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Effectivity in Hedging Longevity Risk

A Life Insurance Scheme of a Child Plan

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Effectivity in Hedging Longevity Risk: A Life Insurance Scheme of A Child Plan

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Abstract:

In this article, we discuss a life insurance scheme-Child Plan. We use the Lee-Carter and the CBD Models to forecast mortality data of US and UK in order to quantify the Plans. We compare the Child Plan with a Single Life Insurance Plan, and find that the Child Plan has a lower price (net premium) and is less sensitive to mortality risk. So, we draw a conclusion that the Child Plan has an inner hedging system owing to a joint death probability design.

We also find that additional benefit conditions-income benefit and both parents insured will increase the price of the Child Plan and cause a higher risk to the insurer. And the older the insurers are, the higher the price and the degree of sensitivity to mortality risks are.

Keywords:

aging, mortality rate, life insurance, Child Plan, hedging

Content

1 Introduction

Child Plan is an Insurance product to ensure children's future by financing the key milestones in their lives (college education, marriage, etc.). Usually, it includes two parts. The first part is the normal maturity benefit: the insured receives the accumulated amount of guaranteed fixed benefit at the maturity date. The second part is the death benefit: if the insured parent dies, the remaining premium payments will stop, and the beneficiary will still receive the guaranteed fixed benefit immediately or on the maturity date. Furthermore, some companies have additional policy that an amount of money which equals to the annualized premium will be paid to the beneficiary starting from the date of death till maturity date. This benefit is known as "income benefit" which is a part of death benefit.

As many articles stated (for example: Bijak, Kupiszewska, et al., 2007; Bovenberg, 2008; Christensen, Doblhammer, et al., 2009), European countries step into an era of aging. Longer life expectancy and lower fertility rate accelerate the aging problem. And an increase of mean age of European mothers at the first childbirth is a reason of lower fertility. For Child Plan, these two factors affect its pricing in opposite directions. Declining death probability of parents reduces the probability of the premium loss for the insurance company, and the delay of the first childbirth means a mortality table with higher age groups should be used in the calculation of premium. And higher age groups in most cases indicate a higher mortality risk.

In this paper, we analyze how the Child Plan reacts to the mortality rate change, and figure out how different benefit conditions work in order to hedge the longevity risk. The layout of this paper is presented as follows.

In chapter 2, we start with a background of the recent aging trend and then outline a basic Child Plan as the benchmark. US data and UK data will be used for comparison. That is because the historical data shows that US and UK both encounter a similar large decline of mortality rate in recent years, whilst the mean age of mothers at the first childbirth in UK increased to 30 while the data in US kept around 25. This five-year gap will lead to a different life table for the insured.

In chapter 3 and chapter 4 of this paper, I fit Lee-Carter model (Lee and Carter, 1992) to US data and fit CBD model (Cairns, Blake, and Dowd, 2006) to UK data respectively, and then discuss the plausibility of the fit. This step help us to get the further forecasting of mortality rate, which will be a quantified tool for us to assume mortality rate change.

In chapter 5, we go back to the benchmark Child Plan and investigate some extensions of this Child Plan. We add two benefit conditions into the Plan: income benefit and both parents insured. We also increase the age of the insured parent(s) to discuss the impact of insured age. We analyze sensitivity to mortality rate of each benefit conditions in Chapter 6. We simulate three mortality risk scenarios and draw sensitivity curves to show the degree of sensitivity.

In conclusion, we find that the Child Plan has an inner hedging system by a joint death probability design. When a mortality-linked issue comes and the mortality rates of the adults and the children change at the same time, the Child Plan will suffer less loss or gain less compared with a Single Insurance Plan. The conditions of income benefit and both parents insured will increase the net premium of the Child Plan respectively, and have larger exposure to mortality risks. Also, the older the insured is, the higher the net premium and degree of sensitivity to mortality risks are, which can be found in US and UK data both.

This article is a discussion about mortality risk in one kind of life insurance plan-Child Plan. The approach to quantify or simulate the mortality risk in this article can also be applied to other mortality linked products like annuity, pension, and other life insurance products. And the sensitivity curve is an effective tool to measure the risk exposure to mortality risks.

2 Background and Benchmark Plan

2.1 Notation

We analyze Child Plan with mortality and survival related data, and we use two mortality models to obtain such data. Thus, it would be better to identify a consistent and clear notation throughout.

Since our original data of mortality related are all from Human Mortality Database, we reference the notation and data explanation of HMD to define the notation in this paper.

Calendar year t is defined as running from time t to time t + 1.

Death rates consist of death counts divided by the exposure-to-risk. We use $m_{x,t}$ as central death rate for age x in calendar year t (shown as “mx” in HMD). More specifically,

$$m_{x,t} = \frac{\#deaths\ during\ calendar\ year\ t\ aged\ x\ in\ last\ birthday}{average\ population\ during\ calendar\ year\ t\ aged\ x\ in\ last\ birthday} \tag{2.1}$$

This central death rate will be used in chapter 3 for Lee-Carter model. But we will use another measure of mortality more frequently, which is called death probability $q_{x,t}$ (or q_x for short, and shown as “qx” in HMD). This is the probability that an individual aged exactly x at exact time t will die between t and t + 1. In some articles this measured data is also called mortality rate. To distinguish with the formal data, we call it death probability or one-year death probability.

Central death rate and death probability are the only two original data in this paper: we use historical date from HMD and future data from the forecast results of Lee-Carter and CBD model. In chapter 2 and 5, there is also a third measure of mortality called survival probability $_Tp_{x,t}$. This is interpreted as the probability that the individual aged x at time t survives at least another T years. But we always get this mortality data by calculation with the following equation:

$$_T p_{x,t} = (1-q_{x,t}) \times (1-q_{x+1,t+1}) \times \cdots \times (1-q_{x+T-1,t+T-1}) = \prod_{i=1}^{T-1} (1-q_{x+i,t+i}) \tag{2.2}$$

2.1.1 Relationship between $m_{x,t}$ and $q_{x,t}$

The death rate, $m_{x,t}$, and the death probability, $q_{x,t}$, are two different ways to measure mortality. The approximate relation between these two coefficients is as following:

$$q_{x,t} \approx \frac{m_{x,t}}{1+(1-a_{x,t}) \times m_{x,t}} \tag{2.3}$$

Here $a_{x,t}$ refers to the average length of survival between ages x and x+n for people dying in the interval during year t. For the 1*1 year period life table we used from HMD, n is 1 and ax equals to 0.5 for age from 1 to 109. This can be intuitively interpreted like this: for an individual die between age x and x+1 (0<x<109), the death could happen in any time of a calendar year with equal probability, thus for a large population of people dying in this year, the middle of year is likely to be the time point that half of the people in this population have died.

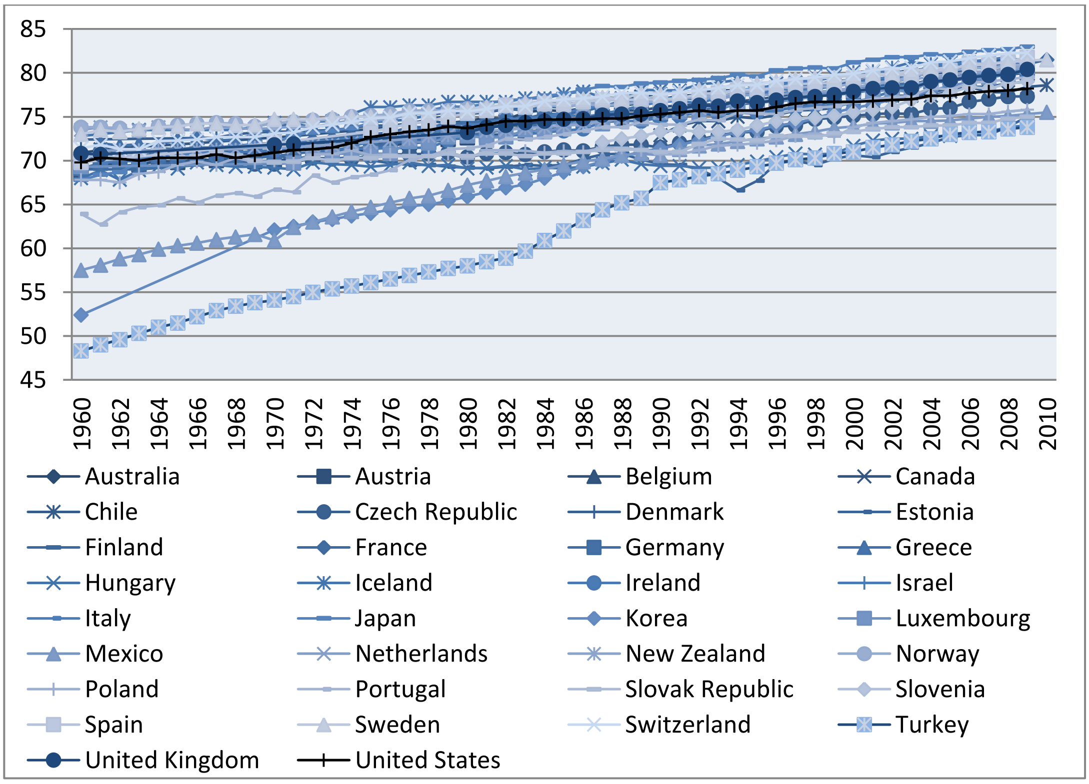
However, when it comes to new born babies younger than one-year old and rather old elders, the results are not the same. Considering the relative high death probability of infants, half year is not the average length of survival for babies dying before one-year-old birthday. In HMD, $a_{x,t}$ usually takes a value below 0.25. Since infant mortality is remarkably improving, $a_{x,t}$ for age 0 are also different among years and shows an increasing trend in recent 50 years. And for age above 110, since there is no yearly data but a summary death mortality rate as age of "110+", the $a_{x,t}$ for this age is thus bigger than 1 year and varies for different years.

In chapter 3, when we get a fitted model of the Lee-Carter approach for central death rate, we use the above approximation method to transfer the results into death probability. Since in this article, we only need the death probability forecast for individuals above one-year old and less than 43 (we assume Plans with latest entry year of 2006 and the latest historical data we have for US is data in year 2007), so we always use $a_{x,t}$ equal to 0.5.

2.2 Aging in UK and US

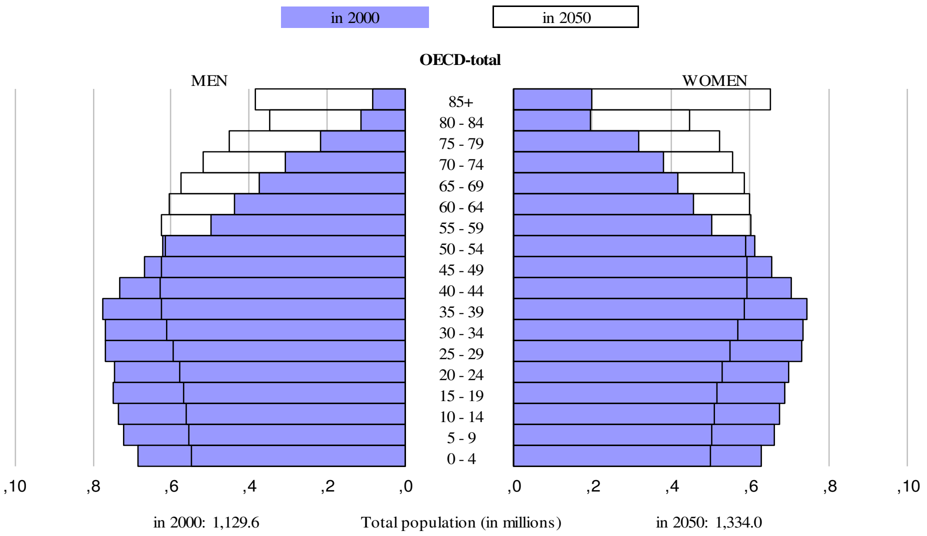
An unprecedented improvement in population longevity has been observed worldwide (see for example, Benjamin and Soliman 1993, McDonald 1997, and McDonald et al. 1998). In 1960, the Average life expectancy at birth varied in OECD countries between 48.3 in Turkey and above 73 in Norway and Netherlands. During the past 50 years, average life expectancy at birth for OECD as a whole has increased by about 2.5 years per decade (Bovenberg, 2008) (Figure 1). This longevity trend, combined with lower fertility rate are the main causes of aging (Bovenberg, 2008) and will change the age structure of the population in OECD countries in 2050 (Figure 2).

Figure 1 34 OECD Countries Life Expectancy at Birth: Total Population



Source: OECD Health Data 2011, November 2011

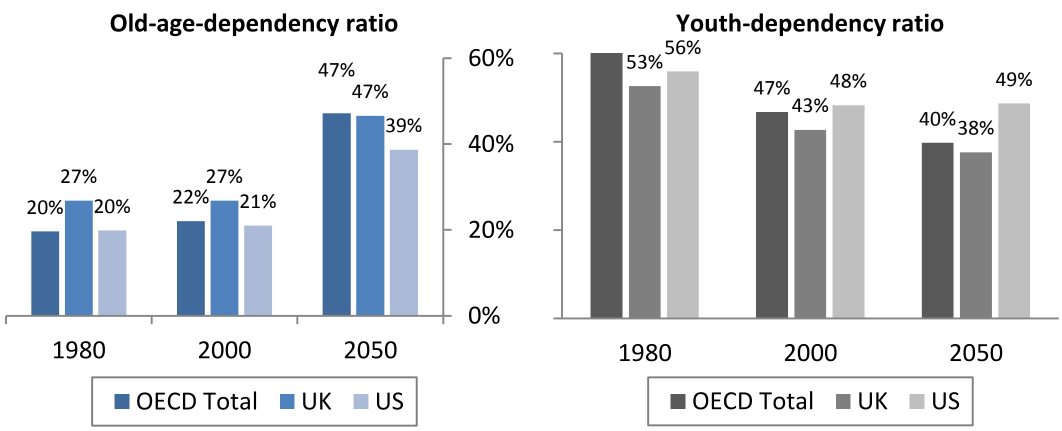
Figure 2 OECD Total Population by Age Group, Gender, in 2000 and 2050, in Percentage of Total Population in Each Group



Source: OECD Demographic and Labour Force database, used in OECD (2007), Society at a Glance: OECD Social Indicators 2006

To measure the degree of aging and the change of the age structure, an indicator named Age-dependency Ratios is applied: the Youth-dependency Ratio (number of individuals aged less than 20 over to the population aged 20 to 64) and the Old-age-dependency Ratio (number of individuals aged 65 over to the population aged 20 to 64). From 1980 to 2050, the Old-age-dependency Ratio is projected to more than double (from 20% to 47%) in OECD area. In UK and US, this ratio is projected to rise sharply from 27% to 47% and 20% to 39%, respectively. Conversely, the Youth-dependency Ratio will decline in the future. By 2050, this ratio is projected to reach a level of 40%, with a decline of 24 percents (around 1/4) in OECD average. The ratio in UK and US will drop less sharply, but will also fall below 50%, with a level of 38% and 49% (Figure 3).

Figure 3 Age-dependency Ratio, in 1980, 2000 and 2050, in OECD Total, UK and US



Source: same to Figure 2

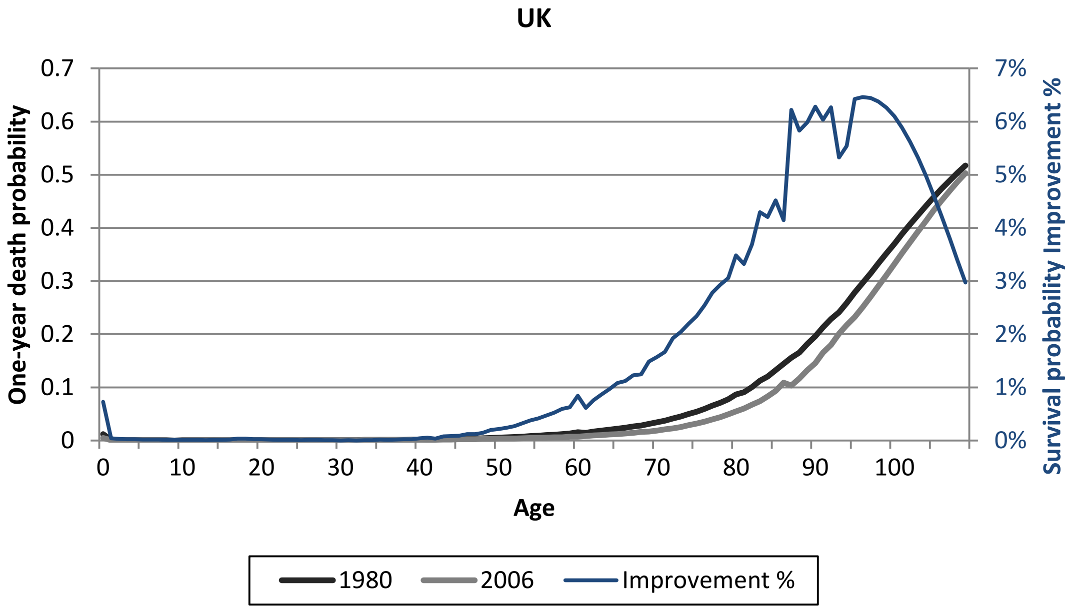
The fall in the youth-dependency ratio may lower public expenditures in education, but these declines are not large enough to offset higher spending towards the elderly[1]. The trend of aging drives a substantial growth of annuity market in life insurance companies. However, mortality improvement also poses a challenge to life insurance companies (Olivieri 2001 and Coppola, Di Lorenzo, and Sibilo 2002), because it delays the payout period and increases the liability for providing the annuity. Therefore, it raises the importance of hedging longevity risk. Many studies have investigated the issue of hedging longevity risk in traditional life annuity, life insurance product or other mortality derivatives (see for example, Milevsky and Promislow 2001, 2002, Blake and Burrows 2001, Charupat and Milevsky 2001…) (Wang, Huang, et al., 2010), in this paper, a sort of insurance scheme named “Child Plan” will be investigated on hedging longevity risk issue.

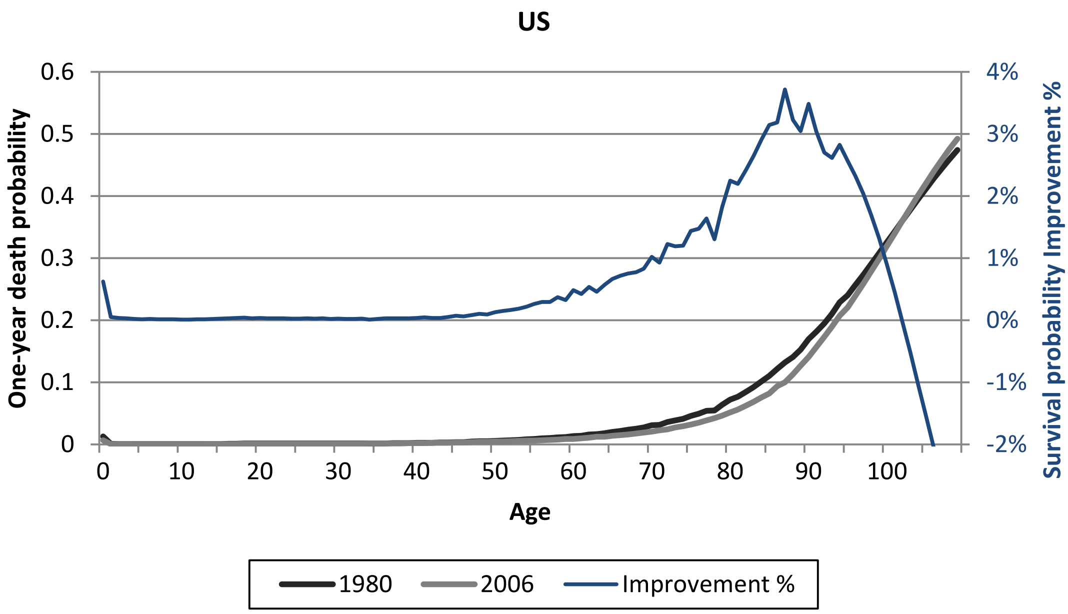
The life expectancy increase didn’t rise smoothly across ages. Compared to the one-year death probability of period life table[2] at 1980 and 2006, it can be found in both UK and US data that the survival probability changes remain less than 0.5% between these two generation from age 1 to age 50, except a significant improvement in survival rate of new born infants (below one-year old). Meanwhile, from age 50 to around 90, the improvement percentage is projected to increase faster and faster to a level of 6% in UK and 4% in US (Figure 4). To sum up, the one-year death probability declines are projected to be significant among the infants and the elderly, and the one-year death probability keep relatively stable for other groups (children, the young and the middle-aged).

[1] SOCIETY AT A GLANCE: OECD SOCIAL INDICATORS – 2006 EDITION

[2] A period life table is supposed to represent the mortality conditions at a specific moment in time, whereas a cohort life table depicts the life history of a specific group of individuals.

Figure 4 One-year Death Probability, in UK and US, birth in 1980 and 2006, and Survival Probability Improvement Percentage[3]





Source: Human Mortality Database

[3] To avoid a negative improvement rate, one-year survival probability, instead of death probability is used to calculate the improvement. The formula is used like this:

$$\frac{\text{death probability at age } x \text{ with birth year 1980} - \text{death probability at age } x \text{ with birth year 2006}}{1 - \text{death probability at age } x \text{ with birth year 1980}} \times 100\%$$

2.3 Benchmark Plan

2.3.1 Plans Outline

In this paper we will discuss lots of plans, and we will make a number of them as following:

Table 1 Child Plans and Contrast plans Outline

No.	Plan Name
P0	Saving Account: deposit in installments and withdraw in lump sum
P1	Single Life Insurance Plan
P2	Child Plan: single female insured without income benefit
P3	Child Plan: single female insured with income benefit
P4	Child Plan: both parents insured without income benefit

The first three plans will be explained in this section later and we will discuss the last two plans in Chapter 5.

2.3.2 Child Plan

A Child Plan usually takes one parent or the parents jointly as Life Assured. The product policies vary in the real market. In this paper, we start the plans with basic policies. Since we only focus on the mortality change among time periods, the gender difference is not a big issue. To simplify the first Child Plan, the situation of insuring one female parent will be discussed. The goal of this plan is to calculate the net premium under different mortality rate tables and to analyze the differences. Before the calculation, there are some assumptions that need to be set up. This plan is a benchmark for further discuss in Chapter 5 and will be labeled as P2.

Child Plan is a joint product that contains life insurance and an individual account. In the life insurance part, the child will be the beneficiary and the life of the parent is insured. In the individual account part, the maturity period is 18 years. The amount of the payout is guaranteed and is assumed at 10,000 USD. The payment period starts at the birth day of the baby and terminates when the child reaches 18-year old or when the situation of death benefit happens. Death benefit refers to the issue that the parent dies before maturity (the child reaches 18-year old) and thereby the premium payment terminates. When the child survives at the policy maturity, the guaranteed benefit will be paid. If death benefit situation doesn't happen, the guaranteed benefit will be paid on the maturity when the child survives. If the child dies before maturity, the premium payment terminates and benefit will not be paid. The annual interest rate is assumed to be fixed at a level of 3%. The premium will be paid on a yearly basis at the beginning of each year. To simplify the plan, the parent and the child are assumed both born in 1^{st} the January. (Table 2)

Table 2 Child Plan (P2) Assumptions

P2: Child Plan	
Item	**Condition**
Annual Premium	to be calculated actuarially
Premium Payment Frequency	Yearly
Annual Interest Rate	3%
Maturity Benefit	It is a fixed benefit guaranteed of 10,000 USD which will be paid at Maturity when the child survives.
	If the child dies before maturity date, the premium payment terminates. No benefit will be paid at end of policy term.
Death Benefit	If the parent dies before maturity date while the child survives then, the premium payment terminates after the parent die. And the guaranteed fixed benefit can still be received at the end of policy term
Income Benefit	No
Entry Age	Child (Nominee): 0
	Mother (Life Assured): equal to the average female age at the first childbirth
Maturity	18 years

2.3.3 Net Premium Measurement

Based on the assumption above, the net premium can be calculated when the death probability is known. In this chapter, 1*1 year period life table is used. It is not accurate to use if longevity risk exists, but it is a very easy way for starting. Later in this paper, we will use cohort life table and show the adjustment. According to the assumptions above, the present value of every dollar guaranteed benefit payout (Y) equals:

$$Y = \left(\frac{1}{1+r}\right)^T \tag{2.4}$$

Here r is the annual interest rate 3%, T refers to maturity period of 18 years. This benefit can only be received when the child survives at maturity. The probability of receiving benefit (Y) can be written as an expression of death probability of the child:

$$P\left[Y = \left(\frac{1}{1+r}\right)^T\right] = {}_T^a p_x = (1 - {}_1^a q_x) \times (1 - {}_1^a q_{x+1}) \times \cdots \times (1 - {}_1^a q_{x+T-1}) \tag{2.5}$$

Where ${}_T^a p_x$ refers to the probability that the child (a) at age x survives at least another T years and ${}_1^a q_x$ refers to the one-year death probability for the child (a) at age x. The expectation of present value of unit benefit payout (Y) equals:

$$EY = \left(\frac{1}{1+r}\right)^T \times {}_T^a p_x \tag{2.6}$$

Similarly, if we assume yearly premium as 1 US dollar, the present value of total premium (Π) is a cumulated value of yearly premium:

$$\Pi = \begin{cases} 1+\frac{1}{1+r}+(\frac{1}{1+r})^2+\cdots+(\frac{1}{1+r})^S, & S<T \\ 1+\frac{1}{1+r}+(\frac{1}{1+r})^2+\cdots+(\frac{1}{1+r})^{T-1}, & S\geq T \end{cases} \quad (2.7)$$

Here S refers to the minimum curtate future lifetime of the child (a) and the parent (b) with beginning age of y. If the parent (b) dies or child (a) dies before maturity (i.e. S<T), the remaining premium before the maturity doesn't need to be paid. If they both survive at least T year (i.e. S≥T), all the premiums of T years have to be paid. To simplify the equation, we use $\ddot{a}_{\overline{n|}}$ to express the present value:

$$\ddot{a}_{\overline{n|}} = \left[1+\frac{1}{1+r}+(\frac{1}{1+r})^2+\cdots+(\frac{1}{1+r})^n\right] \quad (2.8)$$

Therefore, equation of total premium (Π) can be short for:

$$\Pi = \begin{cases} \ddot{a}_{\overline{S|}}, \quad ? & S<T \\ \ddot{a}_{\overline{T-1|}}, ? & S\geq T \end{cases} \quad (2.9)$$

The expected value of total premium (Π) can be expressed by death probability of the parent (b) and child (a):

$$E\Pi = \sum_{k=0}^{T-1}\left(\frac{1}{1+r}\right)^k \times {}_k^b p_y \times {}_k^a p_x \quad (2.10)$$

Likewise, ${}_k^b p_y$ refers to the probability that the parent (b) at age y survives at least another k years. In order to calculate the net premium, we yield the Loss of the insurer (L) given by:

$$L = C\times Y - \pi\times\Pi \quad (2.11)$$

Here C is the amount of the guaranteed benefit payout 10,000 USD and π is the annual premium amount charged by insurance company. When the expected loss of insurer equals to 0 (i.e. EL=0), this annual premium is called net premium. In fact, the real premium charged by insurer to the customer will be added a risk premium called safety loading. This is because the net premium doesn't take into account risk. However, for our cases, we just need an actuarially fair price to judge the characters of Child Plan. That is also the reason why we are setting a plan and assumptions to calculate the net premium other than using actual market product's price. The latter contains safety loading and insurance company's profit which are difficult to be separated. Finally, we get the net premium expression as following:

$$\pi = f_2(r,\ T,\ {}^a p_x,\ {}^b p_y) = C \times \frac{(\frac{1}{1+r})^T \times {}^a_T p_x}{\sum_{k=0}^{T-1} \left(\frac{1}{1+r}\right)^k \times {}^a_k p_x \times {}^b_k p_y} \qquad (2.12)$$

Here the subscript "2" of "f_2" refers to the number of Plan. And the functions to calculate net premium for different plans below is analogous.

2.3.4 Change of Net Premium under Aging Trend

The plan uses the mean age of mothers at the first childbirth as the Entry Age of the parent. In 1980, a woman gave the first childbirth at average age of 25 in UK and 24 in US. However, by 2006, the data increased sharply in UK to a level of 30. In the meantime, the US data only increased slightly to 25. To calculate the net premium, female life table is used to calculate the parent's survival probability and the gender total life table is used for the child.

Based on the death probability decline and childbirth delay, there are 3 groups to be compared. Group A calculates the case that the child born in 1980 and Group B and C refer to child born in 2006. Group A and C uses the mean age of mothers at the first childbirth at that time as the entry age of parent, while the Group B assumes the mean age unchanged and uses the previous age data to calculate (Table 3 left).

On the one hand, because of the death probability declined from 1980 to 2006, the net premium drops in both UK and US (Table 3 right). UK data in Group B dropped slightly by 0.55 USD (about 0.13%) compared with Group A. The US data fell even more slightly about only 0.01%. On the other hand, the mean age of mother at the first childbirth delay increased the death probability of parent, therefore the net premium in Group C rose compared with Group B, with an increase level of 0.33% in UK and 0.06% in US.

Table 3 Net Premium Value, in 1980 and 2006, in UK and US

	Group Description				Result: Premium Value		
	Group No.	A	B	C	A	B	C
UK	Child Birth Year	1980	2006	2006	416.11	415.56	416.95
	Parent Age	25	25	30			
US	Child Birth Year	1980	2006	2006	415.70	415.65	415.91
	Parent Age	24	24	25			

The changes of the premiums are quiet small. It is because the Child Plan embedded an inner hedging system of mortality risk. The benefit payout relies on the probability that the child survives. If the child dies before maturity, the insurance company will gain from the previous premium payments without providing maturity benefit payout. Namely, in this case, $L = -\pi \times \Pi$. When the death probability of the child declines, this gain discussed above is less likely to get. To the contrary, death benefit causes a loss to the insurance. When the death probability of the parent drops, the loss of the death benefit will reduce. However, the survival probability of the parent and the child change in the same direction (Table 4). The benefit and loss will be partly offset.

Table 4 Survival Probability Change between 1980 and 2006

	Mother		Child	
Group No.	A	B	A	B
UK	0.9789	0.9875	0.9819	0.9921
USA	0.9771	0.9852	0.9792	0.9892

2.3.5 Contrast Plans

To prove the existence of the inner hedge system, we introduce another two plans as contrast. The first one is a saving account that is deposited in installments and withdrawn in lump sum. It is totally risk-free with same interest rate of 3%. To be strict, it is not a life insurance plan so we label it as P0. The assumptions are defined in Table 5.

Table 5 Saving Account Plan (P0) Assumptions

P0: Saving Account Plan	
Item	Condition
Annual Deposit	Fixed: 414.65
Deposit Frequency	Yearly
Annual Interest Rate	3%
Maturity Benefit	It is a fixed benefit guaranteed of 10,000 USD which will be withdrawn at Maturity.
Maturity	Same to P2: 18 years

Unlike other life insurance plan, the annual payment (deposit) is fixed with an amount of 414.65, and calculated as following equation:

$$\pi = f_0(r,T) = C \times \frac{(\frac{1}{1+r})^T}{\sum_{k=0}^{T-1}\left(\frac{1}{1+r}\right)^k} \tag{2.13}$$

It is not link to any individual's mortality and is forced to deposit yearly before withdraw. And the deposit amount is the same with UK and US countries.

The second contrast plan is a single Life Insurance Plan and labeled as P1. This Life Insurance Plan insures the life of parent only. Similar to the Child Plan (P2), this product contains yearly premium payment and guaranteed benefit payout in a lump sum at the end of policy period. But the condition of the premium and benefit payout doesn't associate to the death probability of the child or the beneficiary. If the insured dies, the premium will stop and the beneficiary will receive benefit payout at maturity. If the insured survives, guaranteed benefit can also be paid. The assumptions of Life Insurance Plan are summarized as following:

Table 6 Life Insurance Plan (P1) Assumptions

P1: Life Insurance Plan	
Item	Condition
Annual Premium	to be calculated actuarially
Premium Payment Frequency	Same to P2: Yearly
Annual Interest Rate	Same to P2: 3%
Maturity Benefit	It is a fixed benefit guaranteed of 10,000 USD which will be paid at Maturity.
Death Benefit	If the insured dies before maturity date, the premium payment terminates. And the guaranteed fixed benefit can still be received at the end of policy term.
Income Benefit	Same to P2: No
Entry Age	Same to P2
Maturity	Same to P2: 18 years

According to the assumptions above, the net premium can be easy to calculate:

$$\pi = f_1(r, T, {}^{b}p_y) = C \times \frac{(\frac{1}{1+r})^T}{\sum_{k=0}^{T-1}\left(\frac{1}{1+r}\right)^k \times {}^{b}_{k}p_y} \qquad (2.14)$$

For a Life Insurance Plan, the net premium declines due to death probability changes from 1980 to 2006 are much more significant than changes in Child Plan. (Table 7) In UK data, the evolution percentage of Life Insurance Plan is almost triple to that of Child Plan (-0.35% vs. 0.13%) and the US data is even more obvious (-0.28% vs. -0.01%).

Table 7 Premium Comparison of Saving Account, Single Life Insurance Plan and Child Plan

	Saving Account (P0)	Life Insurance Plan (P1)			Child Plan (P2)		
		A	B	Evl. %	A	B	Evl. %
UK	414.65	417.92	416.47	-0.35%	416.11	415.56	-0.13%
USA	414.65	418.12	416.96	-0.28%	415.70	415.65	-0.01%

With Saving Account Plan we can explain why the net premium changes seem so small. Although our life insurance plans are all risky products, they are belong to a life insurance category called Endowment Insurance. An endowment policy is a type of life insurance which is payable to the insured if he/she is still living on the policy's maturity date, or to the beneficiary otherwise. And it is much less risky than normal insurance contract in which the insured will for sure loss premium if he/she survives. When we compare the net premium with P0 and P1, we find that the net premium of P1 isn't much higher than P0. It is because the survival probability of the insured is very high, pretty close to 1 (see Table 4). The price for mortality risk that the insured need to pay will not be too high. If we increase the assumed age of insured, the premium difference will become larger. Therefore, when the dead probability decreases, the net premium of a life insurance product will not drop below the corresponding price of risk-free saving account. Namely, the decline rate from group A to group B of UK in life insurance Plan P1 will not be higher than -0.79% (decline rate from 417.92 to 414.65).

Furthermore, the distance of net premium between P0 and P1 have set the boundary that the Child Plan will be ranged. For example, under age and year choice of group B, the net premium of Child Plan will be neither higher than net premium of P1 (416.76) nor lower than deposit amount of P0 (414.65).

2.3.6 Reference Discount Rate

After the comparison of P0, P1 and P2, we draw a conclusion that amount of yearly deposit of saving account is the bottom line of the net premium. However, the number "414.65" isn't simple, convenient and intuitive enough for the judgment and analysis. So we introduce a better indicator and named it as "reference discount rate".

Here is the interpretation: if the value of net premium is used as the yearly deposit in another saving account with same maturity period and final withdraw as P0, the discount rate of such saving account must be different, and this new interest rate is a risk-free interest rate. So if an individual wants to buy a certain kind of life insurance product, this person can easily know how many points of interest rate he or she will lose if buying a standard saving account product instead. And the losing points of interest rate are the price for the risks covered by insurance contrast.

It should be noticed that the reference discount rate is not a real "discount rate", in this article, the real discount rate always equal to 3% as we assume for our plans. And the solution of reference discount rate (r^*) is the inverse function of following function between π and r^*:

$$\pi = f_0(r^*,\ T) = C \times \frac{\left(\frac{1}{1+r^*}\right)^T}{\sum_{k=0}^{T-1}\left(\frac{1}{1+r^*}\right)^k} \tag{2.15}$$

If we calculate net premium for Single Life Insurance Plan and other Child Plans, we will use functions f_1, f_2, f_3 and f_4 (f_3 and f_4 will be presented in Chapter 5). But the calculation of reference discount rate will always use the same function f_0^{-1}, which is the inverse function of the one calculating net premium in a Saving Account Plan and hasn't any variable of survival probability. With a program in an excel software, we calculate the inverse function $r^* = f_0^{-1}(\pi,\ T)$ and the results are showed in the following table:

Table 8 Reference Discount Rate of Saving Account, Single Life Insurance Plan and Child Plan

	Saving Account (P0)	Life Insurance Plan (P1)			Child Plan (P2)		
		A	B	C	A	B	C
UK	3.00%	2.92%	2.96%	2.92%	2.96%	2.98%	2.94%
USA	3.00%	2.92%	2.94%	2.94%	2.97%	2.98%	2.97%

We can also define reference discount rate into two categories. One is from the premium estimated in the contract before the insurance policy begins. So we call it "Sold Reference Discount Rate". The other one is calculated at the end of policy based on actual death rate and therefore named as "Realized Reference Discount Rate". In Table 8, we can easily know how many interest rate points the insurance company will earn if the insured all survive. That is, the realized reference discount rate equals 3% and the difference from the sold reference discount rate is the benefit for the insurance company. But it is not the profit that an insurance company aims to. As we discussed before, this net premium is not the final offer price, the insurance company will add a safety-loading which is where their majority of income comes from. However, when the mortality rate changes beyond their estimation, the realized reference discount rate will probably be lower than sold reference discount rate and they will suffer a loss.

2.4 Summary

Both net premium analysis and risk-free analysis show a small change of Child Plan compared with single Life Insurance Plan. It is because the Child Plan is a scheme with an easy inner hedging system by a more complicated contract design-joint death probability design. However, in this chapter, we just show a general frame of our research, so we use period life table instead of cohort life table to easily calculate the net premium, which implies that the longevity risk has been ignored in some extent. To attain the cohort life table, death probability forecast model is essential. In the next two chapters, two models will be used to UK and US data.

3 Modeling Mortality in US

3.1 Lee-Carter Model Specification

A variety of approaches have been proposed of modeling mortality rate over time and the leading work is the Lee-Carter method. In their 1992 JASA paper, Lee and Carter postulated a simple relationship for the central death rate $m_{x,t}$ as following:

$$log\,(m_{x,t}) = a_x + b_x k_t + e_{x,t} \tag{3.1}$$

Here the $m_{x,t}$ refers to the central death rate for age x at time t. The a_x coefficients describe the average shape of the age profile, and the b_x coefficients describe the pattern of deviations from this age profile when the parameter k_t varies. To identify the solution of the model, the following normalization constraints are imposed:

$$\begin{aligned} & log\,(m_{x,t}) = a_x + b_x k_t + e_{x,t} \\ s.t. & \\ & \sum_{t=t_1}^{t_n} k_t = 0 \\ & \sum_{x=x_1}^{x_m} b_x = 1 \end{aligned} \tag{3.2}$$

That is because for any solution a, b, k, there exists another solution a-bc, b, k+c and a, bc, k/c for any scalar c. Therefore, k is determined only up to a linear transformation, variable b is determined only up to a multiplicative constant, and variable a is determined only up to an additive constant (Lee and Carter, 1992).

In this paper, we follow the Lee-Carter forecast model to fit US data in order to extend period life table of US data after year 2007. Thus, we can transfer it into a cohort life table to improve the Child Plan.

3.2 Fitting the Lee-Carter Model

We start with the US female data of 1*1 year period format from 1933 to 2007. It is also from Human Mortality Database. But according to Lee-Carter model we use central death rate this time instead of death probability.

Firstly, we built a matrix A with:

$$A = (A_{xt})_{x=x_1,\dots,x_m,\,t=t_1,\dots,t_n} = \log\left(m_{x,t}\right)_{x=x_1,\dots,x_m,\,t=t_1,\dots,t_n} \tag{3.3}$$

It is easy to estimate a_x under first constraint of k variable. That is:

$$\hat{a}_x = \overline{A_x} = \frac{1}{n}\sum_{t=t_1}^{t_n} log\,(m_{x,t}) \tag{3.4}$$

To estimate b_x and k_t, we need Singular Value Decomposition (SVD).

For $A \in \mathbb{R}^{m \times n}$, there exists a factorization as following:

$$\begin{aligned} & A = UDV^T \\ s.t. & \\ & U \in \mathbb{R}^{m \times m} \ \ orthogonal\ matrix \\ & D \in \mathbb{R}^{m \times n} \ \ diagonal\ matrix \\ & V \in \mathbb{R}^{n \times n} \ \ orthogonal\ matrix \end{aligned} \qquad (3.5)$$

If we denote columns of U by $u_i \in R^m$ and columns of V by $v_i \in \mathbb{R}^n$, the equation above can also be rewritten as:

$$A = \sum_{i=1}^{k} d_{ii} u_i v_i^T \qquad (3.6)$$

k refers to the rank of matrix A.

The SVD method can be used to find a least squares solution when applied to the matrix of the logarithm of the rate after the averages over time of the (log) age-specific rates have been subtracted (Lee and Carter, 1992). Therefore, we apply SVD on $(A_{xt} - \hat{a}_x)_{x,t}$ and get the estimators of b_x and k_t as following:

$$\hat{b}_x = \frac{u_1}{\sum_{i=1}^{m} u_i} \qquad (3.7)$$

$$\hat{k}_t = \sum_{i=1}^{m} u_i \times v_1 \times d_{11} \qquad (3.8)$$

Table 9 and Table 10 show the result of our fitted value of coefficients of a_x, b_x and k_t.

Table 9 Fitted Value of a_x and b_x for 1933-2007 of Lee-Carter Model

Age(x)	a_x	b_x	Age(x)	a_x	b_x
0	-4.0934	0.0175	56	-4.8095	0.0080
1	-6.6039	0.0218	57	-4.7336	0.0079
2	-7.1372	0.0191	58	-4.6368	0.0075
3	-7.4400	0.0189	59	-4.5635	0.0077
4	-7.6454	0.0187	60	-4.4672	0.0076
5	-7.7866	0.0183	61	-4.3891	0.0074
6	-7.9020	0.0179	62	-4.2851	0.0072
7	-8.0030	0.0174	63	-4.2149	0.0073
8	-8.0824	0.0168	64	-4.1386	0.0074
9	-8.1562	0.0164	65	-4.0408	0.0074
10	-8.2065	0.0161	66	-3.9719	0.0074
11	-8.1908	0.0159	67	-3.8815	0.0075
12	-8.1183	0.0155	68	-3.7929	0.0075
13	-8.0075	0.0151	69	-3.7113	0.0076
14	-7.8422	0.0141	70	-3.6018	0.0076

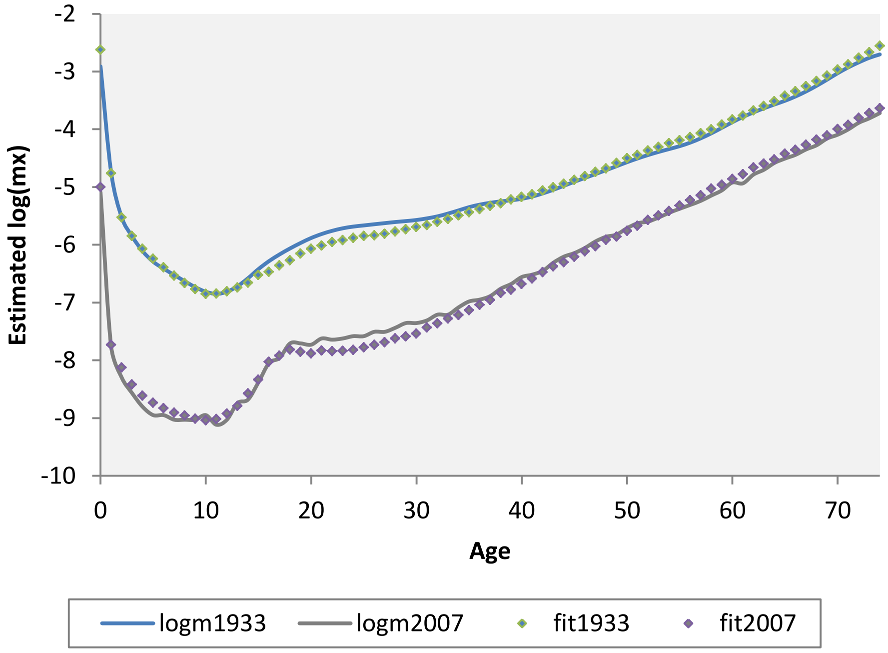
Age(x)	a_x	b_x	Age(x)	a_x	b_x
15	-7.6457	0.0133	71	-3.5234	0.0077
16	-7.4318	0.0114	72	-3.4058	0.0076
17	-7.3231	0.0115	73	-3.3170	0.0077
18	-7.2260	0.0114	74	-3.2247	0.0079
19	-7.2038	0.0125	75	-3.1232	0.0078
20	-7.1899	0.0133	76	-3.0349	0.0078
21	-7.1403	0.0133	77	-2.9461	0.0077
22	-7.1243	0.0139	78	-2.8471	0.0077
23	-7.1052	0.0141	79	-2.7476	0.0076
24	-7.0797	0.0142	80	-2.6524	0.0068
25	-7.0439	0.0142	81	-2.5669	0.0067
26	-7.0094	0.0139	82	-2.4558	0.0065
27	-6.9731	0.0138	83	-2.3509	0.0064
28	-6.9149	0.0136	84	-2.2483	0.0062
29	-6.8807	0.0136	85	-2.1504	0.0059
30	-6.8340	0.0135	86	-2.0520	0.0056
31	-6.7562	0.0130	87	-1.9566	0.0054
32	-6.6907	0.0129	88	-1.8693	0.0051
33	-6.6217	0.0126	89	-1.7733	0.0048
34	-6.5605	0.0126	90	-1.6722	0.0044
35	-6.4848	0.0124	91	-1.6080	0.0039
36	-6.4095	0.0122	92	-1.5166	0.0033
37	-6.3369	0.0120	93	-1.4369	0.0027
38	-6.2429	0.0114	94	-1.3612	0.0021
39	-6.1854	0.0115	95	-1.2852	0.0021
40	-6.1031	0.0111	96	-1.2079	0.0018
41	-6.0269	0.0107	97	-1.1331	0.0015
42	-5.9333	0.0104	98	-1.0609	0.0013
43	-5.8542	0.0100	99	-0.9913	0.0010
44	-5.7853	0.0100	100	-0.9245	0.0008
45	-5.7026	0.0098	101	-0.8605	0.0006
46	-5.6186	0.0096	102	-0.7994	0.0004
47	-5.5336	0.0094	103	-0.7412	0.0002
48	-5.4426	0.0091	104	-0.6860	0.0001
49	-5.3735	0.0093	105	-0.6338	-0.0001
50	-5.2805	0.0092	106	-0.5845	-0.0002
51	-5.2010	0.0090	107	-0.5381	-0.0003
52	-5.1130	0.0088	108	-0.4946	-0.0004
53	-5.0386	0.0087	109	-0.4539	-0.0005
54	-4.9669	0.0086	110	-0.4160	-0.0006
55	-4.8908	0.0083			

Table 10 Fitted Value of k_t for 1933-2007 of Lee-Carter Model

Year	k_t	Year	k_t	Year	k_t
1933	84.5230	1958	5.8245	1983	-30.0169
1934	85.6277	1959	4.4514	1984	-30.3922
1935	83.8017	1960	4.4568	1985	-31.2937
1936	85.0252	1961	1.5681	1986	-31.0086
1937	80.0895	1962	2.5373	1987	-31.7553
1938	72.6189	1963	3.5047	1988	-30.4330
1939	67.3992	1964	2.0563	1989	-31.5531
1940	64.4342	1965	1.1962	1990	-34.6795
1941	59.9336	1966	1.8580	1991	-35.3531
1942	53.7978	1967	-0.5097	1992	-38.1129
1943	56.2651	1968	1.4128	1993	-35.2922
1944	51.0498	1969	0.7945	1994	-36.6766
1945	47.4114	1970	-1.3306	1995	-36.9154
1946	41.8220	1971	-2.3228	1996	-39.2333
1947	36.4637	1972	-3.4035	1997	-41.8263
1948	32.3819	1973	-5.0977	1998	-44.2060
1949	27.9182	1974	-10.0171	1999	-43.9728
1950	24.0438	1975	-14.0740	2000	-45.2831
1951	21.8658	1976	-16.6970	2001	-45.2732
1952	20.8869	1977	-18.1455	2002	-46.0932
1953	15.6471	1978	-19.1975	2003	-46.6063
1954	9.5111	1979	-22.5327	2004	-48.3987
1955	8.0005	1980	-22.4368	2005	-49.4782
1956	6.7686	1981	-25.4420	2006	-50.8710
1957	8.9541	1982	-28.2522	2007	-51.7176

Using these estimators, we choose beginning year 1933 and ending year 2007 as examples to compare the actual data with model estimation (Figure 5). The fit looks good, and the shapes of the fitted value almost match the age profile of actual data.

Figure 5 Comparison of Actual and Estimation for 1933 and 2007 of Lee-Carter Model



3.3 Forecasts for US data

In their paper, Lee and Carter found empirical evidence that the variable k are decreasing approximately linearly.

As a result, Lee-Carter model further assumes the variable k follow a stochastic process as a random walk with drift. Specifically,

$$k_t = C + k_{t-1} + e_t \tag{3.9}$$

The e_t is independent and identically distributed (i.i.d.) with mean zero and variance σ_k^2. The drift is estimated by:

$$\hat{C} = \frac{1}{n-1}\sum_{t_i=t_1}^{t_n}(\hat{k}_{t_{i+1}} - \hat{k}_{t_i}) = \frac{1}{n-1}\sum_{t_i=t_1}^{t_n}\Delta\hat{k}_{t_i} = \frac{\hat{k}_{t_n} - \hat{k}_{t_1}}{n-1} \tag{3.10}$$

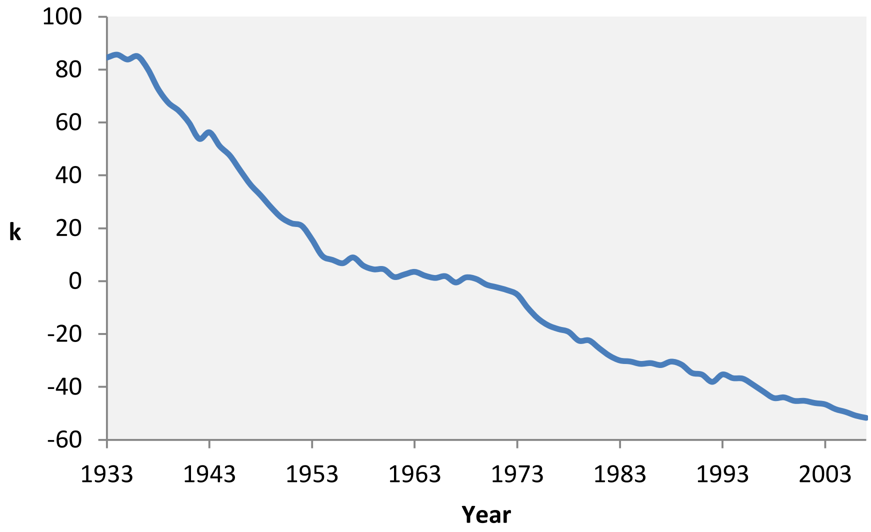
And then we calculate the standard deviation of the error term σ_k^2 by:

$$\hat{\sigma}_k^2 = \frac{1}{n-1}\sum_{t_i=t_1+1}^{t_n}(\Delta\hat{k}_{t_i} - \hat{C})^2 \tag{3.11}$$

The approximate linear decline pattern of k_t also can be observed in our estimation (Figure 6). However, from the figure, we also found that the slope is becoming smaller since around 1953 and remains relatively stable until around 1973. After 1973, estimated k variable goes back decreasing but the decline is not as steep as the first two decades. This slope change suggests a period adjustment for our model. It seems that less historical data will make the forecast better.

We will go back to this discussion later in this article.

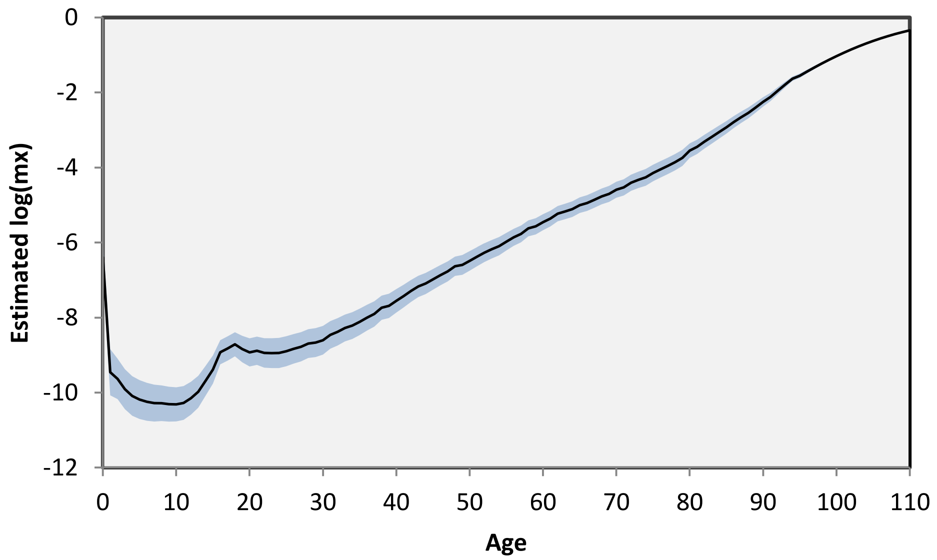
Figure 6 Fitted Value of k_t for 1933-2007 of Lee-Carter Model



Following the Lee-Carter model, we get a yearly decline estimator $\hat{C}$ equal to -1.841, and $\hat{\sigma}_k$ equal to 2.199. This standard deviation is bigger than those calculated in Lee and Carter's paper written in 1992. They got $\hat{C}$ as -0.406 and standard error of equation as 0.550 only for fitting US data from 1930 to 1989 (Lee and Carter, 1992). The reason for the difference is partly because we use one-year death rate data and have 111 observations for b_x while they only use five-year death rate data and have only 23 observations for b_x. Considering the second constraint that sum of b_x always equals to 1, our b_x is likely to be one fifth (23/111=0.207) of theirs. As a result, our solution of k_t may probably be five times to theirs. It is because our solution of a_x is close to theirs due to the same approach of calculate and it restricts the solution of b_x times k_t should also be close to theirs. So the difference of k variables ($\hat{C}$) and the variance $\hat{\sigma}_k$ is also more or less five times of their results.

Since we have got an expression of k variable with time t, we can use the model to extend life table for the future years. Figure 7 show an example of forecast for 2050. To create a cohort life table, a forecast is needed for every year from 2008 to 2050.

Figure 7 Mortality Forecast for 2050, With 95% Confidence Interval of Lee-Carter Model



3.4 Data Period Adjustment

Considering the slope change of k variable in around 1953 and 1973, we repeat the process of Lee-Carter model with data starting from 1953 and 1973 instead of 1933. Table 11 shows how the slope and standard deviation of k variable change. Since the number of b variable for these three choices is the same (111), the slope and standard deviation of k variable are comparable. As we discussed before, recent years' slope (from year 1973 to 2007) is less steep than the historical slope (from year 1933 to 1953), so Data Choice 2 and 3 is better for forecasting future k variable if we assume this slower decline trend continues.

Table 11 K Variable Changes for Processing Different Data Choice of Lee-Carter Model

		Data Choice 1	Data Choice 2	Data Choice 3
K variable	$\hat{C}$	-1.8411	-1.3061	-1.2976
	$\hat{\sigma}_k$	2.1989	1.6697	1.3376

Choice 1 uses central death rate data from year 1933 to 2007; Choice 2 uses data from year 1953 to 2007; Choice 3 uses data from year 1973 to 2007

In order to choose the fittest data period, we compare these three results by using Sum of Square Residuals (SSR)[4] with following equation:

$$SSR_t = \sum_{x=1}^{111} \left(log\left(m_{x,t}\right) - \hat{a}_x - \hat{b}_x \hat{k}_t \right)^2 \tag{3.12}$$

The common period for these three fitting trials is from 1973 to 2007. We separate this 35-year

[4] This SSR is not exactly the same with the common used "SSR" in statistics term. It is because we don't summarize all the residuals of the sample model. Instead, we sum up a certain part of the whole population range in order to compare the situation of this range.

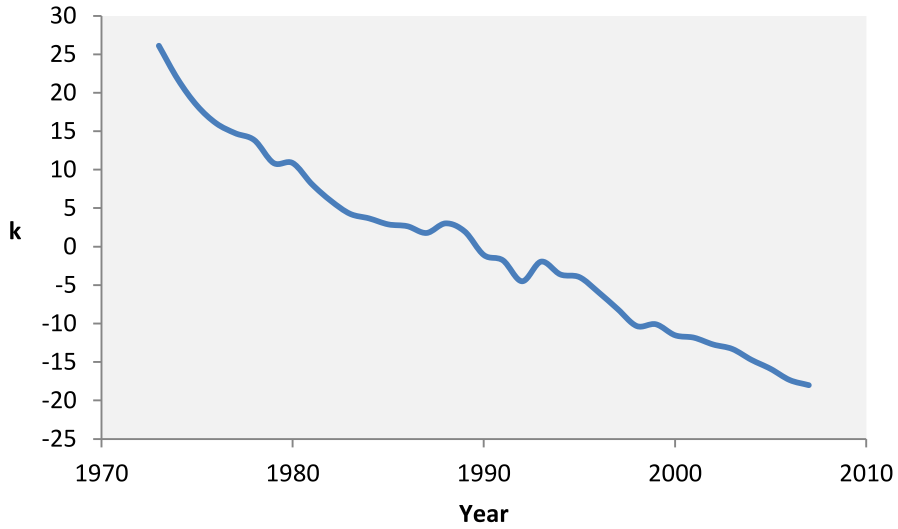
period into four parts: 1973 to 1979, 1980 to 1989, 1990 to 1999 and 2000 to 2007. We firstly calculate every year estimated log($m_{x,t}$) using different estimated variables $\hat{a}_x$, $\hat{b}_x$ and $\hat{k}_t$ that we got from these three trials. After that we calculate the SSR for the yearly fitting results compare with actual data. Then we summarize the yearly SSR into four periods. From the comparison of three data period choice, we find that the third data choice fits the recent year data better than the other two in all four periods (Table 12). Especially for the last period, which is closest to the future, the first and the second trial don't perform well enough and increase the SSR very quickly.

Table 12 Model Fitness Results for Different Data Choice of Lee-Carter Model

		Data Choice 1	Data Choice 2	Data Choice 3
SSR	1970s	2.3649	2.2518	1.8729
	1980s	3.8190	3.5299	2.2620
	1990s	4.3880	2.4686	2.2134
	2000s	6.4656	3.4670	2.3197
	Total	17.0375	11.7173	8.6679

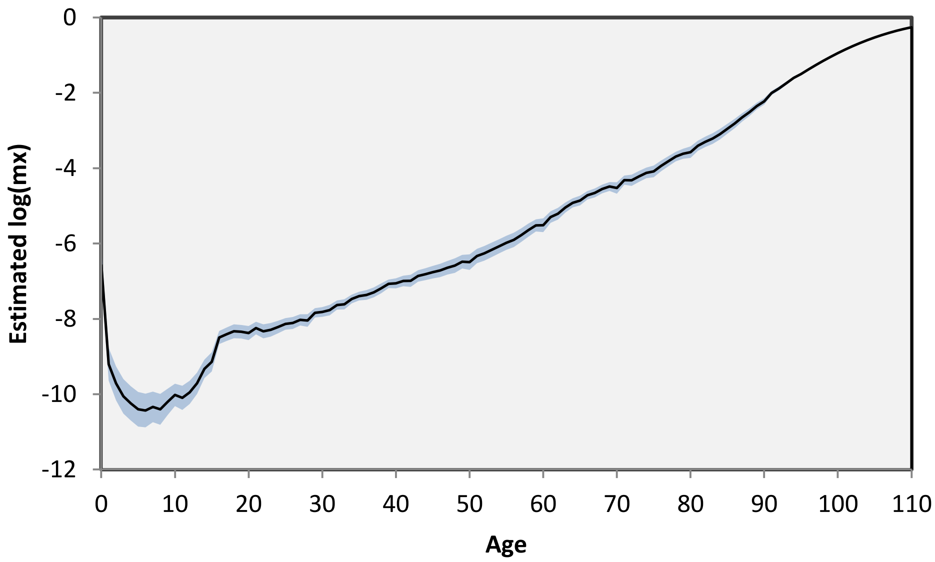
From the two methods comparison above, we decide to adjust our model by using more recent data period (1973-2007) for future forecast. The new pattern of k is smoother decreasing than the previous trial (Figure 8).

Figure 8 Adjusted Fitted Value of k_t for 1933-2007



Since the standard deviation becomes less, the revised mortality forecast for 2050 has a narrower 95% Confidence Interval (Figure 9).

Figure 9 Adjusted Mortality Forecast for 2050, With 95% Confidence Interval



3.5 Cohort Life Table in US

After the data period change, we have new estimation of a, b and k variables of Lee-Carter Model (see detail data in Appendix A). Now we have period life table for US female data from 1933 to 2050, it is long enough to transfer into cohort life table for our Child Plan to calculate net premium. And we also need US Gender Total data to calculate survival probability of children. So we repeat the same process to the data of US Gender Total (see detail data in Appendix B). After changing the life tables of US, we got new results of net premium for the third group we mentioned in chapter 2 (i.e. Group C: Child is born at 2006 when the mother is 25). And we also calculate net premium for a Life Insurance Plan (Table 13). The net premium of Life Insurance Plan decreases because the death probability we use from cohort life table is smaller than that from period life table. Smaller death probability will lead to a bigger survival probability of insured and a higher expected value of premium. To keep the Expected Loss of the insurer equal to zero, the insurer will decrease the annual premium value.

Table 13 Comparison of Net Premium Value and Reference Discount Rate between Period Life Table and Cohort Life Table in US Data

	Life Insurance (P1)		Child Plan (P2)	
	Net premium	Reference discount rate	Net premium	Reference discount rate
Period Life Table	417.22	2.94%	415.91	2.97%
Cohort Life Table	416.72	2.95%	415.63	2.98%

According to the plan assumptions in chapter 2, the difference between Child Plan and Life Insurance Plan is whether another individual's survival (the child) is involved or not in the premium and benefit payout conditions. So we can figure out the effect of this joint death probability design by comparing these two plans.

4 Modeling Mortality in UK

4.1 Cairns-Blake-Dowd Model Specification

After Lee and Carter (1992) developed a stochastic mortality approach fitting to US data, many researchers have extended Lee-Carter model (see, for example Brouhns, Denuit, and Vermunt, 2002; Renshaw and Haberman, 2003; Currie, Durban, and Eilers, 2004). If ranking those models based on the statistical quality of fit, then Cairns, Blake, and Dowd (2006) proposed an extension model (CBD) that fits the UK mortality data best (Cairns, Blake, et al., 2007). CBD model for mortality rate is defined as:

$$logit(q_{x,t}) = k_t^{(1)} + k_t^{(2)}(x - \bar{x}) + e_{x,t} \quad (4.1)$$

In this equation, $\bar{x}$ is the mean age in the sample range, $k_t^{(1)}$ and $k_t^{(2)}$ are stochastic processes that are assumed to be measurable at time t, e_t is the error term. And $logit(q_{x,t})$ refers to logistic regression to death probability $q_{x,t}$:

$$logit(q_{x,t}) = \log(\frac{q_{x,t}}{1 - q_{x,t}}) \quad (4.2)$$

So it can also be rewritten as followed for the mortality curve:

$$q_{x,t} = \frac{e^{k_t^{(1)} + k_t^{(2)}(x-\bar{x}) + e_{x,t}}}{1 + e^{k_t^{(1)} + k_t^{(2)}(x-\bar{x}) + e_{x,t}}} \quad (4.3)$$

4.2 Fitting the CBD Model

Before fitting UK death probability to the CBD model, we need to choose the range of the sample data. Firstly, we choose the age range from 0 to 60. In this paper, we only need mortality data for the children and the middle-aged, so an age range below 60 is sufficient. Secondly, we use the longest sample time range of year 1922 to 2009 we can get from HMD. After that, we do logistic regression to dead probability q_x with age x for each year t and store the results of estimators of $k_t^{(1)}$ and $k_t^{(2)}$ year by year. Table 14 show the fitted value of k variable changed by year.

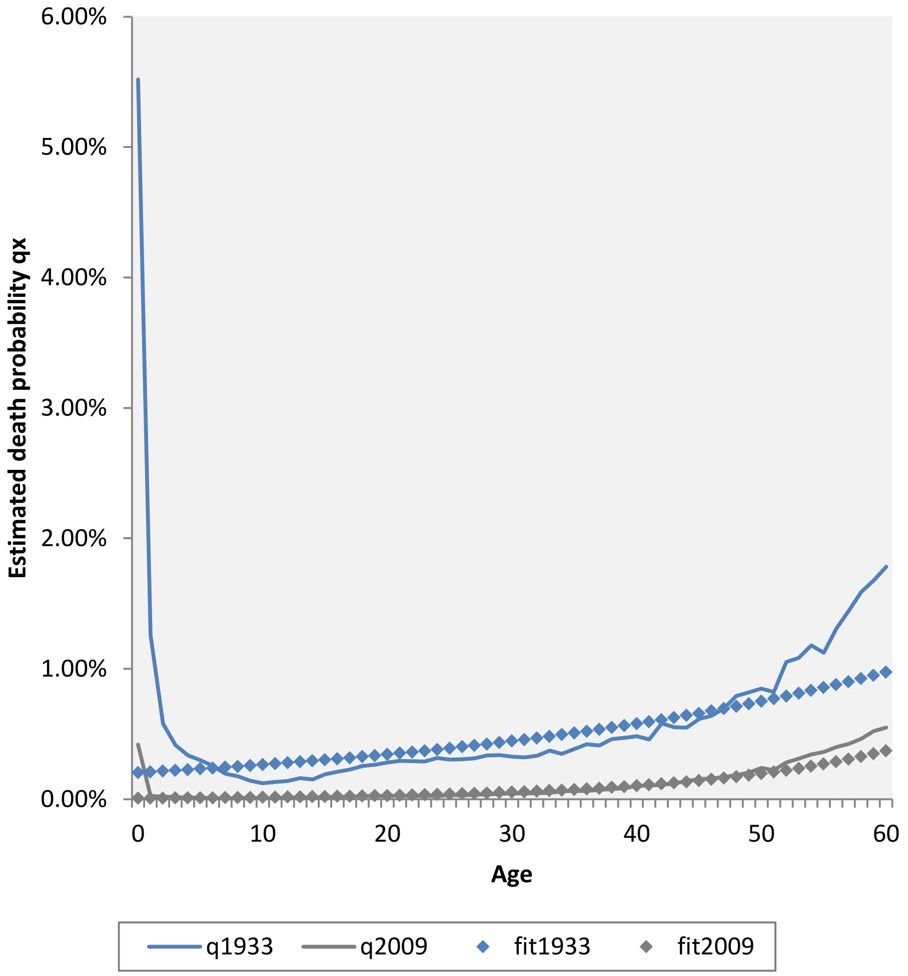
Table 14 Fitted Value of $k_t^{(1)}$ and $k_t^{(2)}$ for 1922-2009 of CBD Model

Year (t)	$k_t^{(1)}$	$k_t^{(2)}$	Year (t)	$k_t^{(1)}$	$k_t^{(2)}$
1922	-5.2227	0.0224	1966	-6.7569	0.0538
1923	-5.3364	0.0236	1967	-6.7950	0.0536
1924	-5.3006	0.0235	1968	-6.7955	0.0542
1925	-5.3183	0.0226	1969	-6.8170	0.0573
1926	-5.3729	0.0240	1970	-6.8214	0.0561
1927	-5.3431	0.0246	1971	-6.8351	0.0557
1928	-5.3813	0.0239	1972	-6.8289	0.0560
1929	-5.3000	0.0236	1973	-6.8475	0.0564

Year (t)	$k_t^{(1)}$	$k_t^{(2)}$	Year (t)	$k_t^{(1)}$	$k_t^{(2)}$
1930	-5.4217	0.0241	1974	-6.8702	0.0569
1931	-5.4058	0.0250	1975	-6.9004	0.0581
1932	-5.4399	0.0252	1976	-6.9083	0.0586
1933	-5.4164	0.0259	1977	-6.9452	0.0588
1934	-5.4592	0.0235	1978	-6.9030	0.0575
1935	-5.5189	0.0263	1979	-6.9668	0.0598
1936	-5.5510	0.0274	1980	-6.9857	0.0589
1937	-5.5544	0.0283	1981	-7.0338	0.0598
1938	-5.6359	0.0273	1982	-7.0609	0.0607
1939	-5.6971	0.0317	1983	-7.0856	0.0601
1940	-5.4961	0.0277	1984	-7.1135	0.0606
1941	-5.5261	0.0243	1985	-7.1202	0.0599
1942	-5.7296	0.0290	1986	-7.1505	0.0604
1943	-5.7252	0.0298	1987	-7.1640	0.0610
1944	-5.7794	0.0306	1988	-7.1798	0.0607
1945	-5.8699	0.0327	1989	-7.1794	0.0599
1946	-5.9575	0.0365	1990	-7.2245	0.0612
1947	-5.9719	0.0364	1991	-7.2228	0.0595
1948	-6.0824	0.0387	1992	-7.2741	0.0618
1949	-6.1401	0.0421	1993	-7.2632	0.0614
1950	-6.2469	0.0449	1994	-7.3237	0.0639
1951	-6.3167	0.0476	1995	-7.3078	0.0633
1952	-6.4538	0.0492	1996	-7.3236	0.0636
1953	-6.4883	0.0491	1997	-7.3346	0.0629
1954	-6.5595	0.0525	1998	-7.3522	0.0633
1955	-6.5871	0.0511	1999	-7.3597	0.0632
1956	-6.6610	0.0536	2000	-7.3956	0.0655
1957	-6.6212	0.0518	2001	-7.3976	0.0631
1958	-6.7111	0.0550	2002	-7.4222	0.0644
1959	-6.6961	0.0528	2003	-7.4143	0.0633
1960	-6.7317	0.0531	2004	-7.4719	0.0652
1961	-6.7257	0.0537	2005	-7.4910	0.0652
1962	-6.7287	0.0539	2006	-7.4841	0.0639
1963	-6.7425	0.0548	2007	-7.5026	0.0648
1964	-6.7549	0.0535	2008	-7.4884	0.0641
1965	-6.7453	0.0532	2009	-7.5143	0.0639

Although we have model data from year 1922, we show the fit result from year 1933 in order to cross-compare the fit result figure of Lee-Carter model (Figure 5). And the latest year data to be displayed is the year 2009 result (Figure 10) which is two year later than the fit result figure of Lee-Carter model. The fit of CBD model is clearly not good. Especially for age 0, the CBD model fails to reveal a relative high death probability of infants and a dramatically sharp decrease to the next age. In adverse, the fitted value of CBD model starts at a very low point and keeps increasing smoothly and strictly. Besides, the fit result of higher age above 50 is also fairly unacceptable. Although, the fitted value has predicted an accelerated increase as age grows, the predicted increase is obviously not quick enough.

Figure 10 Comparison of Actual and Estimation for 1933 and 2007 of CBD Model



The reason for such bad fit is because of the model frame of CBD approach itself. The CBD model just assumes a linear relation between logit(q_x) and age x in each year t. However, other models like Lee-Carter model have variables with age x and are thus able to describe a non-linear shape of age profile. The linear age profile of CBD model is really a strong assumption and thus rather unrealistic. In fact, in their JRI paper of 2006, Cairns, Blake, and Dowd (2006) only consider post-age-60 mortality curve and the model fits good for age above 60. In their later four articles (Cairns, Blake, and Dowd, 2006; Cairns, Blake, et al., 2007; Dowd, Cairns, et al., 2008; Cairns, Blake, et al., 2009), this age range choice hasn't been extended.

However, the latest year data (2009) do show a better result of model fitting than far previous year data (1933). It is the result related to our major work indeed. Therefore, we need to find more details on the fitness progress of CBD mode across the time, so that we can judge whether the CBD model is appropriate for forecasting.

4.3 Fitness Comparison for the CBD Model

The reason we choose age range from 0 to 60 instead of whole range 0 to 110 is because we have learned from the fitting result of last chapter that wider range of data doesn't always result in a better fit. If we choose the whole data, the average age of sample will increase, and then the model quality of fit will be weakened. To check out this hypothesis, we need to fit the model for a second trial and compare the results.

Therefore, we have two trials of CBD model fitting by using Data Choice 1* (age from 0 to 60) and Data Choice 2* (age from 0 to 110+). Furthermore, we found in last section that the CBD model has an innate defect in modeling infant death probability peak, so we attempt to cut down the data of age 0 and check out if there will be a better fit for the rest ages. This is identified as Data Choice 3* (age from 1 to 60).

Similar to last chapter, we use Sum of Square Residuals (SSR) to compare different data range choice. In this case, the equation of SSR is rewritten as following:

$$SSR_t = \sum_{x=1}^{n}\left(q_{x,t} - \frac{e^{\hat{k}_t^{(1)}+\hat{k}_t^{(2)}(x-\bar{x})}}{1+e^{\hat{k}_t^{(1)}+\hat{k}_t^{(2)}(x-\bar{x})}} \right)^2 \tag{4.4}$$

In order to quantify the fitness across the time, we follow the same period division as last chapter and summarize the SSR by following 4 periods (Table 15). However, it should be noted that "1970s" refers to year 1973-1979 for consistency, and the "2000s" includes year 2000 to 2009 in CBD Model and Lee Carter Model. In addition, we only choose age range from 1 to 60 that is the cross among the three trials.

The unit of Table 15 is 1% for sake of clarity. The third trial with the least sample age range excels the other two outstandingly. The total SSR of four periods is merely about one half of the first and the second trial (52% and 58%). Among all of four periods, the third trial keeps an absolutely leading position. The only problem is that we lose the mortality model for age 0. But, in our Child Plan, we in fact don't need forecast data for age 0 since the youngest subject is born in year 2006. Therefore, we will choose the third trial for UK mortality model (see estimators in Appendix D).

Table 15 Model Fitness Results for Different Data Choice of CBD Model, Age Range from 1 to 60

Unit: 0.01		CBD Model			Lee-Carter Model
		Data Choice 1*	Data Choice 2*	Data Choice 3*	Final Choice
SSR	1970s[5]	0.09985	0.10106	0.04553	0.00204
	1980s	0.08846	0.05979	0.06195	0.00101
	1990s	0.04069	0.06271	0.02621	0.00023
	2000s	0.01839	0.05134	0.00986	0.00043
	Total	0.24738	0.27490	0.14356	0.00371

However, this amount of SSR value only has relative meaning for comparing the three trials. To display the fitness result of the whole CBD model, we also make a reference with Lee-Carter model. We use Lee-Carter model to calculate UK female data in order to make sure that the SSR

5 Here 1970s refers to 1973 to 1979 for both CBD and Lee-Carter Model

of both models are comparable.

It can be clearly observed that Lee-Carter model has a much lower SSR result than CBD model, at only around 1.35% to 2.58% of the SSR of CBD model (Table 15). However, we still use CBD model for UK data because it is the most popular model used in many other papers, although CBD is used for population aged above 60 in these papers.

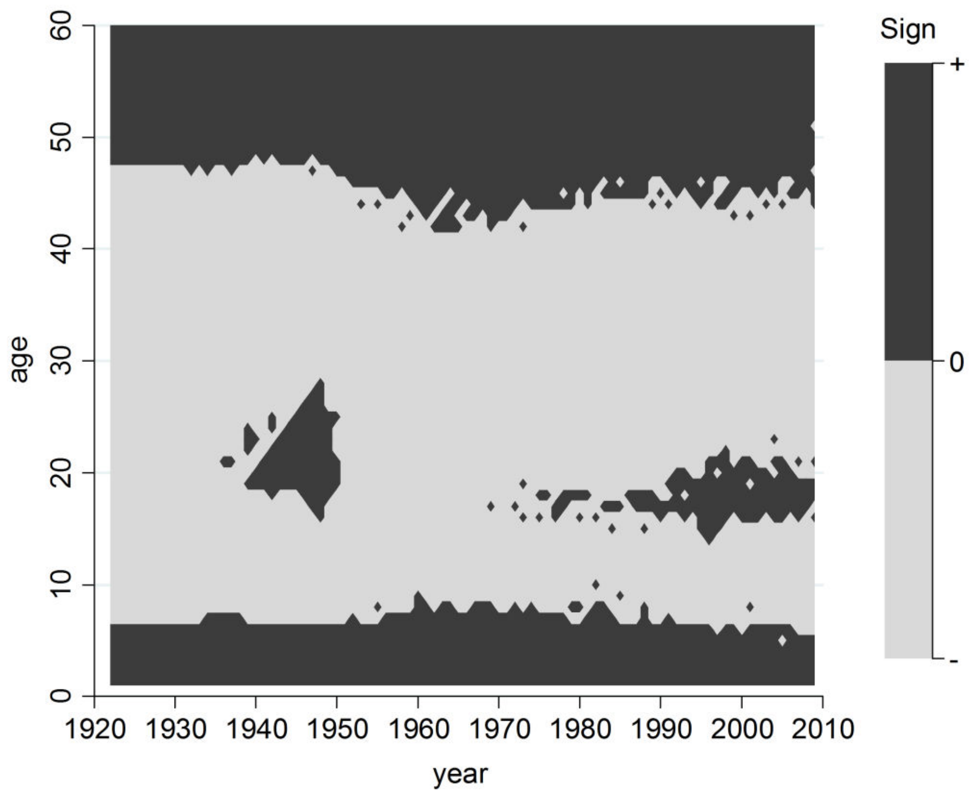
Period SSR Summary Table above is a better approach to provide a standard for us to make data range choice, but it can't show the fitness across age and time. Hence we draw a new matrix graph to show the goodness of fit both at an age and a year level.

Firstly, we consider the direction of model bias. We define the following function of $S_{x,t}$ which describe the sign of model residual:

$$S_{x,t} = \begin{cases} -1, & if\ q_{x,t} < \hat{q}_{x,t} \\ 0, & if\ q_{x,t} = \hat{q}_{x,t} \\ 1, & if\ q_{x,t} > \hat{q}_{x,t} \end{cases} \quad (4.5)$$

Figure 11 shows the sign of the residual of the CBD model trial we choose (Data Choice 3*). Dark grey area ($S_{x,t}$ =1) marks the area that the model is underestimated compared with actual data. At the same time, the light grey area ($S_{x,t}$=-1) shows the opposite situation that the estimated value is bigger than actual data.

Figure 11 Sign of Residual of UK Female Data Fitting by CBD Model



We notice that the function $S_{x,t}$ shows a similar pattern across the years, it is shown as the horizontal extension of the two color blocks. Furthermore, the blocks of color also remain relatively complete in vertical level of age. It almost underestimates all the points above age 50 and age below 5, and in the rest age range overestimates. We suppose that the big color block suggests the goodness of fit is not perfect; otherwise the dark grey and light grey colors will probably mix into smaller blocks.

Secondly, we draw another matrix graph to show the goodness of fit. Similar to SSR Table we keep on using square of residual ($e_{x,t}^2$) as the indicator of the goodness of fit. But in order to present a graph for a better understanding, we make a transform of ($e_{x,t}^2$) as following:

$$\lg r2_{x,t} = \frac{1}{2} \times \log_{10}(e_{x,t}{}^2) \tag{4.6}$$

Hence, we calculate a new variable of "lgr2" ranged from -7.1 to -1.6. It can be interpreted intuitively as the power of 10 that measured the accuracy of the modeling. In Figure 13, we can see its value in a matrix range. If we define a perfect match as a condition that the residual is less than 0.0001, it means lgr2 should be less than -4. In other word, the green and blue blocks in Figure 13 are the perfect match areas.

We also draw a matrix graph for the actual death probability, and we use the following "$\lg q_{x,t}$" for comparison with "lgr2":

$$\lg q_{x,t} = \log_{10}(q_{x,t}) \tag{4.7}$$

However, the death probability itself varies a lot in the matrix (Figure 12). The value of $\lg q_{x,t}$ change from -4.3 to -1.6. So it is not wise to judge the residuals with a boundary of absolute value. If we define a 1% level as a perfect match (i.e., the model residual is less than 1 percent of the actual data), the value of $\lg r2_{x,t}$ should be at least two units smaller than logarithm of death probability ($\lg q_{x,t}$).

Figure 12 Log Death Probability $\log_{10}(q_{x,t})$ for UK Female Data from 1922 to 2009, Age from 1 to 60

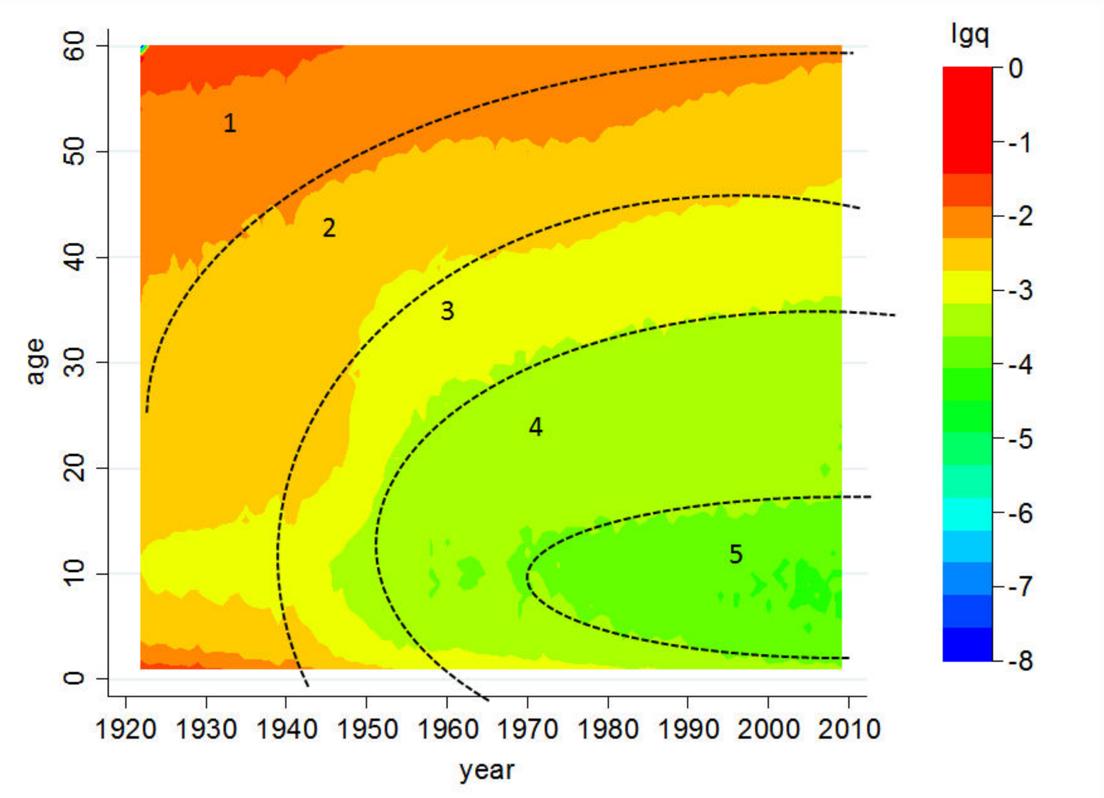
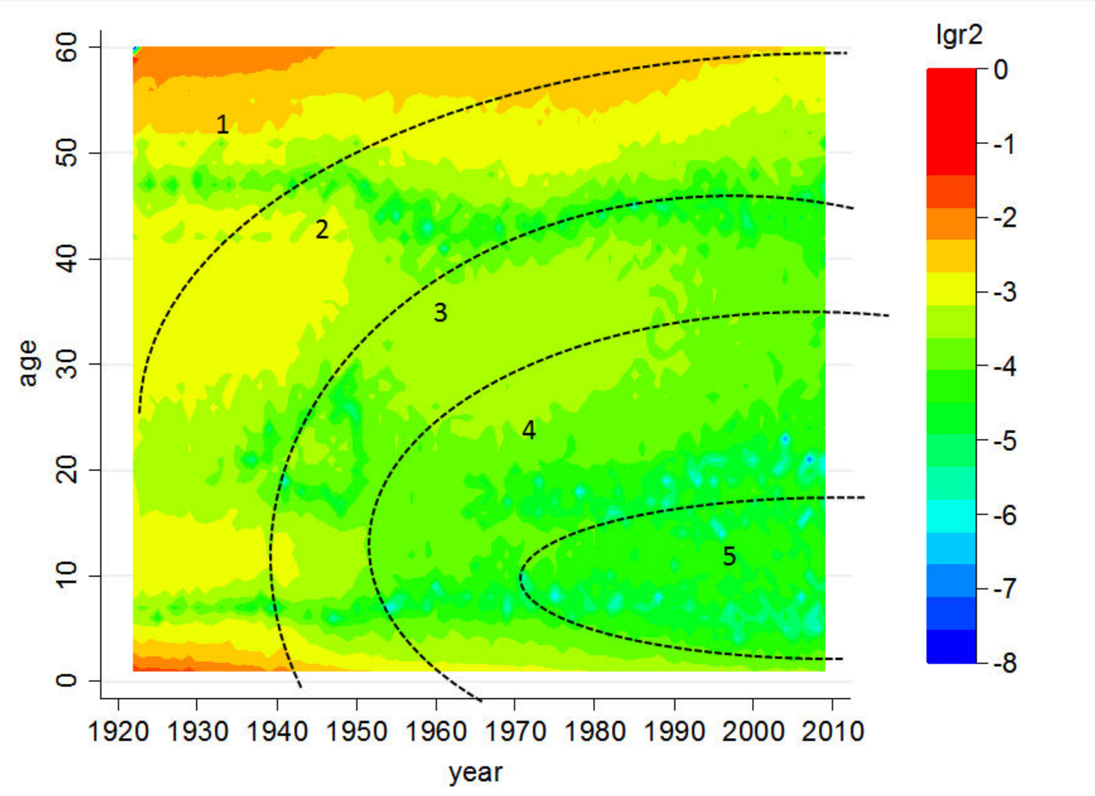


Figure 13 Goodness of Fit of UK Female Data Fitting by CBD Model



In Figure 12, the boundary of different mortality data ($lgq_{x,t}$) level is very clear. But we draw approximate curves for simplicity instead of the real boundary to divide the mortality data into five parts and copy these curves to Figure 13. A bigger area number refers to a relatively higher death probability. We discuss the fitness by these five areas each and visually estimate the average value of each area in Figure 12 and Figure 13 into the following table:

Table 16 Model Fitness Results of CBD Model by Death Probability Volume Area

Area No	lgq	lgr2	Difference
1	-2	-2.7	0.7
2	-2.5	-3.2	0.7
3	-3	-3.5	0.5
4	-3.5	-4	0.5
5	-4	-4.7	0.7
Average	-3	-3.6	0.6

From the table, we notice that the difference is far away from perfect “2”. Our estimation of the average residual weight will be more than 10% (Difference equals “1” means the residual is about 10% of actual data). This might not be acceptable in a relative level. However, in an absolute level, the area 4 and 5 show a small logarithm value (less than -4). It means the difference between actual data and fitted value is only about 0.0001, which can be regarded as an acceptable fit. It is also remarkable that the most fitted area is the recent year data. Therefore, we decide to continue the forecast with CBD model.

4.4 Forecasts for UK Data

According to the CBD model, two variables $k_t^{(1)}$ and $k_t^{(2)}$ follow a random walk with drift. We have already made the regression year by year and get the yearly estimators of these two variables. It shows a relative steady drift respectively (Figure 14 and Figure 15).

Figure 14 Fitted Value of $k^{(1)}$ Variables for 1922 to 2009

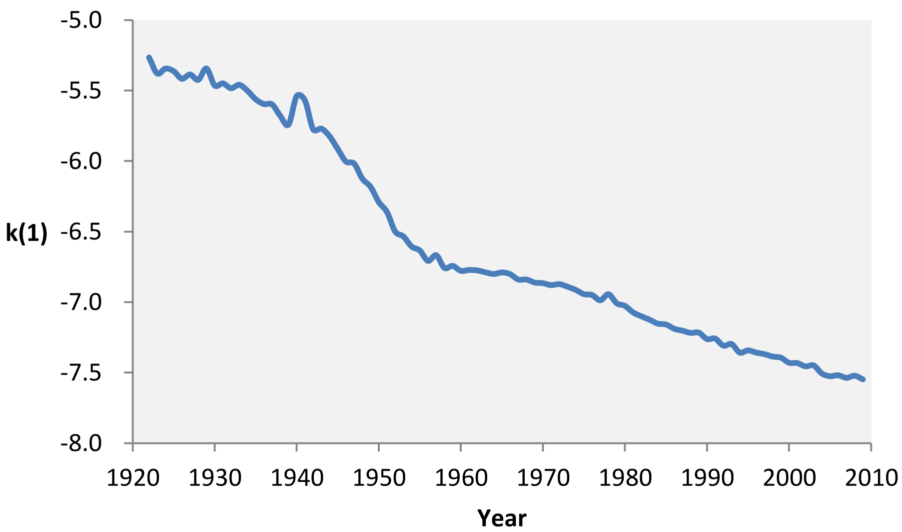
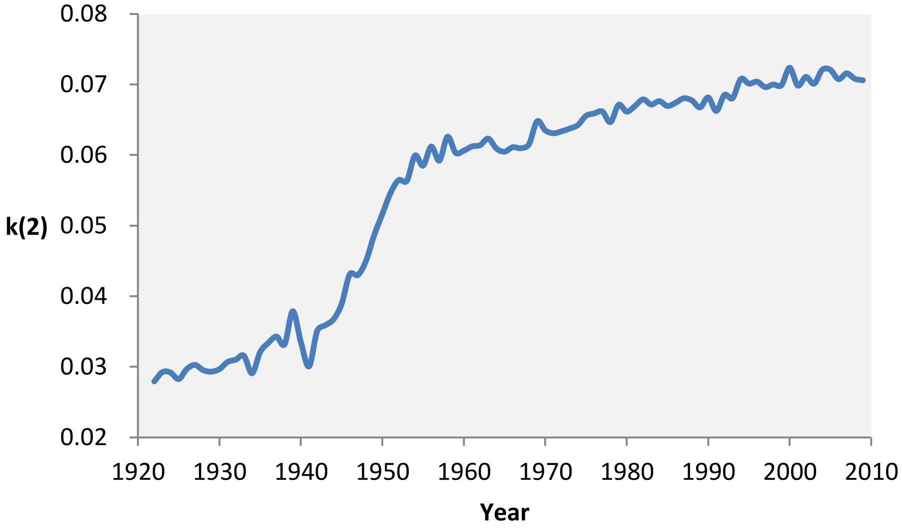


Figure 15 Fitted Value of $k^{(2)}$ Variables for 1922 to 2009



Cairns, Blake, and Dowd modeled $K_t = (k_t^{(1)}, k_t^{(2)})'$ as a two-dimensional random walk with drift (Cairns, Blake, and Dowd, 2006):

$$K_{t+1} = K_t + \mu + CZ_{t+1} \qquad (4.8)$$

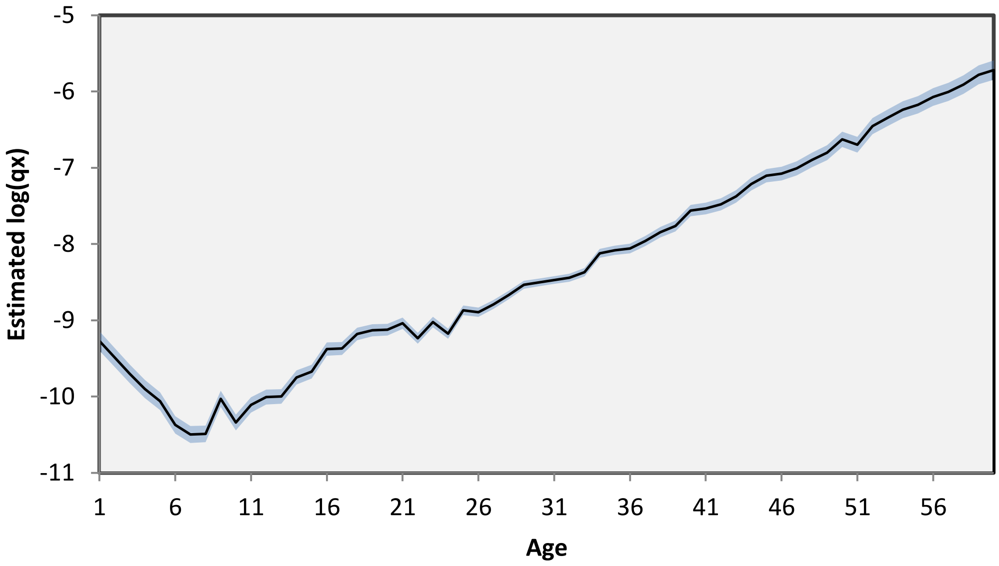
in which μ is a constant 2*1 drift vector, C is a constant 2 * 2 upper triangular ''volatility'' matrix (to be precise, the Choleski ''square root'' matrix of the variance-covariance matrix Σ), and Z_t is a two dimensional standard normal variable, each component of which is independent of the other (Dowd, Cairns, et al., 2008). It is very similar to Lee-Carter model with linear moves assumption of k variable. So we decide to follow the same formula as Lee-Carter Model to get the estimators of the drifts and the variances of the error terms. From Figure 14 and Figure 15 we found both k variables have slowed down the upward or downward trend since about 1970s. To follow a similar time period of US data we fit with Lee-Carter model, we start from year 1973 and get:

$$\hat{\mu} = \begin{pmatrix} -0.018249 \\ 0.0001898 \end{pmatrix} \qquad (4.9)$$

$$\hat{\Sigma} = \hat{C}\hat{C}^T = \begin{pmatrix} 0.0005317 & -0.00002359 \\ -0.00002359 & 0.000001565 \end{pmatrix} \qquad (4.10)$$

We also make an example of forecast for 2050 by using estimators above (Figure 16). And we use data "log(qx)" in order to get a similar shape with Lee-Carter Model (Figure 7). To calculate this forecast, we add estimated 41-year drifts of two k variables to the actual data of year 2009. So the forecast line in Figure 16 is not very smooth. If we choose another approach that uses formula directly to calculate predicted log(qx) for 2050 by estimators of k variables, the forecast line will be too smooth to be real. The reason replies on the assumption in CBD model that the relation between logitq_x and age x in certain year is linear.

Figure 16 Mortality Forecast for 2050, With 95% Confidence Interval of CBD Model



4.5 Cohort Life Table in UK

We follow the same process for UK data of gender total to seek life table for child (see results of estimation in Appendix E). And transfer the period life table to cohort life table and make comparison of the results:

Table 17 Comparison of Net Premium Value and Reference Discount Rate between Period Life Table and Cohort Life Table in UK Data

	Life Insurance (P1)		Child Plan (P2)	
	Net premium	Reference discount rate	Net premium	Reference discount rate
Period Life Table	417.86	2.92%	416.95	2.94%
Cohort Life Table	416.40	2.96%	415.49	2.98%

Similar to US data, the results of net premium decrease and the reference discount rates increase. But the changes in UK data are bigger than US data, with about 4% changes in reference discount rate, while the US data only changes about 1%. It is because the parent age that we use in UK is 5 year higher than that in US. And the cohort effect is thus larger in UK than US.

5 Child Plan Extension

5.1 Set up New Plans

Since we now have the cohort life table for both countries, it is time to do further analysis for Child Plan. First, we need to set up another two plans: Child Plan with income benefit (P3) and Child Plan with both parents insured (P4).

5.1.1 Child Plan with Income Benefit

Income benefit is a special condition as a supplement of the dead benefit. That is: if the insured dies before the end of maturity period, aside from terminating annual premium payment in the rest years of the policy period, the beneficiary will also receive additional payout from the insurer as an "income benefit". Sometimes, the income benefit payout has the same amount as the annual premium payment, but it is not a regulation. In our third plan, we assume that the income benefit equals to the net premium (See Table 18 for all the assumptions). This benefit condition will better protect the beneficiary against a future income flow risk.

Table 18 Child Plan with Income Benefit (P3) Assumptions

P3: Child Plan with Income Benefit	
Item	Condition
Annual Premium	to be calculated actuarially
Premium Payment Frequency	Yearly
Annual Interest Rate	3%
Maturity Benefit	It is a fixed benefit guaranteed of 10,000 USD which will be paid at Maturity when the child survives.
	If the child dies before maturity date, the premium payment terminates. No benefit will be paid at end of policy term.
Death Benefit	If the parent dies before maturity date while the child survives then, the premium payment terminates after the parent die. And the guaranteed fixed benefit can still be received at the end of policy term
Income Benefit	If the parent dies before maturity date, an annual income benefit payment will be received at the same amount of annual premium every year till the end of policy term as long as the child survives.
Entry Age	Child (Nominee): 0
	Mother (Life Assured): equal to the average female age at the first childbirth
Maturity	18 years

To calculate the net premium of P3, we use Π^I as total value of unit payout of income benefit. That is:

$$\Pi^I = \begin{cases} \ddot{a}_{\overline{T-1}|} - \ddot{a}_{\overline{S}|}, & ? \quad S < T \\ 0, & \quad S \geq T \end{cases} \tag{5.1}$$

The expected value of total income benefit can be expressed by death probability of the parent (b) and child (a):

$$E\Pi^I = \sum_{k=0}^{T-1} \left(\frac{1}{1+r} \right)^k \times (1 - {}_k^b p_y) \times {}_k^a p_x \tag{5.2}$$

If we assume the same amount for premium and income benefit payment as π, the total loss of the insurer can be written as:

$$L = C \times Y - \pi \times \Pi + \pi \times \Pi^{1} \tag{5.3}$$

And the net premium can be calculated like this:

$$\pi = f_3(r,\ T,\ {}^{a}p_x,\ {}^{b}p_y) = C \times \frac{(\frac{1}{1+r})^T \times {}_T^{a}p_x}{\sum_{k=0}^{T-1}\left(\frac{1}{1+r}\right)^k \times (2\,{}_k^{b}p_y - 1) \times {}_k^{a}p_x} \tag{5.4}$$

From the equation above, since the survival probability can't be above 1, it can be concluded that the net premium of income benefit product is always higher than normal Child Plan product. It can also be explained by intuitive judgments: the income benefit product should charge a higher price than normal Child Plan because of additional risk protection service. This results can also be seen in Table 19. The net premium value in both US and UK increase significantly from P2 to P3 and are even higher than P1. Although a higher price implies a bigger risk for the insurer, it's too early to tell whether selling a policy of Child Plan with income benefit is a more risky liability than Single Insurance Plan. We will analyze the risks in next chapter.

Table 19 Comparison of Net Premium Value and Reference discount rate among Single Life Insurance Plan (P1), Child Plan (P2) and Child Plan with income benefit (P3), in US and UK Data

Country	Results	P1	P2	P3
US	Net premium	416.72	415.63	417.71
	Reference discount rate	2.95%	2.98%	2.93%
UK	Net premium	416.40	415.49	417.25
	Reference discount rate	2.96%	2.98%	2.94%

5.1.2 Child Plan with Both Parents Insured

In the real market, the number of the insured can be one or more than one. For Child Plan product, it sometimes links both parents together in one policy. In our forth plan, we assume two-year age gap between the father and the mother (the father is older), and the death benefit will be activated by the death of the parent(s) before maturity. All the conditions are assumed below:

Table 20 Child Plan with Both Barents Insured (P4) Assumptions

P4: Child Plan with Both Barents Insured	
Item	Condition
Annual Premium	to be calculated actuarially
Premium Payment Frequency	Yearly
Annual Interest Rate	3%
Maturity Benefit	It is a fixed benefit guaranteed of 10,000 USD which will be paid at Maturity when the child survives.
	If the child dies before maturity date, the premium payment terminates. No benefit will be paid at end of policy term.
Death Benefit	If at least one of the parents dies before maturity date while the child survives then, the premium payment terminates after the death of parent(s). And the guaranteed fixed benefit can still be received at the end of policy term
Income Benefit	No
Entry Age	Child (Nominee): 0
	Mother (Life Assured): equal to the average female age at the first childbirth
	Father (Life Assured): two year older than the mother
Maturity	18 years

In this plan the expected value of total premium can be written as following equation with death probability of child (a), the mother (b) and the father (c):

$$E\Pi = \sum_{k=0}^{T-1}\left(\frac{1}{1+r}\right)^k \times {}_k^b p_y \times {}_k^c p_z \times {}_k^a p_x \tag{5.5}$$

Similarly, ${}_k^c p_z$ refers to the probability that the father (c) at age z survives at least another k years.

We use Lee-Carter model and CBD model once again to obtain the life table for the male in US and UK data (see Appendix C for the fitting results of Lee-Carter model and Appendix F). And the net premium can be calculated like this:

$$\pi = f_4(r,\ T,\ {}^a p_x,\ {}^b p_y,\ {}_k^c p_z) = C \times \frac{\left(\frac{1}{1+r}\right)^T \times {}_T^a p_x}{\sum_{k=0}^{T-1}\left(\frac{1}{1+r}\right)^k \times {}_k^b p_y \times {}_k^c p_z \times {}_k^a p_x} \tag{5.6}$$

Obviously, the price of product of both parents insured will be more expensive than one parent insured product. The results in our plans have shown this (Table 21), the net premiums in both countries increase from around 415 USD to 420 USD. The reference discount rates fall below 2.9%. This increase of net premium is even stronger than the increase impacted by income benefit of P3.

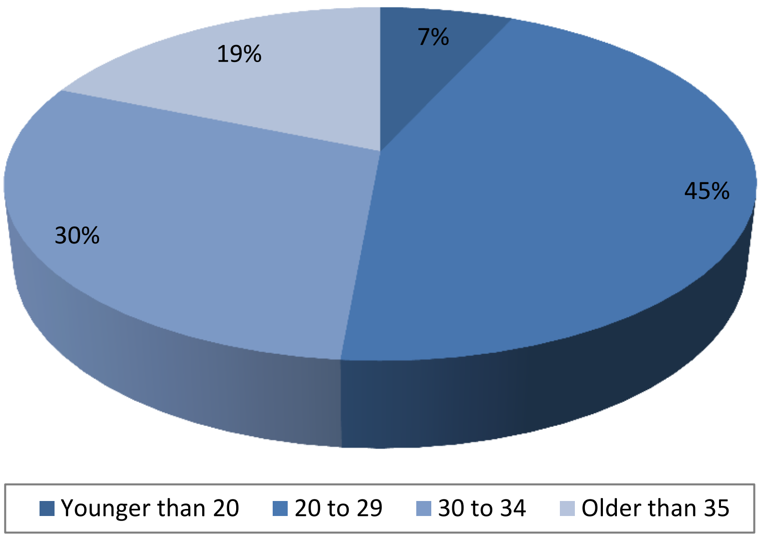
Table 21 Comparison of Net Premium Value and Reference discount rate among Child Plan (P2), Child Plan with Income Benefit (P3) and Child Plan with Both Parents Insured (P4), in US and UK Data

Country	Results	P2	P3	P4
US	Net premium	415.63	417.71	420.65
	Reference discount rate	2.98%	2.93%	2.86%
UK	Net premium	415.49	417.25	419.08
	Reference discount rate	2.98%	2.94%	2.89%

5.2 Older Parent(s) Insured

The entry age of the mother we used for P1 to P4 are assumed as the mean age of mothers at the first childbirth both in UK and US. However, in real world, the proportion of babies with older mother in their late 30s or older has gone up. A figure in 2007 revealed that about one-fifth of the new mothers were over 35 in UK (Figure 17). This older group can't be ignored in Child Plan Insurance products. The higher-age parents have more incentive to purchase an insurance product. On one hand, higher aged parents have lower survival probability and insurance for their children are more essential. On the other hand, higher aged parents are more likely to be higher income group and have financial advantage for the purchase. Therefore in this section, we will use age 35 as the entry age of mother and 37 as the entry age of father, while the child is still assumed born in 2006. Follow the age group number in Chapter 2, this age group will be named as Group D.

Figure 17 UK Age-specific Profiles of Fertility in 2007



From the table below, we can see the difference between average age group and older group. Firstly, since the age gaps between these two groups are 5 year in UK and 10 year in US, the US data decreases more in reference discount rate. Secondly, the declines in P3 and P4 are dramatic in both countries and the reference discount rate of P4 in US data even falls down to 2.71% below 2.80% level.

Table 22 Reference Discount Rate Comparison of Average Age Group (C) and Older Age Group (D) among Single Life Insurance Plan (P1), Child Plan (P2), Child Plan with Income Benefit (P3) and Child Plan with Both Parents Insured (P4), in US and UK Data

Country	Age Group	P1	P2	P3	P4
US	C	2.95%	2.98%	2.93%	2.86%
	D	2.90%	2.92%	2.82%	2.71%
	Diff	0.05%	0.05%	0.11%	0.14%
UK	C	2.96%	2.98%	2.94%	2.89%
	D	2.94%	2.96%	2.90%	2.85%
	Diff	0.02%	0.02%	0.03%	0.04%

6 Sensitivity to Mortality Model Risk

For a Child Plan, 18 years is such a long time challenging for the accuracy of Mortality Model. Both longevity risk and short-term mortality risk exist, so a sensitivity analysis is essential for a Child Plan by a few simulations. This sensitivity analysis will also help us to compare the risk exposure of our plans.

Before the analysis, we quote the instruction of mortality-related risks from Cairns, Blake, and Dowd (2006) for the sake of clarity:

- Longevity risk is uncertainty in the long-term trend in mortality rates and its impact on the long-term probability of survival of an individual. Longevity risk is normally taken to mean the risk that survival rates are higher than anticipated, although we strictly take it to mean uncertainty in either direction.

- Short-term, catastrophic mortality risk should be interpreted as the risk that, over short periods of time, mortality rates are much higher (or lower) than would normally be experienced. Examples of such "catastrophes" include the influenza pandemic in 1918 and the tsunami in December 2004. Once the catastrophe has past, we expect mortality rates to revert to their previous levels and to continue along previous trends.

6.1 Sensitivity Analysis under Lee-Carter Model

Under the Lee-Carter model, we use the following two equations to model central death rate. That is the way how we calculate the results of reference discount rates for Child Plans and their contrast plans in the last chapter. And also, we assume the k variable follow a linear decline over time. This can be interpreted as a natural improvement process with a steady speed to the survival probability:

$$log(\hat{m}_{x,t}) = \hat{a}_x + \hat{b}_x \hat{k}_t \tag{6.1}$$

$$\hat{k}_t = \hat{C} + \hat{k}_{t-1} \tag{6.2}$$

In this section, in order to simulate a scenario of longevity risk and catastrophic mortality risk for US data, we control the k variable of Lee-Carter Model to change the estimation results of survival probability.

6.1.1 Sensitivity Analysis to Longevity Risk under Lee-Carter Model

For longevity risk, we assume that in the future year t, the survival probability has an unanticipated change due to some worldwide issue happens (for example, the effective treatment of aids or a new serious disease spread), and this change will have a permanent influence for the future. However, the natural improvement speed of the survival probability will not change after the issue. It is because this natural improvement of the survival probability is mainly determined by economic improvement, medical technology development, individual awareness to healthcare, and so on. In the Lee-Carter model, the result of simulation is like that: there will be an unanticipated jump in k variable in future year t, while this jump won't be revived in the later years, and the value of yearly drift of k variable will remain the same as before in the future.

We simulate our first two scenarios to show the longevity risk. In these two scenarios, we vary k variable in year t by one unit of standard deviation of error term. And we simulate such change of k_t in both directions (plus or minus $\hat{\sigma}_k$) and collect the differences of the reference discount rate before and after the k_t change. And then we average the differences, and define it as the "Degree of Sensitivity" (also short as "ds"). For the sake of clarity, we use 0.000001 as the unit of ds.

In scenario 1, we assume that the mortality-linked issue only influences the adults, so we only change the k_t for adults (age above 18). We simulate such scenario because not all the mortality-linked issues will have the same influence to adults and the children. For example, some diseases mainly attack adults (such as diabetes mellitus, hypertension, and diseases due to smoking or excessive drinking habits). Under such assumptions, the degree of sensitivity is shown as the following equation:

$$
\begin{aligned}
& ds_t = \frac{f_0^{-1}(\pi_t^+) - f_0^{-1}(\pi_t^-)}{2} \times 1000000 \\
& \text{where} \\
& \quad \pi_t^+ = f_i(r,\ T,\ {}_{k}^{a}p_x,\ {}_{\breve{k}}^{b}p_y,\ {}_{\hat{k}}^{b}p_y{}^+,\ {}_{\breve{k}}^{c}p_z,\ {}_{\hat{k}}^{c}p_z{}^+) \\
& \quad \pi_t^+ = f_i(r,\ T,\ {}_{k}^{a}p_x,\ {}_{\breve{k}}^{b}p_y,\ {}_{\hat{k}}^{b}p_y{}^-,\ {}_{\breve{k}}^{c}p_z,\ {}_{\hat{k}}^{c}p_z{}^-)
\end{aligned}
\qquad (6.3)
$$

Here i refer to the number of plans. k refers to the additional survival years varied from 0 to T, $\breve{k}$ refers to the additional survival years varied from 0 to t-2007 and $\hat{k}$ refers to the additional survival years varied from t-2006 to T. t is the year when the big issue of longevity influence happens.

Obvious, if the big change happens in another year, the result of the reference discount rate change will be different. And therefore we draw a curve of degree of sensitivity of different year from year 2008 (beginning year of forecast data) to year 2024 (the year of maturity benefit payment). We draw different curves for average age group (Figure 18) and older age group (Figure 19) and call such curves as "Sensitivity Curve" in this article.

Figure 18 Sensitivity Curve of Scenario 1 for Average Age Group in US data with $\hat{\sigma}_k$

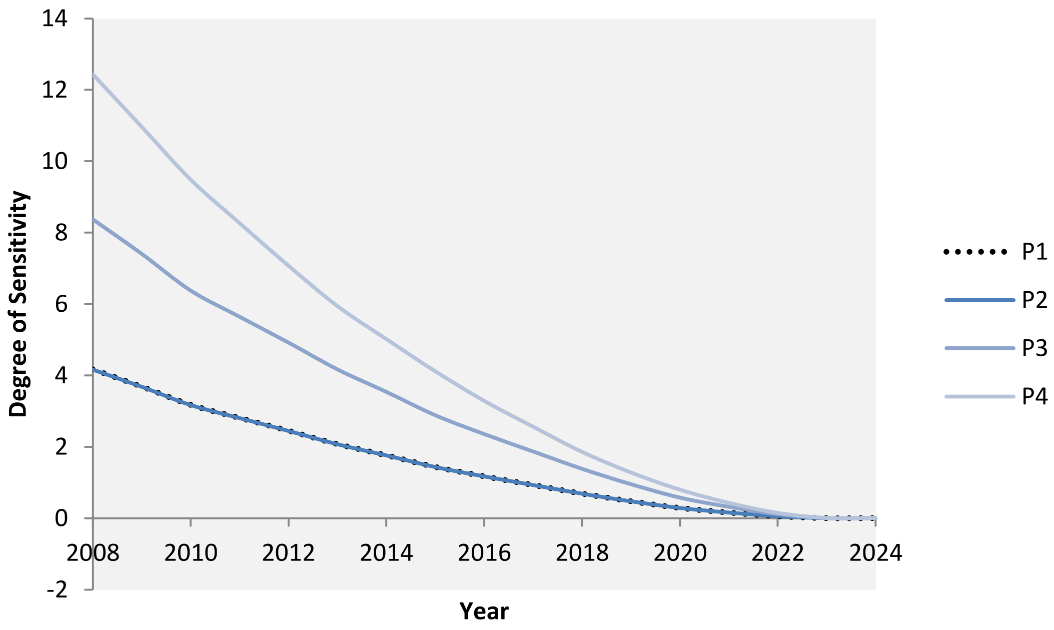
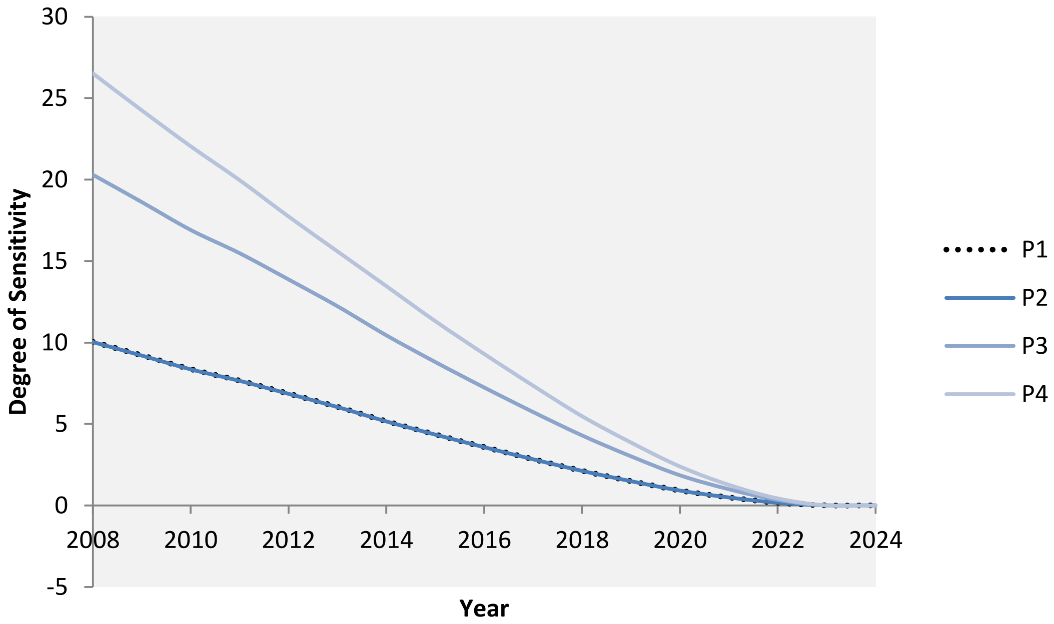


Figure 19 Sensitivity Curve of Scenario 1 for Older Age Group in US data with $\hat{\sigma}_k$



From the two figures above, there are some patterns can be found. Firstly, it is quite clear that the latter the mortality-linked issue happens, the smaller the degree of sensitivity is for all the plans and age groups until the degree of sensitivity falls to zero when the mortality-linked issue happens in 2023. Also, these yearly drops in degree of sensitivity all show an approximate linear tendency. Secondly, the P4-Child Plan with both parents insured in both age groups has the highest degree of sensitivity followed by P3-Child Plan with income benefit. Since we didn't change the survival probability of the child, the degree of sensitivity in P1 and P2 are almost the same for whenever the mortality-linked issue year is. Thirdly, the older age group has much higher degree of sensitivity than the average age group. If the mortality-linked issue happens in the beginning (year 2008), the degree of sensitivity will be more than double in all the four plans for the older age group compared with average age group.

In scenario 2, we assume that the mortality-linked issue has a significant impact to everyone. So we change the k_t for both adults and children. Follow a similar process to draw new sensitivity curves, we calculate the degree of sensitivity like this:

$$\begin{aligned} ds_t &= \frac{f_0^{-1}(\pi_t^+) - f_0^{-1}(\pi_t^-)}{2} \times 1000000 \\ \text{where} \\ \pi_t^+ &= f_i(r,\ T,\ {}_{k}^{a}p_x,\ {}_{k}^{a}p_x{}^{+},\ {}_{k}^{b}p_y,\ {}_{k}^{b}p_y{}^{+},\ {}_{k}^{c}p_z,\ {}_{k}^{c}p_z{}^{+}) \\ \pi_t^+ &= f_i(r,\ T,\ {}_{k}^{a}p_x,\ {}_{k}^{a}p_x{}^{-}\ {}_{k}^{b}p_y,\ {}_{k}^{b}p_y{}^{-},\ {}_{k}^{c}p_z,\ {}_{k}^{c}p_z{}^{-}) \end{aligned} \tag{6.4}$$

In specific, the P1- Single Life Insurance Plan hasn't changed from Scenario 1. And the new sensitivity curves for average group and are shown in Figure 20 and Figure 21. From a general view of the shape of the curves, there is also yearly drop pattern in Scenario 2 but the linear tendency is not significant any more. The degree of sensitivity in Plan 2, 3 and 4 for both age groups all fall below zero in Scenario 2 and rise back to zero in the end. The negative degree of sensitivity is interpreted as an opposite movement for the reference discount rate compared with the change of k variable. For the comparison between age group, the older age group still shows a stronger degree of sensitivity than average group.

However, for the comparison between plans, the situation is different from Scenario 1. In Scenario 1, we find that Plan 2 always show a better performance in hedging longevity risk (smaller degree of sensitivity) than Plan 3 and Plan 4, while the Plan 4 performs worst (the highest degree of sensitivity). However, in Scenario 2, it is not always the case. If the mortality-linked issue happens earlier, the Plan 4 has the biggest influence among the three plans. But then degree of sensitivity of these three plan decline when the mortality-linked issue year comes later and later and fall below zero one by one. The degree of sensitivity will increase although it has an opposite direction.

Figure 20 Sensitivity Curve of Scenario 2 for Average Age Group in US data with $\hat{\sigma}_k$

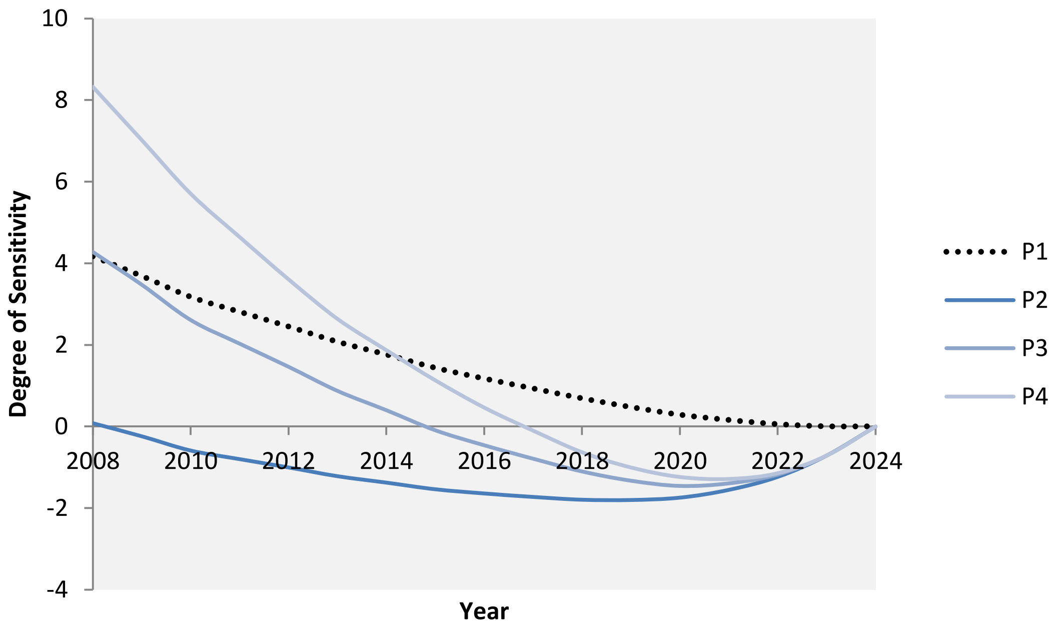
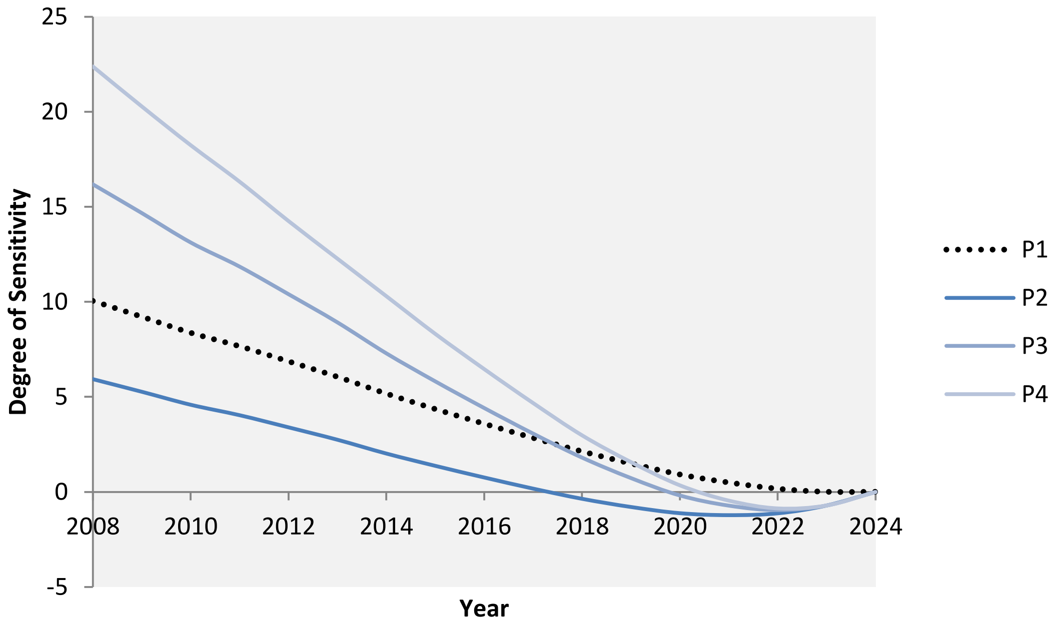


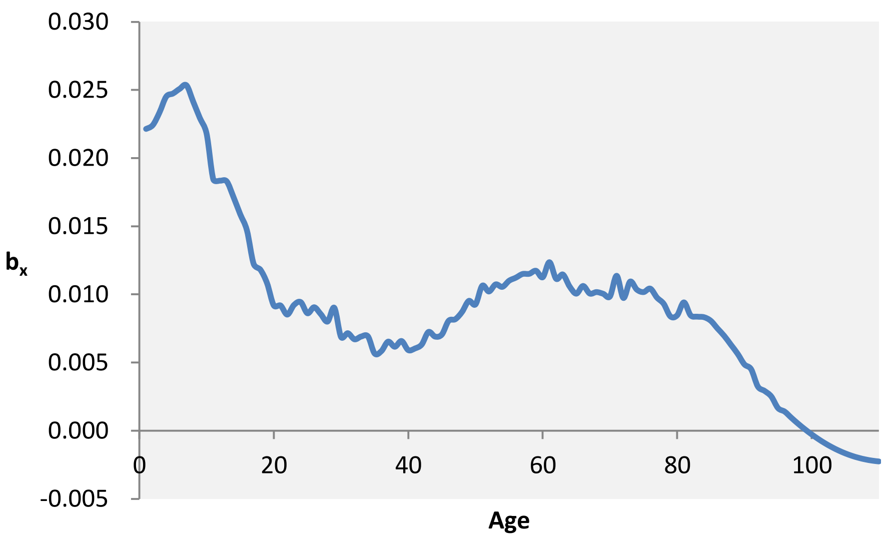
Figure 21 Sensitivity Curve of Scenario 2 for Older Age Group in US data with $\hat{\sigma}_k$



From Scenario 1 to 2, we add survival probability change for children. And from the equation (6.5) of Lee-Carter model, it can be deducted that influence power of mortality-linked issue to different age group varies determined by b variable. However, since almost all the values of b variable for children is higher than adults (Figure 22), the drive due to the influence of the changes in survival probability for children is stronger than the drive from that of the adults' side.

And the influence of the changes in survival probability for children results in an opposite movement of reference discount rate. It is the reason why the negative degree of sensitivity exists. And because of the opposite drive of child survival related policy, Child Plans show an inner hedging system. In Figure 21 of older group, the Plan 2 has a lower absolute value of degree of sensitivity than Plan 1 for the majority years. But this inner hedging system doesn't always perform better. In Figure 20, the influence of the changes in survival probability for children is too strong in Plan 2 for average group that it leads to a higher negative degree of sensitivity on the contrary.

Figure 22 Fitted Value of b_x for US Total Data in 1973-2007 of Lee-Carter Model



6.1.2 Sensitivity Analysis to Short-term Mortality Risk under Lee-Carter Model

To simulate a short-term mortality risk, we assume that in the future year t, the survival probability has a unanticipated change due to some transitory issue happens (for example, influenza pandemic in 1918 and the tsunami in December 2004), but this mortality-linked issue won't have permanent influence for the future. The survival probability will go back to estimated value in the next year and keep the same natural improvement speed as before. Namely, the k variable jump only for one year. The result of the degree of sensitivity is like this:

$$
\begin{aligned}
&ds_t = \frac{f_0^{-1}(\pi_t^+) - f_0^{-1}(\pi_t^-)}{2} \times 1000000 \\
&\text{where} \\
&\pi_t^+ = f_i(r,\ T,\ {}^{a}_{\ddot{k}}p_x,\ {}^{a}_{t-2006}p_x^{+},\ {}^{a}_{\hat{k}+1}p_x,\ {}^{b}_{\ddot{k}}p_y,\ {}^{b}_{\hat{k}-2006}p_y^{+},\ {}^{b}_{\hat{k}+1}p_y,\ {}^{c}_{\ddot{k}}p_z,\ {}^{c}_{\hat{k}-2006}p_z^{+},\ {}^{c}_{\hat{k}+1}p_z) \\
&\pi_t^+ = f_i(r,\ T,\ {}^{a}_{\ddot{k}}p_x,\ {}^{a}_{t-2006}p_x^{-},\ {}^{a}_{\hat{k}+1}p_x,\ {}^{b}_{\ddot{k}}p_y,\ {}^{b}_{\hat{k}-2006}p_y^{-},\ {}^{b}_{\hat{k}+1}p_y,\ {}^{c}_{\ddot{k}}p_z,\ {}^{c}_{\hat{k}-2006}p_z^{-},\ {}^{c}_{\hat{k}+1}p_z)
\end{aligned}
\qquad (6.6)
$$

In Scenario 3, we change k variable in both directions (plus or minus 5 times $\hat{\sigma}_k$) for only one year on children and adults, but with a five times deviation (plus or minus $5\hat{\sigma}_k$). We will no longer simulate a scenario that only impacts adults, because for some catastrophic mortality risk like earthquake or tsunami, people of the different ages suffer the same dangers. Similarly, we also draw two sensitivity curves for average and older age groups:

Figure 23 Sensitivity Curve of Scenario 3 for Average Age Group in US data with $5\hat{\sigma}_k$

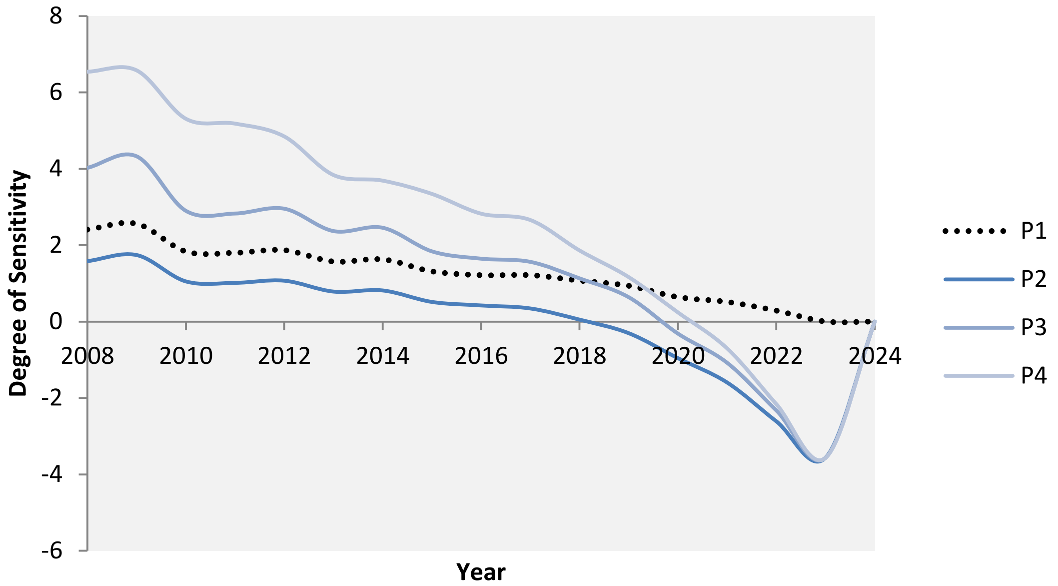
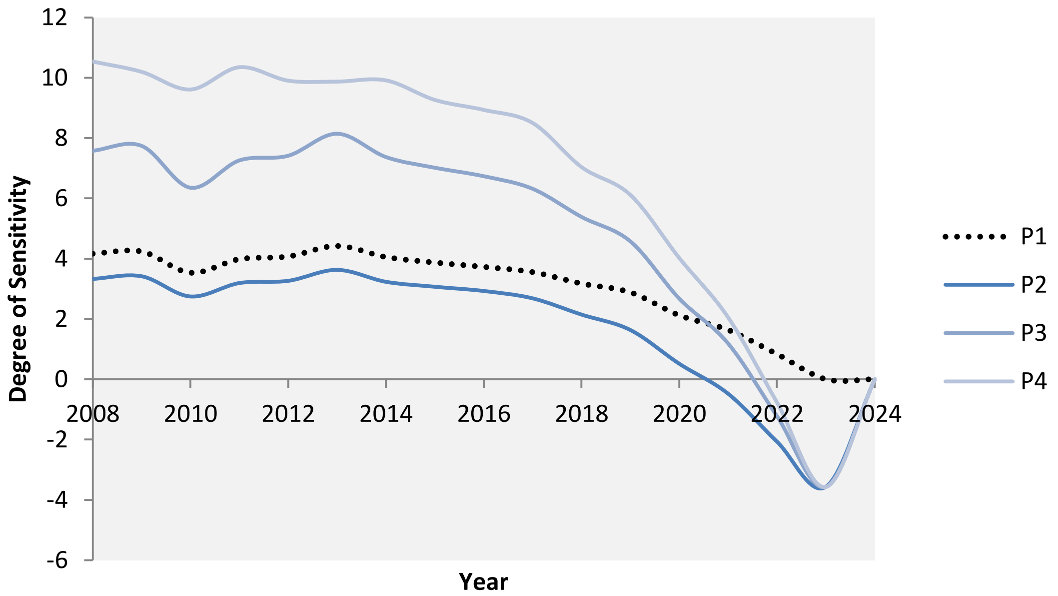


Figure 24 Sensitivity Curve of Scenario 3 for Older Age Group in US data with $5\hat{\sigma}_k$



The curves are not as smooth as Scenario 1 and 2. And similar to Scenario 2, Plan 2, 3 and 4 will fall below zero and back to zero in the ending year.

6.2 Sensitivity Analysis under CBD Model

In Chapter 4, we use the following equations of CBD model to estimate death probability of UK data.

$$\hat{q}_{x,t} = \frac{e^{\hat{k}_t^{(1)}+\hat{k}_t^{(2)}(x-\bar{x})}}{1+e^{\hat{k}_t^{(1)}+\hat{k}_t^{(2)}(x-\bar{x})}} \tag{6.7}$$

$$\hat{K}_{t+1} = \hat{K}_t + \hat{\mu} \tag{6.8}$$

Similar to Lee-Carter model, we can also control k variables to simulate mortality risks.

6.2.1 Sensitivity Analysis to Longevity Risk under CBD Model

We also assume two scenarios to simulate longevity risk: the first one only happens among the adults and the second one influences everyone. Since there are two k variables, we change them at the same time with one unit standard deviation. But $k_t^{(2)}$ will not always have a positive relevance to death probability, it is determined by the sign of $(x-\bar{x})$. So for the age below 29.5 (the average age of our data range), we change the $k_t^{(2)}$ in an opposite way to $k_t^{(1)}$ in order to gain the biggest difference. That is:

$$\begin{aligned}
\hat{q}_{x,t}^{\,+} &= \begin{cases} \dfrac{e^{\hat{k}_t^{(1)}+\hat{\sigma}_{k^{(1)}}+\left(\hat{k}_t^{(2)}+\hat{\sigma}_{k^{(2)}}\right)(x-\bar{x})}}{1+e^{\hat{k}_t^{(1)}+\hat{\sigma}_{k^{(1)}}+\left(\hat{k}_t^{(2)}+\hat{\sigma}_{k^{(2)}}\right)(x-\bar{x})}} & if\ x \geq \bar{x} \\ \dfrac{e^{\hat{k}_t^{(1)}+\hat{\sigma}_{k^{(1)}}+\left(\hat{k}_t^{(2)}-\hat{\sigma}_{k^{(2)}}\right)(x-\bar{x})}}{1+e^{\hat{k}_t^{(1)}+\hat{\sigma}_{k^{(1)}}+\left(\hat{k}_t^{(2)}-\hat{\sigma}_{k^{(2)}}\right)(x-\bar{x})}} & if\ x < \bar{x} \end{cases} \\
\hat{q}_{x,t}^{\,-} &= \begin{cases} \dfrac{e^{\hat{k}_t^{(1)}-\hat{\sigma}_{k^{(1)}}+\left(\hat{k}_t^{(2)}-\hat{\sigma}_{k^{(2)}}\right)(x-\bar{x})}}{1+e^{\hat{k}_t^{(1)}-\hat{\sigma}_{k^{(1)}}+\left(\hat{k}_t^{(2)}-\hat{\sigma}_{k^{(2)}}\right)(x-\bar{x})}} & if\ x \geq \bar{x} \\ \dfrac{e^{\hat{k}_t^{(1)}-\hat{\sigma}_{k^{(1)}}+\left(\hat{k}_t^{(2)}+\hat{\sigma}_{k^{(2)}}\right)(x-\bar{x})}}{1+e^{\hat{k}_t^{(1)}-\hat{\sigma}_{k^{(1)}}+\left(\hat{k}_t^{(2)}+\hat{\sigma}_{k^{(2)}}\right)(x-\bar{x})}} & if\ x < \bar{x} \end{cases}
\end{aligned} \tag{6.9}$$

After that, the calculation of degree of sensitivity is the same as Lee-Carter Model. The UK data we use are from 1973 to 2009, which is two years longer than US data. So we draw the sensitivity curve from year 2010.

From the two figures of Scenario 1 below, we can find the curve shapes are similar as Lee-Carter model. In addition, the patterns are also similar. The Plan 4 shows the highest degree of sensitivity followed by Plan 3, and the curve of Plan 2 almost overlaps that of Plan 1. Furthermore, the older group has higher degree of sensitivity than average age group.

Figure 25 Sensitivity Curve of Scenario 1 for Average Age Group in UK data with $\hat{\sigma}_k$

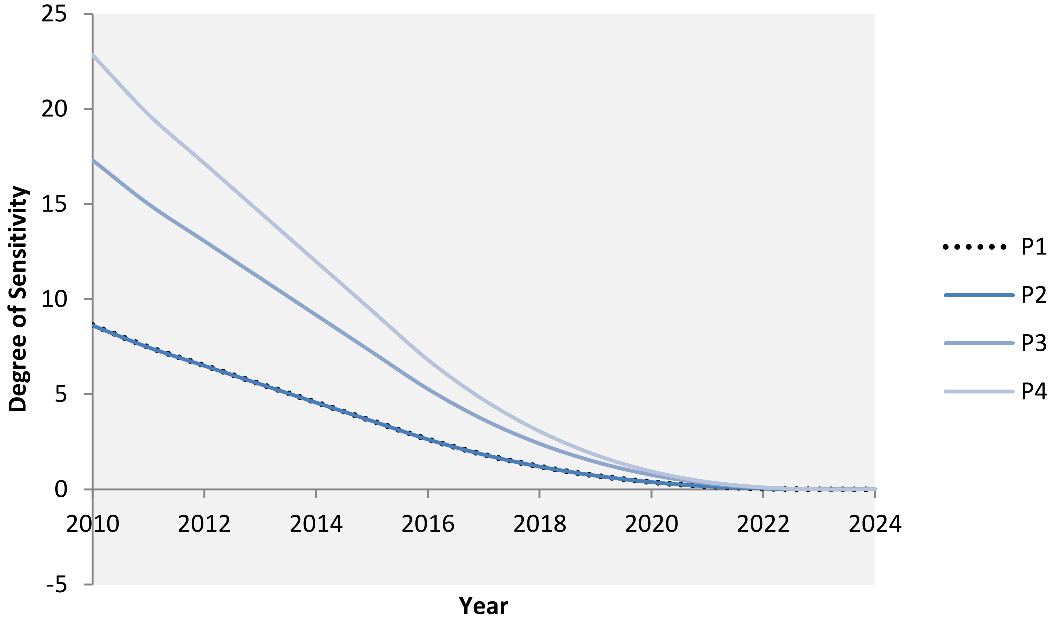
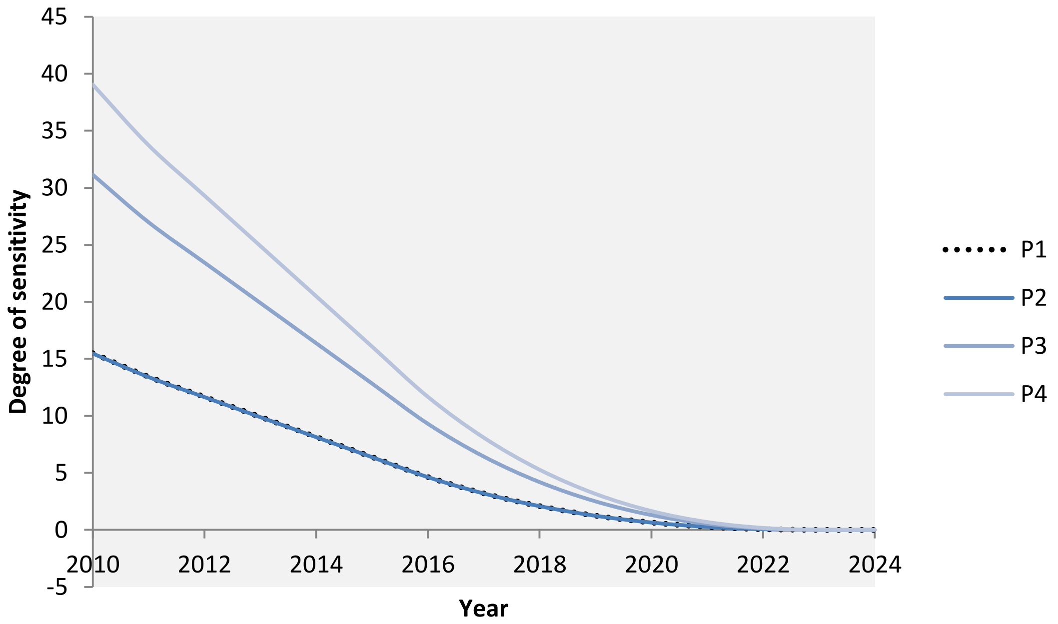


Figure 26 Sensitivity Curve of Scenario 1 for Older Age Group in UK data with $\hat{\sigma}_k$



In Scenario 2, we add mortality rate changes to children. And it should be noticed that the two k variables changes in the opposite direction for life table for children. The figures below have similar shapes and pattern with Lee-Carter model. It also shows an inner hedging system in Child Plan when comparing Plan 1 and Plan 2 in both average age group and older age groups.

Figure 27 Sensitivity Curve of Scenario 2 for Average Age Group in UK data with $\hat{\sigma}_k$

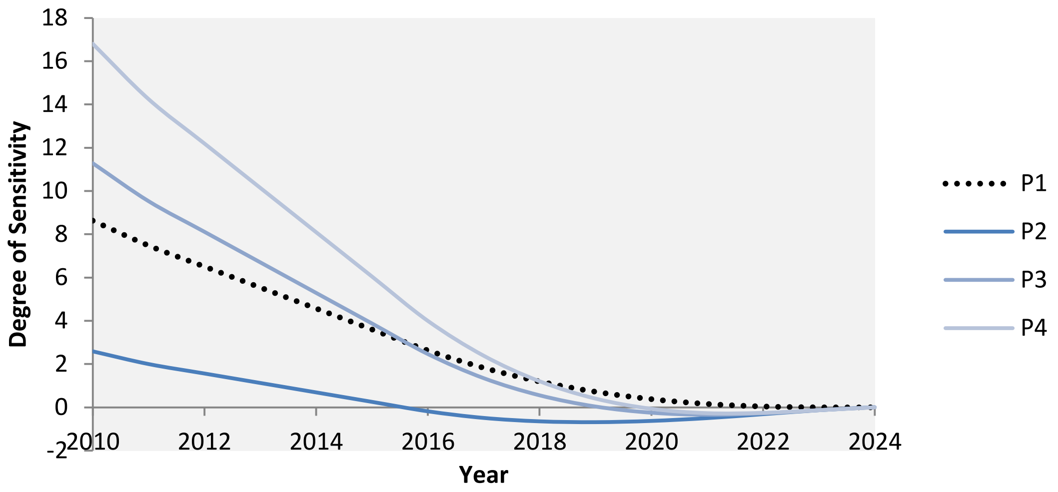
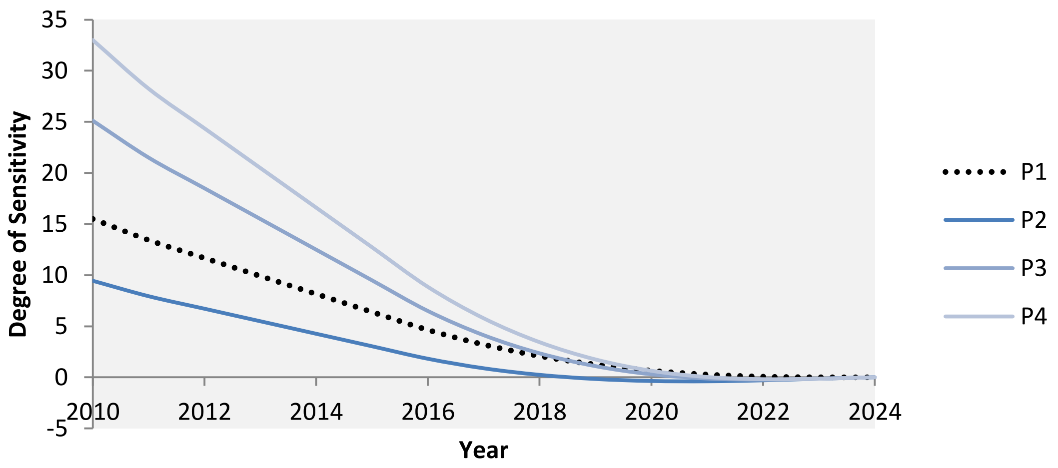


Figure 28 Sensitivity Curve of Scenario 2 for Older Age Group in UK data with $\hat{\sigma}_k$



6.2.2 Sensitivity Analysis to Short-term Mortality Risk under CBD Model

In Scenario 3, we also assume a short-term mortality rate change for everyone. The shapes of the graphs are not smooth (Figure 29 and Figure 30). All the four Plans in both age group simulations preform relatively stable for early mortality-linked issue year, although the degree of sensitivity are so high that reaches 100 for Plan 3 and 4 of older age group. Then all of the eight curves begin to fall in an approximate linear tendency at around year 2015 until drop to zero at the year 2023.

Figure 29 Sensitivity Curve of Scenario 3 for Average Age Group in UK data with $5\hat{\sigma}_k$

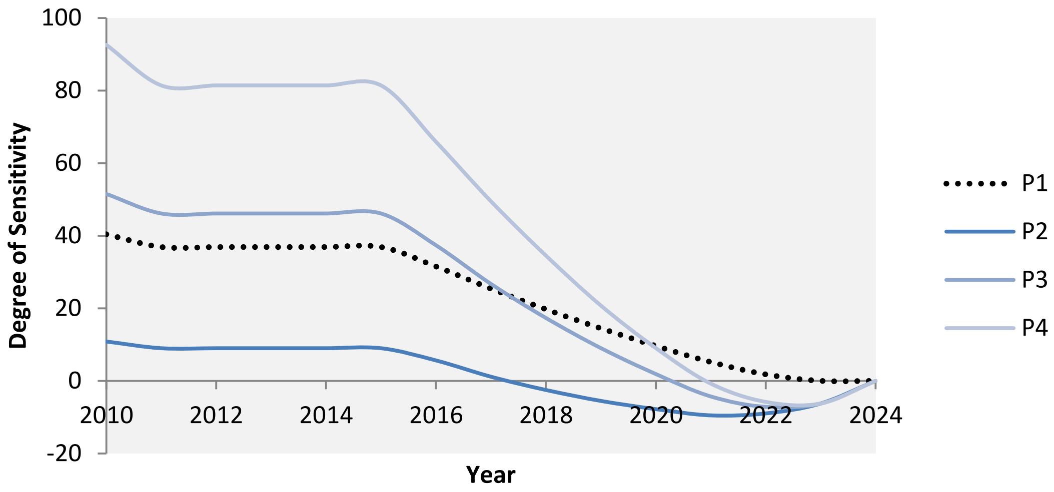
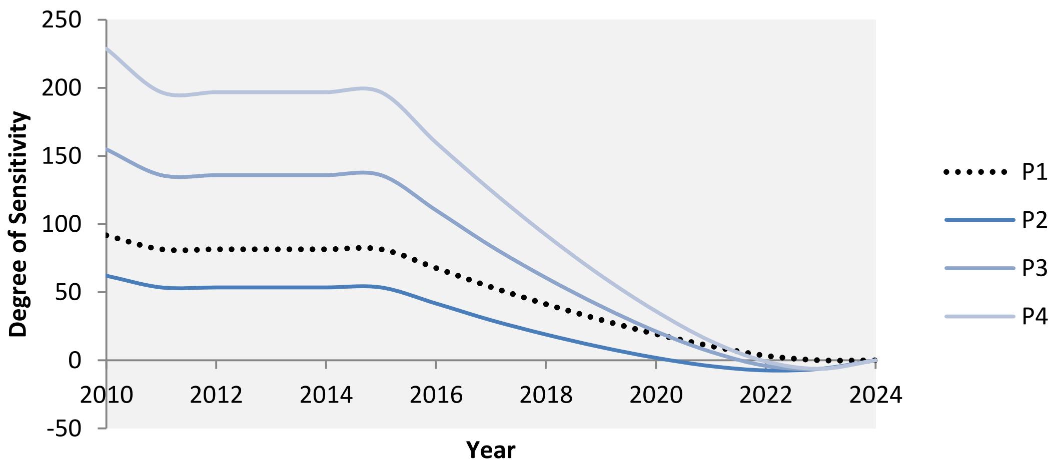


Figure 30 Sensitivity Curve of Scenario 3 for Older Age Group in UK data with $5\hat{\sigma}_k$



6.3 Summary and Discussion on Hedging

Sensitivity curves measure the extent of loss or gain when there is a mortality rate change. Because of the joint death probability design in Child Plans, Child Plan has an inner hedging system which declines its risk exposure compared with Single Insurance Plan (or at least equal to Single Insurance Plan if the children are totally excluded in the mortality-linked issue like in Scenario 1). Also, if there is a mortality-linked issue that increases the mortality rates for both adults and children, the Child Plan will gain less than a Single Insurance Plan.

Moreover, the sensitivity curves of Plan 3 and 4 proved that such additional benefit conditions-income benefit and both parents insured-will increase the risk exposure of Child Plan. In our assumptions of this article, the condition of both parents insured has a larger increase in degree of sensitivity. And we also found that the risk exposure will increase if the age of the insured goes higher.

Considering the sensitivity curves of Child Plans, they all have relatively big exposure to mortality related risks if the mortality-linked issue year comes earlier. And this might cause a great amount of losses if the situation goes worse. However, it is interesting to find that in Scenario 2 and 3, the degree of sensitivity in all of the three Child Plans fall below zero if the mortality-linked issue year comes later. So if the insurer sells two policy of Child Plan with different entry years, there is a hedging opportunity that in an overlapped year the negative exposure of the former policy will hedge the risks with the positive exposure of the latter policy.

Also, all the life insurance plans can be used to hedge risks with annuities, which usually have negative sensitivity curves.

7 Summary and Conclusion

In this article, we discuss a life insurance scheme-Child Plan. We set up assumptions to simulate such products in real market, and then use UK and US data and forecast data from the Lee-Carter Model and the CBD Model to quantify the Plans. During this quantifying process, we found that the fitting result of the CBD model is not as good as Lee-Carter model. But considering most studies using the CBD model for UK data and the bias is acceptable, we decide to continue with the CBD model for UK data.

We compare the Child Plan with a Single Life Insurance Plan, and find that the Child Plan has a lower price (net premium) and is less sensitive to mortality risk. It is because Child Plan has the joint death probability design which creates an inner hedging system within a single policy. When there is an issue that changes the death probabilities of the parent(s) and the child in the same direction, Child Plan will have the influence to its total loss in two directions and this will offset part of the change.

To judge the change of the total loss in a more clear and comparable way, we use the difference of sold reference discount rate and realized reference discount rate instead, and we name this difference “degree of sensitivity”. But the term of total loss and degree of sensitivity have the same meaning to the insurer.

We also discuss two additional benefit conditions: income benefit and both parents insured. We find that these two benefit conditions will increase the price of the Child Plan and cause a higher risk to the insurer. In our assumptions, the condition of both parent(s) insured shows the highest price and the largest degree of sensitivity among three Child Plans.

Furthermore, we extend the Plans with higher age groups and find that: the older the insurers are the higher the price and the degree of sensitivity to mortality risks are. Although we use two different forecast models for UK and US data, we can find this conclusion for both countries. The sensitivity curves can show the quantified results of this age increase of the insured.

Sensitivity curves also show a hedging opportunity to more than one policy of Child Plan with different entry years. Since our different Child Plans all have results of negative degree of sensitivity if the mortality-linked issue comes when the policy is close to the maturity and the survival probability of the children and the parent(s) all have been changed.

The approach we used for quantifying and simulating the mortality risks can be applied for other mortality linked products (other life insurance products, annuity, pension …). We use two classic models-the Lee-Carter model and the CBD model-as our basic forecast tools in this article, but we suggest more extended models can be used in a future research, for example, the Renshaw and Haberman (2006) extension to the Lee-Carter (1992) model that includes a cohort effect, and the extension to the Cairns, Blake & Dowd (2006) model that allows not only for a cohort effect, but also for a quadratic age effect.

In our assumptions of the Child Plan, we assume the interest rate is always fixed at 3%. It is not realistic in the whole 18 years. In fact, the uncertainty of the interest rate is another risk for the life insurance products like Child Plan. We suggest that more studies are needed in the future to combine these two risks together and quantify the plans.

Appendix A

In Appendix A, we show the final results of estimators of Lee-Carter Model for US female data in 1973-2007

Table 23 Fitted Value of a_x and b_x for US Female Data in 1973-2007 of Lee-Carter Model

Age(x)	a_x	b_x	Age(x)	a_x	b_x
0	-4.7500	0.0245	56	-5.0862	0.0111
1	-7.3281	0.0254	57	-5.0059	0.0105
2	-7.7787	0.0262	58	-4.8928	0.0102
3	-8.0884	0.0267	59	-4.8291	0.0094
4	-8.2915	0.0264	60	-4.7287	0.0106
5	-8.4260	0.0268	61	-4.6420	0.0089
6	-8.5154	0.0260	62	-4.5385	0.0091
7	-8.5951	0.0236	63	-4.4705	0.0078
8	-8.6430	0.0238	64	-4.4003	0.0070
9	-8.7041	0.0203	65	-4.3076	0.0075
10	-8.7414	0.0173	66	-4.2353	0.0066
11	-8.7200	0.0187	67	-4.1494	0.0069
12	-8.6283	0.0179	68	-4.0596	0.0067
13	-8.4917	0.0165	69	-3.9805	0.0069
14	-8.2737	0.0143	70	-3.8718	0.0089
15	-8.0449	0.0149	71	-3.7939	0.0071
16	-7.7487	0.0101	72	-3.6819	0.0087
17	-7.6448	0.0104	73	-3.5952	0.0085
18	-7.5389	0.0107	74	-3.5185	0.0082
19	-7.5696	0.0105	75	-3.4199	0.0090
20	-7.5757	0.0109	76	-3.3292	0.0083
21	-7.5286	0.0097	77	-3.2386	0.0078
22	-7.5333	0.0108	78	-3.1486	0.0074
23	-7.5169	0.0105	79	-3.0440	0.0078
24	-7.5041	0.0097	80	-2.9196	0.0089
25	-7.4741	0.0090	81	-2.8232	0.0079
26	-7.4266	0.0093	82	-2.7103	0.0080
27	-7.3868	0.0087	83	-2.6017	0.0083
28	-7.3262	0.0097	84	-2.4937	0.0082
29	-7.3103	0.0072	85	-2.3899	0.0076
30	-7.2615	0.0075	86	-2.2779	0.0073

Age(x)	a_x	b_x	Age(x)	a_x	b_x
31	-7.1700	0.0081	87	-2.1741	0.0065
32	-7.1122	0.0071	88	-2.0735	0.0060
33	-7.0325	0.0079	89	-1.9637	0.0052
34	-6.9768	0.0067	90	-1.8567	0.0050
35	-6.8993	0.0067	91	-1.7644	0.0033
36	-6.8102	0.0075	92	-1.6573	0.0031
37	-6.7336	0.0076	93	-1.5575	0.0025
38	-6.6241	0.0077	94	-1.4639	0.0019
39	-6.5821	0.0066	95	-1.3863	0.0016
40	-6.4862	0.0077	96	-1.2987	0.0011
41	-6.3932	0.0081	97	-1.2139	0.0007
42	-6.2977	0.0094	98	-1.1320	0.0003
43	-6.2077	0.0089	99	-1.0532	-0.0001
44	-6.1359	0.0092	100	-0.9776	-0.0004
45	-6.0476	0.0097	101	-0.9053	-0.0007
46	-5.9548	0.0103	102	-0.8365	-0.0010
47	-5.8690	0.0105	103	-0.7711	-0.0013
48	-5.7569	0.0112	104	-0.7092	-0.0015
49	-5.7055	0.0105	105	-0.6508	-0.0016
50	-5.6071	0.0120	106	-0.5960	-0.0018
51	-5.5125	0.0111	107	-0.5446	-0.0019
52	-5.4241	0.0113	108	-0.4967	-0.0020
53	-5.3434	0.0112	109	-0.4521	-0.0020
54	-5.2726	0.0109	110	-0.4108	-0.0021
55	-5.1787	0.0109			

Table 24 Fitted Value of k_t for US Female Data in 1973-2007 of Lee-Carter Model

Year	k_t	Year	k_t	Year	k_t
1973	26.1065	1985	2.8895	1997	-8.1181
1974	21.7071	1986	2.6513	1998	-10.3274
1975	18.3161	1987	1.7743	1999	-10.0941
1976	16.0221	1988	3.0227	2000	-11.5191
1977	14.7357	1989	1.9673	2001	-11.8434
1978	13.8274	1990	-1.0971	2002	-12.7265
1979	10.8680	1991	-1.7867	2003	-13.3080
1980	10.8638	1992	-4.5055	2004	-14.7183
1981	8.1297	1993	-1.9521	2005	-15.8895
1982	5.9454	1994	-3.6157	2006	-17.3437
1983	4.2666	1995	-3.9697	2007	-18.0113
1984	3.6657	1996	-5.9331		

Appendix B

In Appendix B, we show the final results of estimators of Lee-Carter Model for US Total data in 1973-2007

Table 25 Fitted Value of a_x and b_x for US Total Data in 1973-2007 of Lee-Carter Model

Age(x)	a_x	b_x	Age(x)	a_x	b_x
0	-4.6272	0.0221	56	-4.7919	0.0115
1	-7.2426	0.0224	57	-4.7128	0.0115
2	-7.6400	0.0234	58	-4.6004	0.0117
3	-7.9185	0.0245	59	-4.5403	0.0113
4	-8.1338	0.0247	60	-4.4382	0.0124
5	-8.2639	0.0251	61	-4.3588	0.0111
6	-8.3541	0.0253	62	-4.2543	0.0115
7	-8.4401	0.0242	63	-4.1878	0.0106
8	-8.4595	0.0229	64	-4.1174	0.0101
9	-8.5212	0.0218	65	-4.0277	0.0106
10	-8.5301	0.0184	66	-3.9561	0.0101
11	-8.4813	0.0183	67	-3.8746	0.0102
12	-8.3555	0.0183	68	-3.7855	0.0100
13	-8.1752	0.0171	69	-3.7087	0.0098
14	-7.9257	0.0159	70	-3.6072	0.0114
15	-7.6474	0.0147	71	-3.5345	0.0097
16	-7.2787	0.0122	72	-3.4318	0.0109
17	-7.0807	0.0118	73	-3.3508	0.0104
18	-6.8746	0.0108	74	-3.2768	0.0102
19	-6.8444	0.0092	75	-3.1867	0.0104
20	-6.8413	0.0092	76	-3.1024	0.0098
21	-6.7858	0.0085	77	-3.0169	0.0093
22	-6.7977	0.0092	78	-2.9375	0.0084
23	-6.8003	0.0094	79	-2.8426	0.0085
24	-6.8182	0.0086	80	-2.7243	0.0094
25	-6.8139	0.0091	81	-2.6346	0.0085
26	-6.8009	0.0085	82	-2.5335	0.0084
27	-6.7835	0.0080	83	-2.4337	0.0083
28	-6.7470	0.0090	84	-2.3362	0.0081
29	-6.7560	0.0069	85	-2.2415	0.0075
30	-6.7193	0.0072	86	-2.1394	0.0070

Age(x)	a_x	b_x	Age(x)	a_x	b_x
31	-6.6604	0.0067	87	-2.0441	0.0063
32	-6.6171	0.0069	88	-1.9514	0.0056
33	-6.5628	0.0069	89	-1.8504	0.0049
34	-6.5304	0.0057	90	-1.7537	0.0045
35	-6.4657	0.0058	91	-1.6679	0.0032
36	-6.3990	0.0065	92	-1.5684	0.0029
37	-6.3333	0.0062	93	-1.4743	0.0025
38	-6.2390	0.0066	94	-1.3888	0.0017
39	-6.2111	0.0059	95	-1.3190	0.0014
40	-6.1288	0.0060	96	-1.2380	0.0009
41	-6.0390	0.0063	97	-1.1596	0.0005
42	-5.9556	0.0072	98	-1.0837	0.0001
43	-5.8763	0.0069	99	-1.0106	-0.0003
44	-5.8112	0.0071	100	-0.9403	-0.0006
45	-5.7231	0.0081	101	-0.8730	-0.0009
46	-5.6349	0.0082	102	-0.8087	-0.0012
47	-5.5517	0.0087	103	-0.7476	-0.0015
48	-5.4460	0.0095	104	-0.6895	-0.0017
49	-5.3986	0.0093	105	-0.6347	-0.0018
50	-5.3003	0.0106	106	-0.5830	-0.0020
51	-5.2096	0.0102	107	-0.5345	-0.0021
52	-5.1215	0.0107	108	-0.4891	-0.0022
53	-5.0445	0.0106	109	-0.4467	-0.0023
54	-4.9730	0.0110	110	-0.4073	-0.0023
55	-4.8764	0.0112			

Table 26 Fitted Value of k_t for US Total Data in 1973-2007 of Lee-Carter Model

Year	k_t	Year	k_t	Year	k_t
1973	27.4112	1985	4.3585	1997	-10.0997
1974	23.4356	1986	4.8352	1998	-12.1655
1975	19.9871	1987	4.1479	1999	-12.8992
1976	17.7629	1988	4.2416	2000	-14.3090
1977	16.4107	1989	2.9267	2001	-14.9870
1978	15.4277	1990	0.4539	2002	-15.6697
1979	12.8791	1991	-0.4223	2003	-16.3762
1980	12.6774	1992	-2.9712	2004	-18.5084
1981	9.8404	1993	-1.3752	2005	-18.6728
1982	7.3385	1994	-2.6574	2006	-20.7876
1983	5.5337	1995	-3.7233	2007	-21.8973
1984	4.5248	1996	-6.6710		

Appendix C

In Appendix C, we show the final results of estimators of Lee-Carter Model for US Male data in 1973-2007

Table 27 Fitted Value of a_x and b_x for US Male Data in 1973-2007 of Lee-Carter Model

Age(x)	a_x	b_x	Age(x)	a_x	b_x
0	-4.5328	0.0197	56	-4.5083	0.0125
1	-7.1732	0.0198	57	-4.4272	0.0128
2	-7.5342	0.0207	58	-4.3128	0.0133
3	-7.7901	0.0220	59	-4.2523	0.0130
4	-8.0115	0.0225	60	-4.1458	0.0139
5	-8.1383	0.0226	61	-4.0692	0.0130
6	-8.2338	0.0237	62	-3.9609	0.0133
7	-8.3153	0.0232	63	-3.8928	0.0127
8	-8.3263	0.0218	64	-3.8190	0.0123
9	-8.3812	0.0211	65	-3.7287	0.0129
10	-8.3779	0.0179	66	-3.6548	0.0125
11	-8.3071	0.0176	67	-3.5736	0.0125
12	-8.1649	0.0178	68	-3.4818	0.0124
13	-7.9519	0.0166	69	-3.4031	0.0120
14	-7.6904	0.0157	70	-3.3043	0.0131
15	-7.3846	0.0141	71	-3.2326	0.0116
16	-6.9851	0.0126	72	-3.1349	0.0125
17	-6.7468	0.0119	73	-3.0553	0.0116
18	-6.5043	0.0107	74	-2.9804	0.0115
19	-6.4513	0.0087	75	-2.8945	0.0113
20	-6.4434	0.0088	76	-2.8129	0.0107
21	-6.3840	0.0083	77	-2.7291	0.0102
22	-6.3958	0.0091	78	-2.6566	0.0089
23	-6.4025	0.0095	79	-2.5681	0.0087
24	-6.4287	0.0089	80	-2.4530	0.0095
25	-6.4308	0.0096	81	-2.3673	0.0084
26	-6.4266	0.0090	82	-2.2762	0.0081
27	-6.4147	0.0086	83	-2.1832	0.0078
28	-6.3848	0.0097	84	-2.0950	0.0073
29	-6.4025	0.0078	85	-2.0085	0.0067
30	-6.3693	0.0082	86	-1.9160	0.0059

Age(x)	a_x	b_x	Age(x)	a_x	b_x
31	-6.3232	0.0075	87	-1.8293	0.0053
32	-6.2832	0.0080	88	-1.7445	0.0045
33	-6.2372	0.0077	89	-1.6530	0.0035
34	-6.2130	0.0066	90	-1.5687	0.0030
35	-6.1538	0.0068	91	-1.4910	0.0021
36	-6.0953	0.0075	92	-1.4010	0.0016
37	-6.0338	0.0069	93	-1.3138	0.0013
38	-5.9450	0.0074	94	-1.2393	0.0002
39	-5.9221	0.0069	95	-1.1814	0.0001
40	-5.8456	0.0065	96	-1.1106	-0.0004
41	-5.7553	0.0068	97	-1.0421	-0.0008
42	-5.6767	0.0074	98	-0.9762	-0.0012
43	-5.6013	0.0071	99	-0.9129	-0.0016
44	-5.5389	0.0071	100	-0.8521	-0.0019
45	-5.4477	0.0083	101	-0.7941	-0.0022
46	-5.3601	0.0082	102	-0.7388	-0.0025
47	-5.2759	0.0088	103	-0.6861	-0.0027
48	-5.1721	0.0096	104	-0.6362	-0.0029
49	-5.1246	0.0096	105	-0.5890	-0.0030
50	-5.0240	0.0109	106	-0.5445	-0.0032
51	-4.9336	0.0106	107	-0.5026	-0.0032
52	-4.8431	0.0113	108	-0.4633	-0.0033
53	-4.7660	0.0111	109	-0.4265	-0.0033
54	-4.6912	0.0119	110	-0.3922	-0.0033
55	-4.5901	0.0122			

Table 28 Fitted Value of k_t for US Male Data in 1973-2007 of Lee-Carter Model

Year	k_t	Year	k_t	Year	k_t
1973	28.4857	1985	5.6446	1997	-11.5446
1974	24.5721	1986	6.0721	1998	-13.8005
1975	21.5242	1987	5.6472	1999	-15.2235
1976	19.1551	1988	5.2602	2000	-16.6177
1977	17.9925	1989	3.7547	2001	-17.6591
1978	16.9405	1990	1.6309	2002	-18.1937
1979	14.7565	1991	0.8341	2003	-19.1675
1980	14.2777	1992	-1.5210	2004	-21.5059
1981	11.6922	1993	-0.7317	2005	-21.7618
1982	8.5961	1994	-2.1069	2006	-23.6773
1983	6.8659	1995	-3.6304	2007	-24.8316
1984	5.5503	1996	-7.2794		

Appendix D

In Appendix D, we show the final results of estimators of CBD Model for UK Female data in 1973-2009

Table 29 Fitted Value of $k_t^{(1)}$ and $k_t^{(2)}$ for 1973-2009 UK Female Data of CBD Model

Year (t)	$k_t^{(1)}$	$k_t^{(2)}$	Year (t)	$k_t^{(1)}$	$k_t^{(2)}$
1973	-6.8913	0.0637	1992	-7.3093	0.0685
1974	-6.9140	0.0642	1993	-7.2978	0.0681
1975	-6.9442	0.0655	1994	-7.3589	0.0708
1976	-6.9502	0.0659	1995	-7.3425	0.0701
1977	-6.9877	0.0662	1996	-7.3586	0.0704
1978	-6.9444	0.0647	1997	-7.3694	0.0696
1979	-7.0089	0.0671	1998	-7.3865	0.0700
1980	-7.0271	0.0661	1999	-7.3942	0.0699
1981	-7.0736	0.0669	2000	-7.4306	0.0724
1982	-7.1010	0.0679	2001	-7.4325	0.0698
1983	-7.1252	0.0671	2002	-7.4558	0.0711
1984	-7.1527	0.0676	2003	-7.4495	0.0701
1985	-7.1593	0.0669	2004	-7.5069	0.0721
1986	-7.1894	0.0674	2005	-7.5257	0.0721
1987	-7.2030	0.0680	2006	-7.5193	0.0707
1988	-7.2186	0.0677	2007	-7.5372	0.0716
1989	-7.2170	0.0667	2008	-7.5223	0.0708
1990	-7.2622	0.0682	2009	-7.5483	0.0706
1991	-7.2589	0.0662			

Appendix E

In Appendix E, we show the final results of estimators of CBD Model for UK Total data in 1973-2009

Table 30 Fitted Value of $k_t^{(1)}$ and $k_t^{(2)}$ for 1973-2009 UK Total Data of CBD Model

Year (t)	$k_t^{(1)}$	$k_t^{(2)}$	Year (t)	$k_t^{(1)}$	$k_t^{(2)}$
1973	-6.5754	0.0635	1992	-6.9665	0.0687
1974	-6.6090	0.0645	1993	-6.9620	0.0685
1975	-6.6319	0.0647	1994	-6.9957	0.0694
1976	-6.6399	0.0658	1995	-6.9899	0.0699
1977	-6.6764	0.0664	1996	-7.0137	0.0705
1978	-6.6481	0.0656	1997	-7.0217	0.0696
1979	-6.6834	0.0669	1998	-7.0353	0.0702
1980	-6.7205	0.0669	1999	-7.0520	0.0705
1981	-6.7550	0.0671	2000	-7.0867	0.0717
1982	-6.7816	0.0676	2001	-7.0987	0.0712
1983	-6.8030	0.0671	2002	-7.1092	0.0715
1984	-6.8290	0.0674	2003	-7.1318	0.0719
1985	-6.8399	0.0673	2004	-7.1857	0.0729
1986	-6.8675	0.0682	2005	-7.2110	0.0731
1987	-6.8790	0.0678	2006	-7.1711	0.0714
1988	-6.8778	0.0669	2007	-7.1917	0.0718
1989	-6.8912	0.0663	2008	-7.2027	0.0724
1990	-6.9012	0.0671	2009	-7.2454	0.0737
1991	-6.9119	0.0664			

Appendix F

In Appendix F, we show the final results of estimators of CBD Model for UK Male data in 1973-2009

Table 31 Fitted Value of $k_t^{(1)}$ and $k_t^{(2)}$ for 1973-2009 UK Male Data of CBD Model

Year (t)	$k_t^{(1)}$	$k_t^{(2)}$	Year (t)	$k_t^{(1)}$	$k_t^{(2)}$
1973	-6.3475	0.0645	1992	-6.7412	0.0690
1974	-6.3885	0.0659	1993	-6.7450	0.0692
1975	-6.4076	0.0654	1994	-6.7697	0.0694
1976	-6.4245	0.0664	1995	-6.7646	0.0699
1977	-6.4606	0.0670	1996	-6.7925	0.0708
1978	-6.4431	0.0667	1997	-6.8011	0.0695
1979	-6.4644	0.0676	1998	-6.8155	0.0704
1980	-6.5093	0.0679	1999	-6.8364	0.0711
1981	-6.5372	0.0680	2000	-6.8674	0.0716
1982	-6.5672	0.0683	2001	-6.8911	0.0725
1983	-6.5852	0.0680	2002	-6.8980	0.0724
1984	-6.6125	0.0679	2003	-6.9355	0.0734
1985	-6.6258	0.0681	2004	-6.9766	0.0734
1986	-6.6529	0.0695	2005	-7.0054	0.0737
1987	-6.6630	0.0687	2006	-6.9570	0.0723
1988	-6.6465	0.0670	2007	-6.9746	0.0721
1989	-6.6714	0.0667	2008	-7.0018	0.0739
1990	-6.6683	0.0670	2009	-7.0576	0.0764
1991	-6.6883	0.0671			

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