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Using SAHARA Utility Functions**

MODELING NON-MONOTONE RISK AVERSION USING SAHARA UTILITY FUNCTIONS

AN CHEN*, ANTOON PELSSER[‡], MICHEL VELLEKOOP[§]

ABSTRACT. We develop a new class of utility functions, SAHARA utility, with the distinguishing feature that it allows absolute risk aversion to be non-monotone and implements the assumption that agents may become less risk-averse for very low values of wealth. The class contains the well-known exponential and power utility functions as limiting cases. We investigate the optimal investment problem under SAHARA utility and derive the optimal strategies in an explicit form using dual optimization methods. We also show how SAHARA utility functions extend the class of contingent claims that can be valued using indifference pricing in incomplete markets.

Keywords: *SAHARA utility, optimal investment problem, dual approach, utility indifference pricing*

JEL-Codes: *G11, G13, G22, D52, C61*

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1. INTRODUCTION

The solution to optimal investment problems strongly depends on the risk preferences of the optimizing agent. Realistic but tractable choices for the description of these preferences is therefore important. To obtain closed-form solutions for optimal investment problems, the assumption of Brownian or Geometric Brownian asset dynamics is often coupled to assumptions of constant absolute risk aversion or constant relative risk aversion for all levels of wealth.¹ The seminal work of [19,20] has shown that for these cases the optimal investment strategies may be derived explicitly.

Sometimes it is useful to work with utility functions defined on the whole real line, but closed-form solutions for the associated optimal investment problems have only been formulated for CARA (exponential) utilities. This paper presents a new alternative. Our utility functions are defined for all wealth levels and allow absolute risk aversion to be non-monotone, but show the same analytical tractability as their constant absolute or relative risk aversion counterparts. We call this new class of utility functions the Symmetric Asymptotic Hyperbolic Absolute Risk Aversion (SAHARA) class. The domain of all functions in this class is the whole real line and for every SAHARA utility function there exists a level of wealth, which we call *threshold wealth*, where the absolute risk aversion reaches a finite maximal value. Hence risk aversion increases when we approach the threshold wealth from above, but we see decreasing risk-aversion for increasingly lower levels of wealth below the threshold wealth. SAHARA utility being concave everywhere implies that it still has a higher disutility for the same loss at low wealth than at high wealth. As a result, SAHARA preferences do not imply the notion of forgiveness in bankruptcy introduced by [27,28]. That would imply that preferences are concave in one region and convex in another while in the SAHARA case agents may become less risk-averse but they never become risk-seeking.

An appropriate choice of preferences is particularly important when optimal investment strategies are used to determine the value of contingent claims in incomplete markets. In

¹Utility functions with constant absolute risk aversion or constant relative risk aversion belong to the class of preferences with Hyperbolic Absolute Risk Aversion (HARA, also known as Linear Risk Tolerance-LRT). For a discussion, see the seminal paper by Merton [20] and the references in Table 1 of that paper.

such markets one can define indifference prices as the necessary monetary compensation for a loss in the optimal expected utility of investments due to the addition of the contingent claim to the agent's portfolio. Contingent claims that could lead to a very large loss with a small probability, such as those based on reinsurance contracts, imply the existence of scenarios where wealth levels are pushed to such lows that constant absolute or relative risk aversion can no longer be assumed. Logarithmic and power utility functions do not allow negative values for wealth so possible losses are automatically assumed to be bounded by the current wealth level. But even exponential utility functions cannot be used to determine the indifference price of a shorted call when the underlying asset is assumed to follow Black-Scholes dynamics since lognormal tails for the possibly unbounded loss are too heavy to give a finite price (c.f. e.g. [11]). A natural application of the indifference pricing with SAHARA utility can be adopted by an insurance company which writes unit-linked life insurance contracts.

We therefore analyze the optimal investment problem under SAHARA utility in a complete market setting and apply the results to the indifference pricing problem in incomplete markets. We are able to derive optimal investment strategies which are explicit functions of the key parameters and the level of wealth. In that sense we have the same tractability as in the classical problem analyzed in [19,20]². Such problems have originally been solved by the dynamic programming approach which requires a Markovian assumption on the state process. In the literature, this approach is called the primal formulation and it leads to the Hamilton-Jacobi-Bellman (HJB) equation for the indirect utility function. To solve the optimal investment problem for SAHARA utility, we rely on the dual approach which converts a dynamic optimization problem to a static one. In a complete market setting the dual method has been studied in [3,13,26] and in an incomplete market setting e.g. by [4,11,12,14,29]. Under SAHARA preferences, the dual approach helps to derive explicit solutions thanks to the role of the marginal utility, since the SAHARA preferences are additive in the dual but not in the primal formulation.

²Merton's optimal consumption and investment problem has been further investigated in for example [1,7,21,30].

To be able to compare the optimal strategies for SAHARA and other utility functions, specific choices for the SAHARA parameters need to be made. In our numerical example, we choose these parameters to specify values for quantities with a clear economic meaning, using a probability of falling below the threshold wealth and a desired asymptotic relative risk aversion coefficient.

The remainder of the paper is structured as follows. Section 2 introduces the SAHARA class of utility functions. Optimal investment problem under SAHARA preferences is discussed in Section 3. Section 4 is dedicated to illustrating some of the results obtained in Section 3 and makes a comparison between the optimal strategies for SAHARA, exponential and power utility functions. Section 5 applies the results to an indifference pricing problem in an incomplete market and Section 6 provides some concluding remarks.

2. SAHARA UTILITY

In this section, we introduce the SAHARA class of utility functions and derive some useful properties that we will use in the remainder of the paper. We also show that the popular classes of power and exponential utility functions are limiting cases of SAHARA utility functions.

DEFINITION 2.1. *A utility function U with domain \mathbb{R} is of the SAHARA class if its absolute risk aversion function $A(x) = -U''(x)/U'(x)$ is well-defined on its entire domain and satisfies*

$$A(x) = \frac{\alpha}{\sqrt{\beta^2 + (x - d)^2}} > 0 \tag{1}$$

for a given $\beta > 0$ (the scale parameter), $\alpha > 0$ (the risk-aversion parameter) and $d \in \mathbb{R}$ (the threshold wealth).

Note that α is a dimensionless quantity and that the dimension of β and d are the same as x , i.e. they are monetary. If we move to different scales of wealth then the scale parameter β and the offset d need to be adjusted (for a constant α). Without loss of generality, we will assume that $d = 0$, i.e. the threshold will always be at zero wealth.

Like HARA utility functions, SAHARA utility shows increasing absolute risk aversion when the threshold wealth $x = 0$ is approached from above, which implies that an economic agent tries to avoid falling below this level. However, under SAHARA utility the agent will not try to avoid the threshold wealth at all costs so the level of risk-aversion is still finite at the threshold wealth $x = 0$. Below the threshold wealth the agent becomes less risk averse since he is relatively indifferent between being slightly or severely below the threshold. In this sense, SAHARA captures the phenomenon that an agent may be more inclined to risk more than usual once he has fallen below his threshold wealth level. But the agent with the SAHARA utility remains risk averse beyond this point, unlike the notion of forgiveness in bankruptcy introduced by [27, 28] in which the preferences are concave in one region and convex in another.

The reason why we have chosen such a functional form for the absolute risk aversion is the convenient structure of the convex dual and the inverse marginal utility of SAHARA utility. Our choice leads to preferences that are additive in the dual, not additive in the primal, thanks to the role of the inverse marginal utility. This allows the derivation of explicit solutions in the dual optimal investment problem since we can express the optimal terminal wealth as a weighted difference of two optimal wealths obtained under power utilities.

We now list some properties for our new class of utility functions.

PROPOSITION 2.2. *Let U be a SAHARA utility function with scale parameter $\beta > 0$ and risk aversion parameter $\alpha > 0$. Then*

1. *There exists constants c_1 and c_2 such that $U(x) = c_1 + c_2\hat{U}(x)$ with*

$$\hat{U}(x) = \begin{cases} -\frac{1}{\alpha^2-1} \left(x + \sqrt{\beta^2 + x^2}\right)^{-\alpha} \left(x + \alpha \sqrt{\beta^2 + x^2}\right) & \alpha \neq 1 \\ \frac{1}{2} \ln(x + \sqrt{\beta^2 + x^2}) + \frac{1}{2}\beta^{-2}x(\sqrt{\beta^2 + x^2} - x) & \alpha = 1 \end{cases} \quad (2)$$

where the domain is \mathbb{R} in both cases. We take $c_1 = 0$ and $c_2 = 1$ from now on.

2. *The SAHARA utility function has Reasonable Asymptotic Elasticity (RAE), which is a sufficient condition for the validity of the dual approach (c.f. [16]):*

$$\lim_{x \rightarrow -\infty} \frac{xU'(x)}{U(x)} = 1 + \alpha > 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} = 1 - \alpha < 1.$$

3. The convex dual $\tilde{U}(y) = \sup_{x \in \mathbb{R}} (U(x) - xy)$ and inverse marginal utility function $I = (U')^{-1}$ are

$$\tilde{U}(y) = \begin{cases} \frac{1}{2} \left(\frac{\beta^2 y^{1+1/\alpha}}{1+1/\alpha} - \frac{y^{1-1/\alpha}}{1-1/\alpha} \right) & \alpha \neq 1 \\ \frac{1}{4}(\beta^2 y^2 - 1) - \frac{1}{2} \ln y & \alpha = 1 \end{cases}$$

and

$$I(y) = \frac{1}{2}(y^{-1/\alpha} - \beta^2 y^{1/\alpha}) = \beta \sinh \left(-\frac{1}{\alpha} \ln y - \ln \beta \right) \quad (3)$$

with domain $y \in \mathbb{R}^+$.

4. The risk aversion function $A(x)$ is maximal at the threshold wealth $x = 0$ and decreasing for increasing positive wealth and increasingly more negative wealth. The prudence function $p(x) = -U'''(x)/U''(x)$ equals

$$p(x) = \frac{\alpha \sqrt{\beta^2 + x^2} + x}{\beta^2 + x^2} = \frac{\alpha}{\sqrt{\beta^2 + x^2}} + \frac{x}{\beta^2 + x^2}.$$

It is positive on the whole domain if $\alpha \geq 1$ but for $\alpha \in (0, 1)$ prudence is negative for $x < -\alpha\beta\sqrt{1-\alpha^2}$. Prudence is maximal when $x/\beta = z$ where z solves $\alpha = (1-z^2)/(z\sqrt{1+z^2})$ and is larger than $\sqrt{1/3}$.

PROOF:

1. The first statement follows by integrating $A(x) = -\frac{d}{dx} \ln U'(x)$ twice. The first integration gives

$$U'(x) = \left(x + \sqrt{\beta^2 + x^2} \right)^{-\alpha} = \beta^{-\alpha} e^{-\alpha \operatorname{arcsinh}(x/\beta)} \quad (4)$$

apart from a constant, and a second integration then gives the result.

2. For the second statement we calculate for $\alpha \neq 1$

$$\lim_{x \rightarrow \pm\infty} \frac{xU'(x)}{U(x)} = \lim_{x \rightarrow \pm\infty} \frac{x(1-\alpha^2)}{x + \alpha\sqrt{\beta^2 + x^2}} = \begin{cases} 1 + \alpha & \text{for } x \rightarrow -\infty \\ 1 - \alpha & \text{for } x \rightarrow +\infty \end{cases}$$

which proves the Reasonable Asymptotic Elasticity property. It is easy to check that this statement holds for $\alpha = 1$ too.

3. For the third part we see from (4) that the domain of I and \tilde{U} is \mathbb{R}^+ . We see that $I(\beta^{-\alpha}) = 0$ so $\tilde{U}(\beta^{-\alpha}) = U(0)$ and

$$I(y) = \beta \sinh(-(\ln y)/\alpha - \ln \beta) = \frac{1}{2}(y^{-1/\alpha} - \beta^2 y^{1/\alpha}).$$

Since $\tilde{U}'(y) = -I(y)$ we have for $\alpha \neq 1$

$$\tilde{U}(y) = \frac{1}{2} \left(\frac{\beta^2 y^{1+1/\alpha}}{1+1/\alpha} - \frac{y^{1-1/\alpha}}{1-1/\alpha} \right) + c_3$$

for some constant c_3 and since $\tilde{U}(\beta^{-\alpha}) = U(0) = \alpha\beta^{1-\alpha}/(1-\alpha^2)$ we must have that $c_3 = 0$. For $\alpha = 1$ we find

$$\tilde{U}(y) = \frac{1}{2} \left(\frac{1}{2} \beta^2 y^2 - \ln y \right) + c_4$$

since $\tilde{U}(\beta^{-1}) = \tilde{U}(\beta^{-\alpha}) = U(0) = \frac{1}{2} \ln \beta$ we see that $c_4 = -\frac{1}{4}$.

4. For the last statement one can use the fact that $p(x) = A(x) - \frac{d}{dx} \ln A(x)$ to find the explicit expression for p given here. Putting the derivative of p equal to zero gives $\beta^2 - x^2 = \alpha x \sqrt{x^2 + \beta^2}$ and scaling by $x = z\beta$ gives the equation for z which equals a number smaller than 1 only when z is larger than $\sqrt{1/3}$. ■

Note that the convex dual $\tilde{U}(y)$ possesses an additive separable structure. The additivity of the dual can be interpreted as the sum of demands of two goods sold at fixed relative prices with additive separable utility.

To derive the results above we have assumed that $\beta > 0$ and $\beta < \infty$ in our definition of the SAHARA utilities. Clearly, the case $\beta \downarrow 0$ corresponds to the risk aversion function of the HARA utilities $A(x) = \alpha/x$ for $\alpha \in (0, 1)$ and $x > 0$, while the choice $\alpha = \gamma\beta$ and $\beta \uparrow \infty$ leads to an exponential utility function with constant absolute risk aversion parameter γ .

2.1. Negative prudence. The influence of the risk aversion parameter α and scaling parameter β are illustrated in Figure 1 which show the utility function U , the absolute risk aversion and the prudence as a function of wealth. Note that there is a wealth level at which prudence is maximal just above the threshold wealth while below the threshold wealth we become less prudent. If our risk aversion parameter α is less than one, we may even have negative prudence (c.f. the righthand side graph in Figure 1).

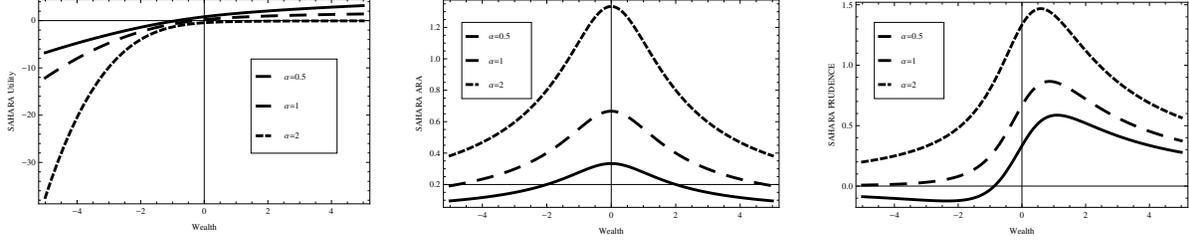


FIGURE 1. SAHARA utility, absolute risk aversion and prudence for $\beta = 1$ and $\alpha = 0.5, 1, 2$.

We see that for $\alpha < 1$ and wealth levels below $-\alpha\beta\sqrt{1-\alpha^2}$ the SAHARA utility exhibits negative prudence, i.e. SAHARA utility may be said to exhibit *precautionary spending*. This is because positive prudence can be interpreted in terms of incentives for precautionary saving, according to [15]. A negative prudence would imply that the agent invests more and more in risky assets and less in the bank account when wealth decreases. How prudence relates to the investment behavior of the agent is discussed more extensively in the next section.

Note that the prudence of a power utility $U(x) = \frac{x^{1-\eta}}{1-\eta}$, $\eta > 0$ is

$$p(x) = \frac{1+\eta}{x}.$$

An agent with such a power utility is always more prudent than the SAHARA-agent if

$$\frac{1+\eta}{x} \geq \frac{\alpha}{\sqrt{\beta^2+x^2}} + \frac{x}{\beta^2+x^2} \Leftrightarrow \alpha \leq \frac{(\eta+1)\sqrt{\beta^2+x^2}}{x} - \frac{x}{\sqrt{\beta^2+x^2}}, \forall x > 0.$$

Since $(\eta+1)\sqrt{\beta^2+x^2}/x - x/\sqrt{\beta^2+x^2}$ is a decreasing function in x , it reaches its minimum as $x \rightarrow \infty$. In other words, a sufficient condition for the prudence of the power utility being greater than or equal to that of the SAHARA utility is

$$\alpha \leq \lim_{x \rightarrow \infty} \frac{(\eta+1)\sqrt{\beta^2+x^2}}{x} - \frac{x}{\sqrt{\beta^2+x^2}} = \eta.$$

For $\alpha > \eta$, we still obtain a greater prudence for the power utility if and only if

$$x > \beta/\sqrt{\eta - (\alpha/2)(\sqrt{\alpha^2 + 4\eta + 4} - \alpha)}.$$

Unfortunately no clear conclusions can be drawn when we compare the prudence of the SAHARA with that of an exponential utility because the latter is a constant.

In the subsequent section, we look at the optimal investment problem under SAHARA utility, where the optimal terminal wealth, the optimal investment policy and the indirect utility function are derived.

3. THE OPTIMAL INVESTMENT PROBLEM

We now formulate the optimal investment problem which dates back to [19, 20]. Assume that we are in a market with a traded asset S satisfying

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W denotes a standard Brownian motion under a probability measure \mathbb{P} , i.e. this asset follows Black-Scholes dynamics with constant instantaneous rate of return $\mu > 0$ and volatility $\sigma > 0$. Also assume the existence of a riskfree asset B which satisfies

$$dB_t = rB_t dt$$

for a deterministic riskfree rate r . An agent with SAHARA utility can only trade in these two assets in a self-financing way starting with initial wealth x_0 . The agent will choose a process θ_t , adapted to the filtration generated by the Brownian motion, which denotes the amount of wealth invested in the risky asset S .

The agent tries to maximize the expected utility of his wealth at a time $T > 0$ so the problem can be stated as

$$\max_{X \in \mathcal{X}(x_0)} \mathbb{E}[U(X_T)] \tag{5}$$

where $\mathcal{X}(x_0)$ denotes the class of all possible wealth processes that can be generated by self-financing strategies θ_t in this market when starting from an initial capital x_0 .

As shown by [3,13], the dynamic stochastic optimal control problem (5) can be formulated as a static optimization problem

$$\max_{X_T} \mathbb{E}[U(X_T)] \quad \text{s.t.} \quad \mathbb{E}[Y_T/Y_0 \cdot X_T] = x_0, \tag{6}$$

where the process Y_t denotes the pricing kernel of the economy. Since we have a complete economy, our pricing kernel is uniquely defined and satisfies $dY_t = -rY_t dt - \lambda Y_t dW_t$,

where $\lambda = (\mu - r)/\sigma$ denotes the market price of risk. Furthermore, the initial value Y_0 serves as a Lagrange multiplier that ensures that the budget equation $\mathbb{E}[Y_T/Y_0 \cdot X_T] = x_0$ is satisfied. In our economy, the pricing kernel Y_T can be alternatively characterised as

$$Y_T = e^{-rT} \left(\frac{S_T}{\hat{C}} \right)^{-\frac{\lambda}{\sigma}},$$

where the constant \hat{C} now acts as the Lagrange multiplier. This alternative formulation reveals explicitly the connection between the pricing kernel Y_T and the so-called growth-optimal portfolio $Z_T = e^{rT} (S_T)^{\frac{\lambda}{\sigma}}$.

Before we proceed to the solution of (6) it is worthwhile to reflect upon the class of wealth processes $\mathcal{X}(x_0)$. As pointed out by [3], the class of self-financing trading strategies in continuous time is so rich that it includes the possibility of so-called doubling strategies (see, e.g. [8]). We therefore need to impose additional constraints on $\mathcal{X}(x_0)$ to rule out these unwanted doubling strategies. The standard approach in the literature is to rule out doubling strategies by imposing a non-negative wealth constraint (see, e.g. [6,18]). This result can be extended by imposing a (negative) lower bound on the wealth processes in $\mathcal{X}(x_0)$ (see e.g. [5]). Imposing lower bounds is appealing since they allow a clear economic interpretation.

In the case of SAHARA utility imposing a lower bound on wealth is much harder since we explicitly allow for negative wealth positions. To make matters worse, due to the declining absolute risk aversion for more negative wealth positions, the agent will follow a strategy that takes on unbounded short positions in the stock whenever S_t approaches zero. And since we may keep a positive position in stock when we have zero wealth, the percentage of our wealth that we invest in stock, θ_t/X_t , cannot be assumed to be finite either. We therefore require throughout the rest of the paper that

$$\theta_t^2 \leq K(1 + X_t^2)$$

for a parameter $K > 0$ which we will allow to depend on the strategy. This allows a position in stock when wealth is zero, but restricts our “leverage” when wealth becomes very large in absolute value (i.e. very positive or very negative). This condition allows us to distinguish between “well-behaved” investment strategies with large short positions

and “pathological” strategies such as doubling or suicide strategies, as the following lemma shows.

LEMMA 3.1. *Under the condition mentioned above, there are no arbitrage possibilities.*

PROOF: It is enough to show that under the new measure \mathbb{Q} defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{rT}Y_T/Y_0$ the discounted gain process $\tilde{X}_t = e^{-rt}X_t - x_0$ is a \mathbb{Q} -martingale. It is certainly a local \mathbb{Q} -martingale since $d\tilde{X}_t = \sigma e^{-rt}\theta_t dW_t^{\mathbb{Q}}$ with $W^{\mathbb{Q}}$ a Brownian motion under \mathbb{Q} so we are done if we prove that $\mathbb{E}^{\mathbb{Q}}[\langle \tilde{X}, \tilde{X} \rangle_t] < \infty$ for all $t \in [0, T]$. Define $H(t) = \mathbb{E}^{\mathbb{Q}}[\tilde{X}_t^2]$, then

$$\begin{aligned} H(t) &= \mathbb{E}^{\mathbb{Q}} \left[\left(\int_0^t \sigma e^{-ru} \theta_u dW_u^{\mathbb{Q}} \right)^2 \right] = \int_0^t \mathbb{E}^{\mathbb{Q}}[\sigma^2 e^{-2ru} \theta_u^2] du \\ &\leq K\sigma^2 \mathbb{E}^{\mathbb{Q}} \int_0^t e^{-2ru} (1 + X_u^2) du \leq K\sigma^2 \left(2 \int_0^t H(u) du + t(2x_0^2 + 1) \right) \end{aligned}$$

which implies by the Gronwall inequality that $H(t) \leq K\sigma^2(2x_0^2 + 1)(t + \int_0^t u e^{2K\sigma^2(t-u)} du)$. But then $\mathbb{E}^{\mathbb{Q}}[\langle \tilde{X}, \tilde{X} \rangle_t] = \int_0^t \mathbb{E}^{\mathbb{Q}}[\sigma^2 e^{-2ru} \theta_u^2] du = H(t)$ which is indeed finite for all $t \in [0, T]$.

■

We will now characterise the optimal wealth strategy and the indirect utility function in the following two theorems.

THEOREM 3.2. *In case of SAHARA utility, the optimal terminal wealth is given by*

$$X_T^* = \frac{1}{2} \left(\frac{S_T}{C} \right)^p - \frac{1}{2} \beta^2 \left(\frac{S_T}{C} \right)^{-p}$$

where $p = \frac{\lambda}{\alpha\sigma}$ and $C > 0$ is a constant given by

$$C = S_0 e^{(r - \frac{1}{2}\sigma^2)T} \left(\beta e^{\text{arcsinh} \frac{x_0}{b(0)}} \right)^{-\frac{1}{p}}.$$

The optimal investment strategy in terms of wealth that should be invested in the risky asset at time $t \in [0, T]$ is given by

$$\theta_t^* = p \sqrt{(X_t^*)^2 + b(t)^2}, \quad b(t) = \beta e^{-(r - \frac{1}{2}\lambda^2/\alpha^2)(T-t)}.$$

PROOF: For our proof, we will closely follow the martingale methodology of [3]. From [3, Theorem 2.1] follows that the terminal wealth under SAHARA utility is given by

$$X_T^* = I(Y_T) = \beta \sinh \left(p \ln \frac{S_T}{C} - \ln \beta \right) = \frac{1}{2} \left(\frac{S_T}{C} \right)^p - \frac{1}{2} \beta^2 \left(\frac{S_T}{C} \right)^{-p}.$$

Note that for every $d > 0$ we have $\mathbb{E}^{\mathbb{Q}}[S_T^d | \mathcal{F}_t] = S_t^d e^{(r + \frac{1}{2}\sigma^2(d-1))d(T-t)}$. We again use \mathbb{Q} to denote the unique equivalent martingale measure defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{rT} Y_T / Y_0$ and we thus have that $\mathbb{E}[(Y_T / Y_t) X_T | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)} X_T | \mathcal{F}_t]$. The optimal wealth process can then be calculated as

$$\begin{aligned} X_t^* &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{2} \left(\frac{S_T}{C} \right)^p - \frac{1}{2} \beta^2 \left(\frac{S_T}{C} \right)^{-p} \middle| \mathcal{F}_t \right] \\ &= e^{(-r + \frac{1}{2} \frac{\lambda^2}{\alpha^2})(T-t)} \frac{1}{2} \left(\left(\frac{S_t e^{(r - \frac{1}{2}\sigma^2)(T-t)}}{C} \right)^p - \beta^2 \left(\frac{S_t e^{(r - \frac{1}{2}\sigma^2)(T-t)}}{C} \right)^{-p} \right) \\ &= b(t) \sinh \left(p \ln \left(S_t e^{(r - \frac{1}{2}\sigma^2)(T-t)} / C \right) - \ln \beta \right). \end{aligned} \quad (7)$$

Imposing the condition $X_0^* = x_0$ we can solve for C .

The trading strategy θ_t^* that replicates the optimal wealth X_t^* can be found by calculating the ‘‘delta-hedge’’ for the wealth $\partial X_t^* / \partial S_t$. This leads to

$$\begin{aligned} \theta_t^* &= pb(t) \cosh \left(p \ln \left(S_t e^{(r - \frac{1}{2}\sigma^2)(T-t)} / C \right) - \ln \beta \right) \\ &= p \sqrt{(X_t^*)^2 + b(t)^2}, \end{aligned} \quad (8)$$

and since this strategy is admissible, we are done. ■

THEOREM 3.3. *In case of SAHARA utility, the indirect utility function associated with the optimal strategy, $u(t, x) = \mathbb{E}[U(X_T^*) | X_t = x]$, is given by the following expression*

$$u(t, x) = \begin{cases} \frac{e^{-(r + \frac{1}{2}\lambda^2/\alpha)(\alpha-1)(T-t)}}{1-\alpha^2} \left(x + \sqrt{x^2 + b^2(t)} \right)^{-\alpha} \left(x + \alpha \sqrt{x^2 + b^2(t)} \right), & \alpha \neq 1, \\ \frac{1}{2} \ln(x + \sqrt{x^2 + b^2(t)}) + \frac{1}{2} (b(t))^{-2} x (\sqrt{x^2 + b^2(t)} - x), & \alpha = 1. \end{cases}$$

PROOF: In principle, we can determine the indirect utility function $u(t, x)$ by a direct evaluation of $\mathbb{E}[U(X_T^*) | X_t = x]$. However, this is a very tedious calculation.

Alternatively, we can exploit the fact that the marginal utility $u_x(t, x)$ is equal to the pricing kernel Y_t for all $0 \leq t \leq T$ (see for example, [3], equation (2.53)). First, we express X_t^* in terms of Y_t as follows:

$$\begin{aligned} X_t^* &= \mathbb{E} \left[\frac{Y_T}{Y_t} \cdot X_T^* | \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\frac{Y_T}{Y_t} \cdot \left(\frac{1}{2} \left((Y_T)^{-1/\alpha} - \beta^2 (Y_T)^{1/\alpha} \right) \right) \middle| \mathcal{F}_t \right] \\ &= b(t) \sinh \left(-\frac{1}{\alpha} \ln Y_t + \left(r/\alpha - \frac{1}{2} \lambda^2 / \alpha \right) (T-t) - \ln \beta \right). \end{aligned}$$

By inverting this equation, we express Y_t as a function of X_t^* and find $u_x(t, x) = Y_t$:

$$\begin{aligned} u_x(t, x) &= e^{(r-\frac{1}{2}\lambda^2)(T-t)} \beta^{-\alpha} e^{-\alpha \cdot \operatorname{arcsinh} \frac{x}{b(t)}} \\ &= e^{(r-\frac{1}{2}\lambda^2)(T-t)} e^{-\alpha(r-\frac{1}{2}\lambda^2/\alpha^2)(T-t)} \left(x + \sqrt{x^2 + b(t)^2} \right)^{-\alpha}. \end{aligned}$$

By integrating in the x -direction we obtain

$$u(t, x) = \begin{cases} \tilde{C}(t) + \frac{e^{-(r+\frac{1}{2}\lambda^2/\alpha)(\alpha-1)(T-t)}}{1-\alpha^2} \left(x + \sqrt{x^2 + b^2(t)} \right)^{-\alpha} \left(x + \alpha \sqrt{x^2 + b^2(t)} \right), & \alpha \neq 1 \\ \tilde{C}(t) + \frac{1}{2} \ln(x + \sqrt{x^2 + b^2(t)}) + \frac{1}{2} (b(t))^{-2} x (\sqrt{x^2 + b^2(t)} - x), & \alpha = 1. \end{cases}$$

If we then impose the boundary condition $u(T, x) = U(x)$ we can conclude that the integration constant $\tilde{C}(t) \equiv 0$ in both cases. ■

Our result for the optimal investment policy θ^* shows that that the economic agent always holds a long position in the risky asset for $\mu > r$. When $\beta = 0$ we find the familiar formula for power utility functions for positive wealth values. Appropriate scaling of wealth by β gives

$$\theta_t^*/\beta = \frac{\lambda}{\alpha\sigma} \sqrt{(X_t^*/\beta)^2 + (e^{-(r-\frac{1}{2}\lambda^2/\alpha^2)(T-t)})^2}$$

and we then see that for $\beta \rightarrow \infty$ the scaled amount put in the risky asset becomes independent of wealth, as in the exponential case. The change in the amount of money invested in the risky asset due to a change in total wealth is controlled by the prudence function and the Sharpe ratio:

$$\frac{\partial \theta^*(X)}{\partial X} = \theta^*(X) p(X) - \frac{\lambda}{\sigma}.$$

For exponential utilities both sides equal zero, for power utilities both are positive constants but for SAHARA utility we may have a negative prudence and both sides are then negative. In that case we put more money in risky assets and less in the bank account when wealth decreases, which shows the “precautionary spending” behaviour associated with negative prudence.

The agent behaves as if he has a power utility for high values of wealth but also when he is far below the threshold wealth since he becomes more and more indifferent between falling below a little or a lot. The minimal amount invested in stock is reached at the

threshold wealth. The amount of money invested in stocks at the threshold wealth equals $\lambda\beta/(\alpha\sigma) \exp[-(r - \frac{1}{2}\lambda^2/\alpha^2)(T - t)]$ which is increasing with the time horizon if λ/α (the Sharpe ratio divided by the risk aversion parameter) is large enough and decreasing otherwise.

4. AN EXAMPLE

In this section we show how the SAHARA parameters α and β can be related to the probability of falling below the threshold wealth. This probability can be calculated as follows:

$$\begin{aligned} \mathbb{P}(X_T^* < 0) &= \mathbb{P}\left(\beta \sinh\left(p \ln \frac{S_T}{C} - \ln \beta\right) < 0\right) = \mathbb{P}\left(p \ln \frac{S_T}{C} < \ln \beta\right) \\ &= N\left(-\lambda\sqrt{T} - \frac{\alpha}{\lambda\sqrt{T}} \operatorname{arcsinh}\left(x_0 e^{(r - \frac{1}{2}(\lambda/\alpha)^2)T} / \beta\right)\right) \end{aligned} \quad (9)$$

with N the cumulative standard normal distribution function. From the first to the second step, we have used the one-to-one correspondence between X_T^* (or S_T) and W_T .

Furthermore, we want the SAHARA parameters α and β to give the desired relative risk aversion. Since this coefficient is a function of wealth, we require the asymptotical relative risk aversion

$$\lim_{x \rightarrow \infty} RRA(x) = \frac{\alpha x}{\sqrt{\beta^2 + x^2}} = \alpha \quad (10)$$

to be within a desired range. This shows how α can already be determined by the desired relative risk aversion, without referring to the value of the scaling parameter β .

Table I illustrates several combinations of (α, β) . The table shows how to select a value for β given a value for α and a given level ε of falling below the threshold wealth. Note that β is expressed as a multiple of x_0 .

Using the chosen parameters, we can plot the optimal wealth and optimal investment strategy under SAHARA utility. Unless stated otherwise, the following parameters are used:

$$X_0 = 1, S_0 = 0.1, r = 0.03, \mu = 0.08, \sigma = 0.15, t = 1/2, T = 1,$$

TABLE I. Combinations of α and β

α	β			
	$\varepsilon = 0.0005$	$\varepsilon = 0.001$	$\varepsilon = 0.005$	$\varepsilon = 0.05$
0.5	$0.23x_0$	$0.27x_0$	$0.39x_0$	$0.83x_0$
1	$0.85x_0$	$0.92x_0$	$1.19x_0$	$2.16x_0$
2	$1.98x_0$	$2.14x_0$	$2.66x_0$	$4.61x_0$
3	$3.06x_0$	$3.29x_0$	$4.07x_0$	$7.00x_0$
4	$4.12x_0$	$4.43x_0$	$5.36x_0$	$9.38x_0$
5	$5.18x_0$	$5.56x_0$	$6.85x_0$	$11.74x_0$

Parameters used are: $r = 0.03$; $\mu = 0.08$; $\sigma = 0.15$; $T = 1$. The parameter ε denotes the probability of the optimal terminal wealth falling below the threshold wealth 0.

where t is the time point when we examine the optimal strategy. In some cases, we add the optimal wealth and optimal strategy for exponential and power utility for comparison³. In particular we set the risk aversion coefficients γ and η

$$\gamma = \eta/X_0 = \frac{\alpha}{\sqrt{\beta^2 + X_0^2}} \quad (11)$$

to match the risk aversion of the two agents at the initial time.

In Figure 2 optimal wealth is plotted as a function of the terminal realization of the risky asset S_T for the three utilities. Naturally, a positive relation between the optimal wealth level X_T^* and S_T is observed for all the cases. As already pointed out, for extreme good or bad market scenarios, the SAHARA agent is less risk averse than the exponential agent. This leads to a larger positive final wealth in a flourishing market, but more investment in the risky asset would also cause a larger negative wealth for the SAHARA agent in an unfavorable financial market. Therefore, Figure 2 demonstrates that the SAHARA strategy over- and underperforms relative to the exponential utility for high and low values of S_T respectively. The optimal terminal wealth from the power utility is always positive and may over- or underperform the SAHARA utility. Note that a choice for power utility can

³With an exponential utility $U(x) = -\frac{1}{\gamma}e^{-\gamma x}$, the optimal $Y_0^* = \exp\{rT - \gamma X_0 \exp\{rT\} - \frac{1}{2}\lambda^2 T\}$ and the optimal terminal wealth is $X_T^* = -\frac{1}{\gamma} \ln Y_T^*$. With a power utility $U(x) = \frac{x^{1-\eta}}{1-\eta}$, the optimal $Y_0^* = X_0^{-\eta} \exp\{-r\eta T + rT + \frac{1}{2}\lambda^2 T + \frac{\eta}{2}\lambda^2 T\}$ and the optimal terminal wealth is $X_T^* = (Y_T^*)^{-1/\eta}$.

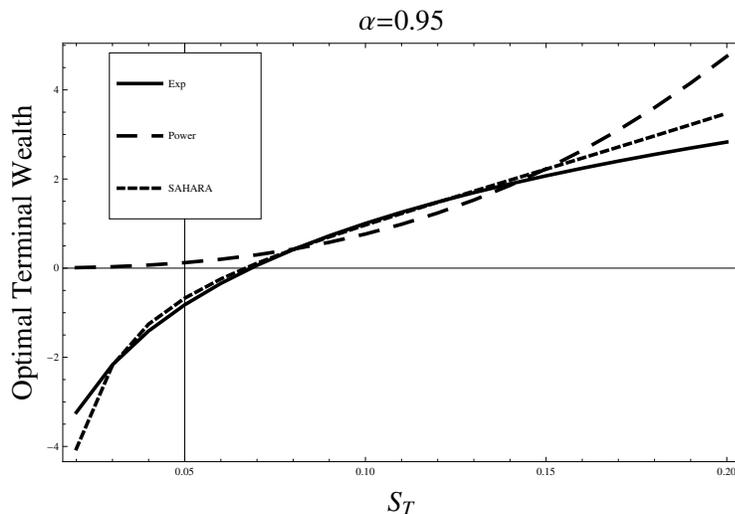


FIGURE 2. Optimal terminal wealth for three utility functions as a function of S_T with $\alpha = 0.95$ and $\varepsilon = 0.01$.

be interpreted as the implementation of a 'portfolio insurance strategy' which makes sure the terminal wealth will always be strictly positive.

Figure 3 reports the optimal amount θ_t^* a SAHARA-investor should invest in the risky asset at time t depending on his wealth at that time, for different volatility levels. First, both graphs exhibit the same shapes in which the SAHARA-investor invests the least amount in the risky asset around the threshold wealth ($X_t = 0$), since SAHARA utility reaches its maximum absolute aversion at that point. Second, the more volatile the risky asset is, the less amount will be invested in the risky asset. Hence, it is observed that the solid curve ($\sigma = 0.15$) lies above the other two. Third, for $\alpha = 2$, less amount is invested in the risky asset than $\alpha = 0.95$. The reason is that the combination of $\alpha = 2$ and $\beta = 2.66$ (i.e. $\varepsilon = 0.005$) leads to a higher absolute risk aversion than the ones generated by $\alpha = 0.95$ and $\beta = 1.12$ (i.e. keeping the same probability of falling below the threshold $\varepsilon = 0.005$).

Figure 4 compares the optimal amount for three utility functions. Under exponential utility, a fixed *amount* is invested in the risky asset. Under power utility, a constant *fraction* of wealth is invested in the risky asset and this has the consequence that the amount in the risky asset is increasing in the wealth level X_t .

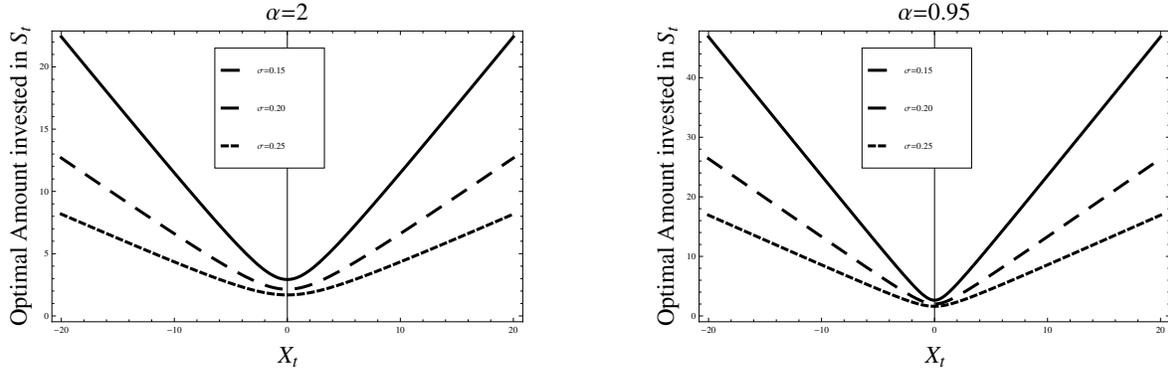


FIGURE 3. Optimal investment amount in risky asset under SAHARA utility for $\alpha = 2$ (left) and $\alpha = 0.95$ (right) with parameter $\varepsilon = 0.005$.

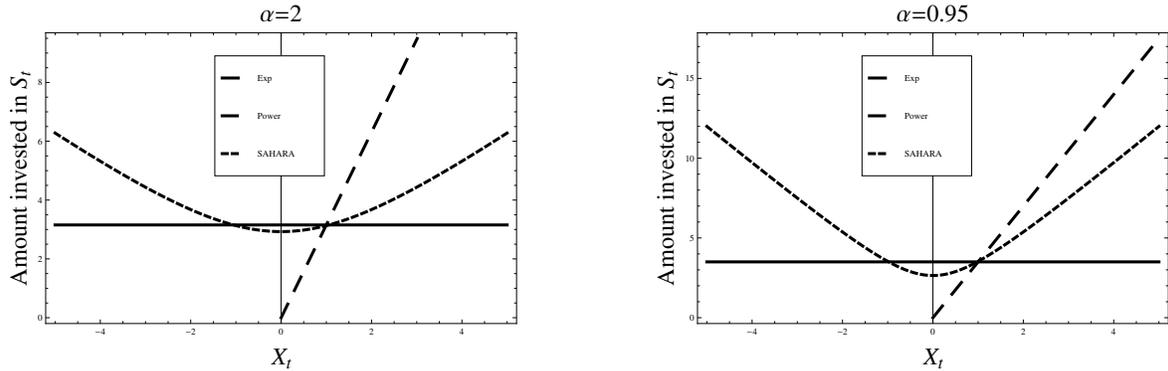


FIGURE 4. Optimal amount for three utility functions for $\alpha = 2$ (left) and $\alpha = 0.95$ (right) with $\varepsilon = 0.005$.

In Figure 5 we plot the densities of the differences in optimized terminal wealth $X_T^{*,\text{SAHARA}} - X_T^{*,\text{exp}}$ and $X_T^{*,\text{SAHARA}} - X_T^{*,\text{pow}}$. We use the same parameter values and the same matching of risk aversion in the initial wealth point as defined in (11). We notice that the distribution for the difference between SAHARA and exponential terminal wealth has a large probability to attain a small negative value and a small probability to attain large positive

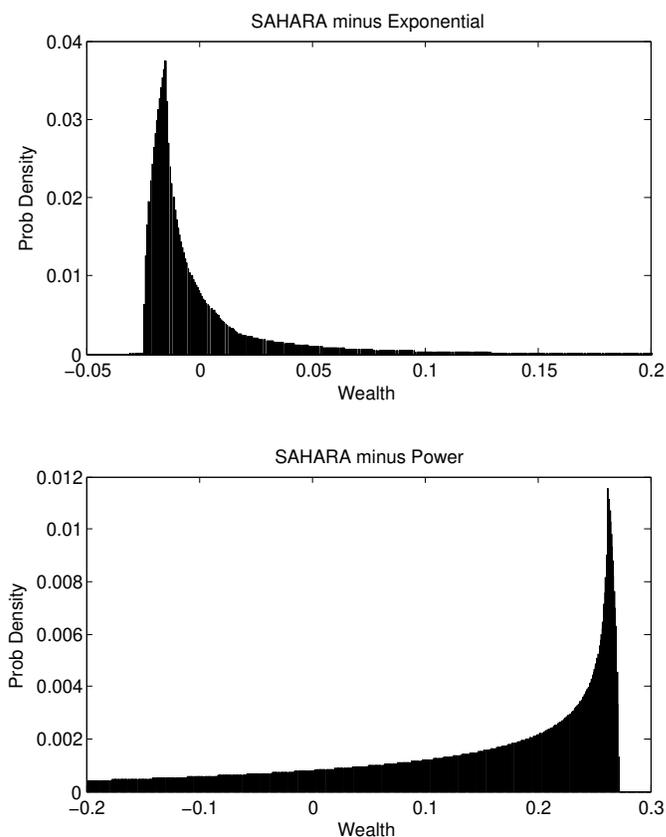


FIGURE 5. Distribution of Differences in Optimal terminal wealth, for $\alpha = 2$ and $\varepsilon = 0.05$.

values. This is due to the fact that for large positive and large negative wealth values the SAHARA utility is less risk averse than the exponential one but for values around zero SAHARA is more risk averse. The difference with power utility optimal wealth shows a large probability of positive values and a relatively fat tail for large negative values. Under optimal investment strategies for power utility, wealth will always remain positive since risk aversion becomes infinite in the threshold wealth. Optimal strategies based on SAHARA utility functions try to avoid the threshold wealth but become less risk averse

once they have crossed it. This explains the fat tails to the left which correspond to very disadvantageous scenarios for the equity process.

5. INDIFFERENCE PRICING

In this section we want to consider the application of the SAHARA utility function to indifference pricing. Indifference pricing has been well-studied in recent years for exponential and power utility functions, see e.g. [2,11,12,17,22,23,24,25].

One of the problems with the “traditional” utility functions is their inability to cope with short positions in unbounded claims. Under power-utility, any possibility of negative wealth is unacceptable, hence short positions with unbounded payoffs cannot be priced at all. Exponential utility is defined for all levels of wealth, so negative wealth is acceptable under exponential utility. However, even under exponential utility an unbounded short position in an asset with a log-normal distribution (or any other distribution with even fatter tails) cannot be priced. This is the well-known “short call” problem, see e.g. [11].

Our class of SAHARA utility functions can be used to price “short call” positions. As the SAHARA utility function behaves asymptotically as a power-function and not as an exponential function, the expectations do converge for log-normal distributions under SAHARA utility.

To illustrate the application of the SAHARA utility function to indifference pricing, we consider the following example. We take a stylized form of a unit-linked insurance contract. Today, at $t = 0$, a policyholder pays an initial amount (say €1) which is invested into the traded asset S . At the maturity date T of the contract the policy holder receives the accumulated value S_T/S_0 of the equity investment, provided the policyholder is still alive at time T . In case the policyholder has died, a cash amount of C_1 is paid to the policyholder’s beneficiaries.

To keep our example as simple as possible, we assume that the uncertainty about the policyholder’s life is not resolved until time T , and that the mortality process is independent from the financial market process S_t . The expected utility of an agent with a

SAHARA utility function after writing this insurance claim is given by

$$q\mathbb{E}[U(X_T - S_T)] + (1 - q)\mathbb{E}[U(X_T - C_1)], \quad (12)$$

where q denotes the probability that the policyholder survives until time T .

The agent will now try to find an optimal investment strategy that leads to an optimal terminal wealth distribution X_T^{**} . We can proceed in a similar way as in Section 3 and find the following first-order condition for the optimal wealth:

$$qU'(X_T^{**} - S_T) + (1 - q)U'(X_T^{**} - C_1) = Y_T. \quad (13)$$

This first-order condition cannot be solved analytically. However, for any given pair of values $\{S_T, C_1\}$, the function $x \mapsto qU'(x - S_T) + (1 - q)U'(x - C_1)$ is monotonically decreasing in x and can assume any value in \mathbb{R} , so a solution to (13) exists and is unique.

In our illustrations that we will give below, we have solved (13) numerically in the following way. We first generate a grid with equally spaced outcomes for the Brownian motion W_T spanning the interval $[-5\sqrt{T}, 5\sqrt{T}]$. For each grid-point W_n we calculate the values $S_n = S_T(W_n)$ and $M_n := Y_T(W_n)/Y_0$ that are generated by the value W_n of the Brownian motion W_T . Then we search numerically for the values X_n^{**} that solve (13) on each grid-point. Given the values of X_n^{**} we obtain on the grid, we can then evaluate the expected utility (12) via numerical integration. Furthermore, we can calculate the initial cost X_0^{**} of the optimal wealth by numerical integration of the budget constraint in $t = 0$.

To fully solve the utility indifference pricing problem we have to equate the level of expected utility that can be achieved before writing the insurance claim, to the level of utility that can be achieved after writing the claim. We can control the level of expected utility in (12) by changing the parameter Y_0 in (13). For this, we need to perform an additional numerical search to find the correct value of Y_0 . The difference in value $X_0^{**} - x_0$ is then the utility indifference price of the insurance claim.

Let us consider a concrete example. For the traded asset we assume $\mu = 0.07, r = 0.04, \sigma = 0.15$. For the SAHARA utility function we take $\alpha = 2, \beta = 2.5$ with an initial

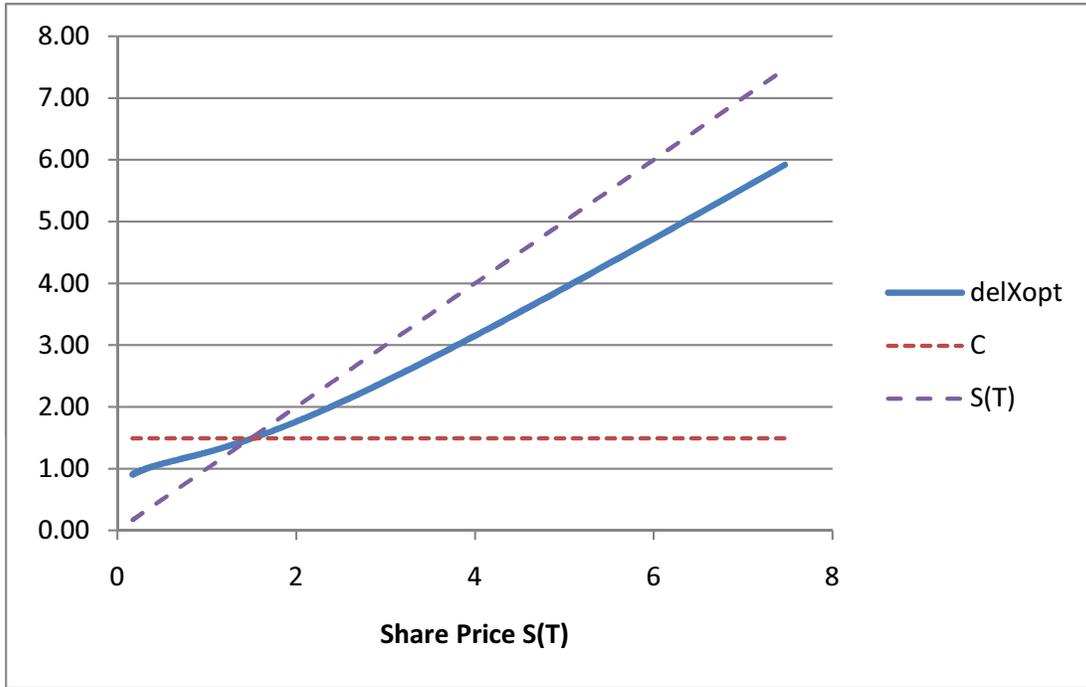


FIGURE 6. Optimal Hedge Portfolio for Unit-Linked Insurance Contract.

wealth position $x_0 = 1$. For the insurance contract we assume $T = 10$, an initial payment of 1, a cash payment of $C_1 = 1e^{rT} = 1.49$ and a survival probability $q = 0.5$. We have constructed the insurance contract in such a way that the market value of the asset payoff and the cash payoff are both equal to 1.

If we calculate for these parameters the optimal investment strategy before writing the insurance claim X_T^* , we can achieve an expected level of utility of -0.112 . This level of expected utility can be matched after writing the the insurance claim by setting $Y_0 = 0.0666$, which corresponds to an initial wealth position of $X_0^{**} = 2.025$. Therefore, the utility indifference price of the insurance contract under SAHARA utility is equal to $X_0^{**} - x_0 = 1.025$. Hence, the extra price that a SAHARA-agent would require as compensation for the unhedgeable mortality risk is 0.025.

We can also investigate the hedge-portfolio that the agent holds to optimally hedge the insurance claim. We therefore plot the difference between $X_T^{**} - X_T^*$ as a function of S_T in Figure 6. The solid line “delXopt” depicts the optimal hedge portfolio $X_T^{**} - X_T^*$. We have also plotted in the same figure the cash payout of the insurance contract (dotted line

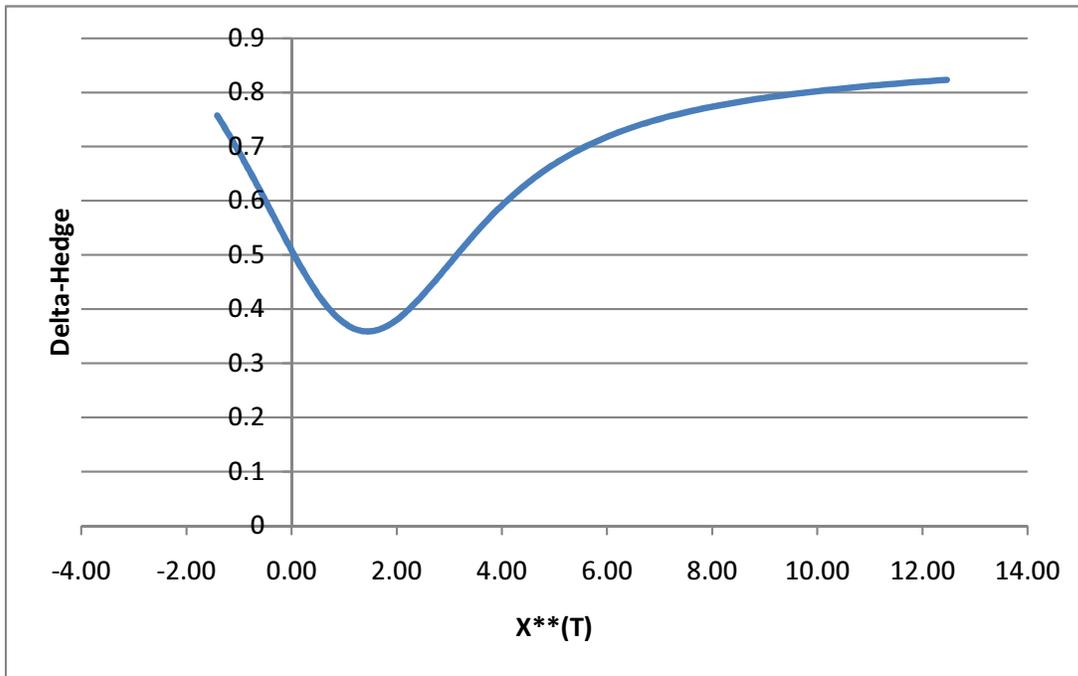


FIGURE 7. Delta-Hedge for Unit-Linked Insurance Contract.

“C”) and the unit-linked payout (dashed line “S(T)”). We see that the optimal hedge tries to strike a balance between the two possible payouts of the insurance contract. Because the agent is more worried about losses than about gains (due to the convex shape of the utility function) the optimal hedge is closer to the highest possible payout.

The optimal hedge portfolio is a non-trivial function of the underlying share-price. In Figure 7 we have plotted the delta-hedge $\partial(X_T^{**} - X_T^*)/\partial S(T)$ against the optimal wealth X_T^{**} . We see that the optimal delta-hedge is driven by the risk aversion (or rather the “risk tolerance” $1/A(x)$). For low levels of wealth the risk tolerance is low, which leads to a relatively low exposure in the share-price. For very low and very high levels of wealth, the agent has a higher risk tolerance and hence the exposure in the share price is increased.

6. CONCLUSION

In the present paper, we introduce the new class of SAHARA utility functions which are quite tractable and incorporate power utility and exponential utility functions as limiting cases. Under SAHARA utility, risk aversion increases when we approach the threshold

wealth from above, but below the threshold wealth an agent becomes more and more indifferent between falling below the threshold a bit or more severely. We solve the optimal investment problem under SAHARA utility in a complete market setting and are able to obtain analytical solutions for the indirect utilities and optimal investment strategy thanks to the convenient form of the inverse marginal utility. We have also shown how indifference prices in incomplete markets can be defined under this assumption. This will be very helpful for financial institutions such as life insurance companies. Many unit-linked types of insurance contracts that are sold by insurance companies contain embedded options such as guarantees which are modelled using an unbounded loss distribution, so models which allow utility to be defined on the whole real line are of particular importance in that case. Even though possible losses will always be bounded in practice, many models in the existing literature do not impose such a restriction. We therefore believe that SAHARA utility functions will prove to be very useful, since they allow an extension of the class of contingent claims which can be valued using indifference pricing methods.

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