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Intergenerational Risk Sharing of Pension Schemes under Modified Utility Function

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Intergenerational risk sharing of pension schemes under modified utility function

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Abstract

In this paper, I want to do some research in the paper: *Intergenerational risk sharing within funded pension schemes* (Jiajia Cui, Frank de Jong and Eduard Ponds, 2011) published on *Journal of Pension Economics and Finance*. The original paper mainly studies the risk sharing between generations in different kinds of funded pension schemes. At first, I build models and simulate them via Matlab software to replicate the path used in this paper, in order to find out the methods and verify the results. After that, I make some changes of the utility function used in the paper for searching the optimization point for some funded pension schemes. Instead of the traditional CRRA (constant relative risk aversion) model, I want to change the CRRA utility function from constant risk aversion rate (γ) to a function containing different risk aversion parameters (γ) for different age cohort. Expectedly, the new optimal points we get might have difference compared with ones in the original paper, and we might get some new observations of performance of different pension schemes. Besides, for participations of different kinds of funded pension schemes, their risk aversion parameter (γ) might change due to the different income styles, and this should be taken into consideration when comparing risk sharing between generations. Moreover, if using this change into models which calculate the investment allocation, the percentage of investment in risky and riskless asset should be change conceptually.

Contents

Acknowledgement	4
1 Introduction	5
2 Replicating models	7
2.1 Basic parameters setting and relevant functions	7
2.2 Risk-allocation rules and optimization method	8
2.3 Optimal pension schemes and welfare evaluation	10
2.4 Optimal individual scheme.....	14
2.5 Figures of intergenerational risk sharing	18
2.6 Market valuation of pension schemes.....	20
3. Modified Utility Function.....	22
3.1 Building models	22
3.2 Analysis of results.....	23
3.2.1 Analysis of results in DB_{CA}	24
3.2.2 Analysis of results in DB_{BA}	26
3.2.3 Analysis of results in DB_H	28
3.3 Analysis of figures.....	30
3.3.1 Analysis of figures in DB_{CA}	30
3.3.2 Analysis of figures in DB_{BA}	33
3.3.3 Analysis of figures in DB_H	35
3.3.4 Optimal individual schemes	38
3.4 Comparison between different pension schemes.....	40
3.4.1 Comparison of optimal parameters and figures.....	40
3.4.2 Comparison of market valuation	44
4. Estimation Error and Extension.....	45
5. Conclusion.....	46
Reference.....	48
Appendix.....	49

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1 Introduction

There have been many literatures to research the optimal designed intergenerational risk sharing (IRS) in the past. Since one can not trade their human capital, there does not exist private insurance based on IRS. Except for that, many papers are published for institutionalizing IRS and completing market such as fiscal policy and public debt arrangement (Fisher (1983) etc.), PAYG social security program (Morton (1983) etc.) and funded pension system (Gollier (2006) etc.). One of the latest papers of the last topic is Jijia Cui, Frank de Jong and Eduard Ponds (2011). This paper researches three different realistic collective defined benefit (DB) pension schemes: benefit adjustment (DB_{BA}), contribution adjustment (DB_{CA}) and hybrid (DB_H), and investment portfolio according to the expected utility of life time consumptions.

This paper plans to keep researching the three collective pension schemes as mentioned above, and try to optimize risk allocation and investment portfolio with changing utility models. Like to Cui.J et al. (2011), most papers use the constant relative risk aversion (CRRA) model to evaluate expected payoff and find out the optimal allocation. Meanwhile, we should notice that people in different generations probably have different risk aversion rates in reality. Even more, the participants in different pension schemes, which are in the same generation, might have different risk aversion rates either. Therefore, I would like to change the CRRA function to a CRRA function group which contains different risk aversion parameters (γ in the function) for different generations and try to find out optimal allocations for the three pension schemes. Besides, I want to regard the results from the ordinary CRRA function as benchmark and find out what are changed due to the model changing.

This paper is based on a static optimization of the pension fund and used contingent claims analysis to value the risk sharing and transfers between generations. The later part have been mentioned by Blake (2002) and some other papers. More important, I will continue the thought of Jijia Cui etc. (2011) which uses optimal individual scheme as the bench mark. "Pension schemes that lead to smoother and higher consumption levels are ranked higher in utility term." (Jijia Cui etc. (2011)) The assumptions of models in this paper are also basically similar with it, i.e. 1) constant real interest rate in reality, 2) wage inflation being identical to the price inflation.

Indeed, there are some materials shows the support viewpoints. One of them is a Netspar discussion paper, *Cognitive aging and risk attitude* (Erik Bonsang and Thomas Dohmen, 2012). In this paper, the authors concluded that they have shown the age related change in risk attitude is largely associated with cognitive aging, via doing research on the SHARE data. Even though there are still some other

areas which the author could not discuss due to the limit of knowledge, such as psychology part, it still provides support and guidance to this paper.

In this paper, we have a few findings. Firstly, with changing risk aversion rates of different generations, we make comparison of results of same collective pension funds, DB_{CA} , DB_{BA} and DB_H . As expected, in contribution adjustment pension funds, changing the risk aversion rate of youth generation have the biggest influence, while in benefit adjustment one, changing the risk aversion rate of old generation have the biggest influence. In the hybrid pension funds, meanwhile, either of both changing could influent the funds. On the other hand, even though not much, changing risk aversion rate of the generation which has constant consumption, i.e. old generation in DB_{CA} and youth generation in DB_{BA} , could also bring some influence to the optimal parameters and the whole pension funds.

Secondly, as the original paper mentioned, the hybrid collective pension funds could share the risk most efficiently, one of which is that it has the smallest fluctuation in consumption before and after retirement. However, with increasing the difference of risk aversion rates among generations, we could find that the fluctuation of benefit adjustment pension funds decreases while the one of hybrid pension funds increases. When reaching a certain level of risk aversion rates, benefit adjustment pension funds performs same or even more efficiently than hybrid one on the side of fluctuation of consumption at retirement time.

Thirdly, during replicating the process of calculating several results provided by the original paper, we have some different results compared with original ones. After careful analysis, we believe some of them are not caused by differences of calculating methods, error of system or numerical methods. Since we could not find an appropriate explanation of them, we could only believe it is just some mistakes of the original paper. For instance, when calculating the start point of benefit level of defined benefit pension funds with benefit adjustment (DB_{BA}) under risk aversion rate equaling to 5, we found that 61.94% instead of 53.6% in original paper. Since there are no random variables or uncertain parameters, and the method to calculate this is quite accurate, we believe that it should be a mistake in original paper.

The paper is organized as follows. For Section 2, I will replicate the model which shown on the paper Intergenerational risk sharing within pension schemes, make clear of the path to get the result and make a check. For Section 3, I will build a modified utility function with different risk aversion rate regarding to different generations, and evaluate the new outcomes under this new model. For Section 4, I will analyze the estimation error and try to find some extension part. For Section 5, there will be a conclusion.

2 Replicating models

2.1 Basic parameters setting and relevant functions

Since we want to replicate the paths of original paper to build up the model, we just use the same parameters and equations as the original ones. Brief introductions of them are as follows:

The constant risk free rate is set as $r = 2\%$. The expected (real) return on stocks is set as $\mu = 6\%$ (This setting is in the line with long-run estimates in Fama and French (2002) since the premium equals to 4%). The subjective discount rate is set as $\delta = 4\%$. The volatility of stock returns is set as $\sigma = 15\%$. Populations are assumed that they start at the age of 25 ($t = 0$), retired at the age of 65 $t = R = 40$ and died at the age of 80, $t = T = 55$. During the first 40 years, people work and earn a flat real labor income, which is normalized to 1. We set the proportional rate, which the investment in the risky market compares with the total asset, is ω . Obviously, we have $0 < \omega < 1$. We set the start point of contribution and benefit level is p and b separately.

As the original paper mentioned, in a traditional average salary defined benefit scheme, the retirement benefit is a fixed fraction of labor income during the working period. The pension benefits are funded by the contributions plus investment proceeds. Under the requirement of actuarially fair contribution principle, market value of the contributions equals to the market value of the benefits. Therefore we have:

$$\int_0^R e^{-rs} p ds = \int_R^T e^{-rs} b ds \quad (1)$$

Therefore, we can express the benefit level (b) via contribution level (p):

$$b = \frac{1 - e^{-rR}}{e^{-rR} - e^{-rT}} p$$

For the liability of Defined Benefit pension schemes of each age group, we could write the following equation group:

$$L_x = \begin{cases} \int_R^T e^{-r(t-x)} b dt - \int_x^R e^{-r(t-x)} p dt & \text{for } x < R \\ \int_x^T e^{-r(t-x)} b dt & \text{for } R < x < T \end{cases}$$

From this function group, we could write a general function of the liability assuming all age groups have equal size:

$$L = \int_0^R \left(\int_R^T e^{-r(t-x)} b dt - \int_x^R e^{-r(t-x)} p dt \right) dx + \int_R^T \left(\int_x^T e^{-r(t-x)} b dt \right) dx \quad (2)$$

Actually, the calculus of equation (2) could be solved as:

$$L = \frac{b}{r^2} \left((T - R)r - e^{-Rr} + e^{-Tr} \right) - \frac{p}{r^2} (rR - 1 + e^{-Rr})$$

Since benefit level (b) could be expressed by contribution level (p) as above, we could calculate the liability if given the number of contribution level (p).

As known, certain part of collective pension fund should be invested in the risky asset market in order to gain higher return rate compared with the risk free rate. As above, we set the pension fund invest ω of total pension funds in the risk asset market. Then we can get the asset function:

$$dA_t = [A_t(r + \omega(\mu - r)) + 40p_t - 15b_t]dt + \omega\sigma A_t dZ_t \quad (3)$$

where p_t and b_t denote the actual contribution and benefit level specified in short time. In the program we write, we use Euler Method to simulate this stochastic differential equation. The surplus term, which is the difference between asset and liability in each period, is defined as:

$$S_t = A_t - L \quad (4)$$

In risky investment policy, there exists the situation of mismatch of risk sometimes. This is done by adjusting contribution (p) and benefit (b). Three defined benefit pension schemes (DB) will be introduced in next paragraphs, which have their own risk allocations rules. These risk-allocation rules specify who of the stakeholders, when and to what extent they are taking part in risk bearing.

2.2 Risk-allocation rules and optimization method

In the defined benefit with contribution adjustments (DB_{CA}) scheme, benefits are fixed at $b_t = b$, but the contributions are adjustable, expressed as:

$$p_t = p - \alpha S_t/R \quad (5)$$

in which coefficient α determines the speed of absorbing the funding imbalance. The lower α is, the more amount of the funding residual is shifted to the future and shared across generations. In this scheme, obviously, working cohorts bear all the funding risk.

In the defined benefit with benefit adjustments (DB_{BA}) scheme, contributions are fixed at $p_t = p$, with adjustable benefits to absorb the risk. In this scheme, retired cohorts bear the funding risk, which the benefit level in each time period could be written as:

$$b_t = b + \beta S_t/(T - R) \quad (6)$$

The lower value of β means the higher degree of intergenerational risk sharing as well.

Both contributions and benefits are adjusted simultaneously to absorb the funding residual in the hybrid defined benefit (DB_H) scheme. The contribution and benefit relate to the funding residual S_t linearly, with fraction α in contribution level for working generations and fraction β in benefit level for retirees:

$$p_t = p - \alpha S_t/R \quad (7)$$

$$b_t = b + \beta S_t/(T - R) \quad (8)$$

Besides, we should set the restriction of $\alpha + \beta > r$ in order to get non-explosive values for the surplus.

As mentioned in the original paper, we choose the target contribution rate (p) together with the risk-allocation rule (α, β) and investment policy (ω), based on the expected life-time utility of the 25-year-old entering participants. If we get the target contribution rate (p), we could have target benefit rate (b) asset (A_t) and liability (L) as explained above. Still, the utility function we used is the constant relevant risk aversion model (CRRA). We could write the optimization problem of pension schemes as:

$$U = \max_{\{p, \alpha, \beta, \omega\}} E_0 \left[\int_0^T e^{-\delta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] \quad (9)$$

with $c_t = 1 - (p - \alpha S_t/R)$, $t < R$

$$c_t = b + \beta S_t / (T - R), \quad R \leq t < T$$

$$0 \leq \omega \leq 1$$

With the method provided by the original paper, we use Monte Carlo method to express the utility function, and Matlab optimization toolbox to calculate the optimal combination of the decision $\{p, \alpha, \beta, \omega\}$. Actually, I use the command `fmincon` to make the optimization. Since this command aims for “find minimum of constrained nonlinear multivariable function” (see document of `fmincon` in Matlab manual), we use this command to minimize the negative utility function in order to find out optimal parameters we need.

2.3 Optimal pension schemes and welfare evaluation

As the original paper assumed, all pension schemes are initially fully funded, i.e. $A_0 = L$. Table 1 is the reports of design parameters $\{p, \alpha, \beta, \omega\}$ under the default parameter ($r = 2\%$, $\delta = 4\%$, $\mu - r = 4\%$).

Also as the original paper, we use certainty equivalent consumption (*CEC*) as the welfare measure to gauge the performance of collective and individual pension schemes. The *CEC* equation could be calculated by:

$$U = \int_0^T e^{-\delta t} \frac{(CEC)^{1-\gamma}}{1-\gamma} dt \quad (10)$$

After make the calculus of the equation above, we get equation (10) about *CEC* calculating:

$$CEC = \left[\frac{U(1-\gamma)\delta}{1-e^{-\delta T}} \right]^{\frac{1}{1-\gamma}} \quad (11)$$

Welfare level is reported as the normalized certainty equivalent consumption in units of annual salary, *CEC*. As CEC^{OI} denotes the welfare level achieved by the optimal individual scheme (OI), the ratio CEC/CEC^{OI} shows the welfare of the collective pension schemes relative to this optimal individual benchmark.

Moreover, we could use the certainty equivalent consumption to examine the programs we write for complicating the path. If all programs work well, we could get the right result of utility after setting

the optimal parameters provided on the original paper. Afterwards, we could use equation (10) to get the *CEC* results by giving the number of utility. If the results are near or equal to the *CEC* results, we could believe that we have written the right programs. Otherwise, we should analyze the source of difference: if it is from the wrong program we wrote or the error of results from original paper.

After using the optimization toolbox to calculate the parameters $\{p, \alpha, \beta, \omega\}$, we also get a table about three collective pension schemes. Table 1 is the parameters we get via using Euler Method to simulate the stochastic differential equation, using Monte Carlo Method to calculate the expectation and using optimization toolbox of Matlab to get the optimal point numerically. All the results are shown on the Table 1. (For the optimal individual (OI) part in the table, we would discussed in 2.4 in the following) Moreover, we also calculate the result with the optimal result given by the original paper. The purpose to do this is to check if the program we wrote performs well and if the results shown in the original paper are correct. The results are shown in Table 2, and we could use the *CEC* results to make comparison with the results in original paper. Considering the volatility of Brownian motion in the stochastic equation, the results perform well by the program we write, and the optimal point in original paper should be mostly correct. Meanwhile, we should also notice that the benefit level of DB_{BA} in risk aversion rate $\gamma = 5$, is 61.94% in our calculation in Table 2, but this result is 53.6% in original paper. As known from equation (1), b could be expressed by a linear function with the single variable p without other uncertain parameters actually, which means the number of b is constant if given p . Therefore, we believe the difference is caused by an error of calculation or writing in original paper.

Table 1. *Optimal pension schemes*

The table shows the optimal pension scheme parameters under the assumption of $\delta = 4\%$ and $\mu - r = 4\%$. p and b represents the optimal contribution level and benefit level separately. The optimal portfolio weight in equities is given by ω . The CEC shows the welfare levels achieved under these optimal collective pension schemes and the ratio CEC/CEC^{OI} shows the relative welfare gain or loss of the collective schemes relative to the optimal individual scheme. Considering the deviation of numerical simulation via Monto Carlo, Euler method and optimization toolbox, we accept most optimal result of the original paper, and also believe the program we write works well in replicating the path of original paper.

γ		OI	DB _{CA}	DB _{BA}	DB _H
3	p	-	16.58%	10.96%	14.00%
	b	-	78.40%	51.82%	66.2%
	α	-	0.0698	-	0.06
	β	-	-	0.03	0.02
	ω	-	100%	91.31%	100%
	CEC	0.9061	0.9316	0.8957	0.9484
	CEC/CEC^{OI}	100%	102.8%	98.8%	104.7%
5	p	-	16.44%	12.99%	13.97%
	b	-	77.74%	61.42%	66.06%
	α	-	0.0497	-	0.0449
	β	-	-	0.0299	0.02
	ω	-	96.04%	58.94%	100%
	CEC	0.8996	0.8902	0.8697	0.9161
	CEC/CEC^{OI}	100%	99.0%	96.7%	101.4%
8	p	-	17.36%	14.0%	15.44%
	b	-	82.09%	66.2%	73.01%
	α	-	0.0475	-	0.0398
	β	-	-	0.02	0.02
	ω	-	67.26%	55%	75.7%
	CEC	0.8806	0.8666	0.8546	0.8847
	CEC/CEC^{OI}	100%	98.4%	97.0%	100.5%

Table 2. Simulation result given the optimal result of original paper

The table shows the results of benefit level (b), certainty equivalent consumption (CEC) and ratio CEC/CEC^{OI} , calculated by the program we write with the parameters $\{p, \alpha, \beta, \omega\}$ as original paper. We noticed that optimal benefit level of DB_{BA} with second risk aversion rate ($\gamma = 5$) is almost 62%, while it shows 53.6% in the original paper. We believe this should be a writing error in original paper.

γ		OI	DB_{CA}	DB_{BA}	DB_H
3	p	-	16.67%	10.96%	14.00%
	b	-	78.82%	53.9%	66.2%
	α	-	0.07	-	0.06
	β	-	-	0.03	0.02
	ω	-	100%	91.31%	100%
	CEC	0.908	0.9310	0.8959	0.9484
	CEC/CEC^{OI}	100%	102.5%	98.7%	104.4%
5	p	-	16.6%	13.1%	14%
	b	-	78.49%	61.94%	66.2%
	α	-	0.05	-	0.045
	β	-	-	0.03	0.02
	ω	-	96 %	60%	100%
	CEC	0.892	0.8913	0.8704	0.914
	CEC/CEC^{OI}	100%	100.0%	97.6%	102.4%
8	p	-	17.5%	14.0%	15.7%
	b	-	82.75%	66.2%	74.24%
	α	-	0.04	-	0.04
	β	-	-	0.02	0.02
	ω	-	67%	55%	76%
	CEC	0.876	0.862	0.8546	0.8849
	CEC/CEC^{OI}	100%	98.4%	97.5%	101.0%

2.4 Optimal individual scheme

Same as the original paper, with the constraints $0 \leq \omega \leq 1$, we could write the utility function of optimal individual scheme as following:

$$U = \max_{\{0 \leq \omega \leq 1, c_t\}} E_0 \left[\int_0^T e^{-\delta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] \quad (12)$$

and subject to the pension wealth dynamics

$$dW_t^{OI} = [W_t^{OI}(r + \omega(\mu - r)) + 1 - c_t]dt + \omega\sigma W_t^{OI} dZ_t \quad \text{for } 0 < t < R \quad (13)$$

$$dW_t^{OI} = [W_t^{OI}(r + \omega(\mu - r)) - c_t]dt + \omega\sigma W_t^{OI} dZ_t \quad \text{for } R \leq t < T \quad (14)$$

$$W_0^{OI} = 0 \quad (15)$$

where W_0^{OI} stands for the accumulated financial wealth of OI scheme.

Based on the four functions above, we could rewrite them as:

$$U_t(W_t) = \max_{\{0 \leq \omega \leq 1, c_t\}} \left[u(c_t) + E_t \left[e^{-\delta} U_{t+1}(W_{t+1}) \right] \right] \quad (16)$$

$$W_{t+1} = (W_t - c_t)[R^f + \omega_t(\tilde{R}_{t+1}^s - R^f)] + 1 \quad \text{for } 1 \leq t < R$$

$$W_{t+1} = (W_t - c_t)[R^f + \omega_t(a_t)(\tilde{R}_{t+1}^s - R^f)] \quad \text{for } R \leq t < T$$

where $W_1 = 1$, $u(c_t) = \frac{(c_t)^{1-\gamma}}{1-\gamma}$, $\tilde{R}_{t+1}^s = \exp\left(\mu + \sigma \int_t^{t+1} dZ_s\right)$, and $R^f = \exp(r)$. For the final period, we have $U(W_T) = u(c_T^*) = \frac{(c_T^*)^{1-\gamma}}{1-\gamma}$ and $c_T^* = W_T$.

With help of Carroll (2006), we set a grid $a_t = W_t - c_t = \{a_t\}_{j=1}^J$, and with that, we could solve the optimal consumptions and portfolio policies as:

$$0 = e^{-\delta} E_t \left[u'(c_{t+1}^*[W_{t+1}]) (\tilde{R}_{t+1}^s - R^f) \right] \quad (17)$$

$$c_t^*(a_t) = I_u \left(e^{-\delta} E_t \left[u'(c_{t+1}^*[W_{t+1}]) \left(R^f + \omega_t^*(a_t)(\tilde{R}_{t+1}^s - R^f) \right) \right] \right) \quad (18)$$

To solve the two equations above and get ω_t^* and c_t^* , given $U(W_T) = u(c_T^*) = \frac{(c_T^*)^{1-\gamma}}{1-\gamma}$ and $c_T^* = W_T$, we could get

$$u'(c_T^*[W_T]) = (c_T^*)^{-\gamma} = W_T^{-\gamma}$$

Since

$$W_{t+1} = (W_t - c_t) \left[R^f + \omega_t^*(a_t)(\tilde{R}_{t+1}^s - R^f) \right] \quad \text{for } R \leq t < T$$

Equation (17) at time T-1 could be solved as:

$$\begin{aligned} 0 &= e^{-\delta} E_{T-1} \left[(c_T^*)^{-\gamma} (\tilde{R}_T^s - R^f) \right] = e^{-\delta} E_{T-1} \left[(W_T^*)^{-\gamma} (\tilde{R}_T^s - R^f) \right] \\ &= e^{-\delta} E_{T-1} \left[\left((W_{T-1} - c_{T-1}) \left[R^f + \omega_{T-1}^*(a_{T-1})(\tilde{R}_T^s - R^f) \right] \right)^{-\gamma} (\tilde{R}_T^s - R^f) \right] \\ &= e^{-\delta} a_{T-1}^{-\gamma} E_{T-1} \left[\left[R^f + \omega_{T-1}^*(a_{T-1})(\tilde{R}_T^s - R^f) \right]^{-\gamma} (\tilde{R}_T^s - R^f) \right] \end{aligned} \quad (19)$$

With ω and a_t , we could solve equation (18)

$$c_t^* = \left(e^{-\delta} E_t \left[u'(c_{t+1}^*[W_{t+1}]) \left(R^f + \omega_t^*(a_t)(\tilde{R}_{t+1}^s - R^f) \right) \right] \right)^{\frac{1}{\gamma}} \quad (20)$$

Then we could solve backward, write a function about ω in Matlab and solve it numerically as the equation above when giving $a_t = W_t - c_t$:

$$0 = e^{-\delta} E_t \left[(c_{t+1}^*[W_{t+1}])^{-\gamma} (\tilde{R}_t^s - R^f) \right] \quad (21)$$

subject to:

$$W_{t+1} = a_t \left[R^f + \omega_t(a_t)(\tilde{R}_{t+1}^s - R^f) \right] \quad \text{for } R \leq t < T$$

With numerical solve for this function via Matlab software, we use cubic spline interpolation with help of W_{t+1} and c_{t+1} calculated by former step of equation (18).

When $1 \leq t < R$,

$$W_{t+1} = (W_t - c_t)[R^f + \omega_t^*(a_t)(\tilde{R}_{t+1}^s - R^f)] + 1 \quad (22)$$

$$0 = e^{-\delta} E_t[(c_{t+1}^*[W_{t+1}])^{-\gamma}(\tilde{R}_t^s - R^f)] \quad (23)$$

subject to:

$$W_{t+1} = a_t[R^f + \omega_t(a_t)(\tilde{R}_{t+1}^s - R^f)] + 1 \quad \text{for } 0 < t < R$$

Calculation of $c_t^*(a_t)$ is same as the procedures mentioned above.

From these steps of calculation, we could get 3 vectors corresponding to ω^* , c^* and W^* in each time spot. After that, with equation (13)~(15) and restriction of $0 \leq \omega \leq 1$, we could make simulation of ω , c and W in each time spot with 10000 trajectories, in which, ω_t and c_t could be calculated via interpolation with help of optimal samples we get before and W_t . We could get three figures of each of them (shown in Figure 1).

Obviously, we could observe the ranges of c and W are larger compared the results in original paper. In fact, we get ω is more than 0.46 after retirement, while it less than 0.4 in original paper. With more assets investing in risky market, the increasing wealth and corresponding consumption would be more predictable. As mentioned in equation (19), the portfolio share in equities at the last time spot is not influenced by the grid of number a_{T-1} . For the reason why we get different results with the original paper, I would only say that there are no errors in modeling and programming, in our viewpoint, after checking the codes and method carefully.

Another observation is the fluctuation of consumption at the time of retirement, while the figure of optimal individual consumption in original paper looks quite smooth at the same time spot. On one hand, since the salary stops at the start of years 40, as shown in equation (13) and (14), it might be reasonable which this fluctuation exists. Same phenomenon is shown in Figure 2, which are figures of consumption of collective pension funds. On the other hand, individuals, who know the stop of salary, may do some preparations for it and make the consumption change smoothly. (Continued on Page 18)

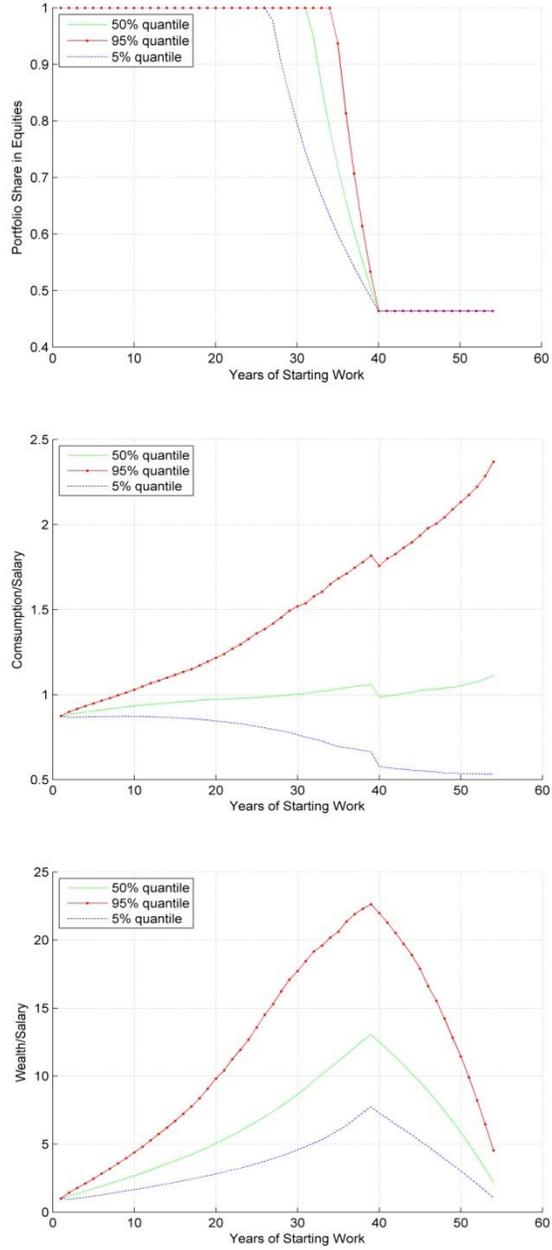


Figure 1. Life cycle profiles of the optimal individual (OI) scheme

Same as the original paper, taking a benchmark investor ($\gamma = 5$ and $\delta = 4\%$), the three pictures in this figure shows quantile distributions of portfolio weights in equities, consumption and wealth in OI scheme

(Continued from Page 18) With the methods mentioned above, we could also calculate the results of optimal individual scheme (OI) under the situation of different risk aversion rate ($\gamma = 2$ and $\gamma = 8$). After getting 10000 trajectories of consumption in each time spot, we could use equation (10) and (11) to calculate the certainty equivalent consumption of optimal individual scheme (CEC^{OI}) and the results are shown in Table 1. The ratio CEC/CEC^{OI} , same with the original paper, represents percentage of welfare of different collective pension funds compared with the optimal individual benchmark.

The CEC test shows, similar with the result in original paper, the hybrid defined benefit scheme (DB_H) provides a welfare gain of 2.3% per annum compared with the OI scheme under risk aversion rate equaling to 5. It indicates that this pension scheme with well-structured intergenerational risk sharing enhance welfare over the full life-cycle. On the other side, the DB_{CA} scheme loses 6% welfare while DB_{BA} scheme loses 2.9% under the same situation. Obviously the hybrid pension scheme outperforms among these three kinds of pension schemes.

In Table 2, however, we prefer to use the CEC^{OI} result from the original paper since we want to evaluate the performance of our program (under the assumption that all CEC^{OI} results are correct in original paper).

2.5 Figures of intergenerational risk sharing

As the original paper mentioned, we plot the 5%, 50% and 95% quantiles of the normalized consumption, c_t , of the optimal collective plans, with the scheme parameters set corresponding to Table 1 with $\gamma = 5$. With the picture, hybrid scheme (DB_H) performs better in distributing shocks over the full lifetime of the individual compared with other two kinds of collective pension funds (DB_{CA} and DB_{BA}). It shows that more risk absorbers could provide more efficient risk sharing, also mentioned on the original paper. We could see that hybrid scheme (DB_H) spreads the funding residual (S_t) over all cohorts, workers and retirees. On the contrary, other two collective pension funds (DB_{CA} and DB_{BA}), with one group of people bearing all risks, performs relatively less efficient in risk sharing.

At first, we assume that the consumption obey normal distribution, and use mean of 10000 trajectories representing the 50% quantile of the consumption, with mean plus or minus 1.96 times of standard deviation representing 5% and 95% quantile of the consumption. Meanwhile, after comparing the figures we got and ones in original paper, we find there exist huge differences between them. After analyzing the consumption matrix, we found that these trajectories do not obey the normal distribution

in fact. Therefore, we rearrange the result matrix from small to big and found the exact number of 5%, 50% and 95% quantile of consumption. The figures are same with ones in original paper and shown as Figure 2. More discussion could be made on the distribution of consumption and find out the reason of it, but there will not be more discussion about it in this paper.

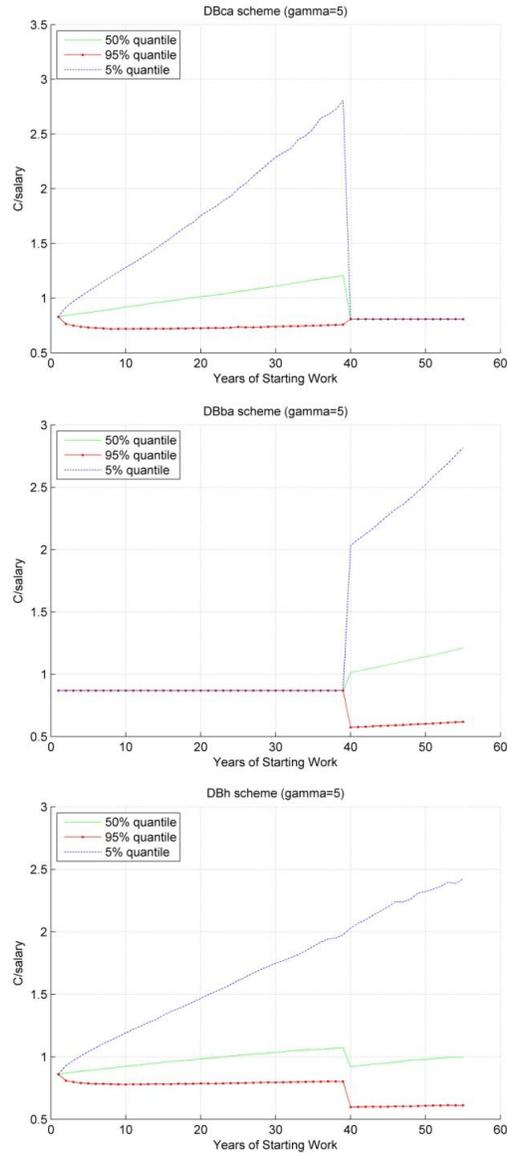


Figure 2. Consumption profiles obtained from the collective schemes

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective schemes (DB_{CA} , DB_{BA} , DB_H). All parameters are provided from Table 1 for $\gamma = 5$. The consumption levels are normalized by annual salary, i.e. c_t/y

2.6 Market valuation of pension schemes

As the original paper mentioned, we need to calculate the market values of intergenerational risk sharing. We use the market value of contingent contributions and benefits of optimal collective pension funds to reach this goal. Actually, even though members in pension funds could hardly trade the pension contract, there still exists the market valuation of given pension deals. Moreover, if the contribution level (p_t) equals to the benefit level (b_t), calculated by equation (1), (7) and (8), we could believe the scheme is a fair deal.

One way to calculate the present value of pension deals is to look at the remaining balance left in individual account at the end of a cohort. A positive remaining balance indicates positive transfers from this generation to other cohorts, while negative one means the opposite direction of cash flow. We could use contingent claim to express this, i.e. any cohort in a collective pension funds writes a call option to share funding surplus and holds a put option to receive protections with other cohorts.

A fair deal means the same present market value between call and put option. With each cohort starts at no surplus or deficit, we could express the dynamic cohort account as:

$$da_t = [a_t(r + \omega(\mu - r)) + p_t - b_t]dt + \omega\sigma a_t dZ_t \quad (24)$$

Here we want to note that the a_t in this chapter and in chapter 3.4.2 represents the value of personal account but not the grid of numbers for calculating parameters of optimal individual schemes in chapter 2.4. For calculating the net present value (*NPV*), we use two methods, the pricing kernel method and the Girsanov theorem. We need a stochastic discount factor M_t when using pricing kernel method:

$$dM_t = -rM_t dt - \lambda dZ_t \quad (25)$$

With this, we could calculate the net present value (*NPV*) as following:

$$NPV = -E_0[M_T a_T] = -E_0[M_T (a_T)^+] + E_0[M_T (-a_T)^+] = -call + put \quad (26)$$

When using Girsanov theorem, we first change the measure from real world to risk-free circumstance, then we rewrite the cohort account and asset function under risk-free measure:

$$da_T = [a_T r + p_T - b_T]dt + \omega \sigma a_T dZ_T \quad (27)$$

After that we could express the net present value (*NPV*) as following:

$$NPV = -E_0[a_t] = -E_0[(a_t)^+] + E_0[(-a_t)^+] = -call + put \quad (28)$$

Moreover, since we use Euler method to simulate stochastic process of asset and pricing kernel with $M = 10000$ trajectories and use Monte-Carlo method to calculate the expectation of contingent claims, we need to know the confidence interval about the results in order to make judgment about whether or not accepting the results. Here, we took 95% confidence interval with the following equation:

$$std(a_t) = var(a_t)/\sqrt{M} \quad (29)$$

$$upper_bound = 1.96std(a_t) \quad (30)$$

$$lower_bound = -1.96std(a_t) \quad (31)$$

The results are shown at Table 3. We could observe that for the pricing kernel method, we got the difference much higher than in original paper. We believe that might be caused by a less accurate numerical method. Here, because we do not know the methods and programs to evaluate the cohort account in original paper, maybe we do not find the most appropriate method for this. More consideration and research could be done in this area. Even though, since we also observed that the differences of DB_{CA} and DB_H are in 95% confidence interval, we could not regret the null hypothesis, i.e. the dynamic cohort account equals to zero. But for DB_{BA} , the result perform quite well if given $b=0.536$, as the original paper mentioned. If we use $b=0.619$, the results show that the dynamic cohort account does not equal to zero significantly. We could get the same results, with smaller confidence interval when using Girsanov theorem to make this calculation.

Besides, we could get that the market values of positive and negative transfer (the implicit call and put option) are potentially large, from 0.2 to 1.4 annual salaries. This rate of the transfers depends on the chosen asset mix, level of contribution and risk-sharing rule. The DB_{CA} scheme has the highest value of transfers, while the DB_{BA} scheme results (with b equals to .536) in the lowest value of transfers.

This means DB_{BA} performs better than DB_{CA} since the latter one could let future generations confronted with larger deficits in pension funds. Meanwhile, under the parameter setting $b=.619$, we do not have this observation.

	DB_{CA}	$DB_{BA}(b = .536)$	$DB_{BA}(b = .619)$	DB_H
call	1.0149	0.1854	0.0885	0.6091
put	1.2887	0.2123	0.4998	0.8929
difference	0.2738	0.0269	0.4114	0.2838
upper bound	1.2925	0.0806	0.0934	1.1712

Table 3. Market value of intergenerational transfer

The table shows the market value of intergenerational transfers which shown in columns ‘call’ and ‘put’ (positive value and negative value), which expressed in equation (13). The upper bound and lower bound define the 95% confidence interval about the null hypothesis of which the difference equals to zero.

3. Modified Utility Function

3.1 Building models

In the paper *Why should Old People Invest Less in Stocks than Young People* (Jagannathan and Kocherlakota, 1996), the authors provide that viewpoint of human wealth provides a very important role in distribute asset in pension funds. In another discussion paper of Netspar *Cognitive aging and risk attitude* (Erik Bonsang and Thomas Dohmen, 2012), the authors shows risk aversion rate would increase due to the increasing age. Combining the two papers, and some other issues, we might guess that the following statement: Since young people have sufficient human wealth, compared with old generations, they have more recovery ability when facing poor financial situation. Therefore, they prefer to take more risks when investment in order to gain more financial return. For the risk aversion rate, we guess that young generations have small risk aversion rate than old ones.

From this viewpoint, we make some change of the constant relevant risk aversion model of utility in order to reflect the real situation more accurately. Here, we plan to increase the risk aversion rate of retirees and decrease the risk aversion rate of youth generation. To simplify the model, we set three periods of time to set different risk aversion rates for generations. As assumption in original paper,

people start work at the age of 25, retire at the age of 65 and die at the age of 80. With this, we set young generation is between 25 and 45, middle generation is between 45 and 65, while old generation is between 65 and 80. We set R_youth as the age of 45 and three risk aversion rates γ_1 , γ_2 and γ_3 for the three generations. Via the analysis above, we have:

$$\gamma_1 \leq \gamma_2 \leq \gamma_3$$

After that, we could rewrite the utility function (12) as following:

$$U = \int_0^{R_youth} e^{-\delta t} \frac{c_t^{1-\gamma_1}}{1-\gamma_1} dt + \int_{R_youth}^R e^{-\delta t} \frac{c_t^{1-\gamma_2}}{1-\gamma_2} dt + \int_R^T e^{-\delta t} \frac{c_t^{1-\gamma_3}}{1-\gamma_3} dt \quad (32)$$

With it, we could optimize the target parameters: contribution rate (p), we could have target benefit rate (b) asset (A_t) and liability (L) via using optimal toolbox of Matlab software:

$$U = \max_{\{p, \alpha, \beta, \omega\}} \int_0^{R_youth} e^{-\delta t} \frac{c_t^{1-\gamma_1}}{1-\gamma_1} dt + \int_{R_youth}^R e^{-\delta t} \frac{c_t^{1-\gamma_2}}{1-\gamma_2} dt + \int_R^T e^{-\delta t} \frac{c_t^{1-\gamma_3}}{1-\gamma_3} dt \quad (33)$$

$$\text{with } c_t = 1 - (p - \alpha S_t/R), \quad t < R$$

$$c_t = b + \beta S_t/(T - R), \quad R \leq t < T$$

$$0 \leq \omega \leq 1$$

Moreover, in former program, we use command `randn(M, N)` to simulate Brownian motion, but it is not appropriate in this section since we want to make comparison between results from different pairs of risk aversion rates. Therefore, we set a certain M-by-N matrix containing pseudorandom values drawn from the standard normal distribution, and use this matrix to simulate stochastic process of asset and make comparison between different risk aversion rates group.

3.2 Analysis of results

At first, we want to make comparison in each of the three collective pension funds (DB_{CA} , DB_{BA} , DB_H). For it, we calculate the optimal parameters $\{p, \alpha, \beta, \omega\}$ using different risk aversion group ($\gamma_1 = \gamma_2 = \gamma_3 = 3$, $\gamma_1 = \gamma_2 = \gamma_3 = 5$, $\gamma_1 = \gamma_2 = \gamma_3 = 8$, $\gamma_1 = 4.5$ $\gamma_2 = 5$ $\gamma_3 = 5.5$, $\gamma_1 = 3$ $\gamma_2 = 5$ $\gamma_3 = 8$).

3.2.1 Analysis of results in DB_{CA}

As the paragraph above mentioned, we use six different combinations of risk aversion rates in optimizing the utility function. Since we fixed the matrix when simulate the stochastic differential equation of asset price, the result we get is not quite same with ones in Part 2. Reason of this less accuracy is that the Matlab code `fmincon` uses numerical method to find the optimal points, i.e. the program redo the whole process a few times to reach the optimal results. In each time of repeat the process in former programs, the code uses a new random matrix while here, the matrix is fixed. We could get this conclusion when comparing Table 4 with Table1.

At first, it is quite clear that the start point of contribution level (p) is lowest when all gammas equal to 5. We get this conclusion from tens of experiment using different matrix of random numbers. It means that people tend to invest the least of their income into pension scheme DB_{CA} not under their least risk aversion rate but in the middle one. Moreover, when compare column 2, 4, 5 in Table 4, we could find that with the increasing difference of risk aversion rates between different generations, all contribution level, risk allocation rule and proportion in risky market increase. This means people tend to invest more in pension schemes, absorbing more risk in current generation and pay more money in risky market.

We try to explain this trend via analyzing the utility and consumption function. Since in contribution adjustment pension scheme, consumption only changes before the age of retirement since contribution level is linked with asset function while benefit level is fixed. Therefore, when decreasing the risk aversion rate of youth generation and increasing the risk aversion rate of old generation, only former part influences the result of utility function. Less risk aversion rate in former means young people are more risk-favor individuals, and therefore, they prefer to invest more in risky market and share more risk in current generations, since young people have more human wealth and less risk aversion.

Besides, we fix two risk aversion rates and let the other one float. No wonder that if the floating risk aversion rate is one of young generation (i.e. generation below 45 years old, following same), slope coefficient α move as expected: if γ_1 increases, α decreases since under higher risk aversion rate, people tend to transfer more risk to future generations. We should also notice that if the floating risk aversion rate is γ_3 , there are also changes on start point of contribution level and slope coefficient α . We also get this conclusion after many observations, but of course, the difference is quite small. We could observe that young generation tends to invest more and transfer more risk to future generations under higher risk aversion rate of old generations. It might be explained by which youth generation prefers to take few risks at present in order to preparing for the coming old generation.

Table 4-1. Simulation result of optimal modified utility function of DB_{CA}

In this table, when make comparison of the first three columns, we could observe that the contribution level (p) is lowest in the middle risk aversion rate. Moreover, when analyzing column 2, 4 and 5, we could observe that with increasing difference of risk aversion rate between generations, people tend to have more contribution level, absorbing more risks in current generation and invest more in risky market.

	$\gamma_1 = 3$	$\gamma_1 = 5$	$\gamma_1 = 8$	$\gamma_1 = 4.5$	$\gamma_1 = 4$
	$\gamma_2 = 3$	$\gamma_2 = 5$	$\gamma_2 = 8$	$\gamma_2 = 5$	$\gamma_2 = 5$
	$\gamma_3 = 3$	$\gamma_3 = 5$	$\gamma_3 = 8$	$\gamma_3 = 5.5$	$\gamma_3 = 6$
p	17.38%	17.09%	17.22%	17.61%	18.14%
b	82.12%	80.81%	82.18%	83.27%	87.55%
α	0.0722	0.0479	0.0402	0.0503	0.0545
β	-	-	-	-	-
ω	100%	98.06%	72.49%	100%	100%
CEC	0.9316	0.8904	0.8659	0.8713	0.8610

Table 4-2. Simulation result of optimal modified utility function of DB_{CA}

In this table, not only could we make comparison between any column and results under constant risk aversion in Table 4-1, but we could also make comparison among the columns as well. Comparing column 1, 3&5 in this table, we could observe that slope coefficient α moves towards the same direction while start contribution level p does opposite with γ_3 , when only letting γ_3 float. If only changing risk aversion rate of youth generation, meanwhile, both α and p increase with smaller γ_1 , as expected. We could observe this when comparing column 1 and 4, or 2 and 5.

	$\gamma_1 = 3$	$\gamma_1 = 2$	$\gamma_1 = 3$	$\gamma_1 = 4$	$\gamma_1 = 3$
	$\gamma_2 = 5$				
	$\gamma_3 = 7$	$\gamma_3 = 8$	$\gamma_3 = 6$	$\gamma_3 = 7$	$\gamma_3 = 8$
p	19.14%	20.26%	18.87%	18.48%	19.35%
b	90.50%	95.80%	89.23%	87.38%	91.50%
α	0.0662	0.0838	0.0671	0.0534	0.0656
β	-	-	-	-	-
ω	100%	100%	100%	100%	100%
CEC	0.8311	0.7842	0.8304	0.8555	0.8355

3.2.2 Analysis of results in DB_{BA}

Same as DB_{CA} pension fund, we use six different combinations of risk aversion rates in optimizing the utility function. And also, since we fixed the matrix when simulate the stochastic differential equation of asset price, the result we get is not quite accurate. Some new conclusion could be get when analyzing Table 5 and use it to make comparison with Table 1.

Compared the first three columns of Table 1 and Table 5 at first, we could observe that the results in Table 5 are quite close to ones in Table 1, which means the fixed matrix of random numbers does not influent much of the results. Besides, the results in both tables show that with risk aversion rates increasing, contribution level increases (p) while risk allocation rule (β) and proportion rate in risky market (ω) decrease. It means that in benefit adjustment pension schemes, individuals tend to contribute more in pension funds, share more risks in future generations and invest less in risky market when having higher risk aversion.

Besides, we make comparison among column 2, 4 and 5. From these three columns, we could observe that with increasing difference of risk aversion rates between different generations, contribution level increases, risk sharing rule and proportion of investment into risky market decreases. Being similar with DB_{CA} , the benefit adjustment pension schemes have the asset-linked consumption only in retirement generation. Therefore, since we increase the risk aversion rate in old generations, people tends to share more risks with future generations and invest less in risky markets. This might be the appropriate explanation for the observations.

When fixing two risk aversion rates and letting the other one float, we get some more observations. If the floating risk aversion rate is γ_3 (i.e. the risk aversion rate of old generation), β and ω move the opposite direction with γ_3 as expected. This means that people tend to transfer more risk to future and invest less in risky market under higher risk aversion rate in old generation. If the floating risk aversion rate is γ_1 (i.e. the risk aversion rate of old generation), β keeps constant, while ω shows different direction in different situations. Column 5 in Table 5-1 and column 3 in Table 5-2 shows that ω decreases when γ_1 decreases. However, when comparing column 1 and 4, or column 2 and 5 in Table 5-2, we could observe that ω increases when γ_1 decreases. More explanations are needed for this.

Table 5-1. Simulation result of optimal modified utility function of DB_{BA}

In this table, we could observe that contribution level increases (p) while slope coefficient (β) and proportion rate in risky market (ω) decrease when risk aversion rate increases. Moreover, when analyzing column 2, 4 and 5, we could observe that with increasing difference of risk aversion rate between generations, people tend to have more contribution level, absorbing less risks in current generation and invest less in risky market.

	$\gamma_1 = 3$	$\gamma_1 = 5$	$\gamma_1 = 8$	$\gamma_1 = 4.5$	$\gamma_1 = 4$
	$\gamma_2 = 3$	$\gamma_2 = 5$	$\gamma_2 = 8$	$\gamma_2 = 5$	$\gamma_2 = 5$
	$\gamma_3 = 3$	$\gamma_3 = 5$	$\gamma_3 = 8$	$\gamma_3 = 5.5$	$\gamma_3 = 6$
p	11.31%	12.96%	14.30%	13.51%	14.01%
b	53.48%	61.28%	67.62%	63.88%	66.25%
α	-	-	-	-	-
β	0.0303	0.0269	0.0244	0.0264	0.0259
ω	94.41%	63.83%	43.53%	59.61%	55.15%
CEC	0.8960	0.8698	0.8552	0.8521	0.8382

Table 5-2. Simulation result of optimal modified utility function of DB_{BA}

In this table, not only could we make comparison between any column and results under constant risk aversion in Table 5-2, but we could also make comparison among the columns as well. Comparing column 1, 3&5 in this table, as expected, we could observe that slope coefficient β moves towards the opposite direction while start contribution level p does same with γ_3 , when only letting γ_3 float. If only changing risk aversion rate of youth generation, meanwhile, p decreases and β keeps constant when increasing γ_1 . But for ω , there seems no constant moving direction for it. We could observe this when comparing column 1 and 4, or 2 and 5 in Table 5-2, and column 5 in Table 5-1 with column 3 in Table 5-2.

	$\gamma_1 = 3$	$\gamma_1 = 2$	$\gamma_1 = 3$	$\gamma_1 = 4$	$\gamma_1 = 3$
	$\gamma_2 = 5$				
	$\gamma_3 = 7$	$\gamma_3 = 8$	$\gamma_3 = 6$	$\gamma_3 = 7$	$\gamma_3 = 8$
p	14.88%	15.61%	14.21%	14.69%	15.44%
b	70.36%	73.81%	67.19%	69.46%	73.01%
α	-	-	-	-	-
β	0.0251	0.0244	0.0259	0.0251	0.0244
ω	48.63%	43.51%	55.16%	48.65%	43.53%
CEC	0.8123	0.7728	0.8098	0.8353	0.8176

3.2.3 Analysis of results in DB_H

From the first columns of Table 6-1, we could observe that with risk aversion rate increasing, risk of the pension funds transfer more to future generations, since both α and β decrease. Besides, the assets are invested less in risky markets, shown by the decline of ω . For the contribution and benefit level, we could see that they are smallest when risk aversion rate equals to 5. With changing the group of risk aversion rates from same ones to different gammas for different generations, observed on Column 4 and 5 in Table 6-1 and whole Table 6-2, the parameter α increases while parameter β decreases. This phenomenon shows that young generation prefers to bear more risk at present while old generation tend to transfer more risk to future generations. The reason for it is that since young generation is more risk-seeking and old generation is more risk-avoiding, the former generation prefers to share more risk at present, and the later one prefers to keep the benefit at present. For the total generations, it is hard to judge if the risk transfer increases or decreases with the difference of risk aversion rates among generations increasing, since we could not evaluate which one influence more for the whole collective pension funds.

Moreover, we take some experiments about changing one risk aversion rate while fixed the other two. Comparing first, fourth and fifth columns of Table 6-2, when the float risk aversion rate increases, the corresponding slope coefficient decreases as expected, which means that the generation tends to transfer more risk to futures. Meanwhile, we also notice that the slope coefficient corresponding to the generation which has fixed risk aversion rate also changes. From the observations, we conclude that if one slope coefficient changes due to the change of risk aversion rate, the other slope coefficient tends to change towards the opposite direction. For instance, comparing first and third columns of Table 6-2, the slope coefficient β in column 3 increases, while the other one α decreases. This means that under the same risk aversion rate, young generation tends to transfer more risk to future generations if the

old generation absorbs more risk. Same phenomenon could be observed on β if α changes. More explanations are needed for this phenomenon.

Table 6-1. Simulation result of optimal modified utility function of DB_H

If we use different risk aversion rate in different generations, the slope coefficient α increases and β decreases while p increases with differences among different risk aversions increasing. This means young generation prefers to absorb more risk at present while old generations do the opposite thing. At the same time, the benchmark of contribution tends to be more.

	$\gamma_1 = 3$	$\gamma_1 = 5$	$\gamma_1 = 8$	$\gamma_1 = 4.5$	$\gamma_1 = 4$
	$\gamma_2 = 3$	$\gamma_2 = 5$	$\gamma_2 = 8$	$\gamma_2 = 5$	$\gamma_2 = 5$
	$\gamma_3 = 3$	$\gamma_3 = 5$	$\gamma_3 = 8$	$\gamma_3 = 5.5$	$\gamma_3 = 6$
p	15.71%	14.97%	15.33%	15.51%	16.01%
b	74.28%	70.79%	72.49%	73.34%	75.70%
α	0.0592	0.0431	0.0308	0.0455	0.0481
β	0.0183	0.0165	0.0118	0.0152	0.0140
ω	100%	100%	96.8%	100%	100%
CEC	0.9497	0.9132	0.8888	0.8927	0.8744

Table 6-2. Simulation result of optimal modified utility function of DB_H

After fixing two risk aversion rates, we could observe that α decreases when β increases with smaller risk aversion rate of old generation. Same thing happens when γ_1 floating. We could observe these when make comparison between column 1, 3 and 4 in Table 6-2. This phenomenon shows that youth generation tends to do the opposite thing with old generation, i.e. they absorb more risk when old generation transfers more, and absorb less risk when old generation transfers less. Meanwhile, old generation do the same thing if youth generation changes first.

	$\gamma_1 = 3$	$\gamma_1 = 2$	$\gamma_1 = 3$	$\gamma_1 = 4$	$\gamma_1 = 3$
	$\gamma_2 = 5$				
	$\gamma_3 = 7$	$\gamma_3 = 8$	$\gamma_3 = 6$	$\gamma_3 = 7$	$\gamma_3 = 8$
p	16.92%	17.70%	16.51%	16.47%	17.26%
b	80.01%	83.31%	78.07%	78.09%	81.61%
α	0.0541	0.0614	0.0536	0.0482	0.0541
β	0.0120	0.0103	0.0136	0.0124	0.0108
ω	100%	100%	100%	100%	100%
CEC	0.8400	0.7881	0.8413	0.8672	0.8414

3.3 Analysis of figures

3.3.1 Analysis of figures in DB_{CA}

In Figure 3-1, we plot the 5%, 50% and 95% quantiles of the normalized consumption, c_t , of the optimal collective plans, with the scheme parameters set corresponding to column 2 & 5 in Table 4-1 and column 1 & 2 in Table 4-2. With these four figures, we could observe that the differences of three trajectories increase with more differences between risk aversion rates of generations. This means under bigger difference between risk aversion rates, the fluctuations become larger.

In Figure 3-2, we make comparison of figures which have floating γ_1 but fixed γ_2 and γ_3 . We use optimal parameters from Table 4 and create two pairs to compare: column 5 in Table 4-1 vs. column 3 in Table 4-2, and column 2 in Table 4-2 vs. column 5 in Table 4-2. Both pairs show the big influence of γ_1 in DB_{CA} pension scheme. Bigger γ_1 means smaller fluctuations.

In Figure 3-3, we compare the trajectories of normalized consumptions with floating γ_3 and fixed γ_1 and γ_2 . We also use two pairs for this figure: column 5 in Table 4-1 vs. column 4 in Table 4-2, and column 1 in Table 4-2 vs. column 3 in Table 4-2. We could observe there are some declines of 95% quantiles when γ_3 increasing, but indeed, this change is not so obvious compared with Figure 2-2.

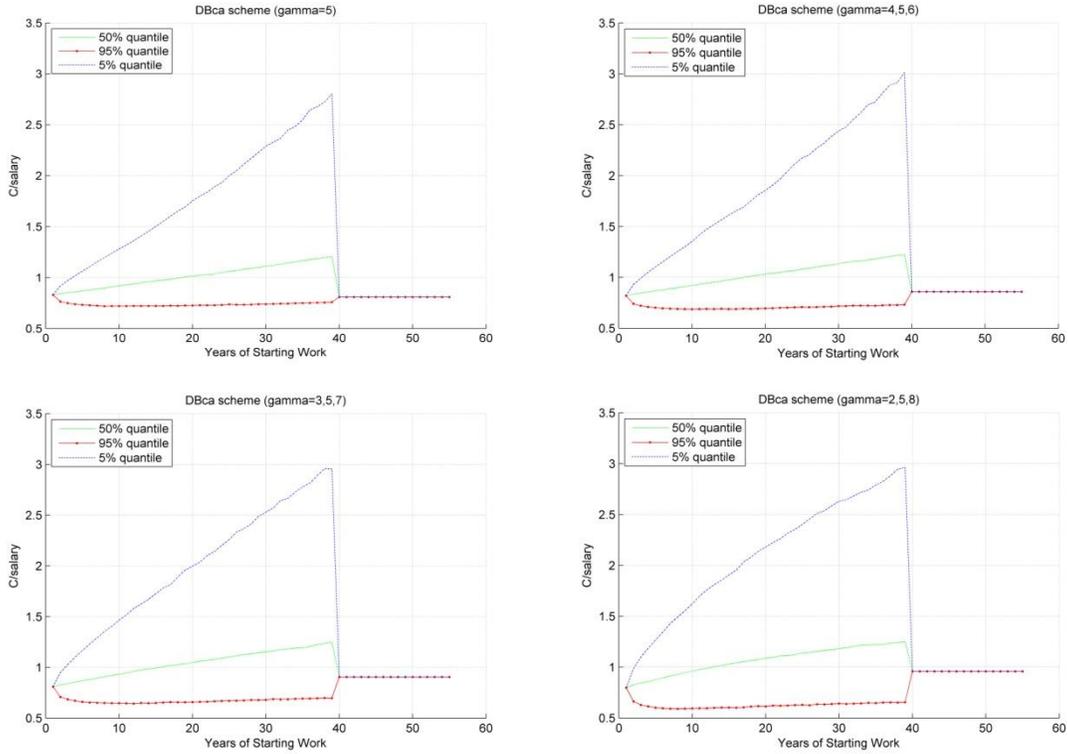
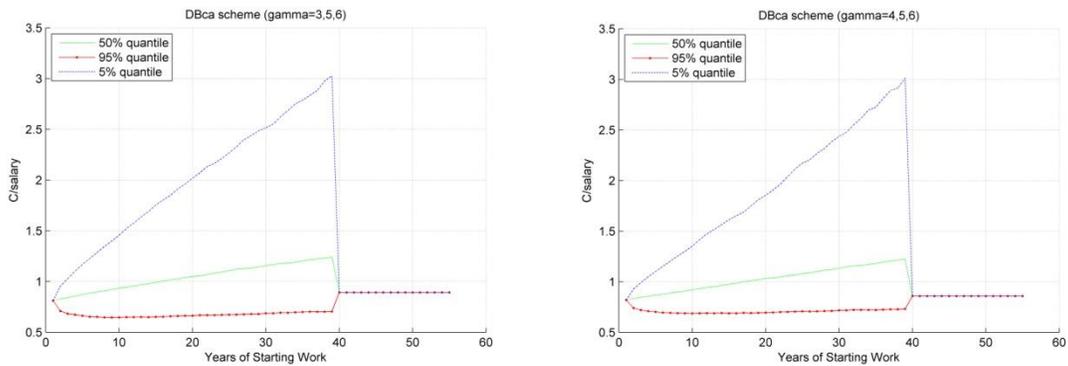


Figure 3-1. Consumption profiles obtained from DB_{CA} under different combination of γ

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective scheme DB_{CA} . All parameters are provided from Table 4 for different combination of γ . The consumption levels are normalized by annual salary, i.e. c_t/y



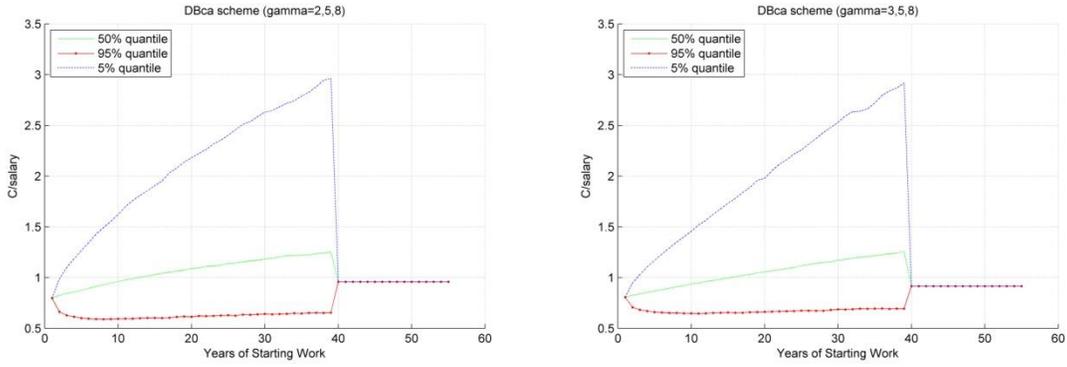


Figure 3-2. Consumption profiles obtained from DB_{CA} under different combination of γ

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective scheme DB_{CA}. All parameters are provided from Table 4 for fixed γ_2 and γ_3 and floating γ_1 . The consumption levels are normalized by annual salary, i.e. c_t/y

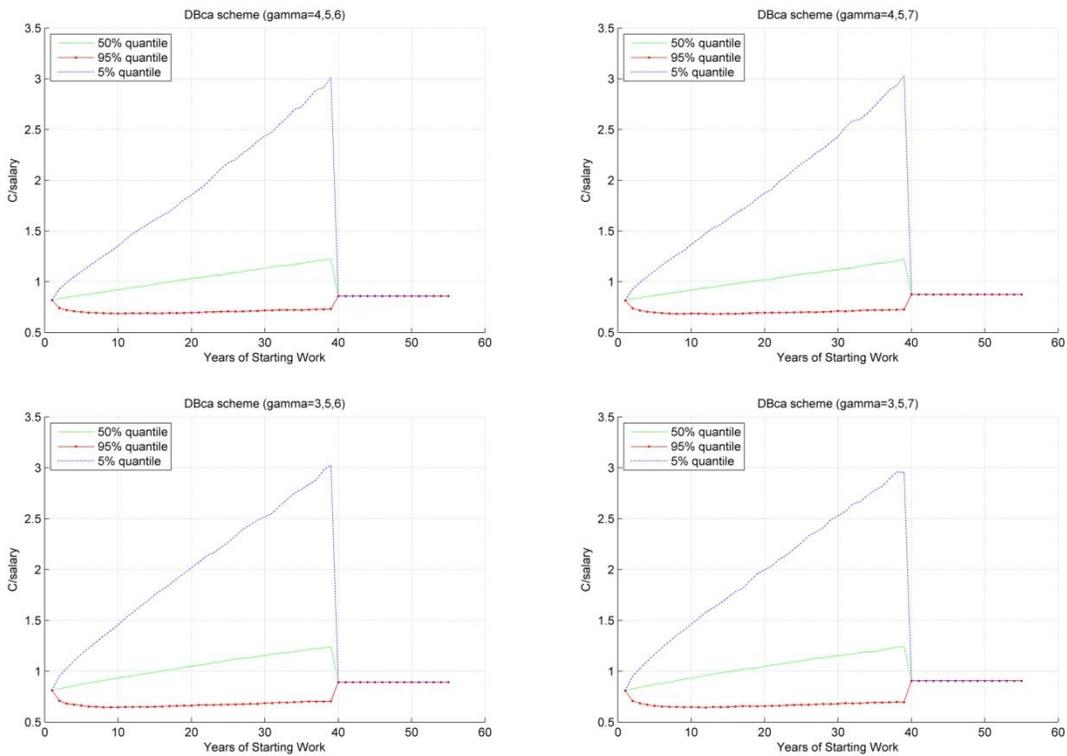


Figure 3-3. Consumption profiles obtained from DB_{CA} under different combination of γ

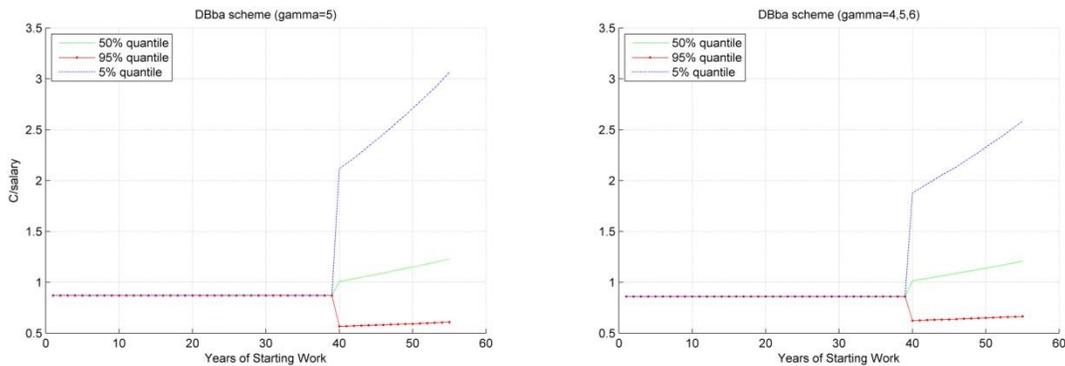
The chart above shows the (5%, 50%, 95%) quantile distributions of consumptions in the optimal collective scheme DB_{CA}. All parameters are provided from Table 4 for fixed γ_1 and γ_2 and floating γ_3 . The consumption levels are normalized by annual salary, i.e. c_t/y

3.3.2 Analysis of figures in DB_{BA}

In Figure 4-1, we plot the 5%, 50% and 95% quantiles of the normalized consumption, c_t , of the optimal collective plans, with the scheme parameters set corresponding to column 2 & 5 in Table 5-1 and column 1 & 2 in Table 5-2. Different with Figure 2, from these four figures, we could observe that the differences of three trajectories decrease with more differences between risk aversion rates of generations. This means under bigger difference between risk aversion rates, the fluctuations become smaller due to higher risk aversion rate of old generation.

In Figure 4-2, we make comparison of figures which have floating γ_1 but fixed γ_2 and γ_3 . We use optimal parameters from Table 5 and create two pairs to compare: column 5 in Table 5-1 vs. column 3 in Table 5-2, and column 2 in Table 5-2 vs. column 5 in Table 5-2. Both pairs show the little influence of γ_1 in DB_{BA} pension scheme. Smaller γ_1 causes tiny upward move for all three trajectories, and for total fluctuations, we could hardly find obvious changes.

In Figure 4-3, we compare the trajectories of normalized consumptions with floating γ_3 and fixed γ_1 and γ_2 . As above, we also use the two pairs: column 5 in Table 5-1 vs. column 4 in Table 5-2, and column 1 in Table 5-2 vs. column 3 in Table 5-2. With increasing γ_3 , we could see obvious decline of fluctuations before and after retirement. Due to higher risk aversion rate, old generation tend to have less risk.



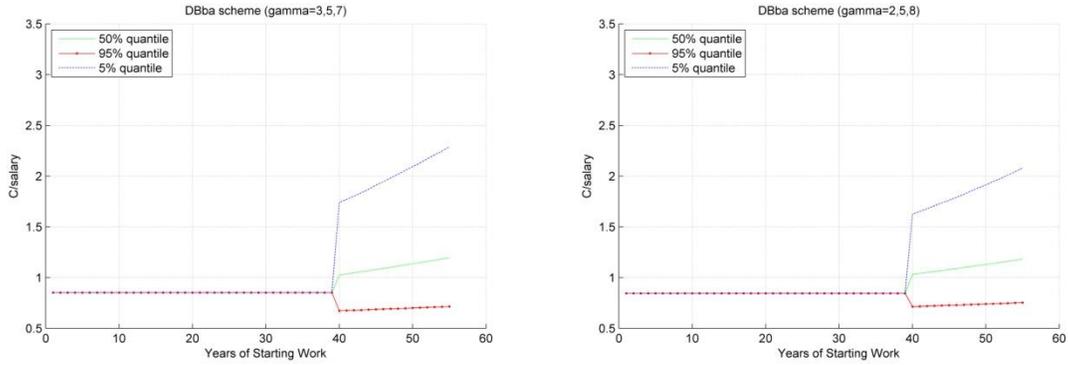


Figure 4-1. Consumption profiles obtained from DB_{BA} under different combination of γ

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective scheme DB_{BA} . All parameters are provided from Table 5 for different combination of γ . The consumption levels are normalized by annual salary, i.e. c_t/y

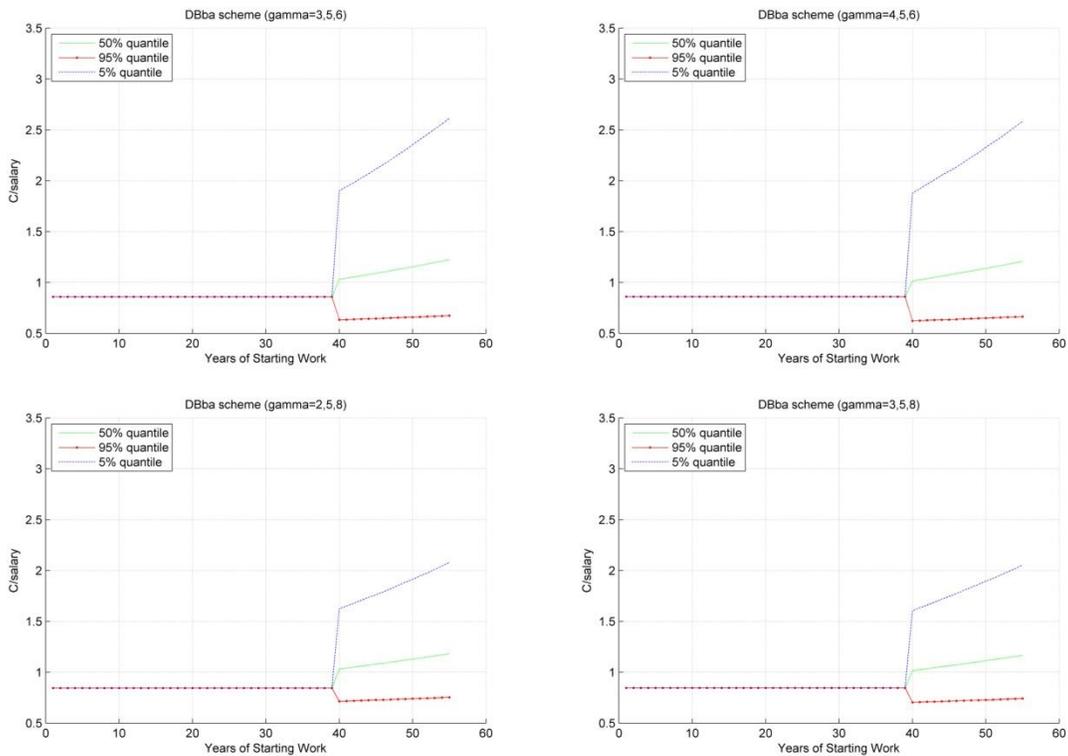


Figure 4-2. Consumption profiles obtained from DB_{BA} under different combination of γ

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective scheme DB_{BA} . All parameters are provided from Table 5 for fixed γ_2 and γ_3 and floating γ_1 . The consumption levels are normalized by annual salary, i.e. c_t/y

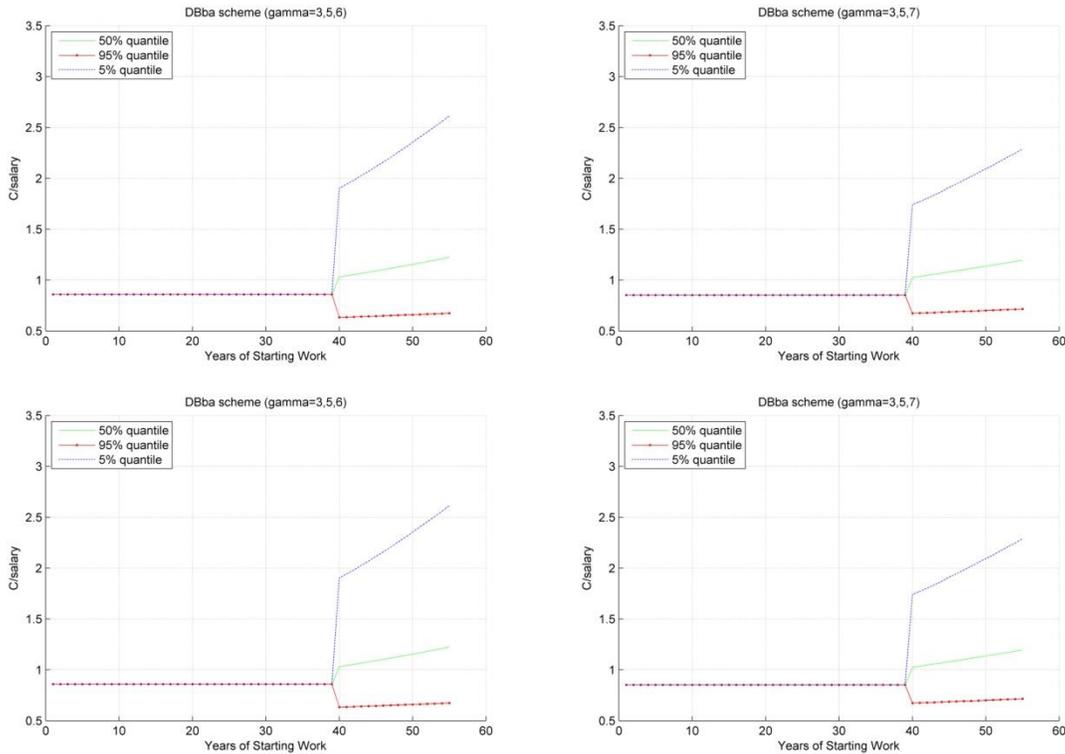


Figure 4-3. Consumption profiles obtained from DB_{BA} under different combination of γ

The chart above shows the (5%, 50%, 95%) quantile distributions of consumptions in the optimal collective scheme DB_{BA} . All parameters are provided from Table 5 for fixed γ_1 and γ_2 and floating γ_3 . The consumption levels are normalized by annual salary, i.e. c_t/γ

3.3.3 Analysis of figures in DB_H

Similar with the two sections above, we plot the 5%, 50% and 95% quantiles of the normalized consumption, c_t , of the optimal collective plans, with the scheme parameters set corresponding to column 2 & 5 in Table 6-1 and column 1 & 2 in Table 6-2. Observing Figure 5-1, we could get the conclusion that with increasing difference of risk aversion rates among different generations, difference of three trajectories before retirement becomes bigger while one after retirement become smaller. Therefore, the fluctuations of 5% and 95% quantile become bigger, but for one of 95% quantile, it is hard to get a certain conclusion.

In Figure 5-2, we make comparison of figures which have floating γ_1 but fixed γ_2 and γ_3 . We use optimal parameters from Table 6 and create two pairs to compare: column 5 in Table 6-1 vs. column 3

in Table 4-2, and column 2 in Table 6-2 vs. column 5 in Table 6-2. As expected, the trajectories before retirement changes due to changing γ_1 . We should also notice, meanwhile, that the trajectories also change with different risk aversion rates of youth generation. As mentioned in 3.2.3, if youth generation decreases their risk aversion rate, old generation tends to be more risk-avoiding.

In Figure 5-3, we compare the trajectories of normalized consumptions with floating γ_3 and fixed γ_1 and γ_2 . We also use two pairs for this figure: column 5 in Table 6-1 vs. column 4 in Table 6-2, and column 1 in Table 6-2 vs. column 3 in Table 6-2. Similar with Figure 4-2, besides the trajectories after retirement change, the ones before retirement also change with changing γ_3 . In fact, in DB_H , the trajectories before and after retirement tend to move towards different direction when changing γ_1 or γ_3 .

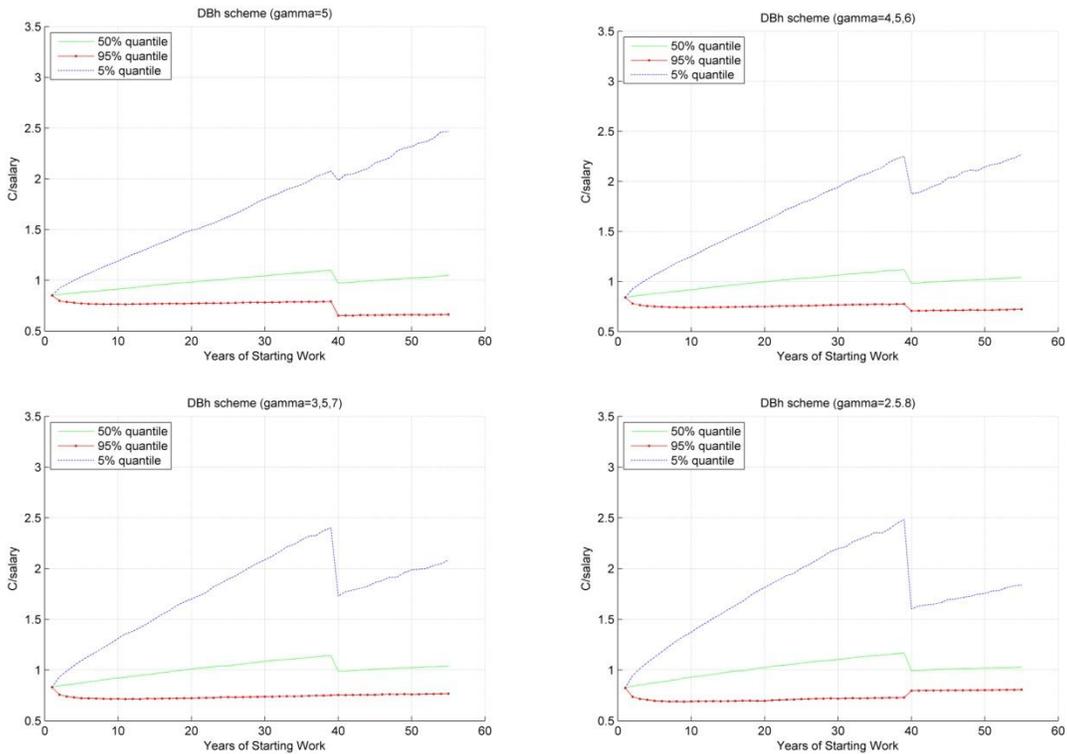


Figure 5-1. Consumption profiles obtained from DB_H under different combination of γ

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective scheme DB_H . All parameters are provided from Table 6 for different combination of γ . The consumption levels are normalized by annual salary, i.e. c_t/y

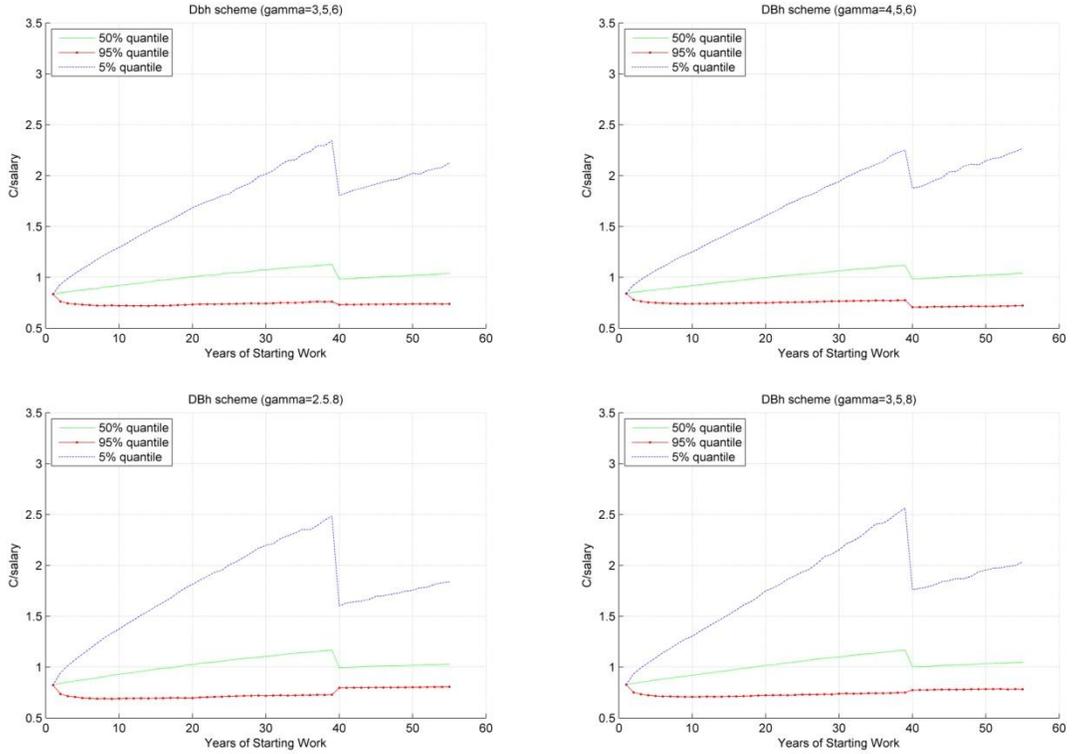
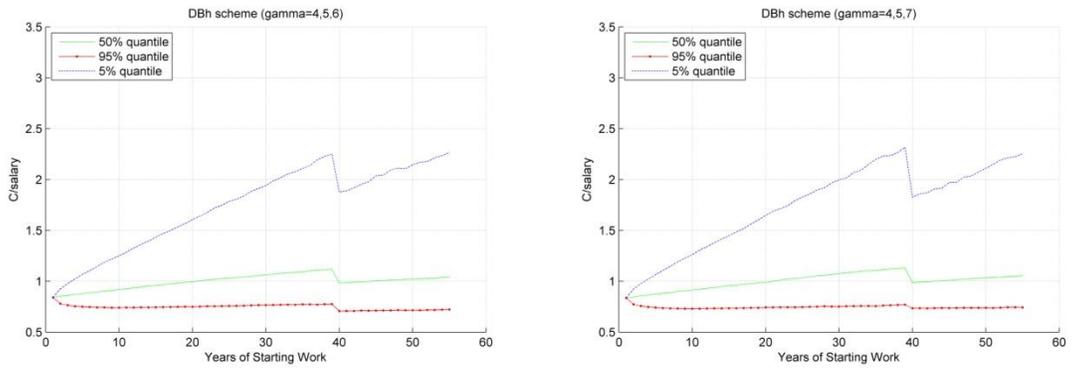


Figure 5-2. Consumption profiles obtained from DB_H under different combination of γ

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective scheme DB_H . All parameters are provided from Table 5 for fixed γ_2 and γ_3 and floating γ_1 . The consumption levels are normalized by annual salary, i.e. c_t/y



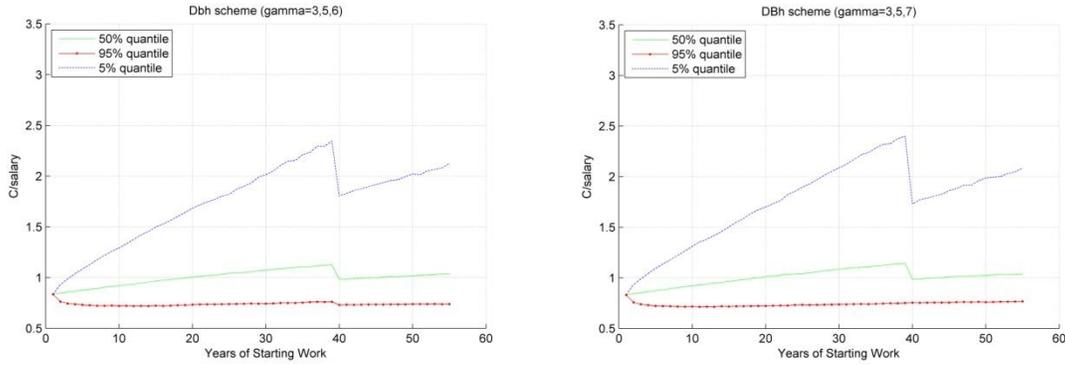


Figure 5-3. Consumption profiles obtained from DB_H under different combination of γ

The chart above shows the (5%, 50%, 95%) quantile distributions of consumptions in the optimal collective scheme DB_H . All parameters are provided from Table 5 for fixed γ_1, γ_2 and floating γ_3 . The consumption levels are normalized by annual salary, i.e. c_t/y

3.3.4 Optimal individual schemes

As part 2.3, the calculation of optimal modified pension schemes follows the same procedures: from equation (12) to equation (23). With these equations, we could get the figures of wealth, consumption and portfolio share in equities of OI under the modified utility model. We could observe the figures of wealth, consumption and portfolio share in equities of optimal individual pension scheme under the risk aversion changing 1 between generations, i.e. $\gamma_1 = 4, \gamma_2 = 5, \gamma_3 = 6$. Compared with Figure 1, we could observe some differences.

At first, the portfolio share in equities (ω in the function), is lower than one in Figure 1 after retirement. This is because people have higher risk aversion rate when retiring, and therefore prefer to invest less in risky market. Interestingly, the result of omega seems near the result shown in Fig.1 in original paper. This observation increases our doubt of the accuracy of that figure, since this figure is drawn under the assumption of $\gamma_3 = 6$ while the figure in original paper under the assumption of $\gamma = 5$.

On another side, the wealth increases a little at the time of retirement ($T = 40$). This phenomenon could be explained by the decreasing risk aversion rate at youth generation ($\gamma_1 = 4$ instead of $\gamma_1 = 5$). Therefore, people tend to invest more in risky market, and have higher average return consequently.

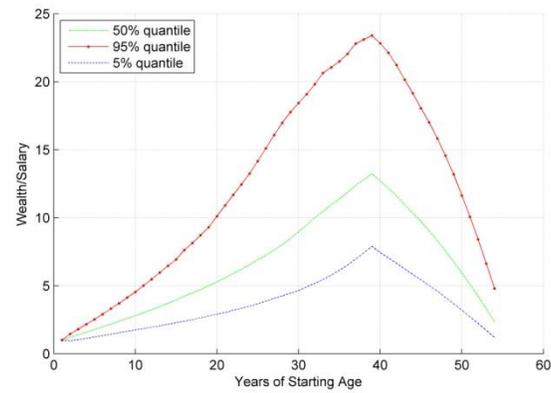
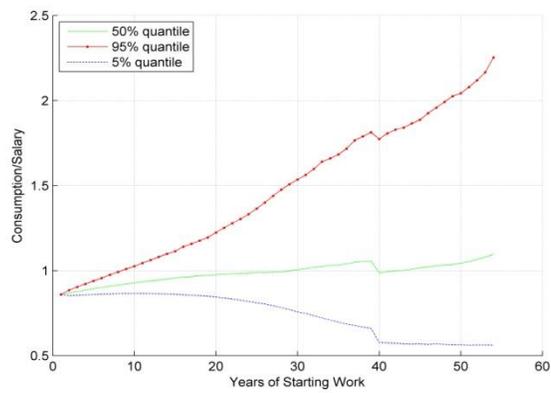
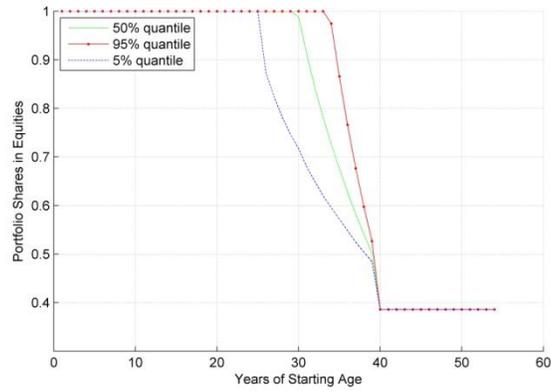


Figure 6. Consumption profiles obtained from the collective schemes

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective schemes (DB_{CA} , DB_{BA} , DB_H). all parameters are provided from Table 7 for $\gamma_1 = 4$, $\gamma_2 = 5$, $\gamma_3 = 6$. The consumption levels are normalized by annual salary, i.e. c_t/y

3.4 Comparison between different pension schemes

3.4.1 Comparison of optimal parameters and figures

To make the comparison among different collective pension schemes, we start at all gammas equaling to 5. After that, we plan to compare the parameters' change under three different combinations of risk aversion rates: $\gamma_1 = 4, \gamma_2 = 5, \gamma_3 = 6$; $\gamma_1 = 3, \gamma_2 = 5, \gamma_3 = 7$; $\gamma_1 = 2, \gamma_2 = 5, \gamma_3 = 8$. The reason to make comparison under these combinations is same changing slope of youth and old generation could help us make comparison more easily. From Table 7, we find that even under different combinations of risk aversion rates, the institutional settings still have significant impact on optimal contribution rates and investment policies. From all gammas equal to 5 to $\gamma_1 = 2, \gamma_2 = 5$ and $\gamma_3 = 8$, the optimal target contribution rate in DB_{CA} always keeps the highest, from 17.09% of annual salary till 20.26% of annual salary. Consequently, the target replacement rate b is also the highest, from 80.81% to 95.08%. On the contrary, DB_{BA} has the lowest contribution and replacement rate, from $b=12.96\%$ and $p=61.28\%$ till $b=15.61\%$ and $p=73.81\%$. For the portfolio choice, DB_{BA} keeps less aggressive than other two collective pension schemes. The risky portfolio weight of the DB_{BA} decreases from 61.28% when all gammas equal to 5, to 43.51% when $\gamma_1 = 2, \gamma_2 = 5$ and $\gamma_3 = 8$. The risky portfolio weights of other two pension schemes, meanwhile, keep 100% for all the four combinations of risk aversion rates.

Figure 7 provides graphical insight into distribution of consumption over time under four combinations of risk aversion rates. Similar as Figure 2, the figure plots the 5%, 50% and 95% quantiles of the normalized consumption of the optimal collective plans, with all parameters set corresponding to Table 7. At the start gamma combination (i.e. $\gamma = 5$), the hybrid scheme DB_H perform much better in distributing shocks over total lifetime than other two pension schemes. Meanwhile, with increasing difference of gammas among generations, we could notice that the fluctuations of DB_{CA} and DB_H , but one of DB_{BA} decreases gradually. Under the situation of $\gamma_1 = 2, \gamma_2 = 5$ and $\gamma_3 = 8$, finally, the fluctuation of DB_H is a little bigger than DB_{BA} in 5% quantile, and almost the same in other two quantiles. And the conclusion “more efficient risk sharing can be achieved by using more risk absorber” in original paper may not exist from this side.

Meanwhile, when observing the *CEC* results of three collective pension schemes, the DB_H always outperforms among them under different combinations of risk aversion rates. The DB_{BA} pension scheme, however, always stays the bottom in ranking. (Continued on page 44)

Table 7. Optimal result of parameters under different combinations of risk aversion rates

The table shows the results of benefit level (b), certainty equivalent consumption (CEC) and ratio CEC/CEC^{OI} with the optimal parameters $\{p, \alpha, \beta, \omega\}$ shown in former tables. We make comparison of four groups of collective pension schemes under different combinations of risk aversion rates.

Pension		$\gamma_1 = 5$	$\gamma_1 = 4$	$\gamma_1 = 3$	$\gamma_1 = 2$
		$\gamma_2 = 5$	$\gamma_2 = 5$	$\gamma_2 = 5$	$\gamma_2 = 5$
		$\gamma_3 = 5$	$\gamma_3 = 6$	$\gamma_3 = 7$	$\gamma_3 = 8$
	p	17.09%	18.14%	19.14%	20.26%
	b	80.81%	87.55%	90.50%	95.80%
	α	0.0479	0.0545	0.0662	0.0838
DB _{CA}	β	-	-	-	-
	ω	98.06%	100%	100%	100%
	CEC	0.8904	0.8610	0.8311	0.7842
	CEC/CEC^{OI}	99.0%	100.5%	101.9%	103.3%
	p	12.96%	14.01%	14.88%	15.61%
	b	61.28%	66.25%	70.36%	73.81%
	α	-	-	-	-
DB _{BA}	β	0.0269	0.0259	0.0251	0.0244
	ω	63.83%	55.15%	48.63%	43.51%
	CEC	0.8698	0.8382	0.8123	0.7728
	CEC/CEC^{OI}	96.7%	97.9%	99.6%	100.2%
	p	14.97%	16.01%	16.92%	17.70%
	b	70.79%	75.70%	80.01%	83.31%
	α	0.0431	0.0481	0.0541	0.0614
DB _H	β	0.0165	0.0140	0.0120	0.0103
	ω	100%	100%	100%	100%
	CEC	0.9132	0.8744	0.8400	0.7881
	CEC/CEC^{OI}	101.5%	102.1%	102.9%	102.1%
	CEC^{OI}	0.8996	0.8565	0.8160	0.7716

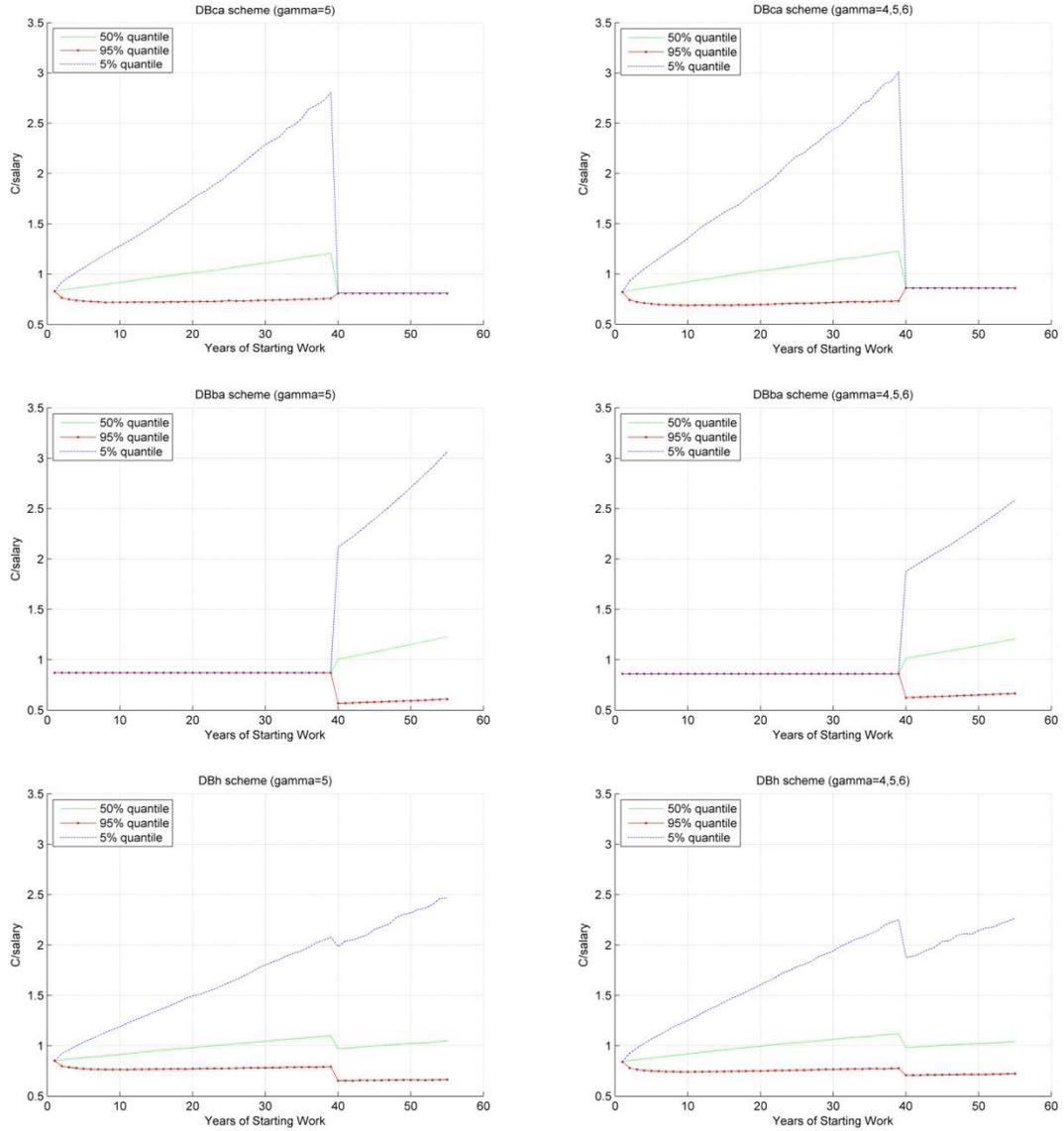


Figure 7-1. Consumption profiles obtained from the collective schemes

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective schemes (DB_{CA} , DB_{BA} , DB_H). all parameters are provided from Table 7 for all $\gamma = 5$ and $\gamma_1 = 4$, $\gamma_2 = 5$, $\gamma_3 = 6$. The consumption levels are normalized by annual salary, i.e. c_t/y

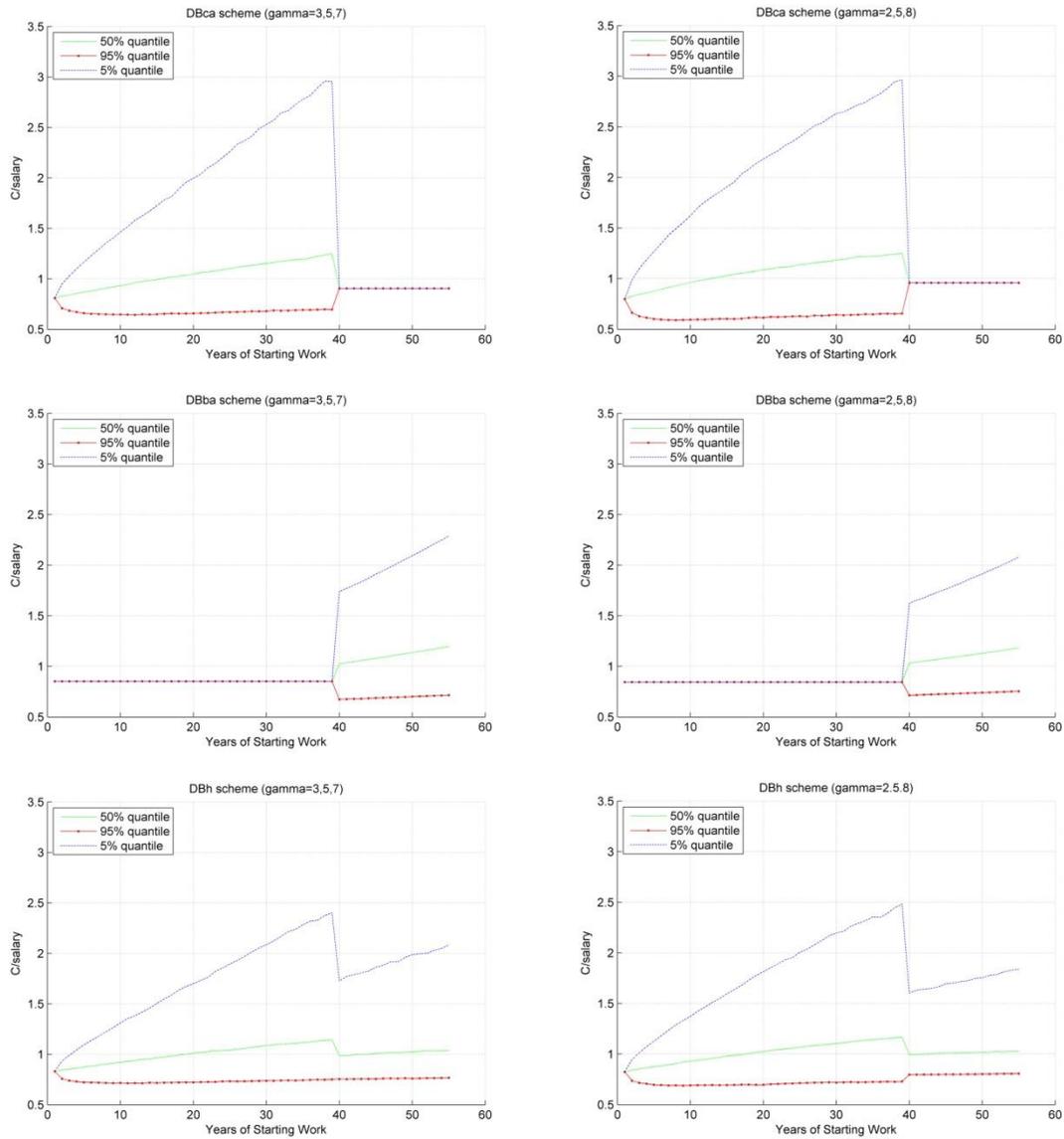


Figure 7-2. Consumption profiles obtained from the collective schemes

The chart above shows the (5%, 50%, 95%) quantile distribution of consumptions in the optimal collective schemes (DB_{CA} , DB_{BA} , DB_H). all parameters are provided from Table 7 for all $\gamma_1 = 3, \gamma_2 = 5, \gamma_3 = 7$ and $\gamma_1 = 2, \gamma_2 = 5, \gamma_3 = 8$. The consumption levels are normalized by annual salary, i.e. c_t/y

(Continued from page 40) From this viewpoint, we should get the conclusion that DB_H performs the

best and could always provide a welfare gain in our risk aversion rates settings.

3.4.2 Comparison of market valuation

For evaluating the market valuation of different kinds of modified collective pension schemes, we still use the pricing kernel method mentioned by 2.4. Since the optimal parameter calculated by fixed random matrix in the former part, we still use the same matrix in the calculation of this part. For DB_{CA} , DB_{BA} and DB_H , we also calculate the NPV (net present value) and confidence interval under four assumptions of different combinations of risk aversion rates: $\gamma_1 = \gamma_2 = \gamma_3 = 5$; $\gamma_1 = 4, \gamma_2 = 5, \gamma_3 = 6$; $\gamma_1 = 3, \gamma_2 = 5, \gamma_3 = 7$; $\gamma_1 = 2, \gamma_2 = 5, \gamma_3 = 8$. We get the following result in Table 8.

At first comparison, we compare Table 8 and Table 2. As mentioned above. Since fixed matrix instead of random matrix of random variables, the optimal parameters of modified collective pension schemes have some inaccuracy compared with the real optimal parameters. Therefore, we could observe the results of all gammas equal to 5 in Table 8 are bigger than ones in Table 3. For DB_{CA} and DB_H pension schemes, the results are in the 95% confidence interval, while the results of DB_{BA} are not. Same as 2.4, we conclude that the cohort accounts of DB_{CA} and DB_H equals to 0, while one of DB_{BA} is not zero significantly.

When observing the results of rest three combinations, we still have the observation that DB_{CA} scheme has the highest value of transfers, while the DB_{BA} scheme results to the lowest one. But in all four combinations of risk aversion rates, dynamic cohort account of DB_{BA} does not equal to zero significantly. Besides, we also mention that with difference of risk aversion rate among generations increasing, the values of transfer of total pension schemes increase. This means that the performance of all these three pensions becomes worse, which could lead future generation confronting larger deficits in pension funds.

Table 8. Market value of intergenerational transfer

The table shows the market value of intergenerational transfers which shown in columns ‘call’ and ‘put’ (positive value and negative value), which expressed in equation (13), under different combinations of risk aversion rates. The upper bound and lower bound define the 95% confidence interval about the null hypothesis of which the difference equals to zero.

Pension		$\gamma_1 = 5$	$\gamma_1 = 4$	$\gamma_1 = 3$	$\gamma_1 = 2$
		$\gamma_2 = 5$	$\gamma_2 = 5$	$\gamma_2 = 5$	$\gamma_2 = 5$
		$\gamma_3 = 5$	$\gamma_3 = 6$	$\gamma_3 = 7$	$\gamma_3 = 8$
DB _{CA}	call	1.0797	1.1788	1.2917	1.4162
	put	1.5685	1.6889	1.8162	1.9549
	difference	0.4888	0.5101	0.5245	0.5387
	upper bound	1.6526	1.8848	2.1819	2.5188
DB _{BA}	call	0.1135	0.0976	0.0835	0.0715
	put	0.4458	0.4571	0.4658	0.4729
	difference	0.3324	0.3595	0.3822	0.4014
	upper bound	0.1148	0.0868	0.0706	0.0601
DB _H	call	0.6471	0.7480	0.8439	0.9355
	put	1.0341	1.1624	1.2824	1.3947
	difference	0.3870	0.4145	0.4385	0.4591
	upper bound	1.2078	1.5219	1.7206	1.9254

4. Estimation Error and Extension

During calculating the optimal parameters under modified utility function, as mentioned in Part 3, we fixed the matrix of random number in order to make more accurate comparison between different

groups of results. When finding out the optimal result via the code `fmincon`, the Matlab program runs a loop for several times to find out the minimum results. We notice equation 3 contains a Brownian motion term, and this term should be given different random instead of fixed numbers each loop in the optimization process in Matlab. Therefore we could observe the big difference of results under same risk aversion rate in Part 2 and Part 3. Since in Part 3, our mainly job is to analyze the differences between results of different kinds of pension schemes but not focus on the numbers themselves, this estimation error may not influence much in our conclusion. If people want to get accurate parameters of pension funds, however, they should avoid fixing the random number matrix.

For extending this topic, I guess this thought would be mainly used in the age-related CRRA utility function. Moreover, we might take some consideration about different risk aversion rate in different situations. For instance, an individual with Defined Benefit pension scheme might have smaller risk aversion rate than one with Defined Contribution pension scheme after retirement. This is because the Defined Benefit pension scheme could provide a constant income each period of time which brings less anxiety of using out of money, while Defined Contribution scheme only provide you a certain amount of money at the beginning of retirement.

Still, we do not know the accurate risk aversion rate of different generations, and there are not much psychological issues about this topic. If we know the more accurate data about this, we could more easily get the good results we want, but not just setting different groups of risk aversion rates and make comparison.

5. Conclusion

Following models and settings of original paper, we use the institutional setting of funded pension schemes to study welfare aspects of intergenerational risk sharing. The benefits and/or contributions of these pension schemes depend on the funding status.

At first, we replicate the models of three collective pension schemes and optimal individual scheme, and get similar results and conclusion with ones in original paper. The hybrid pension scheme has less consumption fluctuation when retiring, and more importantly, it could improve the welfare and have 1%-4% above the OI optimal individual scheme in terms of certainty equivalent consumption under three constant risk aversion rates. It shows that sharing the risk over total generations outperforms compared with sharing the risk only over people before or after retirement. After that, we change the CRRA utility function with different risk aversion rates corresponding to different generations. With

this change, we make comparison of different results of same pension schemes. We get the predictable results that with changing risk aversion of generations, the trajectories of wealth, consumption and portfolio shares in equities increases or decreases. Meanwhile, there are some other changes happen, which even though keeping the risk aversion rate constant in one generation, the trajectories would also be influenced by the change of risk aversion rate in other generations. At last, we make the comparison among different pension schemes under combination of risk aversion rates. From the results we could observe that small fluctuation of consumption is not the advantage of hybrid pension scheme any more, but it still keeps the outperformance in certainty equivalent consumption. Therefore we should believe sharing risks over total generations is the better choice even under different risk aversion rates in different generations.

On another side, we get at least three different results with original paper when replicating models. The first one is the parameter b of DB_{BA} scheme with risk aversion rate equals to 5. The second one is the trajectories of portfolio share in equities in optimal individual scheme. The third one is the market valuation of DB_{BA} scheme. The first difference, after consideration, is quite obviously caused by writing error. But for the latter two differences, since we do not know the model and program of original paper, we could not find out what causes them. Since we do not find problems of our own models and programs after checking, we keep our doubt of those two results in original paper.

In the last part, we make some analysis of estimation error and extension part. Not only might the economic part pay some attention on the modified CRRA utility function, which could reflect the situation more accurately, but the psychological part might make some investigation in this part as well.

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Appendix

In the appendix, I would show some Matlab codes of this paper, which might be helpful to read the total paper.

Code of optimizing parameters of DB_{BA} :

```
clear

%%lower bound
lb = zeros(3,1);lb(2)=.02;
%%upper bound
ub = [1,1,1];

%%start point
x0 = [0.14,0.02,0.55];
%%optimize the utility function
[x,fval] = fmincon(@function_1_3,x0,[],[],[],[],lb,ub)

%%construct the utility function
%%collective DBba

function f = function_1_3(x)
r0 = .02; %interest rate
sigma = .15; %vatility rate of risky asset
gamma = 5; %risk aversion rate
delta = .04;
mu0 = .06; %drift rate of risky asset

N = 55; %number of time
T = 55; %total years in this calculation
dt = T/N;
R = 40; %years of retirement
M = 10000; %number of trajectory

r = r0*ones(M,1);
mu = mu0*ones(M,1);
t = 0:dt:T;

%%static utility function x(1) for p, x(2) for beta, x(3) for omega

p = x(1);
beta = x(2);
omega = x(3);
b0 = (1-exp(-R*r0))/(exp(-R*r0)-exp(-T*r0))*p;

L = b0/(r0^2)*((T-R)*r0-exp(-R*r0)+exp(-T*r0))-p/(r0^2)*(r0*R-1+exp(-R*r0)); %liability is an expression of start benefit and contribution level
A0 = L; %start asset equals to the liability under the assumption of collective pension
asset = A0*ones(M,1);
C = (1-p)*ones(M,1);
utility = C.^(1-gamma)/(1-gamma);

for j=1:N
```

```

S(:,j) = asset(:,j)-L; %surplus function of asset
b(:,j) = b0.*ones(M,1)+beta.*S(:,j)./(T-R); %benefit level
contri(:,j) = p*ones(M,1);
asset(:,j+1)= asset(:,j)+((asset(:,j).*(r+omega*(mu-r)))+40*contri(:,j)-
15*b(:,j))*dt+omega*sigma*asset(:,j)*sqrt(dt).*randn(M,1);%asset function

if j<R
    C(:,j) = 1-contri(:,j);
else
    C(:,j) = b0*ones(M,1)+beta*S(:,j)./(T-R);
end

utility(:,j+1) = utility(:,j)+exp(-delta*j)*(C(:,j).^(1-gamma))/(1-gamma);
end

f = (-1)*mean(utility(:,N+1));

```

Code of CEC calculation of DB_{BA}:

```

%%construct the utility function
%%collective DBba

function f = function_1_3(x)
r0 = .02; %interest rate
sigma = .15; %vatility rate of risky asset
gamma = 8; %risk aversion rate
delta = .04;
mu0 = .06; %drift rate of risky asset

N = 55; %number of time
T = 55; %total years in this calculation
dt = T/N;
R = 40; %years of retirement
M = 10000; %number of trajectory

r = r0*ones(M,1);
mu = mu0*ones(M,1);
t = 0:dt:T;

%%static utility function x(1) for p, x(2) for beta, x(3) for omega

p = x(1);
beta = x(2);
omega = x(3);
b0 = (1-exp(-R*r0))/(exp(-R*r0)-exp(-T*r0))*p;

L = b0/(r0^2)*((T-R)*r0-exp(-R*r0)+exp(-T*r0))-p/(r0^2)*(r0*R-1+exp(-R*r0)); %liability is an expression of start benefit and contribution level
A0 = L; %start asset equals to the liability under the assumption of collective pension
asset = A0*ones(M,1);
C = (1-p)*ones(M,1);
utility = C.^(1-gamma)/(1-gamma);

for j=1:N

    S(:,j) = asset(:,j)-L; %surplus function of asset
    b(:,j) = b0.*ones(M,1)+beta.*S(:,j)./(T-R); %benefit level
    contri(:,j) = p*ones(M,1);

```

```

    asset(:,j+1) = asset(:,j)+((asset(:,j).*(r+omega*(mu-r)))+40*contri(:,j)-
15*b(:,j))*dt+omega*sigma*asset(:,j)*sqrt(dt).*randn(M,1);%asset function

    if j<R
        C(:,j) = 1-contri(:,j);
    else
        C(:,j) = b0*ones(M,1)+beta*S(:,j)./(T-R);
    end

    utility(:,j+1) = utility(:,j)+exp(-delta*j)*(C(:,j).^(1-gamma))/(1-gamma);
end

f = (-1)*mean(utility(:,N+1));

```

Code of simulating optimal individual scheme:

```

%% simulate trajectories of wealth with consumption and omega

clear

M      = 10000;
r0     = 0.02;% risk free rate
mu0    = 0.06;% drift rate of risky asset
delta  = 0.04;% discount rate
sigma  = 0.15;% vatality rate of risky asset
W      = 1*ones(M,2);%start point of wealth
T_R    = 39;%years of retirement
dt     = 1;
T      = 54;

load C_star.mat
load W_prime.mat
load omega.mat

for i=1:T_R

    %using interpolation to calculate c and omega in each period

    c_ori(:,i) = interp1(W_star(:,i),C_star(:,i),W(:,i),'spline');
    c(:,i)     = max(0,c_ori(:,i));

    omega_ori(:,i) = interp1(W_star(:,i),omega(:,i),W(:,i),'spline');
    omega_mod_1(:,i) = min(1,omega_ori(:,i));%for the restriction of omega no
more than 1
    omega_mod(:,i) = max(0,omega_mod_1(:,i));%for the restriction of omega no
less than 0

    r_return(:,i) = r0*ones(M,1)+omega_mod(:,i)*(mu0-r0);%return of investment
portfolio

    W(:,i+1) = W(:,i)+(W(:,i).*r_return(:,i)+1*ones(M,1)-
c(:,i))*dt+sigma*omega_mod(:,i).*W(:,i)*sqrt(dt).*randn(M,1);

end

for i=T_R:T

    %using interpolation to calculate c and omega in each period

```

```

c_ori(:,i) = interp1(W_star(:,i),C_star(:,i),W(:,i),'spline');
c(:,i)      = max(0,c_ori(:,i));

omega_ori(:,i) = interp1(W_star(:,i),omega(:,i),W(:,i),'spline');
omega_mod(:,i) = min(1,omega_ori(:,i));%for the restriction of omega no
more than 1

r_return(:,i) = r0*ones(M,1)+omega_mod(:,i)*(mu0-r0);%return of investment
portfolio

W(:,i+1) = W(:,i)+W(:,i).*r_return(:,i)-
c(:,i)*dt+sigma*omega_mod(:,i).*W(:,i)*sqrt(dt).*randn(M,1);

end

c(:,55)      = W(:,55);%c equals to W at the last time spot

save ('C_sim_gamma_5.mat','c');
save ('W_sim_gamma_5.mat','W');
save ('omega_sim_gamma_5.mat','omega_mod');

```