first element of the state vector X_t. Risk free returns at $t+1 \equiv r_t^R$. It is important to note that the companion form ϕ_x does not include dependency of the short rate on inflation; this ensures that money neutrality holds which is merely a simplification; empirical work shows evidence in support for the role played by inflation in deciding real activity and return on capital in the economy.

$$r_t^R = \delta X_t \qquad \text{where, } \delta = (1,0) \tag{22}$$

Expected Inflation:

Expected next period inflation is determined at the *end* of the previous period and constitutes the second element of the state vector. $E_t(\pi_{t+1}) \equiv \mu_{\pi,t}$.

$$\mu_{\pi,t} = \zeta X_t \qquad \text{where, } \zeta = (0,1) \tag{23}$$

Realised Inflation:

The unexpected part of realised inflation is driven by correlation with the real rate and expected inflation in addition to an uncorrelated shock.

$$\pi_{t+1} = \mu_{\pi,t} + v_{\pi,t+1} \tag{24}$$

$$v_{\pi,t+1} = \boldsymbol{\sigma}'_{\boldsymbol{\pi,x}}\boldsymbol{\epsilon_{x,t+1}} + \sigma_\pi \epsilon_{\pi,t+1} \tag{25}$$

$$\boldsymbol{\sigma}'_{\boldsymbol{\pi,x}} = (\beta_{\pi,r}\sigma_r, \beta_{\pi,\mu_\pi}\sigma_{\mu_\pi}) \tag{26}$$

Pricing Kernel:

Next, we let the pricing factor be affine in the state vector. We require M to price the three financial contracts traded in the market (nominal one-period bond, nominal 10-periods bond, and market-index stock) where we take risk premia to be constant and already specified. As mentioned earlier, we follow Ang et al. (2008) in pricing real rate, and inflation risk, and we add market risk (equity shocks). This leads to the following form for M,

$$M^R_{t+1} = \exp(-\boldsymbol{\delta X_t} - \frac{1}{2}\boldsymbol{\Lambda}'\boldsymbol{\Lambda} - \boldsymbol{\Lambda}'\boldsymbol{\epsilon_{\Lambda,t+1}}) \tag{27}$$

$$M^\$_{t+1} = \frac{M^R_{t+1}}{\exp(\pi_{t+1})} = \exp(-\boldsymbol{\delta X_t} - \frac{1}{2}\boldsymbol{\Lambda}'\boldsymbol{\Lambda} - \boldsymbol{\Lambda}'\boldsymbol{\epsilon_{\Lambda,t+1}} - \pi_{t+1}) \tag{28}$$

$$\boldsymbol{\epsilon}'_{\boldsymbol{\Lambda,t+1}} = (\epsilon_{r,t+1}, \epsilon_{\pi,t+1}, \epsilon_{eq,t+1}) \tag{29}$$

M is log normally distributed with a state dependent mean, and constant risk prices λ, which is a feature of Vasicek-type models. The nominal pricing factor accommodates higher discounting in its mean for inflation risk. This model produces closed-form solutions for the real and nominal term structure.

Risk Prices:

As explained above, we close the model by specifying a form for the risk prices. For the current model, we assume a completely affine structure which, combined with constant factors volatilities, yields a static model with constant expected excess returns on long term bonds.

$$r_e^{(n)} = -\boldsymbol{B_{(n)}}\boldsymbol{\Sigma_x}\boldsymbol{\Lambda} \tag{30}$$

We exogenously specify expected excess returns on two yields of different maturities, and solve for the risk prices of real returns and inflation.

Yield on One Period Nominal Bond:

$$y^{\$}_{(1),t} = A_1 + \boldsymbol{B_1 X_t} \tag{31}$$

$$A_1 = -\Lambda_\pi \sigma_\pi - \Lambda_r \beta_{\pi,r} \sigma_r - \frac{1}{2}(\sigma_\pi^2 + \boldsymbol{\sigma'_{\pi,x} \sigma_{\pi,x}}) \tag{32}$$

$$B_1 = \boldsymbol{\delta} + \boldsymbol{\zeta} = (1,1) \tag{33}$$

$$r^e_{(1)} = -\Lambda_\pi \sigma_\pi - \Lambda_r \beta_{\pi,r} \sigma_r - \frac{1}{2}Var(\pi_{t+1}) \tag{34}$$

Risk premium earned on nominal one period bonds is due to both inflation shock and correlation with the real rate shock. It is important to notice that investors are compensated for inflation by more than just the risk premium; inflation compensation over the real short rate incorporates both the expected inflation and the risk premium.

Yield on 10-Periods Nominal Bond:

The following closed form representation is acquired using induction of the relation $P_t^{(n)} = E_t(M_{t+1}P_{t+1}^{(n-1)})$ which takes advantage of the affine prices.

$$y^{\$}_{10,t} = \frac{A_{10}}{10} + \frac{\boldsymbol{B_{10}}}{10}\boldsymbol{X_t} \tag{35}$$

$$\boldsymbol{B_{10}} = \boldsymbol{\delta} + \boldsymbol{\zeta} + \boldsymbol{B'_9 \phi_x} = (\boldsymbol{\delta} + \boldsymbol{\zeta})(1 - \boldsymbol{\phi_x})^{-1}(1 - \boldsymbol{\phi_x^{10}}) \tag{36}$$

$$A_{10} = A_9 + \boldsymbol{B_9}(1 - \boldsymbol{\phi_x})\bar{\boldsymbol{X}} - \Lambda_r \sigma_r(B_{x1,(9)} + \beta_{\pi,r} + B_{x2,(9)}\beta_{\mu_\pi,r}) - \Lambda_\pi \sigma_\pi - \Psi_{10} \tag{37}$$

$$r^e_{(10)} = -\Lambda_r \sigma_r(B_{x1,(9)} + \beta_{\pi,r} + B_{x2,(9)}\beta_{\mu_\pi,r}) - \Lambda_\pi \sigma_\pi - \Psi \tag{38}$$

$$\Psi_{10} = \frac{1}{2}\sigma_r^2(B_{x1,(9)} + \beta_{\pi,r} + B_{x2,(9)}\beta_{\mu_\pi,r})^2 + \frac{1}{2}\sigma_{\mu_\pi}^2(B_{x2,(9)} + \beta_{\pi,\mu_\pi})^2 + \frac{1}{2}\sigma_\pi^2 \tag{39}$$

Like in the case of one period returns, 10-period nominal bonds earn a risk premium due to inflation shocks and real rate shocks. Compounding over the time to maturity of the real rate shock is taken into consideration. Risk prices of real and inflation shocks follow from the specification of the risk premia. We use estimated values from Campbell and Viceira (2002).

Return on Equity:

Equities here are represented as a market index stock with shocks that are correlated with the real rate. This leads the risk premium earned on stocks to be composed of a compensation against both real rate and equity fluctuations. Returns to the stock are determined at the beginning of each period upon realisation of the current period shock.

$$r^R_{eq,t+1} = r^{eq}_m + v_{eq,t+1} \tag{40}$$

$$v_{eq,t+1} = \sigma_{eq}\epsilon_{eq,t+1} + \beta_{eq,r}\sigma_r\epsilon_{r,t+1} \tag{41}$$

Real Rate Risk Price

$$\Lambda_r = -\frac{r^e_{(10)} - r^e_{(1)} + \Psi_{10} - \frac{1}{2}Var(\pi_{t+1})}{\sigma_r(B_{x1,(9)} + B_{x2,(9)}\beta_{\mu_\pi,r})} \tag{42}$$

$$\tag{43}$$

Although assets that are positively correlated with the real rate shocks earn a positive risk premium due to Λ_r, despite the fact that r^R_t is high in bad times and low in good times. The reason is that the mean of M is determined at t, while ϵ_r occurs at $t + 1$.

Inflation Risk Price

$$\Lambda_\pi = -\frac{r^e_{(1)} + \Lambda_r \beta_{\pi,r}\sigma_r + \frac{1}{2}Var(\pi_{t+1})}{\sigma_\pi} \tag{44}$$

Equity Risk Price

$$\Lambda_{eq} = -\frac{r^{eq}_m - r^R_t + \frac{1}{2}(\sigma^2_{eq} + \beta^2_{eq,r}\sigma^2_r)}{\sigma_{eq}} + \frac{\beta_{eq,r}\sigma_r\Lambda_r}{\sigma_{eq}} \tag{45}$$

4.2   Dynamic Three-Factors Model

Next, we explore the consequences of introducing time varying volatility on risk prices and valuation of contributions and benefits over the time horizon of a pension scheme. In order to do that, we implement the multi-factors model introduced in Ang et al. (2008). They build on the conclusion of Dai and Singleton (2002) that Gaussian models with constant factors volatilities $A_0(X_N)$ and essentially affine risk prices outperform other canonical forms of mixture models $A_m(X_N)$; that is, they give a better fit for the two requirements of linear projection yields (LPY): (1) the distributional characteristics of the historical yield curve and (2) future forecasts of expected excess returns.

We assume three correlated Gaussian factors with homoscedastic variances; the persistence matrix ϕ_x captures the correlation elements between the three factors which we assume to be diagonal. The first two variables are unobservable latent factors. One factor, f_t contributes to the mean values of the term structure, where the other latent factor denoted by q_t is the one that drives the time-varying risk prices. The third state factor is the expected inflation which we use to compute the nominal term structure of bond returns. For simplicity, we divert from Ang et al. (2008) by assuming ϕ_{fq} and $\phi_{\mu_\pi,q}$ to be zero; this leads f_t to be the only factor leading r_t, and isolates q_t from contributing to expected inflation or the real rate dynamics.

Like the previous model, factors are stacked in 3×1 vector that follows an $AR(1)$ process,

$$\boldsymbol{X}_t = \begin{pmatrix} q_t \\ f_t \\ \mu_{\pi,t} \end{pmatrix}$$

$$\boldsymbol{X}_{t+1} = (\boldsymbol{1} - \boldsymbol{\phi_x})\bar{\boldsymbol{X}} + \boldsymbol{\phi_x}\boldsymbol{X_t} + \Sigma_x\boldsymbol{\epsilon_{x,t+1}}$$

where,

$$\bar{\boldsymbol{X}} = \begin{pmatrix} \bar{q} \\ \bar{f} \\ \bar{\mu}_\pi \end{pmatrix}, \ \boldsymbol{\phi_x} = \begin{pmatrix} \phi_{qq} & 0 & 0 \\ 0 & \phi_{ff} & 0 \\ 0 & 0 & \phi_{\mu_\pi,\mu_\pi} \end{pmatrix}, \ \boldsymbol{\Sigma_x} = \begin{pmatrix} \sigma_q & 0 & 0 \\ 0 & \sigma_f & 0 \\ 0 & \beta_{\mu_\pi,f}\sigma_f & \sigma_{\mu_\pi} \end{pmatrix}, \ \boldsymbol{\epsilon_{x,t+1}} = \begin{pmatrix} \epsilon_{q,t+1} \\ \epsilon_{f,t+1} \\ \epsilon_{\mu_\pi,t+1} \end{pmatrix}$$

Real short rate:

Short rate is determined at the end of the previous period; its dynamics depend only on f_t; for simplicity, by denying the role of inflation in the real rate dynamics, we assume that money neutrality holds.

$$r_t^R = \delta_0 + \boldsymbol{\delta_1}\boldsymbol{X_t} \qquad \text{where, } \boldsymbol{\delta_1} = (0,1,0) \tag{46}$$

Expected Inflation:

Expected next period inflation is determined at the *end* of the previous period.

$$\mu_{\pi,t} = \boldsymbol{\zeta_{\mu_\pi}}\boldsymbol{X_t} \qquad \text{where, } \boldsymbol{\zeta_{\mu_\pi}} = (0,0,1) \tag{47}$$

Realised Inflation and Pricing Kernel:

The unexpected part of realised inflation is driven by correlation with f_t and expected inflation in addition to an uncorrelated shock.

$$\pi_{t+1} = \mu_{\pi,t} + v_{\pi,t+1} \tag{48}$$

$$v_{\pi,t+1} = \boldsymbol{\sigma'_{\pi,x}}\boldsymbol{\epsilon_{x,t+1}} + \sigma_\pi\epsilon_{\pi,t+1}, \qquad \text{where } \boldsymbol{\sigma'_{\pi,x}} = (0, \beta_{\pi,f}\sigma_f, \beta_{\pi,\mu_\pi}\sigma_{\mu_\pi}) \tag{49}$$

Pricing Kernel:

In this model, risk in f_t replaces but still reflects real rate risk. In addition, we still also price

inflation risk, and market risk (equity shocks). This leads to the following form for M_t with time varying risk prices,

$$M_{t+1}^R = exp(-\boldsymbol{\delta} \boldsymbol{X_t} - \frac{1}{2}\boldsymbol{\Lambda_t'}\boldsymbol{\Lambda_t} - \boldsymbol{\Lambda_t'}\boldsymbol{\epsilon_{\Lambda,t+1}}) \tag{50}$$

$$M_{t+1}^\$ = \frac{M_{t+1}^R}{exp(\pi_{t+1})} = exp(-\boldsymbol{\delta}\boldsymbol{X_t} - \frac{1}{2}\boldsymbol{\Lambda_t'}\boldsymbol{\Lambda_t} - \boldsymbol{\Lambda_t'}\boldsymbol{\epsilon_{\Lambda,t+1}} - \pi_{t+1}) \tag{51}$$

$$\boldsymbol{\epsilon_{\Lambda,t+1}'} = (\epsilon_{f,t+1}, \epsilon_{\pi,t+1}, \epsilon_{eq,t+1}) \tag{52}$$

Risk Prices:

For this model, we break the connection between factors volatilities and risk prices by introducing state dependent risk prices of the 'essentially affine' form, explained above and first presented in Duffie and Kan (1996).

$$\boldsymbol{\Lambda_t} = \boldsymbol{\lambda_1'}\sqrt{\boldsymbol{S_t}} + \boldsymbol{\lambda_2'}\sqrt{\boldsymbol{S_t^-}}\boldsymbol{X_t}$$

Since we assume Gaussian state factors the first term becomes a constant and frees risk prices to depend on state variables. We assume that only q_t contributes to the varying risk prices,

$$\boldsymbol{\Lambda_t} = \boldsymbol{\lambda} + \boldsymbol{\lambda_1}q_t = \boldsymbol{\lambda} + \boldsymbol{\lambda_1}\boldsymbol{\zeta_q}\boldsymbol{X_t}, \qquad \text{where, } \boldsymbol{\zeta_q} = (1, 0, 0) \tag{53}$$

where $\boldsymbol{\lambda_1}$ is 3×1-vector of constants. The above form generates a dynamic term structure model where risk exposure is constant but expected excess returns are state dependent. We exogenously specify risk prices and let expected excess returns be computed accordingly.

$$r_{e,t}^{(n)} = -\boldsymbol{B_{(n)}}\boldsymbol{\Sigma_x}\boldsymbol{\Lambda_t} \tag{54}$$

Yield on One Period Nominal Bond:

Risk prices used to discount current period shocks are determined at the end of the previous period.

$$y^{\$}_{(1),t} = A_{1,t} + \boldsymbol{B_1}\boldsymbol{X_t} \tag{55}$$

$$A_1 = -\lambda_\pi \sigma_\pi - \lambda_f \beta_{\pi,f} \sigma_f - \frac{1}{2}(\sigma_\pi^2 + \boldsymbol{\sigma'_{\pi,x}}\boldsymbol{\sigma_{\pi,x}}) \tag{56}$$

$$\boldsymbol{B_1} = \boldsymbol{\delta} + \boldsymbol{\zeta_{\mu_\pi}} - (\lambda_{1,f}\beta_{\pi,f}\sigma_f + \lambda_{1,\pi}\sigma_\pi)\boldsymbol{\zeta_q} \tag{57}$$

$$r^{(1)}_{e,t} = -\Lambda_{\pi,t}\sigma_\pi - \Lambda_{f,t}\beta_{\pi,f}\sigma_f - \frac{1}{2}Var(\pi_{t+1}) \tag{58}$$

Yield on 10-Periods Nominal Bond:

The following closed form representation is acquired using induction of the relation $P^{(n)}_t = E_t(M_{t+1}P^{(n-1)}_{t+1})$ which takes advantage of the affine prices.

$$y^{\$}_{10,t} = \frac{A_{10,t}}{10} + \frac{\boldsymbol{B_{10}}}{\boldsymbol{10}}\boldsymbol{X_t} \tag{59}$$

$$\boldsymbol{B_{10}} = \boldsymbol{\delta} + \boldsymbol{\zeta_{\mu_\pi}} + \boldsymbol{B'_9}\boldsymbol{\phi_x} - \left[(\beta_{\pi,f} + B_{x2,(9)} + B_{x3,(9)}\beta_{\mu_\pi,f})\lambda_{1,f}\sigma_f + \lambda_{1,\pi}\sigma_\pi\right]\boldsymbol{\zeta_q} \tag{60}$$

$$A_{10} = A_9 + \boldsymbol{B_9}(\boldsymbol{1} - \boldsymbol{\phi_x})\bar{\boldsymbol{X}} - \lambda_f\sigma_f(B_{x2,(9)} + \beta_{\pi,f} + B_{x3,(9)}\beta_{\mu_\pi,f}) - \lambda_\pi\sigma_\pi - \Psi_{10} \tag{61}$$

$$r^e_{(10)} = -\Lambda_r\sigma_r(B_{x2,(9)} + \beta_{\pi,f} + B_{x3,(9)}\beta_{\mu_\pi,f}) - \Lambda_\pi\sigma_\pi - \Psi \tag{62}$$

$$\Psi_{10} = \frac{1}{2}\sigma_f^2(B_{x2,(9)} + \beta_{\pi,f} + B_{x3,(9)}\beta_{\mu_\pi,f})^2 + \frac{1}{2}\sigma_{\mu_\pi}^2(B_{x3,(9)} + \beta_{\pi,\mu_\pi})^2 + \frac{1}{2}\sigma_\pi^2 \tag{63}$$

Return on Equity:

$$r^R_{eq,t+1} = r^{eq}_m + v_{eq,t+1} \tag{64}$$

$$v_{eq,t+1} = \sigma_{eq}\epsilon_{eq,t+1} + \beta_{eq,f}\sigma_f\epsilon_{f,t+1} \tag{65}$$

4.3 GDP Growth

The relationship between GDP growth and the yield curve is already well established in the literature, which emphasises that growth can be predicted on the basis of features of the yield curve especially the slope. In recessions, premia on long-term bonds are high and yields on short bonds are low, leading to upward sloping yield curves. Premia on long bonds are counter-cyclical because investors are more risk averse in bad times. On the contrary, yields on short bonds are often pro-cyclical as central banks usually lower short yields in recessions to stimulate economic activity. Following Ang et al. (2006), we use the fact that economic growth is predictable on the basis of the short rate and yield spread (a proxy to the curve slope) in order to include it in our factor model. This offers more dynamics in comparison to Lee et al. (2003), and Lee and Tuljapurkar (1998) who model economic growth as an exogenous $AR(1)$ process in the same context of social security

forecasts. There are two main channels through which economic growth impact our simulation of long term pension fund evolution; the direct channel is through its effect on income growth of participants and the value of their contributions. The second less direct effect is through the internal rate of return the fund determines for its notional accounts (i^*); this is usually designed to correlate with real economic growth, as in the Swedish NDC system for instance, see Auerbach and Lee (2009). By assuming constant i^*, we only allow for the first direct channel.

GDP growth is defined as,

$$g_t^{GDP} \equiv logGDP_t - logGDP_{t-1}$$

We use the predictive formula and parameters provided in Ang et al. (2006) to forecast economic growth based on the term spread over 5 years horizon,

$$g_t^{GDP} = \delta_{GDP} + \phi_{GDP}(y_t^{(5)} - y_t^{(1)}) + \sigma_{g_{GDP}}\epsilon_{g_{GDP},t} \tag{66}$$

4.4   Income

A new participant in the system always starts with the same value of real income Y_0, which grows according to a stochastic growth rate, and participants of the same age share the same income value. We ignore wages rigidity, and let income grow with inflation in addition to its own growth rate, which is exposed to its own uncorrelated shock plus equity returns, and GDP growth shocks,

$$Y_{t+1} = Y_t exp(g_Y + \pi_{t+1} + v_{Y,t+1}) \tag{67}$$

$$v_{Y,t} = \sigma_Y \epsilon_{Y,t} + \beta_{Y,g_GDP} + \beta_{Y,M}\sigma_{eq}\epsilon_{M,t} \tag{68}$$

4.5   Population Dynamics

A major elements in the dynamics of the system is the stochastic process modelling population evolution. The stochastic population model follows Lee and Carter (1992) mortality model and the somewhat similar fertility model of Lee (1993). Lee and Carter model the time series of a mortality index for the whole population as a random walk with drift. This index then drives the evolution of age specific mortality rates and thereby survival and life expectancy. This model has been extensively tested, see Lee and Miller (2001), and is widely accepted, see Booth (2006) for a comprehensive survey of demographic forecasting methods. Model parameters are calibrated after

estimation using US data over the interval 1900-1989. We tamper a bit with the estimated fertility parameters to produce different scenarios of population projections.

4.5.1 Mortality

The mortality index is modelled by

$$K_t = K_{t-1} - z + \epsilon_{t,mort} \qquad \text{where } \epsilon_{t,mort} \sim N(0, \sigma_{mort})$$

Age-specific mortality rates are then constructed from,

$$log(m_{x,t}) = a_x^{mort} + b_x^{mort} K_t$$

where b_x's sum up to 1, and a_x sum up to the long run expected overall population mortality rate. We use the parameters values provided in Lee and Carter (1992).

4.5.2 Survival

Age specific mortality rates are then mapped into survival rates $s_{x,t}$ by standard functions reported in Lee and Tuljapurkar (1994):

Ages 0 - 1:

$$s_1 = \frac{2 - m_1}{2 + m_1}$$

Ages 1 - 4:

$$s_2 = \exp(-4m_1 - (4)^3 (0.008) m_1^2)$$

For all other ages:

$$s_x = \exp(-5m_x - (5)^3 (0.008) m_x^2)$$

4.5.3 Fertility

The fertility index is modelled by a mean constrained $ARMA$ process,

$$F_t = cF_{t-1} + \bar{F}(1-c) + u_t + du_{t-1}$$
$$f_t = F_t - \bar{F}$$

Age-specific fertility rates,

$$f_{x,t} = a_x^{fert} + b_x^{fert} f_t$$
$$TFR = \sum a_x^{fert}$$

TFR is the sum of age-specific fertility rates a_x, and represents the unconditional long term total fertility rate. f_t adds to 0 over the horizon, and the sum of all b_x's $= 1$. For a_x and b_x, we use the numbers reported in Lee (1993).

4.5.4 Population Projection

Having derived both mortality and fertility rates, population projections are then drawn as follows, see Lee and Tuljapurkar (1994) for further details:

for female population,

$$N_{f,t} = R_{f,t} N_{t-1}$$

for male population,

$$N_{m,t} = R_{m,t} N_{t-1} + b_{m,t}$$

for total population,

$$N_t = N_{f,t} + N_{m,t}$$

where $b_{m,t}$ is the number of new born males. As explained in Alho and Spencer (2005), R_t is the *Leslie* projection matrix, see Leslie (1945). Named after Patrick H. Leslie, the Leslie matrix is one of the most well known ways to describe the growth of populations. R_t has dimensions of $w+1 \times w$, where w is the range of people alive; We take this to be 110. R_t is zero everywhere except for the first row, the lower first sub-diagonal and the bottom-right corner $(w+1, w)$.

In the case of female population, $R(1, x, t)=$ expected number of girls born during t per woman

in age x, that survives to the beginning of the next year. Not all age groups are fertile, only females of age between 15 and 44 are able to give birth. $R(x, x-1, t)$= proportion of survivors from age $x-1$ at t to age x at $t+1$. $R(w+1, w, t)$= proportion of survivors in age w which we set to zero so that nobody survives after 110. The same story goes for the male population with the difference that males are not fertile, yet births are added to male population every period, $b_{m,t}$. The female to male ratio at birth is constant, and we take it to be 1.05.

There are two important features of this model. The first is regarding the Leslie construction; in case of age-specific time-constant mortality and fertility rates, the population age structure stabilises after a certain period when the initial population composisition adjusts. This is important for pension analysis since this structure is what decides the evolution of the dependency ratio. In the present model, we allow for stochastic time-varying rates, which means the age structure will not be constant. However, variation in age structure over time after some point becomes almost unobservable. The second feature is that, in the current closed population, with long-run total fertility rate of 1.95, population dies over time. Eventually, it goes completely instinct at a point. When immigrants are allowed to flow in, population size adjusts until the net inflow of immigrants is equal to the shortfall in births due to below-replacement fertility; however, we do not feature immigrants in our model.

4.6 Pension System Design

An annual contribution is a fraction of each individual income at time t. Contribution per age group is the multiplication of the individual contribution and the population size of the corresponding age group (j),

$$C_t^{(j)} = \theta Y_t^{(j)} \times N_t^{(j)} \tag{69}$$

Every Individual holds an account which earns a return I_t every year, and grows with her annual contributions. The benefit which a participant is rewarded on retirement is simply equal to the value of her individual account at year of retirement. Liability of the whole system at time t is the sum of all notional individual accounts which the system promises to redeem at the year of retirement for every group,

$$Z_{t+1}^{(j+1)} = I_t(Z_t^{(j)} + C_t^{(j)}) \tag{70}$$

$$X_t = Z_t^{67} \times N_t^{67} \tag{71}$$

$$L_t = \sum_{j=21}^{66} Z_t^{(j)} \times N_t^{(j)} \tag{72}$$

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Funded pension systems, of which NDC can be considered a variation, hold a collective asset that is invested in the financial market. Last period asset is invested at time t and earns a return of $R_{A,t+1}$ at time $t+1$. All working population groups in the range between 21 and 66 pay contributions at the beginning of the new period $t+1$, and benefits are paid to the retiring group of age 67 at $t+1$,

$$A_{t+1} = R_{A,t+1}A_t + \sum_{j=21}^{68} C_{t+1}^{(j)} - X_{t+1} \tag{73}$$

We only implement a static portfolio where re-balancing is not allowed. The system invests in one period nominal bonds, in addition to 10 years nominal bond and equity with weights w_{y10} and w_{eq}, respectively,

$$R_{A,t+1} = Y_{t+1}^{(1)} + w_{y10}(Y_{t+1}^{(10)} - Y_{t+1}^{(1)}) + w_{eq}(R_{t+1}^{eq} - Y_{t+1}^{(1)}) \tag{74}$$

The internal rate of return on individual accounts is computed every year by the pension system based on the fund ratio; this is what creates a direct link between pension benefits and market risks in order to ensure solvency of the system.

$$lnI_t = i^\star + \alpha ln(\frac{A_t}{L_t}) \tag{75}$$

4.7   Market Valuation

Price:

Lastly, we use the simulated processes of benefits, contributions and SDF to compute the present value of the pension plan as perceived by an individual participant of age 21, entering the system at time t, and is assumed to retire at time $t+46$ of age 67 at year of retirement. Discounting a cash flow at time $t+i$ is done by compounding the one period SDF over the time horizon between t and $t+i$. This multiplicative compounding characteristic of M leads to declining present time prices as the time horizon i of the pay off increases. We have discussed this earlier while explaining the implications of a choice of M on the asymptotic decay rate of current prices for long term cash flows,

$$M_{t+i} = M_{t,t+1} \times M_{t+1,t+2} \cdots \times M_{t+i-1,t+i} \tag{76}$$

$$V_t^{21} = E_t(M_{t+46}X_{t+46}) - \sum_{s=0}^{45} E_t(M_{t+s}C_{t+s}), \text{ where } M_t = 1 \tag{77}$$

Expected Returns

From the point of view of an individual of age 21 at time t, participating in a pension scheme is an investment decision with an expected return value; that is, the ratio between the expected payoff (retirement benefit) and the present value of the cash flow paid by the participant. The compensation the participant pays in exchange for the future benefit includes present time value of all contributions in addition to the price V_t paid to buy the pension plan,

$$E_t(R_{t+46}^{Pension}) = \frac{E_t(X_{t+46})}{V_t^{21} + \sum_{s=0}^{45} E_t(M_{t+s}C_{t+s})} = \frac{E_t(X_{t+46})}{E_t(M_{t+46}X_{t+46})} \tag{78}$$

Since $R_{t+46}^{Pension}$ is a long term asset, a potential participant would be keen to compare it with a safer individual investment, namely, the return on an individual savings account that invests in long term nominal yields with descending maturities until retirement. We use our simulation platform below to analyse this scenario. [1] Alternatively, she could invest her individual savings in a risky portfolio; we analyse the case when the risky portfolio is the same as the pension fund portfolio with returns equal to R_p.

Risk Premium

In order to understand the role played by demographics and other parameters in deciding the expected rate of return on an NDC pension scheme, we follow the approach discussed in section 3.2. As we implement the same pricing kernel of Hansen et al. (2008), we can use their characterisation of valuation of assets exposed to long term risk, see equations (18)-(20). We subtract κ from the expected return on a pension plan $E_t(R_{t+46}^{Pension})$; κ, as explained above, is the long term return on a cash flow that does not grow over time and is not exposed to long term risks; this is not to be confused with long term bonds, since, in our model, long term bonds earn a premium and are

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[1] By ignoring the correlation between yields and contributions, the present time expected rate of return on investing annual contributions in yields with descending maturities is equivalent to the present time expected return on investing the the present time discounted value of all contributions in bonds with 46 years maturity; this is a consequence of the local expectations hypothesis, and our assumption that real rate, and inflation risks capture all volatility of long term bonds yields. Therefore, if we are to ignore the channel (yields → growth → income → contributions), the above scenario could be simplified to,

$$E_t(R_{e,t+46}^{Pension}) = E_t(R_{t+46}^{Pension}) - 46 \times exp(y_t^{(46)}) \tag{79}$$

exposed to long term risks. The remaining part of $E_t(R_{t+46}^{Pension})$ is a premium for long term exposure to risk,

$$\kappa_t = \delta_0 + \frac{1}{2}\mathbf{\Lambda}_t'\mathbf{\Lambda}_t \tag{80}$$

$$E_t(\Xi_{t+46}) = -\Lambda_{t,r}^*\Gamma_{t,r} - \Lambda_{t,\pi}^*\Gamma_{t,\pi} - \Lambda_{t,eq}^*\Gamma_{t,eq} \tag{81}$$

Long term risk prices, Λ_t^*, are characterised in (19); however, long run exposure, Γ_t, cannot be analytically expressed in our model. In the next section, we present simulation results, and try to highlight the dynamics controlling this exposure to risk through impulse response and perturbation analyses.

Role of Inflation

Model variables are exposed to inflation and are expressed in nominal terms, which calls for caution in discounting future cash flows. Although the fund asset has been exposed to inflation since it first started, benefits paid at time $t + 46$, from the point of view of a potential participant at time t, are only exposed to 46 years of inflation which is the life time of a notional account. This is reflected in equation (74), where M_t is normalised to 1. This means that expected returns $E_t(R_{t+46}^{Pension})$ are real from the point of view of a participant at time t; however, for us, at time $t = 0$, these expected returns are inflated and nominal. Anyway, our main interest lies in the view of a potential participant at time t.

5 Simulation Results

5.1 Population Projection

Figure (1) shows the population evolution over 500 years. Since we model a closed population with a total fertility rate (1.95), which is below the replacement rate, population decays gradually; for an initial population of around 1500 individuals, extinction takes place around the year 2000. As explained above, populations projected using Leslie matrix (fixed age-specific fertility and mortality rates) eventually stabilises around a certain dependency ratio, since the value of the dependency ratio depends on the fertility and death rates parameters assigned for the different age groups. Using the values reported in Lee (1993) and Lee and Carter (1992) for US population leads to a ratio of almost 1. The mortality index follows a random walk with a drift; we let the drift be equal to zero at the year 100 following Auerbach and Lee (2009). On the other hand, the fertility index follows $AR(1)$ and drives fluctuations around the long term pre-specified TFR of 1.95.
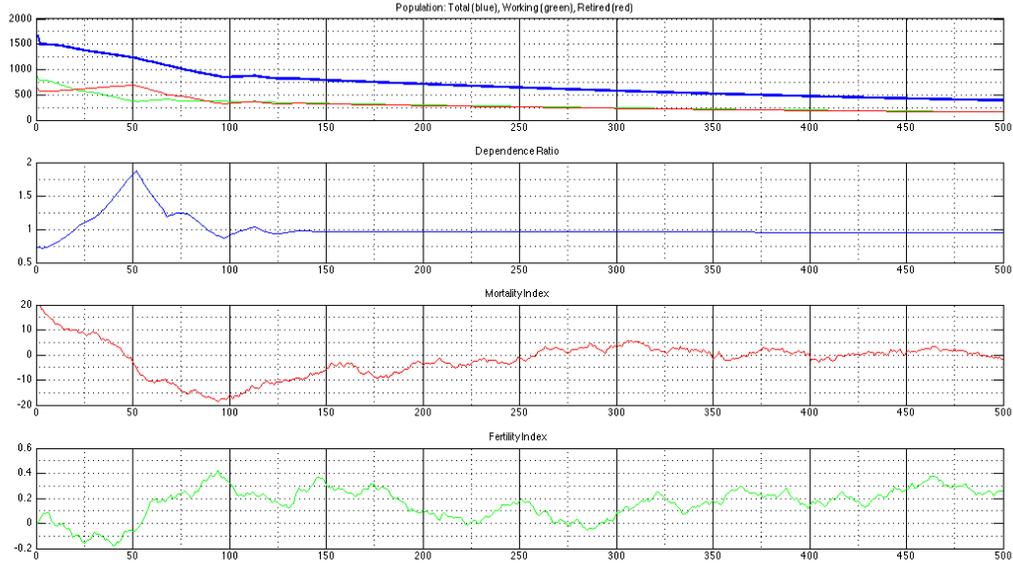
Figure 1: Population Graphs: From top : Total population, Dependency Ratio, Mortality index, and Fertility Index, respectively

5.2 Term Structure

Figure(2) shows the nominal yield curve for maturities between 1 and 46. In order to identify yields, we pre-specify the 20 years nominal bond risk premium to be 0.012. The top part shows risk prices implied by the two-factors static model, where risk prices have constant values except for equity. This is due to the fact that we take the unconditional mean of equity returns to be constant; this leads to time varying risk premium, hence, time varying risk prices. The bottom part reflects the three factors model risk prices as they vary with the factor q_t. Inflation risk price can go positive at some point because it is very close to zero; this means that nominal bonds can earn a negative inflation risk premium in some years. It is important to note that the difference between the two models with respect to nominal bond returns, for our purposes, is not significant. This is a result from the fact that we keep state factors Gaussian in both cases.

5.3 Fund Dynamics

The internal return rule, $I_t = i^* + \alpha ln(A_t/L_t)$, insures that the system reaches a stochastic equilibrium by driving the rate of return on notional accounts to a point where it fluctuates around the fund asset portfolio returns. Two cases are important to mention:

1. $i^* < E(R_p)$ the return on individual accounts I_t starts small allowing the fund asset to grow with a rate higher than growth in liabilities, until $\alpha ln F_t$ drives I_t to point where $I_t = E(R_p)$;

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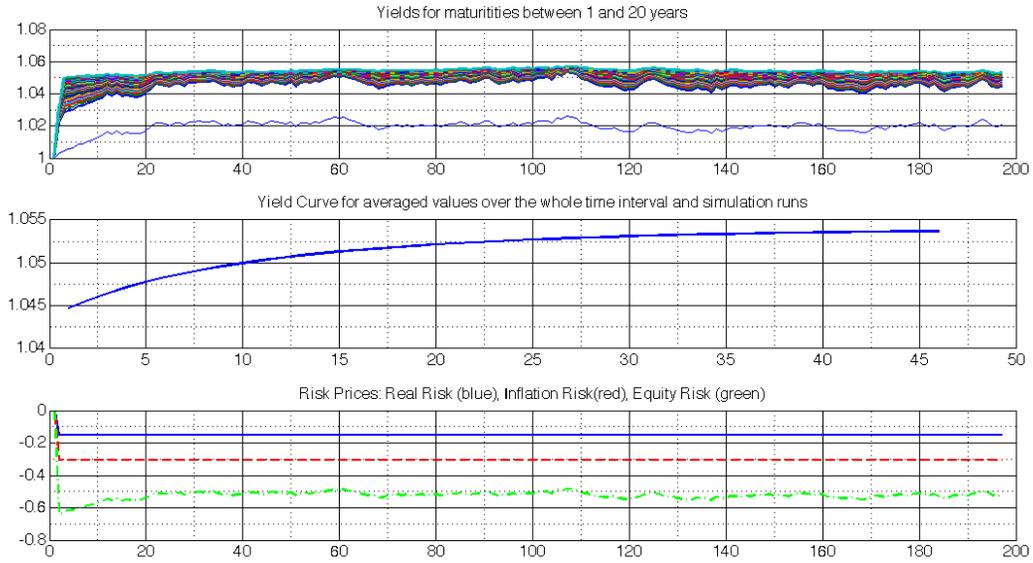
(a) Static Model. Real rate risk (solid), Inflation risk (dashed), Equity Risk (-.)
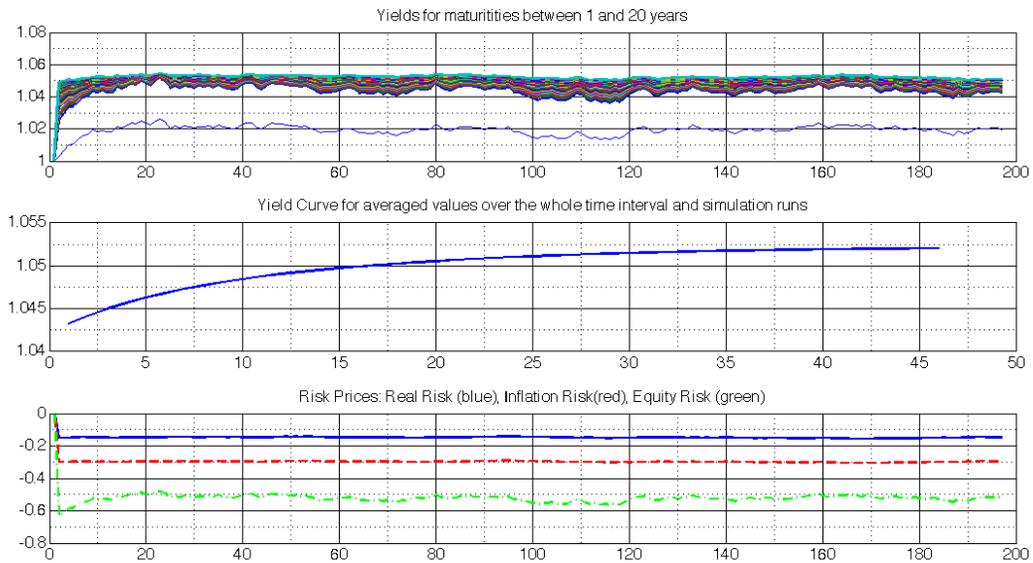
(b) Dynamic Model. Real rate risk (solid), Inflation risk (dashed), Equity Risk (-.)

Figure 2: Nominal Yield Curve for maturities between 1 and 46 for static (top) and dynamic(bottom) models. 20 years nominal bond is pre-specified to earn 0.012 risk premium. Values for the yield curve are averaged over the time horizon and across all simulations. (100 simulations)

this is when fund ratio almost stabilises, but maintains a very slow growth as the asset value grows. In this case, although early retiring groups earn low benefits, the early savings in the asset leads to higher benefits for later generations. On the other hand, higher values of α limit the asset growth in the early stage by spending the savings in the asset value quickly. The comparisons here are to other cases for i^*, see figures (4).

2. $i^* \geq E(R_p)$, fund ratio stabilises at a value lower than 1, and returns on individual accounts are driven down to the same value as asset returns. The early high returns on notional accounts cripple the asset growth and leads benefits to stabilise around a low value. It is important to understand that if α is not high enough to rebalance I_t in time, the asset drops until it is negative and the system collapses. This ensures that the system can temporarily sustain a deficit where all returns on asset are paid to participants and no saving takes place; the fund asset slowly declines and decays in the long run, see figure (3) .
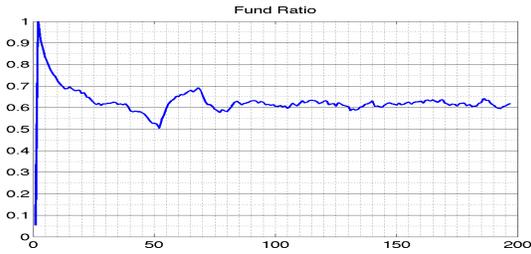
(a) $i^* = 0.2$. Fund Ratio
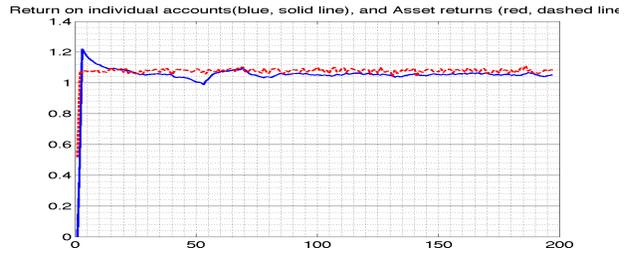
(b) $i^* = 0.2$. Returns on Notional Accounts and Asset
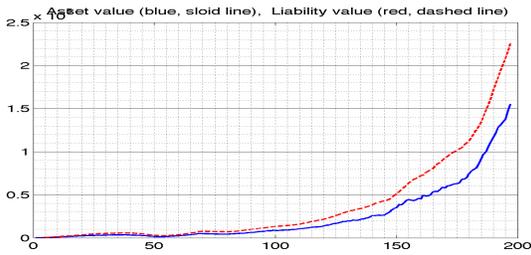
(c) $i^* = 0.2$. Asset Value
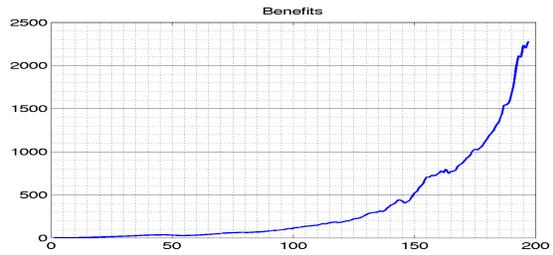
(d) $i^* = 0.2$. Benefits

Figure 3: Fund Dynamics for $i^* = 0.2$. Growth in benefits is due to inflated income and returns. In a model with no inflation, benefits curve converges to stationary state.

The above two cases give a good guide to understand what happens if a break is introduced in the system, and the fund ratio is set, for example, to 5. $\alpha ln F_t$ drives the returns on notional accounts to a higher level allowing retiring generations to benefit from this increase in asset value; because α is not high enough for the first retiring groups after the break to absorb all benefits, the saved amount leads asset value to grow. Afterwards, the rate of return I_t gradually drops as the asset value falls due to higher paid benefits. Eventually, the fund stabilise around a value higher than its initial point, but the fund ratio returns to its former values, see figure (5). Both the asset value and fund ratio slowly increase with the same rates before the break.
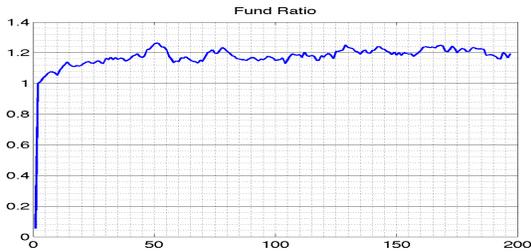
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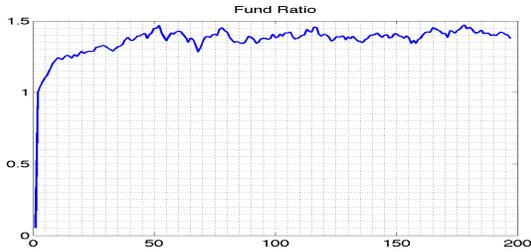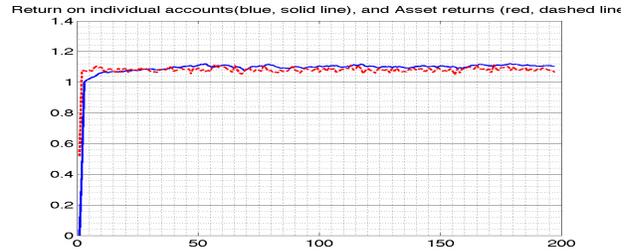
(a) $i^* = 0$. Fund Ratio

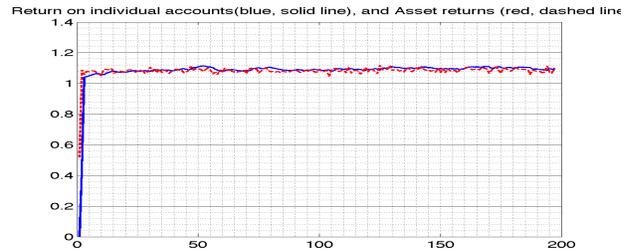(b) $i^* = 0$. Returns on Notional Accounts and Asset
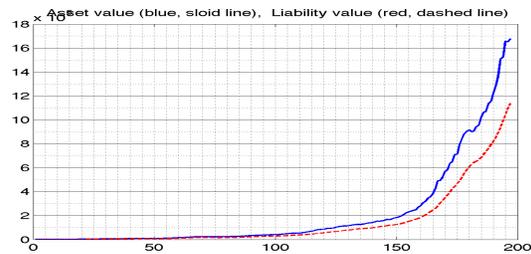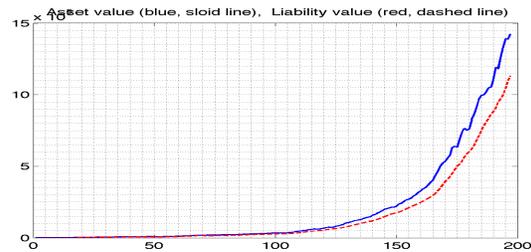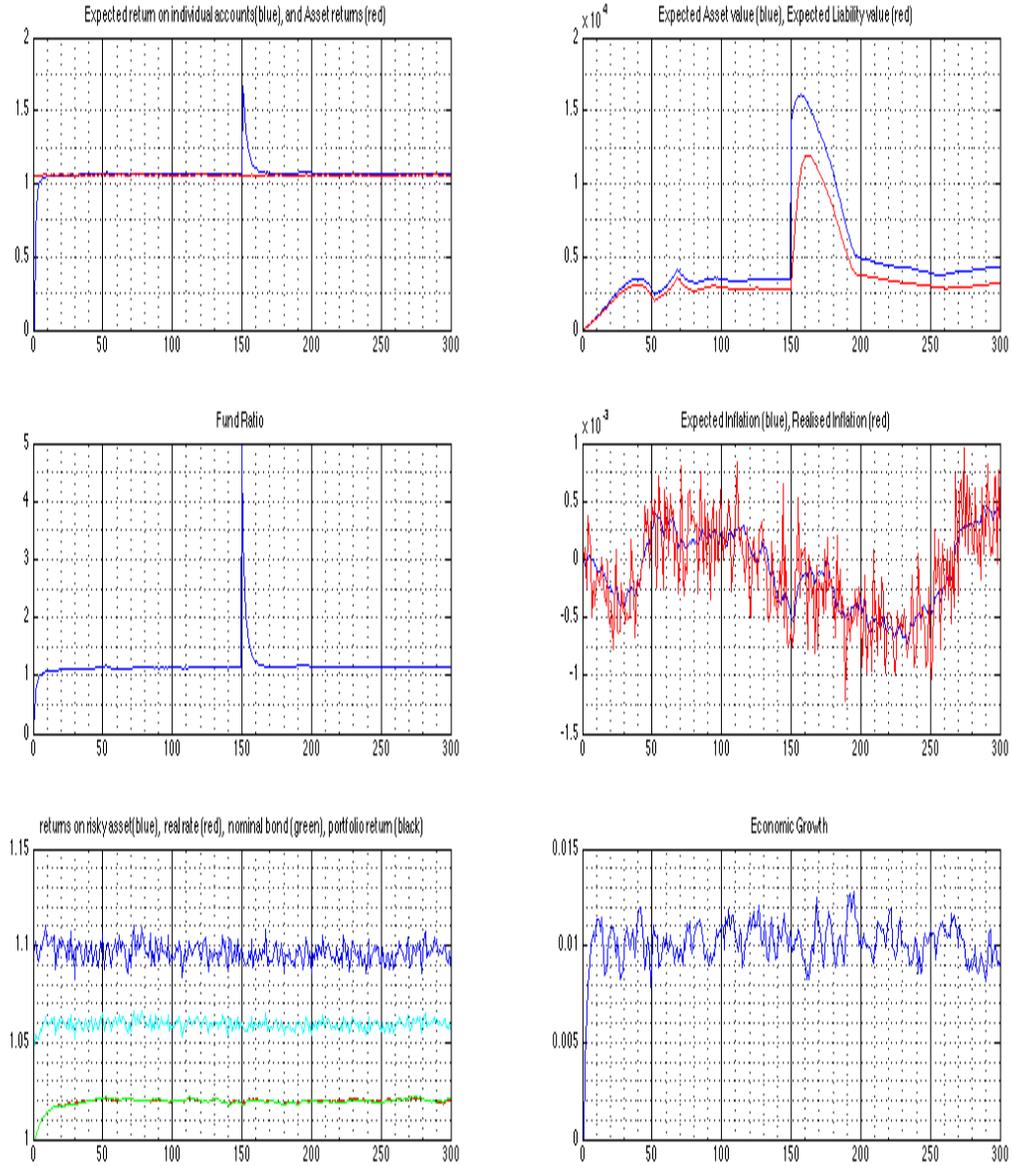

(c) $i^* = 0$. Asset Value
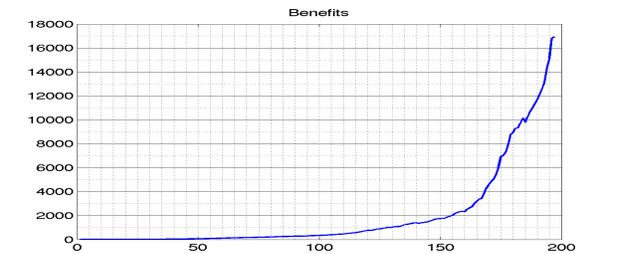
(d) $i^* = 0$. Benefits

(e) $i^* = 0.04$. Fund Ratio

(f) $i^* = 0.04$. Returns on Notional Accounts and Asset

(g) $i^* = 0.04$. Asset Value

(h) $i^* = 0.04$. Benefits

Figure 4: Fund Dynamics for $i^* = 0$ and 0.04. Growth in benefits is due to inflated income and returns. In a model with no inflation, benefits curve converges to almost a state with a very slow rate of change.

Figure 5: System Variables. At year 150, Asset Value is set to equal 5 times the Liabilities. Left column: I_t, R_p, Fund Ratio, and Assets Returns. Right column: Asset and Liabilities values, Inflation, and Economic Growth (1000 Simulations)

6 Valuation Results

We use a simulated sample of 197 years (150 years until dependency ratio stabilises + 47 years the length of one individual participation) to calculate the expectation and volatility of future returns on participation in the pension plan. The first 150 years are non-stochastic evolution of the model with all shocks off; this constitutes our single line of history which gives the initial conditions endogenously determined by the model parameters. Our analysis concerns the group of participants who enter the system at year 151 with age 21, and retire at year 197 with age 67. It could be interesting to discuss other contexts like older generations joining the system at time t and retiring before time $t + 46$; yet, we focus our attention only on the 21 years old generation, since we assume stable systems that have already been operational for years, and leave discussing the transition state for future research.

We compute the expected retirement benefits, and the present time value of the plan. Using both: future payoff and price, we deduce the expected returns on the NDC pension package. The simulation enables us to also compute the volatility of the future benefits with two caveats highlighted in the context: the first being that this volatility constitutes a single observation and does not characterise the distribution or expectation of payoff volatility. Secondly, the observed volatility reflects the short term exposure to risk, which, unlike long run exposure, does not take the growth rate of cash flows into consideration. Throughout, the analysis examines how the above results change for different values of Total Fertility Rate and Fund Ratios[2].

In order to characterise the fund short term exposure to priced risk, we see how the system responds to a market shock for different fertility parameterisation, and fund ratio values. Finally, we introduce time varying risk prices and see how this changes the picture. The results are compared with the case of an individual holding, instead, a savings account which invests in either: (1) yields of nominal long term bonds with a pay off at retirement, or (2) the pension fund portfolio. This provides a benchmark and highlights the gains and risk expected from collective pension plans. Main results are summarised in the conclusion, section 7.

6.1 Benefits, Expected Returns, and Volatility

6.1.1 Future Benefits

Figure (6) presents the expected retirement benefits perceived by the participant *at time of participation* (age 21) for different values of fund ratio and fertility parameters. The three subplots compare between the alternatives of investing in (a) a NDC plan or holding savings account whether (b) risky: *Fund Portfolio* or (c) safe: *long term bonds*[3]. It is clear that benefits received from

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[2]Fund ratio is set at year 151 the same time a new participant joins, by manually fixing the fund asset A_{151}. After that, the system is left to evolve freely.

[3]Long term bonds still carry real rate and inflation risks but they are relatively safer than risky portfolios that invest in the stock market.

the collective system are always higher than individual alternatives. Knowing that a prospective participant in our simulation starts at age 21 with annual nominal income around 20 (real $Y_0 = 1$), and receives a nominal income of 228 the year before retirement, we can scale the lump sum benefits relatively to be in the range between 5 to 11 times the before retirement income.

Benefits are lower, the less funded the system is. For TFR=1.95, benefits drop by around 8% for a decline in the fund ratio from 2 to 1.677; this percentage drop rises to 15%, when the fund ratio drops from 0.677 to 0.33. This reflects a non-linear behaviour of the model; that is, the expected percentage or relative increase in a participant benefit from increasing the fund ratio is higher the less funded the system initial state is. Such strong variations in retirement benefits in response to different initial fund ratios is a result from the simple allocation rule the fund implements (i.e $I_t = i^* + \alpha FR_t$); as a consequence, the fund recovers from shocks in a relatively short time (less than a decade for $\alpha = 0.3$), such that excess benefits from overfunded states or losses from underfunded states are not smoothed over long time intervals.

The fund rule has further implications. As remarked in the footnote before, the fund ratio is set at the year the new participant joins (year 151). This simulates a transitory state of the fund which could be a result of a shock of some sort; otherwise, the fund ratio has a stationary state around a value determined by the system parameters (i^*, α) as explained earlier in section 5.2. This explains why in figure (6.a) a fund ratio of value = 0.3 still offers higher expected retirement benefits than individual savings. From figure (5), we understand that the temporary low fund ratio lasts only for less than a decade before the system adjusts to its stationary value which is above unity (around 1.2). This means that a beneficiary who enrols at bad time of the system will still benefit from the collective nature of the system in the future when the system fund ratio returns to its normal over funded state.

Benefits are also inversely proportional to population fertility. For a just funded system (Fund Ratio = 1), benefits drop by around 12%, and 9% for TFR falling from 2.65 to 1.95 and 1.95 to 1.45, respectively. In general, participants in a NDC system with a growing population enjoy higher values of benefits than systems with decaying populations. This agrees with our intuition that with a growing population, the fund asset increases and benefits to old generations grow. The opposite is what happens for dying populations.
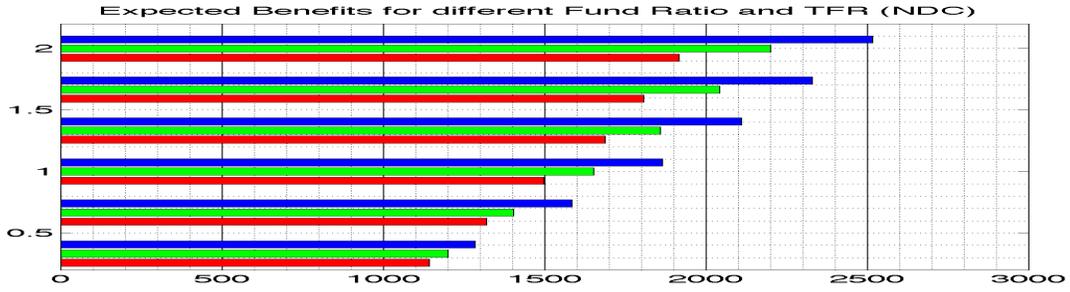
(a) NDC Plans
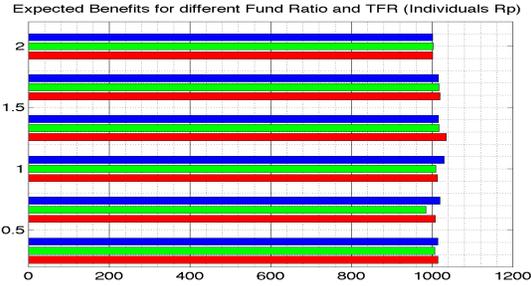
(b) Individual Savings in Rp
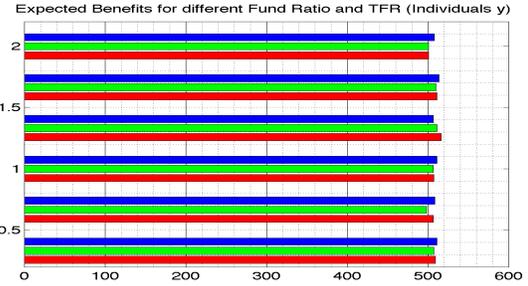
(c) Individual Savings in Nominal Yields

Figure 6: Expected value of benefits earned by participants upon retirement for different values of TFR. Y-axis: Fund Ratio, X-axis: Benefits {Top bar: TFR=2.65, Middle bar: TFR=1.95, Bottom bar: TFR=1.46 } (1000 Simulations)
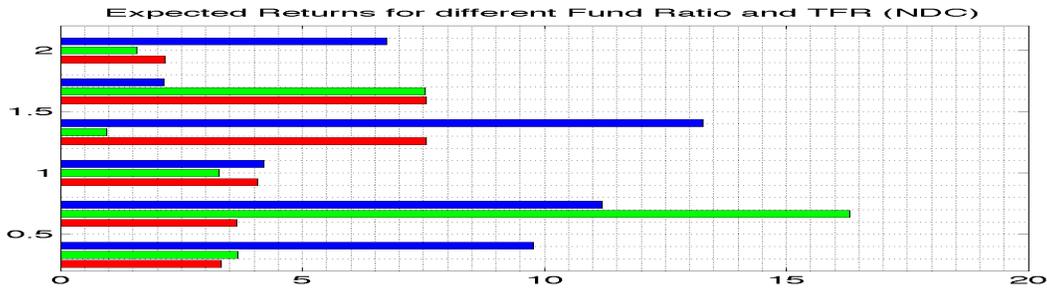
6.1.2 Expected Returns

Having computed the present time valuation, figure (7) presents the derived expected returns on investing in pension plans, according to equation (78), at different values of fund ratio at time of participation (year 151), for different values of long term TFR [4]. Before we discuss any observations, it should be noted that returns on individual savings have no reason to differ over variations of TFR or fund ratio; these variations are an artefact due to the limited number of simulations we are able to run (5000). However, we report them in order to compare between results that belong to the same calibration.

Overall, NDC plans offer lower expected returns than individual savings accounts. This reflects the higher market price assigned to NDC plans. Higher market prices indicate lower long run exposure to priced risks. As we partially discussed above, long run exposure differs from its short run counterpart in many ways; the one which is most relevant here is that the long run growth rate of a future benefit takes part in shaping the long run exposure, *eqn*.19. Therefore, as the impulse response analysis shows later, even though benefits from NDC systems are more exposed to market risk compared with alternative individual systems, the long run exposure is still less due to the higher long run growth rate of NDC systems. In shorter words, the market regards NDC plans as a

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[4]We choose to present expected returns instead of prices because they are easier to understand without a numeraire.

more valuable investment when compared with individual savings accounts, even the ones investing in nominal yields.

Figure (7.a) shows that expected rate of return ranges almost between (1.5 and 16) over the 46 years horizon, which corresponds to approximately (0.8% to 6%) constant compounded annual rate. For TFR=2.65, and TFR =1.46, a fund ratio = 1 provides the least long run exposure to market risk; on the other hand, for TFR =1.95, lowest exposure is achieved at fund ratio = 1.33. In general, however, the results are highly non-linear, making it hard to distinguish a pattern; yet, we can make the observation that elasticity of long run risk exposure (equation (19)) with respect to changes in fertility rates depends on the fund ratio, $\Delta\Gamma/\Delta TFR = f(\frac{A_t}{L_t})$. Sensitivity to changes in fertility is lowest when the system is just funded. On the other hand, the elasticity of risk exposure with respect to changes in the fund ratio, $\Delta\Gamma/\Delta\frac{A_t}{L_t}$, is low for very low and high TFR (1.46, and 2.65), but high for TFR=1.95. To sum this point up, (1) exposure of benefits to market risk in near replacement rate population growth is most sensitive to variations in fund ratio, and (2) this risk exposure in just funded systems is least sensitive to fertility changes. We discuss exposure more in depth in the impulse response analysis below.

(a) NDC Plans
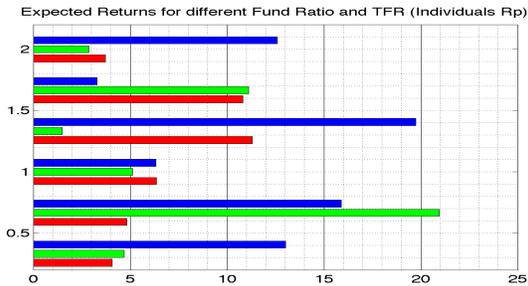
(b) Individual Savings in Rp
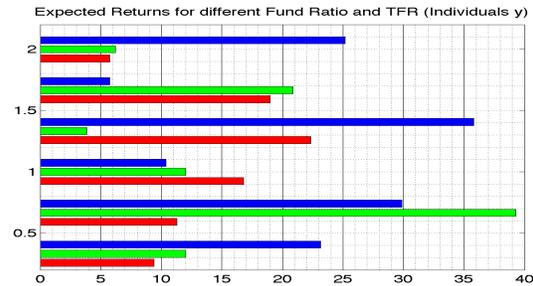
(c) Individual Savings in Nominal Yields

Figure 7: Expected returns on investing in a pension plan for different values of TFR. Y-axis: Fund Ratio, X-axis: Expected Returns {Top bar: TFR=2.65, Middle bar: TFR=1.95, Bottom bar: TFR=1.46 } (1000 Simulations, Static Model)

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6.1.3 Volatility of Returns

Figure (8) shows that NDC plans are actually more volatile than benefits from individual accounts for all the projected scenarios (Volatilities are scaled by the expected benefits in each case). This takes us back to the point we mentioned above and discuss again below about how the high long run growth rate renders the long run exposure; this is why NDC benefit earn less premium despite the higher level of uncertainty associated with them on the short run. There is not much that could be said about elasticities in figure (9) since the values fall very close from each other in the different scenarios. We, however, notice that low fund ratio leads to lower volatility, irrespective of the fertility value; for TFR=1.95, fund ratio dropping from 2 to 0.5 leads to volatility falling by around 13%. Moreover, populations with very low fertility offer less volatile returns. Therefore, uncertainty is higher, the higher the fertility and fund ratio are.

In figure (9), we decompose the realised volatility in order to understand which shocks of the priced risks contribute most to future uncertainty. The simulation results in 70% volatility (scaled by expected benefits every year), of which, real rate fluctuations contribute a little over 25%, and equity fluctuations around 40%. Inflation stochastic process has the minimum effect on future uncertainty of retirement benefits of around 3%.
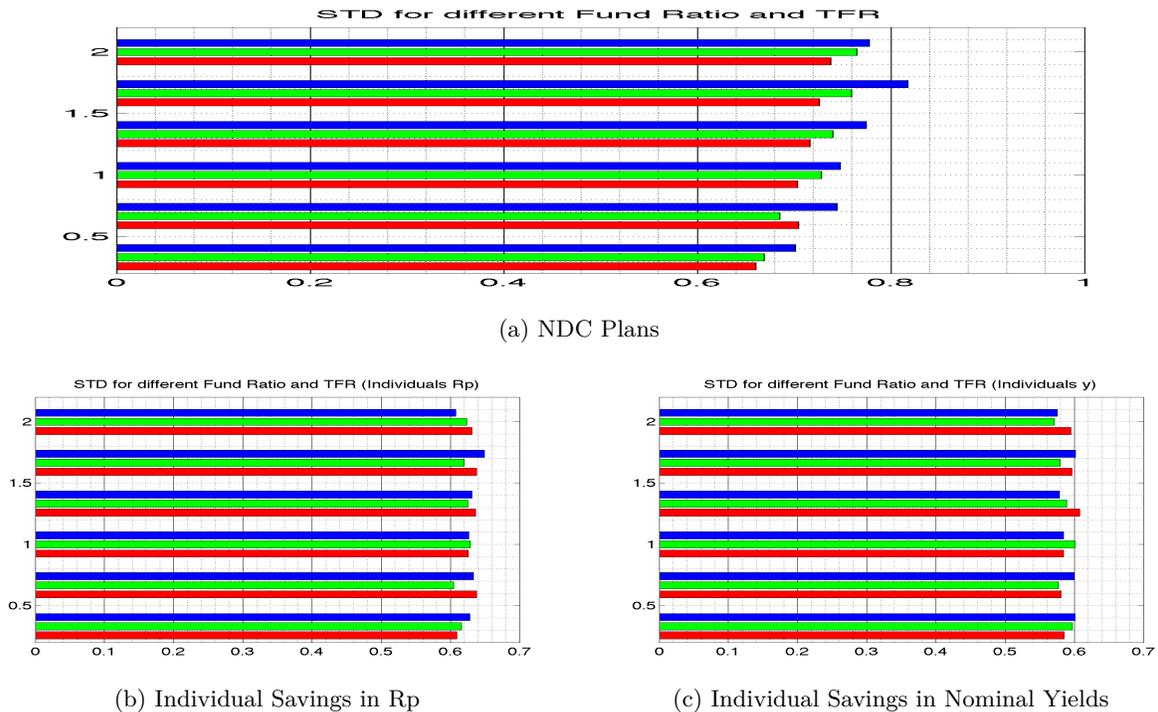

(a) NDC Plans

(b) Individual Savings in Rp

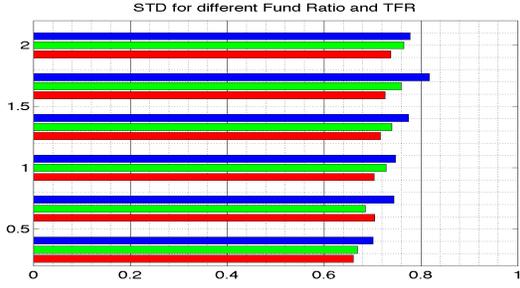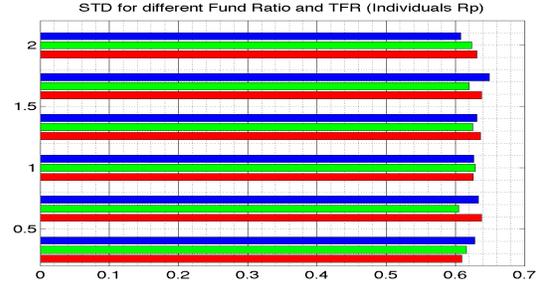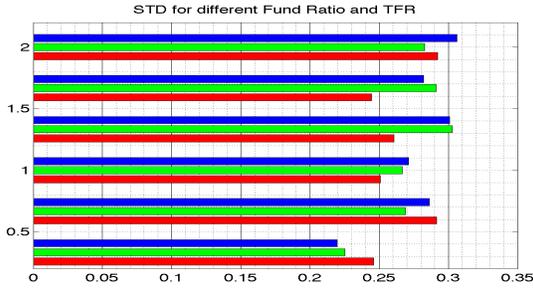(c) Individual Savings in Nominal Yields

Figure 8: Volatility of returns on investing in a pension plan for different values of TFR. Y-axis: Fund Ratio, X-axis: Standard Deviation {Top bar: TFR=2.65, Middle bar: TFR=1.95, Bottom bar: TFR=1.46 } (1000 Simulations, Static Model)
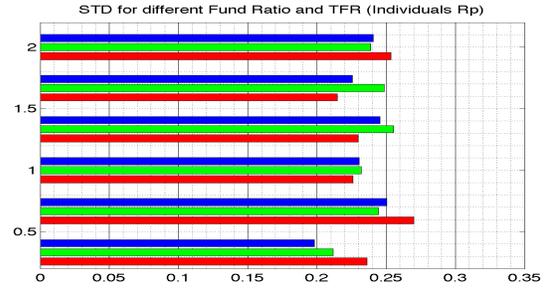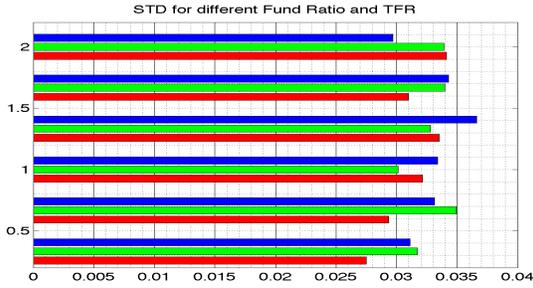
(a) NDC Plans: All Shocks On
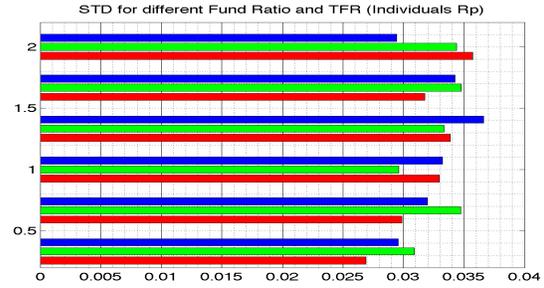
(b) Individual Savings in Rp: All Shocks On
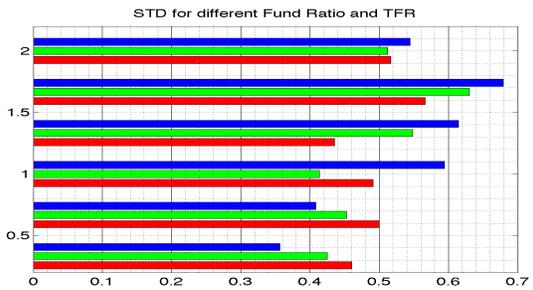
(c) NDC: Only real rate shock On

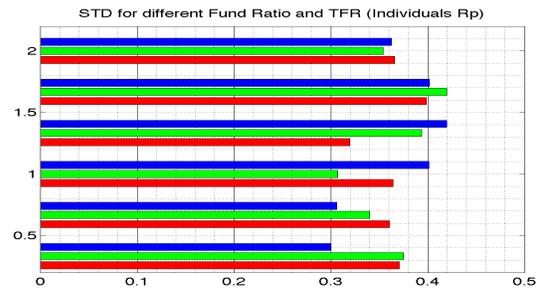(d) Individual Savings in Rp: Only real rate shock On

(e) NDC: Only inflation shock On

(f) Individual Savings in Rp: Only inflation shock On

(g) NDC: Only equity shock On

(h) Individual Savings in Rp: Only equity shock On

Figure 9: Volatility of returns on investing in a pension plan for different values of TFR. Y-axis: Fund Ratio, X-axis: Standard Deviation{Top bar: TFR=2.65, Middle bar: TFR=1.95, Bottom bar: TFR=1.46 } (1000 Simulations, Static Model)

## 6.2 Higher Benefits Volatility, but Higher Price?

We reached the result that funded NDC plans offer less expected returns than individual savings accounts, despite the higher volatility of the returns. However, how could these two facts be conciliated in a No Arbitrage framework? We know that if No Arbitrage holds, then expected returns have to reflect risks involved; higher risk corresponds to a higher expected return, and all bundles of expected returns and volatilities fall on the efficient frontier; that is, *Sharpe* ratios are maximised. Figure (10) shows the pseudo *Sharpe* ratio calculated by dividing long run expected returns over the standard deviation of the simulated realised benefits at year 197. The figures are clearly not arbitrage free.

The answer to this comes through understanding two key points. The first is what we already discussed that prices reflect the *long* run exposure which encodes the growth rate of cash flows in addition to their short term exposure; that is, highly volatile long run cash flows can still be more expensive than a less volatile alternative if it offers a high enough long run growth rate. Second, what a young worker really observes at time t, 46 years before his retirement at $t + 46$, is only the price which encodes the market expectation of benefits and long run exposure to risk. The volatility which we reported so far is just a single realisation which cannot be used to infer the value of expected volatility. This explains why the reported *Sharpe* ratio is a pseudo value. The right approach would have been to use a *long* term equivalence of *Sharpe* ratio which would then require computing the long run expected volatility; this is only possible in linear models where a decomposition like the one in *eqn*.19 is attainable, and second moments are simulated.

Highlighted here is one of the key advantages of market consistent valuation; that is, the calculated price encodes all market information on both growth and risk of a future cash flow. This maximises the long term *Sharpe* ratio of the prospective investor which, consequently, puts the funded NDC plan on the efficient frontier of the different investment options available for her.
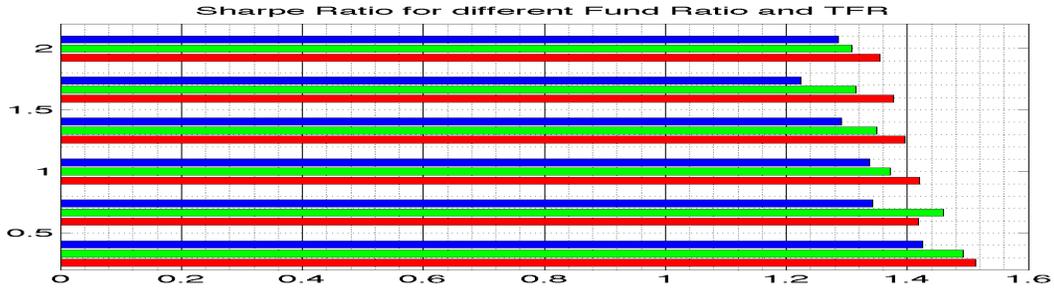
(a) NDC Plans
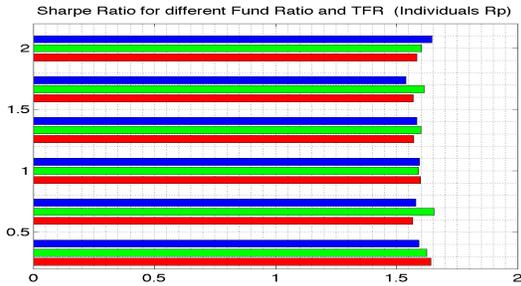
(b) Individual Savings in Rp
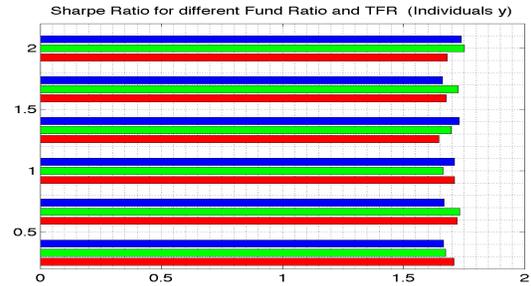
(c) Individual Savings in Nominal Yields

Figure 10: Pseudo Sharpe Ratio of returns on investing in a pension plan, based on computed volatility from simulated realisations of benefits, for different values of Fund Ratio and TFR. {Top bar: TFR=2.65, Middle bar: TFR=1.95, Bottom bar: TFR=1.46 } (1000 Simulations, Static Model)

6.3   Further Analysis of Risk Exposure (*Impulse Response Analysis*)

The above analysis characterised the present time valuation of future retirement benefits, where we made the conclusion that NDC retirement packages are more expensive than savings in individual accounts. Here, we characterise the sources that drive the uncertainty about future returns using impulse response analysis. We give special attention to the interaction of this *short* term exposure with fertility parameters and the fund ratio, which is analytically ambiguous due to the model non-linearity.

We focus mainly on impulse responses of benefits expected value and volatility at time $t + i$ to a financial market shock at time t. Despite the non-linearity of the benefits process, we can safely make the assumption that *real* benefits are, only in approximate sense, covariance stationary over the time horizon of our interest. This assumption is not essential for our results, but it is helpful to intuitively explain the difference between the responses of the expectation of benefits and its volatility. Hence, retirement benefits have a $Wold$'s representation,

$$X_{t+i}^{(t)} = \zeta_{t+i} + \sum_{j=1}^{t+i} \mathbf{\Gamma}_j \boldsymbol{\epsilon}_j \tag{82}$$

Impulse responses of expected benefits basically reflect the net effect of the dynamic multipliers Γ_j, and the impact of the shock on the state variables determining ζ_{t+i}. Perturbing the fertility parameters and fund ratio shows the magnitude and direction of the net dependence on these parameters. This dependence, the part we observe through impulse responses, can take place through two channels: (1) direct channel, where the dynamic multiplier is a direct function of fertility parameters and/or the fund ratio, and (2) indirect channel, where ζ_t or Γ_t are state-dependent, and state variables are functions of the perturbed parameters and exposed to the market shock; the last condition is to make sure that the shock would trigger a change in ζ_{t+i}.

On the other hand, the response in volatility of benefits reflects only the state dependence in Γ_j, which provides a narrower focus than the above. For example, Γ_j could be a function of the asset size, which is exposed to market shocks and depends on the fund ratio and fertility parameters; if this is true, then a market shock would impact the asset size, and consequently the volatility of benefits, and changing the calibration of fund ratio and fertility parameters will change the response. Hansen et al. (2008), whose analysis we explained earlier, do not deal with this matter since they assume constant, state-independent ζ and Γ. Our use of the word 'state variables' here is in its general sense; meaning, all other variables that characterise the system besides the endogenous variable we study, namely future 'benefits', (i.e. Asset Size, Liabilities, Population size ...).

We draw various scenarios to make our analysis as conclusive as possible by perturbing the Fund ratio, and TFR, and computing the response in each case. Since the current model is highly non-linear, we consider both Generalised and Traditional impulse response analysis. Traditional responses for non-linear systems should be studied with caution since it is only true for the conditions it builds on, and should not be used to draw a general conclusion [5]. In the analysis below, the shocks

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[5]The traditional impulse response function describes the effect of a shock of a size δ at time t on the system at time $t+n$, given that no other shocks hit the system. Simply, it reflects the system response when all shocks are turned off except for the one that interests us,

$$I_Y(n, \delta, \omega_{t-1}) = E_t[Y_{t+n}|V_t = 0, V_{t+1} = 0, \ldots, V_{t+n} = 0, \omega_{t-1}]$$
$$-E_t[Y_{t+n}|V_t = 0, V_{t+1} = 0, \ldots, V_{t+n} = 0, \omega_{t-1}]$$

However, the traditional impulse response is best suited for linear systems. In case of non-linear models, I_Y is not independent of past history ω_{t-1}, and varies with the size and sign of the shock δ. In addition, I_Y reflects the system response along one future trajectory only (zero future shocks); future behaviour of non-linear systems could vary under different trajectories.

The generalised impulse response GI is constructed to solve the previous problems, Koop et al. (1996). The problem with treatment of the future is handled by computing the expectations conditional on only the history and/ or shock; future shocks are averaged out. The baseline for the impulse response function is defined as the conditional expectations based on only past history. The present is then perturbed by conditioning on a shock at the present state along with the past history.

take place at year 151, the same time our representative individual enters the system. The first 150 years are non-stochastic with all shocks switched off; this is to let the system build one line of pre-shock history w_{t-1} to condition on, which is determined based on the calibration of system parameters.

6.3.1 Response to a Market Shock

We shock the system with a *negative* shock to the Market index returns (equity) with a magnitude equals to *one* standard deviation. The shock impacts both earnings and contributions. Figure (11) shows the traditional and generalised impulse responses for expected benefits at different values of the fund ratio at the time of the shock (FR=0.5, 1.0, 1.5, and 2) for TFR = 1.95. In figure (12), we repeat the analysis at three values of the long term TFR (1.46, 1.95, and 2.65), while setting the fund ratio at a value of 1 at time of the shock.

Response of Mean Value: As explained above, what we are looking for here is to understand benefits exposure to a financial market risk, and how it depends on the fund ratio and fertility parameters. From figure (11.a), we see that the effect of fund ratio on exposure to risk depends on the time horizon; the higher the fund ratio the less exposed a beneficiary is to shocks that occur between 1 and 15 years before her retirement. In contrary, beneficiaries are more exposed to market shocks that happen 15 to 46 years before retirement, the more funded the system is. This means that the older the participant becomes, he becomes safer the more funded the system is. When participants are still young, they would face less uncertainty if they join less funded systems. Hence, not only is risk exposure a function of the fund ratio, but the elasticity of exposure to changes in the fund ratio is time dependent, $\Delta\Gamma/\Delta\frac{A_t}{L_t} = f(i)$, where $t < i < t+46$; before $t+15$, it is negative, then, it switches signs to positive after $t+15$.

GI responses reflect the same intuition, figure $(11.b)^6$. However, for fund ratios of 1 and 1.5, it

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It should be understood that GI is a random variable if past history and/ or present shock are averaged out,

$$GI_Y(n, V_t, \Omega_{t-1})$$
$$GI_Y(n, V_t, \omega_{t-1})$$
$$GI_Y(n, v_t, \Omega_{t-1})$$

where Ω_{t-1} is the distribution of all possible past histories, V_t is the distribution of all possible present shocks.
However, we are only interested in a single realisation of this random variable conditioning on single set of initial conditions and an uncorrelated shock to fertility of a size equal to one standard deviation,

$$GI_Y(n, v_t, \omega_{t-1}) = E_t[Y_{t+n}|V_t = v_t, \omega_{t-1}] - E_t[Y_{t+n}|\omega_{t-1}] \tag{83}$$

This is simply a version of the traditional impulse response where the future shocks are tuned on. It only solves the problem associated with future trajectories of non-linear models since we average out their effects.

[6] **Monte Carlo Calculation of** GI**:** In order to acquire a set of initial conditions that leave system in a state of stable growth and decay rates, we run the simulation in steady state with all variables being deterministic until year 150. This trajectory constitutes our single line of history ω_{t-1}. In order to compute $Y_{t+n}|_{t-1}$, we turn on all shocks starting year 151. We run this for 5000 times and then take expectations. For the second step, at year 151, we only

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appears that non-linearities are exaggerated; this is not the case for fund ratio =0.5 or 2.0 where the effect seen in traditional response I dominates. Exposure for individual savings accounts to market risk is lower, see figure (11.a). A worker saving in an account that only invests in long term yields is exposed to an equity shock only through the channel of her income which is correlated with the market returns.

The effect of fertility parameters does not express non-linearities, unlike the fund ratio case, figure (12.a). A participant that joins a system with a growing population suffers less losses in response to a market drop. This means that higher fertility parameters provide less exposure to market risk over all time horizons. GI is not much different, figure (12.b), although we can see that the non-linear future stochastic behaviour of the system makes the difference between a near-replacement rate population less distinguishable from a growing population. The results here follow the above findings in figure (8.a) that higher fund ratios lead to higher volatility. It is important to understand that we only talk about short run exposure, we have already established earlier that over funded systems offer higher expected benefits than less funded systems, and that long run exposure of NDC plans is less than individual alternatives.

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turn on the shock of interest with magnitude $= \sigma$. Starting year 152, all shocks are on and we let the system evolve stochastically to get an instance of $Y_{t+n}|V_t = v_t, \omega_{t-1}$. We run this 5000 times and take expectations. This enables me to compute (80).
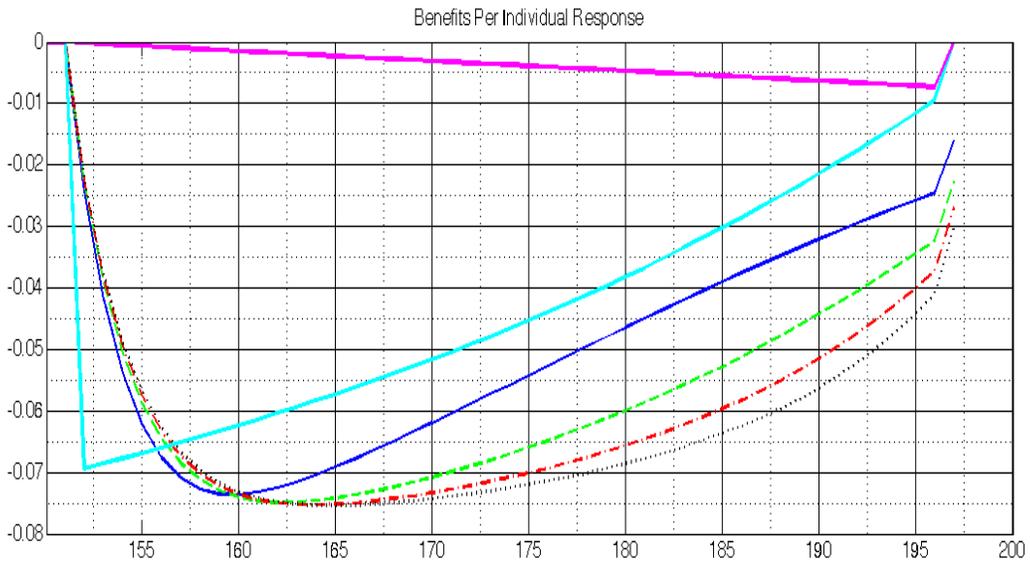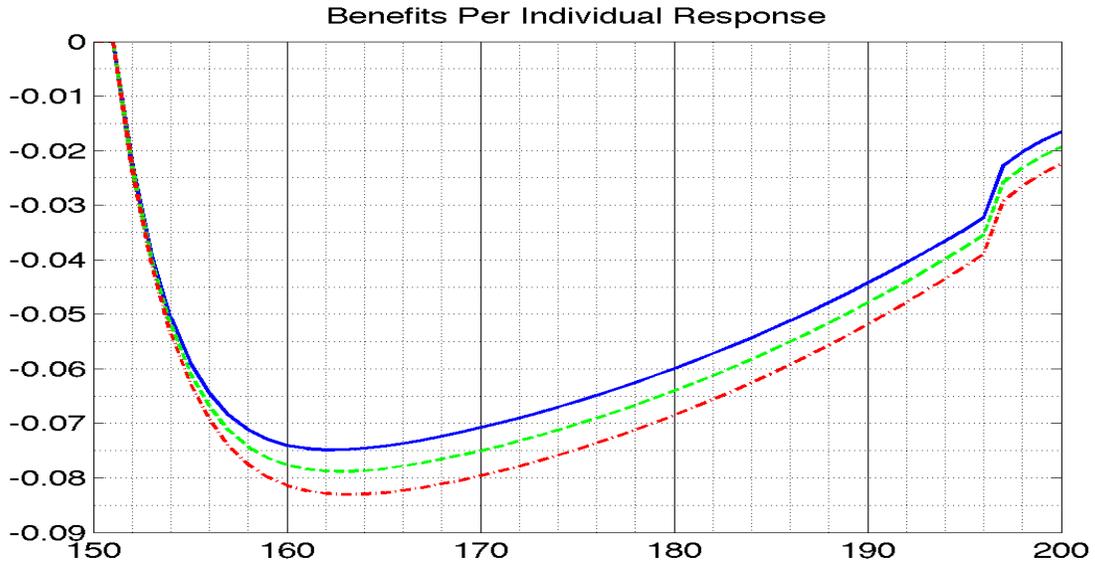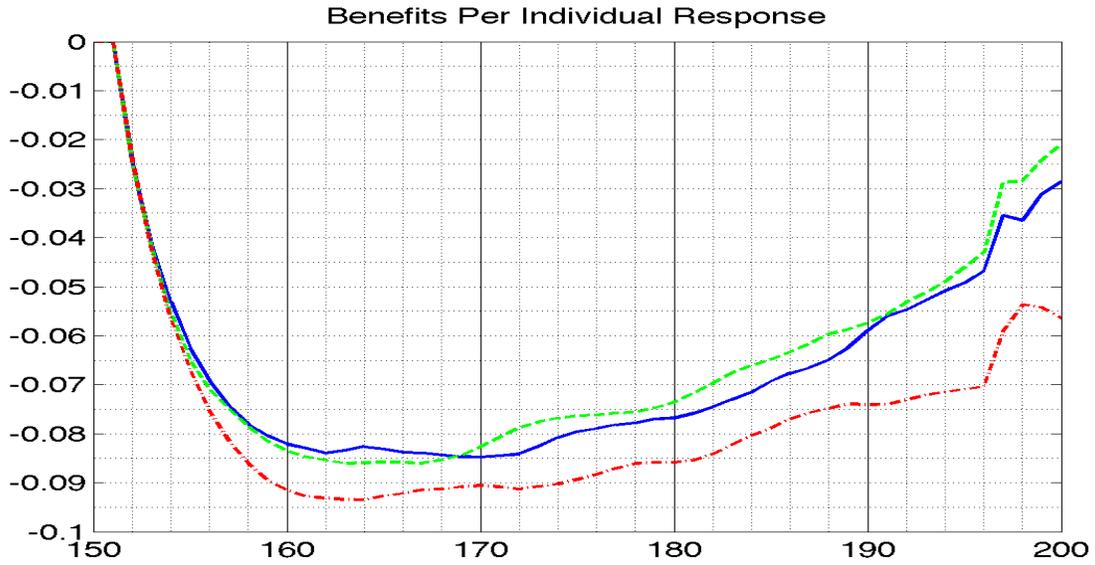
(a) Traditional Impulse Response I
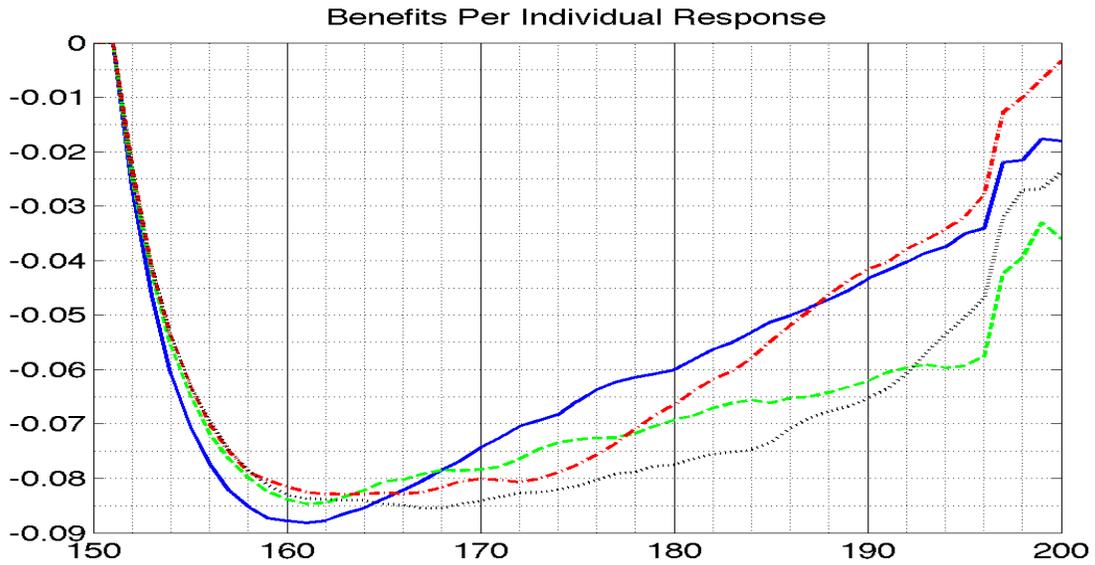
(b) Generalised Impulse Response GI. (5000 Simulations)

Figure 11: Impulse Response of expected benefits to -1 standard deviation equity shock, scaled by benefits level of the no-shock scenario at every year. (TFR=1.95) , Fund Ratio $= 0.5$ (blue, solid), 1.0 (green, dashed), 1.5 (red, -.), 2.0 (black, dotted). The extra two thicker plots in (a) are for individual savings in yields (magenta) and Rp (cyan), from top to bottom, respectively.
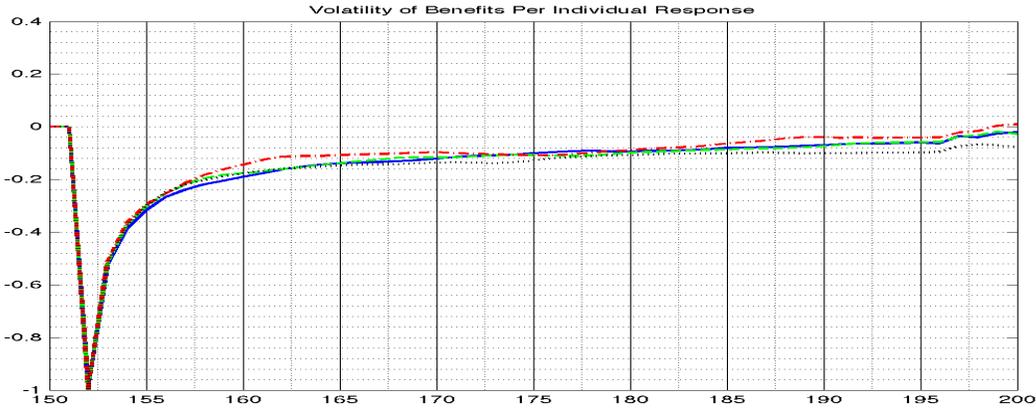
(a) Traditional Impulse Response I

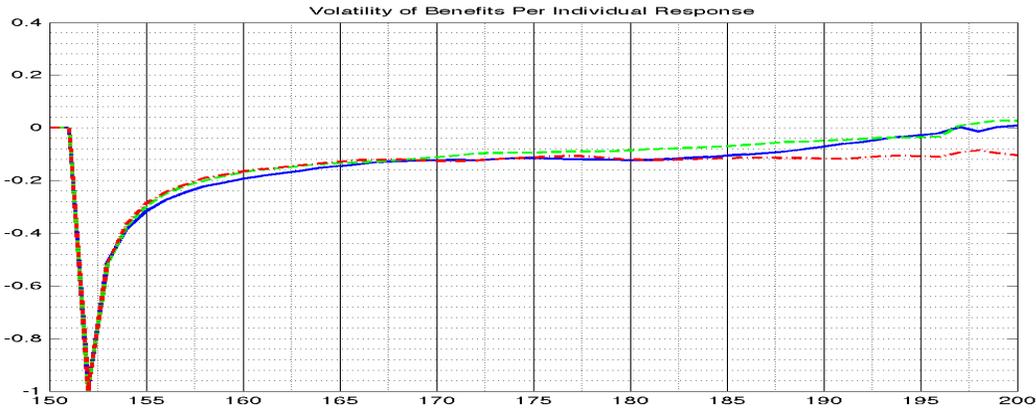(b) Generalised Impulse Response GI. (5000 Simulations)

Figure 12: Impulse Response of Expected Benefits to -1 standard deviation equity shock, scaled by benefits level of the no-shock scenario at every year. (Fund Ratio=1), TFR = 2.65 (blue, solid), 1.95 (green, dashed), 1.46 (red, -.)

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Response of Volatility: In figure (13), we show the response of retirement benefits standard deviation to the market shock at time $t = 151$, for several values of the fund ratio and fertility parameters. The first peak at year 151 is only due to the fact that we turn on all stochastic shocks starting year 151 in the base run of the simulation, where in the second impulse run, shocks are turned on starting year 152; in year 151, at the impulse run, only the market shock is on and set to -1 standard deviation. This has been explained in details in the footnote above.

We can observe a drop in volatility in response to the equity shock by around 18 to 15 %, which reflects the decline in the state variables that drive Γ_j in equation (80). Perturbing the parameters of our interest does not lead to a clear distinguishable pattern. This suggests that either (1) the state variables characterising Γ_j, and that are exposed to the market shock, are not dependent on fertility nor the fund ratio, or (2) the sample size is not big enough to provide convergence towards a clear pattern. We tend to believe more in the first option since variations over all the time horizon are very minimal.

(a) (TFR=1.95) , Fund Ratio = 0.5 (blue, solid), 1.0 (green, dashed), 1.5 (red, -.), 2.0 (black, dotted)

(b) (Fund Ratio=1), TFR = 2.65 (blue, solid), 1.95 (green, dashed), 1.46 (red, -.)

Figure 13: Generalised Impulse Response of Benefits Volatility to -1 standard deviation equity shock, scaled by benefits volatility of the no-shock scenario at every year.

6.3.2   Responses to other shocks

Although it is not the focal point of our research, we report the traditional impulse responses to shocks in (inflation, GDP growth, and income) in order to have a better comprehension of the model dynamics, see figure (14). Responses to real rate and expected inflation shocks are almost negligible. The overall picture is that NDC benefits are better hedged against inflation, GDP growth and income shocks than individual alternatives. The same result from the above analysis holds; that is, systems with lower fund ratio suffer less exposure to shocks. Finally, fertility shocks require long time horizon in order to characterise their full effect; however, we do not believe that a fertility shock has a strong interpretation since population usually grows in a slow and predictable manner.

6.4   Time Varying Risk Prices

Thus far, we have presented all results using the static term structure model with constant factors volatility and constant real rate and inflation risk prices. We have already discussed above that dynamic models that feature time varying risk prices provide a better fit for the observed term structure data; yet, the question here is whether it is of added value for a practitioner to use a more sophisticated model for valuation of the NDC model. We compare the basic results on benefits, returns and volatility with the earlier static model counterparts.

Figure (15) shows that there are no noticeable differences when it comes to expected future benefits nor volatility of returns; however, expected returns in figure 14.d (2.3 to 8.5% constant annual compounding) are much higher than their counterparts of the static model (0.8 to 6% constant annual compounding), and they are much more sensitive to variations in fund ratios and fertility parameters. Since we cannot distinguish a pattern, we can only comment by noting that in a market where information about population growth is not accurate, or reports about the state of the fund are less transparent, dynamic models would give less robust results.
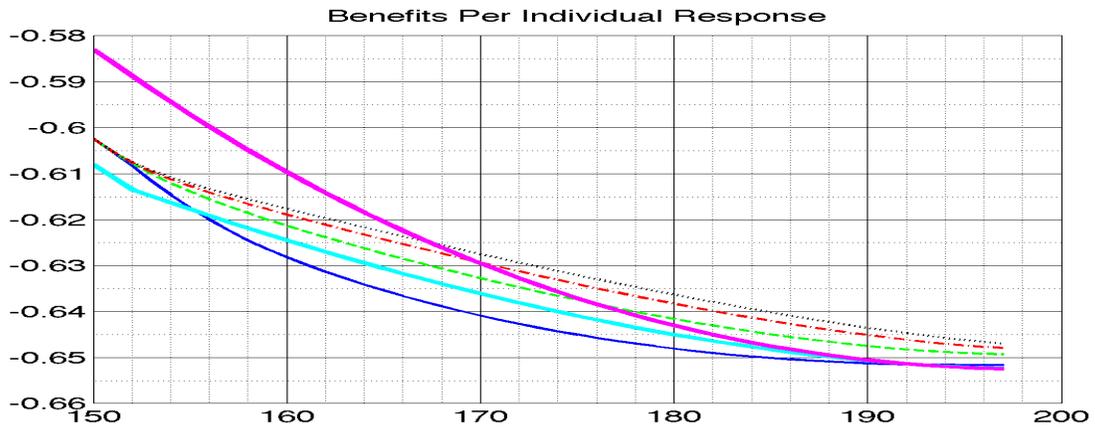
(a) Inflation Shock
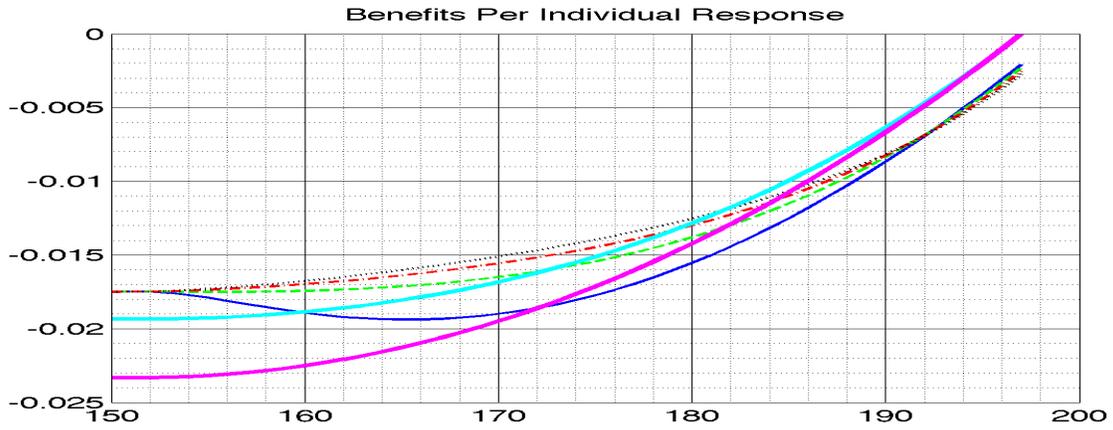
(b) GDP Growth Shock
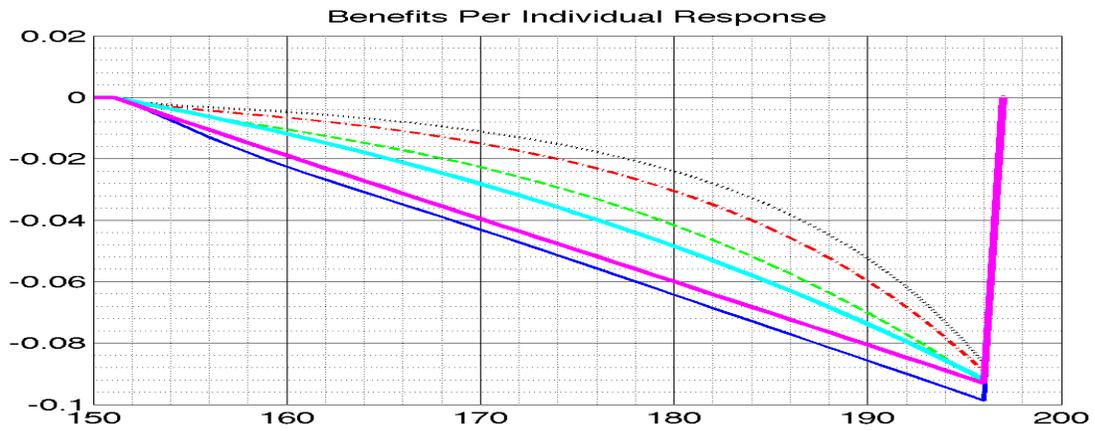
(c) Income Shock

Figure 14: Traditional Impulse Response of Benefits Expected Values to −1 standard deviation shocks, scaled by benefits values of the no-shock scenario at every year. (TFR=1.95) , Fund Ratio = 0.5 (blue, solid), 1.0 (green, dashed), 1.5 (red, -.), 2.0 (black, dotted)
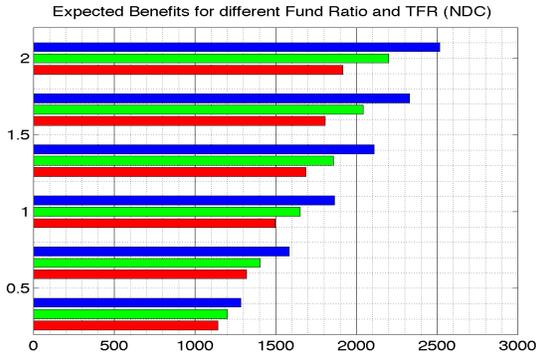
(a) NDC Plans
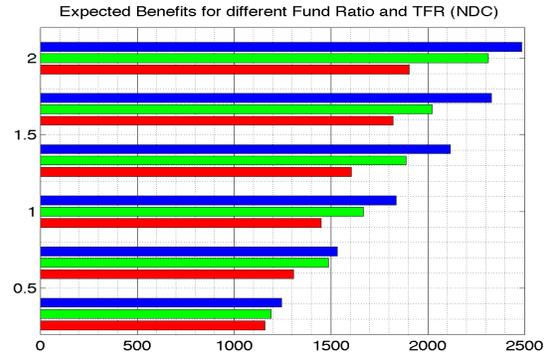
(b) NDC Plans
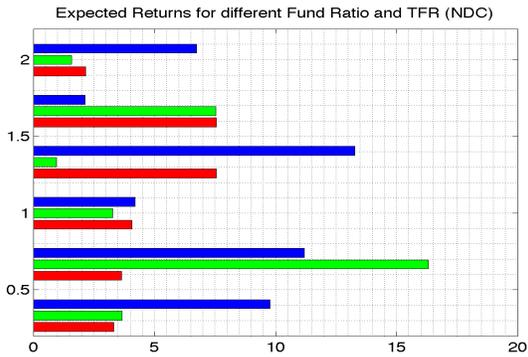
(c) NDC Plans
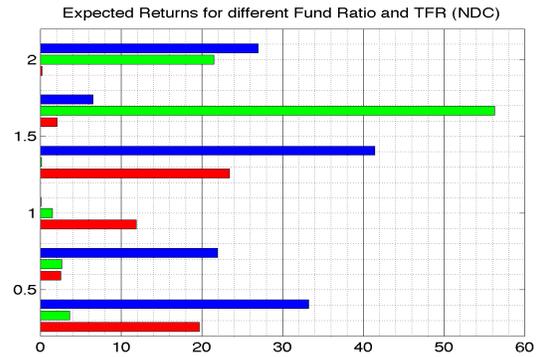
(d) NDC Plans
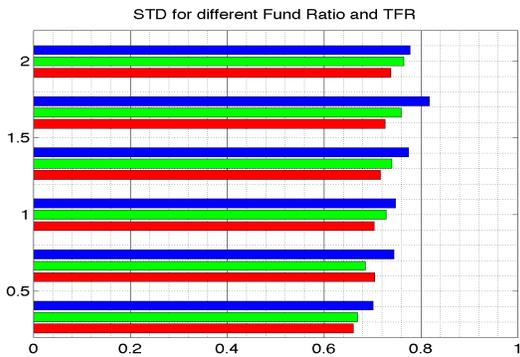
(e) NDC Plans
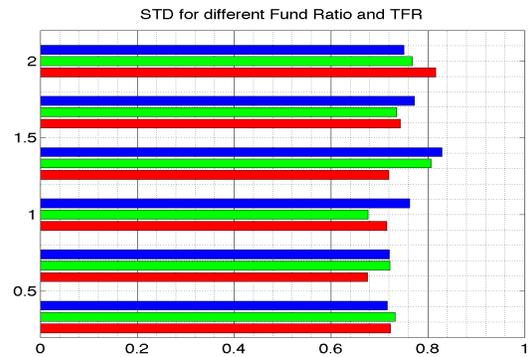
(f) NDC Plans

Figure 15: benefits, returns and volatility for different values of TFR. {Top bar: TFR=2.65, Middle bar: TFR=1.95, Bottom bar: TFR=1.46 } (Left Column: Static Model, Right Column: Dynamic Model) (1000 Simulations)

7 Conclusion

For a young worker of age 21 looking to participate in a pension plan, our findings show that subscribing into a NDC program will have more present time value compared with an individual savings account. For a population growing near the replacement rate (long run TFR=1.95), if she joins a just funded system, she could expect 300% return over the 46 years horizon; this amounts to almost 2.3% compounded annual return. The cost of this retirement benefit is paid for through annual contributions and a front payment at the time of subscription, which represents the present time value of benefits net contributions. This way, the NDC retirement package is a fair deal. Individual savings accounts promise higher returns (3.5 % for investment in fund portfolio, and 5.4% for investment in long term nominal yields), but it is associated with higher long run exposure to risk as well. Although fluctuation in NDC benefits are higher, their growth potential still makes a NDC package more valuable with a lower long run exposure.

For an insurance firm, if information related to population growth, or the fund status is not transparent enough, dynamic models should not be relied on for pricing NDC plans. Our findings show that results from dynamic term structure models are less robust in such cases.

It is also important to note that our model tells a prospective participant the expected benefits value, and the fair price for the package; however, we do not provide her with the expected volatility. Since we use a No Arbitrage framework, she should be able to trust that the *Sharpe* ratio of her investment is maximised, and that the price she pays encodes all relevant information. The model can be extended to characterise the distribution of the second moment of returns, but the computational time needed to complete the simulation would exponentially rise in this case; therefore, we leave this for future research.

Should a prospective participant care about long run TFR and Fund Ratio?

A young worker can expect to earn higher benefits upon retirement if he holds his NDC account in a country with a growing population. In addition, he could feel safer about his benefits since exposure to market risk would significantly decline compared with decaying populations. However, in a market consistent valuation, the price ensures a fair value; this means that in this case of higher fertility, subscribing into a NDC plan would be more expensive. In other words, in two different countries, with different demographic rates, where fair valuation is implemented, NDC plans will be traded at different prices that ensure no arbitrage.

Same story goes for the fund ratio with few remarks. Signing up for a system with a higher fund ratio promises lower long run exposure to market risk with higher expected benefits. Although short run exposure to market risk rises the higher the fund ratio is, long run exposure is lower the higher the fund ratio is due to the higher growth potential of overfunded systems; nonetheless, a just funded system has the advantage of being least sensitive to fertility parameters. This means that in a country where information about population growth is not transparent enough, a just

funded system would ensure less fluctuation of future benefits.

The key point to understand is that the prices we calculated reflect all these information. Therefore, under our pricing, and among the different scenarios we reported, there is no smarter investment than the other; all different scenarios fall on the efficient market frontier, but at different locations depending on their expected returns and *long* run exposure to risk.

Model Calibration

Description	Symbol	Value
State Factors		
Real Rate Unconditional Mean	$\bar{r}$	0.02
Real Rate Persistence	ϕ_r	0.869
Real Rate Volatility	σ_r	0.011
Real Rate Const. Intercept (DTS)	δ_0	0.008
Risk Prices Factor Unconditional Mean	$\bar{q}$	0
Risk Prices Factor Persistence	ϕ_q	0.975
Risk Prices Factor Volatility	σ_q	0.017
Exp Inflation Unconditional Mean	$\bar{\mu}_\pi$	0.02
Exp Inflation Persistence	ϕ_{μ_π}	0.9992
Exp Inflation Volatility	σ_{μ_π}	0.0013
Exp Inflation Correlation with Real Rate	$\beta_{\mu_\pi,r}$	0.0779
Inflation Correlation with Real Rate	$\beta_{\pi,r}$	0.5163
Inflation Correlation with Exp Inflation	β_{π,μ_π}	1.4518
Inflation Volatility	σ_π	0.007
Equity Unconditional Mean	$\bar{R}_{eq}$	0.08
Equity Persistence	ϕ_{eq}	0
Equity Volatility	σ_{eq}	0.15
Equity Correlation with Real Rate	$\beta_{eq,r}$	-3.6366
Yield Curve		
Excess Returns on One Period Nominal Yields	$r^{(1)}$	0.003
Excess Returns on 46 years Nominal Yields	$r^{(46)}$	0.012
Risk Prices (DTS)		
Real Rate Risk Price Const.	$\lambda_{0,f}$	-0.4
Real Rate Risk Price Correlation with Factor q	$\lambda_{1,f}$	-10
Inflation Risk Price Const.	$\lambda_{0,\pi}$	-0.05
Inflation Price Correlation with Factor q	$\lambda_{1,\pi}$	-10

Description	Symbol	Value
GDP Growth		
GDP Growth Unconditional Mean	δ_{gdp}	0.02
GDP Growth Correlation with Yield Spread	ϕ_{gdp}	0.65
GDP Growth Volatility	σ_{gdp}	0.021
Pension Model		
Return on Notional Accounts Const.	i^*	0.02
Fund Ratio Coefficient	α	0.3
Fund Portfolio Weight in Equity Investment	w_{eq}	0.5
Fund Portfolio Weight in 46 years nominal yields Investment	$w_{y^{(46)}}$	0.2
Income		
Initial Minimum Income	$-$	1
Initial Maximum Income	$-$	3
Income Growth Rate Const.	g_Y	0.03
Income Growth Correlation with Equity	$\beta_{Y,eq}$	0.05
Income Growth Correlation with GDP Growth	$\beta_{Y,gdp}$	0.05
Contributions Percentage of Income	θ	0.05
Population		
Number of Living Cohorts	N	111
Age of First Fertile Female	α_{fert}	15
Age of Last Fertile Female	β_{fert}	44
Age of Retirement	$-$	67
Age of First Participation	$-$	21
Initial Population (1 - 15) years	$-$	15%
Initial Population (16 - 64) years	$-$	50%
Initial Population (65 - 111) years	$-$	35%
Immigrants	$-$	0
Fertility Index Persistence	c	0.9676
Fertility Index Shock Persistence	d	0.47978
Fertility Index Volatility	σ_{fert}	0.1106
Mortality Index Serial Correlation	$-$	0.99
Mortality Index Const.	z	0.365
Mortality Index Volatility	σ_{mort}	0.651
Female Birth Ratio	$-$	1.05
Age Specific Fertility Rates	a_x	Lee and Tuljapurkar (1994)
Age Specific Mortality Rates	a_x	Lee and Tuljapurkar (1994)

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