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Pricing Funeral Insurance Contracts Using Option Pricing Techniques and Interest Rate Models

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Imagination is more important than knowledge

ALBERT EINSTEIN

Contents

Abstract

The aim of this paper is to develop an analytical method which is able to price individual insurance contracts. The introduction of Solvency II entailed that insurance companies had to determine the market value of their liabilities. This opened doors for researchers in the section of market consistent valuation. In this paper we will determine the value of a funeral insurance contract with cumulative profit sharing. We will do this with the help of interest rate models and several option pricing techniques. We will determine an upper bound and a lower bound for the price of the insurance contract. Through a Monte Carlo simulation we will show that the bounds give an accurate approximation of the price

1 Introduction

The introduction of Solvency II by the European Union at the end of 2012 caused a landslide in the world of insurance companies. The goal of Solvency II was to improve solvency and risk management of insurance companies to give more financial security to their customers. After having worked with Solvency I for more than 40 years, insurance companies have to adapt in a small time space to Solvency II. The increasing impact of market-, credit-, and operational risk together with the usage of complex derivatives and globalization asked for a new framework. The biggest difference between Solvency I and Solvency II is that insurance companies now have to take into account a broader spectrum of risks for their capital requirements. From now on the insurance companies have to take a look at their balance from an economic point of view. Due to these changes the calculation of the solvency will become more complex. According to Solvency II liabilities have to be in the books with their real market value. Next to that it has also an impact on the amount of buffer capital an insurance company has to hold. Solvency II states that the amount of capital which is required depends on the amount of risk the insurance company takes, so investing in risky products means that your capital requirements will increase. The regulator is forcing the insurance companies to look at risk and return instead of return only. Insurance companies are not clamouring to increase their minimal capital requirement to have enough buffer. Increasing your level of buffer capital is expensive and has an impact on the liquidity position of the insurer. Capital which is used as reserve cannot be used for other activities. As a result of these new rules, we see that insurance companies are shifting their investment strategies to different fields of the market. Instead of investing in stocks they are now investing in "risk free" bonds. The EU gives insurance companies two options, they can either use the standardized model described in Solvency II or use their own model to calculate their minimal capital requirements. The insurance companies prefer using an own model, as they can choose the model (approved by the controller) which minimizes their amount of buffer capital. By having less buffer capital you are creating a competitive advantage towards other players in the market. Next to the changes in the model Solvency II requires some organizational changes from insurance companies. The regulator demands more data and a more frequent reporting of these data. Insurance companies are realizing that Solvency II will have some serious strategic consequences. The transformation to Solvency II creates golden opportunities for Econometricians to develop new (tailored made) models and to tackle the problem of market valuation.

The usual suspect of market valuation is making a replicating portfolio. To valuate the liabilities of for example an insurance company on market value you create a portfolio containing a mix of financial instruments which matches with the cash flows of your liabilities. The market value of the liabilities is than equal to the value of the replicating portfolio. The flaw of this replicating portfolio is that you do not know which combination of financial instruments is linked to a specific insurance contract in the insurer's portfolio. Knowing the market value of a specific insurance contract can be very valuable for insurance companies in the near future. This can be used in calculating the premium and the profitability of such an insurance contract. Next to that we can give a better picture of the risk you are taking as an insurance company in selling a specific contract. Pelsser and Bouwknegt [2002] wrote a paper about the market value of an insurance contract with profit sharing. They looked at a simplified form of an insurance contract with profit sharing. The profit sharing was paid out in cash each year and was not cumulative. In this paper I will have a look at an insurance contract with cumulative profit sharing.

This paper will describe a method to calculate the market value for a specific insurance contract by using option pricing techniques and interest rate models. First, we will introduce and describe the Hull and Whit model as this will be the interest model we will be using. In the second section we will describe Bachelier's option pricing formula and compare it with the Black & Scholes formula. After that we will describe how the return on investment of the insurance company is calculated in my model. We will introduce the random variable ICIR and how it is calculated in a deterministic environment. In the fifth section we will show how the ICIR is calculated in a stochastic environment. The subsequent section will describe how the Bachelier's option pricing formula can be used in determining the value of an insurance contract. Section seven will describe how to deal with cumulative profit sharing with the help of Asian option pricing techniques. In this section we will determine a lower bound for the price of the insurance contract. In the subsequent section we will describe a method to find an upper bound using comonotonicity. Finally we will check the accuracies of the bounds with a Monte Carlo simulation.

2 Hull and White model

To model the interest rate I have chosen to use a one factor Hull and White model which is an extension of the well-known Vasicek model. Hull and White [1990] noted that the Vasicek model implies a poor fitting of the initial term structure of interest rates. The Hull and White model gives a realistic description of the behaviour of interest rates. The mean reversion in the short rate dynamics makes the Hull and White model a more realistic model than other interest rate models like for example the Ho and Lee model [Brigo and Mercurio, 2007]. The mean reversion property of the Hull and White model is also seen in the behaviour of interest rates in the past. There are very strong arguments on the basis of economic theory for the mean-reversion of interest rates. As stressed by Pelsser [2000] the Hull and White model has a rich analytical structure. The Hull and White model is a normal model where the spot interest rate is a quadratic or linear function of the underlying process which gives it normally distributed fundamental solutions. A downside of using the Hull and White model, just like in the Vasicek model, is the possibilty of negative interest rates. Notwithstanding this the Hull and White model is historically one of the most used interest rate models [Brigo and Mercurio, 2007].

Hull and White assume that under the Q-measure the short rate follows the following process.

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma(t)dW_t \tag{1}$$

Where $\theta(t)$ is an arbitrary function of time and a and σ are positive constants [Brigo and Mercurio, 2007]. The term $ar(t)$ is the mean reversion term. In Brigo and Mercurio [2007] it is shown that (1) can be integrated to:

$$r(t) = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}dW_u$$

Where $\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$. $f^M(0,T)$ is the market instantaneous forward rate at time 0 for the maturity T and is equal to $-\frac{\partial lnP^M(0,T)}{\partial T}$, where $P^M(0,T)$ is the market discount factor for the maturity T. It can now be derived that $r(t)$ conditional on $\mathcal{F}_s$ is normally distributed with mean and variance given by

$$\mathrm{E}[r(t)|\mathcal{F}_s] = r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)}$$

$$Var[r(t)|\mathcal{F}_s] = \frac{\sigma^2}{2a^2}(1 - e^{-2a(t-s)})$$

To make life easier I have chosen a variation of the Hull and White model introduced by Pelsser [2000]. In this ”new” model the interest rate is generated by the following process I:

$$dI(t) = -aI(t)dt + \sigma dW_t$$

Where $I(0) = 0$ and $r(t) = I(t) + \alpha(t)$. The advantage of this transformation is that the stochastic process I(t) is now independent of the function $\alpha(t)$ and only depends on the parameters a and σ. The transformed model is easier to analyse than (1). I(t) follows now an Ornstein-Uhlenbeck process. An Ornstein-Uhlenbeck process has very nice properties; one of these is the tendency to drift to its long term mean (mean reverting). It is maybe nice to notice that the Vasicek model is also an Ornstein-Uhlenbeck process. Under the Q-measure I(t) is normally distributed with mean $I(s)e^{-a(t-s)}$ and variance $\frac{\sigma^2}{2a^2}(1 - e^{-2a(t-s)})$ for a given value I(s). Note that the variance of the process I(t) is the same as the variance of the process r(t). With the Hull and White model we can calculate the price of a discount bond with maturity T at time t by

$$D(t,T) = e^{A(t,T)-B(t,T)I(t)} \tag{2}$$

Where

$$A(t,T) = \frac{\sigma^2}{2a^3}\Big(a(T-t) - 2(1-e^{-a(T-t)}) + \frac{1}{2}(1-e^{-2a(T-t)})\Big) - \int_t^T \alpha(t)ds$$

$$B(t,T) = \frac{1}{a}(1-e^{a(T-t)})$$

with

$$\int_t^T \alpha(u)du = -\log\frac{D(0,T)}{D(0,t)} + \frac{\sigma^2}{2a^3}\Big(a(T-t) - 2(e^{-at} - e^{-aT} + \frac{1}{2}(e^{-2at} - e^{-2aT})\Big)$$

As we want to calculate options on pure discount bonds we have to use the T-forward-risk-adjusted measure. Pelsser [2000] shows that the distribution of I(t) under the T-forward-risk-adjusted measure can be calculated by using the Fourier transform.

$$\mathrm{E}^T[I(t)|\mathcal{F}_s] = I(s)e^{-a(t-s)} - \frac{\sigma^2}{a^2}(1-e^{-a(t-s)}) + \frac{\sigma^2}{2a^2}(e^{-a(T-t)} - e^{-a(T+t-2s)}) \tag{3}$$

$$Var^T[I(t)|\mathcal{F}_s] = \frac{\sigma^2}{2a^2}(1-e^{-2a(t-s)}) \tag{4}$$

Note that only the mean has changed due to the change of measure. In appendix A you can see how we derived this correction term for the mean. With the discount factors D(t,T) calculated with the Hull and White model it is very easy to determine the zero rate with the following relationship

$$Z_{t,T} = -\frac{\log D(t,T)}{T-t} \tag{5}$$

Note that it is possible to make a term structure for every t with the Hull and White model.

3 Bachelier

When people are talking about option pricing the first thing which pops up in their minds is the Black-Scholes model. Black and Scholes published their model in 1973 in their paper called "The Pricing of Options and Corporate Liabilities". Many people see Black % Scholes as the pioneers of the option pricing as their model led to a boom in options trading. Many empirical tests have shown that the price calculated by the Black & Scholes model is close to the observed prices. The model was a scientific proof for the day to day business at options markets around the world. What not many people know is that in the absolute beginning of the 20th century somebody wrote a paper about option pricing. Louis Jean-Baptiste Alphonsse Bachelier wrote in 1900 his PHD thesis "Théorie de la Spéculation". Bachelier was the first one to model Brownian motion. He developed the mathematical theory of Brownian motion and linked this to stock prices. He observed that the stock price movements move the same as the motion of small particles suspended in liquid. This assumption was the bases of finding pricing formulas for put and call options. It is maybe hard to believe that the innovative ideas of Bachelier where neglected for more than 50 years. One of the things Bachelier discovered was the distribution of the already above mentioned Ornstein-Uhlenbeck process. [Magdziarz et al., 2011]. Bachelier assumed that the stock price would move according to an arithmetic Brownian motion. Paul Samuelson introduced the idea that stock prices would follow a Geometric Brownian Motion (GBM) based on the theories of Bachelier. Black & Scholes used this GBM to come up with their famous option pricing formulas.

There are some differences and similarities in the approaches of Bachelier on the one hand and Black & Scholes on the other hand. The quotation "The mathematical expectation of a speculator is zero" of Bachelier is the underlying principle of Bachelier's option pricing formulas. He based this on the efficient market hypothesis. The efficient market hypothesis is the use of martingales to describe the stochasticity over time of price movements in ideal markets. Bachelier does not make use of the risk free interest rate in his pricing formulas. This might seem strange for the people who are used to work with Black & Scholes formulas as by no arbitrage arguments the discounted price process should be a martingale under the risk neutral measure. Bachelier applied his formula on forward prices. In the Black & Scholes formula risk free interest rate disappear in the case that forward prices are used. In both the Bachelier and Black & Scholes formula the price process of the underlying security is a martingale under a certain probabilty measure. Although the main underlying ideas are the same for both pricing formulas the underlying arguments are different. Bachelier uses an equilibrium argument and Black & Scholes use the no arbitrage arguments [Schachermayer and Teichmann, 2008].

The reason for choosing Bachelier's option pricing formula is the underlying model. Bachelier's model for the underlying security is given by

$$S_t = S_0 + S_0 \sigma dW_t$$

and the Black & Scholes model for the underlying security is given by

$$S_t = S_0 e^{(\sigma W t - \frac{\sigma^2}{2} t)}$$

You can see that in Bachelier's model S_t is normally distributed and in the Black & Scholes S_t is log-normally distributed. It can be shown with the standard martingale method that the pricing formula for a call option in the Bachelier model is the following formula.

$$
\begin{aligned}
C &= \mathrm{E}[(S_t - K)_+] \\
&= (S_0 - K)N(\frac{S_0 - K}{\sigma\sqrt{t}}) + \sigma\sqrt{t}n(\frac{S_0 - K}{\sigma\sqrt{t}})
\end{aligned}
$$

where the underlying security is S with maturity date t and strike price K. N is in this formula the CDF and n the PDF of a standard normal distribution. Notice that S_0 is the mean of S_t [Schachermayer and Teichmann, 2008]. The underlying security I will use in the rest of my paper has a normal distribution. For this reason I have chosen to use Bachelier's option pricing formula.

4 Deterministic Insurance Company Investment Return (ICIR)

This section will describe how the return on investment of the insurance company is calculated in my model. First of all I want to introduce the random variable ICIR which stands for Insurance Company Investment Return. As already mentioned in my introduction insurance companies have diverged from the path of investing in risky financial products to the path of investing in "risk free" bonds (government bonds). For this reason I assumed that insurance companies only invest in zero-coupon and coupon bearing bonds. First of all I want to make a side note about the risk you are taking by investing in "risk free" bonds. Risk free indicates that you will receive interest payments and the principal at time of maturity with almost absolute certainty. So if you are talking about credit risk you could actually state that a government bond is risk free. But in the case that you are looking at other types of risk a government bond is not totally risk free. A government bond is exposed to interest rate risk as the price of a bond is determined by the level of interest rate. It is maybe nice to note that a stock is less affected by interest rate risk than bonds. A fall of the interest rate level will decrease bond prices and vice versa. The relation between bonds and interest rates makes intuitively sense as a fixed interest payment of for example 3,5% has a higher value if the level of interest rate decreases. Interest rate risk should not be a threshold for insurance companies to invest in bonds. As soon as an insurance company bought a bond they will keep it until maturity and receive all fixed cash flows. They are not interested what happens with the interest rate in-between as this does not affect their cash flows and they do not sell their bonds before maturity.

After our detour I am now coming back to the determination of the ICIR. The first thing we will do is have a look at the input. The term structure of interest I have used is the ECB AAA curve. The curve is generated by using the Nelson-Siegel-Svensson method [Svensson, 1994]. This curve gives the zero spot rates (annually compounded) for each month for a time span of 100 years.

With this curve I determined the discount factors and the forward rates. The forward rates are the one month forward rates per annum determined in the following way

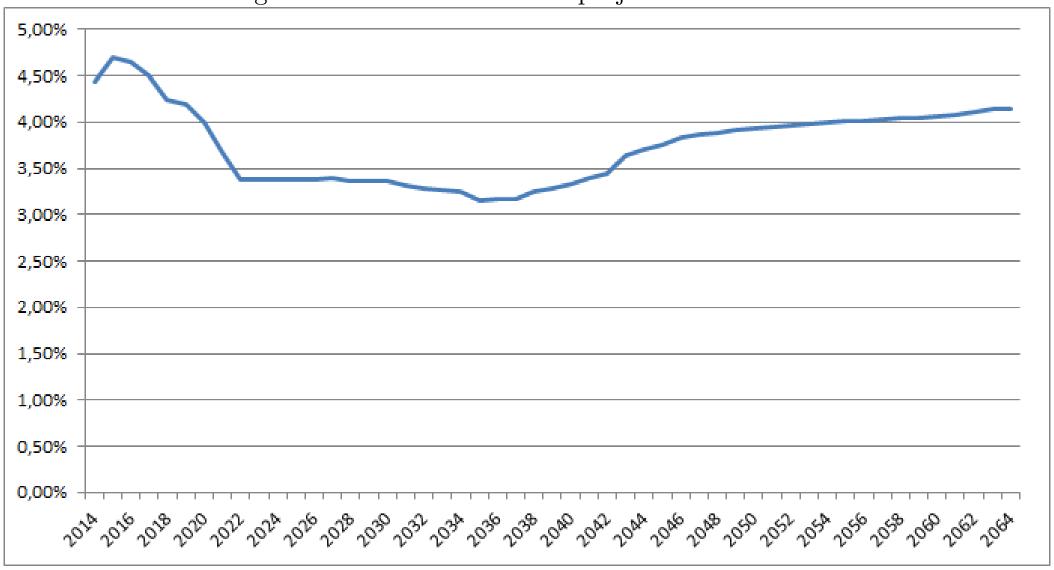
$$f(t,t+\frac{1}{12}) = \Big(\frac{D(0,t)}{D(0,t+\frac{1}{12})}\Big)^{12} - 1$$

Next to the term structure of interest I have chosen a portfolio of 35 government bonds both coupon and zero-coupon bonds with maturities varying from 14 years up to 46 years. All these bonds are real world bonds which can be found at Bloomberg. The portfolio I have chosen is based on a portfolio of a real insurance company of which I do not want to release the name. The actual total amount invested in the bonds is of minor importance as we will be looking at relative returns. The amount invested in each bond was chosen arbitrarily as the correctness of the model is not affected by the investment positions. After having discussed the input we will now have a look at the actual determination of the ICIR. Before we can actually calculate the ICIR I want to introduce the random variable Monthly Insurance Company Investment Return (MICIR). The MICIR is calculated for each month and expressed in per annum rates for the upcoming years. For this paper I have chosen February 2014 as the first month and a time span of 50 years. The MICIR is calculated with the following formula

$$\text{MICIR}_t = \Big(1 + \frac{CP_t + M_t + X_t + Y_t(1+f(t,t+\frac{1}{12}))^{\frac{1}{12}} - 1)}{U_t}\Big)^{12} - 1$$

Where CP_t are the sum of the coupon payments of the all the coupon bearing bonds for month t. M_t is the sum of the "depreciation" of the bonds determined by the accountant at time t. The "depreciation" of a zero-coupon bond is positive as it calculates the part "you get" each month of the principal you will receive at maturity. For a coupon bearing bond the "depreciation" is either positive or negative, this depends on whether the book value is higher or lower than the principal. X_t can be any kind of released realised profits determined by the insurance company at time t. I have chosen to use released realised profits of a real insurance company to get a realistic picture of this variable. Yt is the sum of all face values of all the bonds which have reached maturity before or at time t. U_t is the sum of the book values of all the bonds at time t and the sum of all face values of all the bonds which have reached maturity before or at time t. $f(t,t+\frac{1}{12})$ is the forward rate as specified above. I assume that the sum of the face values (i.e. Yt) is reinvested each month in an account where you will receive the zero rate for that month, in this case the one month forward rate. To get from the MICIR to the ICIR we take the average of all the MICIR in one year (Note that for the first year this will be an average of only 11 months). The ICIR of 2015 are the returns which are made in 2014, the ICIR of 2016 are the returns which are made in 2015, etc. You have maybe noticed that in this stage the MICIR and the ICIR are deterministic which implies of course that they are not stochastic. In Figure 1 you can see the Deterministic ICIR projection. It is maybe nice to highlight that at the end the ICIR tends to go to the Ultimate Forward Rate (UFR). This makes of course absolute sense as after the last maturity date of the bonds, all the money is reinvested in an account where you will receive the forward rate which is after 50 years close to the UFR. The next step is to make the MICIR and so the ISIR a random variable.

Figure 1: Deterministic ICIR projection



5 Stochastic ICIR

In the previous section we have discussed how the MICIR is determined. In this section we will make the MICIR stochastic by replacing one of the deterministic terms in the formula with a stochastic term. The only candidate we can replace with a stochastic term is the forward rate as all the other variables are deterministic for a given portfolio of bonds. In the stochastic ICIR model I replaced the forward rate by the zero rates determined by the Hull and White model denoted by $Z_{t,T}$. I have assumed that in the Hull and White model the constant $a = 0,03$ and $\sigma = 0,01$. This gives the following formula for the stochastic MICIR

$$\text{MICIR}_t = \Big(1 + \frac{CP_t + M_t + X_t + Y_t(1 + Z_{t,T})^{\frac{1}{12}} - 1)}{U_t}\Big)^{12} - 1 \qquad (6)$$

It is easy to show that $Z_{t,T}$ is normally distributed. Recall from the section about the Hull and White model equations (2) and (5). By combining these equations we get the following result

$$
\begin{aligned}
Z_{t,T} &= -\frac{ln(e^{A(t,T)-B(t,T)I(t)})}{T-t} \\
&= -\frac{A(t,T)}{T-t} + \frac{B(t,T)}{T-t} I(t)
\end{aligned} \tag{7}
$$

We see that $Z_{t,T}$ is an affine transformation of the normally distributed random variable $I(t)$. Casella and Berger [2002] state that if we apply an affine transformation on a random variable which has a normal distribution, than the affine transformation is also normally distributed. Knowing this we can find the mean and the variance for $Z_{t.T}$. With (3) and (4) as the mean and the variance of I(t) we find that the mean and the variance of $Z_{t.T}$ are given by

$$\mathrm{E}^T[Z_{t,T}] = -\frac{A(t,T)}{T-t} + \frac{B(t,T)}{T-t}\mu_I$$

$$Var^T[Z_{t,T}] = \frac{B(t,T)}{T-t}^2 \sigma_I^2$$

Where $\mu_I = (3)$ and $\sigma_I^2 = (4)$ By looking at equation (6) we see that the random variable MICIR is not an affine transformation of a normally distributed variable. This could make things complicated, but luckily there is a solution for this problem. We can use a Taylor expansion to make MICIR an affine transformation of $Z_{t,T}$. By applying a 1st order Taylor expansion around the point 0 we obtain

$$\mathrm{MICIR}_t \approx (1+P_t)^{12} + (1+P_t)^{11} Q_t Z_{t,T}) - 1$$

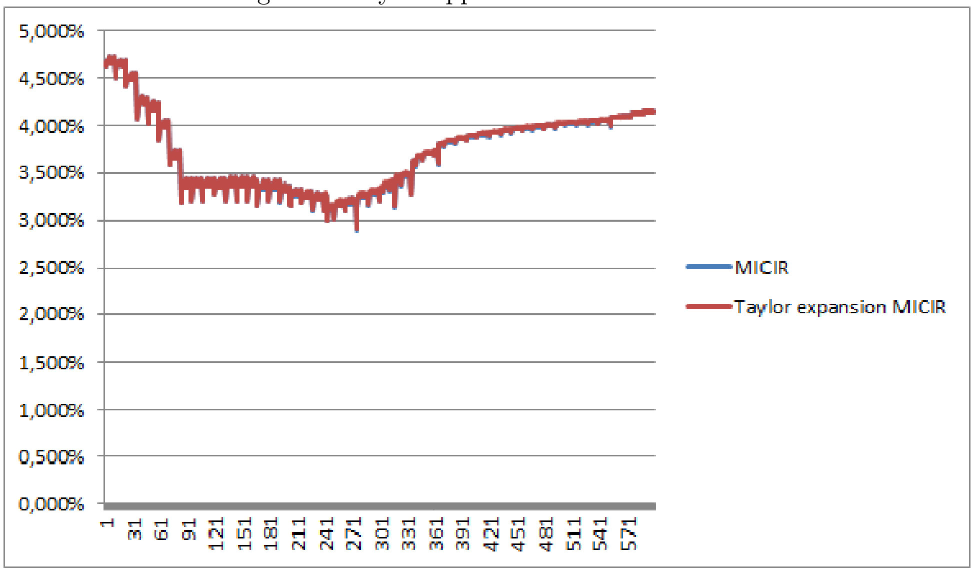
where $P_t = \frac{CP_t+M_t+X_t}{U_t}$ and $Q_t = \frac{Y_t}{U_t}$. We can check the goodness of the approximation by comparing the values of the Taylor expansion of the MICIR with the original values of the MICIR. In Figure 2 you can see that the values of the MICIR and its Taylor expansion coincide almost perfectly. The biggest deviation is 0,00015 which makes the Taylor expansion a really good approximation of the MICIR. For the rest of this paper, MICIR refers to the Taylor expansion of the original MICIR function. The nice thing about this Taylor expansion is that we now have an affine transformation of a normally distributed random variable. We now have all the ingredients to determine the distribution of MICIR. MICIR is normally distributed with the following mean and variance

$$\mathrm{E}^T[\mathrm{MICIR}_t] = K_t - \frac{A(t,T)L_t}{T-t} + \frac{B(t,T)L_t}{T-t}\mu_I$$

$$Var^T[\mathrm{MICIR}_t] = \frac{B(t,T)L_t}{T-t}^2 \sigma_I^2$$

With $K_t = (1+P_t)^{12} - 1$ and $L_(t) = (1+P_t)^{11} Q_t$. Obviously, our next step is to find the distribution of the ICIR. As we now know the distribution of all

Figure 2: Taylor approximation MICIR



individual MICIR for each month of a specific year we are able to determine the distribution of the ICIR. The ICIR for a specific year y is determined by the following formula

$$\text{ICIR}_y = \frac{\sum_{i=y-1}^{y} \text{MICIR}_i}{12}$$

with steps of $\frac{1}{12}$ in the sum. This formula holds for all years accept the first year. (Note that for the first year we will only have 11 months and therefore we will have an average of only 11 elements). From Casella and Berger [2002] we know that the sum of correlated normally distributed variables is also normally distributed. Using this proposition and the mean and the variance for a specific MICIR we get the following result. ICIR for a specific year y is normally distributed with the following mean and variance

$$\text{E}^T[\text{ICIR}_y] = \frac{\sum_i \mu_{\text{MICIR}_i}}{12}$$

$$Var^T[\text{ICIR}_y] = \frac{\sum_i \sum_j \frac{B_i L_i B_j L_j}{(T-t)^2} Cov(I(i), I(j))}{12^2}$$

The $Cov(I(i), I(j))$ can be calculated by using Ito isometry by.

$$\begin{aligned}
Cov(I(i), I(j)) &= \mathrm{E}[(I(i) - \mathrm{E}[Ii])(I(j) - \mathrm{E}[Ii])] \\
&= \mathrm{E}[(\sigma \int_s^i e^{-a(i-u)} dW_u)(\sigma \int_s^j e^{-a(j-v)} dW_v] \\
&= \sigma^2 e^{-a(i+j)} \mathrm{E}[(\int_s^i e^{au} dW_u)(\int_s^j e^{av} dW_v] \\
&= \frac{\sigma^2}{2a} e^{-a(i+j)} (e^{2a \min(i,j)} - e^{2as})
\end{aligned}$$

Remember that the $Var(I(i)) = Cov(I(i), I(i))$. With some simple algebra you can show that the expression for the variance is as described in (5). The Variance can also be found using Ito isometry which states that.

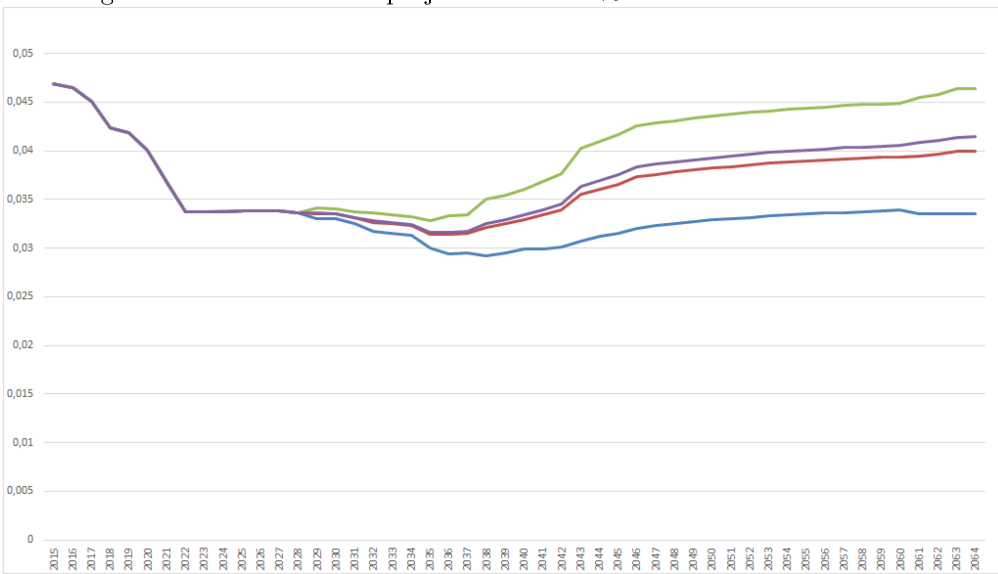
$$\mathrm{E}[\int_0^T (X_t dW_t)^2] = \mathrm{E}[\int_0^T (X_t^2 d_t)]$$

Now that we have the distribution of ICIR we have made the last step in transforming ICIR from a deterministic to a stochastic variable. In Appendix B you can find a timeline which helps to understand how the ICIR is calculated. Before we make a stochastic projection of ICIR I want to address something about the distribution of I_t. To find a proper distribution of the ICIR we want that all the $MICIR_t$ and so all the $I(t)$ have the same measure. This will be especially important when we will use Bachelier's options pricing formula together with the ICIR, as our observation date will not be equal to our payoff date. For the ICIR of a specific year y the forward measure will be the year y for all I(t) of the months between y-1 and y. So the ICIR of the year 2015 will have as payoff date 2015 for all the I(t) of the months between 2014 and 2015. In Figure 3 you can see the stochastic ICIR projection. Where the red line represents the mean of the stochastic projection ICIR, the purple line is the deterministic projection and the green and the blue line represent the 95% confidence interval. What immediately strikes is that until 2028 there is no difference between the deterministic and the stochastic projection. Recall that Yt is the sum of all face values of all the bonds which have reached maturity before or at time t. Before 2028 there is no bond which has yet reached maturity, so nothing has been reinvested before 2028. As the stochasticity of the ICIR totally depends on the random variable $Z_{t,T}$ there is no stochasticity before 2028. The next step is to apply Bachelier's option pricing formula on the ICIR.

6 Applying option pricing techniques on ICIR

Before we can talk about actually applying Bachelier's formula on the ICIR I first want to introduce the insurance contract which us our guinea pig for this new method. First of all I want to stress that I have used a simplified version of

Figure 3: Stochastic ICIR projection with 95% confidence interval



a funeral insurance contract with profit sharing. As a guinea pig I have chosen an insurance contract with profit sharing directly depending on the investment returns of the insurance company. I assumed that the buyer of the insurance contract would get payment X paid by the insurer at some point in the future when death occurs. The insurer promised a yearly profit sharing in percentage in year y denoted by P_y under the following conditions

$$P_y = \max[0,75\mathrm{ICIR}_y - 2,25,0] \tag{8}$$

The profit sharing is processed at the 1st of January of every year. So the profit sharing which you will receive in 2015 is based on the return on investment of the year 2014. From these conditions we can conclude with some very easy calculations that the ICIR should be above 3% to get a profit sharing. Before we can use Bachelier's option pricing formula we have to rewrite condition (8) to

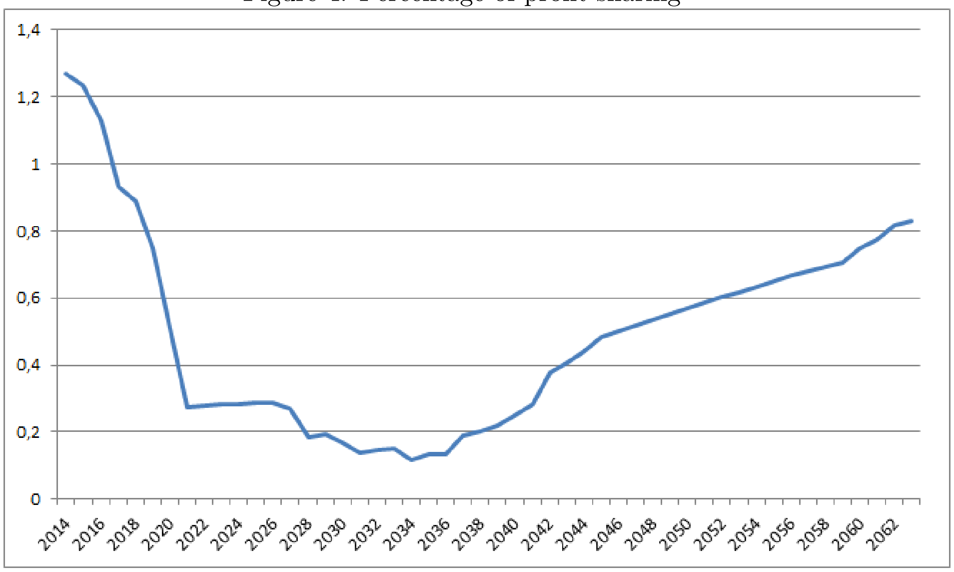
$$P_y = 0,75\max[\mathrm{ICIR}_y - 3,0] \tag{9}$$

As you maybe already recognised $\max[\mathrm{ICIR}_y - 3,0]$ is a standard form of a call option with a strike price of 3. It is now possible to apply Bachelier's option pricing formula on (9). If we take ICIR_y as the underlying security we get the following Bachelier's option pricing formula

$$\begin{aligned} C(\mathrm{ICIR}_y) &= \mathrm{E}[\mathrm{ICIR}_y - K)_+] \\ &= (\mathrm{ICIR}_0 - 3)N(\frac{ICIR_0 - 3}{\sigma\sqrt{t}}) + \sigma\sqrt{t}n(\frac{\mathrm{ICIR}_0 - 3}{\sigma\sqrt{t}}) \end{aligned}$$

Where σ is in this case the standard deviation of ICIR_y. The result of this formula should be interpreted as the expected percentage of profit sharing for year y. You can see the result of applying Bachelier's option pricing formula in Figure 4. So in 2015 the insured amount X would rise with a factor of 1,0126 as

Figure 4: Percentage of profit sharing



we have a profit sharing of 1,26% for 2015. In the case that the profit sharing is not cumulative the shared profit is paid in cash to the insured. In this case we could use the percentages of profit sharing given in Figure 4 to calculate the value and the premium(s) of an insurance contract that pays X at the moment of death. In the case that the profit sharing is cumulative, it is a lot more complicated to calculate the value of the insurance contract in an analytical way. To be able to calculate this we have to pull out all the stops.

7 Asian option pricing techniques on ICIR

If we take a closer look at an insurance contract with cumulative profit sharing we note that the cumulative profit sharing is a path-dependent contingent claim. The path-dependency complicates things a lot. To make the problem a bit clearer we should have a look at the expectation we have to calculate in the case of cumulative profit sharing. From now on we will assume that the person dies after 50 years with probability 1. The expected profit sharing over 50 years, denoted by P^{50} is given by

$$P^{50} = \mathrm{E}\Big[\prod_i \left[1 + 0,75 \max[\mathrm{ICIR}_i - 3, 0]\right]\Big] \tag{10}$$

We see in equation (10) that we are taking the expectation of a product of random variables. The expectation of the sum is equal to the sum of the expectations [Casella and Berger, 2002]. This property does not hold for the expectation of the product (except in the case that we have independent random variables) so to be able to calculate this expectation we have to dig deeper in our toolbox. From Jensen's Inequality we know that

$$\mathrm{E}\Big[\prod_i \big[1+0,75*\max[\mathrm{ICIR}_i-3,0]\big]\Big] \geq \Big[\prod_i \mathrm{E}\big[1+0,75*\max[\mathrm{ICIR}_i-3,0]\big]\Big]$$

Note that as the product is a convex function, we get the $\geq$ sign. One possibility to solve this is by calculating all the cross products. It is quite easy to calculate the expectation of a product of 2 random variables but things already get complicated if you have to look at the product of 3 random variables. So we can exclude the option of looking at the cross products. We could also neglect the cross product of more than two random variables and hope that the neglected cross products do not carry a lot of information. As I am afraid for loss of information I do not think that the latter is a good option. This brings us to look at Asian options or also called Average Rate Options (AROs). Rogers and Shi [1995] introduced a method to approximate the value of an Asian option. Before we will take a look at calculating the value Asian options we can spend a few lines to tell something about these options. Asian Options were introduced to avoid a problem which comes back at European options. Speculators could drive up their profits from the option by manipulating the price of the underlying security near to the maturity date. They also have a smoothing effect in markets where prices are subject to periods of high volatility. Asian options are commonly traded but despite this there is not a simple analytical formula for its value like the Black & Scholes formula for the value of an European option. Asian options are also path-dependent contingent claims, just like the problem we address in this paper. They settle against the arithmetic average of prices calculated over a given time interval [Levy, 1992]. Rogers and Shi [1995] introduced a method to find a lower bound for the price of an Asian option. They state that if we condition a Gaussian process X on a Gaussian variable Z with zero-mean it remains a Gaussian process. To obtain a lower bound they make use of the following inequality

$$\mathrm{E}[A(T)-K]^+ = \mathrm{E}[\mathrm{E}[(A(T)-K)^+|Z]] \geq \mathrm{E}[\mathrm{E}[A(T)-K|Z]^+]$$

Where $A(T) = \frac{1}{n}\sum_{i=1}^{n} S(t_i)$ which represents the discrete arithmetic average. This inequality is the result of using the tower property and Jensen's inequality. The trick that Rogers and Shi apply is to choose a Gaussian variable Z which has a high correlation with $A(T)$. The n random variables in the $A(T)$ are replaced by the $A's$ projection on the single random variable Z. In the ideal situation there would be perfect correlation between Z and $A(T)$ which would result in a zero pricing error [Nielsen and Sandmann, 2003].

We will now apply the idea of Rogers and Shi at our problem. First we have to define our lower bound in our case. In the case of the Asian option the max operator of the sum of the underlying asset is the convex function. In our case the product of the call options is the convex function. An important thing to note is that for the Asian option the distribution of the sum of the underlying asset is unknown. For our problem the distribution of the product of the call options is unknown (as they are not independent). To obtain a lower bound we get the following inequality.

$$\mathrm{E}\Big[\prod_i \big[1+0,75*C_i\big]\Big] = \mathrm{E}\Big[\mathrm{E}\Big[\prod_i \big[1+0,75*C_i\big]|Z\Big]\Big] \geq \mathrm{E}\Big[\prod_i \mathrm{E}\Big[\big[1+0,75*C_i\big]|Z\Big]\Big]$$

where $C_i = max[ICIR_i - 3, 0]$. The next thing we have to do is finding an appropriate variable Z on which we will condition. We want to find a Z which makes, in the ideal situation, the joint distribution perfectly correlated. An appropriate choice for Z is taking Z as $\boldsymbol{\beta' I}$ where $\boldsymbol{I}$ is the random vector containing all $I(t)'s$ and $\boldsymbol{\beta}$ is an arbitrary vector with the same length as the vector $\boldsymbol{I}$. It follows from the distribution of $\boldsymbol{I}$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ that Z has the following distribution: $Z \sim \mathcal{N}(\boldsymbol{\beta'\mu}, \boldsymbol{\beta'\Sigma\beta})$. Now that we know the distributions we can make the conditioning step. We have the following partitioned multivariate normal distribution.

$$\begin{pmatrix} \boldsymbol{I} \\ Z \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\beta'\mu} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Sigma\beta} \\ \boldsymbol{\beta'\Sigma} & \boldsymbol{\beta'\Sigma\beta} \end{pmatrix}\right)$$

To make life easier we will choose $\boldsymbol{\beta}$ in such a way that $\boldsymbol{\beta'\Sigma\beta} = 1$. Now that we know these things we can find the distribution of $\boldsymbol{I}$ conditional on $Z = z$.

$$\boldsymbol{I}|Z = z \sim \mathcal{N}\big(\boldsymbol{\mu} + (z - \boldsymbol{\beta'\mu})\boldsymbol{\Sigma\beta}, \boldsymbol{\Sigma} - \boldsymbol{\Sigma\beta\beta'\Sigma}\big)$$

Note that $(z - \boldsymbol{\beta'\mu}) \sim \mathcal{N}(0,1)$. With this last ingredient we are almost able to determine the lower bound. The last thing we have to take a look at is that we calculate the value of the option under the correct measure. As we are now looking at the final payoff after 50 years and to be precise after $49\frac{11}{12}$ years, we will have to determine the distribution of $I(t)$ under the $49\frac{11}{12}$ years T-forward measure (i.e. in equation (4) $T = 49\frac{1}{12}$). Knowing the correct measure we are able to actually determine the lower bound.

To calculate the lower bound we will replace the random variable $I(t)$ with the conditional random variable $I(t)|Z = z$ to determine the distribution of $ICIR_y$ as described in section 5. As we now know the conditional distribution of $ICIR_y|Z = z$ we can calculate for every $Z = a$, where a is an arbitrary number, the final payment after $49\frac{11}{12}$ years. By applying Bachelier's option pricing formula as described in section 7 on the conditional distribution we are able to calculate the profit sharing for every year. The final payment is than calculated by taking the product of all these yearly profit sharing to get the cumulative profit sharing. Now that we are able to determine the final payment for an arbitrary $Z = a$ we can determine the lower bound by integrating over Z. Remember that $(z - \boldsymbol{\beta'\mu}) \sim \mathcal{N}(0,1)$. We are now able to replace the integration over Z by a numerical integration of a standard normal distribution. For this numerical integration I have chosen to use the Trapezoidal rule from -5σ to 5σ with steps of $0,1\sigma$, where σ is the standard deviation of the standard normal distribution. To optimize the lower bound we have found by applying this method, we want to find the $\boldsymbol{\beta}$ which maximizes the value of the lower bound under the restriction that $\boldsymbol{\beta}'\boldsymbol{\Sigma}\boldsymbol{\beta} = 1$. Note that the highest possible lower bound is the best approximation we can make. After the maximisation over $\boldsymbol{\beta}$ we have found a lower bound of 0,269609. This is the discounted value of the cumulative profit sharing (i.e. If you would receive an insured amount of 1 in $49\frac{11}{12}$ years with cumulative profit sharing it has a present value of 0,269609). The next thing we will do is to look for an upper bound.

8 An upper bound using comonotonicity

Dhaene et al. [2000] found an upper bound for the price of an Asian option. They used comonotonicity and some risk theory one stop-loss order. With these theories they showed that the price of an arithmetic Asian option can be bounded from above by the price of a portfolio of European call options. Also all the other theory I encountered concerning comonotonicity and option pricing was only about convex bound for sums of random variables. For our problem we could not use the technique described by Dhaene et al. but they gave us a push in the right direction. For an overview of the theory of comonotonictity and its applications in finance and insurance of the last years you should read Deelstra et al. [2010]. If you want to learn more about the theory of comonotonicity I would advise you to read Dhaene et al [2002]. The core principle behind our upper bound will be comonotonicity.

The idea we want to exploit is a kind of worst case analysis. In the previous section we tried to make the joint distribution perfectly correlated with the random variable Z and we did not change anything at the marginal distributions. For this upper bound we want to make the marginal distributions of $\boldsymbol{I}$ perfectly correlated. We want that the variables move along with each other. From Casella and Berger [2002] we know that Pearson's correlation coefficient is defined by.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$$

Remember that $\rho(X,Y) = 1$ if and only if there exist numbers $a > 0$ and b such that $P(Y = aX + b) = 1$. Dhaene et al [2002] state that $\rho(X,Y) = 1$ implies comonotonicity of the couple (X,Y). In this case the connected comonotonic support is a straight line. They state that a random vector with marginal distributions that belong to the same location-scale family of distributions is comonotonic if and only if the correlation of each pair of marginal components is equal to 1. They exploit the fact of perfect correlation to get an upper bound in the convex order sense. As we also have a random vector where the marginal distributions belong to the same location-scale family of distributions (Normal distribution) we can also exploit this fact to get an upper bound.

To obtain the upper bound we have to construct our covariance matrix in such a way that the standard deviations of each component in $\boldsymbol{I}$ will stay the same and that all the pairs of components have a correlation of 1. Remember that $\boldsymbol{I} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Recall that the diagonal of the covariance matrix represents the variance of all individual components of $\boldsymbol{I}$. Let's define $\boldsymbol{\sigma}$ as the vector containing the diagonal of the original covariance matrix. We will construct the new covariance matrix by $\boldsymbol{\Sigma_c} = \boldsymbol{\sigma\sigma'}$. With this new covariance matrix we constructed a comonotonic vector of random variables $\boldsymbol{I_c} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma_c})$. To be able to obtain a tight upper bound we will condition this vector on variable $Z_c = \boldsymbol{\beta' I_c}$. By applying this technique we obtain the following inequality

$$\mathrm{E}[f(\boldsymbol{I})] \leq \mathrm{E}[f(\boldsymbol{I_c})] = \mathrm{E}[\mathrm{E}[f(\boldsymbol{I_c}|Z]]$$

To give a clear picture, we have the following partitioned multivariate normal distribution.

$$\begin{pmatrix} \boldsymbol{I_c} \\ Z_c \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\beta'\mu} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma_c} & \boldsymbol{\Sigma_c\beta} \\ \boldsymbol{\beta'\Sigma_c} & \boldsymbol{\beta'\Sigma_c\beta} \end{pmatrix}\right)$$

Again to make life easier we will choose $\boldsymbol{\beta}$ in such a way that $\boldsymbol{\beta'\Sigma_c\beta} = 1$. Knowing this we obtain the following conditional distribution

$$\boldsymbol{I_c}|Z_c = z \sim \mathcal{N}\big(\boldsymbol{\mu} + (z - \boldsymbol{\beta'\mu})\boldsymbol{\Sigma_c\beta}, \boldsymbol{\Sigma_c} - \boldsymbol{\Sigma_c\beta\beta'\Sigma_c}\big)$$

To calculate the upper bound we will replace $I(t)$ with the conditional random variable $I(t)_c|Z_c = z$ to determine the distribution of $ICIR_y$. The rest of the recipe to calculate the upper bound is similar to calculating the lower bound. To optimize the upper bound we want to find the $\boldsymbol{\beta}$ which minimizes the value of the upper bound under the restriction that $\boldsymbol{\beta'\Sigma\beta} = 1$. The conditioning step enables us to find the tightest lower bound. After the minimisation over $\boldsymbol{\beta}$ we have found an upper bound of 0,276296. Again this is the discounted value of the cumulative profit sharing. The final thing we have to check is how accurate these bounds are.

9 Numerical check with Monte Carlo simulation

Our last step is checking the accuracy of the lower and the upper bound. Note that we want to apply a Monte Carlo simulation for simulating data with correlated variables. To be able to simulate this data I have chosen to use a Cholesky decomposition (which is commonly used in generating correlated data). The Cholesky decomposition states that if there exist a symmetric and positive definite matrix $\boldsymbol{A}$, it has a decomposition of the form

$$\boldsymbol{A} = \boldsymbol{CC'}$$

where $\boldsymbol{C}$ is a lower triangular matrix. Remember that a covariance matrix is by definition symmetric and positive-semi definite. In our case the covariance matrix $\boldsymbol{\Sigma}$ is positive definite as there exist a unique Cholesky decomposition.

Note that a symmetric matrix is only positive definite if and only if there exists a unique Cholesky decomposition. So we are able to find a lower triangular matrix in such a way that

$$\boldsymbol{\Sigma} = \boldsymbol{C}\boldsymbol{C'}$$

The nice property of this Cholesky decomposition, is that if we take the matrix $\boldsymbol{C}$ and multiply it with a vector of i.i.d. standard normal variables, we will get a vector of variables that satisfy the original covariance matrix [Miller, 2014].

We used this Cholesky decomposition to simulate our $I(t)$ process 75000 times. For every simulated path we calculated the cumulative profit sharing. We calculated the $ICIR_y$ for each year as described in section 4. Remember that we are now allowed to treat each path as a deterministic path. After calculating the $ICIR_y$ for each year we apply the max operator as described in (8) to find the yearly profit sharing. Next to calculating the cumulative profit sharing we also had to calculate the discount factor for each path as we want to compare the present values. To calculate the discount factor of each path I made use of equation (9). We can calculate the discount factor by using the formula $e^{-\sum_0^{50} Z}$. With this Monte Carlo simulation we obtained a numerical price of 0,26921 with a 95% confidence interval of [0,26642,0,27200]. In the following table you can see an overview of the results we obtained.

lower bound	upper bound	MC mean	MC 95%
0,269609	0,276296	0,26921	[0,26642,0,27200]

In the table you can see that the mean obtained by the Monte Carlo simulation is lower than the lower bound. This does not mean that our lower bound is not correct. As you can see the lower bound is in the 95% confidence interval and is only 0,29 standard deviation away from the mean. This shows that the lower bound is very tight and a very accurate approximation for price of the insurance contract. We clearly have found an upper bound as the upper bound is higher than the upper boarder of the 95% confidence interval. The upper bound is not as tight as the lower bound but can still be very useful in a worst case analysis. From this numerical check we can conclude that the techniques we used, give an accurate approximation of the price of the insurance contract.

10 Conclusion

In this paper we presented techniques to determine an upper bound and a lower bound for the price of an insurance contract with cumulative profit sharing. The idea is that the amount of profit sharing is determined by the return on investment of the insurance company, where we assumed that the insurance company only invested in bonds. We are able to make the return stochastic by using the Hull and White model. To determine the amount of profit sharing for each year we have used Bachelier's option pricing formula. As there is no analytical solution for cumulative profit sharing we had to use alternative techniques to make an approximation for the price of the insurance contract.

We determined a lower bound with the help of Asian option pricing techniques. The upper bound was determined by using comonotonicity. Finally we checked the accuracy of the bounds with a Monte Carlo simulation.

The result of this paper is that we have found analytical ways to get a accurate approximation of the range of the price of the insurance contract. We can now give an approximation of the price of the insurance contract in a fast way without making use of time consuming numerical techniques. We saw that the lower bound is very accurate and gives a tight approximation of the actual price of the insurance contract. The upper bound is very useful in a worst case analysis. For insurance companies a worst case analysis is very important as they can determine a premium which still covers the worst case situation. This upper bound and lower bound give the insurance companies a very clear range of what the present value of their liabilities will be of an insurance contract they sold or want to sell.

In this paper we only had a look at the cumulative profit sharing after 50 years. It is of course not very realistic that you know that people die after 50 years. To get the price of an actual funeral insurance contract we have to determine bounds for all years. If we have the bounds for all years together with the survival probabilities we are able to determine an upper and a lower bound for the price of a (realistic) funeral insurance contract with profit sharing.

These techniques make it easier for the insurance company to determine the price of a funeral insurance contract with cumulative profit sharing. With this price the insurance company can determine the premium and get a clearer picture of the risk they are taking. On top of that this technique is able to determine the market value of the current and future sold funeral insurance contracts. We can conclude that these techniques help insurance companies in determining the premium and profitability of the insurance contract and help them in adapting to Solvency II.

Appendices

A T-forward measure

We want to show that the discounted price process $D(t,T)\frac{1}{M(t)}$ is a martingale. First we have to define the processes $D(t,T)$ and $M(t,T)$ where $D(t,T)$ is the discounted price process and $M(t,T)$ the money market account

$$D(t,T) = e^{A(t,T)-B(t,T)r(t)}$$

$$M(t) = e^{\int_0^t r(u)du}$$
$$\frac{1}{M(t)} = e^{-\int_0^t r(u)du}$$

Note that in this appendix I did not do the transformation of variables which means that I stayed with the original process r(t). By the product rule for Ito processes we get that

$$d(D(t,T)\frac{1}{M(t)}) = D(t,T)d(\frac{1}{M(t)}) + \frac{1}{M(t)}d(D(t,T)) + d(D(t,T))d(\frac{1}{M(t)})$$

It can easily be shown that $d(D(t,T))d(\frac{1}{M(t)}) = 0$ by knowing that $\frac{1}{M(t)}$ is a finite variation process [Bingham and Kiesel, 2004], $dt^2 \to 0$ and $dW_t dt \to 0$. By applying Ito's calculus we obtain the following equations (From now on we will replace $D(t,T)$ and $M(t)$ by D and M. Next to that we define $A(t,T)$,$B(t,T)$ and $r(t)$ as A,B and r)

$$\begin{aligned} d\big(D\frac{1}{M}\big) = D(-r\frac{1}{M})dt + \frac{1}{M}[(D(\frac{\partial A}{\partial t} + e^{-a(T-t)}r) - B(\theta - ar)D + \frac{1}{2}\sigma^2 DB^2]dt \\ + D\frac{1}{M}(-B\sigma)dW_t \end{aligned}$$

For $d(D\frac{1}{M})$ to be an martingale we have to show that the drift term disappears. Before we can do that we have to define A. As we are now using the original Hull and White model without the transformation A is defined as

$$A = \log(P^M(0,T)) - \log(P^M(0,t) + Bf^M(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at})B^2$$

The partial derivative of A with respect to t is given by

$$\begin{aligned} \frac{\partial A}{\partial t} = f^M(0,t) - e^{-a(T-t)}f^M(0,t) + B\frac{\partial f^M(0,t)}{\partial t} + \frac{\sigma^2}{2a}Be^{-a(T-t)} \\ - \frac{\sigma^2}{2a}Be^{-a(T+t)} - \frac{\sigma^2}{2}B^2e^{-2at)} \end{aligned}$$

We know from Brigo and Mercurio [2007] that

$$\theta = \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

By replacing $\frac{\partial A}{\partial t}$ and θ with the terms we have found and using some simple algebra al terms with $f^M(0,t)$ cancel each other out in $d\big(D\frac{1}{M}\big)$. After this we obtain the following expression

$$\begin{aligned} d\big(D\frac{1}{M}\big) = D(-r\frac{1}{M})dt + D\frac{1}{M}[\frac{\sigma^2}{2a}Be^{-a(T-t)} - \frac{\sigma^2}{2a}Be^{-a(T+t)} - \frac{\sigma^2}{2}B^2e^{-2at)} \\ + e^{-a(T-t)}r - \frac{\sigma^2}{2a}Be^{-2at} + arB + \frac{\sigma^2}{2}B^2]dt + D\frac{1}{M}(-B\sigma)dW_t \end{aligned}$$

Recall that $B = \frac{1}{a}(1-e^{a(t-t)}$. If we plug in this expression for B in the formula of $d\big(D(t,T)\frac{1}{M(t)}\big)$ we obtain

$$\begin{aligned} d\big(D\frac{1}{M}\big) = D(-r\frac{1}{M})dt + D\frac{1}{M}[\frac{\sigma^2}{2a^2}e^{-a(T-t)} - \frac{\sigma^2}{2a^2}e^{-2a(T-t)} - \frac{\sigma^2}{2a^2}e^{-a(T+t)} \\ + \frac{\sigma^2}{2a^2}e^{-2aT} - \frac{\sigma^2}{2a^2}e^{-2at} + \frac{\sigma^2}{a^2}e^{-a(T+t)} - \frac{\sigma^2}{2a^2}e^{-2aT} - \frac{\sigma^2}{2a^2} \\ + \frac{\sigma^2}{2a^2}e^{-a(T-t)} + \frac{\sigma^2}{2a^2}e^{-2at} - \frac{\sigma^2}{2a^2}e^{-a(T+t)} + \frac{\sigma^2}{2a^2} - \frac{\sigma^2}{a^2}e^{-a(T-t)} \\ + \frac{\sigma^2}{2a^2}e^{-2a(T-t)} + e^{-a(T-t)}r - e^{-a(T-t)}r + r]dt + D\frac{1}{M}(-B\sigma)dW_t \end{aligned}$$

After some careful cancelation we get that almost all the terms cancel each other out except one. This leads to the following equation

$$\begin{aligned} d\big(D\frac{1}{M}\big) &= D\frac{1}{M} - rdt + D\frac{1}{M}rdt + D\frac{1}{M}(-B\sigma)dW_t \\ &= D\frac{1}{M}(-B\sigma)dW_t \end{aligned}$$

As it has no drift term, we can conclude that the discounted price process is a martingale. Privault [2013] states that by using the Girsanov Theorem [Bingham and Kiesel, 2004] we can show that

$$dW_t^T = dW_t - \frac{1}{D}dDdW_t$$

We know now from the derivation we did a few moments ago that

$$dD = rDdt - DB\sigma dW_t$$

By plugging this into the equation of dW_t^T we obtain

$$\begin{aligned} dW_t^T &= dW_t - rdtdW_t + B\sigma dW_t^2 \\ &= dW_t + B\sigma dt \end{aligned}$$

We now get that under the T-forward measure Q^T

$$dr(t) = [\theta(t) - \sigma^2 B(t,T) - ar(t)]dt + \sigma(t)dW_t^T$$

You can see that we corrected for the term, which is added in the Brownian motion, by subtracting it from the mean. So the term $\sigma^2 B(t,T)$ is the correction for the T-forward measure. If you apply the transformation of variables $r(t) = I(t) + \alpha(t)$ and use that $B(t,T) = \frac{1}{a}(1 - e^{a(t-t)}$ it is really easy to get to the expression given in (4).

B Timeline ICIR

With this timeline (Figure 5) I want to give a better overview of how the ICIR is calculated. First I want to specify some variables. t will be between 0 and $49\frac{11}{12}$. $\tau = \frac{1}{12}$ is the time step between every t. $s = 0$ which represents $\mathcal{F}_{s=0}$. T, in the T-forward measure, is equal to T_1, T_2 etc. for each consecutive year in the case of cash profit sharing. In the case of cumulative profit sharing T is equal to $T_{49\frac{11}{12}}$. At time h_1 we calculate $Z_{t,T}$ in the following way

$$D(\frac{5}{12}, \frac{6}{12}) = e^{A(\frac{5}{12}, \frac{6}{12})} - B(\frac{5}{12}, \frac{6}{12})I(\frac{5}{12})$$

$$D(\frac{5}{12}, \frac{6}{12}) = e^{Z_{\frac{5}{12}, \frac{6}{12}}}\frac{1}{12}$$

$$Z_{\frac{5}{12}, \frac{6}{12}} = -12A(\frac{5}{12}, \frac{6}{12}) + 12B(\frac{5}{12}, \frac{6}{12})I(\frac{5}{12})$$

For time h_2 we calculate it in a similar way

$$D(\frac{15}{12}, \frac{16}{12}) = e^{A(\frac{15}{12}, \frac{16}{12})} - B(\frac{15}{12}, \frac{16}{12})I(\frac{15}{12})$$

$$D(\frac{15}{12}, \frac{16}{12}) = e^{Z_{\frac{15}{12}, \frac{16}{12}}}\frac{1}{12}$$

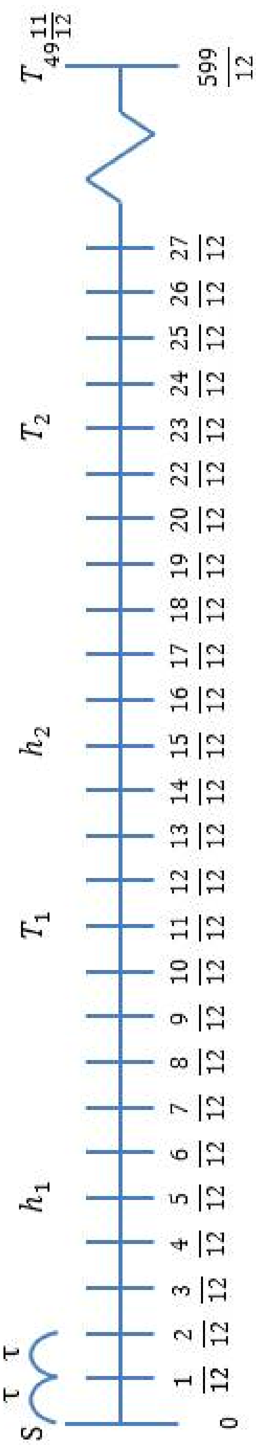
$$Z_{\frac{15}{12}, \frac{16}{12}} = -12A(\frac{15}{12}, \frac{16}{12}) + 12B(\frac{15}{12}, \frac{16}{12})I(\frac{15}{12})$$

We get for every t between 0 and $49\frac{11}{12}$ with $\Delta t = \tau$ the following formula for $Z_{t,T}$

$$Z_{t,t+\tau} = -\frac{A(t, t+\tau)}{\tau} + \frac{B(t, t+\tau)}{\tau}I(t)$$

Where $I(t)$ is distributed with mean (4) and variance (5).

Figure 5: Timeline ICIR



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