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Defined Benefit Pension Schemes: A Welfare Analysis of Risk Sharing and Labour Market Distortions

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Abstract

This paper addresses two policy questions with respect to public defined benefit (DB) pension schemes: Firstly, does a funded DB pension scheme increase welfare? In other words: do the gains from intergenerational sharing of capital market risks outweigh the labour market distortions from pension schemes? Secondly, how large is the commitment problem of pension funds after an adverse capital market shock? The answer to the first question depends on the used welfare measure. If we use risk-neutral weights to aggregate the equivalent variations of different generations in different states of nature then a DB pension scheme is welfare increasing. If we use as weights the stochastic discount factors that corresponds to these states of nature, we conclude the opposite: a DB pension scheme reduces welfare. The probability that future households actually experience a welfare gain if the pension scheme is closed can be as large as 38 per cent. So, a pure DB pension scheme has a large commitment problem: continuity will become at risk in case participation in the pension scheme is not mandatory. These results are most sensitive for the values of the labour supply elasticity, the risk aversion parameter and the mean and the standard deviation of the excess return on equity.

Key words: Public pensions; Macro economic risk; Welfare analysis

JEL code: D61, H55

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1 Introduction

Funded pension schemes are vulnerable to capital market risks. Funded pension schemes are recently faced with a stock market crash, large bond rate spreads and highly volatile exchange rates. Moreover inflation may change in the near future due to expansionary monetary policies. The current situation is especially relevant in the Netherlands, which features one of the biggest funded pension schemes in the world.

From that perspective, two policy questions are addressed in this paper: Firstly, does a defined-benefit (DB) pension scheme increase welfare? In other words: do the gains from intergenerational risk sharing outweigh the labour market distortions from pension schemes? Secondly, how large is the commitment problem of pension funds after an adverse capital market shock?

To answer these questions a model is constructed that has the following features. Firstly, the model describes a pension scheme which is part of a decentralized small open economy with overlapping generations of households. The pension scheme is pure defined benefit, *i.e.* negative shocks lead to premium adjustments, but not to benefit adjustments. Households can save by their own apart from mandatory pension savings. Secondly, the model includes equity, which can be traded. So, we focus on equity return risk[1].

These requirements explain our model choice and point to the first main contribution of this paper to the literature: an answer to the above formulated policy questions using a relatively realistic model. Unlike Teulings and de Vries (2006), Bovenberg, Koijen, Nijman, and Teulings (2007)), who analyze potential welfare gains of first best pension schemes, our paper describes a decentralized economy. Unlike Gollier (2008) we investigate the potential welfare gain of pension schemes when households can save by their own, too. Furthermore, our model distinguishes more than two overlapping generations contrary to for instance Bonenkamp and Westerhout (2010). Lastly, our model holds capital market risk exogenous which is realistic for small open economies, but which deviates from closed economy models like that of Krueger and Kubler (2006).

This explains the choice for a partial equilibrium model for a small open economy with overlapping generations and a DB pension fund. Behaviour of the overlapping generations is modelled in line with Bodie, Merton, and Samuelson (1992). This model is used because it brings about a closed form solution for consumption, which is easy to handle in an overlapping generations framework. Consumption depends on total wealth, the sum of financial, human and pension wealth. As demographic developments are more easy to handle in discrete time, our model reformulates the Bodie et al. (1992) model in discrete time.

A DB pension scheme organizes intergenerational risk sharing. A DB scheme transforms capital market risk of retirees into net labour income risks of workers. Indeed, equity return shocks lead to pension premium changes and make net labour income stochastic even when gross labour income (productivity) is non-stochastic.

The second contribution to the literature is the derivation of the exogenous discrete time

[1]Other macroeconomic tradable risks can easily be handled in our model.

stochastic discount factor which can be used to calculate the current value of uncertain future income. The derived expression is the discrete time equivalent of the stochastic discount factor which can be used to calculate for instance the Black Scholes option formula (see Cochrane (2005)). Because human wealth is uncertain after the introduction of a DB pension scheme, expectations have to be formed conditional on the state of the economy. Parameterized methods are used as in Judd, Maliar, and Maliar (2009) to obtain these conditional expectations.

Unlike Bodie et al. (1992) and Gomes, Kotlikoff, and Viceira (2008) but in line with Mehlkopf (2010), we adopt a specification in which labour supply is not driven by a wealth effect. Our motivation is that wealth effects are usually found to be small when compared to substitution effects (Lumsdaine and Mitchell (1999)), implying that labour supply is relatively unresponsive to changes in financial wealth. Hence, labour flexibility cannot play a role in absorbing capital market shocks and the impact that Bodie et al. (1992) and Gomes et al. (2008) find of labour flexibility on portfolio composition does not arise in our model.

The paper obtains the following conclusions. If we use risk-neutral weights to aggregate the equivalent variations of different generations in different states of nature then a DB pension scheme is welfare increasing. If we use as weights the stochastic discount factors that corresponds to these states of nature, we conclude the opposite: a DB pension scheme reduces welfare. The difference is due to the large uncertainty about future states. The probability that future households actually experience a welfare gain if the pension scheme is closed can be as large as 38 per cent.[2] So, a pure DB pension scheme has a large commitment problem: continuity will become at risk in case participation in the pension scheme is not mandatory. So, our welfare analysis reveals that the first best welfare gain, pointed to by Teulings and de Vries (2006), Bovenberg, Koijen, Nijman, and Teulings (2007) and Gollier (2008), is difficult to achieve in practice. Moreover, our analysis emphasizes the uncertainty of the welfare gain by presenting its distribution and it indicates that abolishing the pension scheme implies large distributional effects between generations. Our results are most sensitive for the values of the labour supply elasticity, the risk aversion parameter and the mean and the standard deviation of the excess return on equity.

The structure of our paper is as follows. The next section sets up our model. Then, we describe various aspects of the life cycle behaviour of households in the baseline. Subsequently, we report the effects of closing the pension scheme. Additional simulations decompose the overall welfare effect of closing the pension scheme into the effects of its constituent elements. We focus on the effects on consumption, labour supply and welfare. We end with some concluding remarks.

[2]Westerhout (2011) explores whether the risk sharing in the pension scheme can be adjusted such as to eliminate this discontinuity risk.

2 The model

2.1 Features model

The model distinguishes markets for goods, labour, bonds and equity and identifies as agents households and a pension fund.

Households decide on private savings (consumption), their portfolio and labour supply (leisure). Households are rational and have rational expectations. They optimize in a consistent microeconomic framework. These features allow for welfare analysis of policy reforms. Households are obliged to participate in the pension funds.

Pension funds decide on pension contributions. Indeed, the pension scheme is of the DB type: annual pension benefits relate to the individual's labour history, but are unrelated to both capital market rates of return and to life expectancy. Shocks to the pension wealth are absorbed by the contributions that the pension fund levies upon working cohorts. The pension contribution rate is uniform, *i.e.* there is no premium differentiation because of age.

The following features are attached to the economy. First, the economy is small relative to the outside world. That is, domestic policies do not affect the interest rate and equity rate of return, which are determined on world capital markets. Second, goods supply and labour demand are perfectly elastic: prices are given. Third, lifetime uncertainty is recognized, but perfect capital markets enable households to insure against longevity risk. Macroeconomic longevity risk is not taken into account. Fourth, the model is stochastic because of uncertain equity returns.

2.2 Equity return

The (gross) return on the risky asset, $\widetilde{R}_s$, follows a lognormal white noise process. The excess return, $\widetilde{e}_s$, on the risky asset is defined as

$$\widetilde{e}_s(t) = \widetilde{R}_s(t) - \widetilde{R}_b \ . \tag{1}$$

In equation (1), index b points to the risk free asset (bonds) and s to the risky asset (equity). The expected value of the excess return on equity, $E[\widetilde{e}_s]$, will be denoted as μ_s; its variance, $E[(\widetilde{e}_s - E[\widetilde{e}_s])^2]$, will be denoted as σ_s^2.

2.3 Population

Households enter the economy at the age of 20 and may work up to the age of 65. From that age onwards, they receive a pension until they die. The time of death is uncertain, but occurs at the age of 100 or before with certainty. We work with periods of five years, so we define the working phase of the life cycle to consist of 9 periods, the retirement phase to consist of 7 periods and the life cycle over which households make decisions to consist of 16 periods. The maximum attainable age, j_e, is 20 periods. Population at the cohort level develops according to $n(t,j) = \zeta(t,j)n(t-1,j-1)$, in which $\zeta(t,j)$ denotes the conditional

(upon being alive at age j at the start of a year t) survival probability. Aggregate variables are obtained by aggregating over cohorts $x(t) = \sum_j n(t,j)x(t,j)$. In the following sections we make use of these aggregation rules without presenting them explicitly. For convenience we will suppress the time subscript in the description of the household model because time and age are one to one related for each individual household.

2.4 Households

2.4.1 Assumptions

An individual of age j maximizes his expected remaining lifetime utility, U, which depends on per-period utility u and on a discount factor. Expectations have to be formed because future consumption and the length of life are uncertain. Lifetime uncertainty is taken into account by assuming that individuals weigh their future per-period utility with survival probabilities. The lifetime utility function reads as

$$U(j) = E_j \sum_{i=j}^{j_e} u(i) \prod_{l=j+1}^{i} \delta(l)^{-1} \ . \tag{2}$$

In this equation E_j denotes the expectations operator, $\delta(l) = \widetilde{\delta}/\zeta(l)$ the per period discount factor, $\widetilde{\delta}$ the time preference factor and $\zeta(l)$ the survival probability.[3] Per-period utility, u, is a function of the consumption of commodities, c, and leisure, v:

$$\begin{aligned} u(i) &= \frac{1}{1-\gamma}\left(c(i) + \alpha_v \frac{v(i)^{1-\beta}}{1-\beta}\right)^{1-\gamma} \quad \text{and} \\ c_l(i) &= -\alpha_v \frac{v(i)^{1-\beta}}{1-\beta} \ , \\ \alpha_v &> 0 \ , \ \beta > 1, \ \gamma > 0 \ . \end{aligned} \tag{3}$$

In this equation $1/\gamma$ is the elasticity of intertemporal substitution (γ is the risk aversion), $1/\beta$ the price elasticity of leisure demand and α_v the utility weight of leisure. The marginal per-period utility of leisure (u_v: the derivative of u with respect to v) becomes infinite as leisure approaches zero. This guarantees positive leisure. The restriction that leisure must be equal to or smaller than the maximum available time (normalized to unity) has to be explicitly enforced, however.

The specification of per-period utility in equation (3), linear in commodity consumption and concave in leisure, was proposed by Greenwood, Hercowitz, and Huffman (1988). The marginal rate of substitution between leisure and consumption (u_v/u_c) does not depend on consumption in this specification. This implies that leisure demand can be determined independent of consumption demand. The implication is that labour supply is unresponsive to changes in financial wealth.

[3]Note, we use as convention $\prod_{l=j+1}^{j} \delta(l) = 1$

This specification of the per-period utility function implies that the consumption of commodities has a minimum that is strictly positive $c(i) > c_l(i) > 0$. Indeed, the marginal utility of the per-period commodity consumption (u_c) becomes infinite as the commodity consumption approaches this minimum level. The positive leisure consumption guarantees a positive minimum commodity consumption as long as the price elasticity of leisure demand is smaller than one ($\beta > 1$). This minimum amount of consumption is age-dependent and decreasing in leisure time. Because of the latter, let us call this labour-induced consumption and denote it as c_l. Let us call $c - c_l$ above labour induced consumption.[4] This implies that households purchase first per-period minimum, labour induced, consumption. The rest of their wealth is then allocated to above labour induced consumption levels.

The intertemporal utility specification implies precautionary saving as the third derivative of the felicity function with respect to consumption is positive. Eeckhoudt and Schlesinger (2008) provides necessary and sufficient conditions on preferences such that changes in risk lead to increases in saving.[5]

The asset accumulation equation describes the development of household financial wealth, w_f^h, through time:

$$w_f^h(i+1) = R_b(i+1)\left(w_f^h(i) + y(i) - c(i)\right) + e_s(i+1)w_s^h(i)\ . \tag{4}$$

Equation (4) signals that households receive non-capital income y, consume c and invest their savings in bonds and equity. Riskless bonds earn a yearly return R_b and equity earns an annual return R_s (with an excess return e_s). w_s^h denotes the household's investment in risky equity. Regarding the timing of transactions, we use as convention that all variables (transactions, demographic changes, stocks) are measured at the start of a period.

The effective rates of return on bonds and equity depend on the household's mortality rate. Hence, the effective rates of return are age-dependent. The reason why is the perfect life insurance market assumption of our model. Pension funds insure households against longevity risk. Households receive an annuity return on their private savings that reflects their mortality risk (Yaari (1965)). As mortality rates are allowed to differ by age, the annuity return will be age-dependent. More precisely, the wealth of the members of a cohort who die,$(1 - \zeta(i))\, w_f^h(i)$, is transferred to the people of the same cohort who survive. This makes the effective rate of return $R_m = \widetilde{R}_m/\zeta$ with $m = b, s$ and the effective excess return $e_s = \widetilde{e}_s/\zeta$. Hence, it is R_b and R_s (and e_s) that appear in our household model, rather than their equivalent variables $\widetilde{R}_b$, $\widetilde{R}_s$ (and $\widetilde{e}_s$).[6]

Non-capital income equals labour income y_w in the working phase of the life cycle, $5 \leq i < j_r$ (where j_r =13 denotes the retirement age) and pension income y_p in the

[4]So, substitution of optimal leisure demand into the utility function leads to a generalized Stone Geary utility function (Stone (1954)) The original Stone Geary utility is based on a Cobb-Douglas form, *i.e.* $\gamma = 1$.

[5]Draper (2008) investigates whether robust control can be used as alternative explanation for precautionary saving. This alternative leads to nearly the same results. Although it has numerical advantages, we prefer here the mainstream approach.

[6]The effective rates play only a role in the household model and not in the pension model.

retirement phase $i \geq j_r$.

$$y(i) = \begin{cases} y_w(i) & \text{for } 5 \leq i < j_r \ , \\ y_p(i) & \text{for } j_r \leq i \leq j_e \ . \end{cases} \tag{5}$$

Labour income depends on working time, the wage rate p_l and the pension premium rate τ_p

$$y_w(i) = (1 - \tau_p(i))\,(1 - v(i))\,p_l(i) \ . \tag{6}$$

Working time is expressed as $1-v$, indicating that we have normalized the time endowment to unity. Future net wage income is uncertain due to uncertain future pension premiums and working time. These uncertainty is due to the pension fund investments in equity which have uncertain returns. This may lead to insufficient funding of the pension rights which induces the pension fund to increase the premium rate. This leads to labour market distortions because it charges labour income without generating any return for labour.

Pension income at the start of the retirement phase is determined on the basis of pension rights that are accumulated during the working phase and indexed with respect to productivity growth during the working period. During the retirement phase pension income increases proportionally with labour productivity

$$y_p(i+1) = \begin{cases} 0 & \text{for } i = 3 \ , \\ \rho\left[y_p(i) + a\,(1 - v(i))\,p_l(i)\right] & \text{for } 4 \leq i < j_r - 1 \ , \\ \rho y_p(i) & \text{for } j_r - 1 \leq i \leq j_e \ , \end{cases} \tag{7}$$

with a the accrual rate and ρ the productivity growth factor. Pension income depends on the labour market history of households. Pension income is non-stochastic.

The household's problem is to maximize expected intertemporal utility (2), subject to the asset accumulation equation (4), his initial amount of financial wealth, w_f^h, and a Kuhn-Tucker condition that ensures that leisure does not exceed the time endowment of the household. Instruments of this optimization problem are the consumption of goods and equity investments in all years of the life cycle and the consumption of leisure in all years of the working phase. The complete solution to this optimization problem can be found in appendix A. Here we only state the first order conditions and rewrite them into equations for equity demand, leisure demand and the consumption of goods.

As explained above, the leisure decision is independent of other decisions in this model (consumption of goods and portfolio decision) and will be discussed first.

2.4.2 Leisure demand

Optimal behaviour implies that the per-period marginal rate of substitution between commodity and leisure consumption (i.e. the per-period utility ratio u_c/u_v) equals the price ratio $(1/\widetilde{p}_v)$. The marginal rate of substitution does not depend on commodity consumption by assumption. So we get the leisure demand relation out of this optimum condition

$$v(i) = \left(\frac{1}{\alpha_v} \widetilde{p}_v(i) \right)^{-\frac{1}{\beta}} \ . \tag{8}$$

where the shadow price of leisure, $\widetilde{p}_v$, is defined as the maximum of the actual price of leisure, p_v, and the utility weight α_v.

$$\widetilde{p}_v(i) = \max \{ \alpha_v, p_v(i) \} \ . \tag{9}$$

This ensures that leisure time does not exceed the time endowment of the household. The price of leisure is the marginal reward of supplying labour, taking into account not just the net wage rate but also future pension income to the extent that it can be imputed to current labour. It will be defined below precisely. Two aspects of leisure demand deserve attention. First, due to our per-period utility function, leisure demand does not depend on the household's financial or total wealth position. This accords with empirical evidence (Lumsdaine and Mitchell (1999)). Second, a Kuhn-Tucker condition ensures that leisure demand does not exceed unity.[7]

The price of leisure consists of three components:

$$p_v(i) = p_l(i) - \tau_p(i) p_l(i) + p_r(i) \ . \tag{10}$$

The first is the wage rate p_l and the second the pension contributions which are proportional to the wage rate. The third component measures the discounted value of future pension income that can be attributed to the marginal hour of work, p_r

$$p_r(i) = a p_l(i) \sum_{h=j_r}^{j_e} \left(\prod_{l=i}^{h-1} \frac{\rho}{R_b(l+1)} \right) \ . \tag{11}$$

This component is also proportional to the wage rate.

A decomposition of the pension premium rate in several components is convenient to explain how the pension scheme interferes with labour supply. In a well-defined special case, the premium rate would equal the build-up rate of new pension rights, $p_r(i) \div p_l(i)$. The price of leisure would then coincide with the wage rate: $p_v(i) = p_l(i)$. This special case involves two conditions: that the pension fund holds zero equity and also that the premium policy is actuarially fair. Both conditions are not met in our model, but the case is interesting as a benchmark.

In our model, the pension fund invests in equity. This changes the premium rate for two reasons. First, the equity premium (the mean excess return on equity) allows a premium reduction. The section on pension fund policies below will document in detail how this component of the premium rate relates to the fund's equity position. Second, shocks in

[7] If labour productivity is below α_v, our model predicts zero labour supply. This shows that households will exit from the labour market in our model not only when labour productivity becomes sufficiently low, but also when the preference for leisure becomes sufficiently high.

the rate of return on equity (defined as realizations in deviation from the mean) imply a positive or negative catching-up component in the premium rate. The section on pension fund policies below will document in detail how this component of the premium rate relates to the fund's financial wealth.

In addition, the pension fund in our model does not levy premiums that are actuarially fair, but levies premiums according to the uniformity principle. This means that people of different age face the same premium rate, although their build-up rates are different. This uniform premium is in fact a tax for young employees and a subsidy for old employees because future pension income that can be attributed to the marginal hour of work, p_r, increases with age. In the section on pension fund policies we will discuss this component in more detail.

2.4.3 The portfolio and asset valuation

Dynamic programming methods are used in appendix A to solve the intertemporal consumption and portfolio choice of households. Here we give a more intuitive presentation.

Households are exposed to equity return risk along different channels. The first, most obvious direct channel is through their own equity investments. The second way is indirectly through the pension fund equity investments. Indeed, equity return shocks are absorbed in the pension premium rate, which changes net income of households. The catching up part of the pension premium has moreover labour supply effects, which is a third channel through which households are exposed to equity return risk.

Households optimize their total equity risk exposure each period which is possible due to the salability of equity. The portfolio share ω_s of the total direct and indirect equity investments in total household wealth (to be defined below) is determined by the following first-order condition

$$0 = E\left(1 + \omega_s(i)e_s(i)\right)^{-\gamma} e_s(i) \ , \tag{12}$$

which is a standard discrete time portfolio equation (known since the publication of Merton (1969) and Samuelson (1969)). Note that ω_s is not age-specific because the distribution of the excess return is invariant over time and thus the same for all ages. So, we will leave out the age subscript onwards.[8] The total risk exposure is a determinant of the development of total wealth. The direct investments in equity are determined out of the development of total wealth and its components.

A second-order approximation of equation (12) leads to the explicit equation $\omega_s = \mu_s \div (\gamma\sigma_s^2)$ which has an intuitive interpretation. The equity investments increase proportionally with the mean of the equity premium μ_s and decrease proportionally with both the variance σ_s^2 and risk aversion parameter γ.

A further interpretation of equation (12) is possible. Income received in different periods may have different utility. Weighting per-period income with marginal utility leads to

[8]Implicit equation 12 is solved with respect to ω_s using the Gauss-Hermite quadrature in the simulation excercise.

comparability. For instance, investing now one unit in bonds implies R_b income next period. Optimal saving behaviour of households imply that the marginal utility of one unit less consumption now equals the expected value of the marginal utility of R_b more consumption tomorrow, *i.e.* $1 = EmR_b$ with m the marginal utility ratio. The same holds for investments in equity: the marginal utility of one unit less consumption has to be equal to the expected value of the marginal utility of the equity return, *i.e.* $1 = EmR_s$. These relations for equity and bonds imply that the discounted value of the excess return on equity is zero $0 = Eme_s$. Comparison with equation (12) reveals that factor $(1 + \omega_s e_s)^{-\gamma}$ has to be proportional to m, the marginal utility ratio of consumption. The proportionality factor can be derived from the bond rate equation $1 = R_b Em$.

The marginal utility ratio is the stochastic discount factor in this consumption based model. The above reasoning shows that only exogenous variables (m is explained by only exogenous variables after substitution of the equation (12) solution with respect to ω_s) determine the marginal utility ratio

$$m(i) = \frac{1}{R_b} \frac{(1 + \omega_s e_s(i))^{-\gamma}}{E\,(1 + \omega_s e_s)^{-\gamma}} \ . \tag{13}$$

Using m implies that certain returns are discounted by the bond rate, because the last term disappears after taking expectations. However, uncertain returns are discounted with a correction which depends on the covariance with the excess return.[9] So, we make use of the exogenous character of the stochastic discount factor in our model. The advantage is that we remain in the Bodie et al. (1992) model set-up, *i.e.* a closed form solution of the model is possible.

The budget constraint of households can now be determined by solving the financial wealth equation (4) forward using the stochastic discount factor. So, we take advantage of the properties of the stochastic discount factor under complete markets, which makes the determination of the market value of assets possible without solving the whole household model.

As explained before, the household decision-making can be interpreted as consisting of two steps. In the first step, households purchase labour induced consumption in all years of their life cycle. In the second step, they allocate the rest of their wealth to above labour induced consumption levels in all years of the life cycle. Therefore, the budget constraint is written as[10]

$$w^h(i) \equiv w_f^h(i) + w_p^h(i) + w_n^h(i) = E_i \sum_{h=i}^{j_e} [c(h) - c_l(h))] \prod_{l=i+1}^{h} m(l) \ . \tag{14}$$

[9]The stochastic discount factor $m(i) = \frac{1}{R_b} - \frac{1}{R_b}\frac{\mu_s}{\sigma_s^2}(e_s(i) - \mu_s)$, proposed by Hansen and Joganathan (Cochrane (2005), page 73) is a linearized version of equation (13). Note, utility parameter γ disappears in this linearized version. The linearized version of the stochastic discount can not be used for large shocks. Indeed, the stochastic discount factor has to be proportional to the marginal utility, which implies positivity. Positive values are not guaranteed in this linearized version, however. The complete market assumption implies an equal value of the stochastic discount factor for all agents because the risky asset prices are the same for all agents. In our model all agents have the same risk aversion.

[10]Note, we use as convention $\prod_{l=i+1}^{i} m(l) = 1$

In this equation, $w^h(i)$ denotes total wealth, corrected for the part of wealth that is needed to finance labour induced consumption. Similarly, the right hand side of equation (14) corrects consumption flows for the parts that are labour induced. Total wealth is defined as the sum of financial wealth w_f^h, the accumulated pension rights

$$w_p^h(i) = R_b(i)\left(w_p^h(i-1) + (1-v(i-1))p_r(i-1)\right) \ , \tag{15}$$

and human wealth, *i.e.* the expected discounted value of future labour income,[11] diminished with labour induced consumption

$$w_n^h(i) = E_i \sum_{h=i}^{j_e} \left[(1-v(h))\, p_v(h) - c_l(h))\right] \prod_{l=i+1}^{h} m(l) \ . \tag{16}$$

The market value of total wealth is determined in equation (14) using the stochastic discount factor approach. Alternatively, a replicating portfolio of equity and bonds could be used to determine this market value. For each asset a replicating portfolio exists. The sum of the equity investments in the replicating portfolios w_s is the total risk exposure, which implies for the development of total wealth over time.

$$w^h(j+1) = R_b(j+1)\left[w^h(j) - x_f(j)\right] + e_s(j+1)w_s(j) \ . \tag{17}$$

The total (direct plus indirect) equity investments is determined by the portfolio share and total wealth diminished with current above labour induced consumption

$$w_s(i) = \omega_s R_b(i+1)\left[w^h(i) - (c(i) - c_l(i))\right] \ . \tag{18}$$

2.4.4 Consumption of goods

Households decide how much to consume above labour-induced consumption. Given the optimal level of leisure, consumption is given by

$$c(i) = c_l(i) + \left(\frac{1}{p_f(i)}\right)^{\frac{\gamma-1}{\gamma}} w^h(i) \ , \tag{19}$$

with

$$p_f(i) = \left[\sum_{h=i}^{j_e-1}\prod_{l=i}^{h-1}\left(\frac{\varphi(l+1)}{\delta(l+1)}\right)^{\frac{1}{\gamma}}\frac{1}{\varphi(l+1)}\right]^{\frac{\gamma}{\gamma-1}} \ ,$$

and the equation for the certainty-equivalent return defined as follows

$$\varphi(i) = \left[E\left(1+\omega_s e_s\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} R_b(i) \ .$$

[11] Human wealth has not to be confused with human capital, which is the stock of personal attributes which lead to the ability to produce economic value. The accumulation of human capital is not modelled as for instance in the Ben-Porath (1967) model.

Without leisure, consumption would be proportional to total wealth as in the standard life cycle model. The life cycle pattern of commodities consumption deviates from the pattern of this standard model, due to the interaction with leisure demand. In particular, the household consumes more (fewer) commodities than prescribed by the standard model in years in which his labour supply is relatively high (low). Our felicity specification thus brings about a positive correlation between consumption and labour supply and, given that labour supply is increasing with the wage rate. Hence, consumption and current income are more strongly correlated than in the standard life cycle model, which may help to solve the excess sensitivity of consumption puzzle (Flavin (1981)).

The price index of total wealth, p_f, is a weighted sum of the number of years households may live. As in the standard life cycle model, the weighting factors refer to two effects. A rate of return higher than the rate of time preference increases savings on account of the substitution effect. The second element of the weighting factor describes the income effect of returns on investments. A high rate of return also adds to consumption possibilities, the income effect. If the intertemporal elasticity of substitution is below unity ($1/\gamma < 1$), the income effect dominates the substitution effect.

Different from the standard life cycle model is the return φ. It is increasing in R_b, in the excess return on equity and in the variance of equity.

Substitution of the optimal consumption equation (19) into the direct utility function leads to the value function

$$V(i) = \frac{1}{1-\gamma}\left(\frac{w^h(i)}{p_f(i)}\right)^{1-\gamma} , \tag{20}$$

which will be used for welfare analysis.

2.5 Pension funds

Pension funds have at the start of a period financial wealth w_f^p, they receive premium income $\tau_p y_{wg}$ from workers ($j < j_r$), pay benefits y_p to retirees and invest in bonds and w_s^p in equity. We do not try to explain the portfolio allocation of pension funds from optimizing behaviour. Rather, we equate this portfolio allocation to the aggregate portfolio allocation of households before the pension funds were introduced. The assets accumulate over time according to

$$w_f^p(t+1) = \widetilde{R}_b(t+1)\left[w_f^p(t) + \tau_p(t)y_{wg}(t) - y_p(t)\right] + \widetilde{e}_s(t)w_s^p(t) , \tag{21}$$

in which the macro variables are obtained by aggregation over the age cohorts. In this equation y_{wg} ($= \sum_j n(t,j)(1-v(t,j))p_l(t,j)$) is gross wage income. The pension benefits for ($j \geq j_r$) are given in pure DB: shocks are absorbed in the current and future premium rates.

The pension fund adjusts gradually its financial wealth towards the accumulated pension rights after equity return shocks using the premium instrument τ_p, which consists of three elements

$$\tau_p(t) = \tau_p^d(t) + \tau_p^c(t) + \tau_p^s(t) . \tag{22}$$

In this equation τ_p^d is a uniform premium to cover the new pension rights. The second element is the catching up premium, τ_p^c, which is used when the funding ratio is below its required level of 100%. The third element is the premium reduction, τ_p^s, due to the expected excess return on equity investments. To define these elements more precisely, note that pension rights[12] evolve according to

$$w_p^h(t+1) = \widetilde{R}_b(t+1)\left[w_p^h(t) - y_p(t) + w_{pn}^h(t)\right] \ , \tag{23}$$

at the aggregate level with $w_{pn}^h(t)$ $(= \sum_j n(t,j)(1 - v(t,j)p_r(t,j))$ the new pension right formed in period t. To determine the uniform premium the new pension rights are charged over the total contribution base $\tau_p^d(t) = w_{pn}^h(t) \div y_{wg}(t)$. The catching up premium is determined by charging a fraction μ of the funding deficit over the total contribution base. This simple premium rule is adjusted by giving the catching up part of the premium an upper bound of 0.5 and a lower bound of -1, leading to $\tau_p^c(t) = \max\left(-1, \min\left(\mu\left(w_p^h(t) - w_f^p(t)\right) \div y_{wg}(t), 0.5\right)\right)$. The catching up premium is the main source for labour market distortions because it charges labour income without generating any return for labour. A premium reduction can be given which is determined by charging the expected excess return on equity investments over the total contribution base $\tau_p^s(t) = -\mu_s(t) w_s^p(t) \div (\widetilde{R}_b(t+1) y_{wg}(t))$.

2.6 Welfare analysis

To evaluate welfare changes due to policy changes we use equivalent variations ev, which are calculated using equation (20). Equivalent variations give the same utility change as the considered policy change, *i.e.*

$$\begin{aligned} &\frac{1}{p_f^0(i)}\left(w^{h0}(i) + ev(i)\right) = \frac{1}{p_f^1(i)} w^{h1}(i) \quad \text{or} \\ &ev(i) = \left(\left[\frac{V^1(i)}{V^0(i)}\right]^{\frac{1}{1-\gamma}} - 1\right) w^{h0}(i) \ , \end{aligned} \tag{24}$$

in which V^0 is utility before the policy change, V^1 utility after the policy change and $w^{h0}(i)$ is households wealth before the policy change. The advantage of using this measure rather than simply calculating the utility changes lies in the obvious way that equivalent variations can be aggregated across generations that live in different periods of time and across states.

With equivalent variations three different welfare concepts can be defined: ex-post, interim and ex-ante welfare (Brunnermeier (2001) and Demange and Laroque (1999)). *Ex-post welfare* evaluates welfare for current and future generations given the true state at that moment, *i.e.* using $ev(t,i)$. For the future generations ($t > 0$) ex-post welfare will be presented at the start of their working life, *i.e.* $ev(t,5)$. *Interim welfare* evaluates

[12] Total pension rights aggregates pension rights of different cohorts: $w_p^h(t) = \sum_j n(t,j) w_p^h(t,j)$. Variable $w_p^h(t,j)$ equals $w_p^h(j)$ in equation (15) at time t.

welfare for future generations ($t > 0$) at the start of their working life ($i = 5$) at time, $\tau \in \{1, \ldots, t-1\}$, using $\prod_{l=\tau}^{t-1} m(l+1) ev(t,5)$, in which m points to the stochastic discount factor. Stochastic discounting implies other discount factors for good and bad states due to risk aversion. Interim welfare gives thus insight into the effects of risk aversion. *Ex-ante welfare* evaluates welfare at date $\tau = 0$ that the reform takes place using $\prod_{l=0}^{t-1} m(l+1) ev(t,5)$ for future generations and $ev(0,5)$ for current generations. The ex-ante welfare indicator thus equals the ex-post welfare indicator for current generations.

These welfare concepts differ according to the dates at which welfare is evaluated and the information available at those dates. Ex-ante welfare evaluates welfare for all current and future cohorts, *i.e.* it includes the transition generations. The distribution of the ex-post equivalent variations quantifies the commitment problem. The difference between distribution of the interim and ex-post equivalent variations presents the effects of risk aversion.

To evaluate the overall ex-ante welfare effect of a policy measure we take as stand that insurance between good and bad states is possible. The expected ex-ante welfare measure, $E_o \prod_{l=0}^{t-1} m(l+1) ev(t,5)$, is than the relevant welfare measure. To determine whether a policy is ex-ante *potentially Pareto efficient* (welfare improving), it suffices to aggregate the equivalent variations of all current and future cohorts and over different states of the economy $\sum_i n(0,i) ev(0,i) + \sum_t E_0 \prod_{l=0}^{t-1} m(l+1) n(t,5) ev(t,5)$, in which E_0 points to expectations from date 0 at which the reform takes place.[13] Gollier (2008) considers ex-ante Pareto-efficiency the most relevant concept for measuring the efficiency of intergenerational risk sharing.

3 Calibration and expectations

This section discusses the values for the exogenous variables and the model parameters. Moreover, it discusses the method used to generate conditional expectations.

3.1 Calibration

The population cohorts 1 up to 15 are all of equal size, n, which is normalized to 10. The cohort size aged 16 and over declines with 2 per cohort, so cohort 16 has size 8, cohort 17 size 6 and so on, reflecting an increasing death probability. The conditional (upon being alive at age j at the start of a year t) survival probability $\zeta(j)$ is assumed to be constant over time. Total available time a year is scaled to one and utility parameter α_v is used to calibrate leisure time v at 0.5 during the working ages in between age 20 and 65. Then gross wage rate p_l is scaled to 2. The risk-free rate $\left(\widetilde{R}_b - 1\right)$ takes a value of 0.10 (which is

[13] It is also possible to use certainty-equivalent wealth of future cohorts before $W_0^{h0}(t,5) \equiv \left[E_0 \prod_{l=0}^{t-1} m(l+1) w_t^{h0}(t,5)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$ and after the policy change $W_0^{h1}(t,5) \equiv \left[E_0 \prod_{l=0}^{t-1} m(l+1) w_t^{h1}(t,5)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$ as starting point for welfare analysis (Gollier (2008)). This measure can also be defined for current cohorts.

0.02 a year). This value is equal to the post-war EU average of about 2% for government bonds (Broer (2010)).The productivity growth rate is set at $(\rho - 1) = 0.085$, (0.017 a year) which is about the same as the long run estimate for the US presented in Broer (2010) to which the EU converges in the long-run. The equity premium is set to $\mu_s = 0.03$ (a year) and the standard deviation of the excess return on equity to $\sigma_s = 0.15$ (a year). The value of the expected excess return is a "reasonable" lower bound for the equity premium, based on fundamentals (Broer (2010)) and in between the benchmark values derived in the meta study of van Ewijk, de Groot, and Santing (2010). The standard deviation is about equal to the standard deviation of US stocks over the last 50 years (Cochrane (2005)).

Table 1: Model parameters

μ=0.5	β=3	γ=3	$-1<\tau_c<0.5$	a=.50/45	μ_s=0.03	σ_s=.15

The intertemporal substitution parameter is fixed at 0.33 ($\gamma = 3$). This value is within the range obtained by Epstein and Zin (1991). The rate of time preference $(\widetilde{\delta} - 1)$ is calibrated such that the individual consumption growth rate equals the productivity growth rate. The price elasticity of leisure equals $-1/\beta = -1/3$ which corresponds fairly well with the results from the meta-analysis of Evers, de Mooij, and van Vuuren (2005).

The pension fund charges a fraction $\mu = 0.5$ of the funding deficit over the total contribution base. In the simulations with a pension fund the accrual rate $a = .50/45$ a year, *i.e.* the replacement rate, the ratio between pension income and labour income, is 50 per cent, which can be built up in the 45 years of working life. Pension funds equate the portfolio composition of their wealth to the portfolio share of the aggregate of households for the economy without pension funds. This boils down to a portfolio share of equity of about 68 per cent.

3.2 Parametrized expectations

The consumption decision depends on the expected discounted value of labour income diminished with labour-induced consumption:

$$w_n^h(i) = E_i \sum_{h=i}^{j_e} \left[(1 - v(h))\, p_v(h) - c_l(h)\right] \prod_{l=i}^{h-1} m(l+1) \equiv E_i x(i) \ . \tag{25}$$

Households have expectations conditional on the state of the economy.[14] These expectations depend on the state of the economy only and are time invariant. Rational expectations imply that the agents understand the working of the economy, i.e. they have a very good model to predict the future, given the state of the economy. To get a good model of the working of the economy households use Kalman filters, *i.e.* households project x on the state of the economy at time i. This method is known as parameterized expectations in

[14]There is no instantaneous perfect correlation between labour income and capital market risk which makes it impossible to derive an analytical expression for the expected value.

the literature (see Heer and Maussner (2005) and Judd et al. (2009)). The number of state variables relevant for expectation formation is small through writing the model in this 'nearly' closed form which is convenient for simulation purposes. With a stable population the funding ratio of pension funds, $q = w_p^h/w_f^p$ is the most obvious relevant statistic for the state of the economy. For instance assume as projection the linear regression model $x = \alpha_0 + \alpha_1 q + \varepsilon$ in which α the regression coefficient and ε the error term. After recursive estimation, using the Kalman filter, the expectations can be generated as $Ex = \alpha_0 + \alpha_1 q$. This simple linear regression model is obvious too simple. So, we used a fifth-order Hermite polynomials of this state variable as Judd et al. (2009) advocates. The model uses rational expectations, *i.e.* the maximum likelihood estimators of the coefficients, which are obtained after learning the coefficients in a first estimation round in which is assumed that households learn from younger cohorts and from other simulated paths. Appendix B details on this subject.

4 Stochastic Simulations

This section addresses the two policy questions of this paper:

- Does a small open economy generate, under capital market risk, with a defined-benefit (DB) pension scheme more welfare than without a pension scheme?
- How large is the commitment problem of pension funds after capital market shocks?

This section simulates the effects of closing a pension fund. The state of the economy is fixed at the closing moment of the pension fund. The pension fund is fully funded at that moment. Households obtain a transfer equal to their pension rights. The economy with a pension fund is the situation before the policy change at calculating the equivalent variations.

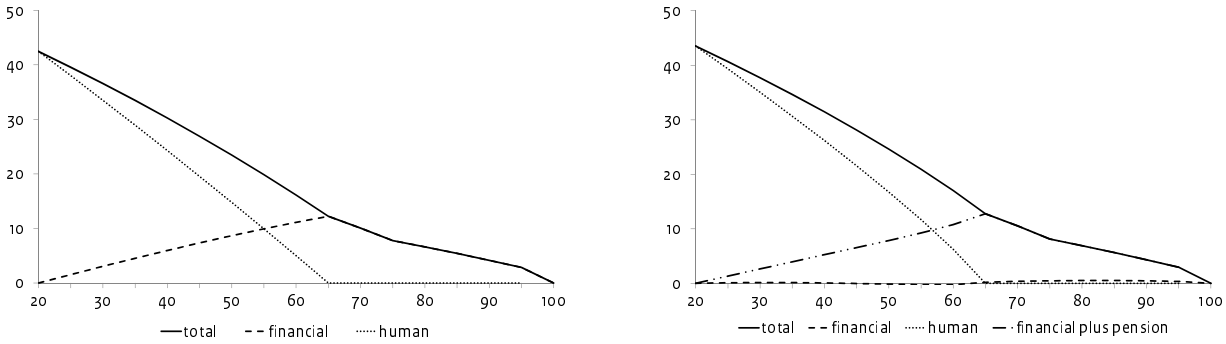
We start with comparing the unconditional expected developments of consumption, income and wealth in the steady state. This analysis gives a first indication of the potential welfare effects because welfare is determined by consumption and leisure. The consumption effects are split up into intergenerational risk sharing effects and effects of labour market distortions. Next, the ex-post distribution of the equivalent variations is presented, which quantifies the commitment problem of pension funds. Lastly the expected ex-ante welfare is presented which indicates whether closing the pension fund is a potential change for the worse in Pareto efficiency.

The ex-ante welfare calculations are based on 10,000 different stochastic paths (and their antithetic counterparts). Fifteen stochastic paths could not be used because the modelled feedback mechanism did not deliver stable developments. So we ended up with 9,985 stochastic paths (and the antithetic counterparts).

A sensitivity analysis is presented at the end of this section. The influence of parameter changes on the steady state results are presented. For each parameter change new maximum likelihood estimators are determined for the conditional expectation equations. Next, we determined the unconditional expected steady state results.

4.1 Consumption and wealth in the steady state

Figure 1: Wealth profile without (left) and with pension funds (right)



The unconditional expected development of wealth in the steady state for the situation without (left panels) and with pension funds (right panels) are presented in Figure 1. For the purpose of illustration all presented variables are scaled for the productivity level. The x-axis gives the age of a representative household and the y-axis the wealth components. The right panel of Figure 1 reveals that the pension rights profile resembles the profile of financial wealth in case pension funds do not exist. Indeed, private saving is negligible in case a pension scheme exists. Financial wealth, $E_0 w_f^h(i)$, and pension rights, $E_0 w_p^h(i)$ are accumulated during the working phase and decumulated in the retirement phase. Human wealth, the discounted value of net labour income $E_0 \sum_{h=i}^{j_e} d_o(h) \left[(1 - v(h)) p_v(h)\right]$, is highest when households enter the labour market and falls gradually to zero over the working phase. Financial wealth and pension rights display the usual inverted U-pattern. The small kink at age 75 is due to intergenerational transfers beginning at that age, while the small kink at age 95 indicate the loss of income afterwards.

Figure 2: Consumption and income profile without (left) with pension funds (right)

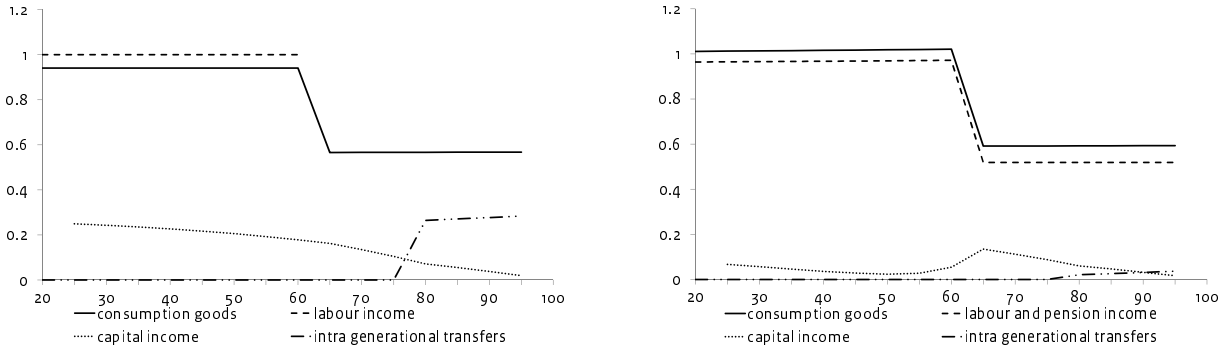


Figure 2 presents the development of expected consumption, and income. Scaled consumption $E_0 c(i)$ is constant during the working ages. Unscaled consumption increases as the return on savings is larger than the time preference. Upon retirement, consumption drops because leisure increases. There are three sources of income: labour and pension

income $E_0 y(i)$, capital income $E_0 R_b(i) w_f^h(i-1) + e_s(i) w_s^h(i-1)$ and intergenerational transfers $E_0 \left(1 - \zeta(i)\right) w_f^h(i)$. Labour income is generated during the working ages. Capital income diminishes over the life cycle. Indeed, the equity investments are largest early in life in the absence of capital market restrictions. Intergenerational transfers are due to inheritance. Wealth of those who die is distributed over households with the same age.

Expected consumption is over the whole life cycle lower in the economy without a pension fund than in the economy with a pension fund. This consumption decline indicates that an economy without a pension fund generates less welfare on average than an economy with a pension fund. The expected consumption decline is for workers about 7.4 per cent and for pensioners about 4.3 per cent (see Figure 3). Figure 3 also presents the three different factors that determine this consumption decline on balance. These three factors are: diminished risk sharing across generations; diminished labour market distortions due to the abolition of the uniform premium and other labour market effects. The other labour market effects can be split up into substitution and catching up premium effects.

The intergenerational risk sharing effect is generated by the premium reduction ($\tau_p^s(i)$ see equation (10). The distortion effects are generated by the difference of the uniform premium from the actuarial fair premium ($\tau_p^d(i) - \tau_p^a(i)$) and by the catching up premium ($\tau_p^c(i)$).

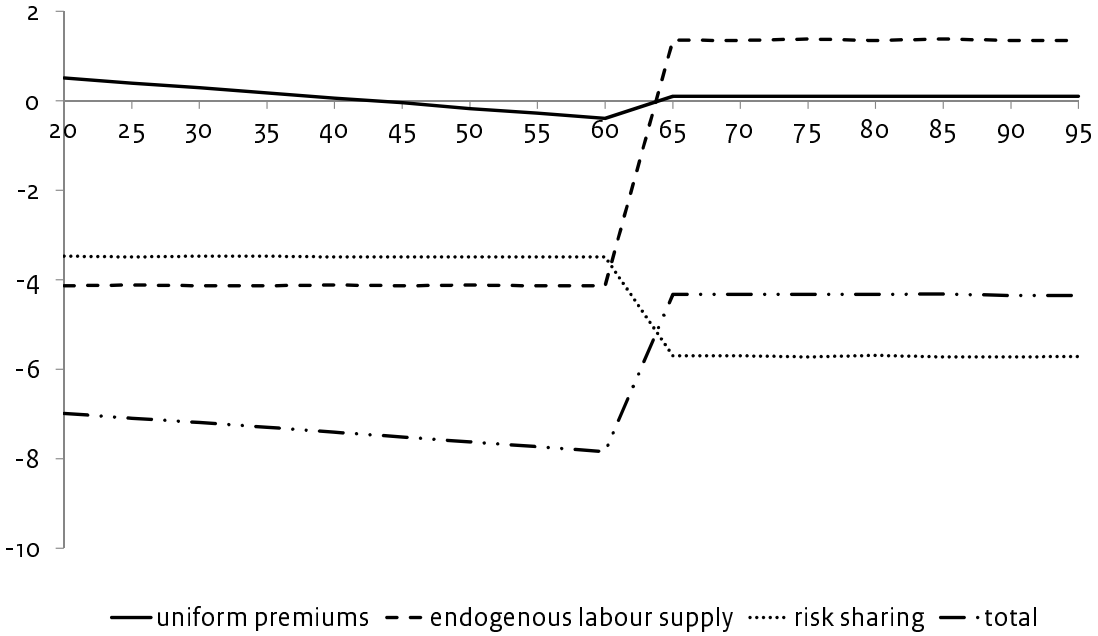
The biggest part of the consumption decline is due to the termination of intergenerational risk sharing, *i.e.* the income effect of closing the pension fund. Indeed, the discounted value of new pension rights per hour worked is larger than the pension premiums per hour worked. This implies that closing the pension fund leads to less income over the life cycle. The implied consumption decline is 5.7% for pensioners and 3.5 per cent for workers. The different percentage changes can be attributed to an equal absolute development in combination with larger consumption of workers relative to pensioners in the base path.

The influence on consumption of the uniform contribution rate, the second distinguished factor, appears to be small. The consumption of young workers increases a little bit, while that of older workers decreases by abolishing the uniform contributions. This shift from the old to the young is due to uniform premiums combined with a constant pension accrual rate, which implies a subsidy of young to old workers. Terminating this subsidy leads to less employment of the older and to more employment of the younger workers. The consumption change is due to labour induced consumption, *i.e.* consumption related to employment[15]. The other labour supply effects of closing the pension scheme are important for consumption. Work becomes less attractive relative to leisure after closing the pension scheme because the discounted value of new pension rights per hour worked are expected to be larger than the pension premiums per hour worked. Workers are partly compensated through more leisure. Total consumption smoothing leads to more good consumption by pensioners.

The catching up premium does not play a role in the expected development, because it will be zero on average. However, the catching up premium has important welfare effects.

[15]See Draper and Armstrong (2007) for a further explanation.

Figure 3: Percentage change consumption over the lifecycle

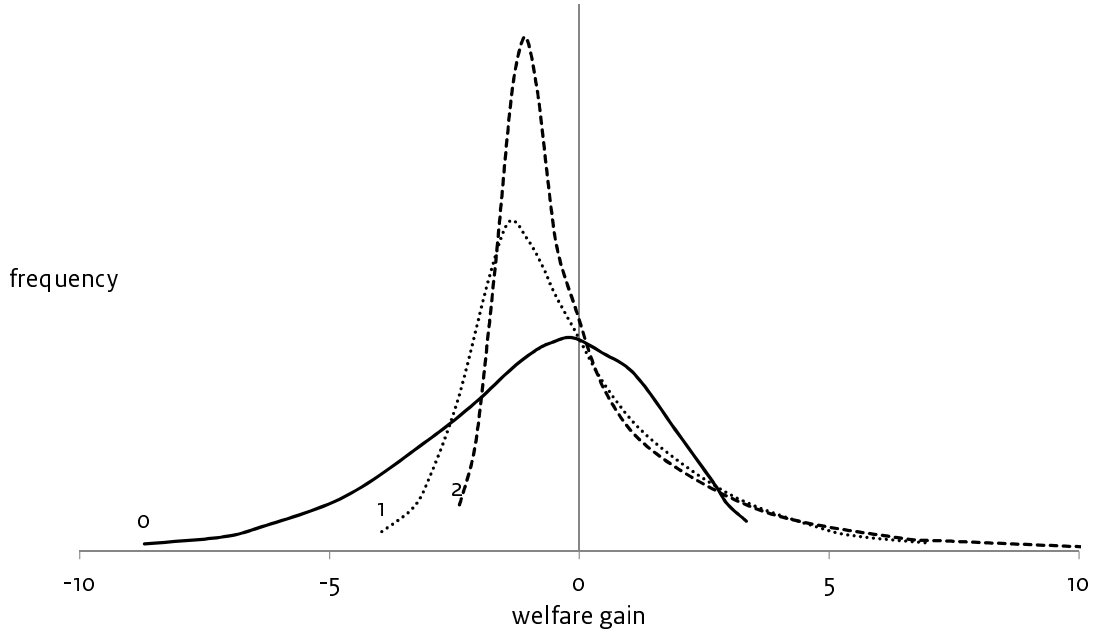


4.2 Interim and ex-post welfare distribution

Figure 4 presents the *ex-post* distribution (indicated with 0) of the equivalent variations and the interim distributions one and two periods ahead (indicated with 1 and 2, respectively) of a new born generation. A positive value of the equivalent variations means that closing the pension fund is welfare improving. The expected value of the distribution is -1.0, which implies that closing the pension fund is on average welfare decreasing. The ex-post distribution illustrates that this pension scheme has a large commitment problem. Indeed, the ex-post probability that closing the pension scheme is welfare improving is 38 per cent (the surface right of the y-axis and below the 0 distribution curve). The probability of a welfare loss is only 62 per cent.

The bad states are characterized by a low funding rate of pension funds leading to large pension premiums and low net income of households. In those bad states consumption will be lower with a pension scheme than without, leading to larger stochastic discount factors m. So, bad states obtain extra weight through discounting, leading to a relative shift to the right of the interim welfare distributions. This shift lead to a sign switch in the expected value of the distribution. The expected value of the interim one period ahead distribution is still negative (-0.1), but the mean of the two period ahead distribution is positive (0.3). We may conclude that although the pension fund has ex-post on average a positive welfare contribution, these welfare effects are too uncertain to bring about a positive welfare assessment for future generations earlier in time.

Figure 4: Ex-post (0) and interim distributions equivalent variations of a twenty year old generation one (1) and two (2) periods ahead



4.3 Ex-ante Pareto efficiency with insurance

The *ex-ante* expected welfare indicator, $E_0 \prod_{l=0}^{t-1} m(l+1) ev(t,5)$, is presented in Figure 5. The x-axis gives the year at which the household starts working relative to the current year, 0. The y-axis gives the value of the ex-ante welfare indicator which is direct comparable with labour income, which is scaled to one in the base year. The numbers correspond to the expected values of the n period ahead distributions as discussed in section 4.2. For instance, the expected values -1.0, -0.1 and 0.3 for respectively the ex-post and interim one and two periods ahead distributions are presented in Figure 5 at age -20, -15 and -10. The ex-ante welfare calculations are based on 10,000 different stochastic paths (and their antithetic counterparts). This number of paths leads to the dotted line and is obvious too small to get a good approximation for the true expected value for each cohort. To get a better approximation a third-order polynomial was estimated for age cohort 15 to 1015 (the solid line).

The ex-ante welfare indicator, the current market value of the equivalent variations reveal that current generations experience a large welfare loss in case the pension fund is closed (for instance an equivalent of about 10% total wealth of the 55 year old cohort). However, the market value of the equivalent variations for future generations is positive, which indicate an ex-ante welfare gain. The bad states become dominant over the good states for those generations by weighting the equivalent variations with the stochastic discount factor. Aggregating over all cohorts indicate that closing the pension fund is welfare improving, *i.e.* the positive effects for future generations outweigh the negative

effects for current generations. However, the total welfare gain of closing the pension is very small and amounts only 10.5. So, the redistribution effects dominate the efficiency effects.

Figure 5: Expected discounted value equivalent variations by birth year

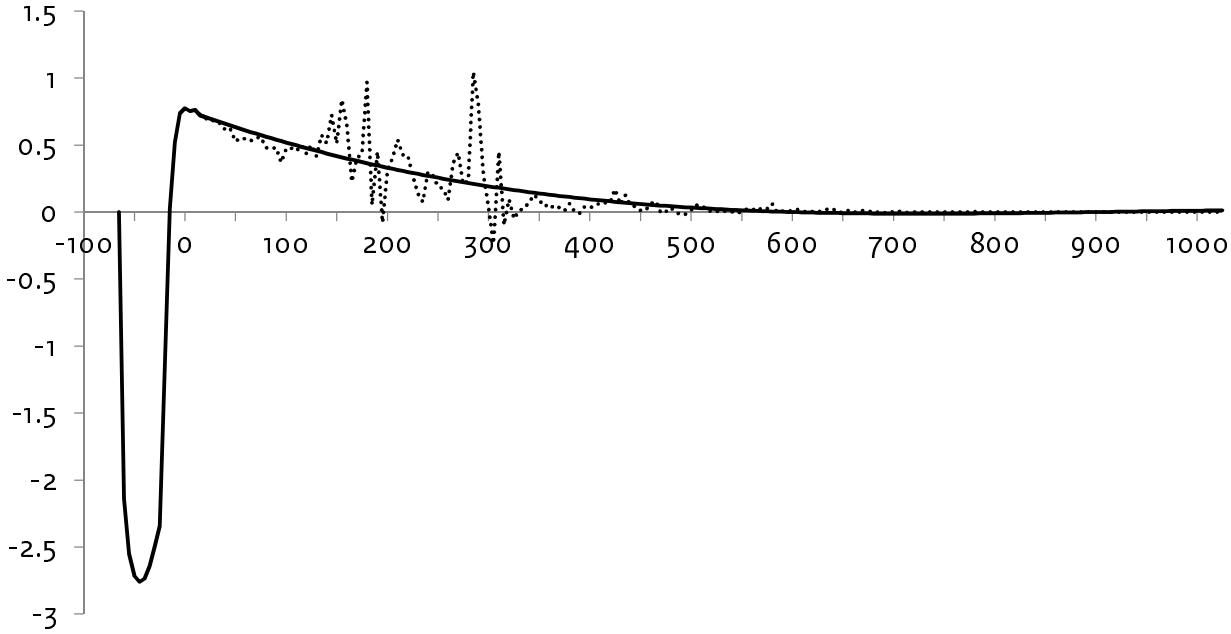


Figure 6: Decomposition expected discounted value equivalent variations by birth year

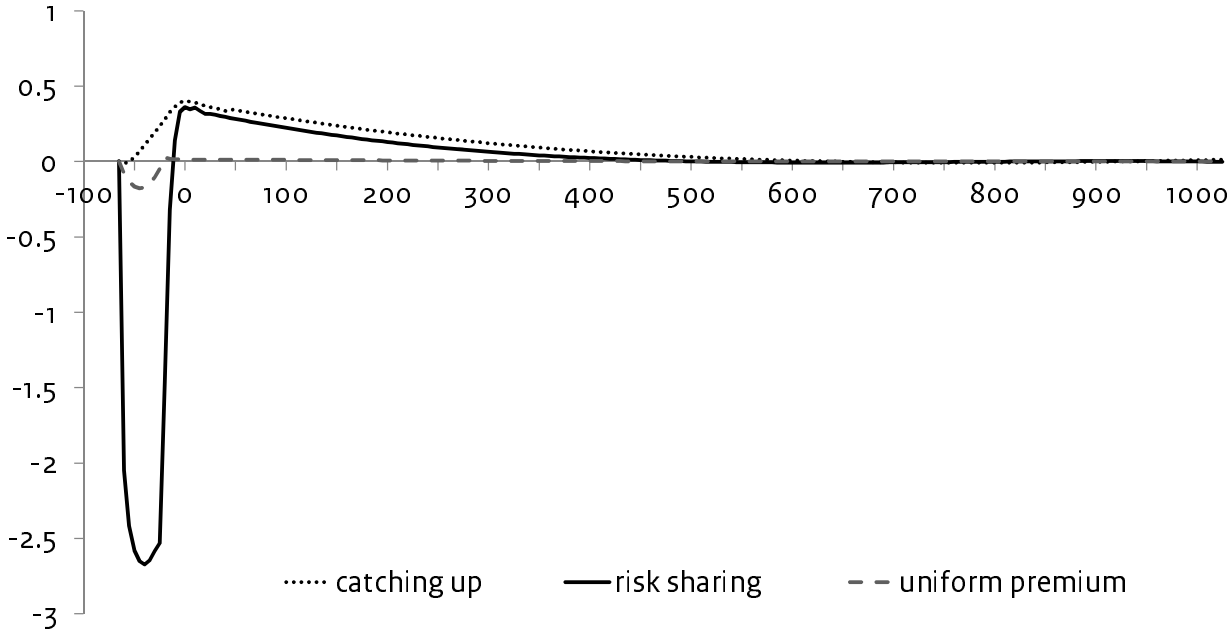


Figure 6 presents a decomposition of the ex-ante welfare effects into: risk sharing, catching up and uniform premium effects.[16] The results are mainly driven by the first two elements.

So, the overall conclusion has to be that a small open economy does not generate, under capital market risk, with a defined-benefit (DB) pension scheme more welfare than without a pension scheme. Pension funds will have large commitment problems after capital market shocks. To assess whether our results depend on the specific model parameters, a sensitivity analysis seems necessary.

4.4 Sensitivity analysis

Table 2 presents the influence of parameter changes on the long run results.[17] The second column gives the outcomes for the base run.[18]. A shorter recovery period for funding deficits (μ) is not very effective, because our model distinguishes five-years periods. The labour supply elasticity ($-1/\beta$) appears to be important for the welfare effects of a pension fund. A larger labour supply elasticity implies a large decrease of the mean of the ex-post equivalent variation distribution when the pension closes and to considerable increase of the probability that the ex-post welfare decreases. Note the large consumption effects relative to the situation without pension fund. An increase of the risk aversion parameter (γ) has the opposite effect: most households prefer the situation without pension funds. Ewijk et al. (2009) reports also (for first best pension funds relatively to a no pension scheme) that a larger risk aversion leads to smaller welfare gains. The next simulation reveals that the exact upper and lower bound of the catching up premium (τ_c) has minor effects on the simulation outcomes. A smaller pension fund (a) is also not important for the qualitative outcomes. However, a larger risk premium (μ_s) leads to a large probability that closing the pension fund leads to lower utility. This result is in line with Ewijk et al. (2009).The expected pension premium is strongly dependent of the excess return parameter. A smaller standard deviation of the risk premium (σ_s) is important for utility of the pension fund.

The overall conclusion is that our results our are most sensitive to the assumptions made about the labour supply elasticity, the risk aversion parameter and the distribution parameters of the excess return.

[16]The expected values for future generations are determined using a third-order polynomial, as in Figure 5.

[17]To evaluate the sensitivity of the ex-ante welfare indicator brings about a large computation burden. So, the sensitivity analyses is limited to the long run.

[18]The parameter values for the base run are summarized in Table 1.

Table 2: Sensitivity simulation results for parameter changes [a]

	base	μ=.75	β=2	γ=5	$\lvert\tau_c\rvert < .75$	a=.40/45	μ_s =.04	σ_s=.135
Funding ratio								
- E	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
- σ	0.28	0.24	0.28	0.28	0.28	0.27	0.28	0.25
Pension premium								
- E	9	10	8	9	9	7	3	8
- σ	27	34	27	27	27	22	28	24
Consumption at age 20								
- E	5.05	4.92	6.27	4.61	5.05	4.99	5.73	5.30
- σ	0.73	0.82	0.84	0.68	0.73	0.58	0.79	0.67
- %	7.5	4.6	42.8	−0.3	7.5	6.2	15.2	10.4
Ex-post distribution								
- E	−1.06	−0.20	−5.14	2.11	−1.05	−0.87	−3.30	−2.15
- Probability welfare loss	62	49	97	16	62	63	91	82

[a] E points to the expected value, σ to the standard deviation, % to the percentage change relative to the situation without pension fund

5 Conclusions

Two policy questions are addressed in this paper: Firstly, does a defined-benefit pension scheme increase welfare? In other words: do the gains from intergenerational risk sharing outweigh the labour market distortions from pension schemes? Secondly, how large is the commitment problem of pension funds after an adverse capital market shock?

If we use risk-neutral weights to aggregate the equivalent variations of different generations in different states of nature then a DB pension scheme is welfare increasing. If we use as weights the stochastic discount factors that corresponds to these states of nature, we conclude the opposite: a DB pension scheme reduces welfare. The difference is due to the large uncertainty about future states. The probability that future households actually experience a welfare gain if the pension scheme is closed can be as large as 38 per cent. So, a pure DB pension scheme has a large commitment problem: continuity will become at risk in case participation in the pension scheme is not mandatory. We present now some further details.

The expected value of the equivalent variations at a certain moment in time reveals that closing the funded DB pension scheme may decrease welfare by diminishing valuable intergenerational risk sharing.[19] However, our analysis has shown that this expected value

[19]The long run welfare effects are much smaller than those obtained in an earlier version of this paper

is quite misleading. This is indicated by the distribution of the equivalent variation over all possible states at that moment in time. The probability that closing the pension fund is welfare improving is 38 per cent. This leads to the conclusion that a pure DB system has a large commitment problem.

Current generations experience a large welfare loss in case a pension scheme, without a funding deficit, is closed. However, the market value of the equivalent variations for future generations is positive, which indicate a welfare gain after closing the pension scheme. The bad states become dominant over the good states for those generations by weighting the equivalent variations with the stochastic discount factor. Aggregating over all cohorts indicate that closing the pension fund is Pareto efficient, *i.e.* the positive effects for future generates outweigh the negative effects for current generations using market valuation.

These results are most sensitive for the values of the labour supply elasticity, the risk aversion parameter and the mean and the standard deviation of the excess return on equity. A larger labour supply elasticity and mean (just as a lower standard deviation) of the excess return on equity increases the probability of a welfare loss after closing the pension scheme. A larger risk aversion decreases the probability of a welfare loss after closing the pension scheme

Our paper can be extended in several directions. An important reason to extend the model is the political unsustainability of the investigated pure DB system. Gollier (2008) presents a second-best intergenerational risk-sharing DC scheme with a positive effect on the welfare of all current and future generations. An extension of our DB scheme with DC elements by introducing conditional and age-specific indexation may lead to these desired properties in our setup.

(Draper and Westerhout (2010)). Important reasons are: the use of the non-linear stochastic discount factor instead of a linear approximation; the rigour use of the complete market hypotheses; testing whether the solution is in the permitted domain. Moreover, the welfare effects of the annuity market are no longer investigated.

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A Derivation household behaviour

Section A.1 derives intertemporal budget equation (14) in the main text. Section A.2 derives leisure demand equation (8). The derivation of consumption equation (19), portfolio equation (12) and stochastic discount factor (13) is the subject of section A.3.

A.1 Financial wealth, pension rights and human wealth

This section derives the intertemporal budget restriction. Assume the stochastic discount factor m is exogenous and $1 = EmR_b$ and $0 = Eme_s$ hold. These assumptions will be verified later on (see section A.3.3). Multiply equation (4) in the main text with the stochastic discount factor, take expectations and reorder to obtain

$$w_f^h(j) = -y(j) + c(j) + E_j m(j+1) w_f^h(j+1) \tag{A.1}$$

Forward solution leads to

$$w_f^h(j) = -E_j \sum_{i=j}^{j_e} (y(i) - c(i)) \prod_{l=j+1}^{i} m(l) \tag{A.2}$$

i.e. financial wealth equals the discounted value of the difference between future consumption and income in case households don't leave bequests. Non-capital income, $y(j)$ consists of net wage income $y_w(j) = (1 - \tau_p(j))(1 - v(j))\, p_l(j)$ in the working ages ($j < j_r$) and pension income $y_p(j)$ in the retirement period ($j \geq j_r$). The discounted value of pension income can be split up into pension rights related to past work, w_p^h, and future work

$$E_j \sum_{i=j_r}^{j_e} y_p(i) \prod_{l=j+1}^{i} m(l) \equiv w_p^h(j) + E_j \sum_{i=j}^{j_e} (1 - v(i))\, p_r(i) \prod_{l=j+1}^{i} m(l) \tag{A.3}$$

in which p_r denotes the discounted value of future pension income that can be attributed to the marginal hour of work as defined in equation (11) in the main text. Use the price of leisure definition $p_v(i) = (1 - \tau_p(i)) p_l(i) + p_r(i)$ and equation (A.3) to write the discounted value of income as

$$E_j \sum_{i=j}^{j_e} y(i) \prod_{l=j+1}^{i} m(l) = E_j \sum_{i=j}^{j_e} (1 - v(i))\, p_v(i) \prod_{l=j+1}^{i} m(l) + w_p^h(j) \tag{A.4}$$

Substitute this result (A.4) into equation (A.2)

$$w_f^h(j) + w_p^h(j) = -E_j \sum_{i=j}^{j_e} ((1 - v(i))\, p_v(i) - c(i)) \prod_{l=j+1}^{i} m(l) \tag{A.5}$$

Reorder to obtain equation (14) in the main text.

A.2 Leisure demand

This section derives leisure demand equation (8) in the main text. Leisure demand is a static decision due to the assumptions made. Define the value of per-period total expenditures as the sum of good consumption and leisure consumption

$$X(j) = v(j)p_v(j) + c(j) \tag{A.6}$$

The per-period decision problem is

$$\max_{c(j),\, v(j)} \; u(j) = \frac{1}{1-\gamma}\left(c(j) + \alpha_v(j)\frac{v(j)^{1-\beta}}{1-\beta}\right)^{1-\gamma} \tag{A.7}$$

given the restriction (A.6) and given leisure less than the total available time, which is normalized to one, *i.e.* $v(j) < 1$. The Lagrangian of this static problem reads as

$$L = u(j) + \lambda(j)\left[X(j) - v(j)p_v(j) - c(j)\right] - \lambda(j)\mu(j)\left[v(j) - 1\right] \tag{A.8}$$

with λ and μ Lagrangian parameters. First-order conditions are

$$\frac{\partial L}{\partial c(j)} = \frac{\partial u(j)}{\partial c(j)} - \lambda(j) = 0 \tag{A.9}$$

$$\frac{\partial L}{\partial v(j)} = \frac{\partial u(j)}{\partial v(j)} - \lambda(j)p_v(j) - \lambda(j)\mu(j) = 0 \tag{A.10}$$

The Kuhn Tucker condition is

$$\left[v(j) - 1\right]\mu(j) = 0 \tag{A.11}$$

Define the price of leisure, inclusive the shadow price as

$$\widetilde{p}_v(j) \equiv p_v(j) + \mu(j) \tag{A.12}$$

Combining the two first-order conditions (A.9) and (A.10) leads, after substitution of both marginal utilities, to the following leisure demand equation:

$$v(j) = \left(\frac{1}{\alpha_v(j)}\widetilde{p}_v(j)\right)^{-\frac{1}{\beta}} \tag{A.13}$$

In case $v(j) = 1$ then $\widetilde{p}_v(j) = \alpha_v(j)$. This is leisure demand equation (8) in the main text.

A.3 The intertemporal consumption problem

This section derives consumption equation (19), portfolio equation (12) and stochastic discount factor (13) in the main text. Define above labour induced consumption $x_f(j)$

$$x_f(j) \equiv c(j) - c_l(j) \tag{A.14}$$

in which labour induced consumption c_l is defined in (3). Substitute this definition in equation (14) and write it in difference equation format

$$w^h(j) = x_f(j) + E_j\left(w^h(j+1)m(j+1)\right) \tag{A.15}$$

The market value of total wealth is determined using the stochastic discount factor approach. Alternatively, a replicating portfolio of equity and bonds could be used to determine this market value. For each asset a replicating portfolio exists. The sum of the equity investments in the replicating portfolios w_s is the total risk exposure, which implies for the development of total wealth over time

$$w^h(j+1) = R_b(j+1)\left[w^h(j) - x_f(j)\right] + e_s(j+1)w_s(j) \tag{A.16}$$

The instantaneous utility function of the dynamic allocation problem can be written as

$$u(j) = \frac{1}{1-\gamma}x_f(j)^{1-\gamma} \tag{A.17}$$

Substitution into the intertemporal utility function gives

$$U(j) = \frac{1}{1-\gamma}E_j\sum_{i=j}^{j_e}x_f(i)^{1-\gamma}\prod_{l=j}^{i}\frac{\delta(j)}{\delta(l)} \tag{A.18}$$

The recursive restatement of the maximum problem is

$$V(j) = \max\left[\frac{1}{1-\gamma}x_f(j)^{1-\gamma} + \delta(j+1)^{-1}E_jV(j+1)\right] \tag{A.19}$$

Subject to the intertemporal budget equation. Assume, the value function can be written as

$$V(j) = \frac{p_f(j)^{\gamma-1}}{1-\gamma}w^h(j)^{1-\gamma} \tag{A.20}$$

with $p_f(j)$, constants for $\forall_j$ to be defined later on. The assumption will be checked in section A.3.5.

A.3.1 First-order conditions

Substitute the value function assumption (A.20) into the Bellman equation (A.19) just as the expression for total wealth (A.16) to write the maximum problem as

$$V(j) = \max \Bigg[u(j) + \delta(j+1)^{-1} \frac{p_f(j+1)^{\gamma-1}}{1-\gamma} \times \\ \times E_j \left[R_b(j+1) \left[w^h(j) - x_f(j) \right] + e_s(j+1) w_s(j) \right]^{1-\gamma} \Bigg] \tag{A.21}$$

First order conditions are

$$\frac{\partial V}{\partial x_f(j)} = \frac{\partial u(j)}{\partial x_f(j)} - \delta(j+1)^{-1} E_j \frac{\partial V(j+1)}{\partial w^h(j+1)} R_b(j+1) = 0 \tag{A.22}$$

$$\frac{\partial V}{\partial w_s(j)} = \delta(j+1)^{-1} E_j \left(\frac{\partial V(j+1)}{\partial w^h(j+1)} e_s(j+1) \right) = 0 \tag{A.23}$$

$$\frac{\partial V}{\partial w^h(j)} = \delta(j+1)^{-1} E_j \left(\frac{\partial V(j+1)}{\partial w^h(j+1)} R_b(j+1) \right) \tag{A.24}$$

A.3.2 Portfolio decision

Substitute (A.16) into the implicit portfolio equation (A.23) gives

$$0 = E_j \left(R_b(j+1) \left[w^h(j) - x_f(j) \right] + e_s(j+1) w_s(j) \right)^{-\gamma} e_s(j+1) \tag{A.25}$$

Divide (A.25) by the sum of the certain terms $R_b(j+1)\left(w^h(j) - x_f(j)\right)$

$$0 = E \left(1 + \omega_s e_s\right)^{-\gamma} e_s \tag{A.26}$$

with

$$\omega_s = \frac{w_s(j)}{\left[R_b(j+1) \left(w^h(j) - x_f(j) \right) \right]} \tag{A.27}$$

Equation (A.26) is the implicit equity demand equation (12) in the main text. The direct investments in equity can be obtained by inverting the budget equation

$$w_s^h(i) = \frac{w_f^h(i+1) - R_b(i+1) \left(w_f^h(i) + y(i) - c(i) \right)}{e_s(i+1)} \tag{A.28}$$

with

$$w_f^h(j) = w^h(j) - \left(w_p^h(j) + w_n^h(j) \right) \tag{A.29}$$

A.3.3 The unique stochastic discount factor

Equation (A.23) and (A.24) can be written as $0 = Eme_s$ and $1 = EmR_b$ respectively with m the relative utility. Equation (A.26) makes identification of the stochastic discount factor possible

$$m = \kappa \left(1 + \omega_s e_s\right)^{-\gamma} \tag{A.30}$$

with κ a proportionality factor. This proportionality factor can be derived from $1 = R_b Em = \kappa R_b E \left(1 + \omega_s e_s\right)^{-\gamma}$. This implies for the stochastic discount factor

$$m = \frac{1}{R_b} \frac{\left(1 + \omega_s e_s\right)^{-\gamma}}{E \left(1 + \omega_s e_s\right)^{-\gamma}} \tag{A.31}$$

which is equation (13) in the main text. The expected value can be obtained using numerical integration. Because ω_s is determined by exogenous variables in equation (A.26), m can be considered as given for the households.

A.3.4 Consumption-saving decision

Starting point for the non-labour related consumption decision is equation (A.22) Substitution of the marginal utility and the derivative of the value function gives

$$\begin{aligned} x_f(j)^{-\gamma} = p_f(j+1)^{\gamma-1} \frac{R_b(j+1)}{\delta(j+1)} \left[R_b(j+1)\left(w^h(j) - x_f(j)\right)\right]^{-\gamma} \times \\ \times E\left[(1 + \omega_s e_s)\right]^{-\gamma} \end{aligned} \tag{A.32}$$

make use of definition

$$\eta \equiv \left(E \left(1 + \omega_s e_s\right)^{-\gamma}\right)^{-\frac{1}{\gamma}} \tag{A.33}$$

to write the non-labour-related total consumption equation as

$$x_f(j) = \left(\frac{1}{p_f(j)}\right)^{1-\frac{1}{\gamma}} w^h(j) \tag{A.34}$$

with the price index of total wealth defined as

$$p_f(j)^{\frac{\gamma-1}{\gamma}} = p_f(j+1)^{\frac{\gamma-1}{\gamma}} \left(\frac{R_b(j+1)}{\delta(j+1)}\right)^{\frac{1}{\gamma}} \eta^{-1} R_b(j+1)^{-1} + 1 \tag{A.35}$$

Equation (A.34) is consumption equation (19) in the main text,

A.3.5 Value function

We check now the assumption of the value function. Total wealth develops according to

$$w^h(j+1) = w^h(j)R_b(j+1)\left(1 - \left(\frac{1}{p_f(j)}\right)^{1-\frac{1}{\gamma}}\right)[1 + \omega_s e_s(j+1)] \tag{A.36}$$

Use this relation, substitute the assumption for the value function for $j+1$ and the consumption relation into the Belmann equation to obtain

$$\begin{aligned} V(j) &= u(j) + \delta(j+1)^{-1}E_j V(j+1) \\ &= \frac{1}{1-\gamma}p_f(j)^{\gamma-1}w^h(j)^{1-\gamma} \end{aligned} \tag{A.37}$$

Note, $p_f(j)$ are constants (*i.e.* exogenous given) for $\forall_j$.

B Parametrized expectations

B.1 Conditional expectations human wealth

This section follows Judd (1999), Heer and Maussner (2005) and Judd et al. (2009). Variable[20] $w_n(j)$ is the conditional expectations given the information at age j, *i.e.* the state of the economy at age j

$$w_n(j) = E_j \sum_{i=j}^{j_e} [(1 - v(i))p_v(i) - c_l(i)] \prod_{l=j+1}^{i} m(l) \tag{B.1}$$

Define $y = \frac{u-\mu_u}{\sigma_u}$ with $u \equiv \sum_{i=j}^{j_e} [(1 - v(i))p_v(i) - c_l(i)] \prod_{l=j+1}^{i} m(l)$, μ_u average and σ_u standard deviation. The state variables are collected in x. Assume one relevant state variable only: the transformed funding ratio of pension funds $x = \frac{v-\mu_v}{\sigma_v}$ and $v = q$. The funding ratio is for each age cohort at a certain point in time equal and defined by $q(t) = w_p^h(t)/w_f^p(t)$ The conditional expectations $E\{y \mid x\}$ is a function of x, $\psi(x)$, such that $E\{(y-\psi(x))g(x)\} = 0$ for all g. We seek a function $\widehat{\psi}(x; a)$ which approximates $\psi(x)$. For instance $\widehat{\psi}(x; a) = \alpha_1 x + \alpha_2 x^2$. A constant term is not necessary due to the transformation. Assume both y and x depend on random variable $\widetilde{R}_s$. The least square parameters are

$$\min_a \left(\sum (y_i - \widehat{\psi}_i(x; a)) \right)^2 \tag{B.2}$$

We use the Monte Carlo approach to generate pairs of $y(j)$ and $x(j)$. Then we regress in our example

$$y(j) = \alpha_1(j)x(j) + \alpha_2(j)x(j)^2 + \varepsilon \tag{B.3}$$

The conditional expectation equals

$$E_j y(j) = \alpha_1(j)x(j) + \alpha_2(j)x(j)^2 \tag{B.4}$$

which is the same as (B.3) except for the error term. The conditional epectations of human wealth become

$$w_n = u = \mu_u + \sigma_u E_j y \tag{B.5}$$

Judd et al. (2009) advocates to use Hermite or Chebyshev polynomials in case higher order approximations are necessary instead of ordinary polynomials. Next section details on the construction of a complete set of Hermite polynomials

[20]Note, $w_n = w_h^h - w_l^h$ in the main text. So we take both variables togeter to explain the procedure. In the simulations we modelled both separately.

B.2 Hermite polynomial representation

To approximates $\psi(x)$ we consider Hermite $h(m)$ polynomials.

$$\begin{aligned} h_i(0) &= 1 \\ h_i(1) &= x_i \\ h_i(m) &= x_i h_i(m-1) - (m-1)h_i(m-2) \text{ and } m > 1 \end{aligned} \tag{B.6}$$

We construct a complete set of polynomials of degree p in n variables using the ordinar polynomials

$$z = \left\{ \prod_{i=0}^{n} h_{l_i}(m) \mid \sum_{i=1}^{n} l_i = j,\ j = 1 \dots p \right\} \tag{B.7}$$

In fact this set is not the complete set because one is not included. One is not included due to the normalization of the variables. In the same way a complete set of Ordinary polynomials can be defined.With one exogenous variable and ordinary polynomials we get $\{x, x^2\}$ as in the example in previous section. The complete set of polynomials of degree 2 in 2 variables is

$$z = \{h_1(1), h_2(1), h_1(1)h_2(1), h_1(2), h_1(2)\} \tag{B.8}$$

B.3 Kalman filter

Rational expectations imply that the agents understand the working of the economy, i.e. they have a very good model to predict the future, given the state of the economy. To get a good approximation a Kalman filter approach seems necessary. This section follows Harvey (1986) page 106 up to 110. Equation (B.3) can be summarized by

$$y_t = z_t' \alpha_t + \xi_t \tag{B.9}$$

with $\xi_t \sim WS(0, \sigma^2)$ in which WS indicates the variable has a mean and a variance; WS stands for 'wide sense'. Suppose constant parameters

$$\alpha_t = \alpha_{t-1} \tag{B.10}$$

We use a_t for the the minimum mean square linear estimator (MMSLE) of α_t at time t. The covariance matrix of a_{t-1} is $\sigma^2 P_{t-1}$. The covariance matrix of the estimation error is

$$a_{t-1} - \alpha_t \sim WS(0, \sigma^2 P_{t-1}) \tag{B.11}$$

The error made in prediction y_t at time $t-1$ is

$$\upsilon_t = z_t'(\alpha_t - a_{t-1}) + \xi_t \tag{B.12}$$

with variance

$$Var(\upsilon_t) = \sigma^2 \left(z_t' P_{t-1} z_t + 1\right) \equiv \sigma^2 f_t \tag{B.13}$$

Updating rule for the covariance matrix

$$P_t = P_{t-1} - P_{t-1} z_t z_t' P_{t-1} / f_t \tag{B.14}$$

The updating rule for the state vector is

$$a_t = a_{t-1} + P_{t-1} z_t \upsilon_t / f_t \tag{B.15}$$