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# Integrating Longevity Risk, Macroeconomic Fluctuations, and Financial Risk

## A VAR Approach

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# Integrating Longevity Risk, Macroeconomic Fluctuations, and Financial Risk: a VAR Approach

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**Abstract**

The uncertainty in future mortality trends, i.e. the longevity risk, can cause substantial problems for the pension and insurance industry. This paper proposes to forecast the distribution of future mortality rate jointly with variables related to macroeconomic conditions and financial market variations by a vector autoregressive(VAR) model. The method is illustrated by empirical implementations based on U.S data from 1970 to 2007. Based on the empirical results, two numerical examples are analyzed to asses the importance of longevity risk. Firstly, I investigate the effects of raising retirement age on the fair premiums needed for a pension policy and I find a substantial tradeoff between the two values. Next, I examine the impact of longevity risk on the funding ratio uncertainty. The results show that while the micro-longevity risk vanishes for funds of large size, the macro-longevity risk still remains, although its role becomes less important when financial market risk is introduced.

# Contents

# 1 Introduction

The 20th century has seen a remarkable increase in average human lifetime compared to previous centuries. For most developed countries, mortality rates have fallen dramatically at all ages. By the beginning of the 21st century, the average life span has reached about 70 years, while in the middle of the 20 century the number was 60-65 and in the middle of the 18th century the number was around 40-45. The average lifetime among early humans was considered to be between 20 and 30 years, as suggested by archaeological evidence. Roughly speaking, the average life span increased by 25 years in the 10,000 years before the middle of 18th century and increased by another 25 years between the late of 18th century and the 20th century. The recent longevity improvement is rather impressive. For more details of the mortality trend, see Pitacco et al. (2009).

Improved longevity has caused a number of social and economic problems. For example, many of the current pension systems failed in predicting such a fast increase in human lifetime and their solvency is vulnerable to this risk. According to Biffis and Blake (2009), every additional year of life expectancy at age 65 is estimated to add at least 3 percent to the present value of UK pension liabilities. The problem will be mitigated if the trend of increase in longevity can be perfectly anticipated. However, this is not the case. The uncertainty in the distribution of future mortality rate is referred to as (macro) longevity risk, which is an aggregate risk and makes the problem more complex.[1] Hári et al. (2008a) showed that uncertainty in the future longevity is likely to cause significant risks in portfolios of pension annuities as evidenced by increased volatility of the future funding ratio.

Meanwhile, in recent years, both public and private sectors have put great emphasis on risk management. For example, the Solvency II project, which aims at redesigning financial regulation of insurance companies in Europe, has required a risk-based capital approach and has encouraged development of internal models. At the same time, there is increasing awareness that uncertain future longevity contributes nontrivially to the riskiness of many pension and insurance products. Thus, longevity risk should be taken into account, in combination with traditional risk factors, in predicting future scenarios for pension policy, insurance products, individual financial situation, and so forth. This is also the focus of this paper.

To quantify and manage longevity risk, a suitable stochastic mortality

[1] Micro-longevity risk refers to the risk that, given a distribution, the actual death rates might vary from predictions due to a small portfolio.

model, capable of forecasting the distribution of future mortality rates is necessary. One of the most widely used methods is the Lee-Carter model (Lee and Carter (1992)), which assumes that there is an underlying time-dependent latent process determining the evolution of mortality rates over time. The parsimonious model has performed quite well in many applications. However, the Lee-Carter model is purely statistical and tends to ignore the role of some social and economics factors, which are also potential determinants of mortality rates. Among these factors, the Gross Domestic Product (GDP) as one of the most popular indicators of macroeconomic conditions, is supposed to play a significant role by some literature. In this paper, I investigate the relationship between the GDP growth rates and mortality rate for the US. Furthermore, I propose to apply a vector autoregressive(VAR) model to quantify the joint effects of uncertain longevity, financial risk, and macroeconomics fluctuations. A detailed empirical illustration with US data is given. Once the future distributions of mortality rate, payoff of financial assets, and other state variables are predicted from the VAR model, quantitative analysis can be performed in a large number of contexts. Two numerical examples with real life implications are provided. Firstly, a non-negligible tradeoff between fair premium and retirement age is found for a typical defined benefit pension plan. Secondly, I find that longevity risk contributes greatly to the uncertainty of liability for various pension and insurance products and the risk cannot be eliminated by increasing the size of the pool. However, its role becomes less important when financial risk is considered.

This paper is organized as follows. In the next section I review the literature on mortality modeling with a focus on the Lee-Carter model. I also investigate the relationship between the trend in mortality rates and macroeconomics fluctuations by applying time series analysis. The third section illustrates the VAR approach to forecast future scenarios of financial markets, macroeconomics conditions, and mortality rates.

In the fourth section, I use numerical examples to show possible applications of the VAR approach in section 3 to quantify longevity risk. First, since increasing retirement age is proposed as a remedy for the heavy burden of public expenditure on pensions in several countries, I examine the trade-off between retirement age and fair premium for pension plans. Second, I study the impact of longevity risk on the characteristics of the future funding ratio for a typical pension product. The results of the study can provide important insights into the risk management in the pension and insurance industry. The fifth section concludes.

# 2 Mortality Modeling and Macroeconomics Fluctuations

## 2.1 Mortality Modeling

Mortality forecasting is studied in different disciplines such as demology and actuarial science, with numerous models. The development and current progress of mortality modeling is well reviewed in Tabeau (2002), Booth (2006), Cairns et al. (2007), Cairns et al. (2008), and Yang et al. (2010). See also the recent books by Girosi and King (2008) and Pitacco et al. (2009). These models can be classified by the factors used to describe their dynamics, by whether they are formulated in discrete or continuous time, or by other features. Traditional models treat the distribution of mortality as static or deterministic and provide only point estimates of future mortality rates. On the contrary, modern methods, expressing mortality as a function of time, allow for uncertainty in projected rates and can provide quantifications of estimation inaccuracy. The stochastic feature of these models is closer to reality compared to the deterministic models. Among stochastic mortality models, the discrete-time two-factor (age and time) model proposed by Lee and Carter (1992) is one of the earliest and still the most widely used methods. The Lee-Carter model is famous for its succinct form, yet robust performance. Some recent applications, variants, and extensions of Lee-Carter models can be found in Lee (2000) , Brouhns et al. (2002), Booth et al. (2006) and Hári et al. (2008b).

Before introducing the Lee-Carter model in detail, explaining some notations and concepts commonly applied in mortality modeling is beneficiary. It is widely documented that the characteristics of mortality rates are differentiated by age, sex, geographical region, or other features. Thus, the population under study may be disaggregated into various homogeneous groups. Denote one of such groups by $g$. One key concept is the central death rate with age $x$ of group $g$ in year $t$ defined by $m^g_{x,t} = D^g_{x,t}/E^g_{x,t}$, where $D^g_{x,t}$ is the number of deaths in group $g$ at age $x$ and time $t$ and $E^g_{x,t}$ the exposure-to-risk, is the number of person-years in group $g$ at age $x$ and year $t$. The central death rates $m^g_{x,t}$ are often recorded as annual data, with ages ranging from 0 to some maximum number, such as 110. The number of deaths $D^g_{x,t}$ and exposure-to-risk $E^g_{x,t}$ can be achieved or estimated from a demographic database. The one-year death probability $q^g_{x,t}$ can be easily calculated given the corresponding $m^g_{x,t}$ and some assumptions. One option is to assume that the exposure is linear in $x$, which results in the following relationship

$$q^g_{x,t} = \frac{m^g_{x,t}}{1 + \frac{1}{2} m^g_{x,t}}. \tag{2.1}$$

Alternatively, one may assume that the central death rate equals so-called force of mortality to get

$$q^g_{x,t} = 1 - \exp{(-m^g_{x,t})}. \tag{2.2}$$

Despite of their different forms, the actual values of death rates from equation (2.1) and equation (2.2) are usually very close. I use equation (2.2) throughout the paper. Given $q^g_{x,t}$, the the corresponding one-year survival probability can be easily obtained, which is $p^g_{x,t} = 1 - q^g_{x,t}$. The multi-year survival probability at time $t$ can be easily obtained by the product of one-year survival rates. For example

$$_\tau p_{x,t} = p_{x,t} \cdot p_{x+1,t+1} \cdot p_{x+2,t+2} \cdots p_{x+\tau-1,t+\tau-1}, \tau = 1, 2, 3, \cdots \tag{2.3}$$

where $_\tau p_{x,t}$ denotes the probability that a life aged $x$ at time $t$ will survive at least $\tau$ years. In mortality modeling, the central death rate for a group $g$, $m^g_{x,t}$, is often treated as an endogenous variable, which is decomposed into a systematic part, say $\widetilde{m}^g_{x,t}$, and a remaining idiosyncratic part. For forecasting, the systematic part is projected into the future. Then, future death rates can be obtained using the forecasted systematic part. Since mortality models typically consider a fixed group $g$, I shall suppress the superindex $g$ in the remaining sections whenever no ambiguity is caused.

Having explained the above concepts, I can formally define the the Lee-Carter model, which postulates

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t} \tag{2.4}$$

with time-invariant parameters $\alpha_x$ and $\beta_x$, and a homoskedastic error term $\epsilon_{x,t}$ with mean 0 and variance $\sigma^2_\epsilon$, where $\kappa_t$, often labeled as mortality index in the literature, is a one-dimensional time-dependent latent process that quantifies the variation in the level of mortality over time. The parameter $\alpha_x$ describes the age pattern of the log central death rate, averaged over time, while the parameter $\beta_x$ captures the sensitivity of the log central death rate at age $x$ to variations in $\kappa_t$. Finally, the error term $\epsilon_{x,t}$ represents the age and time specific variations not captured by the systematic trend.

In equation (2.4), the parameters cannot be identified without additional constraints. Lee and Carter (1992) assume that $\sum_t \kappa_t = 0$ and $\sum_x \beta_x = 1$, where $\kappa_t$ is summed over all time periods and $\beta_x$ is summed over all available ages in the sample. Under this normalization, $\beta_x$ can be interrelated as the proportion of change in the overall central death rate contingent on age $x$. The model cannot be estimated by simple regression since $\kappa_t$ is unobservable. Instead, Lee and Carter (1992) proposes a solution based on singular value decomposition (SVD) and an adjustment of the mortality index so that fitted deaths match observed total deaths in any year. The details can be found in Lee and Carter (1992) and there are also several modifications proposed, such as Wilmoth (1993), Lee and Miller (2001), Booth et al. (2001), and Girosi and King (2005). In most studies, the latent variable $\kappa_t$ is modeled as a ARIMA$(p, d, q)$ process with best fitting form $(p, d, q)$=(0,1,0), which is a random walk with drift, i.e.,

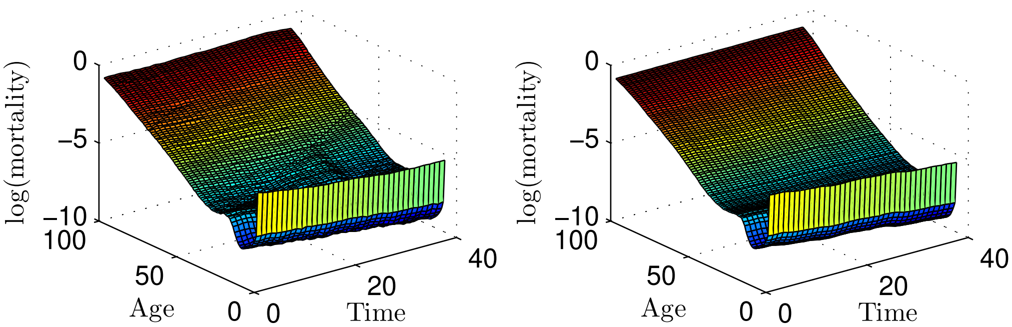
$$\kappa_t = \theta + \kappa_{t-1} + \delta_t \tag{2.5}$$

where $\theta$ is a drift term and $\delta_t$ is a white noise error term. After estimating the coefficients, the fitted central death rate is given by:

$$\hat{m}_{x,t} = \exp(\hat{\alpha}_x + \hat{\beta}_x \tilde{\kappa}_{x,t}) \tag{2.6}$$

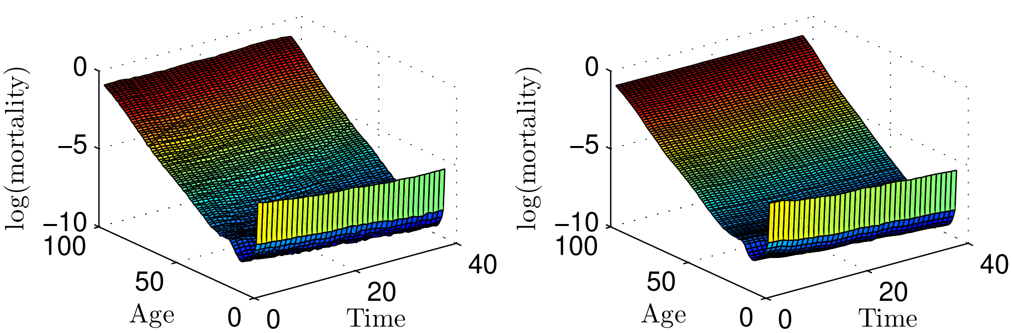
In the following I use the Lee-Carter method and data from the Human Mortality Database to model the dynamics of mortality trend in U.S.. To capture the relative recent trend in mortality evolution, I use a sample starting from 1970 and ending in 2007 with age ranging from 0 to 99. In addition, males and females are estimated separately as they show different mortality patterns.

Figure 2.1 and Figure 2.2 show the raw central death rates and fitted rates from the Lee-Carter model for males and females, respectively. The graphs indicate the typical pattern of mortality as a function of age and time. For a given year, the mortality rate is rather high at very young ages, reflecting the frailty of infants. Approximately, after age 10, the mortality rate begins to decline sharply, and then increases gradually, with a hump around age 20 which is observed in several countries for both genders. Turning to the time axis, it can be seen that the mortality rate shows a decreasing trend over time for most ages, revealing the improvement of longevity. It can also be observed that the Lee-Carter fitted mortality rates are close to the raw data for both genders. The mortality index $\kappa_t$ is plotted in Figure 2.3. The level of mortality shows a decreasing trend over time for both genders.




(a) Raw rates (b) Fitted rates

Figure 2.1: Log mortality and Lee-Carter fit for U.S. males (1970-2007, age 0-99)




(a) Raw rates (b) Fitted rates

Figure 2.2: Log mortality and Lee-Carter fit for U.S. females (1970-2007, age 0-99)

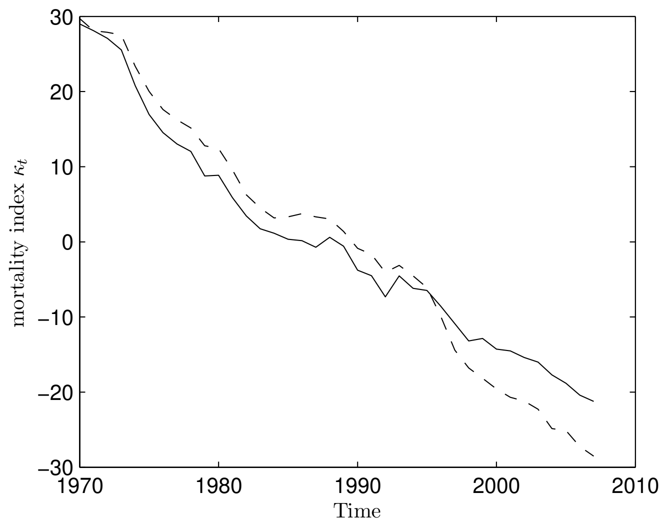



Figure 2.3: LC Mortality index for U.S.(1970-2007). The solid line depicts the results for males and the dashed line for females.

## 2.2 Lee-Carter Mortality Index and Macroeconomics Fluctuations

The Lee-Carter model is an extrapolative method that focuses on the historical patterns of the mortality rate. However, changes in the trend of mortality might be caused by some exogenous variables, such as medicine, lifestyles, and economy, which are not incorporated in the Lee-Carter model. One factor that may have impact on the mortality rate is the macroeconomic condition of an area, which has been studied in the literature with mixed results. Recently, using data for six OECD countries over the period 1950 to 2006, Hanewald (2011) studies the impact of macroeconomic fluctuations on mortality dynamics in the Lee-Carter mortality forecasting model, finding that the mortality index $\kappa_t$ in the Lee-Carter model and GDP levels are significantly correlated in some time periods and countries. In this section, I further investigate the relationship between the mortality index in the Lee-Carter model and GDP level using U.S data from 1970 to 2007. The mortality part is similar to the one in section 2.1. The model is estimated for U.S. males and females separately with ages ranging from 0 to 99. The U.S. annual real GDP levels are obtained from the Federal Reserve Economic Data. Figure 2.4 shows the time series for U.S. real GDP in billions of chained 2005 dollars, Historically, the U.S. economy has maintained

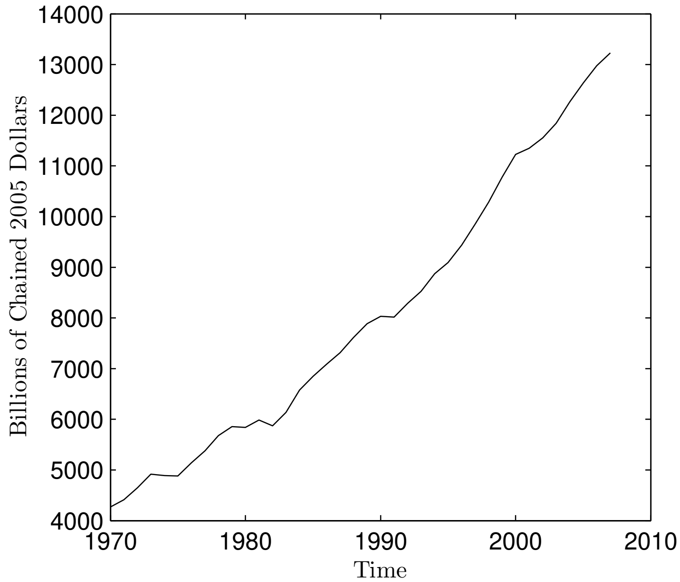



Figure 2.4: U.S. real GDP level (1970-2007)

a overall GDP growth.

At first, all time series are analyzed in levels. The correlations between real GDP levels and mortality rates are rather high, −0.98 for males and −0.94 for females. However, this relationship might be misleading for non-stationary series. I apply two methods to test the stationarity of time series. One is the Augmented Dickey-Fuller (ADF) test and the other is the Kwiatkowski, Phillips, Schmidt, Shin (KPSS) test. The former is for testing a null hypothesis that there is a unit root while the latter has a null hypothesis that an observable time series is stationary around a deterministic trend. By analyzing the data under both tests, we can have a more comprehensive view of the time series data.

The Augmented Dickey-Fuller Unit Root Test indicates that the time series for real GDP and mortality indices are non-stationary while a following step shows that their first differences are all stationary, which can be seen from Table 2.1. The results of KPSS test are shown in Table 2.2. The stationarity of the real GDP is still rejected at 5% significance level. Thus, we cannot only look at the correlations of the two series. However, the KPSS test fails to reject the stationarity of the logarithm of the real GDP, which is consistent with the finding that the GDP is characterized by exponential growth and is trend stationary. for males, the stationarity of mortality in-

Table 2.1: Augmented Dickey-Fuller Unit Root Test

| Variables | Test Statistics | P-values |
|---|---|---|
| realGDP | 2.9914 | 1 |
| $\Delta^{*}$(realGDP) | -3.7760 | 0.0068 |
| $\ln realGDP^{**}$ | -4.255394 | 0.0095 |
| $\kappa_t$(male) | -0.4929 | 0.8814 |
| $\Delta(\kappa_t$(male)) | -4.3921 | 0.0013 |
| $\kappa_t$(female) | -2.0741 | 0.2558 |
| $\Delta(\kappa_t$(female)) | -5.7638 | 0 |

* $\Delta$ stands for first difference

** $\ln real_GDP$ is tested with a time trend

dex at level cannot be rejected under KPSS test. For females, the null of stationarity is rejected at significance level 10%. The results implicate some degree of trend stationarity in the mortality index.

As there may exist long-term relationship between GDP levels and mortality index, I further make the hypothesis that the two series are cointegrated. This hypothesis can be tested by regressing $\kappa_t$ on real GDP levels by the ordinary least squares approach and testing the stationarity of the residuals. If the residuals are stationary then we cannot reject the hypothesis that the two series under analysis are cointegrated. Formally, we are testing the stationarity of the residuals from the model

$$\kappa_t = \beta_0 + \beta_1 realGDP_t + \beta_2 realGDP_{t-1} + \ldots + \beta_{k+1} realGDP_{t-k} + \epsilon_t \quad (2.7)$$

where lagged real GDP are included in order to incorporate the possible lagged effects of macroeconomics fluctuations on mortality trend, which are also suggested in the literature.

As suggested in Verbeek (2004), for residuals, the commonly used unit root tests such as Augmented Dickey-Fuller method are not appropriate, which tend to underestimate the possibility of the existence of unit root, as the OLS estimator will make the residuals as stationary as possible. Thus, I use the critical values provided by Davidson and MacKinnon (1993), shown in Table 2.3, which are more negative so as to take into account the potential bias.

Table 2.4 shows the OLS regression results of $\kappa_t$ on real GDP for specifications of equation (2.7) with different number of lags. The $R^2$ is rather high and coefficients are significant in most cases, indicating a strong relationship between the mortality index and real GDP levels. However, to

Table 2.2: Kwiatkowski Phillips Schmidt Shin Test

| | | |
|---|---|---|
| Asymptotic critical values | 1% level | 0.216 |
| | 5% level | 0.146 |
| | 10% level | 0.119 |
| Variables | Test Statistics | Bandwidth |
| realGDP | 0.1936 | 5 |
| $\Delta realGDP^*$ | 0.1682 | 10 |
| $\ln realGDP$ | 0.0637 | 1 |
| $\kappa_t$(male) | 0.0831 | 4 |
| $\Delta\kappa_t$(male) | 0.059 | 2 |
| $\kappa_t$(female) | 0.1377 | 8 |
| $\Delta\kappa_t$(female) | 0.1155 | 8 |

* Δ stands for first difference

Table 2.3: Asymptotic Critical Values Residual Unit Root Test for Cointegration (with Intercept)(Davidson and MacKinnon, 1993)

| Number of Variables | Significance Level | | |
|---|---|---|---|
| | %1 | %5 | %10 |
| 2 | -3.90 | -3.34 | -3.04 |
| 3 | -4.29 | -3.74 | -3.45 |
| 4 | -4.64 | -4.10 | -3.81 |
| 5 | -4.96 | -4.42 | -4.13 |

exclude the possibility of spurious regression, we need to test the stationarity of residuals. The unit root tests are performed for the residuals from the regressions and the test statistics are reported in the last row of Table 2.4. For the corresponding critical values, see Table 2.3. As the results suggest, although the unit root hypothesis cannot be rejected for some specifications, the test statistics approach the critical values when higher number of lags are included. While the test statistics at most reaches the significance level at 10%, considering the relatively short time period included in these regressions, the results still indicate some relationship between mortality rate and macroeconomic fluctuations for both genders. Table 2.5 presents the results of the above analysis using the logarithm of U.S. real GDP instead of level data. The regression results again indicate significant correlations between real GDP and mortality index, with even higher adjust $R_2$ values in all specifications. In the case with log GDP, the nonstationarity of the residuals are rejected at 10% significance level for most of the specification. The results strongly support the possibility of cointegration between macroeconomic indicators and mortality trends. Thus, it is worth considering the possible relationship when forecasting future mortality trends.

Table 2.4: Regression Results of $\kappa_t$ on Real GDP with Different Lags

| Specification | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
|---|---|---|---|---|---|---|---|---|
| | $\kappa_{t,male}$ | $\kappa_{t,female}$ | $\kappa_{t,male}$ | $\kappa_{t,female}$ | $\kappa_{t,male}$ | $\kappa_{t,female}$ | $\kappa_{t,male}$ | $\kappa_{t,female}$ |
| Cont. | 49.9104 | 39.8616 | 48.9822 | 38.3033 | 48.0482 | 36.6142 | 46.7913 | 34.7132 |
| | (3.9105) | (5.6128) | (3.9195) | (2.5038) | (3.818) | (2.4053) | (3.4835 ) | (4.6468) |
| $realGDP_t$ | -0.0062 | -0.0050 | -0.0062 | -0.0036 | -0.0073 | -0.0047 | -0.0078 | -0.0060 |
| | (0.0004) | (0.0006) | (0.0045) | (0.0061) | (0.0042) | (0.0058) | (0.0042) | (0.0043) |
| $realGDP_{t-1}$ | | | 0.0001 | -0.0013 | 0.0058 | 0.0056 | 0.0058 | 0.0067 |
| | | | (0.0047) | (0.0062) | (0.0047) | (0.0093) | (0.0047) | (0.0052) |
| $realGDP_{t-2}$ | | | | | -0.0046 | -0.0056 | -0.0012 | -0.0031 |
| | | | | | (0.0039) | (0.0060) | (0.0046) | (0.0050) |
| $realGDP_{t-3}$ | | | | | | | -0.0028 | -0.0022 |
| | | | | | | | (0.0043) | (0.0045) |
| adjusted $R^2$ | 0.9583 | 0.8860 | 0.9578 | 0.8886 | 0.9584 | 0.8974 | 0.9621 | 0.9000 |
| unit root of ${\epsilon_t}^*$ | -2.5159 | -2.2952 | -3.0020 | -2.7496 | -3.5670 | -3.3529 | -3.8782 | -4.1433 |

Note: Newey-West standard errors in parentheses.

* t-Statistics are reported. For the corresponding critical values, see Table 2.3.

Table 2.5: Regression Results of $\kappa_t$ on Log Real GDP with Different Lags

| Specification | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
|---|---|---|---|---|---|---|---|---|
| | $\kappa_{t,male}$ | $\kappa_{t,female}$ | $\kappa_{t,male}$ | $\kappa_{t,female}$ | $\kappa_{t,male}$ | $\kappa_{t,female}$ | $\kappa_{t,male}$ | $\kappa_{t,female}$ |
| Cont. | 446.7029 | 364.9358 | 444.5286 | 356.1853 | 442.3825 | 346.1744 | 437.0395 | 334.3775 |
| | (13.6226) | (28.5978) | (14.66927) | (28.4341) | (14.8657) | (26.6251) | (13.9965 ) | (23.4493) |
| $\log realGDP_t$ | -49.9914 | -40.8407 | -27.7407 | -9.2237 | -35.9345 | -18.2046 | -30.4332 | -22.1886 |
| | (1.4825) | (3.1238) | (24.0925) | (29.8284) | (23.9564) | (27.6503) | (22.5203) | (21.7618) |
| $\log realGDP_{t-1}$ | | | -22.0859 | -30.7609 | 29.9265 | 27.6618 | 16.8195 | 21.7618 |
| | | | (24.4319) | (31.2090) | (23.8285) | (28.9408) | (22.3214) | (25.1530) |
| $\log realGDP_{t-2}$ | | | | | -43.7146 | -48.4785 | -0.4108 | -13.0395 |
| | | | | | (19.0725) | (26.7852) | (25.9062) | (26.7230) |
| $\log realGDP_{t-3}$ | | | | | | | -35.2136 | -29.3290 |
| | | | | | | | (24.8042) | (24.5742) |
| adjusted $R^2$ | 0.9789 | 0.9449 | 0.9771 | 0.9432 | 0.9772 | 0.9469 | 0.9772 | 0.9533 |
| unit root of ${\epsilon_t}^*$ | -3.52 | -3.11 | -3.88 | -3.75 | -3.67 | -3.22 | -3.57 | -4.13 |

Note: Newey-West standard errors in parentheses.

* t-Statistics are reported. For the corresponding critical values, see Table 2.3.

# 3 A VAR Approach to Forecast Future Scenarios

Forecasting future scenarios is a critical step in the management of pension and insurance products. Common areas of application based on scenarios forecasting include pricing strategies, capital allocation, setting target rates for return, and policy evaluation, among other things. A widely used approach to study the interactions among a group of variables of interest and to make forecasts based on their relationships is the Vector Autoregressive (VAR) model. Compared to univariate time series models, the VAR not only captures the correlation of the variables through time (autocorrelation), but also the possible correlation between variables ((cross)correlation). Furthermore, VAR captures the conditional long-run dynamics from multiple time series (Hoevenaars et al. (2003)). Compared to some nonparametric and semi-parametric models, the VAR model is succinct and easy to implement. Due to its many merits, the VAR approach has been applied widely in pension and insurance analysis. However, to the knowledge of the author of this article, none of the previous literature takes into account the mortality factor in the analysis based on a VAR model. As the problem of longevity risk keeps increasing, the mortality rate is also an important factor that should be modeled and forecasted. Thus, I propose a simple but original VAR model in this section to incorporate longevity risk into modeling the dynamics of a group of variables.

## 3.1 Model and Data

To capture the recent mortality trend I only use 30 years of time series data. Besides, the highest frequency of GDP data is quarterly. The use of multiple lags in time series models decreases the degrees of freedom, which might erode the accuracy of estimation, especially when the sample size is not large, as is the case with data sets in the paper. As a result, I postulate a unrestricted the VAR(1) model:

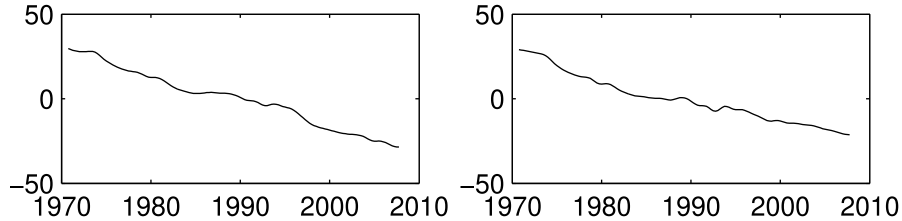
$$y_t = c + B \cdot y_{t-1} + \epsilon_t \tag{3.1}$$

where $y_t$ is a vector of state variables, $c$ is a vector of intercept terms, $B$ is a matrix of parameters, and $\epsilon_t$ a vector of error terms from a joint normal distribution, with mean values equal to zero, and covariance matrix denoted as $\Sigma$.

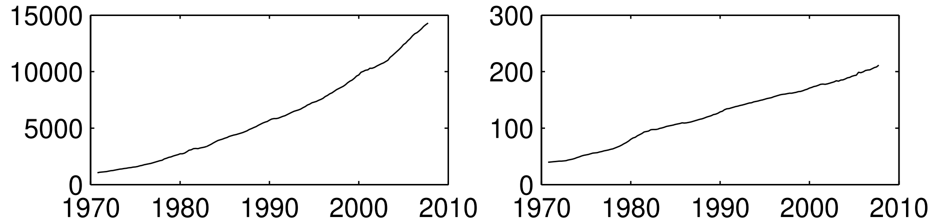
The selection of state variables in $y_t$ is based on the literature of economic forecasts as well as on the relevance for pension analysis. Commonly used variables for forecasting future economic dynamics in a VAR framework include stock market returns, bond market returns, the short-term interest rate, the long-term interest rate, and the inflation rate, among other things. (Campbell and Viceira (1999, 2005); Boender et al. (2006)). The return on risky assets mainly affects the asset part of a pension fund, while the interest rate affects both the asset part and the liability part. I include the short-term interest rate as well as the long-term interest rate. In addition, the inflation rate links the nominal data and real data. The link between GDP and financial markets is also documented in the literature. See, for example, the review by Stock and Watson (2003). Finally, the mortality index from the Lee-Carter model is included to quantify longevity risk in pension and insurance portfolios. Specifically, I use quarterly U.S. financial time series collected from FRED and CRSP for the period 1970.4 to 2007.4. Following standard practice, I proxy the short-term interest rate by the secondary market rate of the 3-month T-bills ($i_t$), the long-term interest rate by the 10-Year Treasury Constant Maturity Rate($l_t$), the large-cap equities market by the S&P 500 price index ($ps_t$), the bond market by the 10-year bond level index ($pb_t$), the inflation by the Consumer Price Index for All Urban Consumers ($cpi_t$), and the macroeconomic fluctuation by the nominal Gross Domestic Product ($gdp_t$). In addition, the mortality index for males ($\kappa_{t,male}$) and females ($\kappa_{t,female}$) in section 2 are disaggregated to quarterly data by cubic spline technique in order to match the other series. Figure 3.1 shows the plots of these series. I also perform some transformations on the raw data following common practice, as explained in Table 3.1. A list of all the state variables and their descriptions can be seen in Table 3.1. The corresponding descriptive statistics are shown in Table 3.2.

Table 3.1: Description of State Variables in VAR

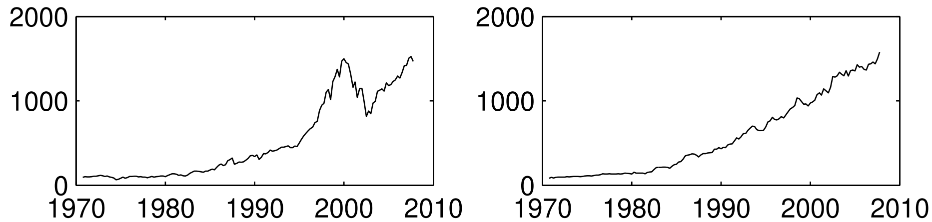
| Variable | Description |
|---|---|
| $rgdp_t$ | Nominal Gross Domestic Product Annualized Growth Rate $((\ln(gdp_t) - \ln(gdp_{t-1})) \times 4 \times 100$ |
| $i_t$ | short interest rate:3-Month Treasury Bill: Secondary Market Rate (percentage) |
| $l_t$ | long interest rate: 10-Year Treasury Constant Maturity Rate (percentage) |
| $\pi_t$ | Annualized Inflation Rate $(\ln(cpi_t) - \ln(cpi_{t-1}) \times 4 \times 100$ |
| $s_t$ | S&P 500 Index continuous compound annualized return $(\ln(ps_t) - \ln(ps_{t-1})) \times 4 \times 100$ |
| $b_t$ | 10 year bond continuous compound annual return $(\ln(pb_t) - \ln(pb_{t-1})) \times 4 \times 100$ |
| $\Delta\kappa_{t,male}$ | First Difference of Mortality index for males from Lee-Carter Model |
| $\Delta\kappa_{t,female}$ | First Difference of Mortality index for females from Lee-Carter Model |



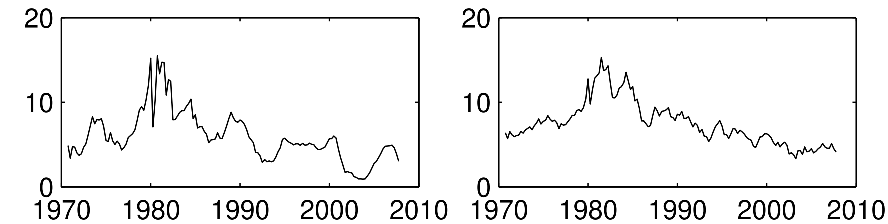

(a) $\kappa_{male}$ (b) $\kappa_{female}$




(c) Nominal GDP (d) CPI




(e) S&P500 price (f) 10-year bond price




(g) 3 month T-bill (h) 10 year interest rate

Figure 3.1: Plots of state variables (1970-2007)

Table 3.2: Descriptive Statistics of State Variables

| | $\Delta\kappa_{male}$ | $\Delta\kappa_{female}$ | $rgdp$ | $s$ | $b$ | $\pi$ | $i$ | $l$ |
|---|---|---|---|---|---|---|---|---|
| Mean | -1.5735 | -1.3574 | 0.0705 | 0.0749 | 0.0793 | 0.0451 | 0.0591 | 0.0749 |
| Std.dev. | 1.3319 | 1.4717 | 0.0367 | 0.3201 | 0.1713 | 0.0329 | 0.0291 | 0.0256 |
| Minimum | -4.8161 | -5.1935 | -0.0123 | -1.2107 | -0.3155 | -0.0174 | 0.0090 | 0.0333 |
| Maximum | 1.4644 | 3.6809 | 0.2272 | 0.7818 | 0.6109 | 0.1631 | 0.1549 | 0.1532 |
| Skewness | -0.3761 | 0.1993 | 1.0822 | -0.8699 | 0.6098 | 1.1027 | 0.9584 | 0.8846 |
| Kurtosis | 2.8041 | 4.2063 | 5.2385 | 5.0256 | 3.4953 | 4.0849 | 4.4217 | 3.3785 |

$\Delta\kappa$ is the first difference of Lee-Carter mortality index, $rgdp$ is the nominal GDP growth rate, $s$ and $b$ are stock annualized returns and bond annualized returns respectively, $\pi$ is the inflation, $i$ is the short term interest, and $l$ is the long term interest rate

Table 3.3: Augmented Dickey-Fuller Unit Root Test For State Variables

| Variable | Test Statistic | P-value |
|---|---|---|
| $rgdp_t$ | -5.2940 | 0 |
| $i_t$ | -2.0433 | 0.2682 |
| $l_t$ | -1.4329 | 0.5647 |
| $l_t - i_t$ | -4.2633 | 0.0007 |
| $\pi_t$ | -2.2193 | 0.2004 |
| $s_t$ | -11.6285 | 0 |
| $b_t$ | -12.7093 | 0 |
| $\Delta\kappa_{t,male}$ | -12.7093 | 0.0254 |
| $\Delta\kappa_{t,female}$ | -3.0348 | 0.0342 |

Table 3.3 shows the ADF results of unit root test for all the state variables. Most of them are stationary at the 5% significant level with three exceptions: the short rate, the long rate, and the inflation. However, the yield spread, the different between long rate and short rate, is stationary. Besides, the time period for estimation is relatively short. Thus, the non-stationarity of those series will be tolerated in my case.

## 3.2 Estimation Results

The estimation of the model is based on the maximum likelihood method and the results are shown in Table 3.4. The interest rates and mortality index are quite persistent, which is consistent with previous studies. The coefficients are 0.85, 0.91, and 0.81, respectively. The $R^2$ of stock return and bond return are relatively low compared to the $R^2$ of other state variables. The correlation of the residuals is shown in Table 3.5.

Table 3.4: VAR Estimation Coefficients, 1970 - 2007

| | Coefficients on lagged variables | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | (1)$\Delta\kappa_{male,t}$ | (2)$\Delta\kappa_{female,t}$ | (3)$rgdp_t$ | (4)$s_t$ | (5)$b_t$ | (6)$\pi_t$ | (7)$i_t$ | (8)$l_t$ |
| cont. | -0.2003 | -0.1327 | 0.0349 | -0.0608 | -0.0562 | 0.0216 | 0.0007 | 0.0028 |
| | ( 0.1602) | (0.2176) | (0.0102) | (0.0987) | (0.0508) | (0.0072) | (0.0037) | (0.0022) |
| $(1)_{t-1}$ | 0.9665 | 0.0684 | 0.0032 | -0.0709 | -0.0058 | 0.0052 | 0.0010 | 0.0004 |
| | (0.0516) | (0.0701) | (0.0033) | (0.0318) | (0.0164) | (0.0023) | (0.0012) | (0.0007) |
| $(2)_{t-1}$ | -0.0677 | 0.8084 | -0.0016 | 0.0648 | -0.0034 | -0.0052 | -0.0004 | -0.0002 |
| | (0.0486) | (0.0660) | (0.0031) | (0.0300) | (0.0154) | (0.0022) | (0.0011) | (0.0007) |
| $(3)_{t-1}$ | -0.7977 | -2.8970 | 0.2975 | 0.1353 | -0.5598 | 0.1155 | 0.0512 | 0.0123 |
| | (1.2960) | (1.7601) | (0.0825) | (0.7987) | (0.4107) | (0.0585) | (0.0296) | (0.0175) |
| $(4)_{t-1}$ | 0.0469 | -0.0483 | 0.0173 | -0.0159 | -0.0772 | -0.0006 | 0.0063 | 0.0040 |
| | (0.1344) | (0.1825) | (0.0086) | (0.0828) | (0.0426) | (0.0061) | (0.0031) | (0.0018) |
| $(5)_{t-1}$ | 0.4027 | 0.2536 | -0.0153 | 0.3818 | -0.0898 | -0.0121 | -0.0003 | -0.0004 |
| | (0.2561) | (0.3479) | (0.0163) | (0.1578) | (0.0812) | (0.0116) | (0.0058) | (0.0035) |
| $(6)_{t-1}$ | -0.2948 | 0.3386 | 0.2176 | 0.6083 | -0.2392 | 0.3453 | 0.0041 | 0.0151 |
| | (1.8472) | (2.5087) | (0.1176) | (1.1384) | (0.5854) | (0.0833) | (0.0422) | (0.0250) |
| $(7)_{t-1}$ | -9.0555 | -9.8190 | -0.2045 | -1.8728 | -1.6215 | 0.7082 | 0.8471 | 0.0593 |
| | (3.6127) | (4.9063) | (0.2300) | (2.2263) | (1.1449) | (0.1630) | (0.0825) | (0.0488) |
| $(8)_{t-1}$ | 8.5524 | 9.8443 | 0.2536 | 2.0676 | 3.7139 | -0.5322 | 0.0686 | 0.8957 |
| | (3.6721) | (4.9871) | (0.2338) | (2.2630) | (1.1638) | (0.1657) | (0.0838) | (0.0496) |
| $R^2$ | 0.8603 | 0.789 | 0.2186 | 0.0784 | 0.1323 | 0.5335 | 0.8471 | 0.9309 |

Note: Standard errors are in parentheses.

Table 3.5: VAR Correlation of Residuals, 1970 - 2007

| | $\Delta\kappa_{male}$ | $\Delta\kappa_{female}$ | $rgdp$ | $s$ | $b$ | $\pi$ | $i$ | $l$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta\kappa_{male}$ | 0.4974 | | | | | | | |
| $\Delta\kappa_{female}$ | 0.7884 | 0.6755 | | | | | | |
| $rgdp$ | -0.0804 | -0.1389 | 0.0317 | | | | | |
| $s$ | 0.0863 | 0.0796 | -0.0422 | 0.3065 | | | | |
| $b$ | 0.2218 | 0.1514 | -0.3934 | 0.1285 | 0.1576 | | | |
| $\pi$ | -0.0317 | -0.0604 | 0.3853 | -0.2322 | -0.3400 | 0.0224 | | |
| $i$ | -0.1305 | -0.1061 | 0.3491 | -0.1302 | -0.5394 | 0.3922 | 0.0114 | |
| $l$ | -0.1704 | -0.1152 | 0.4175 | -0.1861 | -0.9127 | 0.4136 | 0.7010 | 0.0067 |

Note: The correlation matrix shows standard deviations on the diagonal

### 3.3 Forecast Through VAR

Future economic scenarios can be generated from the VAR model through simulations. The idea is drawing random variables from the probability distribution of the error terms given the variables of the last period. Mathematically this look as follows:

$$y_{t+1} = \hat{c} + \hat{B}y_t + L'x \tag{3.2}$$

where $y_{t+1}$ is a $(n \times 1)$ vector with next period scenario values of n state variables, $\hat{c}$ and $\hat{B}$ are estimated constant and coefficients of the VAR, $x$ is a $(n \times 1)$ vector containing random variables from a standard normal distribution, and $L$ is the Choleski decomposition of the estimated covariance matrix of the error term, $\hat{\Sigma}$ , such that $L'L = \hat{\Sigma}$. Through forward iterations, we can get the simulation results for time period $t + \tau$, i.e.,

$$\hat{y}_{i,t+\tau} = \left[\sum_{j=0}^{\tau-1} \hat{B}_j\right] \hat{c} + \hat{B}_j y_t + \sum_{j=0}^{\tau-1} \hat{B}_j L' x_{i,t+\tau-j}, \tag{3.3}$$

where $i$ is one of the $n$ simulated results. Furthermore, quantities related to mortality can be predicted given forecasted mortality index $\kappa_t$, according to equations (2.1), (2.2), (2.3), (2.6), etc. The VAR system in this paper has potential applications in many contexts related with pension and insurance industry, which will be illustrated in the following section.

If we treat the estimated parameters as the true parameters and investigate the possible variations caused by error terms, we only consider process risk, the risk that actual results can vary from predictions based on random chance, even when the underlying system is fully known. However, the method underlying the estimates might not fully capture the true characteristics of the variables, leading to so-called model risk. A particular kind of model risk is the parameter risk which results from sampling inaccuracy, given a selected model (class). In this paper I focus on process risk. Model risk, although also an important part in quantitative analysis, is beyond the scope of this paper. For more discussions on model risk in longevity modeling, see Stevens et al. (2010).

# 4 Quantification of Longevity Risk

### 4.1 Retirement Age and Fair Premiums in Pension Policy

To deal with the deficit of public finance caused by increasing longevity and the financial crisis, several countries have increased, or are planning to increase the official retirement age, so that the pension funds can delay the payment to policyholders for several years. In this section, I quantify the effects of increasing retirement age on fair premium needed for pension plans by means of numerical examples.

### 4.1.1 Background Setting

The pension system analyzed in this paper is analogous to the current Dutch pension system, with necessary modifications and simplifications as mentioned below. In nominal terms the pension fund has a defined benefit and defined contribution structure. In the pension scheme, a policyholder begins to work at age 25 and pays a constant premium $\pi$ to the insurer at the end of each working year until age $R-1$, where $R$ is the retirement age set by the government. In my analysis, the range of $R$ is from 65 to 70. Policyholders build up the right to receive 70% of their annual wages until death. The payment is made at the end of each year after retirement. The last payment is made in the year the policyholder dies. The annual salary is normalized to 1 for each policyholder and there is no pay rise. The maximum age for the individual is 100 years old (the death probability conditioning on being survival at age 100 is one). The number of policyholders in the pool is $I$. I consider two parallel pools, consisting of solely males and females at age 25 in the beginning of the year 2008, respectively. Since I consider a government pension fund in this section, I assume that the size of the pool is large enough, so that the micro-longevity risk is eliminated. I further assume that every policyholder is homogeneous in terms of the future distributions of mortality rates. For simplicity, there are no new entrants into the pool after the beginning time.

### 4.1.2 Fair Premium in the Context of Longevity Risk

At time $t$, the discounted value of the premium income of the pool, denoted by $\Pi$, is

$$\Pi = I \cdot \pi \cdot \sum_{\tau=1}^{R-25} \cdot \exp(-r_t^\tau \cdot \tau) \cdot \frac{1}{I} \sum_{i=1}^{I} 1_{\{T_i \geq \tau\}}, \tag{4.1}$$

where $1_{\{\{T_i \geq \tau\}\}}$ is the indicator function which is 1 if the remaining life time of the person $i$ is at least $\tau$ years and 0 otherwise, $r_t^\tau$ is the $\tau - year$ yield at time t. In reality, we do not know exactly the remaining life time of a policyholder and we can only calculate the expected value. From equation (4.1), assuming that the pool is large enough so that $\frac{1}{I}\sum_{i=1}^{I} 1_{\{T_i \geq \tau\}}$ converges to

${}_{\tau}p_{x,t}$, the expected value of the discounted premium income $E_t\Pi$ is

$$E_t\Pi = I \cdot \pi \cdot \sum_{\tau=1}^{R-25} E_t[({}_{\tau}p_{x,t}) \cdot \exp(-r_t^{\tau} \cdot \tau)], \tag{4.2}$$

where $E_t$ is the expectation conditional on the information at time $t$. The conditional expectation in equation 4.2 explicitly takes into account the possible change of distributions of underlying variables, which is in accordance with the existence of longevity risk. In contrast, traditional actuarial science practices assume that the future survival rates ${}_{\tau}p_{x,t}$ is known at current time. Similarly, the expected liability of the insurer at time $t$ related to a person $i$, denoted by $E_tL$, is

$$E_tL = I \cdot 0.7 \cdot \sum_{\tau=R}^{T_{max}} \cdot E_t[({}_{\tau-25}p_{x,t}) \cdot \exp(-r_t^{\tau+1-25} \cdot (\tau+1-25))], \tag{4.3}$$

The fair premium in the context of longevity risk, $\pi_{fair}$, that the insurer should charge is the premium that makes the expected discounted future premium income equal the expected discounted future liability, considering the possible change of mortality trend in the future. Formally, the quantity can be solved from the equation

$$E_t\Pi = E_tL, \tag{4.4}$$

where $E_t\Pi$ and $E_tL$ are given in equation (4.2) and equation (4.3), respectively.

### 4.1.3 Term Structure Modeling

An important variable in calculating discounted value is the term structure of interest rates. A flat term structure, while often applied for its simplicity, is not supported by the data and often does not satisfy the arbitrage free condition in pricing. Nowadays, there is an increasing trend to apply market valuation methods to pension and insurance liability. As a result, I do not only consider the flat term structure, but also the widely used Vasicek model (Vasicek (1977)) to incorporate the mean reversion feature of the interest rates.

The Vasicek specification for the term structure assumes that the instantaneous spot rate, so-called short rate, follows an Ornstein- Uhlenbeck process

$$dr(t) = \alpha(\mu - r(t))dt + \sigma dW(t), \tag{4.5}$$

where $\alpha$ and $\mu$ are strictly positive and $W(t)$ is a one dimensional Wiener process. The solution to the above SDE between time periods $s$ and $t$, with $0 \leq s < t$ , is

$$r(t) = \mu(1 - e^{-\alpha(t-s)}) + r(s)e^{-\alpha(t-s)} + \sigma e^{-\alpha t} \int_s^t e^{\alpha u} dW_u \tag{4.6}$$

Following Brigo et al. (2008), the discrete time version of equation (4.6), with constant time step $\Delta t = t_i - t_{i-1}$ is:

$$r(t_i) = c + br(t_{i-1}) + \delta\epsilon, \tag{4.7}$$

where

$$c = \mu(1 - e^{-\alpha \Delta t}),$$
$$b = e^{-\alpha \Delta t},$$

and

$$\delta = \sigma\sqrt{(1 - e^{-2\alpha\Delta t})/2\alpha}.$$

Here $\epsilon$ is a white noise with standard normal distribution.

The discrete time version of Vasicek model (4.7) is an AR(1) process and estimators for $c$ , $b$, and $\delta$ are available from time series regression of the short term interest rates. An additional parameter of the Vasicek model is the market price of risk, $\lambda$, which is to adjust the stochastic equation in (4.5) for the risk neutral process. Under the Vasicek model, the solution for the yield at time t with time to maturity, $T$, is :

$$R(t,T) = [A(t,T) - B(t,T)r(t)]/(-(T-t)), \tag{4.8}$$

where

$$B(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha},$$
$$A(t,T) = (B(t,T) - (T-t))(\widetilde{\mu} - \frac{\sigma^2}{2\sigma^2}) - \frac{\sigma^2}{4\alpha}B(t,T)^2$$

and

$$\widetilde{\mu} = \mu - \frac{\lambda}{\alpha}. \tag{4.9}$$

The market price of risk, $\lambda$, cannot be estimated merely by the time series of the short rate. However, as I also have the time series of 10-Year Treasury Constant Maturity Rate (the 'long rate'), $\lambda$ can be calibrated according to equation (4.8) at $T = 10$. To be more specific, $\lambda$ is calibrated by minimizing the sum of square errors between the historical long rates and the theoretical

Table 4.1: Estimates of the Vasicek Model

| Parameter | Estimation | Standard error* |
|---|---|---|
| $\hat{\alpha}$ | 0.6637 | 0.4602 |
| $\hat{\mu}$ | 0.0046 | 0.2204 |
| $\hat{\sigma}$ | 0.0246 | 0.0043 |
| $\hat{\lambda}$ | -0.0318 | 0.0293 |

* Stand errors are obtained by bootstrapping from the residuals.

long rates, based on the Vasicek model given the observed short rates and all other parameters.

The time series parameters of the Vasicek model are obtained from the VAR(1) model. $\lambda$ is calibrated using the historical long term interest rate, which is also one of the state variables in the VAR(1) model. The results are shown in Table 4.1. According to 4.9, The drift term under the risk neutral probability, $\tilde{\mu}$, is 0.0525, which is also the long term average. Given the three month T-bill rate to equal 0.03 at the end of the sample period (the fourth quarter of 2007), it can be expected that the predicted yield curve at that time point will be upward sloping, which is confirmed by Figure 4.1.

### 4.1.4 Numerical Results

Given the estimation results of VAR in section 3 and data at $t = 0$, the end of the sample period (2007 Q4), I simulate future state variables from $t = 1$ to $t = 75$ for 1000 times. Thus, the simulation procedure forecasts the financial and longevity variations from the beginning of the pension fund to the time when the participants achieve the maximum attainable age. As the mortality rates are usually analyzed on an annual basis while the model is estimated by quarterly data, a future step is taken after simulation from the original VAR model. I transform the simulated variables into annual data by selecting the variables at the fourth quarter of each year. Finally, $E_t\Pi$ and $E_tL$ are calculated by averaging the simulation results. I also report the the quantiles of the simulation results.

Figure 4.2 shows the forecast results for the mortality index $\kappa_t$ of both genders. The lower bounds (the 2.5% quantile) and the upper bounds (the 97.5% quantile), are constructed from sorting the simulated paths. In a similar way, Figure 4.3 plots the simulated logarithm of central death rate for 65-year-olds. In general, a nontrivial longevity improvement is predicted by the model while the uncertainty of forecast increases with the length of

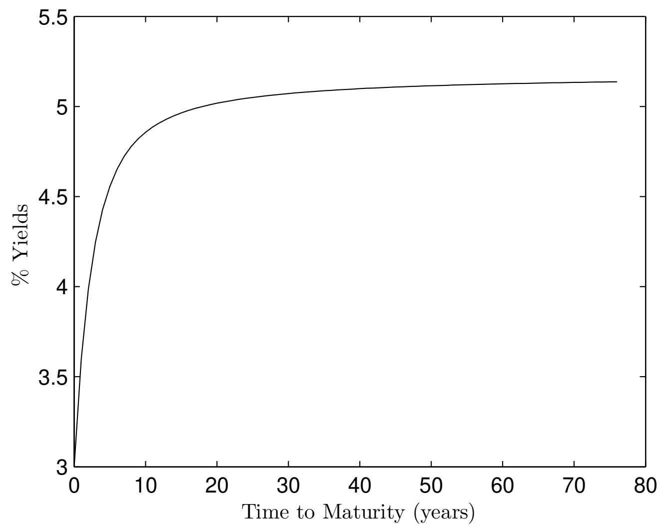
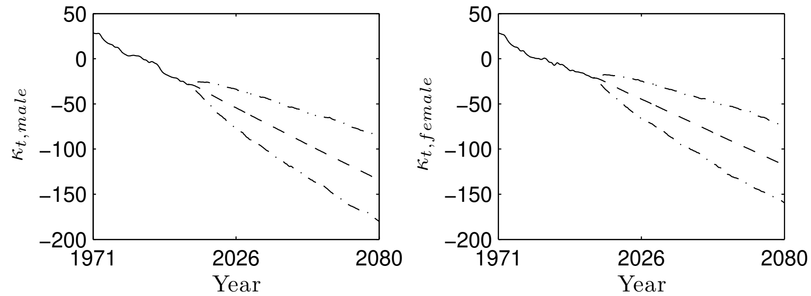



Figure 4.1: The term structure of interest rates, 2007 Q4. The figure depicts the term structure of interest rate in the fourth quarter of 2007 based on the Vasicek model, parameters are given in Table 4.1

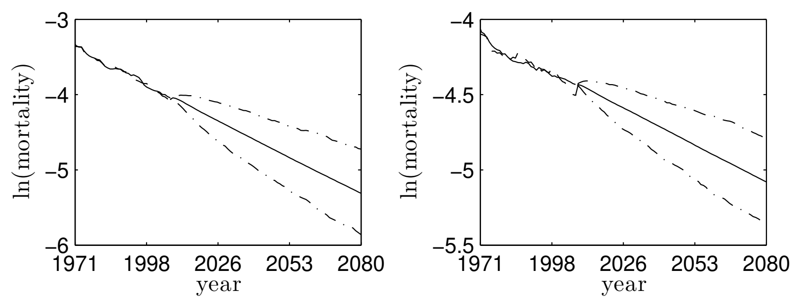
prediction period. Besides, the mortality of males seems to decrease at a higher speed, as reflected by the more negatives slope of $\kappa_{t,male}$.

I investigate the relationship between retirement age and fair premium under two assumptions of the term structure separately, a fixed interest (3%) which equals the value at the end of the sample period, and, a term structure implied by Vasicek model as illustrated in Figure 4.1. Figure 4.4 and Figure 4.5 plot the results. Detailed numbers can be seen in Table 4.2. The premiums for females are higher than those for males for all retirement ages and term structure assumptions, which can be expected as females have a higher life expectancy on average. In every case, the fair premium decreases by approximately one sixth from age 65 to age 67, and by another one sixth from age 67 to age 70 , demonstrating a remarkable tradeoff between the retirement age and the liability of the pension fund, which is a natural consequence of the fact that by raising retirement, more premiums can be collected and less benefits are paid. Under the flat term structure, the fair premiums vary around 0.1 while under the Vasicek term structure, the fair premiums vary around 0.05. The tradeoff is smaller under the Vasicek model. The reason is that the Vasicek model in this case implies higher interest rates in the long run, which discount the liability to a larger extent. In conclusion, raising the retirement age is an effective way to lessen the financial burden



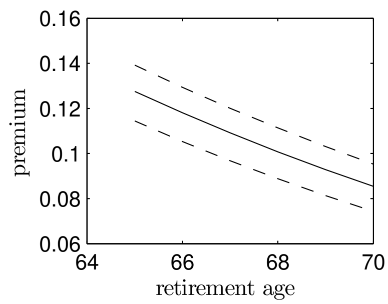

(a) Males (b) Females

Figure 4.2: Estimated and simulated LC mortality index (1970-2082). The solid line is the estimated $\kappa_t$ from sample data. The dashed line is the mean of simulated results. The dash-dot lines are lower (2.5%) and upper (97.5%) forecast bounds respectively.

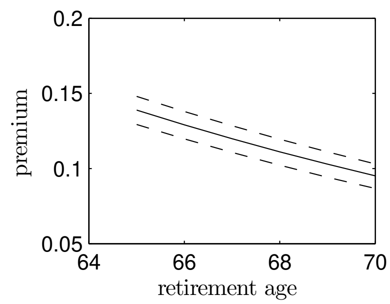



(a) Males (b) Females

Figure 4.3: Historical and simulated logarithm of central death rate for 65-years-olds (1970-2082). The dashed line is the observed data. The dashed line is the estimated and predicted (mean) results. The dash-dot lines are lower (2.5%) and upper (97.5%) forecast bounds respectively.




(a) Males




(b) Females

Figure 4.4: Fair premium and retirement age (flat term structure). The figure shows the fair premiums corresponding to different retirement ages based on projections from VAR under a constant interest rate (3%). The solid line depicts the results based on the predicted means. The dashed lines are lower (2.5%) and upper (97.5%) bounds respectively.

caused by improving longevity, while there are variations in the results among different assumptions related to the future interest rates.

## 4.2 The Characteristics of Future Funding Ratio

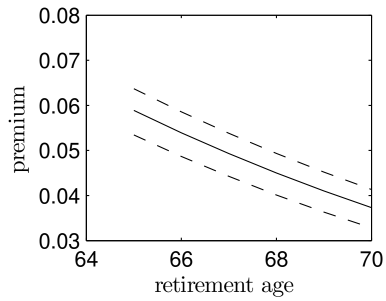
The funding ratio is one of the most popular measures of solvency used by pension funds and insurance companies. Formally, the funding ratio at time $T$ ($FR_T$) , is the ratio between market value of asset at time $T$ ($A_T$) and the market value of liabilities at time $T$ ($L_T$):

$$FR_T = \frac{A_T}{L_T} \tag{4.10}$$

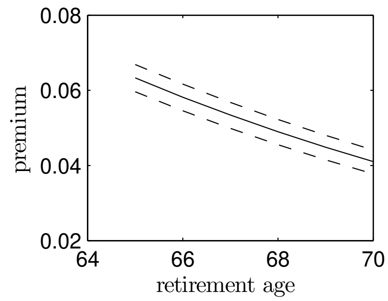
In this part I will quantify the effects of longevity risk on the probability distribution of the funding ratio for a given time horizon $T$. The steps in this section are similar to those in Hári et al. (2008a). However, Hári et al. (2008a) use different methods to predict mortality trend and financial variables.

### 4.2.1 Basic Setting

I consider a nominal defined benefit fund where a policyholder builds up the right to receive an annual benefit of 1 Euro after retirement until death. The payment is made at the end of each year after the policyholder reaches




(a) Males




(b) Females

Figure 4.5: Fair premium and retirement age (Vasicek term structure). The figure shows the fair premiums corresponding to different retirement ages based on projections from VAR under Vasicek interest model as shown in Figure 4.1. The solid line depicts the results based on the predicted means. The dashed lines are lower (2.5%) and upper (97.5%) bounds respectively.

Table 4.2: Fair Premium and Retirement Age

| | age | 65 | 66 | 67 | 68 | 69 | 70 |
|---|---|---|---|---|---|---|---|
| | | | | flat term structure | | | |
| $\pi_{male}$ | mean | 0.1275 | 0.1181 | 0.1092 | 0.1008 | 0.0929 | 0.0854 |
| | $Q(0.025)$ | 0.1144 | 0.1053 | 0.0968 | 0.0888 | 0.0814 | 0.0743 |
| | $Q(0.025)$ | 0.1392 | 0.1294 | 0.1201 | 0.1114 | 0.1031 | 0.0953 |
| $\pi_{female}$ | mean | 0.1389 | 0.1291 | 0.1198 | 0.1111 | 0.1029 | 0.0952 |
| | $Q(0.025)$ | 0.1294 | 0.1197 | 0.1107 | 0.1022 | 0.0942 | 0.0867 |
| | $Q(0.025)$ | 0.1480 | 0.1379 | 0.1284 | 0.1195 | 0.1111 | 0.1031 |
| | | | | Vasicek term structure | | | |
| $\pi_{male}$ | mean | 0.0589 | 0.0539 | 0.0493 | 0.0450 | 0.0410 | 0.0373 |
| | $Q(0.025)$ | 0.0534 | 0.0487 | 0.0443 | 0.0401 | 0.0363 | 0.0328 |
| | $Q(0.025)$ | 0.0637 | 0.0586 | 0.0538 | 0.0493 | 0.0452 | 0.0413 |
| $\pi_{female}$ | mean | 0.0633 | 0.0582 | 0.0534 | 0.0490 | 0.0449 | 0.0410 |
| | $Q(0.025)$ | 0.0596 | 0.0546 | 0.0499 | 0.0456 | 0.0415 | 0.0378 |
| | $Q(0.025)$ | 0.0669 | 0.0616 | 0.0568 | 0.0523 | 0.0480 | 0.0441 |

the age of 65. The last payment is made in the year the policyholder dies. The maximum age for the individual is 100 years old (the death probability conditioning on being survival at age 100 is one). The pension fund consists solely of 65-year-old males (females) in the beginning of the year 2008. The number of policyholders is $I$. The current asset level of the fund is set to be equal to the current market value of the liabilities, so the funding ratio at time $t = 0$ is 1. Besides, there is no new entrants into the fund and no premiums are paid or rights are built up after the beginning time.

Analogous to Section 4.1, I project the VAR system into the future and get the distributions of future state variables. The future mortality rates at time $t$ are determined by the mortality index $\kappa_t$, which is forecasted by the VAR model. The term structure of interest rates is the same as the Vasicek term structure used in Section 4.1. For simplicity, I also assume that the future term structures are in consistent with the forward rates implied from the current term structure, so I do not consider the interest rate risk in this case. Assets can be invested into zero coupon bonds, a stock index and a long term bond. The returns on zero coupon bonds are in line with the term structure. The returns on the stock index and the long term bond are calculated based on the simulated S&P 500 stock index and the 10 year bond price index from the VAR model.

### 4.2.2 The Funding Ratio Distribution

Denote $I$ the number of participants in the fund at time $t = 0$, $x_i$ the age at time $t = 0$ of policyholder $i$, and, $T_i$ the random remaining lifetime of participant $i$, $i = 1, \ldots, I$. The value of the assets at time $t + 1$ equals the gross return of the assets at time $t$, subtracting the benefits paid to the policyholders at time $t + 1$, i.e.

$$A_{t+1} = A_t(1 + R_{t+1}) - \sum_{i=1}^{I} 1_{(T_i \geq t)} \cdot 1 \tag{4.11}$$

where $R_{t+1}$ denotes the net return on the assets between time $t$ and $t + 1$, $1_{(T_i \geq t)}$ is an indicator function which is equal to 1 if the remain lifetime of a policyholder $i$ at time 0 ($T_i$) is at least $t$.

The market value of the pension fund's liability at time $T$ is the sum of the present values of the future cash stream over all participants who are still alive at time $T$, which is a random variable given by:

$$L_T = \sum_{i=1}^{I} 1_{(T_{xi} \geq T)} \cdot \sum_{\tau=0}^{100-(65+T)} \cdot E_T[{}_{\tau}p_{65+T,T} \cdot P_T^{\tau+1}] \cdot 1 \tag{4.12}$$

where ${}_{\tau}p_{65+T,T}$ is the $\tau$ year survival probability of an individual aged $65+T$ at time $T$, and, $P_T^{\tau+1}$ denotes the market value at time $T$ of a zero-coupon bond maturing at time $T + \tau + 1$.

Simulation of the market value of the liabilities ($L_T$) at time $T$ involves:

1. Simulation of future state variables from the VAR model for $t = 1, \ldots, T$ , denote the number of simulations $N1$.

2. Calculation of the $T$ - year survival probabilities of the policyholders according to the simulated mortality index ($\kappa$) and estimated coefficients from Lee-Carter Model .

3. Simulation of the number of living policyholders at time $T$ , which is $\sum_{i=1}^{I} 1_{(T_{xi} \geq T)}$, given the simulated $T - year$ survival probabilities. The death rates of policyholders are assumed to be independent. Thus, $\sum_{i=1}^{I} 1_{(T_{xi} \geq T)}$ can be simulated from a binomial distribution. Denote the number of simulations in this step $N2$ .

4. Determination of

$$\sum_{\tau=0}^{100-(65+T)} E_T[{}_{\tau}p_{65+T,T} \cdot P_T^{\tau+1}] \cdot 1 \tag{4.13}$$

   by projecting the VAR model into the future, conditioning on simulated information at time $T$. Thus, the value of (4.13), which represents the expected liability for one policyholder, is the same for every living policyholder for a certain mortality trend scenario at time $T$ simulated from the VAR.

5. For each of the $N1$ paths generated in step 1, there are $N2$ scenarios of the number of living policyholders from step 4. As a result, there are in total $N1 \times N2$ possible scenarios for future liabilities[2].

[2] I set $N1 = 1000$ and $N2 = 1000$.

### 4.2.3 Numerical Results

I investigate the distribution of the funding ratio over a one year time, that is, $T = 1$. To examine the effect of portfolio size, I let the fund size vary in a rage from 500 to 10000. I study two types of longevity risk, the micro-longevity risk and the combination of micro- and macro-longevity. In the former case, I let the mortality index $\kappa_t$ move deterministically alone the simulated mean values. While, in the latter case, I introduce the randomness of the mortality trend by simulating future $\kappa_t$ for 1000 times.

At first, I focus on the effect of longevity risk by perfectly hedging the market risk. To be specific, the initial assets are exclusively invested into zero bonds which match the initial expected liabilities. I further assume that, at each time, the surplus is reinvested in, and the deficit is financed with, 1-year zero-coupon bonds. Next, I study the joint effects of longevity risk and market risk by investigating three investment strategies: (1) assets are invested exclusively in the long term bond $(100\% - 0\%)$; (2) half of the assets is invested in the long term bond and the other half in the stock index $(50\% - 50\%)$; (3) assets are invested exclusively in the stock index $(0\% - 100\%)$.

The standard deviation of the funding ratio relative to its expectation, $(S_T Dev[FR_T]/E[FR_T])$, as a measure of the riskiness in future funding ratio, is calculated in each case. The 2.5% quantile $(Q(0.025))$ and the 97.5% quantile $(Q(0.975)$ are also reported. The results for males and females are shown in Table 4.3 and Table 4.4 respectively.

Without market risk and macro-longevity risk, $S_T Dev[FR_T]/E[FR_T]$ decreases to 0.001 in a large pool for both genders, which indicates the pooling effects when the size increases. However, macro-longevity risk remains significant as the fund size increases, with the relative standard deviation of the funding ratio varying around 0.01 for both males and females. According to the reported quantiles, if only micro-longevity exists, with probability 2.5%, a fund of size 10000 will encounter underfunding of around 0.3% at a 1-year horizon. In contrast, the figure increases to 2% if macro-longevity risk is also considered. Thus, the risk of macro-longevity is remarkable. In addition, the longevity risk is less severer for the fund consisting of females than the one consisting of males, which is consistent with the finding that death rates of females are less volatile, especially at old ages.

The total riskiness in the future funding ratio increases dramatically if financial market risk is also taken into account. The relative standard deviation of the funding ratio jumps to 0.04 if assets are invested exclusively in long term bonds, and further increase to 0.05 when the assets are composed

Table 4.3: Distribution of Future Funding Ratio at $T = 1$, 65-year-old Males

| | Micro | | | Micro+Macro | | |
|---|---|---|---|---|---|---|
| | 500 | 5000 | 10000 | 500 | 5000 | 10000 |
| | Perfect hedge of market risk | | | | | |
| $S_T Dev[FR_T]/E[FR_T]$ | 0.006 | 0.002 | 0.001 | 0.013 | 0.011 | 0.012 |
| $Q(0.025)$ | 0.989 | 0.996 | 0.997 | 0.976 | 0.979 | 0.977 |
| $Q(0.975)$ | 1.013 | 1.004 | 1.003 | 1.027 | 1.023 | 1.022 |
| | 100% bond - 0% stock | | | | | |
| $S_T Dev[FR_T]/E[FR_T]$ | 0.043 | 0.043 | 0.044 | 0.046 | 0.047 | 0.047 |
| $Q(0.025)$ | 0.881 | 0.886 | 0.884 | 0.880 | 0.880 | 0.880 |
| $Q(0.975)$ | 1.049 | 1.051 | 1.050 | 1.055 | 1.056 | 1.059 |
| | 50% bond - 50% stock | | | | | |
| $S_T Dev[FR_T]/E[FR_T]$ | 0.050 | 0.051 | 0.051 | 0.052 | 0.054 | 0.054 |
| $Q(0.025)$ | 0.886 | 0.878 | 0.884 | 0.881 | 0.874 | 0.879 |
| $Q(0.975)$ | 1.072 | 1.070 | 1.080 | 1.079 | 1.078 | 1.084 |
| | 0% bond - 100% stock | | | | | |
| $S_T Dev[FR_T]/E[FR_T]$ | 0.082 | 0.087 | 0.086 | 0.084 | 0.088 | 0.087 |
| $Q(0.025)$ | 0.839 | 0.827 | 0.826 | 0.834 | 0.825 | 0.825 |
| $Q(0.975)$ | 1.164 | 1.160 | 1.153 | 1.164 | 1.155 | 1.156 |

Table 4.4: Distribution of Future Funding Ratio at $T = 1$, 65-year-old Females

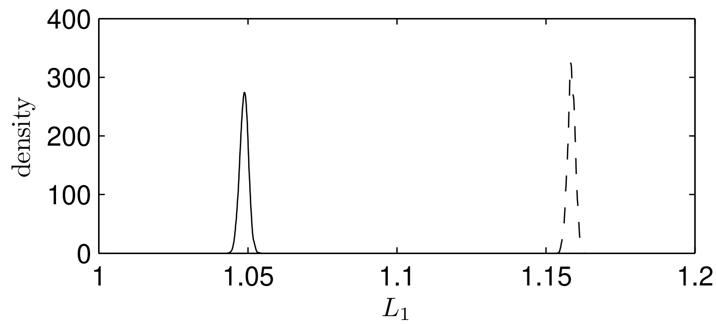
| | Micro | | | Micro+Macro | | |
|---|---|---|---|---|---|---|
| | 500 | 5000 | 10000 | 500 | 5000 | 10000 |
| | | | Perfect hedge of market risk | | | |
| $S_T Dev[FR_T]/E[FR_T]$ | 0.005 | 0.002 | 0.001 | 0.010 | 0.008 | 0.008 |
| $Q(0.025)$ | 0.992 | 0.997 | 0.998 | 0.980 | 0.984 | 0.983 |
| $Q(0.975)$ | 1.010 | 1.003 | 1.002 | 1.019 | 1.016 | 1.014 |
| | | | 100% bond - 0% stock | | | |
| $S_T Dev[FR_T]/E[FR_T]$ | 0.043 | 0.043 | 0.044 | 0.044 | 0.045 | 0.046 |
| $Q(0.025)$ | 0.975 | 0.980 | 0.977 | 0.882 | 0.884 | 0.884 |
| $Q(0.975)$ | 1.158 | 1.160 | 1.159 | 1.050 | 1.051 | 1.053 |
| | | | 50% bond - 50% stock | | | |
| $S_T Dev[FR_T]/E[FR_T]$ | 0.050 | 0.051 | 0.051 | 0.050 | 0.052 | 0.052 |
| $Q(0.025)$ | 0.980 | 0.971 | 0.978 | 0.884 | 0.875 | 0.883 |
| $Q(0.975)$ | 1.183 | 1.182 | 1.192 | 1.073 | 1.074 | 1.082 |
| | | | 0% bond - 100% stock | | | |
| $S_T Dev[FR_T]/E[FR_T]$ | 0.082 | 0.086 | 0.085 | 0.082 | 0.087 | 0.085 |
| $Q(0.025)$ | 0.928 | 0.915 | 0.914 | 0.837 | 0.823 | 0.827 |
| $Q(0.975)$ | 1.285 | 1.280 | 1.272 | 1.162 | 1.157 | 1.153 |

of 50% long term bond and 50% stocks. Finally, this figure increases to around 0.08 if 100% of the assets are invested on stocks. The findings imply that stock returns analyzed in this paper are more volatile than bond returns. However, stock investment also increases the performance of the fund in good states, as reflected by the 97.5% quantile. Thus, a good balance in the asset composition is necessary. In the presence of market risk, adding macro-longevity risk only increases the risk measure marginally. However, macro-longevity is still present in each case. Furthermore, as I only consider a one-year period, it can still be expected that in the long run longevity risk might contribute nontrivially to the overall risk.

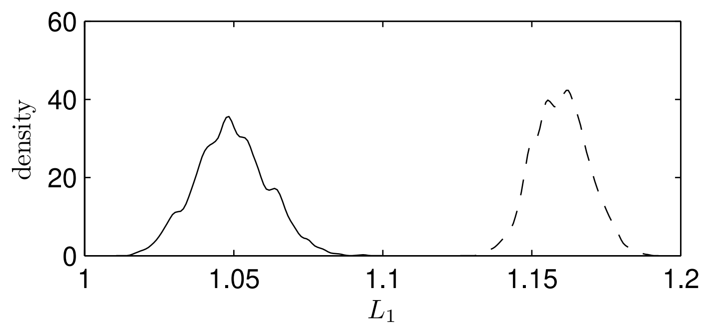
Figure 4.6 and Figure 4.7 show the kernel density estimates of the distribution of future liabilities and assets for different risk sources. The corresponding descriptive statistics are illustrated in Table 4.5. The values are in the unit of $10^5$ Euro. For simplicity, I only report the results for the pool of 10000 participants. As the liabilities are affected by longevity risk and assets are affected by investment risk, the behaviors of liabilities and assets well illustrated the influence of different risks. The standard deviation of the liabilities with micro-longevity risk is around 0.001. The figure increases by about 10 times when macro-longevity risk also considered, indicating a great level of uncertainty when the future trend of mortality is treated as random. The statement can also be supported by the fact that the shape of the distribution of future liabilities with macro-longevity risk is much wider than the one with only micro-longevity risk. Compared to the labilities, the assets are much more volatile and the uncertainty in assets increases with the proportion of stock investment, which are also implied by the statistics and figures. The standard deviation of the assets with 100% bond investment is already around 0.45, three times higher than the standard deviation of liabilities with macro-longevity risk. The figure doubles if the assets are invested exclusively in stocks. Correspondingly, the the distribution of future assets is wider in comparatione with the distribution of liabilities and the width increases when more assets are invested in stocks.

# 5 Conclusion and Recommendation

This paper has investigated the relationship among longevity risk, macroeconomic fluctuations, and financial risk based on U.S data. Correlations have been found between the mortality index from Lee-Carter model and U.S GDP. A VAR(1) model is proposed to forecast mortality trends, macroeconomic conditions and financial market variations simultaneously. Based on

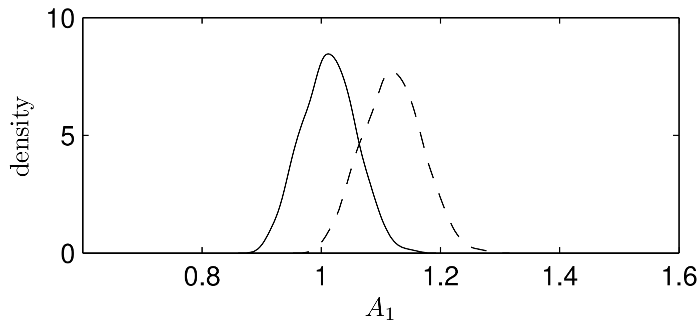



(a) micro-longevity risk

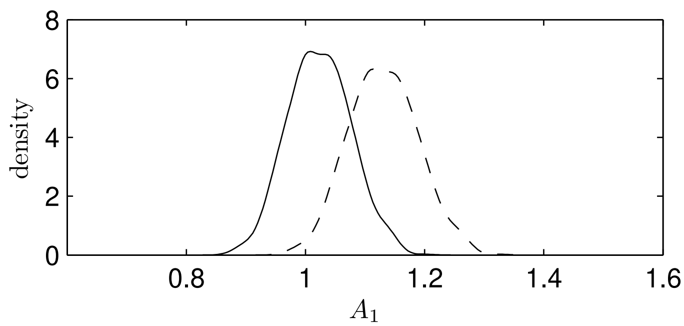



(b) micro- and macro- longevity risk

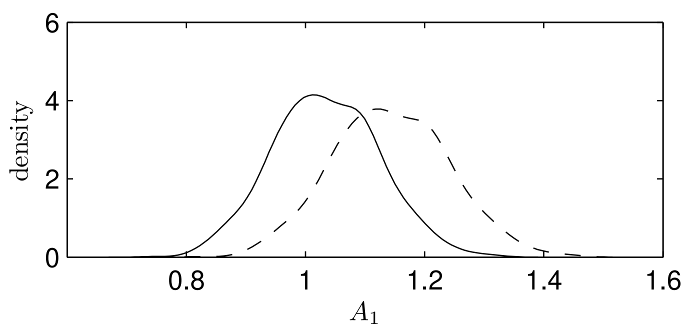
Figure 4.6: Distribution of future liabilities at $T = 1$. The figure shows the kernel density estimates of the distribution of future liabilities for. The solid line depict the results for males and the dashed lines are for females. The liabilities are in the unit of $10^5$ Euro.




(a) $100\%bond - 0\%stock$




(b) $50\%bond - 50\%stock$




(c) $0\%bond - 100\%stock$

Figure 4.7: Distribution of future assets at $T = 1$. The figure shows the kernel density estimates of the distribution of future assets for different investment strategies. The solid line depict the results for males and the dashed lines are for females. The assets are in the unit of $10^5$ Euro.

Table 4.5: Descriptive Statistics of $L_1$ and $A_1$ (unit: $10^5$ Euro)

| | Mean | Min | Max | Std.dev. | Skew. | Kurt. |
|---|---|---|---|---|---|---|
| Male | | | | | | |
| $L_1$(Micro) | 1.0486 | 1.0446 | 1.0529 | 0.0014 | -0.0745 | 2.9540 |
| $L_1$(Micro+Macro) | 1.0487 | 1.0009 | 1.0848 | 0.0122 | -0.1393 | 3.2092 |
| $A_1(100\% - 0\%)^*$ | 1.0129 | 0.8981 | 1.1553 | 0.0446 | 0.0977 | 2.7258 |
| $A_1(50\% - 50\%)^{**}$ | 1.0235 | 0.8706 | 1.2004 | 0.0524 | 0.0856 | 2.7947 |
| $A_1(0\% - 100\%)^{***}$ | 1.0341 | 0.7427 | 1.3172 | 0.0885 | 0.0967 | 2.8850 |
| Female | | | | | | |
| $L_1$(Micro) | 1.1585 | 1.1551 | 1.1628 | 0.0013 | 0.0146 | 2.6977 |
| $L_1$(Micro+Macro) | 1.1593 | 1.1239 | 1.1927 | 0.0095 | -0.1661 | 3.0721 |
| $A_1$(100% - 0%) | 1.1192 | 0.9935 | 1.2753 | 0.0489 | 0.0977 | 2.7258 |
| $A_1$(50% - 50%) | 1.1309 | 0.9634 | 1.3247 | 0.0575 | 0.0856 | 2.7947 |
| $A_1$(0% - 100%) | 1.1425 | 0.8232 | 1.4527 | 0.0970 | 0.0967 | 2.8850 |

* 100% bond - 0% stock
** 50% bond - 50% stock
*** 0% bond - 100% stock

the empirical results, I found that longevity risk plays an important role in various pension schemes. Firstly, in the setting of the typical defined benefit pension plan in this paper, raising the retirement age by 2 years can decrease the fair premium by approximately one sixth for both genders. The results hold for either the flat term structure and the Vasicek term structure. However, if we assume a higher level of interest rate in the future, the effects of delaying retirement age on reducing premiums are relatively smaller. Secondly, longevity risk also substantially increases the uncertainty in the future funding ratio and cannot be eliminated by pooling effects, although its role is overshadowed by financial risk when more risky investment of the asset is evolved.

The approach of this paper can be applied to many related problems in pension and insurance industry. For simplicity and due to lack of data, I have considered only one lag in the VAR model. I also only consider the GDP level as the proxy for macroeconomic conditions. It is straightforward to add more lags and other related variables in the model. In terms of econometric methodology, the model in this paper assumes linear relationship between variables and Gaussian distribution of error terms, which are easy to implement and interpret, but might fail to consider some non-Gaussian fluctuations in state variables, such as the stylized fat tails in stock return distribution. Another caveat has to do with the possible model risk. When studying the implications of the model, I have treated the parameters as known. In fact these parameters may be higher uncertain, and the effect of model risk is worth investigating. Finally, the results of the paper are based on U.S data, which represents the situations of a developed country. However, longevity risk also exist in other developed countries and developing countries and I leave this to future research.

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