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Abstract

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1 Introduction

Conventional wisdom and existing research suggests that learning about fixed but unknown ‘structural’ parameters has minor asset pricing implications, and because of this most of the literature focuses on learning about stationary latent state variables.¹ To see this, assume the logarithm of consumption growth is normally distributed, $\Delta \ln(C_t) = y_t \sim \mathcal{N}(\mu, \sigma^2)$, and that the ‘structural’ parameter μ is unknown. Agents update normally distributed initial beliefs, $\mu \sim \mathcal{N}(\mu_0, \sigma_0^2)$, using Bayes rule which implies that the posterior of μ is $p(\mu|y^t) \sim \mathcal{N}(\mu_t, \sigma_t^2)$, where μ_t and σ_t^2 are given by standard recursions and y^t is data up to time t . If the representative agent has power utility preferences, the ‘equity’ premium on a single-period consumption claim is $\gamma(\sigma^2 + \sigma_t^2)$. Since σ_t^2 decreases rapidly over time, the effect of parameter uncertainty on the equity premium is generally small to begin with and then quickly dies out.² Thus, parameter uncertainty seems to have a negligible effect when allowing for a minimal amount of learning.

In this paper, we show that this conventional wisdom does not hold generally when the representative agent has Kreps-Porteus preferences and a preference for the timing of the resolution of uncertainty. A key feature of parameter uncertainty and rational learning is that mean parameter beliefs, or posteriors, are martingales. To see this, note that $\mu_t = E(\theta|y^t)$, where θ is a fixed parameter, is trivially a martingale by the law of iterated expectations. This implies that shocks to beliefs are permanent, affecting the conditional distribution of consumption growth indefinitely into the future. Parameter uncertainty thus generates a particularly strong form of long-run consumption risks (see Bansal and Yaron (2004)). For agents who care about the timing of the resolution of uncertainty, assets whose payoffs are affected by unknown parameters may therefore be particularly risky.

The goal of this paper is to quantify the asset pricing implications of structural parameter uncertainty when the temporal resolution of uncertainty matters.³ We consider first the simplest setting where aggregate log consumption growth is i.i.d. normal and the represen-

¹See Veronesi (1999), Brennan and Xia (2001), among others.

²Weitzman (2007) argues that uncertainty about consumption volatility can be large and economically important. Bakshi and Skouliakis (2010) show that this is due to the prior distribution on the variance, and that the impact of volatility uncertainty with a prior with bounded support is negligible.

³While we employ a Bayesian approach to parameter learning, the literature on robustness and ambiguity aversion (e.g., Hanson (2007), Hansen and Sargent (2010)) also consider the asset pricing implications of parameter or model uncertainty. In these models, the representative agent prices claims relative to a ‘worst-case’ probability measure, which is often given exogenously. Relative to this literature, the Bayesian approach entails a complete description of the state space and its associated probabilities. Thus the sequential rational learning problem, which is what leads to subjective long-run consumption risk, is explicit.

tative agent is unsure about the true mean growth rate. We consider cases with unbiased beliefs, where priors are centered at the true values, to focus particularly on the impact on asset prices of priced parameter uncertainty (unlike, e.g., the analysis in Sargent and Cogley (2008)). We are particularly interested in studying the dynamics of central asset pricing quantities like the equity premium and return volatility, as well as short- and long-term real yields on default free bonds. We price equity as a levered consumption claim assuming Epstein-Zin preferences with a preference for early resolution of uncertainty. Although this model is too simple along many dimensions to be considered realistic, the learning dynamics reveal a number of interesting findings.

We find that parameter uncertainty has a quantitatively large and long-lasting impact on the equity premium. As a benchmark, the average excess return on a levered consumption claim in the known-parameter benchmark case is roughly 1.7% per year, whereas over a 100 year sample in a reasonably calibrated parameter learning case the average excess equity returns are 4.4%. The equity premium does decline over the sample – in the first 10 years it is about 11%, while after 50 years is about 4.5%. Even after 100 years, the equity premium is 3%. This magnitude may at first seem almost implausibly large to the reader, as the agent after 100 years of learning is quite confident in her mean belief about the consumption growth rate. However, it is a direct effect of the combination of permanent shocks to long-run growth expectations and the preference for early resolution of uncertainty. The representative agent experiences a large amount of risk even after 100 years of learning as expected consumption growth shocks, while small when viewed over a quarter, last forever and therefore have a large impact on the continuation utility.

These results show that the asset pricing implications of rational parameter learning can be quantitatively significant for a very long time, despite the fact that the posterior standard deviation of the mean growth rate declines rapidly. In fact, after 50 years, the standard deviation of shocks to mean beliefs is 5.8 times smaller than at the beginning of the sample, but the equity premium falls by a factor of 2, only. The standard deviation of the log pricing kernel – the price of risk – falls by a factor slightly less than 2 over the same period. Over the next 50 years, the standard deviation of shocks to mean beliefs falls by a factor of 1.9, while the price of risk drops by a factor of about 1.2. Two observations can be made here. First, the standard deviation of shocks to mean beliefs about the mean growth rate declines much faster in the beginning of the sample than after some time has elapsed. This is a standard result from Bayesian updating. Second, the price of risk in the economy declines at a much slower rate. The latter seems puzzling, but is in fact an endogenous

outcome of the deterministically decreasing variance of beliefs and can be understood as follows.

The effect on the continuation utility of shocks to beliefs is nonlinear. In particular, in the beginning of the sample, when there is a lot of parameter uncertainty, discount rates are high. Therefore, shocks to the beliefs about the mean growth rate are relatively quickly discounted in terms of their effect on wealth (utility). Towards the end of the sample, when there is less parameter uncertainty, discount rates are lower and so shocks to the mean belief about the growth rate have a larger effect on wealth. Since shocks to wealth appear in the pricing kernel when agents have a preference for early resolution of uncertainty, this increase in the sensitivity of wealth to updates in mean beliefs affects the volatility of the pricing kernel. Overall, while the magnitude of the shocks to mean parameter beliefs decreases rapidly with rational learning, the sensitivity of the continuation utility to such shocks is endogenously increasing. The net effect is a relatively slow decline in the risk premium and the price of risk.

Parameter learning also induces excess return predictability. This occurs both because the conditional risk premium declines over time and because of a small-sample correlation between future returns and the price-dividend ratio. The rationale for the latter is described in Timmermann (1996) and Lewellen and Shanken (2002). While the model with parameter learning features subjective long-run risks, there is no consumption growth predictability in the model. In particular, the price-dividend ratio does not predict long-horizon consumption growth in population or in small samples. This behavior is different from existing long-run risk models, which have been critiqued by Beeler and Campbell (2011) on the grounds that they assume a high value of agents' elasticity of intertemporal substitution, typically well above one, while Hall (1988) and several authors after him have estimated the elasticity of substitution to be close to zero. We run the same regressions as in Hall (1988) on simulated data from our models and show that we can replicate these low estimates even though the representative agent in fact has a high elasticity of intertemporal substitution. Again, the reason is that the asset prices, and in this case the risk-free rate, respond to agents' perceived consumption growth rate and not to the ex-post true growth rate. Thus, the parameter learning model is not subject to Beeler and Campbell's (2011) main critiques of long-run risk models.

We consider three other cases of parameter uncertainty. First, consider the case of unknown variance in the simple i.i.d. consumption growth case. Weitzman (2007) argues in a power utility setting that unknown variance can explain an arbitrary risk premium, thus

labeling the standard puzzles as 'anti-puzzles'. Bakshi and Skouliakis (2010), however, argue that this result is sensitive to the choice of prior and that with a reasonable upper bound on the support for the variance parameter, the asset pricing implications of learning about the variance parameter are negligible. In any case, we note that the effect of parameter learning we document is very different from that highlighted in these papers. In particular, Weitzman's results come from a very fat lower tail of the distribution of marginal utility, induced by a subjective t -distribution for consumption growth – uncertainty about the variance parameter induces a fat-tailed subjective consumption growth distribution. In our case, however, the primary driver of the risks induced by parameter learning is not the shape of the subjective conditional consumption growth distribution. It is the permanent shocks generated by updating beliefs which are priced risks when agents have a preference for early resolution of uncertainty. We show that dynamic learning about the variance parameter can indeed have quantitatively significant asset pricing implications, but only for a relatively short period of time (less than 20 years even when starting from a relatively high degree of initial parameter uncertainty). The reason the effect is less long-lived than the case of learning about the mean growth rate is that learning is more rapid for the variance parameter compared to the mean parameter, and the variance of consumption growth is a second order moment whose changes in relative terms affect wealth less than changes in the mean growth rate does.

Second, we consider learning about parameters governing the consumption dynamics in disasters. Learning about rare events is slow as there are, by definition, few historical observations to learn from, which implies the asset pricing effects of parameter learning last for a very long time, certainly a much longer time than for what we have reliable data available. Third, we consider an economy with structural breaks. In particular, we assume there is a small probability each quarter that the mean growth rate of the economy is redrawn from a given distribution. Such structural breaks restart the parameter learning problem and makes parameter uncertainty a perpetual learning problem. These alternative learning models largely exhibit the same properties as the case where investors learn about the unconditional mean growth rate. Parameter uncertainty significantly increases the risk premium, return volatility, the amount of return predictability, and the equity return Sharpe ratio, due to the learning-induced long-run risks which avoid excess consumption growth predictability.

The paper proceeds as follows. In Section 2 we describe in general how parameter learning is a natural source of long-run consumption risks. In Section 3, we describe the simple model with unknown mean growth rate, as well as cases where there is uncertainty about

the variance of consumption growth. Section 4 considers the case of learning about disasters. Section 5 considers an economy with structural breaks.

2 Parameter learning as a source of long-run risks

In a setting with parameter uncertainty, the process of rational updating of beliefs via Bayes rule provides a natural source of ‘long-run’ risks. Intuitively, this occurs because optimal beliefs have the property that forecast errors are unpredictable, which implies that shocks to beliefs are permanent. Formally, long run risks arise due to various martingale properties associated with conditional probabilities.

To see this, note that rational learning about parameters from observed data requires that agents update their posterior beliefs using the rules of conditional probabilities, aka, Bayes rule. Denoting the posterior density at time t as $p(\theta|y^t)$, Bayes rule implies that

$$p(\theta|y^{t+1}) = \frac{p(y_{t+1}|\theta)p(\theta|y^t)}{p(y_{t+1}|y^t)}. \quad (1)$$

Bayes rule also implies the laws of conditional expectations and, in particular, the law of iterated expectations. To see the implications, let $\mu_t = \mathbb{E}[\theta|\mathcal{F}_t]$ denote the posterior mean at time t . By the law of iterated expectations,

$$\mathbb{E}[\mu_{t+1}|\mathcal{F}_t] = \mathbb{E}[\mathbb{E}[\theta|\mathcal{F}_{t+1}]|\mathcal{F}_t] = \mathbb{E}[\theta|\mathcal{F}_t] = \mu_t, \quad (2)$$

which implies that μ_t is a martingale. Thus,

$$\mu_{t+1} = \mu_t + \eta_{t+1}, \quad (3)$$

where $\mathbb{E}[\eta_{t+1}|\mathcal{F}_t] = 0$ and $\mathbb{E}[\eta_{t+1}E[\theta|\mathcal{F}_t]|\mathcal{F}_t] = 0$. From this, it is clear that the shocks to beliefs, η_{t+1} , are not just persistent, but *permanent*. This martingale property holds more generally as posterior probabilities ($\mathbb{P}[\theta \in A|y^t]$), expectations of functions of the parameters ($\mathbb{E}[h(\theta)|y^t]$), and likelihood ratio statistics are all martingales. Thus, rational learning about parameters, or even model specifications themselves, induces a belief process with permanent shocks.⁴

⁴This property is well known and has a range of implications. Hansen (2007) noted this property and considered the implications with a robust decision maker.

This paper considers economies where a representative agent derives utility from consumption, but where the parameters determining consumption dynamics are unknown to the agent. The agent updates beliefs via Bayes rule. Throughout the paper, we consider Epstein-Zin utility, V , over consumption, C :

$$V_t = \left\{ (1 - \beta) C_t^{1-1/\psi} + \beta (E_t [V_{t+1}^{1-\gamma}])^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}}, \quad (4)$$

where γ is relative risk aversion, ψ is the elasticity of intertemporal substitution, and β is the discount rate. The stochastic discount factor (SDF) in this economy is

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\beta \frac{PC_{t+1} + 1}{PC_t} \right)^{\theta-1}, \quad (5)$$

where PC_t is the wealth-consumption ratio at time t and where $\theta = \frac{1-\gamma}{1-\psi^{-1}}$. The first component of the pricing kernel, $\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$, is of the usual power utility form. With a preference for the timing of the resolution of uncertainty, (i.e., if $\theta \neq 1$; see Epstein and Zin (1989)), the SDF has a second term, $\left(\beta \frac{PC_{t+1} + 1}{PC_t} \right)^{\theta-1}$, providing the conduit through which long-run risks impact asset prices.

Learning about parameters governing consumption dynamics impacts marginal intertemporal rates of substitution in this economy. In particular, belief shocks generate permanent shocks to the conditional distribution of future aggregate consumption, impacting the price-consumption ratio due to changes in growth expectations and/or discount rates. From Equation (5) it is immediate that these shocks are priced risk factors in this economy. An alternative, equivalent expression for the stochastic discount factor in this economy is helpful for intuition. In particular, express the second risk factor in terms of the value function normalized by consumption, $VC_t \equiv V_t/C_t$:

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \left(\frac{VC_{t+1}}{E_t [VC_{t+1}^{1-\gamma} (C_{t+1}/C_t)^{1-\gamma}]^{1/(1-\gamma)}} \right)^{\frac{1}{\psi}-\gamma}. \quad (6)$$

Since VC_t is a function of the conditional subjective distribution of future consumption growth, it responds to shocks to this distribution.

Note that alternative preference specifications featuring a preference for early resolution of uncertainty will be affected by parameter learning similarly to the Epstein-Zin case we

consider, as these alternative utility specifications also lead to a pricing kernel where continuation utility is a priced risk factor. The quantitative effects will of course depend on the utility specification and parameter assumptions. Examples include Kreps-Porteus preferences more generally, as well as the smooth ambiguity aversion preferences of Klibanoff, Marinacci, and Mukerji (2009) and Ju and Miao (2012). See Strzalecki (2011) for a theoretical discussion of the relation between ambiguity attitude and the preference for the timing of the resolution of uncertainty.

In the following, we quantify the asset pricing implications of long-run risks in a number of different model specifications. We first consider the simplest possible model, where consumption growth is truly i.i.d. lognormal, but the mean growth rate is unknown. This case gives most of the intuition needed in a transparent way and, surprisingly, works remarkably well in terms of matching a number of stylized facts. We then move on to more complicated consumption dynamics, including learning about rare events and learning in an economy with structural breaks.

3 Case 1: i.i.d. log-normal consumption growth

Assume that aggregate log consumption growth is i.i.d. normal:

$$\Delta c_{t+1} = \mu + \sigma \varepsilon_{t+1}, \quad (7)$$

where $\varepsilon_{t+1} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$. This is a natural starting point for consumption-based asset pricing models (see, e.g., Hall (1978)), and the i.i.d. nature of the exogenous endowment process also means that any time-variation in the risk-free rate, the risk premium, and/or the wealth-consumption ratio is due to endogenous learning dynamics.

The representative agent does not know the mean growth rate, but starts the sample with a prior: $\mu \sim N(\mu_0, \sigma_0^2)$. We later truncate this prior to ensure finite utility, but for now consider the untruncated case for ease of exposition. The volatility parameter, σ , is for now assumed known. The agent updates beliefs sequentially upon observing realized consumption growth using Bayes rule:

$$\mu_{t+1} = \frac{\sigma_t^2}{\sigma_t^2 + \sigma^2} \Delta c_{t+1} + \left(1 - \frac{\sigma_t^2}{\sigma_t^2 + \sigma^2}\right) \mu_t, \quad (8)$$

$$\frac{1}{\sigma_{t+1}^2} = \frac{1}{\sigma_t^2} + \frac{1}{\sigma^2}. \quad (9)$$

In the agent's filtration, aggregate consumption dynamics are:

$$\Delta c_{t+1} = \mu_t + \sqrt{\sigma^2 + \sigma_t^2} \tilde{\varepsilon}_{t+1}, \quad (10)$$

where $\tilde{\varepsilon}_{t+1} \sim \mathcal{N}(0, 1)$. Further, note that:

$$\begin{aligned} \mu_{t+1} &= \frac{\sigma_t^2}{\sigma_t^2 + \sigma^2} \Delta c_{t+1} + \left(1 - \frac{\sigma_t^2}{\sigma_t^2 + \sigma^2}\right) \mu_t \\ &= \frac{\sigma_t^2}{\sigma_t^2 + \sigma^2} \left(\mu_t + \sqrt{\sigma^2 + \sigma_t^2} \tilde{\varepsilon}_{t+1}\right) + \left(1 - \frac{\sigma_t^2}{\sigma_t^2 + \sigma^2}\right) \mu_t \\ &= \mu_t + \frac{\sigma_t^2}{\sqrt{\sigma^2 + \sigma_t^2}} \tilde{\varepsilon}_{t+1}. \end{aligned} \quad (11)$$

In words, in the agent's filtration the mean expected consumption growth rate is time-varying with a unit root. Comparing this to the consumption dynamics in Bansal and Yaron (2004), note that learning induces truly long-run risk in that shocks to expected consumption growth (in the agent's filtration) are permanent versus Bansal and Yaron's persistent, but still transitory, shocks. The process does not explode, however, as the posterior variance is declining over time and will eventually (at $t = \infty$) go to zero. Note also that actual consumption growth is not predictable given its i.i.d. nature (Eq. (7)). Thus, the long-run consumption risks that arise through parameter learning do not imply excess consumption growth predictability – a critique often levied against long-run risk models (see, e.g., Beeler and Campbell (2011)).

Learning increases consumption growth volatility: from the agent's perspective, the consumption growth variance is $\sigma^2 + \sigma_t^2$. Setting $\sigma_0^2 = \sigma^2$ as an upper bound, learning can maximally double the subjective conditional consumption growth variance.⁵ The posterior standard deviation decreases quickly, as shown in Figure 1. After ten years of quarterly consumption observations, the agent perceives the standard deviation of consumption growth to be only 1.012 times greater than the objective consumption growth standard deviation. This fact may explain why prior literature working with power utility preferences have not considered learning about the mean unconditional growth rate an important consideration for asset pricing.⁶ In particular, with power utility preferences the conditional volatility of

⁵If you start with a diffuse prior ($\sigma_{-1}^2 = \infty$), you will after having observed *one* consumption growth outcome have $\sigma_0^2 = \sigma^2$.

⁶Many papers consider learning about a stationary, time-varying mean (e.g., Veronesi (1999, 2000)). Veronesi (2002) considers learning about a stationary mean where the bad state occurs only once every 200

the log pricing kernel is $\gamma\sqrt{\text{Var}_t(\Delta c_{t+1})}$, and so, after ten years, learning will increase the maximum Sharpe ratio by only a tiny fraction.

[FIGURE 1 ABOUT HERE]

However, with a preference for early resolution of uncertainty ($\gamma > 1/\psi$), the agent strongly dislikes shocks to *expected* consumption growth as in Bansal and Yaron (2004). In particular, Bansal and Yaron show that even with a very small persistent component in consumption growth, the volatility of the pricing kernel can increase significantly relative to the power utility case. We make use of the same mechanism here. While the posterior variance decreases quickly, it takes a long time to converge to zero (see Figure 1). The second component of the pricing kernel (see Eq. 3), $\left(\beta \frac{PC_{t+1}+1}{PC_t}\right)^{\theta-1}$, then adds volatility in the following way. An increase in expected mean consumption growth, which occurs upon a higher than expected consumption growth realization, increases the wealth-consumption ratio when $\psi > 1$. In our main calibrations, $\gamma > 1$ and $\psi > 1$, which implies that $\theta < 0$, such movements in PC_{t+1} increase the total volatility of the pricing kernel. Since the shocks to mean consumption growth are permanent, they have a large impact on the wealth-consumption ratio. In the following, we gauge the quantitative implications of parameter uncertainty in this general equilibrium model.

The dividend claim

In our main analysis, we assume that the market return is a levered consumption claim:

$$R_{M,t} = \left(1 + \frac{D}{E}\right) R_{C,t}, \quad (12)$$

where $R_{C,t+1} = \frac{C_{t+1}}{C_t} \frac{1+PC_{t+1}}{PC_t}$ is the return to the consumption claim. The aggregate debt-to-equity ratio (D/E) in the U.S. postwar data is about 0.5, so the return we report is 1.5 times the return to the consumption claim. The rationale for looking directly at a levered consumption claim is two-fold. First, the dynamics of the consumption claim are more directly related to the learning problems we consider. Any dynamics in the idiosyncratic component

years on average and then lasts on average for 20 years. He uses CARA preferences and focuses on small sample "Peso" explanations of the high historical stock returns. We have not found a paper that explicitly analyses general equilibrium implications of parameter uncertainty in a power utility model, but it is our impression that the intuition given in the text is a 'folk theorem' known to many in the profession.

of dividends obfuscates this relation. Second, while it is straightforward to price a claim to an exogenous dividend stream in our setup, different but common assumptions regarding the dividend dynamics can give quite different asset pricing results. For instance, if one as in Abel (1999) models dividends as simply $\lambda\Delta c_t$, with $\lambda = 3$, the dividend claim would be much more sensitive to fluctuating expectations of the long-run mean of the economy than the consumption claim is. If, instead, one models dividends as cointegrated with consumption, as in most DSGE models, this long-run sensitivity is the same as for the consumption claim. While it is important to understand the joint (long-run) behavior of dividends and consumption, this is not the focus of this paper. We simply note that our definition of market returns is conservative in terms of its exposure to long-run risks relative to the long-run risk model of Bansal and Yaron (2004). Since we assume no idiosyncratic risk, the volatility of the market return will be low. However, the risk premium of this claim, which derives from the covariance of returns with the pricing kernel, is a quantity we can reasonably compare to the average excess equity returns in the data. We will in a separate section consider a couple of different specifications of the dividend growth process to show how different assumptions about dividend dynamics affect the risk premium and return volatility.

3.1 Results

We calibrate the true consumption dynamics to match the mean and volatility of time-averaged annual U.S. log, per capita consumption growth, as reported in Bansal and Yaron (2004): $E_T[\Delta c] = 1.8\%$ and $\sigma_T(\Delta c) = 2.72\%$. This implies true (not time-averaged) quarterly mean and standard deviation of 0.45% and 1.65%, respectively. The models are calibrated at a quarterly frequency. For the cases with parameter uncertainty, the prior beliefs about μ are assumed to be distributed as a truncated normal. The truncation ensures that utility is finite. The lower bound is set at a -1.2% annualized growth rate, while the upper bound is set at a 4.8% annualized growth rate. The prior beliefs are assumed to be unbiased.⁷ Our baseline model has $\beta = 0.994$, $\gamma = 10$ and $\psi = 2$.

⁷Note that the updating equations for the mean and variance parameters for the prior are the same regardless of whether the distribution is truncated or not – the truncation only affects the limits of integration and not the functional form of the priors. Thus, we retain the conjugacy of the standard normal prior. We solve the models numerically, working backwards from the known-parameters boundary values on a grid for μ and time t (or, equivalently, a grid for the posterior standard deviation, σ_t ; see Johnson (2007)).

3.1.1 The effect of parameter uncertainty over time

First, we show how parameter uncertainty affects asset pricing moments over time. Note that the updating equation for the variance of beliefs (see Equation 9) is deterministic, and so this exercise captures the non-stationary aspect of parameter learning. At this point, we do not calibrate the prior dispersion, but simply start with a maximum standard deviation of prior beliefs, σ_0 , set to 1.65% – i.e., equal to σ . This is the same as assuming investors at the beginning of the sample has observed only one consumption growth realization with a completely diffuse earlier prior. The prior mean belief is set to the true value of mean quarterly consumption growth, 0.45%.

Table 1 shows the ensuing decade by decade asset pricing moments averaged across 20,000 simulated 100-year economies that all start from the same initial prior. The prior standard deviation at the beginning of each decade is given in the second column of the table, as implied by the deterministic updating equation given in Equation (9). For instance, after 10 years of learning, the prior standard deviation over the mean drops from 1.65% to 0.26%, after 50 years the standard deviation of beliefs is 0.12% and after 100 years it is 0.09%. Thus, while the standard deviation of beliefs decreases very quickly the first 10 years, the decrease is quite slow thereafter.

[TABLE 1 ABOUT HERE]

Column 3 gives the annualized conditional volatility of the log pricing kernel, $\sigma_t(m_{t+1})$, which is a measure of the maximal Sharpe ratio attainable in the economy. The conditional volatility of the log pricing kernel is on average 1.05 in the first decade, 0.87 in the second decade, 0.61 in the fifth decade, and 0.48 in the tenth decade. This is compared to the conditional volatility of the log pricing kernel in the benchmark economy with known parameters, which is only 0.33. Thus, after 50 years of learning, the volatility of the pricing kernel is twice as high as in the fixed parameters case, while after 100 years of learning it is one and a half times as high as in the fixed parameter benchmark case. Clearly, parameter uncertainty in this economy has long-lasting effects.

The slow decrease in the volatility of the pricing kernel is striking compared to the very fast decline that occurs in a power utility model. The reason the decrease is so slow is that the sensitivity of the continuation utility to shocks to growth expectations is endogenously increasing over time, offsetting the decline in the posterior variance. The intuition is

straightforward: when the prior variance is high, discount rates are endogenously high and so the wealth-consumption ratio is less sensitive to shocks to growth rates. As parameter uncertainty decreases, discount rates decrease and get closer to the expected growth rate, and thus the sensitivity of the wealth-consumption ratio to shocks to the expected consumption growth rate is higher. We explain these general equilibrium dynamics in detail in Section 3.1.4.

Columns 4 – 7 in Table 1 show the mean risk-free rate, the difference between the 10-year zero-coupon default-free real yield and the short-term risk-free rate, the average market excess return and return volatility. Though the mean belief about the growth rate averaged across the 20,000 samples is at its true value, the risk-free rate is increasing through time. This is due to a decrease in the pre-cautionary savings component as the amount of risk decreases deterministically as the agent beliefs about the mean growth rate become more precise. This upward drift in the risk-free rate is reflected in yield spreads, which are positive the first 50 years or so of learning and effectively zero thereafter. This is notably different from the standard long-run risk models, which have strongly negatively sloped real yield curves (see Beeler and Campbell, 2012).

The annualized market risk premium is 11% in the first decade, 4.5% in the fifth decade, and 3% in the tenth decade, compared with 1.7% in the known parameters benchmark economy. A similar decreasing pattern holds for the standard deviation of market returns. Even after 100 years of learning, the excess volatility is still a sizable 24% of fundamental volatility; 6.2% versus the benchmark economy’s 5%.

3.1.2 Average moments over a long sample

Given that parameter uncertainty has a long-lasting impact on standard asset price moments, we next evaluate the asset pricing implications of the model, given a plausibly calibrated prior, for the standard long-sample asset price moments the literature typically considers. In particular, Table 2 shows 100-year standard sample moments averaged across the simulated economies, as well as the corresponding moments in the U.S. data taken from Bansal and Yaron (2004). We set the standard deviation of initial prior beliefs about the mean growth rate to 0.26%, which corresponds to a standard deviation of the annual growth rate of 1.04%. The Shiller data has real per capita consumption data available from 1889. The standard error of the estimated mean annual growth rate using this data up until a hundred years ago, in 1910, is in fact slightly higher at 1.12%. The prior mean beliefs are set equal to the

true mean of consumption growth.

[TABLE 2 ABOUT HERE]

The third columns of Table 2 shows that the model with parameter uncertainty (*unknown* μ) yields a 100-year average excess annual market returns of 4.4%, compared to the 1.7% of the benchmark fixed parameter model (column 4; *known* μ). The risk premium in the data is higher still at 6.3% per year. The average annual volatility of the log pricing kernel in the learning model is 0.60, compared to 0.33 in the known parameters case. While the historical Sharpe ratio of equity returns is 0.33, the annual correlation between equity returns and consumption growth in the Shiller data is about 0.55 and so the pricing kernel need to have a volatility greater than or equal to 0.6 ($=0.33/0.55$) to match this value. Due mainly to no idiosyncratic component of dividends, the equity return volatility is too low in all the models relative to the data. The return volatility of the learning model is 7.35% versus the benchmark "fundamental" volatility of the known parameter case of 5%. Thus, the *excess volatility* (Shiller, 1980), measured as the ratio of standard deviation of returns in the learning case versus the standard deviation of returns in the no-learning case minus one, is 0.47 in the learning model. As reported by Bansal and Yaron (2004), the corresponding ratio of standard deviation of returns relative to the standard deviation of dividend growth minus one is 0.70. Thus, while the learning model does not generate quite as much excess volatility in relative terms, it goes a long way towards what is in the data. Due to the i.i.d. consumption growth assumption, the known parameter benchmark case features no excess volatility. Finally, the risk-free rate is low in the learning model and not too volatile, due to the high level of intertemporal elasticity of substitution, while the yield spread is on average slightly positive due to the on average upwards trend in real rates as agents become more sure of the mean growth rate. In sum, in terms of these unconditional sample moments, the simple learning model does quite well.

The two rightmost columns in Table 2 show the same moments for a model where the agent has power utility and thus is indifferent to the timing of the resolution of uncertainty. In this case, risk aversion is still 10, but the EIS is 0.1. The annual equity premium with no learning is 1.7%, but the equity premium with learning is -1.4% . This is due to the low EIS as an increase in investors perception of the expected growth rate in this case decreases the price-consumption ratio sufficiently to make stock returns negatively correlated with

consumption growth (see Veronesi (2000)). Also, note that the learning does not increase the volatility of the log pricing kernel relative to the known parameters case in the 100-year sample, at least not to the second decimal, as expected. The indifference to the timing of the resolution of uncertainty means that the fact that shocks to growth expectations are permanent is immaterial for the conditional volatility of this investor’s intertemporal marginal rates of substitution.

3.1.3 Predictability of returns, not consumption

The fixed parameter benchmark case features no predictability of excess returns or consumption growth by construction since consumption growth is assumed to be i.i.d. However, in the data excess equity market returns are predictable. A standard predictive variable is the price-dividend ratio. On the other hand, as emphasized by Beeler and Campbell (2012), aggregate consumption growth is not predicted by the price-dividend ratio in U.S. data. Further, Lettau and Ludvigson (2001) show that a measure of the wealth-consumption ratio also predicts excess returns but not long-horizon consumption growth. The latter point has been a bit of a sticking point for long-run risk models that rely on a small, but highly persistent component in consumption growth, as these models counterfactually imply that the price-dividend and price-consumption ratios should predict future, long-horizon consumption growth.

In the model presented here with parameter learning, there is no consumption growth predictability. The agent will ex post perceive the mean of consumption growth as changing, but in reality it is not (by assumption), and so the price-consumption ratio in the models with parameter uncertainty will not in population predict future consumption growth. Nevertheless, there is, in small samples, a correlation between the current price-consumption ratio and future consumption growth: if early consumption growth realizations happened to be high relative to the remainder of the sample, the price-consumption ratio will be negatively correlated with future consumption in-sample. Table 3 shows forecasting regression results for consumption growth and excess returns. The reported statistics are sample medians from the 20,000 simulated 100-year economies discussed previously.

[TABLE 3 ABOUT HERE]

Panel A of Table 3 shows that this small-sample correlation is not significant at the 1- or 5-year consumption growth forecasting horizons for the median economy. The average

standard errors reported are Newey-West with lags accounting for autocorrelation on account of quarterly overlapping observations. Panel A also reports the risk-free rate regression of Hall (1988) on the simulated data. In particular, we regress quarterly consumption growth on the lagged risk-free rate. In a model with constant volatility of the pricing kernel, the coefficient on the real risk-free rate is a measure of the elasticity of intertemporal substitution, which in our model is 2. However, the reported median regression coefficient is -0.02 and insignificant, and the R^2 is low. This magnitude of the regression coefficient is consistent with what Beeler and Campbell (2012) show empirically. They also note that simulated data from the long-run risk model of Bansal and Yaron (2004) yields estimates of the EIS well in excess of 1. In the learning model, consumption growth is in fact unpredictable. The variation in the risk-free rate is due to time-variation in agents' perceived mean consumption growth rate, which is a function of their current beliefs. Thus, the long-run risk that arises through this learning channel does not result in counter-factual estimates of the EIS using the Hall-type regressions, even though the representative agent's elasticity of intertemporal substitution is in fact very high. In sum, the model with parameter uncertainty is a long-run risk model that addresses two of the main critiques Beeler and Campbell (2012) levy against long-run risk models.

Panel B of Table 3 addresses excess equity return predictability at the 1- and 5-year horizon. Here, the price-consumption ratio significantly predicts both 1- and 5-year equity returns with R^2 's of 7% and 31%, respectively, over the median 100-year economy. These R^2 values are close to those reported in Beeler and Campbell (2012) who use the price-dividend ratio as the predictive variable. While not reported, the R^2 of the predictability regression is higher in the first 50 years than in the last 50 years as the effect of parameter uncertainty slowly wanes. This is broadly consistent with the evidence on excess return predictability using the price-dividend ratio as the predictive variable (see, e.g., Lettau and van Nieuwerburgh (2008)). Note that the return predictability arises both because excess returns are in fact predictable and because of an in-sample correlation between the price-dividend ratio and future returns. The in-sample relation is the same as that for consumption growth – if consumption has happened to be high, returns will also have been high, while the price-dividend ratio will have increased as investors' mean belief about the growth rate increases. Going forward, then, the returns are lower in an in-sample sense, and so there is a negative relation between the price-dividend ratio (or wealth-consumption ratio) and future excess returns (see also Timmermann (1996)). This evidence also implies that out-of-sample predictability is much lower than in-sample predictability, consistent with the

empirical findings of Goyal and Welch (2006). Figure 2 show these dynamics by plotting a representative sample path of the ex ante annualized risk premium versus the ex post risk premium as predicted by the forecasting regression in Panel B of Table 3.

[FIGURE 2 ABOUT HERE]

3.1.4 Inspecting the mechanism

There are two particularly surprising results regarding the asset pricing implications of parameter learning when agents have a preference for early resolution of uncertainty. The first is that the volatility of the pricing kernel decreases at a much slower rate than the posterior variance of beliefs. The second is that after 100 years of learning, when the shocks to growth expectations are tiny – with a standard deviation of only 0.0041% per quarter – these long-run shocks increase the volatility of the pricing kernel by a factor of almost 1.5 relative to the known parameter benchmark economy. Here, we explain the economic rationale for both of these results in more detail.

[FIGURE 3 ABOUT HERE]

Long-lasting effects of learning. The analysis of the learning model points to a nonlinear relation between the level of parameter uncertainty, as measured by the level of the variance of beliefs over time (see Figure 1), and the impact of parameter learning as measured by standard asset pricing moments. Figure 1 shows that the posterior standard deviation initially decreases very rapidly – after 50 years it is 14 times smaller than the initial maximum prior dispersion of 1.65%. The top plot of Figure 3, however, shows that the standard deviation of the log pricing kernel – the price of risk – drop by a factor of about 2 over the same period. Over the next 50 years, the posterior standard deviation drops by a factor of 1.4, while the price of risk drops by a factor of about 1.3.

To understand these dynamics better, it is useful to consider the two components of the pricing kernel, as given in Equation (5), separately. In particular, the middle and bottom plots in Figure 3 show the annualized standard deviation of the two components of

the log pricing kernel separately as a function of time. The "Power utility component" is $\ln \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$, while the continuation utility component is $\ln \left(\beta \frac{PC_{t+1}+1}{PC_t} \right)^{\theta-1}$. The top plot shows that the volatility of the "power utility component" is always very close to the known-parameter benchmark price of risk, $\gamma \times \sigma = 0.33$. In the very beginning of the sample, the volatility is only slightly higher, which reflects the fact that subjective consumption growth volatility is slightly higher due to parameter uncertainty. This is the standard intuition we get from the power utility model: parameter learning has only a small, and highly transient, impact on the price of risk.

The bottom plot shows the conditional volatility of the "continuation utility component" as a function of time. With known parameters, this component has a conditional volatility equal to zero. In the parameter uncertainty case, however, the conditional volatility starts at about 0.8 and ends, after 100 years of learning, at about 0.15. Casual intuition would suggest a much quicker decline. In particular, from Equation (11), we have that the volatility of the shocks to the mean parameter belief, in the untruncated normal case, is $\frac{\sigma_t^2}{\sqrt{\sigma_t^2 + \sigma^2}}$, which decreases by a factor of 10 over the 100 year sample calibrated with an initial quarterly prior dispersion of 0.26%. Thus, it would seem as though the amount of long-run risk decreases by a factor of 10 over the sample. Applying the intuition from the Bansal and Yaron model, this should relatively directly be reflected in a corresponding decrease in the volatility of the continuation utility component of the pricing kernel. However, in the case with parameter learning, the volatility dynamics are non-stationary which lead to an endogenous time-dependence in discount rates. In particular, endogenously high discount rates in the beginning of the sample make the consumption claim (total wealth) of relatively short *duration*. Thus, a given shock to mean parameter beliefs, μ_t , has a lower effect on total wealth early in the sample than later in the sample, when discount rates are lower and total wealth has relatively high duration. This intuition is confirmed in the top plot in Figure 4, which shows that the average path of the price-consumption ratio is increasing over time, indicating that discount rates decrease over time.⁸ The middle plot of Figure 4 shows the numerical derivative of the log price-consumption ratio with respect to the mean parameter belief, μ_t , evaluated at the true mean, μ . This sensitivity is increasing over time. As mentioned, the volatility of the shocks to μ_t is rapidly decreasing over the sample. The net outcome of the two effects is shown in the bottom plot in Figure 4, which shows that $\frac{\sigma_t^2}{\sqrt{\sigma_t^2 + \sigma^2}} \times \frac{\partial p_{ct}}{\partial \mu_t} |_{\mu_t = \mu}$ over time is decreasing, but at a slow rate corresponding more closely to

⁸Since the subjective growth rate averaged across the 20,000 simulated economies is approximately constant, the increase in the P/C-ratio must come from a decrease in the discount rate.

the slow decline in the price of risk as shown in Figure 3.

[FIGURE 4 ABOUT HERE]

In sum, the asset pricing implications of parameter learning are large and long-lived due to the interaction of permanent shocks to beliefs about growth rates (subjective long-run consumption risk) and an endogenously increasing sensitivity of continuation utility with respect to these updates in beliefs.

Dynamics in the context of the Bansal and Yaron model. The approximate analytical solution to the Bansal and Yaron (2004) model provides a useful way to gain further intuition for the mechanics of the parameter learning case. Consider the homoskedastic case of the Bansal and Yaron model:

$$\Delta c_{t+1} = \mu + x_t + \sigma \varepsilon_{t+1}, \quad (13)$$

$$x_{t+1} = \rho x_t + \varphi \sigma \eta_{t+1}, \quad (14)$$

where both ε and η are i.i.d. normal shocks. We can, for intuition, think of these consumption dynamics as approximating the subjective consumption dynamics of the parameter learning case if we set ρ very high, say $\rho = 0.9999$, where x_t measures the time-variation in the long-run growth rate. The approximate solution to this model yields:

$$pc_t = A_0 + A_1 x_t. \quad (15)$$

Thus, the sensitivity of the log price-consumption ratio to x_t is $A_1 = \frac{1-1/\psi}{1-\kappa_1\rho}$, where $\kappa_1 = \frac{\exp(\rho c)}{1+\exp(\rho c)}$ is an equilibrium quantity. The question is how this sensitivity depends on changes in the amount of long-run risk, as given by the parameter φ in the Bansal and Yaron model. With the parameters we consider, where $\psi = 2$ and $\gamma = 10$ (and so $\theta < 0$), we get that $\frac{d\kappa_1}{d\varphi} < 0$ and so $\frac{dA_1}{d\varphi} < 0$. That is, the unconditional level of the price-consumption ratio increases when the amount of long-run risk, φ , decreases. This in turns means that the sensitivity of the price-consumption ratio to changes in x_t increases as φ decreases, analogously to what we find in the parameter learning case.

Next, we turn to the level effect of the very small volatility of the long-run shocks the learning model implies after 100 years. After this long of a history of learning, the decrease

in the posterior variance is very slow. Therefore, we can reasonably look at the magnitude of the long-run risk effect using the Bansal and Yaron model, which has constant volatility of long-run shocks, as a laboratory, assuming that $\rho = 0.9999$. In particular, the shocks to the log stochastic discount factor in the Bansal and Yaron economy is given by:

$$m_{t+1} - E_t[m_{t+1}] = -\gamma\sigma\varepsilon_{t+1} - (\gamma - 1/\psi) \kappa_1 \frac{\varphi}{1 - \kappa_1\rho} \sigma\eta_{t+1}. \quad (16)$$

In the learning case, the two shocks are perfectly positively correlated (see Equations (10) and (11)). Thus, we have that:

$$\sigma_t(m_{t+1}) = \left(\gamma + (\gamma - 1/\psi) \kappa_1 \frac{\varphi}{1 - \kappa_1\rho} \right) \sigma. \quad (17)$$

To mimic our quarterly calibration after 100 years of learning, we set $\rho = 0.9999$, $\gamma = 10$, $\psi = 2$, $\beta = 0.994$, $\sigma = 0.0165$, $\mu = 0.0045$ and $\varphi = 0.00411\%/ \sigma = 0.2491\%$. Given these parameters, we find the equilibrium $\kappa_1 = 0.9955$. This yields $\sigma_t(m_{t+1}) = 0.2495$ which means the *annualized* log volatility is 0.499 versus 0.33 in the benchmark, known parameters case. Thus, the very high persistence of the shocks and the fact that the long-run risk shocks are perfectly correlated with the shocks to realized consumption growth combine to generate approximately a 1.5 time increase in the volatility of the log pricing kernel, relative to the benchmark case where there is no long-run risk. This is very close to the magnitude we find in the numerical solution for the non-stationary learning problem after 100 years of learning.

Robustness of results to alternative dividend dynamics.

The equity claim considered so far has simply been a levered consumption claim. It is, however, common in the literature to specify exogenous dividend dynamics that feature a high loading on a consumption shock as well as idiosyncratic shocks. Here we consider two alternatives: Case 1:

$$\text{Case 1 : } \Delta d_{t+1} = \lambda \Delta c_{t+1} + \delta (c_t - d_t) + \sigma_d \varepsilon_{d,t+1}, \quad (18)$$

$$\text{Case 2 : } \Delta d_{t+1} = \mu_0 + \lambda (\Delta c_{t+1} - \mu_0) + \sigma_d \varepsilon_{d,t+1}. \quad (19)$$

Case 1 has dividends as cointegrated with consumption over long-horizons. The leverage parameter λ is set to 3, and the quarterly idiosyncratic shock volatility is set to 5.75%. The autocorrelation of the consumption-dividend ratio is calibrated to NIPA data from

1929 – 2010, which yields the error-correction variable $\delta = 0.003$. Case 2 does not impose cointegration between consumption and dividends, but it does impose that dividend and consumption growth have the same unconditional growth rate. In this case, $\lambda = 2.25$ and $\sigma_d = 4.5\%$. The leverage parameter and the volatility of idiosyncratic risk are in both cases calibrated to (roughly) match an annual dividend growth volatility of 11.5%, as reported in Bansal and Yaron (2004) and a annual correlation between consumption growth and stock returns of 0.55, which is the same as that in the Shiller data for the 100-year period 1910 – 2010.

[TABLE 4 ABOUT HERE]

Table 4 shows the risk premium, return volatility, and Sharpe ratio over the same 100-year samples as those shown earlier. The risk premiums in both cases are higher than for the levered consumption claim, as given in Table 2 – 6.5% for Case 1 and 5.1% for Case 2. The return volatility is still somewhat too low at 14.7% and 13.0%, though above the volatility of dividend growth, which is 11.6% and 11.8%. Given the added idiosyncratic risk, the return volatilities of these alternative equity claims are of course quite a bit higher than that for the levered consumption claim. The Sharpe ratio of returns are in both cases a little higher than that in the data. Notably, at the end of the 100-year sample, the conditional, annualized risk premium on both claims is close to 4.1% and 3.9% versus 2.2% and 2.5% in the known mean benchmark case. In sum, the results reported for the market return as a levered consumption claim are robust to common alternative specifications of the dividend dynamics.

3.2 Unknown variance

In the preceding, the variance parameter σ^2 was assumed known to investors. It is straightforward to relax this assumption, though as pointed out in Weitzman (2007) and Bakshi and Skouliakis (2010), it is necessary to truncate also the support for σ^2 in order to ensure finite utility. Weitzman (2007) argues that learning about the variance parameter can lead to arbitrarily high risk premiums as the subjective distribution for consumption growth becomes fat-tailed. He further argues that learning about the mean, as in the preceding section, does not increase the fatness of the tails of the conditional consumption growth distribution and therefore cannot help in explaining asset pricing puzzles. Clearly, the latter intuition does

not hold when considering a utility function that allows for a preference for early resolution of uncertainty.⁹

Bakshi and Skouliakis (2010) argue that Weitzman's results, which are developed under power utility, are not robust to reasonable truncation limits for σ^2 . However, given that we focus primarily not on the fatness of the tails, but on permanent shocks to the conditional consumption growth distribution induced by the learning process itself, uncertain variance can potentially still have important asset pricing implications. In the following, we show that quantitatively large asset pricing implications of learning about the variance parameter indeed can arise, but that interesting asset pricing effects of learning about the variance parameter are shorter-lived than those documented for the uncertain mean case.

We assume that the joint prior over the mean μ and the variance σ^2 is Normal-Inverse-Gamma:

$$p(\mu, \sigma^2 | y^t) = p(\mu | \sigma^2, y^t) p(\sigma^2 | y^t), \quad (20)$$

where

$$p(\sigma^2 | y^t) \sim IG\left(\frac{b_t}{2}, \frac{B_t}{2}\right), \quad (21)$$

$$p(\mu | \sigma^2, y^t) \sim N(a_t, A_t \sigma^2). \quad (22)$$

Given that log consumption growth is normally distributed, these prior beliefs lead to posterior beliefs that are of the same form (conjugate priors). The updating equations for investors' beliefs are:

$$A_{t+1}^{-1} = 1 + A_t^{-1}, \quad (23)$$

$$\frac{a_{t+1}}{A_{t+1}} = \frac{a_t}{A_t} + y_{t+1}, \quad (24)$$

$$b_{t+1} = b_t + 1, \quad (25)$$

$$B_{t+1} = B_t + \frac{(y_{t+1} - a_t)^2}{1 + A_t}. \quad (26)$$

In terms of pricing, note that this system can be reduced to three state-variables: a_t , B_t , and t , given initial priors. We solve the model numerically and, as before, use the closed-form solution for the known parameters cases as the boundary values in a recursion that is solved

⁹In fact, with a truncated normal as the prior, the tails of the subjective distribution are actually less fat than for a normal distribution with the same dispersion, but due to the updating that generates long-run risks, the asset pricing implications were shown to be nontrivial.

backwards in time on a grid for a_t and B_t . In order for the Inverse Gamma distribution to have a finite mean and variance, which is convenient, we set the max prior uncertainty as $b_0 = 5$. As mentioned, we need to truncate the distribution for σ^2 and we choose wide bounds: $\overline{\sigma^2} = 100 * \sigma^2$, $\underline{\sigma^2} = \sigma^2/100$. As before, the true quarterly variance is calibrated as $\sigma^2 = (1.65\%)^2$, and the model is solved at the quarterly frequency. The other parameters of the model are the same as in the case where the mean was the only unknown parameter: $a_0 = \mu = 0.45\%$, $A_0 = 1$, $\gamma = 10$, $\psi = 2$, and $\beta = 0.994$. We set $b_0 = 5$ and $\frac{B_0}{b_0-2} = \sigma^2$. The latter implies that the initial truncated prior for the variance is unbiased, with a standard deviation of $(1.85\%)^2$.

[FIGURE 5 ABOUT HERE]

Figure 5 shows the conditional annualized volatility of the log pricing kernel as the average per quarter across 20,000 simulated economies over a 100 year sample. We plot three cases. Learning about the mean only, as discussed in the previous section, learning about the variance only, and learning about the mean and the variance parameters. First, consider the dashed line, which shows the case when learning about the variance only. The volatility of the pricing kernel is very high in the first decade, but then comes down quite quickly towards the benchmark, known parameter value of 0.33.¹⁰ Pretty much all of this pattern comes from the continuation utility component of the pricing kernel and not from the power utility component. Thus, we confirm the results in Bakshi and Skouliakis (2010), the fatness of the tails given reasonable bounds on the variance parameter is not sufficient to strongly affect asset prices. However, the large updates in beliefs about the variance that occur in the first 10 years does have significant impact on the volatility of the pricing kernel through the effect on the continuation utility. After this, the impact of shocks to beliefs about the variance parameter have a very small impact. The dash-dotted line shows the case of unknown mean *and* variance. Here, we see that adding unknown variance yields a pricing kernel that is on average always more volatile than in the known variance, unknown mean case. However, there are only large differences in the first decade, relative to the case with only unknown mean (dotted line).

¹⁰The somewhat uneven line for the variance cases in the 5 first years is due to the truncation bounds slightly affecting the form of the subjective distribution for the variance parameters when the level of uncertainty is very high.

The risk premium for a 100 year long sample that start with priors corresponding to tossing out the 10 first years plotted in Figure 5, is 1.8% for the case of unknown variance but known mean, relative to 1.7% for the benchmark known parameters economy. In the case of unknown mean and variance, the average risk premium over this sample is 4.9% compared to 4.4% for the case of unknown mean and known variance.

In sum, unknown variance has more of a second-order effect on asset pricing moments, unless uncertainty is very large, as would be the case in the decade after a structural break for instance. There are two reasons for this more short-lived effect. First, Bayesian learning implies that learning about variance is much faster than learning about the mean. Second, the variance is a second order moment, so generally less important for the continuation utility than changes in the mean.

4 Case 2: Learning about rare events

Uncertainty about parameters that govern rare events is likely to be large, as rare events by their very nature yield few historical observations available for agents to learn from. In recent work, Barro, Nakamura, Steinsson, and Ursua (2011), hereafter BNSU, estimate that consumption disasters occur with a probability of 2.8% per year using the longest consumption series as available from a wide cross-section of countries. This enables them to estimate disaster parameters with some degree of accuracy. For instance, the standard error of their estimate of the probability of a world disaster is 1.6%. Consumption volatility in disasters is estimated to be 12%, and a disaster is estimated to on average lead to a -15% permanent negative shock to consumption. The latter quantity is estimated with a standard error of 4.2%.¹¹ So, even after using all historical data available in both the time-series and the cross-section of countries, there is quite a bit of uncertainty about the parameter estimates.

We will consider a simpler model for consumption disasters relative to BNSU, but the parameters and the associated parameter uncertainty is calibrated to their estimates as far as possible. In particular, let:

$$\Delta c_{t+1} = g_{t+1} (1 - D_{t+1}) + z_{t+1} D_{t+1}, \quad (27)$$

¹¹In BNSU, the disaster state lasts on average for almost 6 years. It has a mean growth rate of -2.5% per year, which is estimated with a standard error of 0.7%. We "truncate" the 6 years into one quarter-long disaster event, which has a mean of $-2.5\% * 6 = -15\%$ with estimation error $6 * 0.7\% = 4.2\%$. Thus, consumption growth is still i.i.d. in this model.

where

$$g_{t+1} = \mu_N + \sigma_N \varepsilon_{t+1}, \quad (28)$$

$$z_{t+1} = \mu_D + \sigma_D \varepsilon_{t+1}, \quad (29)$$

where ε is i.i.d. standard normal and where D_{t+1} is 1 with probability λ and 0 with probability $1 - \lambda$. We assume that investors observe D_{t+1} . Given the very large average initial consumption decline in a disaster, as estimated by BNSU, learning whether you are in a disaster or not would not add much as the agent would be able to tell pretty much immediately anyway. There are two other simplifying assumptions here. First, true consumption growth is still i.i.d., whereas BNSU estimate the average disaster lasts for 6 years. Second, we only consider the permanent shocks to consumption and not the transitory effects BNSU also consider. Keeping the i.i.d. nature of consumption growth as in our initial case means that any asset price dynamics comes from the learning channel alone.

We set the disaster probability, mean and volatility to $\lambda = 2.8\%$, $\mu_D = -15\%$, $\sigma_D = 12\%$, respectively. We calibrate the mean and volatility in the good state such that we match the same unconditional consumption moments as before. In particular, the unconditional mean and volatility of non-time-averaged quarterly consumption are 0.45% and 1.65%, respectively. We consider two cases of parameter uncertainty; first, about the mean in the disaster state, μ_D , and then about the probability of a disaster, λ .

4.1 Uncertain mean of disaster state

We take the estimation uncertainty from BNSU of the disaster mean of 4.2% as the initial prior standard deviation here. Thus, this exercise is forward-looking in the sense that this is the best estimate available using all data up until now. The prior is assumed to be unbiased with a mean of -15% .¹² Clearly, the agent will only learn about the disaster mean when a disaster state occurs, which is what makes learning much slower in this case. With 2.8 disasters per 100 years, learning is loosely speaking 35 times slower than in the simple i.i.d. case considered previously. Since learning in this case only happens quite rarely, there is little dynamics induced by learning in terms of excess volatility, return predictability, and time-variation in the price-consumption ratio. In particular, the price-consumption ratio is constant between disasters and return volatility only reflects realized consumption growth

¹²As before, the prior is a truncated normal. The upper and lower truncation bounds are set at ± 3 standard deviations of the initial prior (4.2%) around the mean of the initial prior (-15%).

during normal times. Given this, it is clear that outside of disasters, parameter uncertainty about the disaster mean will mainly give an increase in the level of Sharpe ratios and the risk premium. In a disaster, however, there is quite a bit of learning as the initial prior uncertainty is large.

The two middle columns of Table 5 shows average moments from 20,000 simulated economies with 100 year samples given the initial prior, as well as the known disaster mean benchmark model. The preference parameters are $\gamma = 7$ and $\beta = 0.993$ and $\psi = 2$, similar to the parameters used in BNSU. The time-preference parameter β is calibrated to roughly match the real risk-free rate. This parameter is important as it determines the effective duration of a permanent shock in terms of its effect on the continuation utility.

[TABLE 5 ABOUT HERE]

In the uncertain disaster mean case, the risk premium is 5.1% versus 3.3% in the known disaster mean case. The volatility of the log pricing kernel is 1.23 and 0.82, respectively. Thus, parameter uncertainty increases these quantities by about 50%. Note that this increase is very large relative to the small increase in subjective consumption growth volatility in the disaster state. In particular, for the initial observation, subjective consumption growth volatility in the disaster state is $\sqrt{0.042^2 + 0.12^2} = 0.127$ – only slightly higher than the objective volatility of 0.12. Again, it is the long-run risk aspect of the shocks to beliefs that makes the price of risk increase by as much as it does. The risk-free rate is low as in the data by construction and lower than in the known parameter case as there is more risk with uncertain parameters. The excess volatility is very small for the parameter uncertainty case, as learning only occurs very rarely when disasters occurs. Related, there is no economically significant return predictability in this model (not reported). Thus, uncertainty about parameters that one can only learn about during the rare event itself can have a large effect on the level of the risk premium and maximum Sharpe ratio, but will not lead to interesting dynamics in the price of risk and/or the risk premium in normal times. On the other hand, for these type of learning problems, it will take a very long time for agents to learn and thus for the asset pricing implications to become economically insignificant. In particular, the conditional volatility of the log pricing kernel after 100 years averaged across the 20,000 simulated economies is 0.96 times the initial conditional volatility of the log pricing kernel.

4.2 Uncertain probability of disasters

The posterior standard deviation of the BNSU estimate of the probability of a world disaster is 1.6%. We calibrate the model at the quarterly frequency, and so set $\lambda = 0.7\%$ with a prior standard deviation of 0.4%. The Beta-distribution yields a conjugate prior for a probability, and so we assume that the prior at time t is $\lambda \sim \beta(a_t, b_t)$. Here a_t is the number of times a disaster has happened, while b_t is the number of times the normal state has occurred. The 0.4% standard error reported by BNSU is roughly consistent with having observed a total of 400 quarterly observations. Thus, we set $a_0 = 2.8$ and $b_0 = 397.2$. This means the mean belief is unbiased and equal to $0.7\% = \frac{a_0}{a_0 + b_0}$.

The two rightmost columns of Table 4 report the average 100-year sample moments across 20,000 simulated economies for the model with uncertain disaster probability and the benchmark case of known parameters. The preference parameters are $\gamma = 7$ and $\psi = 2$, as in the case of unknown disaster mean, and $\beta = 0.99$ is set to roughly match the level of the real risk-free rate. The average annualized volatility of the pricing kernel and risk premium are 1.5 and 7.3%, respectively. In the benchmark, known disaster probability case, these quantities are 0.8 and 3.3%. Thus, learning about the disaster probability appears to have a stronger effect than learning about the disaster mean. Further, the volatility of returns is 5.0% versus 4.1% in the benchmark case, and so the excess volatility measure is 0.21 – still short of what is in the data (0.7), but more than 4 times that of the case of uncertain disaster mean.

There are two reasons why uncertainty about the disaster probability helps more with explaining standard asset pricing moments than learning about the disaster mean. First, the subjective distribution about the disaster probability is positively skewed with high kurtosis. Thus, there is a non-trivial probability assigned to the disaster probability being relatively high. This is very risky for the agent as disasters are very bad events. Second, the updating is continuous. In normal times, there are no disasters, which is reflected in b_t increasing while a_t stays constant. If a disaster occurs, b_t is constant but a_t increases. Thus, each period agents revise their subjective beliefs about the disaster probability, which leads to time-variation in both the expected consumption growth rate and the equity premium. The latter effects lead to excess volatility in stock returns. In fact, Table 5 shows that excess returns are predictable in this model over the 100-year samples, while consumption growth is not – much like what was the case for when agents learn about the mean in the initial simple i.i.d. model. Figure 6 shows the conditional moments of the model over time averaged across the simulated

economies. The annualized conditional volatility of the pricing kernel decreases from about 1.6 to 1.0 over the sample, while the annualized conditional risk-premium decreases from about 10% to 5%.

[FIGURE 6 ABOUT HERE]

The average moments do not reveal all of the dynamics of the disaster models, however. In particular, while it is clear there is a decrease in the price of risk and the risk premium on average due to decreased parameter uncertainty, the actual sample paths look more interesting. For each disaster, the subjective belief of the disaster probability increases markedly, which is reflected in the price-consumption ratio, the Sharpe ratio and the risk premium. As long as a disaster does not occur, the subjective mean of the disaster probability decreases. Thus, there is a "saw-tooth" pattern in asset prices and beliefs when learning about the disaster probability. Figure 7 shows a representative sample path for the conditional risk premium in this model, which shows that the "saw-tooth"-pattern in beliefs is reflected in the risk premium. Each vertical increase in the risk premium happens when a disaster occurs.

[FIGURE 7 ABOUT HERE]

5 Case 3: Structural breaks

With parameter learning, rational agents will eventually learn any fixed parameter. Of course, "eventually" may be in a really long time, but still such parameter learning does not embody the notion of a "new" paradigm which anecdotally may be an important component of agents' belief formation (see the discussion in Hong, Stein, and Yu (2007)). Structural breaks, studied earlier in the context of asset pricing by for instance Timmermann (2001) and Pastor and Veronesi (2001), is a way to make parameter learning a recurring problem. In this section, we consider a structural breaks version of the simple i.i.d. consumption growth economy.

In particular, we assume that log aggregate consumption growth within paradigm s is given by:

$$\Delta c_{t+1} = \mu_s + \sigma \varepsilon_{t+1}, \tag{30}$$

where ε is i.i.d. standard normal, σ is the constant volatility parameter, and where μ_s is a paradigm-specific mean growth rate, where s denotes the s 'th paradigm. Each period there is a constant probability λ that there is a structural break. If a structural break occurs, a new mean growth rate μ_{s+1} is drawn from a normal distribution with mean μ and standard deviation σ_μ . The agent is assumed to know when a new paradigm has been drawn, but not the value of the mean of that paradigm, μ_{s+1} . Note that the fact that a redraw occurs does not imply that the new mean is far from the current mean. In fact, the assumption of a normal distribution for the μ_s 's along with a relatively small σ_μ means that the most likely outcome is around the unconditional mean. Likely candidates for times of a redraw includes the beginning and end of world wars, technological revolutions (e.g., the dot-com era), as well as the recent financial crises. The assumption of a constant λ is a simplification that it is easy to extend. However, this assumption makes the analysis cleaner as any dynamics in the price of risk and the risk premium will in this case come from the learning channel alone.

This model in many ways looks much like the original long-run risk model of Bansal and Yaron (2004). In both cases, the true conditional mean of consumption growth is time-varying, but very persistent. There are two main differences. First, in the structural breaks model, the mean is constant within each regime, which means that the agents in the economy face a paradigm-specific parameter learning problem. The parameter learning induces quite different dynamics relative to what learning about the long-run risk component in the Gaussian state space model of Bansal and Yaron would. Second, we calibrate the regimes to have an expected duration of 50 years. Thus, these are in fact even longer-run risks than those assumed in Bansal and Yaron (2004).

5.1 Results from a calibrated model

We calibrate this model at the quarterly frequency. Thus, we set $\mu = 0.45\%$, $\sigma_\mu = 0.25\%$, $\lambda = 0.5\%$, $\sigma = 1.65\%$. We truncate the normal distribution for the redraw at $\pm 4 \times \sigma_\mu$ around the unconditional mean, μ . We assume the meta-parameters μ and σ_μ are known to isolate the effect of the structural breaks assumption. We set the preference parameters as in the initial i.i.d. case: $\beta = 0.994$, $\gamma = 10$, and $\psi = 2$. We also consider a case with $\psi = 1.5$ to show the effect of decreasing the preference for early resolution of uncertainty by lowering the elasticity of intertemporal substitution.

[TABLE 6 ABOUT HERE]

Table 6 shows the 100-year sample moments averaged across 20,000 simulated economies. For the model with $\psi = 2$ and unobserved paradigm means, the risk premium is 4.7% per year and the annualized volatility of the log pricing kernel is 0.72. The volatility of returns is 6.5% which implies an excess volatility of 0.22 relative to the 5% volatility of cash flow growth. The risk-free rate is 1.2% with a volatility of 0.2%. Thus, the structural breaks model does much better than the known parameters model with i.i.d. consumption growth, as given in Table 2. However, when compared to the case of structural breaks with a known paradigm mean, the learning model has a *lower* price of risk and risk premium. In particular, for the known means structural breaks case, the annualized volatility of the log pricing kernel is 1.25 and the annualized risk premium is 6.8%. The volatility of returns is lower, however, at 5.4%.

Given the results in the first simple i.i.d. consumption growth case, where parameter learning gives a risk premium of 4.4% versus only 1.7% in the known mean benchmark case, the fact that learning in the structural breaks case decreases the price of risk and the risk premium relative to if the paradigm mean is observed might seem surprising. However, the reason is straightforward. In the simple i.i.d. case, the agent was learning about a fixed quantity, and so shocks beliefs gave a permanent shock to consumption growth expectations. Thus, in this case learning yields slightly higher short-run risk and much higher long-run risk. However, in the structural breaks case, while the individual paradigm means are constant, expected true consumption growth follows a stationary, mean-reverting process. In this case, the optimal learning smooths the beliefs about the conditional mean consumption growth relative to the unconditional, known mean μ . Thus, learning now yields *less* long-run risk and slightly more short-run risk. This is perhaps easiest to understand by referring to well-known Kalman filter results. Consider the Bansal and Yaron (2004) model, which is a Gaussian state-space model. If the conditional mean process (x_t) is unobserved, the filtered \hat{x}_t -process will have the same autocorrelation coefficient, but lower volatility than the true x_t process. In other words, *less* long-run risk.

The rightmost columns of Table 6 shows the case when $\psi = 1.5$, all else equal. Now, the price of risk and the risk premium decline for both the case of unknown and known paradigm means. In the learning case, the risk premium is now 4.0%, while it is 5.2% in the observed means case. Thus, as expected, a decrease in the preference for early resolution of uncertainty decreases the risk price for shocks to beliefs about future consumption dynamics.

5.1.1 Forecasting regressions and risk dynamics

While learning unconditionally decreases the price of risk in the structural breaks case relative to the case of known paradigm means, it does induce interesting dynamics. In particular, the known paradigm means case has constant price of risk and risk premiums. In the learning case, on the other hand, the risk premium and price of risk increases at the onset of a new paradigm as parameter uncertainty increases. Figure 8 shows a representative sample path for the annualized risk premium over a 100 year period. In this sample path, there are three breaks, around 5 years, 60 years, and 95 years. In each case, the risk premium shoots up from 2 – 3% to more than 8%, as parameter uncertainty jumps up due to the redraw of the paradigm mean. The lower plot of Figure 8 shows that the price-consumption ratio also typically decreases at the onset of a new regime. However, note that the decrease depends on the consumption growth realizations early in the new paradigm. There are cases where a high initial consumption growth realization causes the price-consumption ratio to move up on account of the ensuing high subjective belief about the paradigm mean, even though the risk premium still increases.

[FIGURE 8 ABOUT HERE]

Table 7 shows the forecasting regressions from the structural breaks model. As before, Panel A considers consumption growth predictability. In the structural breaks case, consumption growth is in fact predictable over very long horizons. However, as the regressions show, using the price-consumption ratio or the real risk-free rate as the predictive variables lead to no significant predictability. Again, the estimate of the EIS from the risk-free rate regression is negative and comparable to that found in the same regression run on the historical data, even though the EIS is in fact 2. First, the actual predictability in consumption growth is quite small over conventional forecasting horizons and, second, the subjective estimates of the growth rate are quite volatile, especially at the start of a new paradigm, and so the relation between the risk-free rate, which reflects the subjective beliefs, and future consumption growth, which reflects the truth, is very weak.

[TABLE 7 ABOUT HERE]

Panel B of Table 7 shows that the price-consumption ratio is, in the median simulated economy, insignificantly related to future 1- and 5-year excess returns. However, as also shown in Panel B, if one conditions the start of a 50-year sample period as being the start of a new paradigm, the price-consumption ratio reemerges as a significant return predictor also in the structural breaks model. The reason for this is that the structural breaks with the different growth paradigms create time-variation in the price-consumption ratio that is unrelated to the risk premium. By conditioning on a redraw of the paradigm mean at the beginning of the samples across the simulated economies, the true mean of consumption growth is constant for on average the next 50 years, and so the relation between the price-consumption ratio and future returns reemerges as in the "Case 1"-model considered earlier with parameter uncertainty about the unconditional mean growth rate. This is consistent with Lettau and van Nieuwerburgh (2008) who show that if one estimates structural breaks in the aggregate price-dividend ratio and removes the paradigm means from this ratio, the resulting adjusted price-dividend ratio is a much stronger predictor of future excess returns than the actual price-dividend ratio.

In sum, the structural breaks model delivers high Sharpe ratios, a high risk premium, excess return volatility, and excess return predictability. It does this with a high elasticity of intertemporal substitution, but still replicating the Hall (1988) regressions in the data. Further, the price-consumption ratio does not significantly predict future consumption growth up to the 5-year forecasting horizon. In this sense, the structural breaks model also addresses many of the critiques of the long-run risk models raised by Beeler and Campbell (2012).

The mean and volatility of the redraw distribution, μ and σ_μ , are both assumed known in this model, however. This is quite unrealistic as there effectively is only one observation every 50 years about this distribution. If one were to add parameter uncertainty about, say, μ as well, the asset pricing implications of the model are likely to take on aspects of the initial parameter uncertainty problem first considered in this paper. We conjecture that this would add risk, excess return volatility, and further return predictability.

5.2 Structural Breaks: This time is different

In this section we consider an economy where the representative agent suffers from a "this time is different"-bias (see Reinhart and Rogoff (2009)). In particular, let the true consump-

tion dynamics be as in the structural breaks economy considered in the previous section:

$$\Delta c_{t+1} = \mu_s + \sigma \varepsilon_{t+1}. \quad (31)$$

Different from the previous economy with structural breaks, however, the agent with the "this time is different"-bias prices assets as if the current paradigm will last forever. When a new paradigm arises (a new μ_s is drawn), the agent observes this event, which was unanticipated for him/her, and restarts the learning problem assuming this new regime now will last forever as "this time is different." Thus, this really is an economy that repeats the "Case 1" economy, where the representative agent is learning about a fixed growth rate, in each paradigm.

[TABLE 8 ABOUT HERE]

Table 8 shows unconditional 100 year moments from such an economy. The risk premium and return volatility are now 4.9% and 7.6%, respectively. The volatility of the log pricing kernel is 0.65. Thus, relative to the structural breaks case with no "this time is different"-bias, the return volatility is quite a bit higher (7.6% versus 6.5%). The risk-free rate has about the same level, but the risk-free rate volatility is also higher (0.6% versus 0.2%). The table also gives the moments for the case of no parameter uncertainty, but still a "this time is different"-bias (column with 'known μ '-header). As before, the risk premium is 1.67%, the volatility of the log pricing kernel is 0.33, and there is no excess volatility in this economy. The final two columns of Table 7 shows the corresponding cases when the representative agent has power utility preferences ($\gamma = 10 = 1/\psi$). As explained in the "Case 1" economy with parameter learning, the equity premium is negative as the wealth effects dominates and price-consumption ratio declines upon a high consumption growth realization. The low elasticity of intertemporal substitution gives rise to a risk-free rate puzzle. In sum, the preference for early resolution of uncertainty and a high elasticity of intertemporal substitution are necessary elements for the "This time is different"-economy to match standard asset pricing moments.

The high excess volatility in the "This time is different"-economy comes from repeating the learning problem in each paradigm and also the belief that the in each paradigm the new mean of consumption growth now will remain constant forever. Naturally, this leads to high return predictability, as shown in Panel B of Table 9. In particular, the R^2 's for the 1- and 5-year excess return forecasting horizons are about the same as in the data. Panel A

of Table 9 shows that again, there is no significant consumption growth predictability when using the price-consumption ratio or the risk-free rate as predictors. Thus, the estimate of the elasticity of intertemporal substitution from the Hall (1988) regressions are close to zero as in the data also for this model.

[TABLE 9 ABOUT HERE]

6 Conclusion

This paper finds that structural parameter uncertainty – that is, uncertainty about fixed parameters governing the exogenous aggregate endowment process of the economy – can have long-lasting, quantitatively significant asset pricing implications. This conclusion relies on rational learning, which implies that posterior probabilities regarding fixed quantities are martingales, and that agents have a preference for early resolution of uncertainty. For such agents, the updating of beliefs, with its associated permanent shocks to the conditional distribution of future consumption growth, constitutes an additional risk.

Bayesian learning is fast, but we show that asset pricing implications of such learning nevertheless can be long-lasting. The reason is an endogenous interaction between the increased precision of beliefs and the sensitivity of marginal utilities to shocks to these beliefs. For example, as agents become more certain about the mean growth rate of the economy, discount rates decrease due to reduced parameter uncertainty. However, the decreased discount rates makes the continuation utility more sensitive to shocks to the mean growth rate. The net effect is a much slower decrease in the volatility of the pricing kernel than in the posterior variance of the parameters. We show that even after 100 years of learning, the risk premium and maximal Sharpe ratio in the economy remains well above the fixed parameter benchmark economy.

When learning about structural parameters, the subjective probabilities agents assign to different states of the world deviate from the objective probabilities, thus throwing a wedge between real outcomes and asset prices not present in the known parameter benchmark economy. We show that with parameter uncertainty the long-run consumption risks that arise endogenously through learning do not imply excess consumption predictability or a high sensitivity of expected consumption growth to fluctuations in the real risk-free rate. In

particular, the models we present, which have an elasticity of intertemporal substitution well above one, replicates the regression-based evidence Hall (1988) interprets as evidence for a low elasticity of intertemporal substitution.

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Table 1 - Decade by decade

Table 1: This table gives average annualized sample moments from 20,000 simulations of 400 quarters of data from the model where the representative agent has a preference for early resolution of uncertainty ($\gamma = 10$ and $\psi = 2$). The sample moments are broken into decades, however, to illustrate the effect of parameter learning over time. In particular, the second column shows the prior dispersion parameter at the beginning of each decade ($\sigma_t(\mu)$). The remaining columns show the volatility of the log pricing kernel, the risk-free rate, the difference between the 10-year zero-coupon yield and the short-term risk-free rate, the equity premium, and finally equity return volatility.

	Prior uncertainty $\sigma_t(\mu)$	SDF volatility $\sigma_T[m_{t,t+1}]$	Asset price moments [$\gamma = 10, \psi = 2$]			
			$E_T[R_{f,t}]$	$E_T[y_t^{10}-r_{f,t}]$	$E_T[R_{M,t}-R_{f,t}]$	$\sigma_T[R_{M,t}-R_{f,t}]$
Decade 1	1.65%	1.05	-1.3%	1.4%	11.0%	10.7%
Decade 2	0.26%	0.87	-0.2%	0.6%	8.1%	9.4%
Decade 3	0.18%	0.75	0.6%	0.2%	6.3%	8.4%
Decade 4	0.15%	0.67	1.0%	0.1%	5.2%	7.8%
Decade 5	0.13%	0.61	1.3%	0.1%	4.5%	7.3%
Decade 6	0.12%	0.57	1.5%	0.0%	4.0%	7.0%
Decade 7	0.11%	0.54	1.7%	0.0%	3.6%	6.7%
Decade 8	0.10%	0.51	1.8%	0.0%	3.4%	6.5%
Decade 9	0.09%	0.49	1.8%	0.0%	3.2%	6.4%
Decade 10	0.09%	0.48	1.9%	0.0%	3.0%	6.2%
Known parameters	0.00%	0.33	2.5%	0.0%	1.7%	5.0%

Table 2 - 100 year sample moments

Table 2: This table gives average annualized sample moments from 20,000 simulations of 400 quarters of data from each model. The initial prior is centered around the true mean with a standard error of 0.26%, which corresponds to the dispersion that would obtain if one started with a flat prior and had learned for ten years. $E_T[x]$ denotes the sample mean of x , $\sigma_T[x]$ denotes the sample standard deviation of x , and m is the log stochastic discount factor, R_M denotes the simple "market" return, defined as 1.5 times the return to the consumption claim. R_f is the real simple risk-free rate, y_{10} is the continuously compounded annual yield on a zero-coupon default-free bond. "Excess volatility" is defined as the relative amount of return volatility in excess of the volatility of cash flow growth. The values in the "Data" column are taken from Bansal and Yaron (2004) and correspond to U.S. data from 1929 to 1998. In their data, dividend growth volatility is 11.5%, while return volatility is 19.4% which means "excess volatility" is $19.4/11.5 - 1 = 0.70$. All statistics are annualized.

	Data	Preference for early resolution of uncertainty ($\gamma = 10, \psi = 2$)		Indifferent to the timing of resolution of uncertainty ($\gamma = 10, \psi = 1/\gamma$)	
		Unknown μ	Known μ	Unknown μ	Known μ
$\sigma_T [m_t]$	≥ 0.6	0.60	0.33	0.33	0.33
$E_T [R_{M,t} - R_{f,t}]$	6.33	4.42	1.67	-1.39	1.67
$\sigma_T [R_{M,t} - R_{f,t}]$	19.42	7.35	5.00	5.31	5.00
Excess volatility	≈ 0.70	0.47	0.00	0.06	0.00
$E_T [R_{f,t}]$	0.86	1.34	2.51	15.1	15.1
$\sigma_T [R_{f,t}]$	0.97	0.67	0.00	2.12	0.00
$E_T [y^{10} - r_f]$	≈ 0	0.09	0.00	-1.26	0.00

Table 3 - Forecasting regressions

Table 3: This table shows the results from forecasting regressions of 1- and 5-year log consumption growth and excess market returns on the lagged log price-consumption ratio, as well as a regression of one quarter ahead consumption growth on the log risk-free rate. The β 's reported are the median regression coefficient across 20,000 simulated paths from the model with $\gamma = 10$ and $\psi = 2$. Each sample path is 100 years long. The initial prior is centered around the true mean with a standard error of 0.26%, which corresponds to the dispersion that would obtain if one started with a flat prior and had learned for ten years. The median Newey-West t -statistic is also reported, where the number of lags equals the number of overlapping observations. The regressions use quarterly simulated data, so for the annual forecasting horizon there are 3 lags used. * denotes significance at the 10% level, ** denotes significance at the 5% level, and *** denotes significance at the 1% level. Finally, the median R^2 is also reported for each regression. The "data" columns are taken from Beeler and Campbell (2011), who use U.S. data from 1930 to 2008.

Forecasting horizon	Data		Median model outcomes		
	β^{data}	$\bar{R}^{2,\text{data}}$	β^{model}	(s.e.)	$\bar{R}^{2,\text{model}}$
Panel A: Consumption growth predictability					
Consumption growth vs. P/C-ratio: $\Delta c_{t,t+j} = \alpha + \beta pc_t + \varepsilon_{t,t+j}$					
1 year	0.01	6.0%	-0.01	(0.05)	0.0%
5 years	-0.00	0.0%	-0.12	(0.10)	3.0%
Consumption growth vs. risk-free rate: $\Delta c_{t,t+j} = \alpha + \beta r_{f,t} + \varepsilon_{t,t+j}$					
1 quarter	-0.12	not reported	-0.02	(0.49)	0.0%
Panel B: Excess return predictability					
Excess returns vs. P/C-ratio: $r_{t,t+j} - r_{f,t,t+j} = \alpha + \beta pc_t + \varepsilon_{t,t+j}$					
1 year	-0.09*	4.4%	-0.25***	(0.08)	7.4%
5 years	-0.41***	26.9%	-1.16***	(0.30)	30.9%

Table 4 - Alternative dividend claims

Table 4: This table gives average annualized sample moments from 20,000 simulations of 400 quarters of data from the model with $\gamma = 10$ and $\psi = 2$. The initial prior is centered around the true mean with a standard error of 0.26%, which corresponds to the dispersion that would obtain if one started with a flat prior and had learned for ten years. $E_T[x]$ denotes the sample mean of x and $\sigma_T[x]$ denotes the sample standard deviation of x . R_M denotes the simple "market" return, defined according to the "Case 1" and "Case 2" specifications of dividends, as given in the paper. In "Case 1" dividends are cointegrated with consumption, while in "Case 2" consumption and dividend growth are only constrained to have the same unconditional growth rate. The values in the "Data" column are taken from Bansal and Yaron (2004) and correspond to U.S. data from 1929 to 1998. All statistics are annualized.

Moment	Data	Case 1		Case 2	
		Unknown μ	Known μ	Unknown μ	Known μ
$E_T [R_{M,t} - R_{f,t}]$	6.33	6.5	2.2	5.1	2.5
$\sigma_T [R_{M,t} - R_{f,t}]$	19.42	14.7	11.6	13.0	11.8
Sharpe ratio	0.33	0.44	0.25	0.39	0.21
$E_t [R_{M,t+1} - R_{f,t+1}]$ at end of sample		4.1	2.2	3.9	2.5

Table 5 - 100-year sample moments from disaster models

Table 5: This table gives average annualized sample moments from 20,000 simulations of 400 quarters of data from two version of parameter uncertainty in the consumption disaster model. $E_T[x]$ denotes the sample mean of x , $\sigma_T[x]$ denotes the sample standard deviation of x , and m is the log stochastic discount factor, R_M denotes the simple "market" return, defined as 1.5 times the return to the consumption claim. R_f is the real simple risk-free rate, y_{10} is the continuously compounded annual yield on a zero-coupon default-free bond. "Excess volatility" is defined as the relative amount of return volatility in excess of the volatility of cash flow growth. The values in the "Data" column are taken from Bansal and Yaron (2004) and correspond to U.S. data from 1929 to 1998. In their data, dividend growth volatility is 11.5%, while return volatility is 19.4% which means "excess volatility" is $19.4/11.5 - 1 = 0.70$. All statistics are annualized.

	Data	Learning about disaster mean ($\beta = 0.993, \gamma = 7, \psi = 2$)		Learning about disaster probability ($\beta = 0.99, \gamma = 7, \psi = 2$)	
		Unknown μ_D	Known μ_D	Unknown λ	Known λ
$\sigma_T [m_t]$	≥ 0.6	1.23	0.82	1.48	0.81
$E_T [R_{M,t} - R_{f,t}]$	6.33	5.14	3.31	7.33	3.30
$\sigma_T [R_{M,t} - R_{f,t}]$	19.42	4.19	4.03	4.95	4.10
Excess volatility	≈ 0.70	0.04	0.00	0.21	0.00
$E_T [R_{f,t}]$	0.86	0.82	1.87	0.87	3.08
$\sigma_T [R_{f,t}]$	0.97	0.06	0.00	0.43	0.00
$E_T [y^{10} - r_f]$	≈ 0	0.00	0.00	0.05	0.00

Table 6 - Forecasting regression from disaster model

Table 6: This table shows the results from forecasting regressions of 1- and 5-year log consumption growth and excess market returns on the lagged log price-consumption ratio, as well as a regression of one quarter ahead consumption growth on the log risk-free rate. The regression β 's reported are the median regression coefficient across 20,000 simulated paths from the model with learning about the disaster probability with $\beta = 0.99$, $\gamma = 7$ and $\psi = 2$. Each sample path is 100 years long. The median Newey-West t -statistic is also reported, where the number of lags equals the number of overlapping observations. The regressions use quarterly simulated data, so for the annual forecasting horizon there are 3 lags used. * denotes significance at the 10% level, ** denotes significance at the 5% level, and *** denotes significance at the 1% level. Finally, the median R^2 is also reported for each regression. The "data" columns are taken from Beeler and Campbell (2011), who use U.S. data from 1930 to 2008.

Forecasting horizon	Data		Median model outcomes		
	β^{data}	\bar{R}^2, data	β^{model}	(s.e.)	\bar{R}^2, model
Panel A: Consumption growth predictability					
Consumption growth vs. P/C-ratio: $\Delta c_{t,t+j} = \alpha + \beta pc_t + \varepsilon_{t,t+j}$					
1 year	0.01	6.0%	-0.01	(0.04)	0.3%
5 years	-0.00	0.0%	-0.28	(0.27)	3.6%
Consumption growth vs. risk-free rate: $\Delta c_{t,t+j} = \alpha + \beta r_{f,t} + \varepsilon_{t,t+j}$					
1 quarter	-0.12	not reported	-0.27	(0.31)	0.0%
Panel B: Excess return predictability					
Excess returns vs. P/C-ratio: $r_{t,t+j} - r_{f,t,t+j} = \alpha + \beta pc_t + \varepsilon_{t,t+j}$					
1 year	-0.09*	4.4%	-0.57***	(0.14)	10.0%
5 years	-0.41***	26.9%	-2.67***	(0.56)	39.7%

Table 7 - 100 year sample moments for Structural Breaks case

Table 7: This table gives average annualized sample moments from 20,000 simulations of 400 quarters of data from models with structural breaks. $E_T[x]$ denotes the sample mean of x , $\sigma_T[x]$ denotes the sample standard deviation of x , and m is the log stochastic discount factor, R_M denotes the simple "market" return, defined as 1.5 times the return to the consumption claim. R_f is the real simple risk-free rate, y_{10} is the continuously compounded annual yield on a zero-coupon default-free bond. "Excess volatility" is defined as the relative amount of return volatility in excess of the volatility of cash flow growth. The values in the "Data" column are taken from Bansal and Yaron (2004) and correspond to U.S. data from 1929 to 1998. In their data, dividend growth volatility is 11.5%, while return volatility is 19.4% which means "excess volatility" is $19.4/11.5 - 1 = 0.70$. All statistics are annualized.

	Data	$\gamma = 10, \psi = 2$		$\gamma = 10, \psi = 1.5$	
		Unknown μ	Known μ	Unknown μ	Known μ
$\sigma_T [m_t]$	≥ 0.6	0.72	1.25	0.67	0.99
$E_T [R_{M,t} - R_{f,t}]$	6.33	4.65	6.81	3.95	5.17
$\sigma_T [R_{M,t} - R_{f,t}]$	19.42	6.45	5.44	5.94	5.21
Excess volatility	≈ 0.70	0.29	0.09	0.19	0.04
$E_T [R_{f,t}]$	0.86	1.24	-0.38	1.70	0.75
$\sigma_T [R_{f,t}]$	0.97	0.20	0.40	0.21	0.18

Table 8 - Forecasting regressions, structural breaks case

Table 8: This table shows the results from forecasting regressions of 1- and 5-year log consumption growth and excess market returns on the lagged log price-consumption ratio, as well as a regression of one quarter ahead consumption growth on the log risk-free rate. The β 's reported are the median regression coefficient across 20,000 simulated paths from the structural breaks model with $\gamma = 10$ and $\psi = 2$. Each sample path is 100 years long unless otherwise specified. The median Newey-West t -statistic is also reported, where the number of lags equals the number of overlapping observations. The regressions use quarterly simulated data, so for the annual forecasting horizon there are 3 lags used. * denotes significance at the 10% level, ** denotes significance at the 5% level, and *** denotes significance at the 1% level. Finally, the median R^2 is also reported for each regression. The "data" columns are taken from Beeler and Campbell (2011), who use U.S. data from 1930 to 2008.

Forecasting horizon	Data		Median model outcomes		
	β^{data}	$\bar{R}^{2,\text{data}}$	β^{model}	(s.e.)	$\bar{R}^{2,\text{model}}$
Panel A: Consumption growth predictability					
Consumption growth vs. P/C-ratio: $\Delta c_{t,t+j} = \alpha + \beta pc_t + \varepsilon_{t,t+j}$					
1 year	0.01	6.0%	0.10	(0.08)	1.4%
5 years	-0.00	0.0%	0.21	(0.33)	4.2%
Consumption growth vs. risk-free rate: $\Delta c_{t,t+j} = \alpha + \beta r_{f,t} + \varepsilon_{t,t+j}$					
1 quarter	-0.12	not reported	-0.06	(0.30)	0.0%
Panel B: Excess return predictability					
Excess returns vs. P/C-ratio: $r_{t,t+j} - r_{f,t,t+j} = \alpha + \beta pc_t + \varepsilon_{t,t+j}$					
100 year sample medians:					
1 year	-0.09*	4.4%	-0.26	(0.18)	1.9%
5 years	-0.41***	26.9%	-0.99	(0.66)	7.3%
50 year sample medians, conditioning on structural break at beginning of sample					
1 year	-0.09*	4.4%	-0.51**	(0.24)	5.9%
5 years	-0.41***	26.9%	-1.88***	(0.69)	20.7%

Table 9 - 100 year sample moments when This Time Is Different

Table 9: This table gives average annualized sample moments from 20,000 simulations of 400 quarters of data from models with structural breaks, but where investors suffers from a "This time is different"-bias. In particular, investors believes upon a structural break that the new regime will last forever. The table shows average moments for both the case of early resolution of uncertainty and the standard power utility case, where investors are indifferent to the timing of the resolution of uncertainty. $E_T[x]$ denotes the sample mean of x , $\sigma_T[x]$ denotes the sample standard deviation of x , and m is the log stochastic discount factor, R_M denotes the simple "market" return, defined as 1.5 times the return to the consumption claim. R_f is the real simple risk-free rate, y_{10} is the continuously compounded annual yield on a zero-coupon default-free bond. "Excess volatility" is defined as the relative amount of return volatility in excess of the volatility of cash flow growth. The values in the "Data" column are taken from Bansal and Yaron (2004) and correspond to U.S. data from 1929 to 1998. In their data, dividend growth volatility is 11.5%, while return volatility is 19.4% which means "excess volatility" is $19.4/11.5 - 1 = 0.70$. All statistics are annualized.

	Data	$\gamma = 10, \psi = 2$ Unknown μ	Known μ	$\gamma = 10, \psi = 1/\gamma$ Unknown μ	Known μ
$\sigma_T [m_t]$	≥ 0.6	0.65	0.33	0.33	0.33
$E_T [R_{M,t} - R_{f,t}]$	6.33	4.91	1.67	-1.81	1.67
$\sigma_T [R_{M,t} - R_{f,t}]$	19.42	7.61	5.00	6.74	5.00
Excess volatility	≈ 0.70	0.52	0.00	0.35	0.00
$E_T [R_{f,t}]$	0.86	1.16	2.51	15.12	15.14
$\sigma_T [R_{f,t}]$	0.97	0.63	0.00	1.92	0.00

Table 8 - Forecasting regressions: This time is different

Table 10: This table shows the results from forecasting regressions of 1- and 5-year log consumption growth and excess market returns on the lagged log price-consumption ratio, as well as a regression of one quarter ahead consumption growth on the log risk-free rate. The β 's reported are the median regression coefficient across 20,000 simulated paths from the structural breaks model, where investors suffer from a "This time is different"-bias. In particular, investors believes upon a structural break that the new regime will last forever. The preference parameters are $\beta = 0.994$, $\gamma = 10$ and $\psi = 2$. Each sample path is 100 years long unless otherwise specified. The median Newey-West t -statistic is also reported, where the number of lags equals the number of overlapping observations. The regressions use quarterly simulated data, so for the annual forecasting horizon there are 3 lags used. * denotes significance at the 10% level, ** denotes significance at the 5% level, and *** denotes significance at the 1% level. Finally, the median R^2 is also reported for each regression. The "data" columns are taken from Beeler and Campbell (2011), who use U.S. data from 1930 to 2008.

Forecasting horizon	Data		Median model outcomes		
	β^{data}	\bar{R}^2_{data}	β^{model}	(s.e.)	\bar{R}^2_{model}
Panel A: Consumption growth predictability					
Consumption growth vs. P/C-ratio: $\Delta c_{t,t+j} = \alpha + \beta pc_t + \varepsilon_{t,t+j}$					
1 year	0.01	6.0%	-0.01	(0.03)	0.5%
5 years	-0.00	0.0%	-0.13	(0.13)	3.9%
Consumption growth vs. risk-free rate: $\Delta c_{t,t+j} = \alpha + \beta r_{f,t} + \varepsilon_{t,t+j}$					
1 quarter	-0.12	not reported	-0.02	(0.50)	0.0%
Panel B: Excess return predictability					
Excess returns vs. P/C-ratio: $r_{t,t+j} - r_{f,t,t+j} = \alpha + \beta pc_t + \varepsilon_{t,t+j}$					
100 year sample medians:					
1 year	-0.09*	4.4%	-0.28***	(0.08)	7.5%
5 years	-0.41***	26.9%	-1.19***	(0.31)	26.8%

Figure 1 - Posterior Standard Deviation

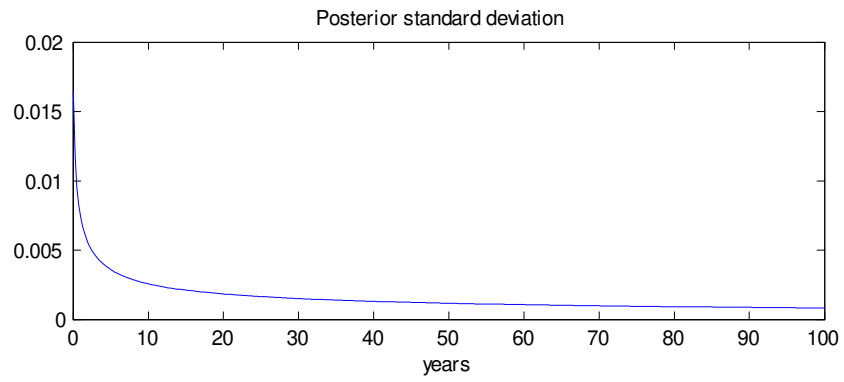


Figure 1: The graph shows the posterior standard deviation on the vertical axis and the time elapsed in years on the horizontal axis. The initial prior standard deviation is 0.0165.

Figure 2 - Ex ante vs ex post return predictability

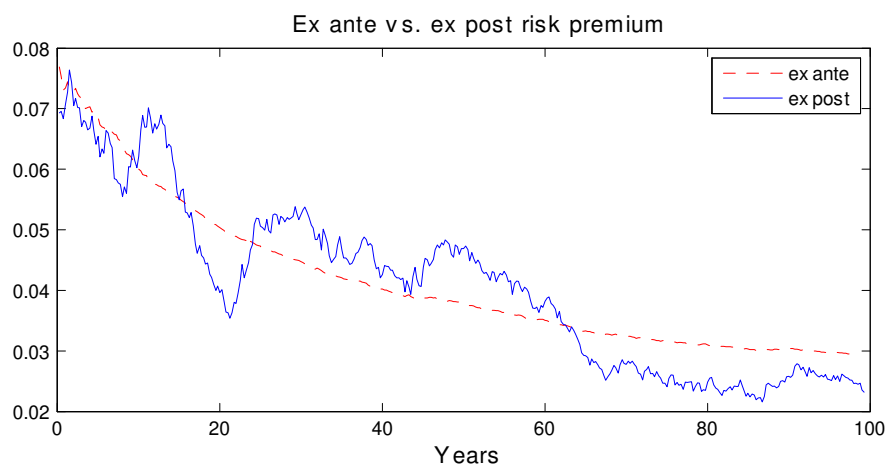


Figure 2: The graph shows a representative example of a sample path of the ex ante subjective risk premium, and the ex post estimated risk premium obtained from an ex post regression of annual excess returns on the lagged price-consumption ratio. The sample length is 100 years and the initial prior is unbiased with dispersion equal to 0.26%. The solid line shows the ex post estimate, while the dashed line shows the ex ante value.

**Figure 3 - Conditional Volatility of the Pricing Kernel:
Preference for early resolution of uncertainty**

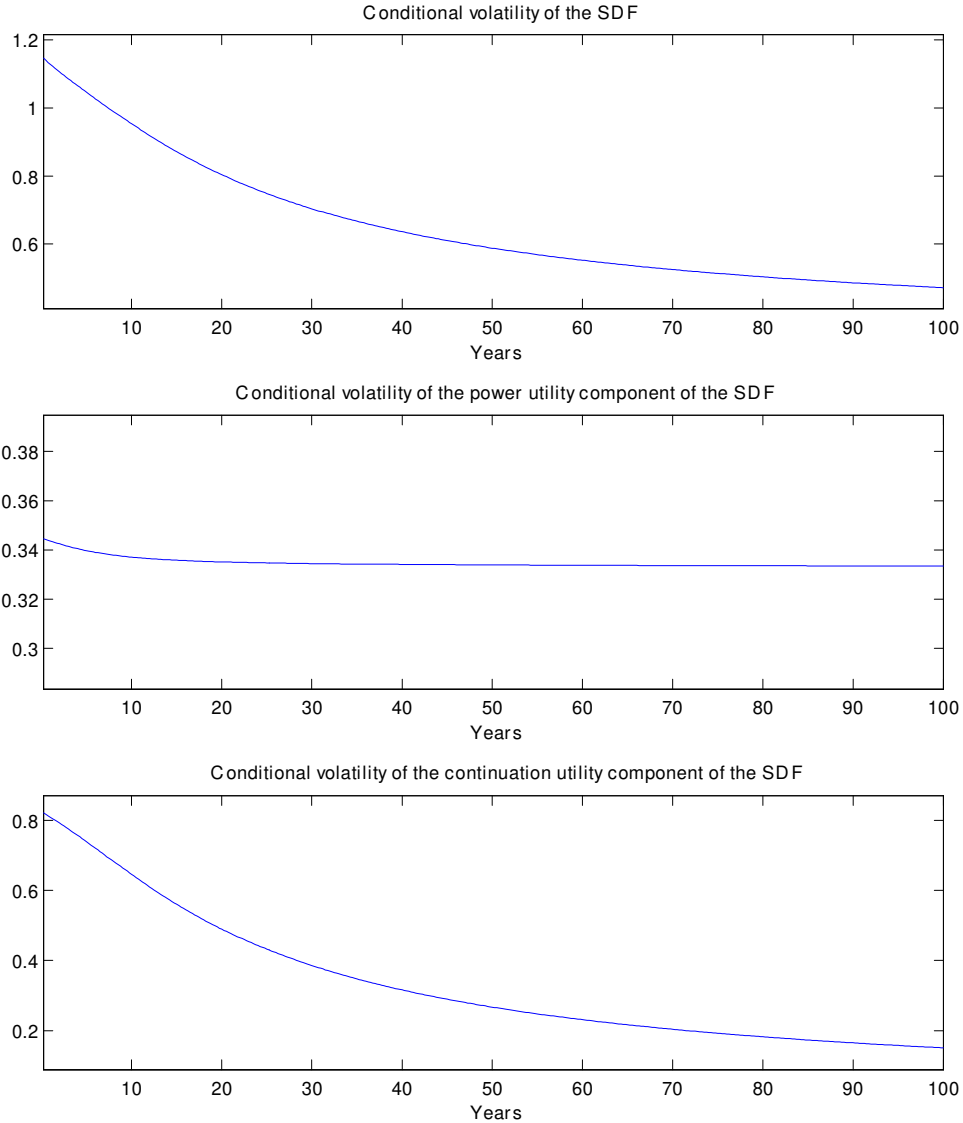


Figure 3: The top plot shows the subjective conditional annualized volatility of the Epstein-Zin stochastic discount factor with preference parameters $\gamma = 10, \psi = 2$ over a 100 year sample period. The plot shows the average conditional volatility across 20,000 simulated economies at each time t . The initial prior is unbiased with dispersion over mean consumption growth of $\sigma_{t=0} = 0.0165$. The middle plot shows the same for the "power utility component" of the stochastic discount factor ($\beta * \exp(-\gamma \Delta c_{t+1})$), while the bottom plot shows the conditional annualized volatility of the "continuation utility component" of the stochastic discount factor ($(\beta \frac{PC_{t+1}+1}{PC_t})^{\theta-1}$).

**Figure 4 - Sensitivity of log P/C ratio to changes in mean beliefs:
Preference for early resolution of uncertainty**

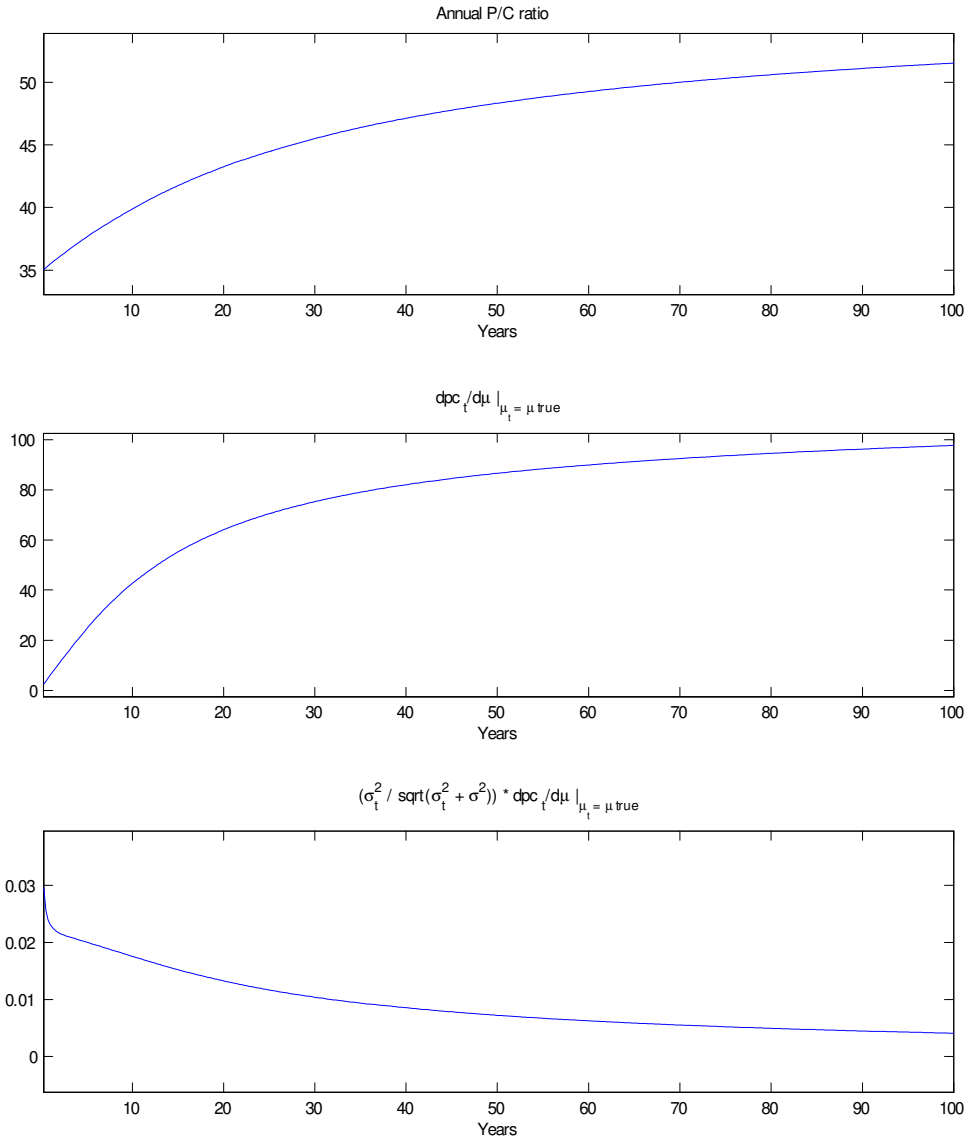


Figure 4: The top plot shows the annual wealth-consumption ratio (P/C) over a 100 year period, starting with an unbiased prior and initial dispersion $\sigma_{t=0} = 0.0165$. The plot gives the average outcome over 20,000 simulated economies with preference parameters $\gamma = 10, \psi = 2$. The middle plot shows the derivative of the log wealth-consumption ratio (pc) with respect to the mean beliefs about the consumption growth rate, evaluated at the true mean of consumption growth, versus years passed since the initial prior. The bottom plot shows the same derivative multiplied by the standard deviation of shocks to beliefs about the *mean* consumption growth rate, assuming a normal untruncated prior.

**Figure 5 - Conditional Volatility of the Pricing Kernel:
Cases with unknown variance**

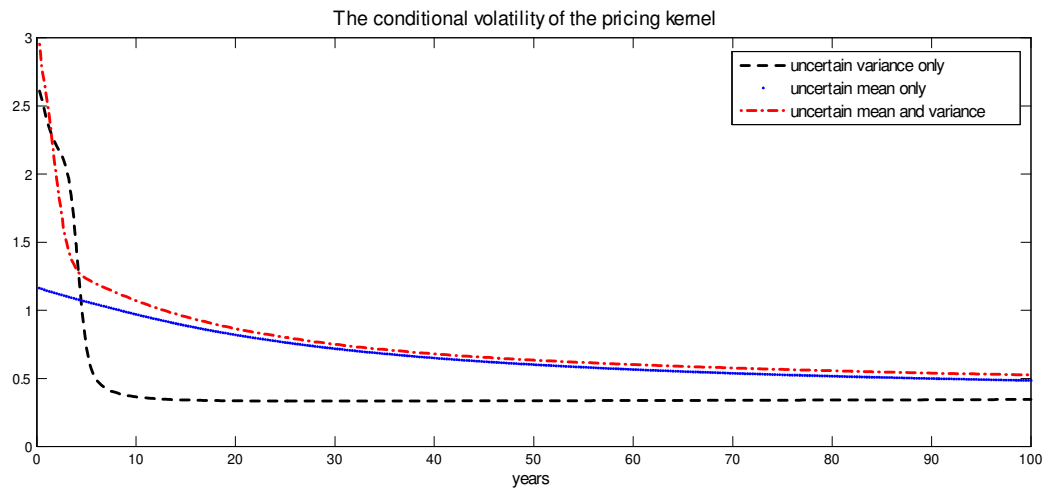


Figure 5: The graph shows the subjective conditional annualized volatility of the Epstein-Zin stochastic discount factor with preference parameters $\gamma = 10$, $\psi = 2$ and $\beta = 0.994$ over a 100 year sample period, averaged across 20,000 simulated economies at each time t . The dashed lined corresponds to the case of unknown variance only, the dotted line corresponds to the case of unknown mean only, while the dash-dotted line corresponds to the case of unknown mean and variance.

Figure 6 - Average conditional moments for case of learning about disaster probability

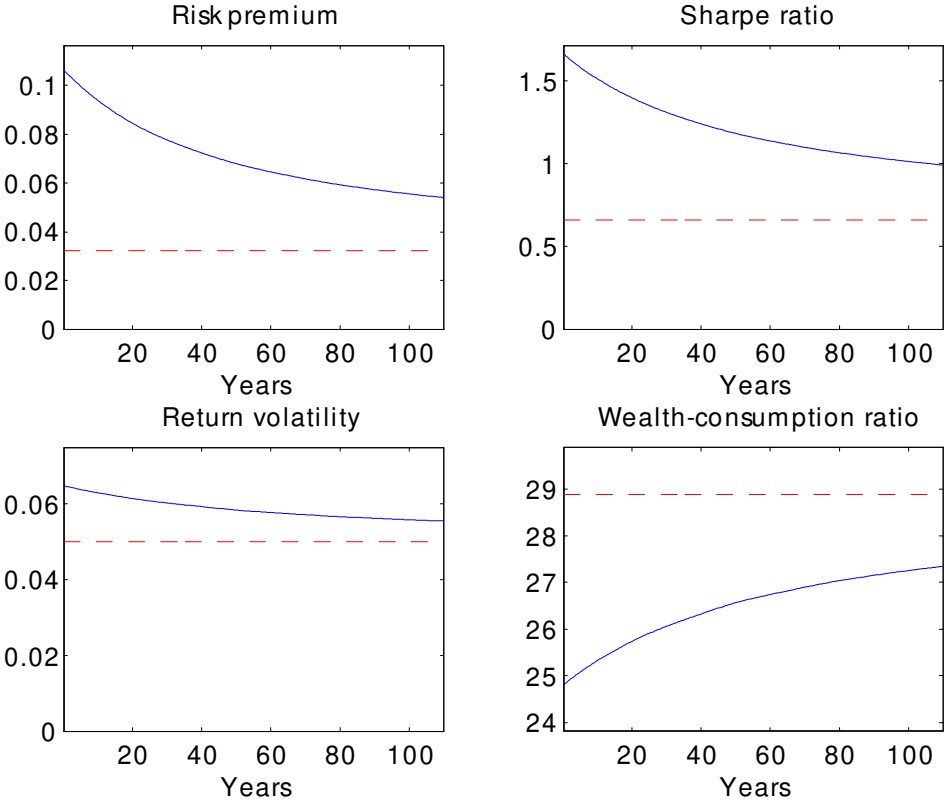


Figure 6: The figure shows annualized conditional asset pricing moments averaged across 20,000 simulated economies where the disaster probability is uncertain. Preference parameters $\gamma = 7, \psi = 2$.

Figure 7 - Sample path of the risk premium for case of learning about the disaster probability

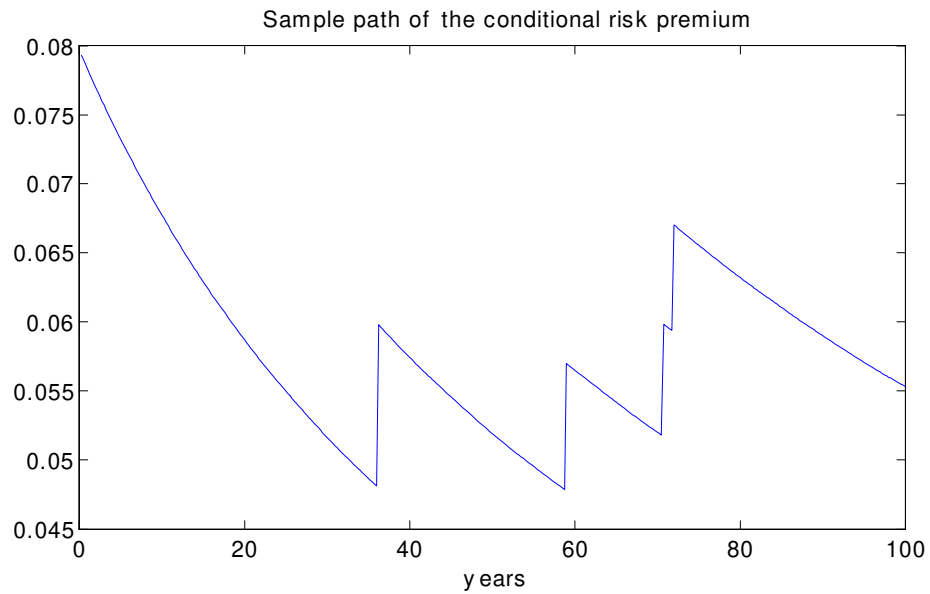


Figure 7: The figure shows a representative sample path of the annualized conditional risk premium from the model with learning about the disaster probability.

Figure 8 - Sample path of the risk premium and P/C ratio for structural break case

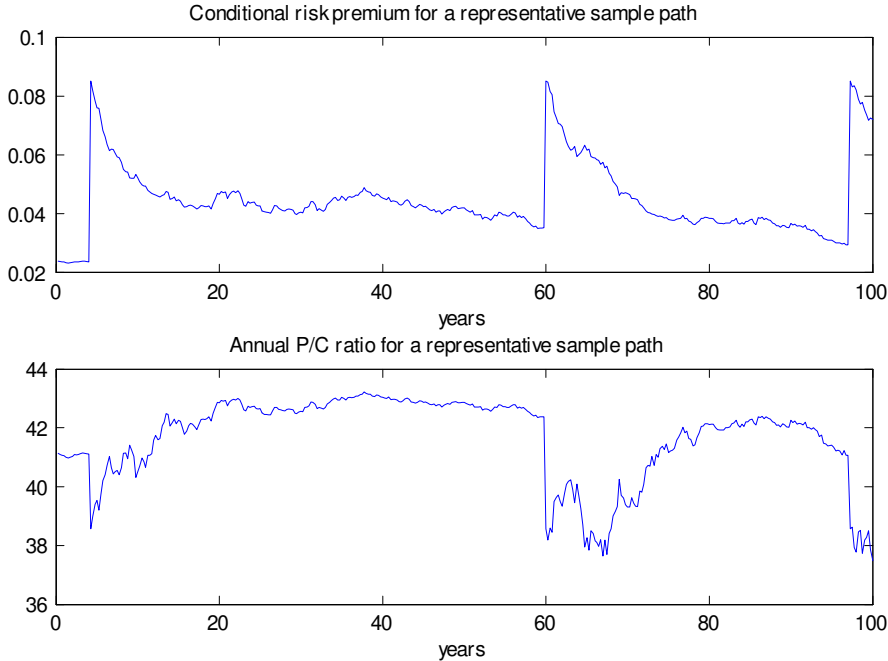


Figure 8: The figure shows a representative sample path of the annualized conditional risk premium in the top plot and the annual price-consumption ratio in the bottom plot – both from the model with structural breaks and learning about the mean within each paradigm.