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"Buy an annuity cheap, and make your life interesting to yourself and everybody else that watches the speculation."

Jonas Chuzzlewit , *Martin Chuzzlewit*,
by Charles Dickens (1843 - 1844)

During the past three years I have worked as a Ph. D. student at Tilburg University on this dissertation. I started my project with an empty hard disk, an almost-empty desk, and the assignment to do some research. Now, my hard disk is full, my desk is nothing more than piles of papers, and the project is 'finished' or as finished as something in science can be. It is my pleasure to take the opportunity to thank those who made this dissertation possible.

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CHAPTER 1

INTRODUCTION

"...but if you observe, people always live for ever when there is an annuity to be paid them; and she is very stout and healthy, and hardly forty.
An annuity is a very serious business; it comes over and over every year, and there is no getting rid of it."

Mrs Dashwood, *Sense and Sensibility*, by Jane Austen (1811)

1.1 Introduction

The first babyboomers are retiring while pension wealth is low because of the financial crisis, putting additional pressure on annuity providers. As a consequence, old-age income security systems are under review all over the world. There are discussions about, for instance, the retirement age, and who should bear the investment and longevity risk. In this thesis, I will address several issues concerning old-age income security, mainly focussing on annuities. A (whole life) annuity is defined as a contract which states that the annuity provider pays an individual or group of individuals an income each period until death in return for a lump sum (or premium) paid in advance (Cannon and Tonks, 2008). I will first discuss a brief history of annuities.

During his incumbency as Lord Chief Justice of the King's Bench, 1613 – 1620, Sir Edward Coke defined an annuity as "a yearly payment of a certain sum of money granted to another in fee, for life or years, charging the person of the grantor only" (Kopf, 1927). However, annuities have existed well before this period.

As early as 2500 B.C., there may already have been some form of annuities in Babylon. From the researches of Trenerry, one may develop the idea that there was a fairly wide-spread practice of granting a series of periodic payments secured by land or other property. Annuities may have been adopted from the commercial codes of the Hindus and Chinese, antedating the Babylonian era (Kopf, 1927). Kopf (1927) also reports that annuities where already sold in Egypt roughly 1000 year Before Christ. Annuities, then called *annua*, existed also in the ancient Roman times, (see James (1947) quoted in Poterba (2001)). The first recorded life table for the purpose of computing the value of annuities was ascribed by Aemilius Macer in 220 AD to Domitius Ulphianus (Haberman, 1996). Macer notes that a more practical formula was often used: for someone over the age x of 30, the purchase price was calculated as $60 - x$ (Cannon and Tonks, 2008). The price of annuities has increased significantly, partly due to a steep increase in life expectancy. In the Roman times, the remaining life expectancy for a male aged 65 was 5.3 years, whereas the life expectancy for a male aged 65 in the Netherlands in 2009 is 17 years according to Statistics Netherlands. For an overview of the history of annuities see Cannon and Tonks (2008) and the references in it.

Annuities are appealing to individuals because they insure individuals against the risk of outliving their assets, i.e., becoming old is seen as a risk. As off retirement, individuals finance consumption by depleting their assets. Without an annuity and in case the age of death is certain, the individual could compute the time profile of consumption that would exhaust his wealth when he died. However, the age of death is generally uncertain. Moreover, because of inflation it is also not certain how much consumption he can finance from his assets. In addition, the individual should be able to compute the time profile of consumption and act accordingly. The latter may be a problem. (Orszag and Stiglitz, 2001) quote the chairman of the U.S. Securities and Exchange Commission as stating that over 50 percent of Americans did not know the difference between a stock and a bond. Consequently, they may not be able to compute the time profile of consumption. If the individual buys an annuity, a periodic payment is made conditional on being alive. So, annuities solve the problem of planning consumption when the future lifetime is uncertain. Another advantage of annuities is that, if costs are low, the return on an annuity exceeds the return on the equivalent non-annuitized asset for those who are alive because the annuity provider pools mortality risk across individuals, see for instance, Poterba (2001); Davidoff, Brown, and Diamond (2005).

There have been many developments in the annuity market since the ancient Roman times. I will discuss several developments below. Annuities have become more freely available, and there are currently many different varieties of annuities available. They differ with respect to the beneficiaries, the timing, and the benefit level. The most common annuity is a *single-life annuity*, which makes a periodic payment to the beneficiary as long as he or she is alive. For households, a *joint & survivor annuity* is developed, which makes a periodic payment as long as both spouses are alive and a (reduced) periodic payment when only one of the spouses is alive. Concerning the timing there are immediate annuities and deferred annuities. *Immediate annuities* make a periodic payment immediately after the purchase of the annuity whereas *deferred annuities* make a periodic payment beginning at some future date. Both annuities pay out conditional on the life of the beneficiary. Concerning the benefit levels, three different types can be considered: single-level annuities, inflation-linked annuities, and variable annuities. The benefit level of *single-level annuities* remains constant over time whereas the benefit level of *inflation-linked* annuities increases annually with inflation. The benefit level of *variable annuities* depends on the performance of the underlying portfolio. In the latter case, the insurer can guarantee a minimum payment. For an overview of all annuity products, see Cannon and Tonks (2008).

There are also varieties in who provides the annuities and how they are paid for. Annuities can be provided by the state, by pension funds (through employer-sponsored schemes), and by insurance companies. In most countries the state provides a basic state pension for its citizens. The state pension may be financed either by a pay-as-you-go system or it may be funded. Pay-as-you-go pensions are paid out of current revenue. The current working population pays for the current retired population. Funded pensions are paid from accumulated funds. In employer-sponsored schemes, there are generally two types of pension schemes, namely: defined benefit (DB) and defined contribution (DC). In case of pure DB scheme, the contribution is flexible and the employee's pension benefit is an annuity. In a DC scheme, the contribution is fixed and the pension benefit can either be a lump sum or can be converted to an annuity. Participation in employer-sponsored schemes may be mandatory. Annuities may also be voluntarily bought by individuals from insurance companies. Although research has derived conditions under which annuities should be attractive to individuals (see Yaari, 1965; Diamond, 2005), most individuals do not annuitize voluntarily. This is generally referred to as "the annuity puzzle". There have been many explanations for this puzzle. A will discuss a few of them. First, individuals may not want to annuitize fully because they would like to leave a bequest. However, there is conflicting evidence about the importance of bequests to individuals (see Cannon and Tonks, 2011). Second, the price of the annuity may be too high. Mitchell, Poterba, Warshawsky, and Brown (1999) found that the expense loading, which covers among others administrative costs and a risk premium, is about 7.3%. Third, individuals might not want to annuitize fully because they might want to have precautionary savings to cover for instance unexpected health expenditures (Peijnenburg, Nijman, and Werker, 2011). Health care cost have increased rapidly in the last decades, especially in the U.S.. Fourth, individuals may have less desire for annuities because they can share part of the risk within the family (Kotlikoff and Spivak, 1981). Finally, means-tested benefits may reduce further annuity demand (Bütler and Teppa, 2007).

In this thesis I will address three issues concerning annuities. First I consider individuals who have the option to defer the payment of their pension benefits. Currently, individuals in many countries can choose the age as off which they would like to receive their pension benefits. Take the basic state pension as an example. In the U.S., individuals can claim benefits before, at, or after the full retirement age, currently set at age 66. In case benefits are claimed before the full retirement age the annual benefit received is reduced and in case benefits are claimed after the full retirement age the annual benefit received is increased. In the U.S. and the U.K., the annual old-age Social Security benefit is increased with 8% and 10.4% respectively, for each year benefit claiming is delayed after the full retirement age. In the Netherlands, there is a proposal to increase the annual benefit of the *AOW* pension with 6.5% for each year that benefit claiming is delayed after the full retirement age. In Chapter 2, I discuss under which conditions insurers can offer a product which dominates the option to delay benefit claiming as offered in many state pension schemes. I derive preference-free conditions under which insurers can offer this dominating product. The conditions depend on the term structure of interest rates, the expense loading, and individual characteristics which influence the survival probabilities. The conditions are irrespective of the individual's utility function. This chapter is based on Sanders, De Waegenaere, and Nijman (2011b).

In Chapter 3, I estimate the effect of imperfections in the annuity market on the optimal annuity portfolio of a couple. Consider a traditional household where only the husband has accrued pension rights and suppose that the annuity prices must be based on gender-neutral survival probabilities. In many employer-sponsored pension schemes, there will be a drop in income when the husband dies but there will be no a drop in income when the wife dies. When both spouses have the same utility function, this will not be optimal. The couple may want to receive a higher income when both spouses are alive and a reduced income when only one of the spouses is alive. The reason for this is that the cost of living for an individual is lower than for a couple. In Chapter 3 I discuss several of these imperfections in the annuity market. We quantify the welfare losses couples bear because of these imperfections. We also investigate the impact of gender-neutral pricing.[1] Since the wife generally outlives the husband, annuity prices based on the wife's survival probabilities are typically higher than annuity prices based on the husband's survival probabilities. Consequently, the gender-neutral priced annuities are priced favorable for the wife and less favorable for the husband, increasing demand for the annuity which pays out when only the wife is alive and decreasing demand for the annuity which pays out when only the husband is alive. This chapter is based on Sanders, De Waegenaere, and Nijman (2011a).

In Chapter 4, I estimate the joint survival probabilities of spouses, taking into account the possible dependence between the remaining lifetimes of spouses. Previous literature suggests that the husband's and wife's remaining lifetimes are positively correlated. The dependence may affect the price of for instance, a joint & survivor annuity, which in turn affects the value of the liabilities of annuity providers. I use a sub-sample of a rich data-set containing all Dutch individuals from the period January 1995 until January 2008 to estimate the joint survivor probabilities. Both parametric models and semi-nonparametric models are used for the estimation. Estimation results are used to determine the actuarially fair value of different annuities. This chapter is based on Sanders and Melenberg (2011).

[1] The relevance of this issue is underlined by the recent European court ruling that as off December 2012 insurers are not allowed to differentiate annuity prices based on gender.

CHAPTER 2

WHEN CAN INSURERS OFFER PRODUCTS THAT DOMINATE DELAYED OLD-AGE PENSION BENEFIT CLAIMING?

> "I advise you to go on living solely to enrage those who are paying your annuities. It is the only pleasure I have left."
>
> François Marie Voltaire

2.1 Introduction

This chapter is based on Sanders, De Waegenaere, and Nijman (2011b). In many countries, individuals can decide either to claim their Social Security old-age pension benefits once the minimal retirement age has been reached, or to delay benefit claiming. In case of delay, the individual is offered the same choice next period and so on, until either the maximum age at which benefits can be claimed has been reached or benefits have been claimed. When an individual defers pension receipts, the benefit level is subjected to an actuarial adjustment for each year that benefit claiming is delayed.[1] In many cases, the adjustment is a constant fraction of the benefit level at the normal retirement age, irrespective of age, gender, and other individual characteris-

[1] Such possibilities exist in Social Security pension systems in, e.g., the U.S., the U.K., the Netherlands, Japan, Germany, France, Australia. (see Queisser and Whitehouse, 2006).

tics. In the U.S. and the U.K., the benefit levels increase by respectively 8% and 10.4% for each year benefit claiming is delayed (see Diamond, 2005; Queisser and Whitehouse, 2006). In the Netherlands a proposal has been put forward to increase the benefit level by only 6.5% for each year of delay. As argued by, e.g., Horneff, Maurer, and Stamos (2008), governments seem to want simple and standardized rules for annuitization applied to a large heterogeneous group of retirees, which may be the reason for choosing a fixed instead of an age-dependent accrual.

The adjustment of the benefit level in case of delayed benefit claiming is typically not actuarially neutral in the sense that the expected present value of the missed benefits in case benefit claiming is delayed is typically not equal to the expected present value of the additional benefits received once benefits are claimed (see, e.g., Coile, Diamond, Gruber, and Jousten, 2002; Duggan and Soares, 2002; Brown, 2003; Desmet and Jousten, 2003; Sun and Webb, 2009). This lack of actuarial neutrality occurs for several reasons. First, the expected present value of the missed and additional benefits in case of delayed benefit claiming depends on the term structure of interest rates. Higher short-term interest rates typically decrease the expected present value of the missed benefits relative to the expected present value of the additional benefits. The opposite holds for high long-term interest rates. The adjustment of the benefit level in case of delayed benefit claiming, however, is typically fixed for a number of years and therefore not adjusted for changes in the term structure of interest rates. Second, an age-independent accrual leads to actuarial unfairness because, as age increases, the number of years over which the increased benefit level should be paid out decreases, and the level of the missed benefits due to deferral of one more year increase. Finally, the expected present value of the missed and additional benefits in case of delayed benefit claiming depend on survival probabilities, which in turn, depend on individual characteristics such as gender and socio-economic status. Thus, heterogeneity among participants leads to actuarial nonequivalence at the individual level (see Brown, 2003; Desmet and Jousten, 2003).[2]

[2] The actuarial nonequivalence is well-documented in the literature. For example, Duggan and Soares (2002) calculate actuarially fair adjustment factors when benefits are claimed at ages 62 to 70, and find that results depend strongly on both gender and discount rate. They also find that the annual accrual for delayed benefit claiming of 8%, given in the U.S., is too low in most cases. Desmet and Jousten (2003) show that there is a high degree of heterogeneity among participants of a large public pension system, so that benefit adjustments that are based on the "average" participant can lead to large degrees of actuarial unfairness at the individual level .

As argued by Duggan and Soares (2002) actuarially nonequivalent benefit adjustments may have unintended consequences in the sense that they affect claiming behavior. Coile, Diamond, Gruber, and Jousten (2002) and Sun and Webb (2009) consider optimal claiming of Social Security benefits in the U.S., and argue that even when the adjustment of the benefit level is lower than actuarially fair, delaying benefit claiming can be attractive to risk averse individuals from a utility perspective. This occurs because a risk-averse individual attaches more value to the increased longevity insurance due to the higher benefit level.[3] Coile, Diamond, Gruber, and Jousten (2002) find that delaying Social Security annuitization for a period of time after the minimal retirement age is optimal in a wide variety of cases under expected utility maximization. Sun and Webb (2009) find that, for plausible preference parameters, the optimal age to claim Social Security benefits for single individuals is between 67 and 70.

Our goal in this paper is to show that the actuarial unfairness inherent in many public pension systems implies that an individual who wishes to defer the receipt of pension benefits can be better off by claiming Social Security benefits immediately and using them to buy annuity products. We consider settings where individuals are allowed to work after they have claimed pension benefits. This is for instance allowed in the U.S. and the Netherlands.[4] Consider, for example, a man aged 66 who would like to receive pension benefits as of age 67. He can do so by deferring benefit claiming with one year, which implies that his benefit level will be increased. At age 66 the individual can finance his current consumption with, for instance, labor income, savings, or by borrowing money (at some interest rate). Suppose now that the level of the accrual is actuarially unfair for this particular man in the sense that the expected present value of the missed benefits at age 66 is higher than the expected present value of the additional benefits received as of age 67. If the difference is sufficiently large, insurers may be able to offer a deferred annuity

[3]There is an extensive literature that characterizes individuals' optimal behavior with regard to the timing and level of annuitization of their wealth (see, e.g., Yaari, 1965; Brugiavini, 1993; Brown, 2001; Milevsky, 2001; Brown, 2003; Davidoff, Brown, and Diamond, 2005; Gupta and Li, 2007; Horneff, Maurer, Mitchell, and Dus, 2006; Milevsky and Young, 2007a,b; Gerrard, Højgaard, and Vigna, 2010, to name just a few). Our focus is on claiming behavior in Social Security systems with delay options.

[4]Since the elimination of the Earnings Test for individuals over the full retirement age in the U.S., (see Benítez-Silva and Heiland, 2008), there are no complications anymore to continue working while receiving pension benefits. For a detailed analysis of how the Earnings Test works (see Michaud and van Soest, 2008).

that starts to pay out as of age 67, with a benefit level that is higher than the accrual offered by the pension provider, and for a periodic premium that is lower than the benefits received in case they are claimed at age 66. If this is the case, the individual is better off by claiming benefits at age 66, and using these benefits to buy the deferred annuity than to delay benefit claiming until age 67. The individual's consumption at age 66 can again by financed by either labor income, savings, or borrowed money. Therefore, for our analysis, consumption at age 66 is irrelevant, since the individual can use the same source of financing in case he defers benefit claiming until age 67 as when he claims benefits at age 66 and uses the benefits received to buy an annuity product at an insurer. Stated differently, the individual's preference order does not change if we do not take the ages before benefits are received into account because the individual's behavior can be the same in both cases.

In this paper we characterize conditions under which insurers can offer *super-replicating* annuity products. The annuity product is super-replicating if it satisfies two conditions. First, it can be bought for a periodic premium that is at most equal to the benefit level obtained in case Social Security benefits are claimed immediately. Second, upon annuitization it yields a benefit level that combined with the Social Security income is at least equal to the benefit level received in case Social Security benefits would have been claimed at that age. If these two conditions are satisfied, deferred benefit claiming is dominated because the individual is better off by claiming benefits immediately and using them to buy the annuity product. An important aspect of this approach is that because the annuity product is super-replicating, there is preference-free dominance of immediate benefit claiming. All that is required for the individual to prefer claiming benefits immediately and using them to buy the annuity product is that more is preferred to less. To characterize such preference-free dominance conditions, we consider two cases. First we consider the case where an individual at a given age decides as of which age he would like to receive his pension benefits, and derive conditions under which insurers can offer deferred annuities that the individual prefers above deferring benefit claiming. Next, we determine conditions under which insurers can offer super-replicating annuity options for those individuals who want to defer receipt of pension benefits until an unspecified age. The individual who buys the annuity option can, year by year, decide whether he wants to annuitize, or defer annuitization for at least one more year.

Whether insurers will be able to offer super-replicating annuity products depends on the degree of actuarial unfairness in the Social Security system, as well as on how insurers price annuity products. Two factors are important. First, annuities offered by insurance companies are typically also actuarially unfair in the sense that the premium includes a load to cover costs. Second, in contrast to Social Security providers, insurers can adjust premium conditions to the prevailing term structure of interest rates. Moreover, they can to some extent differentiate premiums based on individual characteristics that affect survival probabilities. We first consider the case where insurers can differentiate premiums on the basis of age and gender only, and characterize conditions on the level of the premium load and the term structure of interest rates under which they can offer super-replicating annuity products to men and women, respectively. We find that there is ample room for insurers to profitably offer annuity products that men prefer above deferring benefit claiming. For women it is less likely that dominating strategies exist. We then consider the case where insurers can also differentiate premiums based on factors that are correlated with educational level. This additional flexibility increases the room for insurers to offer super-replicating annuity products, in particular to individuals with lower educational levels. This occurs because individuals with lower educational levels have lower life expectancy, and therefore the accruals offered by the social security system are more unfair for them.

Our results potentially have important implications because the existence of super-replicating annuity products can alter claiming incentives and may thereby distort benefit acceptance decisions. Specifically, it can imply that individuals may decide not to defer benefit claiming, even though they do wish to defer annuitization. This can affect the long-run program costs of public pensions (see Hurd, Smith, and Zissimopoulos, 2004). Benefit claiming decisions are not only important for public pensions but also for defined benefit (DB) pensions. It is not uncommon that participants in a (DB) pension plan can, at least to some extent, choose at which age they claim benefits. The annual benefit level is then adjusted to the age at which benefits are first claimed. When the adjustments are not actuarially neutral with respect to the age at which benefits are claimed, participants may choose to strategically exploit outside options offered by insurance companies. This may affect claiming behavior, which in turn affects the plan's liabilities.

The remainder of this paper is organized as follows. Section 2.2 discusses factors that generate actuarial unfairness in Social Security pension systems with delay options. In Sections 2.3 and 2.4 we consider the case where insurers differentiate premiums based on gender only, and characterize conditions under which they can offer super-replicating annuity products for men and women, respectively. We also quantify the potential gains for both individuals and insurers. Section 2.3 considers individuals who wish to defer the receipt of pension benefits to a specific age. Section 2.4 extends the analysis to cases where the individual wishes to defer the receipt of pension benefits to an unspecified age. In Section 2.5 we illustrate the potential gains when insurers can, in addition to gender, also differentiate premiums on the basis of factors correlated with educational level. We end with the conclusions in Section 2.6.

2.2 Actuarial unfairness

Existing literature shows that the option to delay Social Security benefit claiming is often actuarially unfair in the sense that the expected present value of the additional benefits in case of deferred benefit claiming is strictly lower than the expected present value of the missed benefits (see, e.g., Coile, Diamond, Gruber, and Jousten, 2002; Duggan and Soares, 2002; Brown, 2003; Desmet and Jousten, 2003; Sun and Webb, 2009). This unfairness implies that individuals who wish to defer the receipt of pension benefits may be better off by claiming benefits immediately, and using them to buy annuity products at the market. Coile, Diamond, Gruber, and Jousten (2002) show that despite the actuarial unfairness, risk-averse individuals may want to delay benefit claiming to increase longevity protection. Our goal is to characterize under which preference-free conditions insurers can offer annuity products that individuals prefer above deferring benefit claiming.

We focus on cases in which an individual wishes to delay the receipt of pension benefits beyond the so-called full retirement age, which we denote by $\underline{x}$.[5] Each year, the individual decides either to claim old-age pension benefits immediately, or to delay benefit claiming for a period of at least one year.[6] In case of delay, the individual is offered the same choice next year and so on, until either the maximum age at which benefits can be claimed has been reached or benefits have been claimed. We denote the maximum age at which benefits can be claimed by $\overline{x}$. When the individual claims benefits, he receives them in the form of a *whole life annuity* that periodically pays a fixed amount

[5] In many countries (including, e.g., the U.S.), individuals can also claim pension benefits at an earlier age than the full retirement age, in which case the benefit level is adjusted downwards. Our focus is on delayed benefit claiming.

[6] It is not uncommon that individuals can decide on a monthly basis to claim benefits or delay benefit claiming. For expositional convenience, we assume that the decision is made annually.

as long as he is alive. Without loss of generality, we normalize the annual benefit level in case benefits are claimed at the full retirement age to 1. For each year of delay, the benefit level increases by a fixed amount a, for some $a > 0$. Therefore, in case benefit claiming is deferred until age $y > \underline{x}$, the annual benefit level is equal to $1 + (y - \underline{x})a$.

Whether insurers will be able to offer more attractive delay options clearly depends on the degree of actuarial unfairness in the Social Security system. This degree of unfairness depends not only on the accrual a, but also on the term structure of real interest rates and individual characteristics that affect survival probabilities (see, e.g., Duggan and Soares, 2002). First, higher long-term interest rates lead to less expensive annuities, which may result in an opportunity for insurance companies to outperform the Social Security provider. Second, the delayed retirement credit does not differ with individual characteristics (such as, e.g., gender) even though survival probabilities do differ with these characteristics. This leads to actuarial nonequivalence at the individual level. Thus, even if the system would be fair for the “average” individual, it would be unfair to certain groups of individuals (see, e.g., Brown, 2003; Desmet and Jousten, 2003). Insurers can, at least to some extent, differentiate premiums and may therefore be able to offer more attractive delay options to those individuals for which the Social Security system is actuarially unfair.

To illustrate that the degree of actuarial unfairness can be significant, and that it depends strongly on both the term structure of real interest rates and individual characteristics, we determine the *money’s worth* of deferring the receipt of pension benefits. The money’s worth of the option to delay benefit claiming is defined as as the ratio of the expected present value of the additional benefits received as of claiming age over the expected present value of the missed benefits (see, e.g., Sun and Webb, 2009). Let us denote $R^{(\tau)}$ for the τ-years real interest rate, and ${}_{\tau}p_x$ for the probability that an individual with age x survives at least the first τ years. Now consider an individual aged x who wants to defer the receipt of pension benefits to age y. Because the missed benefit equals $1 + a(x - \underline{x})$ at ages $x, \cdots, y - 1$, and the additional benefit equals $a(y - x)$ annually as of age y, the money’s worth of deferring benefit claiming from age x to age y, denoted by $MW(y, x)$, is given by:

$$MW(y,x) = \frac{a(y-x)\left(\sum_{\tau=y-x}^{\infty} \frac{{}_{\tau}p_x}{\left(1+R^{(\tau)}\right)^{\tau}}\right)}{(1+a(x-\underline{x}))\left(\sum_{\tau=0}^{y-x-1} \frac{{}_{\tau}p_x}{(1+R^{(\tau)})^{\tau}}\right)}.$$

Figure 2.2 displays the money’s worth of delaying benefit claiming from

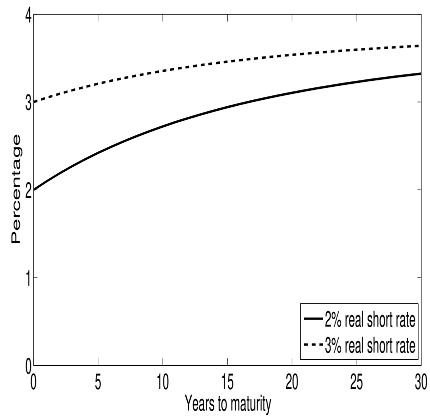


Figure 2.1 – Term structures of real interest rates (in percentages), generated by a one-factor Vasicek model with parameters given in Table 2.5 in Appendix 2.B.

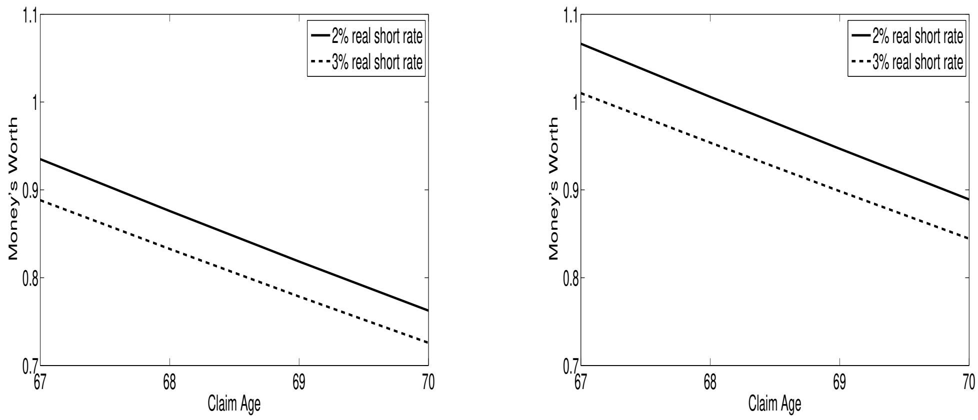
age 66 to age y, for $y = 67, \cdots, 70$, for men and women, and for the two term structures of real interest rates displayed in Figure 2.1.[7] The solid (dashed) lines correspond to the lower (upper) term structure. We consider the U.S. setting in which the annual accrual offered by the Social Security system equals 8% (i.e., $a = 0.08$), and the full retirement age equals 66 (i.e., $\underline{x} = 66$).[8] Survival probabilities are those of U.S. males (females) for the years 2000 up to and including 2004, as reported in the Human Mortality Database.[9]

The option to defer benefit claiming to age $y > 66$ is actuarially unfair if the corresponding money's worth is below one, because this indicates that the expected present value of the additional annuity received as of age y in return for delaying benefit claiming is strictly lower than the present value of the missed benefits at ages $66, \cdots y-1$. Figure 2.2 shows that the degree of actuarial unfairness can be substantial, and that it depends strongly on the

[7]The term structures are generated by a one-Vasicek model with parameters as displayed in Table 2.5 in Appendix 2.B, and with a short rate of 2% (solid lines) and 3% (dashed lines), respectively. In Sections 2.4 and 2.5 is needed to determine the prices of call options and to generate the profit distributions. For consistency, we use the same term structure of interest rates in this Section.

[8]Because there is an earnings test for claiming benefits before the full retirement age (i.e., between the age of 62 and 65) (see e.g., Song and Manchester, 2007), we focus on individuals who wish to delay benefit claiming beyond the full retirement age of 66. However, the analysis can be easily extended to individuals who want to claim before the full retirement age.

[9]Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on $05-01-2009$). The survival probabilities are displayed in Figure 2.9 in Appendix 3.B.



(a) Men **(b)** Women

Figure 2.2 – The money's worth of deferring Social Security benefit claiming from age 66 to age y (i.e., $MW(y, 66)$) as a function of y, for men (a) and women (b), and for two term structures of interest rates, generated by a one-factor Vasicek model with parameters given in Table 2.5 in Appendix 2.B. The solid lines (dashed lines) correspond to a real short rate of 2% (3%). The annual accrual is $a = 8\%$, and the full retirement age is set at $\underline{x} = 66$. The survival probabilities are those of U.S. males (females) for the period $2000 - 2004$.

term structure of real interest rates as well as on individual characteristics such as gender and the preferred deferral period. First, comparing the solid and the dashed lines shows that the deferral option is more unfair when interest rates are high. When interest rates are higher (dashed lines), the money's worth shifts downwards for both men and women, and for all deferral periods. Higher long term interest rates decrease the value of the additional benefits relative to the value of the missed benefits, and therefore make deferral more actuarially unfair. The figure also shows that the system is more unfair for men than for women, and more unfair for those who wish to defer for a longer period. Because women have higher life expectancy than men, they are expected to receive the increased benefit for a longer period of time. Therefore, the money's worth of deferring benefit claiming is significantly lower for men than for woman. Consider, for example, the case where the real interest rate is upward sloping from 2% for the real short rate to just above 3.3% for a maturity of 30 years (Figure 2.1, solid line).[10] The money's worth for men is below one for all deferral periods. For women, the money's worth is above one for deferral of at most two years, but strictly below one for longer deferral

[10]The results in this case are similar to those reported in Sun and Webb (2009) using survival probabilities of the Social Security administration, and a flat term structure of 3%.

periods. Finally, for both men and women and for both term structures, the money's worth of deferring benefit claiming is decreasing in the length of the deferral period. Stated differently, the system is more unfair for those who would like to delay benefit claiming more than for those who would like to delay benefit claiming just a couples of years.

The above results suggest that the degree of actuarial unfairness in the Social Security system is substantial, in particular for those who wish to defer benefit claiming for a longer period. In the next sections we show that this unfairness implies that individuals who wish to defer the receipt of pension benefits may be better off by claiming benefits immediately, and using them to buy annuity products at the market.

2.3 Dominating strategies using deferred annuities

In this section we characterize conditions under which the market can offer annuity products that are preferred by individuals above deferring pension benefit claiming. The annuity products must be attractive for both insurers and participants, implying that insurers should be able to offer them on profitable terms and individuals should achieve a higher benefit level by buying these products than by deferring benefit claiming. Conditions will be determined under which this holds. When these conditions are satisfied, claiming benefits early and using them to buy a deferred annuity dominates deferring benefit claiming in the sense that the former strategy is preferred to the latter, irrespective of the individual's preference relation.[11] An example of such a preference-free choice is given below.

> Suppose that a man with current age 66 would like to receive pension benefits as of age 67. Furthermore, assume that the benefit level of his pension when he claims benefits immediately equals 100 and that when the man delays benefit claiming by one year, his future benefit level will be increased by 8%. Thus, when he defers pension benefit claiming from age 66 to

[11]When the insured claims benefits, they generally are taxed. However, in case the income is used as a premium for annuities, they are in many cases received taxfree and then taxed when the annuity pays out. In the U.S. there are some qualified retirement accounts in which individuals can invest taxfree. The wealth invested can then be used to finance annuities, where the payments of the annuities are taxed (see Brown, Mitchell, Poterba, and Warshawsky, 2001). We assume a tax system were both premiums and returns on the premiums for annuities are exempted from taxation, and only the annuity payments are taxed.

age 67, he will receive an annual benefit of 108 as of age 67. Now suppose that the man is able to buy a deferred annuity at the market which gives an annual benefit of 9 as of age 67 for a price of 100. When he claims benefits immediately and uses the benefits to finance this deferred annuity, he will receive an annual benefit level of 109 as of age 67. We will therefore argue that, independent of the individual's preferences, claiming benefits at age 67 is dominated by claiming benefits at age 66 and using the benefits as a premium for a deferred annuity that starts to pay out at age 67. The different strategies are displayed in Table 2.1.

Strategy	Annual Cash flow at age				
	66	67	68	69	70 ....
Claim 66	100	100	100	100	100 ...
Claim 67	–	108	108	108	108 ...
Claim 66, buy deferred annuity	–	109	109	109	109 ...

Table 2.1 – The annual payments in a stylized example for a man with age 66, for an accrual a of 8%, and for different strategies.

First note from Table 2.1 that for the last two strategies, the man needs an alternative source of income to finance consumption at age 66. For both strategies, he can finance consumption at age 66 with for instance labor income, or alternatively he may rely on his own savings or borrow money at some interest rate. Since he can finance consumption at age 66 with the same income source for both strategies, it does not affect the relative attractiveness of both strategies in a utility framework. Therefore, for our analysis it is irrelevant how consumption before the individual receives annuity income is financed.

From Table 2.1 it is clear that claiming benefits at age 67 is dominated by claiming benefits at age 66 and using the benefits received that year to buy a deferred annuity that starts to pay out at age 67. Of course this is just a stylized example and we still have to analyze the conditions under which insurers can indeed offer a higher benefit level. In the next subsection we determine sufficient conditions under which the market can outperform the option to delay as offered by the Social Security Administration.

2.3.1 Characterizing conditions for dominance

In this section we consider an individual who, at a given age x (e.g., the full retirement age), decides as of which age he would like to receive his pension benefits, and derive preference-free conditions under which insurers can offer deferred annuities that the individual prefers above deferring benefit claiming.

For an individual aged x, deferring benefit claiming to age y can be considered as buying a deferred real annuity. The premium equals the missed benefits at ages $x, \cdots, y-1$. In return for this premium, a deferred annuity with a benefit level of $(y-x)a$ as of age y is received. For example, in case benefit claiming is deferred to age $\underline{x}+1$, a premium of 1 (i.e., the benefit level in case the individual would have claimed at age $\underline{x}$) is used to finance a deferred annuity with start age $\underline{x}+1$, and benefit level a. If the expected present value of the additional benefits is lower than the premium paid (i.e., when the money's worth of this deferred annuity is less than one), the deferred annuity offered by the pension provider is actuarially unfair, and so the market may be able to outperform the pension provider by offering a more attractive deferred annuity.

Suppose that an individual with age x would like to receive pension benefits as of age y, with $y > x$. He could do so by deferring benefit claiming until age y, in which case the benefit level will equal $1+(y-\underline{x})a$. Alternatively, however, the individual could claim benefits at age x, and (conditional on being alive) use the benefits received up to age y as periodic premiums to finance a deferred annuity that starts to pay out at age y.[12] Let $b_{y,x}$ denote the benefit level offered by the insurer. Then the aggregate benefit level received as of age y equals the sum of the Social Security benefits that were claimed at age x, $1+(x-\underline{x})a$, and the payoff from the deferred annuity, $b_{y,x}$, i.e.,

$$B_{y,x} := 1+(x-\underline{x})a + b_{y,x}. \tag{2.1}$$

This strategy is preferred if insurers can offer a deferred annuity with a benefit level $b_{y,x}$ that is strictly higher than the accrual offered by the Social Security

[12] Alternatively, the individual could use only part of the claimed benefits to buy a deferred annuity. It can be verified that assuming that the claimed benefits are fully used is without loss of generality. Deferring benefit claiming is dominated by claiming immediately if and only if this is the case when the claimed benefits are fully used. Note that for both strategies, the individual needs additional financial resources to fund his consumption at age x until age $y-1$. However, since the same financial resource can be used for both strategies, this does not affect the relative attractiveness of both strategies.

system, i.e., if

$$b_{y,x} > (y-x)a. \tag{2.2}$$

Indeed, (2.1) and (2.2) imply that the aggregate benefit level is strictly higher than the benefit level received in case Social Security benefit claiming is deferred to age y, i.e., $B_{y,x} > 1 + (y - \underline{x})a$.[13]

Whether insurers will be able to offer deferred annuities that individuals prefer above deferring benefit claiming clearly depends on the prices charged for deferred annuities. The annuity insurers offer is in general not actuarially fair because insurers impose a premium load. The load may include costs for administration and adverse selection, but also a risk premium, and is typically expressed as a percentage l of the premium (see, e.g., Mitchell, Poterba, Warshawsky, and Brown, 1999). Now consider an individual who claims benefits at age x, and uses the benefits received at ages $x, \cdots, y-1$, as periodic premiums to finance a deferred annuity that starts to pay out at age y. Then, the benefit level $b_{y,x}$ that insurers would offer follows from setting the expected present value of the premium net of cost loading equal to the expected present value of the payments of the deferred annuity, i.e.,

$$(1-l)\left(1+(x-\underline{x})a\right)\left(\sum_{\tau=0}^{y-x-1}\frac{{}_{\tau}p_x}{\left(1+R^{(\tau)}\right)^{\tau}}\right) = b_{y,x}\left(\sum_{\tau=y-x}^{\infty}\frac{{}_{\tau}p_x}{\left(1+R^{(\tau)}\right)^{\tau}}\right). \tag{2.3}$$

Combined with (2.2), this implies that claiming benefits immediately and using them to buy a deferred annuity dominates deferring benefit claiming if

$$b_{y,x} := \frac{(1-l)\left(1+(x-\underline{x})a\right)\left(\sum_{\tau=0}^{y-x-1}\frac{{}_{\tau}p_x}{\left(1+R^{(\tau)}\right)^{\tau}}\right)}{\left(\sum_{\tau=y-x}^{\infty}\frac{{}_{\tau}p_x}{\left(1+R^{(\tau)}\right)^{\tau}}\right)} > (y-x)a. \tag{2.4}$$

Whether this condition can be satisfied depends on the term structure of real interest rates as well as on the premium load l. In the next subsection, we investigate the effect of the term structure of real interest rates and the premium load on the existence of dominating strategies.

2.3.2 Effect of term structure and premium load

In this subsection conditions are characterized under which insurers can profitably offer deferred annuities that individuals prefer above deferring benefit

[13]For simplicity we ignore default risk of the insurer and the state.

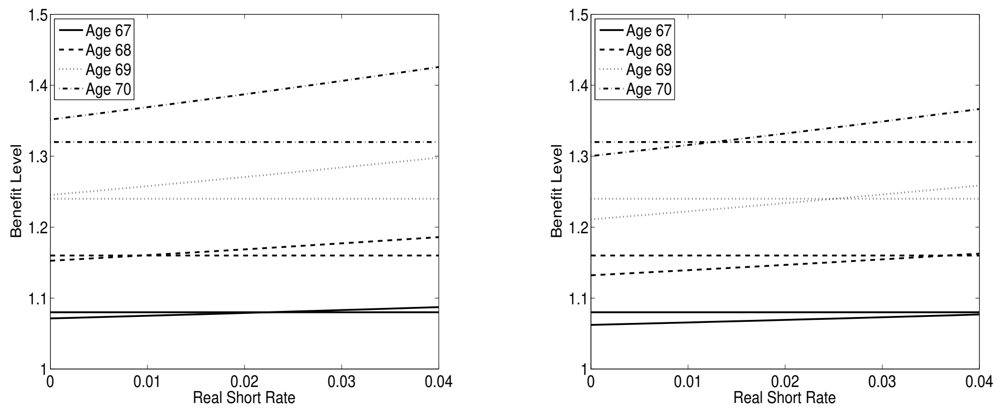
claiming. To do so, we compare the benefit levels individuals can obtain by either delaying benefit claiming or by claiming immediately and using the benefits to buy a deferred annuity at the market. We first consider a base case in which the term structure of real interest rates is as displayed in Figure 2.1, solid line. It is upward sloping from 2% for the real short rate to just above 3.3% for a maturity of 30 years. The premium load equals 7.3%, i.e., $l = 0.073$.[14] We then investigate the sensitivity of the results with respect to changes in the term structure of real interest rates or in the premium load.

Annuity	Claim age (x) Men					Claim age (x) Women				
Age (y)	66	67	68	69	70	66	67	68	69	70
66	1.00					1.00				
67	1.08	1.08				1.07	1.08			
68	**1.17**	**1.17**	1.16			1.15	1.16	1.16		
69	**1.27**	**1.27**	**1.26**	1.24		1.23	**1.24**	**1.25**	1.24	
70	**1.39**	**1.38**	**1.37**	**1.35**	1.32	**1.33**	**1.34**	**1.34**	**1.33**	1.32

Table 2.2 – The aggregate benefit level received as of age y for an individual aged x, when Social Security benefits are claimed at age x and used to finance a deferred annuity that starts to pay out at age y ($B_{y,x}$, off-diagonal elements), and when claiming Social Security benefits is deferred to age y (diagonal elements). The left (right) panel corresponds to men (women). The bold entries represent dominating strategies. The annual accrual a equals 8% and the load l equals 7.3%. The survival probabilities are those of U.S. males (females) for the period 2000-2004. The term structure of real interest rates is as displayed in Figure 2.1, solid line.

Table 2.2 displays the benefit levels for the base case. For any given age $y = 66, \cdots, 70$, the diagonal displays the benefit level received as of age y when Social Security benefits are claimed at age y, and the off-diagonal elements (i.e., for $x < y$) yield the benefit level the individual receives as of age y when he claims Social Security benefits at an earlier age x, and uses them to finance a deferred annuity that starts to pay out at age y. If the latter exceeds the former (bold entries), deferring benefit claiming is suboptimal. For example, in case a man aged 66 would like to receive pension benefits as of age 68, the dominating strategy he can follow is claiming benefits immediately and using these benefits to buy a deferred annuity which starts paying off at age

[14]For most maturities the interest rate is lower than the 3% real interest rate as assumed in for instance Sun and Webb (2009), and Coile, Diamond, Gruber, and Jousten (2002). The load is taken from the 1999 annuity value per premium dollar computed on an after tax basis in Mitchell, Poterba, Warshawsky, and Brown (1999).



(a) Men **(b)** Women

Figure 2.3 – The aggregate benefit level received as of age y, as a function of the real short rate at age 66 , when Social Security benefits are claimed at age 66 and used to finance a deferred annuity that starts to pay out at age y ($B_{y,66}$, upward sloping lines), and when claiming Social Security benefits is deferred to age y (horizontal lines). The left (right) panel corresponds to men (women). The annual accrual a equals 8% and the load l equals 7.3%. The survival probabilities are those of U.S. males (females) for the period $2000-2004$. The term structure of real interest rates corresponding to a specific real short rate is generated with a one-factor Vasicek model, with parameters given in Table 2.5 in Appendix 2.B.

68. Men with age 67 or higher and women with age 68 or higher are better off by claiming benefits immediately and using them to buy a deferred annuity than by delaying benefit claiming, regardless of how long they wish to defer the receipt of their pension benefits.

The above results correspond to the term structure as displayed in Figure 2.1, solid line. Higher long-term interest rates make deferred annuities cheaper, and so it becomes more likely that insurers will be able to offer deferred annuities that individuals prefer above deferring benefit claiming. The opposite holds for lower long-term interest rates. To investigate the sensitivity of our results with respect to changes in the term structure of real interest rates, we use a one-factor Vasicek model (Vasicek, 1977). In this one-factor model, the term structure is fully determined by the short rate, and so the sensitivity of the results with respect to the term structure of real interest rates can be investigated by varying the short rate. Details on the one-factor Vasicek model can be found in Appendix 2.B.

Figure 2.3 displays the benefit level that an individual aged 66 can obtain as of age y, for $y = 67, \cdots, 70$, as a function of the real short rate, and for two strategies: claiming benefits immediately and using them to finance a deferred annuity that starts to pay out at age y (upward sloping lines), and deferring benefit claiming until age y (horizontal lines).

The figure shows that for each annuity age y, there exists a critical value of the real short rate at which the individual is indifferent between these two strategies. Whenever the short rate is higher than this critical value, annuities are relatively cheap, and insurers can profitably offer annuities that yield higher benefit levels than the accrual offered by the Social Security (upward sloping line higher than horizontal line). Thus, deferring benefit claiming is dominated by claiming benefits immediately and using them to buy a deferred annuity. Below the critical real short rate, deferring pension benefit claiming is preferred above buying additional annuities at the market. Second, the figure shows that dominating strategies are more likely to exist for men than for women. For a man aged 66 who would like to receive pension benefits as of age 67 (solid lines), claiming benefits early to finance a deferred annuity dominates delayed benefit claiming in case the real short rate is above 2.25%. For a woman, the critical real short rate for deferral of one year equals 4.7%, which is quite high. As a result, dominating strategies are not likely to exist in this case. Finally, the figure shows that for both men and women, the critical real short rate decreases when the age as of which they would like to receive pension benefits increases. For men (women), it decreases to -1.8% (1.2%) for deferral to age 70 (dashed-dotted lines). This occurs because the system is more unfair for those who would like to delay benefit claiming more than for those who would like to delay benefit claiming just a couples of years (recall that the money's worth of deferring benefit claiming decreases when the deferral period increases, see Figure 2.2). Consequently, there is more room for dominance for individuals who wish to delay the receipt of pension benefits for a longer period.

The above results correspond to settings where the premium load equals 7.3%. It is immediately clear from (2.4) that a higher premium load reduces the benefit level that insurers can offer for a given premium, and therefore makes it less likely that insurers are able to offer deferred annuities that individuals prefer above deferring benefit claiming. In order to investigate the sensitivity of our results to the level of the premium load, we determine the load such that the individual is indifferent between deferring benefit claiming, and claiming immediately and buying a deferred annuity. Consider an individual aged x would like to receive pension benefits as of age y, with $y > x$. The individual is indifferent between the two strategies if they yield the same benefit level, i.e., if

$$b_{y,x} = (y-x)a.$$

Therefore, it follows from (2.4) that the indifference load $l_{\max}$ is given by:

$$l_{\max} = 1 - \frac{(y-x)a\left(\sum_{\tau=y-x}^{\infty} \frac{{}_{\tau}p_x}{\left(1+R^{(\tau)}\right)^{\tau}}\right)}{(1+(x-\underline{x})a)\left(\sum_{\tau=0}^{y-x-1} \frac{{}_{\tau}p_x}{\left(1+R^{(\tau)}\right)^{\tau}}\right)} = 1 - MW(y,x).$$

As long as the premium load is strictly lower than $l_{\max}$, the market can offer deferred annuities that (combined with the Social Security benefits claimed at age x) give a higher benefit level than the benefit level offered by the Social Security provider in case benefit claiming is delayed until age y. Thus, deferring benefit claiming is dominated by claiming immediately.

Table 2.3 displays the maximum load under which claiming Social Security benefits and using them to buy a deferred annuity dominates deferring benefit claiming, for all possible combinations of the claim age x and the annuity age $y > x$.

Annuity	Claim Age (x) Men				Claim Age (x) Women			
Age (y)	66	67	68	69	66	67	68	69
67	6.51				−6.64			
68	12.41	16.39			−0.06	4.29		
69	18.14	21.82	24.90		5.30	9.86	13.72	
70	23.72	27.10	29.93	32.10	11.06	15.31	18.91	21.97

Table 2.3 – The maximum load $l_{\max}$ (in percentages) under which, at age x, deferring benefit claiming to age $y > x$ is dominated by claiming Social Security benefits at age x and using them to buy a deferred annuity that starts to pay out at age y. The left (right) panel corresponds to men (women). The accrual is set at $a = 8\%$. The survival probabilities are those of U.S. males (females) for the period $2000 - 2004$. The term structure of real interest rates is as displayed in Figure 2.1, solid line.

For men aged 66 who would like to receive pension benefits as of age 67, the load insurance companies can impose should be below 6.5%. However, for men who wish to defer the receipt of pension benefits until at least age 68, loads can be imposed that are significantly higher than the benchmark level of 7.3%. For women aged 66 who would like to receive pension benefits as of age 67 or 68, dominating strategies will not exist because a negative load is needed. This occurs because for them the option to defer benefit claiming in the Social Security system is more than actuarially fair (i.e., the money's worth is higher than one; see Figure 2.2, solid line). For women aged 68 or women who would like to defer benefit claiming for a longer period, the loads are also significantly higher than the benchmark level.

2.4 Dominating strategies using annuity options

In the previous section we characterized conditions under which it is optimal for the individual to claim pension benefits at an earlier age than the age as of which he wants to receive annuity benefits, and use the pension benefits to buy a deferred annuity. We considered the case where an individual at a given age decides as of which age he would like to receive pension benefits, so that a deferred annuity with the corresponding deferral period can be bought. This section considers an individual who wishes to defer the receipt of pension benefits until an unspecified age. We develop an annuity product, called an *annuity option*, in which the individual can, year by year, decide whether he wants to annuitize or defer annuitization for at least one more year. We characterize conditions under which insurers can offer annuity options that super-replicate those offered by the Social Security provider.

2.4.1 Super-replicating annuity options

In this subsection we design an *annuity option* that super-replicates the option to delay benefit claiming in the Social Security system. The individual who buys this option pays a periodic premium (in case he is still alive) until the time he decides to annuitize, and from there on receives annuity payments from the insurer. The level of the periodic premium depends on the age at which the product is bought. The level of the annuity payment depends on the age at which the option to annuitize is exercised, as well as on the age at which the option is bought. Let us denote:

- x for the age at which the insured buys the annuity option;
- $Y \in \{x+1, ..., \overline{x}\}$ for the age at which the insured annuitizes. Y is unknown until it is reached, we denote y for any given realization of Y;
- $\pi(x)$ for the premium paid at ages $z \in \{x, ..., Y-1\}$, conditional on being alive, and given that the annuity option was bought at age x;
- $b_{y,x}$ for the benefit level of the annuity, conditional on annuitizing at age y, and given that the annuity option was bought at age x. We assume

that:

$$0 = b_{x,x} \leq b_{x+1,x} \leq \cdots \leq b_{\overline{x},x}.$$

At each age $z \in \{x+1, ..., \overline{x}-1\}$, the individual decides either to pay a premium of $\pi(x)$ and defer annuitization for at least another year, or to stop paying premium and annuitize. When he annuitizes, he receives an immediate annuity from the insurance company with a benefit level $b_{y,x}$ that depends on his current age y, and the age x at which he bought the annuity option. The benefit levels are determined at the moment the annuity option is bought.

This annuity option (weakly) dominates the option to delay benefit claiming in the Social Security system if the periodic premium is at most equal to the benefit level obtained in case benefits are claimed at age x, and, *for each possible annuity age* y, the level of the annuity payment is at least equal to the accrual offered by the Social Security system in case benefit claiming would have been delayed to that age, i.e.,

$$\pi(x) \leq 1 + (x - \underline{x})a, \tag{2.5}$$

$$b_{y,x} \geq (y - x)a, \text{ for all } y = x+1, \cdots, \overline{x}. \tag{2.6}$$

If these two conditions are satisfied with at least one strict inequality, then for an individual aged x who did not yet claim pension benefits, further deferring benefit claiming is dominated by claiming benefits (of $1 + (x - \underline{x})a$) and using (part of) these benefits to pay the periodic premiums for the annuity option. Indeed, (2.5) implies that the benefits are sufficient to pay the periodic premium, and (2.6) implies that, for any given annuity age y, the aggregate benefit level (from Social Security benefits claimed at age x and from the annuity option),

$$B_{y,x} := 1 + (x - \underline{x})a + b_{y,x},$$

is weakly higher than the benefit level received in case Social Security benefit claiming is deferred to age y. Condition (2.6) needs to be satisfied for each $y \in \{x+1, .., \overline{x}\}$ because the individual can decide on a yearly basis to delay Social Security benefit claiming. Thus, for dominance the benefit level obtained by buying the annuity option needs to be at least as high as benefit level obtained by delaying Social Security benefit claiming for each age $y \in \{x+1, .., \overline{x}\}$.

Whether insurers will be able to offer super-replicating annuity options clearly depends on how they are priced. Because the risk associated with uncertainty in the age at which the individual will exercise the option to annuitize cannot be hedged, the payoffs of the annuity option cannot be replicated by payoffs from existing assets. In the following subsection we determine conditions under which there exists a selffinancing strategy that super-replicates the payoffs of the annuity option. The strategy is selffinancing if any new assets or annuity payments can be financed from revenues from previously bought assets combined with the premium received from the individual. If these conditions are satisfied, insurers can offer annuity options that satisfy the dominance conditions (2.5) and (2.6), while making nonnegative profits in each future year.

2.4.2 The financing strategy of the insurer

The risk due to the uncertainty with respect to the age at which individuals exercise the annuity option is idiosyncratic because it depends on, for instance, the state of the economy. Therefore, we first design a strategy such that at every possible exercise date, the insurer holds a portfolio of zero-coupon bonds with a market price equal to the price of the annuity in case the insured annuitizes at that date. To hedge the interest risk in the bond portfolio, the insurer buys a porfolio of call-options. If the insured does not annuitize, the payoff of the bond portfolio is used to finance a new bond portfolio. If the insured annuitizes, the bond portfolio is sold to finance the immediate annuity. Formally, suppose that an individual aged x buys an annuity option at time $t = 0$, and consider the following strategy:

- At age x, the insurer knows that the benefit level of the annuity will be at least $b_{x+1,x}$. He buys a portfolio of zero-coupon bonds which cash flow matches the expected payments (plus cost loading) of the annuity in case the insured annuitizes at age $x + 1$. We assume that ${}_{110-x+\tau}p_x = 0$ for $\tau \geq 0$.
$$\left(\frac{b_{x+1,x}}{1-l}\right)\ {}_sp_x, \text{ in years } s = 1, ..., 110 - x.$$
- At age $x < z < Y$, the insured does not yet annuitize, and the insurer knows that the benefit level upon annuitization will be at least $b_{z+1,x}$, i.e., the benefit level increases by at least
$$\widetilde{b}_{z,x} = b_{z+1,x} - b_{z,x}.$$
Therefore, the insurer buys, in addition, a portfolio of zero-coupon bonds with cash flows $\left(\frac{\widetilde{b}_{z,x}}{1-l}\right)\ {}_sp_x$, in years $s = z - x + 1, \cdots, 110 - x$. Define

$\tilde{b}_{x,x} = b_{x+1,x}$. Combined with bonds bought at ages $x, \cdots, z-1$, this implies that he holds a portfolio of zero-coupon bonds with cash flows

$$\left(\sum_{\tau=x}^{z} \frac{\widetilde{b}_{\tau,x}}{1-l}\right) {}_sp_x = \left(\frac{b_{z+1,x}}{1-l}\right) {}_sp_x, \text{ in years } s = z-x+1, ..., 110-x.$$

He receives a cash flow of $\left(\frac{b_{z,x}}{1-l}\right) {}_{z-x}p_x$ from previously bought bonds, as well as a premium payment equal to $\pi(x)$ from every insured that survived. Combined, the expected cash inflow equals

$$\left(\pi(x) + \frac{b_{z,x}}{1-l}\right) {}_{z-x}p_x.$$

- At age $z = Y$, the insured annuitizes. The insurer holds a portfolio of bonds, bought at ages $x, \cdots, Y-1$, with aggregate payoff

$$\left(\frac{b_{Y,x}}{1-l}\right) {}_sp_x, \text{ in years } s = Y-x, ..., 110-x.$$

  The market price of this bond portfolio equals the price of the annuity that pays off $b_{Y,x}$ in every future year that the insured is alive.

This strategy yields the desired payoff as of age Y. For it to be selffinancing, however, revenue at each age before annuitization needs to be sufficient to finance the new bond portfolio. In every year in which the insured has not yet exercised the annuity option, the insurer receives revenue which consists of the premium paid by the insured and the cash flow of previously bought bonds which mature. From this revenue, he needs to finance a bond portfolio. The strategy therefore involves losses when the price of the bond portfolio exceeds the revenue. Moreover, for ages $z > x$, the price of the bond portfolio that needs to be bought at age z depends on the term structure of real interest rates in year $t = z - x > 0$. To eliminate this interest rate risk, the insurer can, for each age $z = x+1, \cdots, \overline{x}-1$, buy a call option with maturity date $t = z - x$ on the corresponding bond portfolio. To minimize the price of the call options while still guaranteeing that revenue is sufficient to buy the bond portfolio, we set the strike price $K(z,x)$ of the call option on the bond portfolio that needs to be bought at age z equal to revenue received at that age, i.e.,

$$K(z,x) = \left(\pi(x) + \frac{b_{z,x}}{1-l}\right) {}_{z-x}p_x. \tag{2.7}$$

In the following table we summarize the insurer's revenue and expenses at each age, with and without call options. We denote $P_{Calls}(x)$ for the date $t = 0$

price of the portfolio of call options. Moreover, to avoid overloaded notation, we denote $P_{Bonds}(z, x)$ for the date $t = z - x$ price of the bond portfolio that needs to be bought at age z.

		Age $z = x$	Age $z \in [x+1, Y-1]$
Revenue		$\pi(x)$	$\left(\pi(x) + \frac{b_{z,x}}{1-l}\right) {}_{z-x}p_x$
Expenses	without options	$P_{Bonds}(x, x)$	$P_{Bonds}(z, x)$
	with options	$P_{Calls}(x) + P_{Bonds}(x, x)$	$\min\{P_{Bonds}(z, x), K(z, x)\}$
Profit	without options	$+/-$	$+/-$
Sign	with options	$+/-$	$+$

Table 2.4 – The insurer's revenue and expenses at age z (i.e., in year $t = z - x$), for $z = x, \cdots, Y-1$, for an insured who buys the annuity option at age x and exercises it at age Y, and for two financing strategies: the case where the insurer buys call options and the case where he does not buy call options. The last two rows display the sign of the corresponding profit (revenue minus expenses).

With call options, expenses at age x increase, but expenses at ages $z \in [x + 1, Y - 1]$ (weakly) decrease because the required bond portfolio can be bought at the minimum of the market price and the strike price of the call option. Moreover, because the strike price of the call option on the bond portfolio that needs to be bought at a given age is set equal to the revenue at that age, the revenue always weakly exceeds the expenses at any age $z > x$. Thus, the insurer can offer the annuity option at a nonnegative profit in every year if and only if revenue exceeds expenses at age x, i.e., if and only if

$$P_{Calls}(x) + P_{Bonds}(x, x) \leq \pi(x). \tag{2.8}$$

Our goal is to characterize conditions on the premium load l, and the term structure of real interest rates under which the dominance conditions (2.5) and (2.6), and the profit condition (2.8) are satisfied. When these conditions are satisfied, insurers can profitably offer annuity options such that individuals who wish to defer the receipt of pension benefits until an unspecified age are better off by claiming benefits and using them to buy the annuity option. This approach is conservative in the sense that it assumes that the insurer wishes to eliminate all interest rate risk. If insurers are willing to bear some risk, the conditions under which they can offer super-replicating annuity options will become less strict. The insurer buys call options for each possible exercise age of each individual.

2.4.3 When can the insurer profitably offer a super-replicating annuity option?

In this section we first determine conditions on the term structure of real interest rates under which insurers can profitably offer super-replicating annuity options. We then investigate the sensitivity of these results to the level of the premium load charged by insurers. Finally, we quantify the potential gains for insurers from offering super-replicating annuity options.

To characterize conditions on the term structure of real interest rates and the profit loading under which insurers can profitably offer super-replicating annuity options, we consider the annuity option that *replicates* the option to defer benefit claiming in the Social Security system, i.e., we set

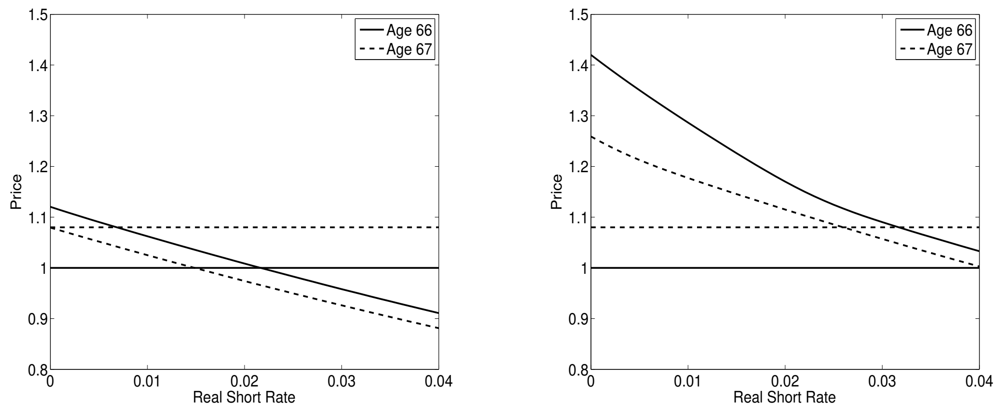
$$\pi(x) = 1 + (x - \underline{x})a, \tag{2.9}$$

$$b_{y,x} = (y - x)a, \text{ for all } y = x + 1, \cdots, 70. \tag{2.10}$$

and investigate under which conditions an insurer who uses the selffinancing strategy defined in Subsection 2.4.2 makes a strictly positive profit in the year in which the annuity option is sold (i.e., $P_{Calls}(x) + P_{Bonds}(z, x) < \pi(x)$). If this is the case, the insurer can profitably offer the replicating annuity option because, as can be seen from Table 2.4, the revenue weakly exceeds expenses in all future years. Moreover, since the profit in the first year is strictly positive, either the annual premium $\pi(x)$ could be decreased or the benefit level for at least one annuity age y could be increased, so that a super-replicating annuity option can be offered while still making a positive profit. An individual who wishes to defer annuitization until an unspecified age is then better off by claiming benefits and using them to buy that annuity option than by further delaying benefit claiming. Indeed, either the individual has strictly more wealth before annuitization (if $\pi(x) < 1 + (x - \underline{x})a$), or the benefit level as of annuitization is strictly higher for at least one annuity age (if $b_{y,x} > (y - x)a$).

Figure 2.4 displays the insurer's revenue (horizontal lines) and expenses (downward sloping lines) in the year in which the annuity option is sold, as a function of the real short rate at that time.[15] The revenue equals the premium paid by the individual. The expenses are equal to the price of the portfolio of call options and bonds that needs to be bought at the time the contract is

[15] Recall that in the one-factor Vasicek model, the term structure is fully determined by the short rate, and so the sensitivity of the results with respect to the term structure of real interest rates can be investigated by varying the short rate. Details on the Vasicek model as well as on how the price of the portfolio of call options and bonds is determined can be found in Appendices B and C.



(a) Men **(b)** Women

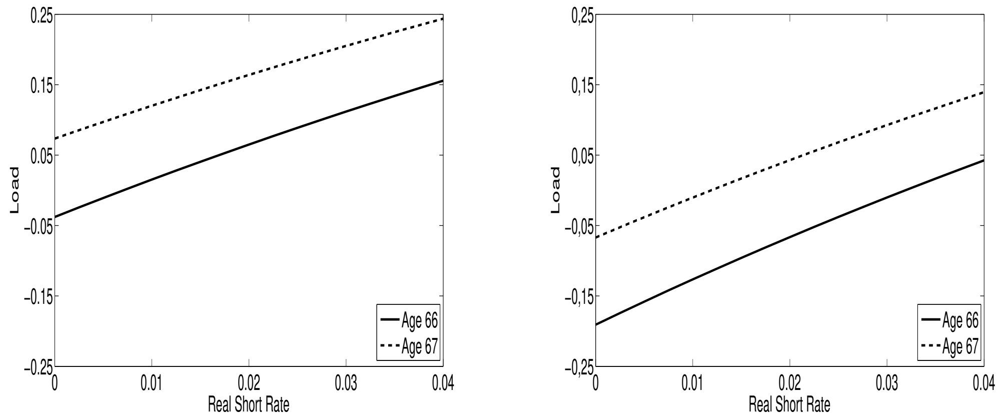
Figure 2.4 – The insurer's revenue ($\pi(x)$, horizontal lines) and expenses ($P_{Call}(x) + P_{Bonds}(x,x)$, downward sloping lines) in the year in which the annuity option is sold, as a function of the real short rate at that time. The solid (dashed) lines correspond to an individual who buys the annuity option at age $x = 66$ ($x = 67$). The accrual a offered by the Social Security system is set at 8%, and the profit load l equals 7.3%. The left (right) panel corresponds to men (women). The survival probabilities are those of U.S. males (females) for the period $2000 - 2004$. The term structure of real interest rates corresponding to a specific real short rate is generated with a one-factor Vasicek model, with parameters given in Table 2.5 in Appendix 2.B.

sold (for details, see Table 2.4). The figure considers two cases. The solid lines correspond to an annuity option sold to an individual aged 66, for a periodic premium of 1 (i.e., the Social Security benefit level if benefits are claimed at age 66). The dashed lines correspond to an annuity option sold to an individual aged 67, for a periodic premium of 1.08 (the Social Security benefit level if benefits are claimed at age 67).

First consider men who buy the annuity option at age $x = 66$ (left panel, solid lines). The figure shows that there exists a critical value of the real short rate at which the insurer's expenses in the first year are equal to the revenue (the premium received from the insured). When the real short rate is above the critical value of 2.25%, the portfolio of call options and bonds becomes less expensive, i.e., the expenses decrease. This implies that the insurer can profitably offer a super-replicating annuity option. Men aged 66 who wish to defer benefit claiming until an unspecified age are then better off by claiming benefits immediately and using them to buy that annuity option. Indeed, that strategy yields a higher benefit level, regardless of when they will decide to annuitize. When the annuity option is bought at age of 67 instead of age 66 (left panel, dashed lines), the conditions for dominance are even more likely to be fulfilled. The reason is that the maximum premium the insurer can ask (the benefit level in case Social Security benefits are claimed at age 67) increases from 1 to 1.08, and the minimum benefit level that he needs to offer when the individual annuitizes at age y (the accrual offered by the Social Security benefits when benefit claiming is deferred to age y) decreases from $(y-66)a$ to $(y-67)a$. Therefore, the insurer's revenue increases (the horizontal line shifts upwards), and the expenses decrease (the downward sloping line shifts downwards). For men aged 67 (left panel, dashed lines), a positive real short rate is enough for them to prefer buying an annuity option above deferring benefit claiming. For women (right panels), dominating strategies are less likely to exist. Because they have higher life expectancy, the option offered by the Social Security provider is less unfair for them. For women aged 66, the real short rate would need to be well above 4%, which is unlikely to be the case. For women aged 67, insurers can offer annuity options that they prefer above deferring benefit claiming if the real short rate is above 2.75%.

The above results correspond to a premium load of 7.3%. In order to investigate the sensitivity of our results with respect to the level of the premium load, we determine the maximum value of the premium load under which an insurer who follows the selffinancing strategy described in Subsection 2.4.2 can profitably offer the replicating annuity option. Specifically, we determine the load such that the insurer's expenses in the first year equal the premium received from the insured in that year, i.e., $P_{Calls}(x) + P_{Bonds}(z,x) = \pi(x)$. Whenever the load charged by insurers is strictly lower than this maximum load, the market can offer super-replicating annuity options that individuals strictly prefer above deferring Social Security benefit claiming.

Figure 2.5 displays the maximum feasible load as a function of the real short rate. The solid (dashed) lines correspond to a super-replicating annuity option sold to an individual aged 66 (67). The left panel corresponds to men; the right panel corresponds to women. Because higher values of the real short rate make annuities less expensive, the maximum feasible load is increasing in the real short rate. There is more room for insurers to offer annuity products that individuals prefer above deferring benefit claiming when interest rates are high. For men who buy the annuity option at age 66, the feasible load is above 7% whenever the short rate is at least 2%. For women, the maximum load is negative for most realistic values of the real short rate, indicating that dominating strategies are not likely to exist. However, when the product is bought at age 67, the maximum feasible load increases significantly for both men and women.



(a) Men **(b)** Women

Figure 2.5 – The maximum load $l_{\max}$ (in percentages) under which insurers can offer a super-replicating annuity option to men (left panel) and to women (right panel) aged 66 (solid lines) and aged 67 (dashed lines), as a function of the real short rate. The accrual a is set at 8%. The survival probabilities are those of U.S. males (females) for the period $2000 - 2004$. The term structure of real interest rates corresponding to a specific real short rate is generated with a one-factor Vasicek model, with parameters given in Table 2.5 in Appendix 2.B.

The above results were determined for the case where the insurer uses a conservative financing strategy in which all interest rate risk is eliminated. Insurers, however, may be willing to take some risk, which implies that there may be more room to offer super-replicating annuity options. To conclude this section, we therefore quantify the potential gains for insurers from offering super-replicating annuity options under the two financing strategies described in Subsection 2.4.2: eliminating all interest rate risk by buying a portfolio of call options, and accepting some interest rate risk. In both cases, the financing strategy is such that upon annuitization, the insurer holds a portfolio of bonds with a market price equal to the market price of the annuity. Therefore, the insurer's profit consists of profit made in all years prior to annuitization.

As an illustration, we consider a super-replicating annuity option sold to an individual aged 66 for a periodic premium equal to 1 (the Social Security benefits claimed at age 66), with benefit levels given by:

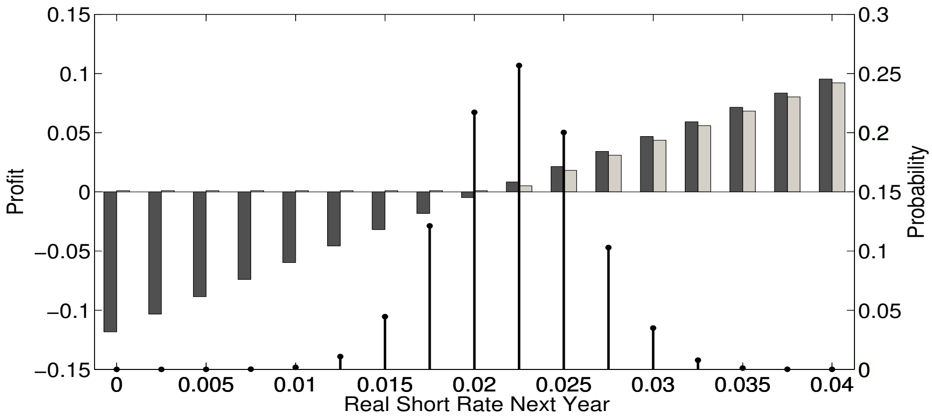
$$\begin{aligned} b_{y,x} &= 0.08, && \text{for } y = 67, \\ &= 0.08 + (y-67)0.09, && \text{for } y = 68, \cdots, 70. \end{aligned} \tag{2.11}$$

Thus, the benefit level received upon annuitization is strictly higher than the accrual offered by the Social Security system as soon as annuitization is delayed until at least age 68. We determine the probability distribution of the present value of the insurer's profits in all years prior to annuitization, in case the individual exercises the option to annuitize at age 68. The profit in the first year depends on the short rate at the time the annuity option is sold (i.e., when the individual turns 66); the profit made in the year in which the insured turns 67 depends on the short rate next year (see Table 2.4 for details on these profits). The former is known when the contract is offered, but the latter is stochastic.

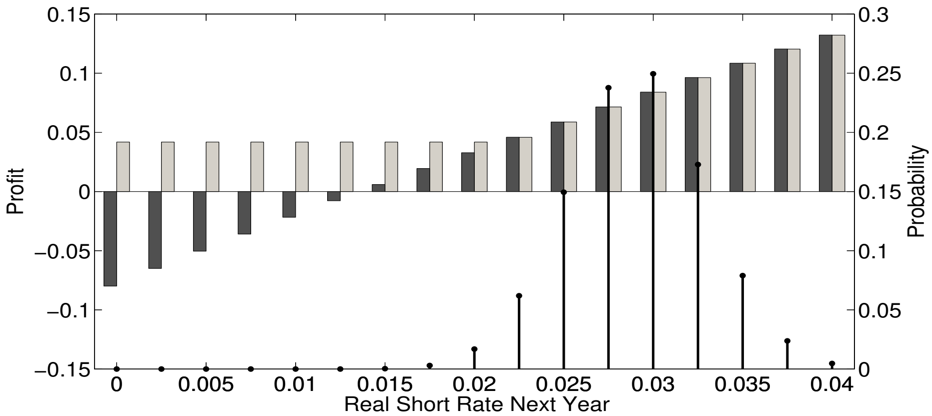
Figure 2.6 displays the probability distribution of the insurer's profit for two values of the short rate at the time the contract is sold. The upper (lower) panel corresponds to the case where the real short rate at the time the contract is sold equals 2.25% (3%). In each case, the figure displays the present value of the insurer's profit as a function of the real short rate next year (bars), as well as the probability that the real short rate next year falls into the corresponding bracket (stems). It considers two financing strategies: buying call options (light grey bars) and not buying call options (dark grey bars). The premium load is set equal to 7.3%. Profit values are displayed on the left y-axis, probability values are displayed on the right y-axis.

The figure shows that for both financing strategies and for both values of the short rate at the time the contract is sold, the insurer's profit is (weakly) increasing in the real short rate next year. This occurs because a higher short rate next year makes the bond portfolio that needs to be bought at age 67 less expensive. Comparing the upper and the lower panel shows that profits are also increasing in the current real short rate. A higher real short rate at the time the contract is bought (lower panel) makes the portfolio that needs to be bought in the first year less expensive, and, in addition, makes it more likely that the short rate in the second year is also higher, so that the bond portfolio that needs to be bought at age 67 also becomes less expensive. When the short rate at the time the contract is sold equals 3%, the insurer's profit is very likely positive even when interest rate risk is not hedged.

We now discuss the effect of the financing strategy. When call options are bought, the first year profit is strictly lower, but the second year profit is weakly higher because the bond portfolio that needs to be bought at age 67 can then be bought at the minimum of the market price and the strike price of the call option. Because the market price of the bond portfolio is decreasing in the short rate, there exists a critical value of the short rate in the second year such that the present value of profits with call options is lower (higher) when the short rate is below (above) the critical value. Specifically, when the



(a) real short rate of 2.25% at age 66



(b) real short rate of 3% at age 66

Figure 2.6 – The present value of the insurer's profit for a man who buys the annuity option at age 66 and exercises it at age 68. The bars represent the present value of the insurer's profit as a function of the real short rate next year, for two financing strategies: buying call options (light grey bars) and not buying call options (dark grey bars). The stems represent the probability that the real short rate next year falls into the corresponding bracket.Profit values are displayed on the left y-axis, probability values are displayed on the right y-axis. The upper (lower) panel corresponds to the case where the real short rate at age 66 equals 2.25% (3%). The benefit levels of the annuity option are as given in (2.11). The accrual offered by the Social Security system is set at 8%. The premium load is set equal to 7.3%. The survival probabilities are those of U.S. males (females) for the period 2000 − 2004. The term structure of real interest rates corresponding to a specific real short rate is generated with a one-factor Vasicek model, with parameters given in Table 2.5 in Appendix 2.B.

short rate in the second year is above 2.125%, the market price of the bond portfolio is lower than the strike price of the call option. Therefore, the profit made in the second year is the same for the two financing strategies, and so the present value of profits is lower when call options are bought. The difference (the price of the call options) is about 0.5% of the annual premium when the current short rate is 2.5% (upper panel, dark grey bars), and negligibly small in case the current short rate is 3% (lower panel). When the short rate in the second year falls below the critical level of 2.125%, the price of the bond portfolio is strictly higher than the strike price. Therefore, the second year's profit is zero in case the insurer bought call options, but strictly negative in case he did not. So, without call options the present value of profits can be negative, but the size and likelihood of such losses depend strongly on the current real short rate. When the current short rate is 2.25% (upper panel), a loss is made whenever the short rate falls below the critical level of 2.125%. In contrast, when the current short rate is 3% (lower panel), the profit made in the first year is significantly higher, and high enough to compensate for the loss made in the second year as long as the short rate is in the second year is not below 1.375%. The probability that the short rate falls below this level is negligibly small, so that the insurer almost surely makes no losses, even when interest rate risk is not hedged.

2.4.4 How much can individuals gain from buying annuity options?

The previous subsection shows that, depending on the real short rate and the premium load, insurers can make significant profits from offering a replicating annuity option. This suggests that they may also be able to offer annuity options with benefit levels that are significantly higher than those offered by the Social Security system, while still making a nonnegative profit. In this subsection we quantify the potential gains for individuals from such super-replicating annuity options.

Recall that in case of delayed Social Security benefit claiming, the accrual received for an additional year of delay equals a for every year of delay. Such a fixed accrual implies that the deferral option is more unfair for those who wish to defer for a longer period (recall that the money's worth is decreasing in the length of the deferral period, see Figure 2.2). This occurs because the expected number of years over which the additional benefit payment should be made decreases when benefit claiming is delayed further. Consequently, insurers might be able to offer annuity options in which the accrual received for an additional year of delay increases each year. To illustrate the potential gains for insureds, we consider the case where the insurer offers an annuity option with the following conditions:

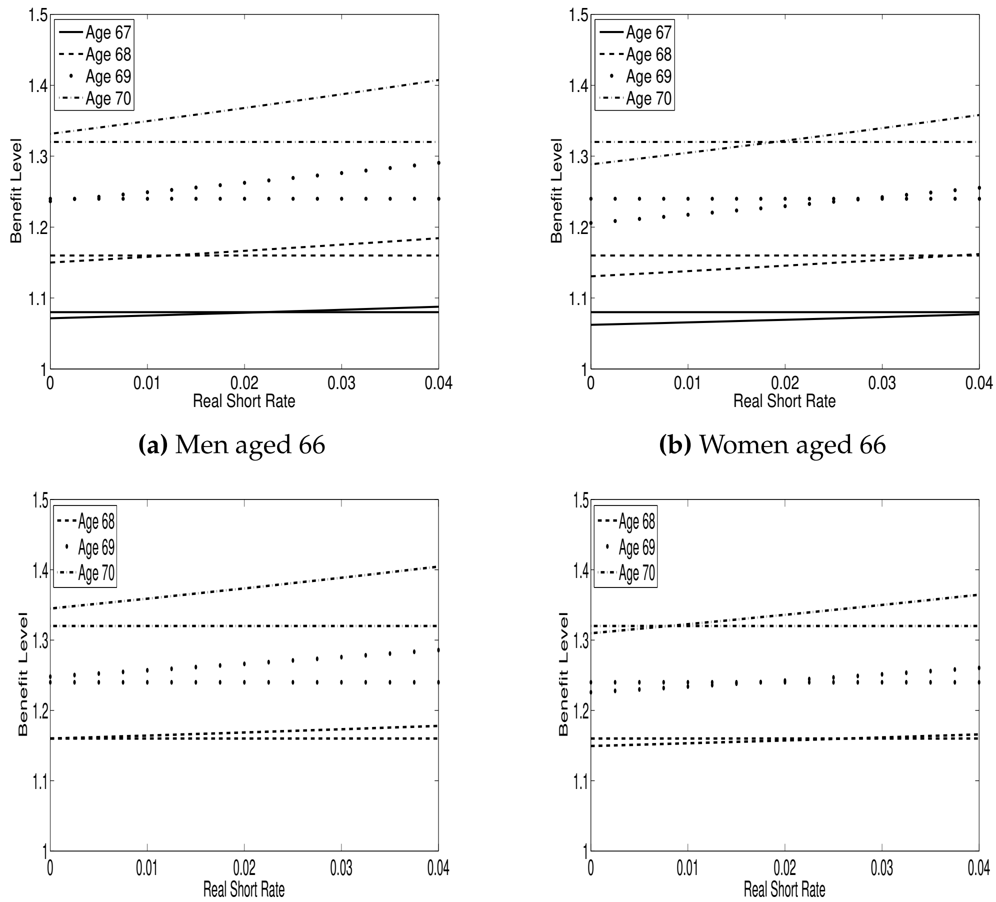
$$\pi(x) = 1 + (x - \underline{x})a, \tag{2.12}$$

$$b_{y,x} = \sum_{\tau=0}^{y-x-1} (1+c)^{\tau} b, \tag{2.13}$$

for some $b, c \in (0, 1]$. Thus, the annual premium for the annuity option is equal to the Social Security benefits received in case they are claimed at age x, and the accrual received for an additional year of delay increases by c% each year. Consider, for example, an individual aged $x = \underline{x} = 66$ who would like to defer the receipt of pension benefits. If he claims Social Security benefits immediately and uses them to buy the annuity option, he will receive an annual benefit level as of age 69 of $1 + \left[1 + (1+c) + (1+c)^2\right] b$. In contrast, if he delays Social Security benefit claiming until age 69, he receives $1 + 3a$.

Figure 2.7 displays the total benefit levels received as of age $y = 67, \cdots, 70$, as a function of the real short rate, for two strategies: the case where the individual claims benefits at age x and uses them to buy the annuity option (upward sloping lines), and the case where he defers benefit claiming until age y (horizontal lines). The upper (lower) panels correspond to $x = 66$ ($x = 67$). The benefit levels offered in the annuity option are as defined in (2.13). To illustrate the potential gains for individuals, we choose $c = 10\%$, and let b be the level that insurance companies can offer in a competitive market in which excess profits are driven to zero (i.e., condition (2.8) is satisfied in equality). The accrual offered in the Social Security system is set at $a = 8\%$ and the premium load is set at $l = 7.3\%$.

The figure shows that insurance companies are able to offer an attractive alternative to the option to defer pension benefit claiming as offered by the Social Security provider when the real short rate is sufficiently high. Strict dominance occurs when for every given annuitization age y, the benefit level received in case the annuity option is bought is higher than when benefit claiming is deferred (i.e., the upward sloping line is above the horizontal line for all annuity ages y). In order to have strict dominance a real short rate of 2.25% is needed for men and of 4% for women. However, some individuals may know for sure that they do not wish to annuitize before a certain age. In such cases, insurers are able to offer attractive annuity options even when the short rate is lower. Suppose, for example, that an individual with age 66 knows that he would like to defer annuitization until at least age 68. Then, dominating annuity options exist already when the real short rate is above 1% for men and 3.5% for women. When the individual knows he would like



(a) Men aged 66 **(b)** Women aged 66

(c) Men aged 67 **(d)** Women aged 67

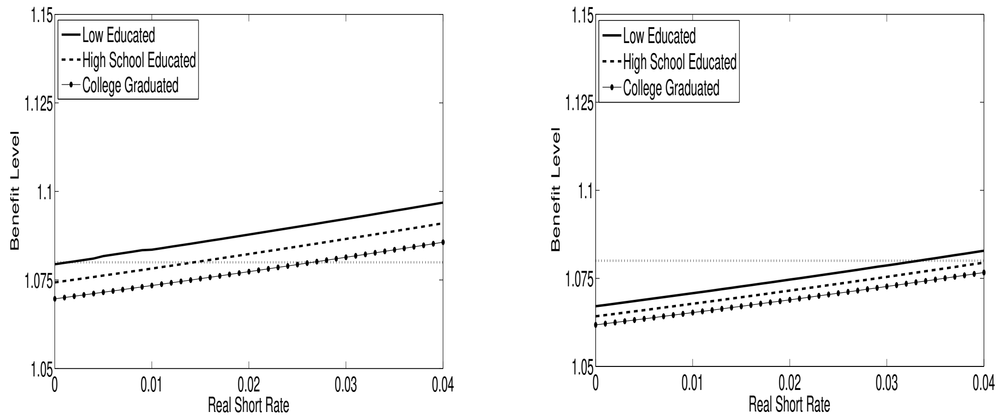
Figure 2.7 – The aggregate benefit level received as of age y, as a function of the real short rate at age x, when Social Security benefits are claimed at age x and used to buy the annuity option ($B_{y,x}$, upward sloping lines), and when claiming Social Security benefits is deferred to age y (horizontal lines). The upper (lower) panel corresponds to $x = 66$ ($x = 67$). The left (right) panel corresponds to men (women). The annual accrual a equals 8% and the load l equals 7.3%. The benefit levels of the annuity option are as defined in (2.13) with $c = 0.1$. The survival probabilities are those of U.S. males (females) for the period $2000 - 2004$. The term structure of real interest rates corresponding to a specific real short rate is generated with a one-factor Vasicek model, with parameters given in Table 2.5 in Appendix 2.B.

to defer until at least 69, the critical values of the real short rate decrease to 0.25% for men, and 2.75% for women. There is even more room for insurers to offer attractive annuity options when the option is bought at age 67 (lower panels). An individual can, for example, defer social security benefit claiming until age 67 , and then use the claimed benefits to buy an annuity option. In this case, insurers can offer a product that dominates further delay of pension benefit claiming irrespective of the real short rate for men. The reason is that a higher annual premium is paid (1.08 instead of 1) and that the minimum required benefit level upon annuitization (the accrual offered by the Social Security benefits when benefit claiming is further deferred to age y) decreases.

2.5 Dominating Strategies using differentiated survival probabilities

In the previous sections we characterized conditions under which insurers can offer super-replicating annuity products, taking into account that they can differentiate premium and benefit levels on the basis of gender. There is strong empirical evidence, however, that mortality probabilities also depend substantially on individual characteristics such as, for example, educational level. This heterogeneity leads to actuarial nonequivalence at the individual level (see, e.g., Brown, 2003; Desmet and Jousten, 2003). In contrast to Social Security providers, insurers may, at least to some extent, be able to differentiate premiums on factors that affect survival probabilities. If this is the case, there is more room to offer super-replicating annuity products for those individuals for which the accruals offered in the Social Security system are more unfair. To illustrate the potential effects, we characterize conditions under which insurers are able to offer the super-replicating annuity option defined in (2.13) to groups of individuals who differ in educational level. Three educational levels are distinguished: less than high school, high school plus up to three years of college, and college graduates. We use relative mortality factors differentiated to age, gender, and educational level determined by Brown (2003) to calculate the differentiated survival probabilities (see Appendix 2.D). As in the previous section we consider the case where excess profits are driven to zero, i.e., the benefit level b is such that the insurer's profit in the first year is zero.

Figure 2.8 displays the benefit level that an individual aged 66 can obtain as of age 67, as a function of the real short rate at age 66 , and for two strategies: claiming benefits immediately and using them to buy the annuity option (upward sloping lines), and deferring benefit claiming until age 67 (horizontal lines). It distinguishes three educational levels: low education (solid lines), high school education (dashed lines), college graduate (dashed-dotted lines). The figure shows that the critical level of the real short rate above which insurers can offer super-replicating annuity options is increasing in the educational



(a) Men **(b)** Women

Figure 2.8 – The benefit level $(B_{67,66})$ as a function of the real short rate for different groups who buy an option to annuitize at age 66 and annuitize at age 67. The horizontal line denotes the benefit level when benefits are claimed at age 67. A factor c of 10% and a load l of 7.3% were assumed. The survival probabilities are those of U.S. males (females) for the period $2000 - 2004$. The term structure of real interest rates corresponding to a specific real short rate is generated with a one-factor Vasicek model, with parameters given in Table 2.5 in Appendix 2.B.

level. Because individuals with lower educational levels have lower life expectancy, they expect to receive the additional benefits offered by the Social Security system for a shorter period of time, which implies that the system is more unfair for them. The differences for men are large. For men with low education, the critical short rate is 0.2%. For college graduates, it increases to 2.7%.

2.6 Conclusions

In many countries accruals to annual pension benefits are offered to those who claim benefits later. Typically, these accruals are fixed for a number of years, and are independent of both interest rates and individual characteristics such as gender. In addition, the accrual received for an additional year of delay is typically a fixed percentage of the benefit level in case benefits are claimed at the full retirement age. The actuarially fair value of the additional deferred annuity that the individual receives in case he delays benefit claiming, however, depends nontrivially on the length of the deferral period, the term structure of real interest rates, and individual characteristics that affect survival probabilities. As a consequence, public pension systems with fixed accruals are not actuarially fair, and the degree of unfairness varies over time (as it depends on the term structure of real interest rates). Moreover, the degree of unfairness depends on individual characteristics.

We show that the actuarial unfairness implies that individuals who wish to defer the receipt of pension benefits may be better off by claiming benefits and using them to buy annuity products at the market. Conditions under which it is optimal for them to do so are investigated in a preference-free setting assuming only that more is preferred to less. We first quantify the degree of unfairness in the public pension system on the basis of the market term structure of real interest rates, generated by a Vasicek term structure model. We then characterize conditions under which insurers can offer attractive deferral options without taking any interest rate risk. Our results suggest that there is a broad range of settings (for market conditions, required premium loads, and individual characteristics) in which insurers can profitably offer deferral options that are more actuarially fair than those offered by the public pension provider. Individuals can exploit these options by claiming benefits early, and using them to buy annuity products from insurers. The potential gains for individuals and insurers increase when market conditions are more favorable (e.g., when interest rates are relatively high), and when insurers have more flexibility to differentiate premium and benefit levels on the basis of individual characteristics. If individuals choose to strategically exploit outside options offered by insurance companies, this will affect benefit claiming behavior, which in turn affects long run program costs.

Acknowledgments

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2.A Survival Probabilities

Throughout the paper, we use the one-year mortality probabilities differentiated to age and gender reported by the Human Mortality Database for U.S.

males (females) for the year 2000 up to and including 2004.[16] Let q_x denote the probability that an individual with age x dies within one year. The probability that an individual is alive over τ years conditional on being alive at age x is given by:

$$_{\tau}p_x = \prod_{v=1}^{\tau}(1 - q_{x+v-1})$$

Figure 2.9 displays the cumulative survival probabilities, conditional on being alive at age 66 , i.e., $_{\tau}p_{66}$, as a function of τ.

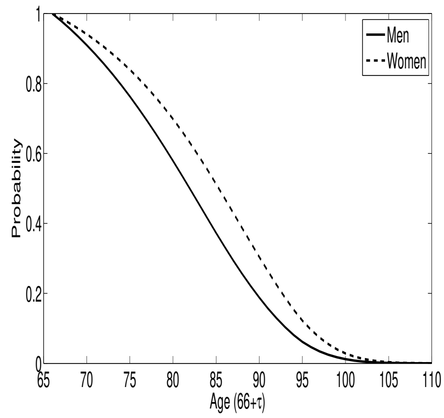


Figure 2.9 – The cumulative survival probabilities ($_{\tau}p_{66}$), as a function of age ($66 + \tau$) for men (solid line) and women (dashed line) respectively with age 66.

2.B The One-factor Vasicek Model

The Vasicek model assumes that the instantaneous real short rate at time t, r_t, is generated by:

$$dr_t = \kappa[\theta - r_t]dt + \sigma dW_t, \;\; r(0) = r_0,$$

where W_t is a Wiener process, θ denotes the long-run mean, κ the parameter of mean reversion, and σ the volatility.

The time-t price of a zero-coupon bond which matures at time T, denoted by $P(r_t, t, T)$, is given by:

$$P(r_t, t, T) = \exp\{A(t, T)\} \; \exp\{-B(t, T)r_t\},$$

[16] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 05-01-2009).

with

$$B(t,T) = \frac{1 - \exp\{-\kappa(T-t)\}}{\kappa}, \tag{2.14}$$

$$A(t,T) = [B(t,T) - (T-t)] \left(\frac{\kappa(\kappa\theta + \lambda\sigma) - \sigma^2/2}{\kappa^2} \right) - \frac{\sigma^2}{4\kappa} B(t,T)^2, \tag{2.15}$$

and where λ denotes the market price of risk. Then, the time-t real interest rate for a maturity of $T-t$ years given that the short rate at time t equals r_t, is given by:

$$R(r_t, t, T) = \frac{-\log P(r_t, t, T)}{T-t}.$$

Throughout the paper we use the parameter values displayed in Table 2.5.

Vasicek model	
κ	0.1
θ	0.02
σ	0.004
λ	0.5

Table 2.5 – The parameter values of the Vasicek model for interest rate.

The long-term average θ is set equal to 2%. Moreover, the market price of risk λ is set equal to 0.5. This reflects a setting in which the real interest rate for a maturity of six years is 0.5% higher than the short rate. The benchmark case displayed in Figure 2.1, solid line, corresponds to the case where the real short rate equals the long-term average θ.

2.C Pricing call options on bond portfolios

In this subsection we determine the price $P_{Calls}(x)$ of the portfolio of call options that the insurer buys in order to eliminate interest rate risk. Jamshidian (1989) has derived an exact formula to price options on (coupon-bearing) bonds, assuming that interest rates are generated by a one-factor Vasicek model. The pricing problem is further addressed in Hull (2003) and Brigo and Mercurio (2001). Let us denote $P(r,t,s)$ for the date-t price of a zero-coupon bond with maturity date s, given that the real short rate at time t equals r. The date-0 price of a call option with strike price K and maturity date t, on a zero-coupon bond with maturity s and principal L, is given by:

$$C(s,t,K,L) = LP(r_0, 0, s)\Phi(h) - KP(r_0, 0, t)\Phi(h - \sigma_P),$$

where $\Phi()$ denotes the standard normal cumulative distribution function, r_0 denotes the real short rate at time 0, and h and σ_P are respectively given by:

$$h = \frac{1}{\sigma_P} \ln\{\frac{LP(r_0, 0, s)}{P(r_0, 0, t)K}\} + \frac{\sigma_P}{2}, \tag{2.16}$$

$$\sigma_P = \frac{\sigma}{\kappa}(1 - \exp(-\kappa(s-t)))\sqrt{\frac{1-\exp(-2\kappa t)}{2\kappa}}, \tag{2.17}$$

respectively, where κ denotes the parameter of mean-reversion and σ denotes the volatility of real short rate.

Recall that, for each age $z = x+1, \cdots, \overline{x}-1$, the insurer needs to buy a call option with strike price $K(z,x)$ given by (2.7), on a portfolio of zero-coupon bonds with maturity dates $s = z-x+1, ..., 110-x$, and with corresponding principals $L_{z,x,s} = \left(\frac{\widetilde{b}_{z,x}}{1-l}\right) {}_s p_x$. The price of this call option is equal to the price of a *portfolio* of call options, one for each individual zero-coupon bond, where the strike prices $K(z,x,s)$ of the individual call options are such that $\sum K(z,x,s) = K(z,x)$, and they all have the same exercise region, i.e.,

$$\begin{aligned} & K(z,x,s) = L_{z,x,s} P(r^*, z-x, s) \\ \text{with } r^* \text{ such that: } & \sum_{s=z-x+1}^{110-x} L_{z,x,s} P(r^*, z-x, s) = K(z,x). \end{aligned} \tag{2.18}$$

Given that a call option is needed for every age $z = x+1, \cdots, \overline{x}-1$, the price of the portfolio of call options equals:

$$P_{Calls}(x) = \sum_{z=x+1}^{\overline{x}-1} \sum_{s=z-x+1}^{110-x} C(s, z-x, K(z,x,s), L_{z,x,s}), \tag{2.19}$$

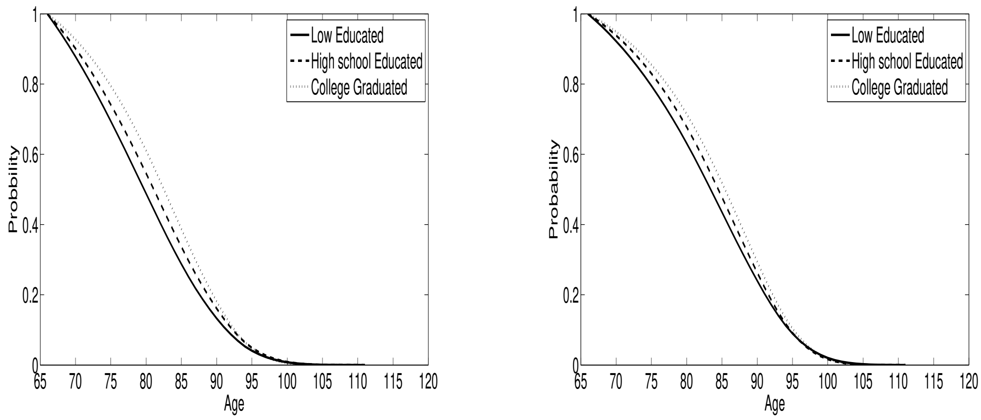
Now, the price of the portfolio of call options follows from (2.19), with $L_{z,x,s} = \left(\frac{\widetilde{b}_{z,x}}{1-l}\right) {}_s p_x$, and with $K(z,x,s)$ determined by (2.7) and (2.18).

2.D Differentiated survival probabilities

In this Appendix we discuss how we determine survival probabilities differentiated by age, gender, and educational level using the relative mortality factors from Brown (2003). He constructs age-specific relative mortality factors for black, white, and Hispanic men and woman, where the white and black groups are then further differentiated on the basis of education. Three educational levels are distinguished for whites, namely: less than high school, high school plus up to three years of college, and college graduates. To obtain survival probabilities differentiated by educational level, we multiply the relative mortality factors for white men and women with different educational level with the mortality probabilities from the Human Mortality database as described in Appendix 3.B. Let $c_x^{(e)}$ denote the relative mortality factor of an individual with age x with educational level e. The probability that an individual with educational level e is alive over τ years conditional on being alive at age x is given by:

$$_{\tau}p_x^{(e)} = \prod_{v=1}^{\tau}(1 - q_{x+v-1}c_{x+v-1}^{(e)})$$

The differentiated cumulative survival probabilities for whites are displayed in Figure 2.10.



(a) White Men **(b)** White Women

Figure 2.10 – The cumulative survival probabilities differentiated to gender and educational level for men (left) and women (right), conditional on being alive at age 66.

CHAPTER 3

WELFARE IMPLICATIONS FOR COUPLES OF ANNUITY MARKETS INCOMPLETENESS

"Sex differences in mortality are well established facts. In the human population of developed countries where infectious diseases are not significant causes of death, the penalty for maleness is that almost every important disease has a higher mortality rate in males than in females"

Leonard Hayflick (1982)

3.1 Introduction

This chapter is based on Sanders, De Waegenaere, and Nijman (2011a). Most individuals save for retirement while working to smooth consumption over time. At retirement, the savings can be annuitized such that a periodic payment is received until death. An individual with a partner may not only want to protect himself from outliving his assets, but he may also want to protect his partner from outliving her assets. Consequently, a periodic payment must be made until both individuals have deceased. To support living after retirement, couples can typically buy a single-life annuity, a joint annuity, a survivor annuity, or a portfolio of these annuities. A single-life annuity insures the individual against outliving one's assets, i.e., a periodic payment is made to the individual as long as he or she is alive. A joint annuity makes a periodic payment as long as both spouses are alive and a survivor annuity makes a periodic payment as of the moment one of the spouses has deceased and the other spouse is still alive. In this paper, we first investigate the effect of market imperfections on couples' optimal annuity decisions, and then use these results to investigate the welfare losses they bear because of the imperfections.

Although there is an extensive literature investigating annuity demand for singles, the literature about annuity demand for couples is much smaller. Kotlikoff and Spivak (1981) analyze annuitization decisions in families. They demonstrate that implicit risk-sharing arrangements within marriage and the family can substitute to a large extent for the purchase of annuities in public markets. Brown and Poterba (2000) estimate the "annuity equivalent wealth", which determines the amount of wealth that couples would need in the absence of actuarially fair annuity markets in order to achieve the same utility level as in case they would have access to these markets. An annuity is actuarially fair when the price of the annuity equals the expected discount present value of the annuity. They find that a couple consisting of a man aged 65 and woman aged 62 who have access to an actuarially fair annuity market would require between 18 and 39 percent more wealth in case no such market is available to obtain the same utility level. Dushi and Webb (2004) find that it is optimal for married couples to delay annuitization because of the actuarial unfairness of annuities. Fitzgerald (1989) examines the wealth allocation decision of married couples between consumption for the surviving widow should the husband die, and consumption for the couple should both live. He finds that the annuitized consumption of widows of older husbands is generally higher than if their husbands had lived. He also finds that younger widows tend to be relatively worse off in terms of annuitized consumption than older widows. In Schmeiser and Post (2005) self-annuitization strategies are investigated. Self-annuitization means that the wealth is not annuitized but invested in an investment fund from which a payout stream is generated. Self-annuitization strategies create opportunities for the retiree to leave a bequest. However, the retiree bears the risk of outliving his assets. Therefore, Schmeiser and Post (2005) propose a family strategy where heirs bear both the chances of receiving a bequest and the risk that the retiree outlives his assets. Post, Gründl, and Schmeiser (2006) demonstrate that the family self-annuitization strategy is optimal for a wide range of parameter values. Vidal-Meliá and Lejárraga-García (2006) extend the model of Brown and Poterba (2000) by adding a bequest motive. Hurd (1999) investigates consumption levels of couples subject to their bequest motive.

In this paper, we investigate the effect of market imperfections on couples' optimal annuity decisions, and on the welfare losses they bear because of the imperfections. We consider three sources of market imperfections. First, couples typically only have access to annuities that yield flat benefit levels, i.e., the payment of the annuity is state dependent but not time dependent. Second, there are several reasons why the survival probabilities used by the annuity provider for pricing annuities can differ from the survival probabilities used by the couple to determine the utility they derive from a specific annuity portfolio. Such differences can occur, for example, due to information asymmetry between the annuity provider and the couple, or due to gender neutral pricing in collective pension plans. Third, many individuals have accrued pension rights in the form of a single-life annuity or a single-life with survivor annuity while working. It is common practice that they can, at retirement, exchange (part of) the single-life annuity for a survivor annuity or vice versa, but the exchange in many cases is subject to restrictions.[1] A typical restriction is that the insured can exchange (part of) his single-life annuity for a survivor annuity for his spouse, but cannot exchange part of his single-life annuity for a single-life annuity for his spouse or for a survivor annuity for himself. This implies that when only the husband has accrued pension rights, there is no reduction in the couple's income upon the spouse's death, creating an asymmetric form of protection, where primary longevity protection is for the participant and secondary protection is for the spouse.

We investigate the effect of these market imperfections on the couples' optimal annuity decisions, and on the welfare they derive from buying annuities. We do so by comparing the optimal consumption pattern given the imperfections to the optimal consumption pattern in a benchmark case with a complete annuity market with time- and state-dependent Arrow annuities (see, e.g., Davidoff, Brown, and Diamond (2005)).

First, we find that how the absence of time-dependent annuities affects the couple's optimal consumption pattern depends strongly on whether and how the survival probabilities used by the annuity provider differ from those

[1] Johnson, Ucello, and Goldwyn (2003) measure the share of married retirees with pension annuities who forgo survivor protection. They examine which factors influence the decision between a single-life annuity and a joint & survivor annuity, finding that retirees are more likely to reject survivor protection when the spouse has access to alternative sources of survivor protection, the pension wealth is limited, they expect to outlive their spouse, and the relationship with the spouse is weak. Previtero (2010) finds that those retiring after stock market increases of six to 12 months are much more likely to select the lump sum option rather than lifetime income.

used by the couple. Absent such differences, the couple prefers a flat consumption level over time, and so the fact that only annuities with flat benefit levels are available does not affect their optimal consumption pattern. In contrast, when the survival probabilities used by the annuity provider differ from those used by the couple, these differences affect the relative attractiveness of survival annuities and single-life annuities. As a consequence, the optimal consumption pattern is no longer necessarily flat over time, and so the couple may bear welfare losses in case only flat annuities are available. However, differences between the survival probabilities used by the annuity provider and by the couple can also imply that some annuities are priced relatively attractively, which can lead to welfare gains.

Second, for couples with illiquid pension wealth in the form of pre-accrued pension rights, restrictions imposed on the exchange of pension rights affect the couple's optimal consumption pattern in case one of the spouses has accrued significantly more pension rights than the other spouse. These restrictions then imply that upon the death of the spouse that has accrued less pension rights, the surviving spouse's consumption is sub-optimally high as compared to the case where all wealth is liquid. The welfare losses are strongly affected by differences in survival probabilities used by the couple and the provider of the annuities, and by whether the couple holds illiquid longevity insurance in the form of a state pension. When the couple is also entitled to receive a state pension, welfare losses increase when both spouses have accrued pension rights, but decrease when only one of the spouses has accrued pension rights.

Finally, many pension schemes offer a default option, and existing literature suggests that many couples tend to follow the default. A common default is for the spouse to receive 50% of the participant's monthly benefit upon the participant's death. Our results, however, suggest that the optimal survivor fraction is higher than 50%.[2] We quantify the welfare losses that couples bear in case they choose the default option.

The remainder of this paper is structured as follows. In Section 3.2, the optimal portfolio of annuities for a couple is determined in a benchmark case with a complete market with time-dependent Arrow annuities. In Section 3.3, we determine the optimal portfolio of annuities when only annuities with a flat benefit level over time are available. We distinguish the case where all pension wealth is liquid, and the case where the couple has pre-accrued pen-

[2]Smith (2003) finds that the survivor's living expenses are between 60% and 80% of the couple's expenses before the first death.

sion rights that cannot freely be reallocated. In Section 3.4 we determine the welfare losses the couple bears due to these market imperfections, and show how these welfare losses are affected by differences in survival probabilities used by the annuity provider and by the couple. We also determine the welfare losses a couple bears in case they choose the default option. Section 4.8 concludes.

3.2 Optimal annuity portfolio in complete markets

In this section, we consider optimal annuity decisions by couples in the benchmark case in which the couple has access to a complete market of time- and state-dependent annuities. In Subsection 3.2.1 we first discuss how we model the lifetime expected utility of couples. We analyze optimal consumption patterns in Subsection 3.2.2. Using these optimal consumption patterns, we determine the optimal survivor fraction in Subsection 3.2.3. The survivor fraction of a spouse is defined as the ratio of the consumption of that spouse in case the other spouse has deceased over the aggregate consumption of the couple in case they are both alive.

3.2.1 The couple's lifetime expected utility

We use the utility framework of Brown and Poterba (2000) to model the lifetime expected utility of couples. They have extended the utility framework of Kotlikoff and Spivak (1981) by allowing for consumption externalities between spouses, or "joint consumption". Consumption externalities arise when the purchase of a good by one spouse is utility increasing for the other spouse, or vice versa. For example, two spouses have to pay only one mortgage or only one newspaper. The joint consumption can also be interpreted as public consumption.

The couple consists of a husband (m) and wife (f). We assume that the couple does not have a bequest motive, i.e., they do not derive utility from income after decease. Let us further denote:

- $C_t(mf)$ for the joint real consumption in period t of the husband and wife when both spouses are alive in period t and $C_t(m)$ ($C_t(f)$) for the real consumption in period t of the husband (wife) when the husband (wife) is alive in period t but the wife (husband) has deceased;
- T for the maximum remaining lifetime of the longest living spouse;

- $p_t(m)$ ($p_t(f)$) for the (subjective) probability that the husband (wife) is alive in period t given that the husband (wife) is alive in the current period. The couple may not know their actual survival probabilities but they can maximize their lifetime expected utility using their subjective survival probabilities.

Following Brown and Poterba (2000), we assume that the couple has intertemporal separable utility, and that husband and wife have the same CRRA utility function. We choose a CRRA utility function because we can normalize the wealth level without loss of generality. Specifically, for all periods $t \in \{0, \cdots, T\}$, the utility that a spouse (husband or wife) derives from consuming C in period t, when the other spouse consumes C' in that period is given by:

$$U(C, C') = \begin{cases} \frac{(C+\mu C')^{1-\gamma}}{1-\gamma}, \text{ for } \gamma > 1, \\ \log(C + \mu C'), \text{ for } \gamma = 1, \end{cases}$$

where μ is the degree of joint consumption and γ is the parameter for risk aversion. μ could also be interpreted as the economies of scale parameter. The utility of the couple in period t when one spouse consumes C and the other spouse consumes C' is the sum of the utility of the husband and wife, i.e., $U(C, C') + U(C', C)$. For computational ease, we assume that the utility function of the husband and wife are identical. For more general utility functions, we refer to Appendix 3.C.1.

To determine the lifetime expected utility of the couple, we distinguish four possible states: both spouses are alive, only the husband is alive, only the wife is alive, both spouses have deceased. The utility in case both spouses have deceased is not defined. In case both spouses are alive, they need to optimally divide the aggregate consumption $C_t(mf)$ between themselves. Because the husband and the wife have the same concave utility function, they optimally consume equal amounts when they are both alive, that is, $C = C' = \frac{1}{2}C_t(mf)$.[3] The couple's lifetime expected utility is given by:

$$L = \sum_{t=0}^{T} 2U(\frac{1}{2}C_t(mf), \frac{1}{2}C_t(mf))\frac{p_t(m)p_t(f)}{(1+\rho)^t} + \sum_{t=0}^{T} U(C_t(m), 0)\frac{p_t(m)(1-p_t(f))}{(1+\rho)^t}$$
$$+ \sum_{t=0}^{T} U(C_t(f), 0)\frac{(1-p_t(m))p_t(f)}{(1+\rho)^t}, \tag{3.1}$$

where ρ is the rate of time preference. The lifetime expected utility of the couple is the discounted value of the sum over time of the couple's utility when

[3]Maximizing $U(C, C') + U(C', C)$ subject to $C + C' = C_t(mf)$ yields $C = C' = C_t(mf)/2$.

	all households	renters	wife aged 60 or older
mean	0.69	0.69	0.66
5% quantile	0.50	0.48	0.49
95% quantile	0.85	0.85	0.77

Table 3.1 – The mean degree of joint consumption and the 5% and 95% quantiles for all households, households who rent, and households where the wife is aged 60 or older.

both spouses are alive multiplied with the probability that both spouses are alive in that period, the husband's utility when only the husband is alive multiplied with the probability that only the husband is alive in that period, and the wife's utility when only the wife is alive multiplied with the probability that only the wife is alive in that period. Throughout the paper we assume that the couple's rate of time preference equals the interest rate.

In Subsections 3.2.2 and 3.2.3 we analyze the couple's optimal consumption pattern in case they have access to a complete market of time- and state-dependent annuities. In the remainder of this section, we first discuss how the parameter values used in the numerical examples are calibrated. All numerical results are displayed for a couple consisting of a husband and wife aged 65 and 62 respectively at the moment annuities are purchased. Following Brown and Poterba (2000), we set the degree of risk aversion equal to 2 or 5. The degree of joint consumption (μ) is calibrated with data from the Longitudinal Internet Studies for Social Sciences (LISS) panel, which is a large socio-economic survey representative of the U.S. population of ages 16 and older.[4] Some 5000 households (comprising 8000 individuals) received a questionnaire about, among others, intrahousehold consumption. See van Soest, Vermeulen, and Schollier (2010) for a more detailed description of the data-set. Table 3.1 displays summary statistics of the degree of joint consumption of all households in the data-set. Detailed information about the data-set and how the degree of joint consumption is determined can be found in Appendix 3.A.

We find that the mean degree of joint consumption equals 70%. However, there is a wide variation in the degree of joint consumption between couples. The 5% quantile is 50% whereas the 95% quantile is 85%. Among couples were the wife is at least 60 years old, the degree of joint consumption is lower. One reason is that couples who own a house may already have paid off their

[4]The data are freely available for non-commercial research purposes. See www.liissdata.nl for more information.

mortgages, decreasing the public consumption. In Brown (2001) the degree of joint consumption is determined based on the work done on 'equivalence scales' in household consumption by Citro and Michael (1995). Brown (2001) finds that the degree of joint consumption equals 62.45%, which is slightly below the values we find. In our numerical result, we set the degree of joint consumption (μ) equal to either 50% or 70%.

3.2.2 Optimal consumption patterns

In this subsection, we analyze optimal consumption patterns in case the couple has access to a complete market of "Arrow annuities" (see, e.g., Davidoff, Brown, and Diamond, 2005). Whereas standard annuities yield payments that are state-dependent but not time-dependent (i.e., the payment depends only on whether or not the insured and/or his spouse are alive), the payment of an Arrow annuity can be both time- and state-dependent.

We use the framework of Davidoff, Brown, and Diamond (2005) for a specific utility function. In contrast with Davidoff, Brown, and Diamond (2005) we take into account couples instead of singles. A *single-life Arrow annuity* makes a periodic payment to an individual in a given period if the individual is alive in that period. Because our focus is on annuity decisions by couples, we consider in addition *joint Arrow annuities* that make a periodic payment to a couple in a given future period if both spouses are alive in that period, and *survivor Arrow annuities* that make a payment in a given future period to one of the spouses if that spouse is alive, and the other spouse has deceased.

We allow for differences between the survival probabilities used by the annuity provider for pricing the annuities and the survival probabilities the couple uses to maximize their lifetime expected utility. There are several reasons why the survival probabilities used by the annuity provider may differ from the survival probabilities used by the couple. First, there may be information asymmetry between the couple and the annuity provider regarding factors that affect the couple's mortality probabilities such as, for instance, their health status.[5] Second, the annuity provider may not be allowed to differentiate premium based on characteristics such as ethnicity, educational level, or even gender, although survival probabilities do depend on these characteristics. This is particularly relevant for individuals who have accrued pension

[5]Salm (2010) finds that consumption and saving choices vary with surjective mortality probabilities in a way that is consistent with a life cycle model of consumption and savings behavior.

rights in collective pension plans. As argued above, upon retirement the participant is often allowed to exchange rights for one type of annuity for rights for another type of annuity, under the restriction that the actuarial value of the pension rights before and after exchange is the same. To determine the actuarial value, however, pension funds are not allowed to let the survival probabilities depend on characteristics other than age.

Let $p_t^*(m)$ ($p_t^*(f)$) denote the probability that the husband (wife) is alive in period t given that the husband (wife) is alive in the current period, used by the annuity provider to price annuities. The price of a joint Arrow annuity which pays one unit of consumption in period t when both spouses are alive in that period is then given by:

$$P_{joint}^*(t) = (1+l)\frac{p_t^*(m)p_t^*(f)}{(1+r)^t}, \tag{3.2}$$

where l represents the expense loading and r denotes the discount rate, i.e., the price of the annuity is the discounted value of the probability that both spouses are alive in period t, increased with the loading. The price of a survivor Arrow annuity that pays off one unit of consumption in period t if the spouse with gender g is alive and the spouse with gender g' has deceased is given by:

$$P_{surv,g}^*(t) = (1+l)\frac{p_t^*(g)(1-p_t^*(g'))}{(1+r)^t}, \text{ for } g, g' \in \{m, f\}. \tag{3.3}$$

We refer to a survivor annuity that pays off when the husband is alive and the wife has deceased as the husband's survivor annuity and a survivor annuity that pays off when the wife is alive and the husband has deceased as the wife's survivor annuity. Moreover, we denote $P_{joint}(t)$ and $P_{surv,g}(t)$ for the corresponding annuity prices in case the annuity provider would use the couple's survival probabilities and rate of time preference to price annuities, i.e.,

$$P_{joint}(t) = (1+l)\frac{p_t(m)p_t(f)}{(1+\rho)^t}, \tag{3.4}$$

$$P_{surv,g}(t) = (1+l)\frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t}, \text{ for } g, g' \in \{m, f\}. \tag{3.5}$$

Following Davidoff, Brown, and Diamond (2005), we assume that couples cannot invest in risky assets, and that the mortality credit is sufficiently large such that buying annuities dominates saving against the risk-free rate r. This implies that it is optimal for the couple to annuitize all wealth.[6] The budget constraint is then given by:

$$W = (1+l)\sum_{t=0}^{T}\left(C_t(mf)P^*_{joint}(t) + C_t(m)P^*_{surv,m}(t) + C_t(f)P^*_{surv,f}(t)\right), \quad (3.6)$$

where W equals the wealth of the couple at current time 0, $C_t(mf)$ is the benefit level of the joint annuity at time t, $C_t(m)$ is the benefit level of the survivor annuity for the husband at time t, and $C_t(f)$ is the benefit level of the survivor annuity for the wife at time t. Note that the optimal fraction of wealth invested in the different Arrow annuities does not depend on the initial wealth level W because of the CRRA utility function. Without loss of generality, we can therefore normalize W. In all our numerical examples, the total wealth of the couple at the moment they annuitize is set at $W = 100$.

We maximize the couple's lifetime expected utility in (3.1) subject to the budget constraint in (3.6) and subject to the constraints that consumption should be non-negative in all states. This yields that, for any $\gamma \neq 1$, the optimal consumption in year t in case both spouses are alive is given by

$$C_t(mf) = 2\,(1+\mu)^{\frac{1}{\gamma}-1}\,\lambda^{-\frac{1}{\gamma}}\left(\frac{P_{joint}(t)}{P^*_{joint}(t)}\right)^{\frac{1}{\gamma}}, \quad (3.7)$$

and the optimal consumption in year t if the spouse with gender g is alive and the spouse with gender g' has deceased, for $g \in \{m, f\}$, is given by

$$C_t(g) = \lambda^{-\frac{1}{\gamma}}\left(\frac{P_{surv,g}(t)}{P^*_{surv,g}(t)}\right)^{\frac{1}{\gamma}}, \quad (3.8)$$

where λ is the Lagrange multiplier. λ is chosen such that (3.6) is satisfied.[7] The results for more general utility functions is displayed in Appendix 3.C.1.

[6] Annuitizing wealth is preferred to saving against the risk-free rate r when the following inequality holds:

$$(1+l)\frac{1-(1-p^*_t(m))(1-p^*_t(f))}{(1+r)^t} \leq \frac{1}{(1+r)^t}.$$

The left hand side of the inequality equals the price of one consumption unit in period t by buying "Arrow annuities" for each state for which at least one spouse is alive, and the right hand side equals the price of one consumption unit in period t *in each state* by saving. The "Arrow annuities" are more attractive when it is less expensive to finance future consumption by buying these annuities than by saving, i.e., when the mortality credit is sufficiently large to compensate for the profit loading l.

[7] The expression for λ and the solution for $\gamma = 1$ are displayed in Appendix 3.C.

In the remainder of this section, we investigate how the optimal consumption pattern is affected by differences in the survival probabilities used by the couple and the annuity provider, respectively. In Section 4, we will use these results to determine the effect of differences in survival probabilities on the welfare losses couples bear due to market incompleteness.

When the survival probabilities used for pricing equal the couple's (subjective) survival probabilities (i.e., when $p_t^*(m) = p_t(m)$, and $p_t^*(f) = p_t(f)$, for all t), the optimal consumption pattern is flat over time, i.e., $C_{t+1}(mf) = C_t(mf)$ and $C_{t+1}(g) = C_t(g)$, for $g, g' \in \{m, f\}$. Specifically, the optimal consumption pattern is given by:

$$C_t(mf) = 2\,(1+\mu)^{\frac{1-\gamma}{\gamma}}\,\lambda^{-\frac{1}{\gamma}}, \tag{3.9}$$

$$C_t(g) = \lambda^{-\frac{1}{\gamma}}, \text{ for } g \in \{m, f\}, \tag{3.10}$$

where λ is independent of t.

When the couple's (subjective) survival probabilities differ from the survival probabilities used for pricing annuities, the optimal consumption patterns are not necessarily flat over time anymore. We can investigate this by expressing $C_{t+1}(mf)$ and $C_{t+1}(g)$ in terms of $C_t(mf)$ and $C_t(g)$ respectively. This gives:

$$C_{t+1}(mf) = C_t(mf)\left[\frac{P_{joint}(t+1)/P^*_{joint}(t+1)}{P_{joint}(t)/P^*_{joint}(t)}\right]^{\frac{1}{\gamma}}, \tag{3.11}$$

$$C_{t+1}(g) = C_t(g)\left[\frac{P_{surv,g}(t+1)/P^*_{surv,g}(t+1)}{P_{surv,g}(t)/P^*_{surv,g}(t)}\right]^{\frac{1}{\gamma}}, \text{ for } g, g' \in \{m, f\}. \tag{3.12}$$

For any given period, an annuity is priced attractively in case the price based on the couples survival probabilities $(p_t(m), p_t(f))$ is higher than the price based on the survival probabilities used by the insurer $(p_t^*(m), p_t^*(f))$, i.e., when $P_{joint}(t) > P^*_{joint}(t)$ for a joint annuity and $P_{surv,g}(t) > P^*_{surv,g}(t)$ for a survivor annuity. It follows immediately from (3.11) and (3.12) that consumption is increasing from period t to $t+1$ in the state where both spouses are alive when $P_{joint}(t+1)/P^*_{joint}(t+1) > P_{joint}(t)/P^*_{joint}(t)$, i.e., when the joint annuity is priced relatively attractively in period $t+1$ compared with period t. Similarly, consumption is increasing from period t to $t+1$ in the state where only the spouse with gender g is alive when $P_{surv,g}(t+1)/P^*_{surv,g}(t+1) > P_{surv,g}(t)/P^*_{surv,g}(t)$.

We illustrate the effect of differences in survival probabilities on the couple's optimal consumption pattern in Figure 3.1. In the beginning of this section, several reasons are given for differences between the survival probabilities used by the annuity provider for pricing annuities and those used by the couple to evaluate their lifetime expected utility. In our numerical examples, we focus on the case where the annuity provider uses gender-neutral survival probabilities. In many countries, regulators impose that survival probabilities used for pricing annuities are gender-neutral.[8] Moreover, it is common that the gender-neutral survival probabilities in case of collective pension plans depend on the distribution of men and women in the pension fund. We therefore consider the case where the gender-neutral survival probabilities used for pricing annuities are given by:

$$p_t^*(m) = \delta \cdot p_{m,m,t} + (1-\delta) \cdot p_{f,m,t}, \quad \text{for all } t; \tag{3.13}$$

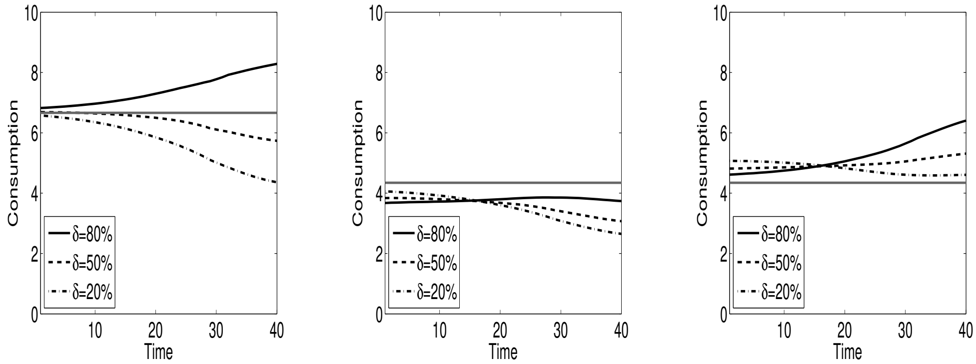
$$p_t^*(f) = \delta \cdot p_{m,f,t} + (1-\delta) \cdot p_{f,f,t}, \quad \text{for all } t, \tag{3.14}$$

where $\delta \in [0,1]$ reflects the gender composition of the fund, and $p_{g,\overline{g},t}$ for $g, \overline{g} \in \{m, f\}$ denotes the t-year survival probability of an individual with gender g with the same age as the insured with gender $\overline{g}$. Note that $p_t^*(m)$ differs from $p_t^*(f)$ only due to the age difference between man and wife. In all our numerical examples, we consider the case where the survival probabilities $p_{g,\overline{g},t}$ used by the annuity provider to determine the gender-neutral survival probabilities in (3.13) and (3.14), as well as the survival probabilities $(p_t(m), p_t(f))$ used by the couple to determine the expected lifetime expected utility are derived from age- and gender-specific population mortality probabilities taken from the Human Mortality database.[9] This implies that differences between the survival probabilities used by the couple and the annuity provider arise only due to gender-neutral pricing.

Figure 3.1 displays the optimal consumption patterns for different gender-neutral survival probabilities used for pricing the annuities. The left figure displays the joint consumption level of the couple in case they are both alive, the middle (right) figure displays the consumption level of the husband (wife) in case the wife (husband) has deceased. Because the total wealth W is normalized at 100, the consumption level in a given period and a given state can

[8]The European court has recently decided that as off December 2012 insurers are not allowed to differentiate annuity prices based on gender.

[9]Specifically, for $g \in \{m, f\}$, we let $p_t(g) = (1 - q_x^{(g)})(1 - q_{x+1}^{(g)}) \cdots (1 - q_{x+t-1}^{(g)})$, where x denotes the age of the insured with gender g, and $q_{x+s}^{(g)}$ denotes the one-year death probability of an individual with gender g and age $x + s$, as reported in the Human Mortality Database. The corresponding t-year survival probabilities $p_t(g)$, are displayed in Appendix 3.B. Likewise, we let $p_{g,\overline{g},t} = (1 - q_{\overline{x}}^{(g)})(1 - q_{\overline{x}+1}^{(g)}) \cdots (1 - q_{\overline{x}+t-1}^{(g)})$, where $\overline{x}$ denotes the age of the insured with gender $\overline{g}$.



(a) $C_t(mf)$ **(b)** $C_t(m)$ **(c)** $C_t(f)$

Figure 3.1 – Optimal consumption as a function of time, for a couple consisting of a husband and wife with age 65 and 62, respectively, at the moment they buy the annuities. Panel (a) displays optimal consumption when both spouses are alive. Panel (b) displays optimal consumption when only the husband is alive, and panel (c) displays optimal consumption when only the wife is alive. The parameter of risk aversion γ equals 2 and the parameter of joint consumption μ equals 70%. The discount rate r equals 3%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for three values of δ. The black solid lines correspond to $\delta = 80\%$, the dashed lines correspond to $\delta = 50\%$, and the dotted lines correspond to $\delta = 20\%$. The grey lines represent the optimal consumption patterns for the case where $(p_t^*(m), p_t^*(f))$ equals $(p_t(m), p_t(f))$ for all t.

be interpreted as the percentage of total wealth consumed in that period and that state.

Because of the differences between the survival probabilities used by the couple and the annuity provider (i.e., differences between $(p_t(m), p_t(f))$ and $(p_t^*(m), p_t^*(f))$), the optimal consumption patterns are not flat over time. From the first order conditions in Appendix 3.C.1, it follows immediately that the optimal consumption patterns are in general not flat over time anymore. Moreover, comparing figures (b) and (c) shows that consumption when only the wife is alive is higher than consumption when only the husband is alive. Since the survival probability of a man is typically lower than the survival probability of a woman with the same age, the annuity prices under gender-neutral pricing for the husband (wife) are relatively high (low). Therefore, the consumption of the wife when the husband has deceased is optimally higher than the consumption of the husband when the wife has deceased. Comparing different line types in each of the three figures shows that the optimal consumption patterns depend strongly on the differences between $(p_t(m), p_t(f))$ and $(p_t^*(m), p_t^*(f))$. For $\delta = 80\%$, consumption when both spouses are alive is increasing over time whereas for $\delta = 50\%$ or $\delta = 20\%$, consumption when both spouses are alive is decreasing over time.

3.2.3 Optimal survivor fraction

Many collective plans include some form of survivor protection. In most cases, however, there are implicit or explicit constraints on the survivor fraction, i.e., the ratio of insured rights for survivor protection over insured rights for joint-life protection. In this section, we analyze the optimal survivor fraction in case the couple has access to a complete and frictionless market of Arrow annuities. These results will then be used as a benchmark to quantify the welfare losses couples bear from such restrictions.

We will denote the optimal survivor fraction for the husband and wife by $\alpha_t(m)$ and $\alpha_t(f)$ respectively. The survivor fraction of a spouse in period t is the consumption when only that spouses is alive relative to the joint consumption when both spouses are alive, that is:

$$\alpha_t(g) = \frac{C_t(g)}{C_t(mf)}, \text{ for } g \in \{m, f\}. \tag{3.15}$$

To obtain the optimal survivor fraction ($\alpha_t(g)$ for $g \in \{m, f\}$), (3.7) and (3.8) are substituted into (3.15). This yields the following proposition.

Proposition 1 *The optimal survivor fraction of a spouse with gender $g \in \{m, f\}$ is given by:*

$$\begin{aligned}\alpha_t(g) &= \frac{1}{2}(1+\mu)^{1-\frac{1}{\gamma}} \left[\frac{p_t^*(g')(1-p_t(g'))}{p_t(g')(1-p'^*_t(p'))}\right]^{\frac{1}{\gamma}}, \\ &= \frac{1}{2}(1+\mu)^{1-\frac{1}{\gamma}} \left[\frac{P^*_{joint}(t)P_{surv,g}(t)}{P_{joint}(t)P^*_{surv,g}(t)}\right]^{\frac{1}{\gamma}}, \textit{ for } g \in \{m, f\}.\end{aligned} \tag{3.16}$$

The optimal survivor fraction of a spouse is independent of the interest rate r and the expense loading l. In addition, the optimal survivor fraction of a spouse is increasing in the degree of joint consumption (μ) for $\gamma > 1$, and independent of the degree of joint consumption for $\gamma = 1$. Moreover, it holds that:

(i) if annuity provider and couple use the same survival probabilities, then

- *$\alpha_t(g)$ is independent of time and gender;*
- *$\alpha_t(g)$ is increasing in γ when $\mu > 0$, and independent of γ when $\mu = 0$.*

(ii) if annuity provider and couple use different survival probabilities, then

- *$\alpha_t(g)$ can depend on time and gender;*
- *$\alpha_t(g)$ is increasing in γ if $\frac{P^*_{joint}(t)P_{surv,g}(t)}{P_{joint}(t)P^*_{surv,g}(t)} < 1+\mu$, and decreasing in γ if $\frac{P^*_{joint}(t)P_{surv,g}(t)}{P_{joint}(t)P^*_{surv,g}(t)} > 1+\mu$.*

Proof. (3.16) follows immediately from substituting (3.7) and (3.8) into (3.15). Because $\left[\frac{P^*_{joint}(t)P_{surv,g}(t)}{P_{joint}(t)P^*_{surv,g}(t)}\right]^{\frac{1}{\gamma}} > 0$, it follows from (3.16) that $\alpha_t(g)$ is increasing in μ when $\gamma > 1$.

(i) When the annuity provider and the couple use the same survival probabilities and interest rate, i.e., when $P^*_{joint}(t) = P_{joint}(t)$, and $P^*_{surv,g}(t) = P_{surv,g}(t)$ for all t and $g \in \{m, f\}$, it follows from (3.16) that $\alpha_t(g)$ simplifies to $\frac{1}{2}(1+\mu)^{1-\frac{1}{\gamma}}$, which is independent of time and gender. Moreover, it is increasing in γ when $\mu > 0$, and independent of γ when $\mu = 0$.

(ii) When $P^*_{joint}(t) \neq P_{joint}(t)$, and $P^*_{surv,g}(t) \neq P_{surv,g}(t)$ for some t it follows from (3.16) that $\alpha_t(g)$ can depend on t and on g. The effect of γ on $\alpha_t(g)$ follows from the fact that
$\alpha_t(g) = \frac{1}{2}(1+\mu)\left[\frac{P^*_{joint}(t)P_{surv,g}(t)}{P_{joint}(t)P^*_{surv,g}(t)}/(1+\mu)\right]^{\frac{1}{\gamma}}$.

■

The above proposition shows that differences in the survival probabilities used by the annuity provider and by the couple affect the optimal survivor fraction in a nontrivial way. First consider the case where annuity provider and couple use the same survival probabilities and discount rate $((p_t(m), p_t(f)) = (p^*_t(m), p^*_t(f))$ for all t). It then follows immediately from (3.16) that

$$\alpha_t(g) = \alpha = \frac{1}{2}(1+\mu)^{1-\frac{1}{\gamma}}, \text{ for } g, g' \in \{m, f\}. \tag{3.17}$$

When there is no joint consumption, i.e., when $\mu = 0$, it follows from (3.17) that the optimal survivor fraction is independent of time, and equal to 50% for both genders. The reason is that risk averse couples prefer to smooth consumption over time. Because they each consume half of the consumption in case they are both alive, consumption remains flat when one of them deceases only if the survivor fraction is 50%.[10] The optimal survivor fraction increases when the degree of joint consumption μ increases for $\gamma > 1$ The reason is that when one of the spouses dies, the surviving spouse no longer receives the consumption externalities from the former spouse's consumption. Because a risk-averse couple prefers to smooth consumption over time, couples with a higher degree of joint consumption optimally choose a higher consumption when only one of the spouses is alive. This effect is stronger to the extent that the couple is more risk averse. We illustrate these effects Figure 3.2.

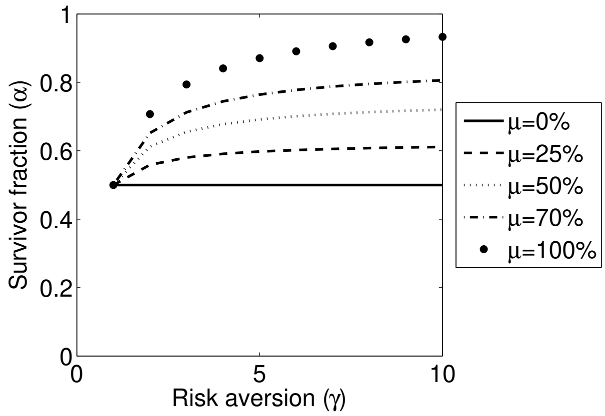


Figure 3.2 – The optimal survivor fraction α from (3.17), as a function of the parameter of risk aversion γ, for different degrees of joint consumption μ, for the case where annuity provider and couple use the same survival probabilities, i.e., $p_t^*(m) = p_t(m)$ and $p_t^*(f) = p_t(f)$ for all t. The survival probabilities used by the couple and by the annuity provider are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$.

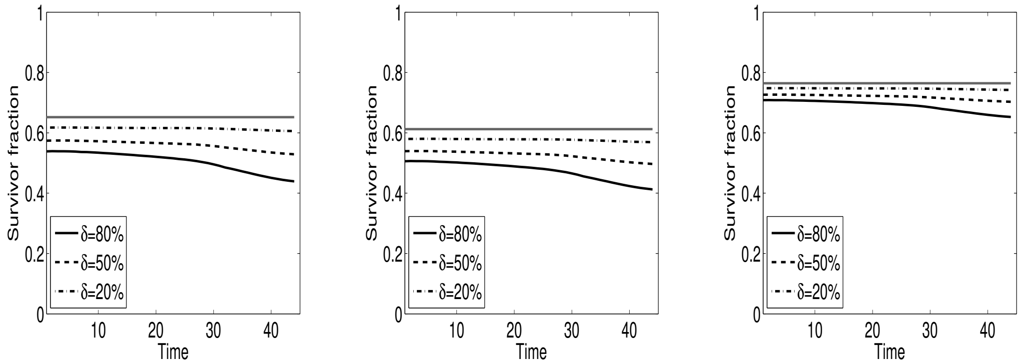
In Figure 3.2 the effect of the degree of joint consumption (μ) and the parameter of risk aversion (γ) on the optimal survivor fraction (α) from (3.17) is displayed when $p_t(m) = p_t^*(f)$, and $p_t(f) = p_t^*(f)$ for all t. A frequently offered option is a joint annuity in combination with a 50% survivor annuity. When there is no joint consumption, a 50% survivor benefit is optimal. For all other parameter values, the optimal survivor fraction is higher.

Now consider the case where annuity provider and couple use different survival probabilities. First, it is seen immediately from (3.16) that in general the optimal survivor fraction is no longer flat over time. The reason is that the relative attractiveness of consumption in the different states and periods

[10]Without the joint consumption, the couple solves the same maximization problem as two singles solve.

is also affected by whether or not the corresponding annuity is priced attractively. Consider for example the case where there is no joint consumption, i.e., $\mu = 0$. Then, it follows from (3.2)-(3.5) and (3.16) that the optimal survivor fraction in period t is strictly larger than 50% if $\frac{P^*_{surv,g}(t)}{P^*_{joint}(t)} < \frac{P_{surv,g}(t)}{P_{joint}(t)}$, and strictly lower than 50% if $\frac{P^*_{surv,g}(t)}{P^*_{joint}(t)} > \frac{P_{surv,g}(t)}{P_{joint}(t)}$. To understand the intuition, first note that if $\frac{P^*_{surv,g}(t)}{P^*_{joint}(t)} < \frac{P_{surv,g}(t)}{P_{joint}(t)}$, the price the annuity provider charges for the survivor annuity, relative to the price of the joint annuity, is strictly smaller than the price ratio that would result if the annuity provider would have used the couple's survivor probabilities. Thus, differences in survival probabilities in this case make survivor annuities more attractive. As a consequence, the optimal survivor fraction is higher than when there are no differences in survival probabilities. The opposite holds when $\frac{P^*_{surv,g}(t)}{P^*_{joint}(t)} > \frac{P_{surv,g}(t)}{P_{joint}(t)}$. Second, differences in survival probabilities used by the annuity provider and the couple imply that the optimal survivor fraction is no longer necessarily increasing in the degree of risk aversion γ. For example, when $\frac{P^*_{surv,g}(t)}{P^*_{joint}(t)} < \frac{P_{surv,g}(t)}{P_{joint}(t)}$ and $\mu = 0$, the attractive pricing of the survivor annuity relative to the joint annuity implies that the optimal survivor fraction $\alpha_t(g)$ is higher than 50% (see (3.16)). Because more risk averse couples attach more value to flat consumption patterns, and because with $\mu = 0$ a flat consumption pattern is obtained when $\alpha_t(g) = 50\%$, an increase in the degree of risk aversion implies a decrease in the optimal survivor fraction.

Figure 3.3 displays the husband's optimal survivor fraction as a function of time for different gender-neutral survival probabilities and different degrees of risk aversion and degrees of joint consumption. Because of the unfavorable annuity prices for men, the husband's optimal survivor fraction in case of gender-neutral pricing is below his optimal survivor fraction when his survival probabilities would be used. Moreover, the optimal survivor fraction is not flat over time anymore. In Appendix 3.D, the wife's optimal survivor fractions are displayed. Because of the favorable annuity prices for the wife, the optimal survivor fraction of the wife is even above 100% for some parameter values, meaning that the total consumption of the couple is optimally higher in states where only the wife is alive than in states where both spouses are alive. In addition, the husband's optimal survivor fraction is decreasing as a function of time whereas the wife's optimal survivor fraction is increasing as a function of time.



(a) $\gamma = 2, \mu = 70\%$ **(b)** $\gamma = 2, \mu = 50\%$ **(c)** $\gamma = 5, \mu = 70\%$

Figure 3.3 – The husband's optimal survivor fraction as a function of time, for a couple consisting of a husband and wife with age 65 and 62, respectively, at the moment they buy the annuities, for three combinations of the degree of risk aversion γ and the degree of joint consumption μ: $\gamma = 2$ and $\mu = 70\%$ (Figure (a)), $\gamma = 2$ and $\mu = 50\%$ (Figure (b)), and $\gamma = 5$ and $\mu = 70\%$ (Figure (c)). The discount rate equals 3%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for three values of δ. The black solid lines correspond to $\delta = 80\%$, the dashed lines correspond to $\delta = 50\%$, and the dotted lines correspond to $\delta = 20\%$. The grey lines represent the optimal consumption patterns for the case where $(p_t^*(m), p_t^*(f))$ equals $(p_t(m), p_t(f))$ for all t.

3.3 Optimal annuity portfolios in incomplete annuity markets

Compared to the benchmark case discussed in Section 3.2, there are two potentially important sources of welfare losses. First, individuals typically only have access to annuities that yield flat benefit levels over time, i.e., the payment of the annuity is state dependent, but not time dependent. Second, when the couple has illiquid pension wealth in the form of pre-accrued pension rights, there is typically only a limited set of annuity portfolios they can choose from. For example, it is common in many collective pension schemes that individuals who have accrued the right to receive a single-life annuity can exchange part of that annuity for a survivor annuity for their spouse. However, they are typically not allowed to exchange their single-life annuity for a single-life annuity for their spouse.

To determine the relative importance of these two sources of market imperfections, we first determine the couple's lifetime expected utility in case all pension wealth is liquid, and the only restriction is that annuities yield a flat payment over time (i.e., time independent). We then determine the couple's lifetime expected utility in case the couple has pre-accrued pension rights which can be reallocated to a limited set of actuarially equivalent portfolios of annuities.

3.3.1 The effect of flat annuities

When only annuities with flat benefit levels over time are available, the couple can obtain consumption patterns that are state dependent but not time dependent. To compare with the setting where couples have pre-accrued pension rights, we consider settings where the couple fully annuitizes at time $t = 0$. The couple can save in all periods. Then, in all numerical examples considered, it is optimal for the couple to consume all annuity income each period.[11] Let $C(mf)$ denote the real aggregate consumption of the two spouses when both spouses are alive, and $C(m)$ $(C(f))$ the real consumption of the husband (wife) when only the husband (wife) is alive. The couple can obtain this state dependent consumption pattern by buying a joint annuity that pays $C(mf)$ in every year that both spouses are alive, a survivor annuity for the husband that pays $C(m)$ in every year that the husband is alive and the wife has died, and a survivor annuity that pays $C(f)$ in every year that the wife is alive, and the husband has died. The prices of the joint annuity and of the two survivor

[11]Optimizing (3.23) subject to (3.24) yields the optimal consumption pattern under the assumption that the couple consumes all annuity income in each period, i.e., it ignores the possibility to change the consumption pattern through saving. However, if $r = \rho$ and condition (3.18) is satisfied, saving is not optimal. Specifically, let $\tilde{p}^{(g)}_{x+t}$ denote the probability that an individual with age $x + t$ and gender g survives at least one year for $t \geq 0$. Let $C(mf), C(m), C(f)$ be the solution of (3.23) subject to (3.24). Let x be the husband's age at time $t = 0$ and y be the wife's age at time $t = 0$. Then, when the rate of time preference equals the interest rate, there will be no savings in period t when the when the following inequality holds:

$$(1+\mu)\left(\frac{1}{2}(1+\mu)\right)^{-\gamma} > \tilde{p}^{(m)}_{x+t}(1-\tilde{p}^{(f)}_{y+t})\alpha(m)^{-\gamma} + \tilde{p}^{(f)}_{y+t}(1-\tilde{p}^{(m)}_{x+t})\alpha(f)^{-\gamma} + \tilde{p}^{(m)}_{x+t}\tilde{p}^{(f)}_{y+t}(1+\mu)\left(\frac{1}{2}(1+\mu)\right)^{-\gamma}, \tag{3.18}$$

where $\alpha(g) = \frac{C(g)}{C(mf)}$ for $g \in \{m, f\}$. It can be verified that this condition is satisfied for all $t \geq 0$ in all the numerical examples that we present. The derivation of the condition is presented in Appendix 3.C.2. We also present the more general conditions in case $r \neq \rho$.

annuities, respectively, are given by:

$$P^*_{joint} = (1+l)\sum_{t=0}^{T}\frac{p^*_t(m)p^*_t(f)}{(1+r)^t}, \tag{3.19}$$

$$P^*_{surv,g} = (1+l)\sum_{t=0}^{T}\frac{p^*_t(g)(1-p^*_t(g'))}{(1+r)^t}, \text{ for } g,g' \in \{m,f\}. \tag{3.20}$$

Based on the couple's rate of time preference and (subjective) survival probabilities, the annuity prices are given by:

$$P_{joint} = (1+l)\sum_{t=0}^{T}\frac{p_t(m)p_t(f)}{(1+\rho)^t}, \tag{3.21}$$

$$P_{surv,g} = (1+l)\sum_{t=0}^{T}\frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t}, \text{ for } g,g' \in \{m,f\}. \tag{3.22}$$

The couple's lifetime expected utility and budget constraint are respectively given by:

$$L = 2U(\frac{1}{2}C(mf),\frac{1}{2}C(mf))\sum_{t=0}^{T}\frac{p_t(m)p_t(f)}{(1+\rho)^t} + U(C(m),0)\sum_{t=0}^{T}\frac{p_t(m)(1-p_t(f))}{(1+\rho)^t}$$
$$+ U(C(f),0)\sum_{t=0}^{T}\frac{(1-p_t(m))p_t(f)}{(1+\rho)^t}. \tag{3.23}$$

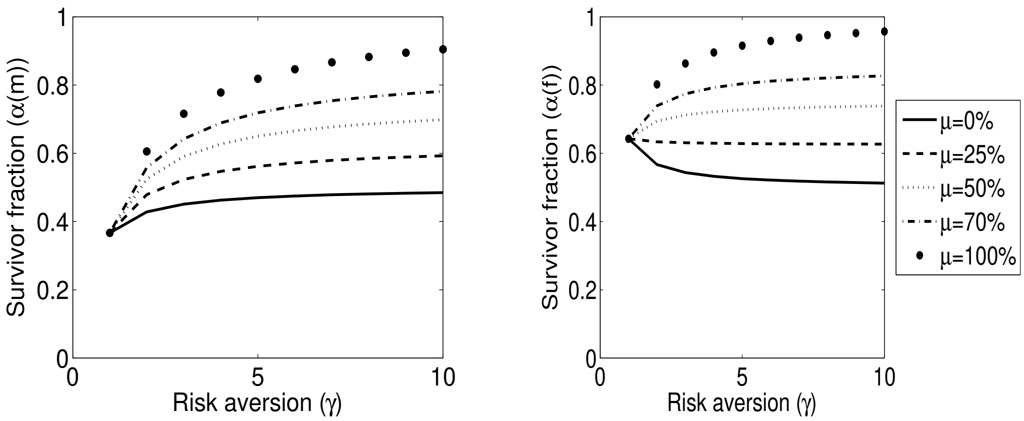
$$\frac{W}{1+l} = C(mf)P^*_{joint} + C(m)P^*_{surv,m} + C(f)P^*_{surv,f}. \tag{3.24}$$

The maximization problem and budget constraint are the same as in (3.1) and (3.6), respectively, except that the annuity levels (and with that the consumption levels) are time-independent. The solution of (3.23) subject to (3.24) is displayed in Appendix 3.C.

The optimal survivor fraction of a spouse with gender $g \in \{m,f\}$ is given by:

$$\alpha(g) = \frac{1}{2}(1+\mu)^{1-\frac{1}{\gamma}}\left[\frac{P_{surv,g}/P_{joint}}{P^*_{surv,g}/P^*_{joint}}\right]^{\frac{1}{\gamma}}, \text{ for } g,g' \in \{m,f\}.$$

It follows from (3.25) that the husband's (wife's) optimal survivor fraction is decreasing in the ratio of the price of the survivor annuity over the price of the joint annuity based on the survival probabilities uses by the annuity provider ($P^*_{surv,g}/P^*_{joint}$), and decreasing in the price ratio that would result in case the annuity provider would have used the couple's survival probabilities ($P_{surv,g}/P_{joint}$). In contrast to the benchmark case with a complete annuity market with time-dependent Arrow annuities (see (3.16)) the optimal survivor fraction for a spouse now depends on the survival probabilities of both spouses. Moreover, it is also affected by the discount rate r and the rate of time preference ρ.



(a) Husband **(b)** Wife

Figure 3.4 – The optimal survivor fraction $\alpha(g)$ from (3.25), as a function of the parameter of risk aversion γ, for different degrees of joint consumption μ. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for $\delta = 50\%$.

Figure 3.4 displays the optimal survivor fraction when gender-neutral survival probabilities are used to price the annuities and when only annuities with a flat benefit payment over time are available. Because of the favorable annuity pricing for the wife, the optimal survivor fraction for the wife is higher compared to the case of gender-dependent pricing. Likewise, the optimal survivor fraction of the husband decreases compared to the case of gender-dependent pricing because of the unfavorable annuity pricing for the husband. For more risk averse couples, i.e., when γ is higher, the optimal survivor fractions are closer to the optimal survivor fractions in case $(p_t(m), p_t(f))$ equals $(p_t^*(m), p_t^*(f))$ for all t. Stated differently, differences between survival probabilities used by the couple and by the annuity provider have less effect on the couple's optimal consumption pattern when the couple is more risk averse. There is a tradeoff between benefiting from the relatively inexpensive annuity prices for the wife and smoothing consumption over the different states. The more risk averse the couple is, the more the couple wants too smooth consumption over the different states and the less they benefit from the favorable annuity prices for the wife.

3.3.2 The effect of exchange restrictions in case of pre-accrued pension rights

As argued above, another source of market incompleteness for individuals with pre-accrued pension rights is that there are restrictions on the exchange of pension rights. One of the possible restrictions is that the single-life annuity for the husband can only be exchanged for a survivor annuity for the wife and vice versa. Because of these restrictions, the couple may not be able to obtain the same portfolio of annuities as in case all wealth was liquid. For example, if all pension wealth is generated by one individual, death of his or her partner will not reduce income, leading to a different optimal annuity portfolio compared with the benchmark case of liquid wealth. In this subsection, we determine the couple's optimal annuity portfolio given the restrictions on the exchange of pre-accrued pension rights. We assume that both spouses can have pre-accrued pension rights and they are allowed to use these rights to finance a single-life annuity on their own life and/or a survivor annuity on their spouse's life. The price of a single-life annuity that pays off one unit of consumption in every year that the individual with gender g is alive is given by:

$$P^*_{sl,g} = (1+l)\sum_{t=0}^{T} \frac{p^*_t(g)}{(1+r)^t}, \text{ for } g \in \{m, f\}.$$

The price of the survivor annuity is as given in (3.22).

Let $W(m)$ $(W(f))$ be the actuarial value of the husband's (wife's) accrued pension rights net of the expense loading, based on the survival probabilities $(p^*_t(m), p^*_t(f))$. We assume that the couple can exchange the accrued pension rights to obtain a different portfolio of annuities under the following two restrictions. First, the portfolios of annuities before and after exchange are actuarially equivalent, based on the survival probabilities used by the annuity provider. So whether the rights are initially accrued as a single-life annuity or as a single-life with a survivor annuity is not of importance, as long as the price of the portfolio before and after exchange is the same, based on the survival probabilities $(p^*_t(m), p^*_t(f))$. Second, short selling of annuities is not allowed, i.e., the couple is not allowed to sell an annuity on the husband's life nor on the wife's life. This implies that the couple maximizes the lifetime expected utility as given in (3.23) subject to the following constraints:

$$W(m) = C_{sl}(m)P^*_{sl,m} + C_{surv}(f)P^*_{surv,f}, \tag{3.25}$$

$$W(f) = C_{sl}(f)P^*_{sl,f} + C_{surv}(m)P^*_{surv,m}, \tag{3.26}$$

$$C(mf) \leq C_{sl}(m) + C_{sl}(f), \tag{3.27}$$

$$C(g) \leq C_{sl}(g) + C_{surv}(g), \text{ for } g \in \{m, f\}, \tag{3.28}$$

$$C_{sl}(m), C_{sl}(f), C_{surv}(m), C_{surv}(f) \geq 0, \tag{3.29}$$

where $C_{sl}(g)$ is the benefit level of the single-life annuity after exchange, and $C_{surv}(g)$ is the benefit level of the survivor annuity after exchange. Equation (3.25) (equation (3.26)) ensures that the actuarial value of the portfolio of the husband's (the wife's) annuities equals the actuarial value of the pre-accrued pension rights of the couple. Equation (3.27) ensures that in states where both spouses are alive, they do not consume more than the benefit level of the two single-life annuities. So, when for instance the wife's consumption level is higher than the benefit level of her single-life annuity, the husband's consumption level should be below the benefit level of his single-life annuity such that their total consumption level does not exceed their total benefit level. Equation (3.28) ensures that when only one of the spouses is alive, that spouse does not consume more than the benefit level of both his own single-life annuity and his survivor annuity. Equation (3.29) ensures that short selling of annuities is not allowed.[12]

The short sales constraints in (3.29) imply that the couple cannot freely diversify their wealth over the three states.[13] Suppose for example that only the husband has accrued pension rights (so $W(m) > 0, W(f) = 0$). The couple can allocate their wealth over a single-life annuity for the husband and a survivor annuity for the wife only. So, their wealth should be allocated over two annuities, although there are three states (both spouses alive, only husband alive, only wife alive). This implies that when both spouses are alive, they

[12]There will be no savings when (3.18) holds. In all numerical results displayed, this inequality is satisfied for all $t \geq 0$.

[13]We allow that both spouses allocate their accrued pension rights such that the benefit level of the survivor annuity exceeds the benefit level of the single-life annuity, which is typically not allowed. However, in all our numerical cases, it is not optimal to have an increasing income after one of the spouses deceases in case the annuity level is independent of time. Therefore, imposing the additional restrictions that $C_{surv}(m) \leq C_{sl}(f)$ and $C_{surv}(f) \leq C_{sl}(m)$ would not change the numerical results. Another difference is that we allow the couple can choose any possible survivor fraction $(\alpha(m)(\alpha(f)))$, where in reality, only a limited number of survivor fractions may be offered.

consume $C(mf) = C_{sl}(m)$, and when only the husband is alive, he also consumes $C(m) = C_{sl}(m)$. Stated differently, when only the husband has accrued pension rights, the following holds:

$$C(m) = C(mf),$$

which implies that $\alpha(m) = \frac{C(m)}{C(mf)} = 100\%$, i.e., the survivor fraction of the husband equals 100%. Because each spouse consumes half of the aggregate consumption in case they are both alive, this implies that consumption of the husband doubles upon the decease of the wife. We know from Figure 3.2, that this is not optimal for most parameter values. Absent the short sales constraint, the husband could reduce the income he receives when his wife dies by selling income in that state, i.e., by choosing $C_{surv}(m) < 0$. The following proposition shows that when the short sales constraints in (3.29) do not bind, the optimal consumption pattern in case of pre-accrued rights equals the optimal consumption pattern in case of liquid wealth. Put differently, because of (3.29) the results differ from the case where all wealth is liquid.

Proposition 2 *Let $\frac{W}{1+l} = W(m) + W(f)$. Then, the consumption pattern ($C(mf)$, $C(m)$, $C(f)$) that maximizes (3.23) subject to (3.25)-(3.28) also maximizes (3.23) subject to (3.24).*

Proof. First, note that in the optimum for (3.23) subject to (3.25)-(3.28), the constraints (3.27) and (3.28) are binding. Therefore, it is sufficient to show that for any ($C(mf)$, $C(m)$, $C(f)$) that satisfies (3.24), there exist $C_{sl}(m)$, $C_{sl}(f)$, $C_{surv}(m)$, $C_{surv}(f)$ such that (3.25)-(3.28) are satisfied in equality, and vice versa.

Let ($C(mf)$, $C(m)$, $C(f)$) be such that (3.24) is satisfied. It can be verified easily that there exist unique $C_{sl}(m)$, $C_{sl}(f)$, $C_{surv}(m)$, and $C_{surv}(f)$ such that (3.27) and (3.28) are satisfied in equality, and (3.25) is satisfied. So it remains to show that $C_{sl}(m)$, $C_{sl}(f)$, $C_{surv}(m)$, and $C_{surv}(f)$ also satisfy (3.26). This can be seen as follows:

$$\begin{aligned}
W(f) &= \frac{W}{1+l} - W(m) \\
&= C(mf)P^*_{joint} + C(m)P^*_{surv,m} + C(f)P^*_{surv,f} \\
&\quad -C_{sl}(m)P^*_{sl,m} - C_{surv}(f)P^*_{surv,f} \\
&= \left(C_{sl}(m) + C_{sl}(f)\right) P^*_{joint} + \left(C_{sl}(m) + C_{surv}(m)\right) P^*_{surv,m} \\
&\qquad + \left(C_{sl}(f) + C_{surv}(f)\right) P^*_{surv,f} - C_{sl}(m)P^*_{sl,m} - C_{sl}(f)P^*_{sl,f} \\
&= C_{sl}(m)\left(P^*_{joint} + P^*_{surv,m} - P^*_{sl,m}\right) + C_{sl}(f)\left(P^*_{joint} + P^*_{surv,f}\right)
\end{aligned}$$

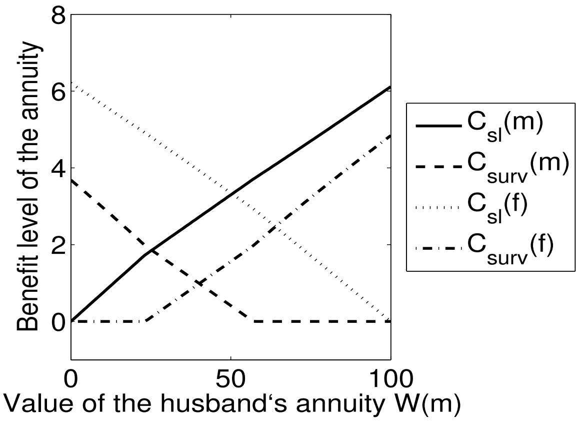


Figure 3.5 – The optimal annuity portfolio as a function of the actuarial value of the accrued pension rights of the husband ($W(m)$) for a man aged 65 and a wife aged 62. The actuarial value of the annuity portfolio of the wife is given by $W(f) = 100 - W(m)$. The degree of joint consumption μ equals 70% and the risk aversion γ equals 2. The discount is set at 3%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for $\delta = 50\%$.

$$
\begin{aligned}
&+ \quad C_{surv}(m) P^*_{surv,m} \\
&= \quad C_{sl}(f) P^*_{sl,f} + C_{surv}(m) P^*_{surv,m}.
\end{aligned}
$$

Using similar arguments, it can be shown that if there exist there exist $C_{sl}(m)$, $C_{sl}(f)$, $C_{surv}(m)$, and $C_{surv}(f)$ such that ($C(mf)$, $C(m)$, $C(f)$) satisfies (3.25)-(3.28) in equality, then it also satisfies (3.24). This concludes the proof. ■

The above proposition implies that if the short sales constraints in (3.29) are not binding, then the optimal consumption pattern in case of pre-accrued wealth equals the optimal consumption in case of liquid wealth. Stated differently, the restrictions on the exchange of pension rights in this case do not affect the couple's optimal utility.

In the remainder of this section, we first show the effect of restrictions on the exchange of pension rights on the optimal annuity portfolio, and on the corresponding optimal consumption patterns. We then show the effect on the optimal survivor fractions.

Figure 3.5 displays the optimal annuity portfolio for a couple as a function of the actuarial value of the accrued rights of the husband, for $W(f) = 100 - W(m)$. The figure shows that whether the short sales constraints are binding depends on the division of pre-accrued pension wealth between the two spouses.

First consider the case where the actuarial values of the pre-accrued pension rights of the husband and wife are the same (i.e., $W(m) = W(f)$). Then, the short sales constraints are not binding, and so the optimal solution in case of pre-accrued wealth equals the optimal solution in case of liquid wealth. To understand the intuition, suppose for example that the husband would forego survivor protection for his spouse, i.e., the couple would choose $C_{surv}(f) = 0$. Then, in case of decease of the husband, the wife would consume less than half of the aggregate consumption when they are both alive. To see why, first note that because the wife is younger than the husband, the gender-neutral survival probabilities used to determine the actuarial value of her pension rights are higher than the gender-neutral survival probabilities used to determine the actuarial value of the husband's pension rights. Combined with (3.25) and (3.28), this implies that $C_{sl}(m) \geq C_{sl}(f)$, i.e., the benefit level of the single-life annuity of the wife is lower than the benefit level of the single-life annuity of the husband. Therefore, in case of decease of the husband, the wife would consume $C(f) = C_{sl}(f)$, which is less than half of the aggregate consumption (because $C_{sl}(f) \leq \frac{C_{sl}(m)+C_{sl}(f)}{2} = \frac{C(mf)}{2}$). Stated differently, the wife's survivor fraction would be less than 50%. We know from Figure 3.4b, however, that it is optimal for a wife to have a survivor fraction that is higher than 50%. Therefore, the husband optimally chooses for a single-life with survivor annuity. Whether it is also optimal for the wife to opt for a single-life and survivor annuity in general depends on the degree of joint consumption (μ) and risk aversion (γ). Because of the gender-neutral pricing, single-life annuities are priced relatively favorably for women, and unfavorably for men. Consequently, the couple can benefit from the gender-neutral pricing by increasing the benefit level of the annuities for the wife at the cost of annuities for the husband. For the parameter values used in Figure 3.5, however, it is optimal for the wife to buy survivor protection for the husband. Thus, the short sales constraints are not binding in this case, and it follows from Proposition 2 that the optimal annuity portfolio is the same as when these restrictions were not imposed.

Now consider instead the case where the actuarial value of the accrued pension rights of one spouse is relatively low compared to the actuarial value of the accrued pension rights of the other spouse. Then, some short sales constraints are binding. Specifically, consider the case where only the husband has accrued pension rights, i.e., $W(m) = 100$, and $W(f) = 0$. As argued above, this would imply that the survivor fraction of the husband would be equal to 100%. We know from Figure 3.4a that this is sub-optimally high. As

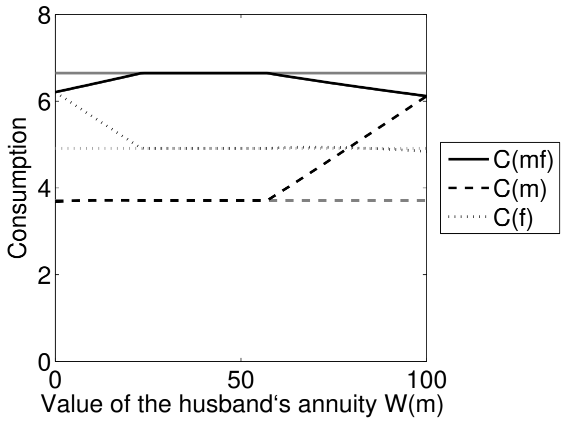


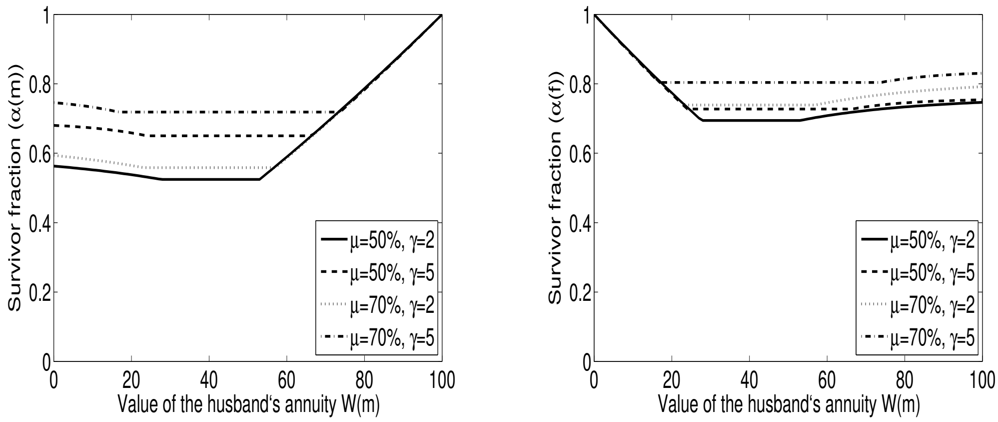
Figure 3.6 – The optimal consumption in each state as a function of the actuarial value of the annuity portfolio of the husband ($W(m)$) for a man aged 65 and a wife aged 62. The actuarial value of the annuity portfolio of the wife is given by $W(f) = 100 - W(m)$. The light grey lines represent the optimal consumption levels in case all wealth was liquid. The degree of joint consumption μ is set at 70% and the risk aversion γ is set at 2. The discount rate r is set at 3%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for $\delta = 50\%$.

a consequence, the wife would optimally short sell survivor protection for the husband. This, however, is not allowed. When the actuarial value of the accrued pension rights of the wife are well below the actuarial value of the accrued pension rights of the husband, the optimal strategy for the wife is to choose a single-life annuity only, and the husband's survivor fraction is sub-optimally high compared to the case where all pension wealth is liquid. Likewise, if the wife has accrued most of the pension rights, the husband optimally chooses for a single-life annuity only. The relative actuarial value as of which a spouse opts for a single-life with survivor annuity instead of a single-life annuity, however, depends on the gender of the spouse. As argued above, because gender-neutral pricing implies that single-life annuities are priced relatively favorable for women, and unfavorable for men. Therefore, the range of values for which it is optimal for the wife to buy survivor protection for the husband is smaller than the range of values for which it is optimal for the husband to buy survivor protection for the wife.

The above analysis shows that if one spouse has accrued significantly more pension rights than the other spouse, short sales constraints imply that the survivor fraction of the spouse that has accrued most of the pension rights is sub-optimally high compared to the case where all pension wealth is liquid. As a consequence, consumption in case both spouses are alive would be sub-optimally low. We illustrate this in Figure 3.6. Figure 3.6 displays the optimal consumption levels in the three states: both spouses alive, only the husband alive, and only the wife alive. The light-grey lines correspond to the consumption levels in case all wealth was liquid. When both spouses have accrued about the same pension wealth, short sales constraints are not binding, and we know from Proposition 2 that the optimal solution in case of pre-accrued pension rights is the same as the optimal solution in case of liquid wealth. However, when one of the spouse has accrued significantly more pension rights than the other spouse, consumption in case they are both alive is sub-optimally low, and consumption in case the spouse with the lowest (highest) pension rights deceases is sub-optimally low (high).

The combinations of pre-accrued pension wealth $(W(m), W(f))$ for which a short sales constraint is binding depend on the degree of risk aversion and the degree of joint consumption. It follows from (3.25) that for $\gamma > 1$, a higher degree of joint consumption leads optimally to a higher consumption in states where only one of the spouses is alive relative to consumption when both spouses are alive. Consequently, a higher degree of joint consumption reduces the combinations of $(W(m), W(f))$ for which the short sales constraints are binding. Moreover, it follows from (3.25) that a higher degree of risk aversion increases the survivor fraction if $\frac{P^*_{surv,g}/P^*_{joint}}{P_{surv,g}/P_{joint}} < 1 + \mu$, and decreases the survivor fraction if $\frac{P^*_{surv,g}/P^*_{joint}}{P_{surv,g}/P_{joint}} > 1 + \mu$. A higher degree of risk aversion therefore reduces the combinations of $(W(m), W(f))$ for which the short sales constraints are binding when $\frac{P^*_{surv,g}/P^*_{joint}}{P_{surv,g}/P_{joint}} < 1 + \mu$. We illustrate this in Figure 3.7.

In Figure 3.7 the optimal survivor fractions are displayed as a function of the actuarial value of the pre-accrued pension rights of the husband $(W(m))$. The optimal survivor fraction with pre-accrued pension rights differs from the optimal solution in case all wealth is liquid, unless the actuarial value of the rights accrued by both spouses is about equal. The range of values of the accrued pension rights for which this holds depends on the degree of risk aversion and the degree of joint consumption. Consider for example the case where the degree of risk aversion equals 2, and the degree of joint consumption equals 50%. Then, as long as $W(m)$ is between 28% and 53% of the total actuarial value of the couple's pre-accrued pension rights, the husband's optimal survivor fraction equals 52%. It can be verified from (3.25) that this fraction is optimal also in case pension wealth is liquid. Outside this range,



(a) husband **(b)** wife

Figure 3.7 – The optimal survivor fraction as a function of the actuarial value of the husband's annuity portfolio ($W(m)$) for a man aged 65 and a wife aged 62. The actuarial value of the wife's annuity portfolio is given by $W(f) = 100 - W(m)$. The discount rate is set at 3%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) with $\delta = 50\%$.

however, the husband's survivor fraction in case of illiquid pension wealth is suboptimally high compared to the case of liquid pension wealth. Now consider instead the case where the degree of risk aversion is 5, and the degree of joint consumption equals 70%. Then, the lower (upper) bound for of values of the pre-accrued pension rights of the husband $W(m)$ for which exchange restrictions do not affect the optimal consumption patterns decreases (increases) to 17% (73%).

The above results show that exchange restrictions can lead to significantly higher survivor fractions when one of the spouses has accrued significantly more pension rights than the other spouse. As a consequence, the lifetime expected utility of the couple in case of pre-accrued pension rights may be lower than the lifetime expected utility of the couple in case of liquid wealth. In the next section, we will quantify the welfare losses. We will first analyze the optimal annuity portfolios in case the utility functions of the husband and wife are not identical

3.3.3 The impact of different degrees of risk aversion and joint consumption on the optimal survivor fraction

Throughout the paper, the assumption was made that both the husband and wife have *CRRA* utility functions and that they have the same degree of risk aversion and the same degree of joint consumption. In this subsection, we investigate the impact of different degrees of risk aversion and joint consumption for the spouses. The the husband's and wife's utility function are respectively given by:

$$U_m(C, C') = \frac{\left(C + \mu_m C'\right)^{1-\gamma_m}}{1-\gamma_m},$$
$$U_f(C, C') = \frac{\left(C + \mu_f C'\right)^{1-\gamma_f}}{1-\gamma_f},$$

where (μ_m, μ_f) are the degree of joint consumption of the husband and wife, respectively, and (γ_m, γ_f) are the degree of risk aversion of the husband and wife, respectively. The couple maximizes the lifetime expected utility, given by:

$$L = (Um(\tilde{C}(mf), \tilde{C}(fm)) + U_f(\tilde{C}(fm), \tilde{C}(mf))) \sum_{t=0}^{T} \frac{p_t(m)p_t(f)}{(1+\rho)^t} + U_m(C(m), 0) \sum_{t=0}^{T} \frac{p_t(m)(1-p_t(f))}{(1+\rho)^t} + U_f(C(f), 0) \sum_{t=0}^{T} \frac{(1-p_t(m))p_t(f)}{(1+\rho)^t}, \tag{3.30}$$

where $\tilde{C}(mf)$ $(\tilde{C}(fm))$ is the husband's (wife's) consumption when both spouses are alive. In case only annuities with a flat benefit level are available, the budget constraint is given by:

$$\frac{W}{1+l} = (\tilde{C}(mf) + \tilde{C}(fm))P^*_{joint} + C(m)P^*_{surv,m} + C(f)P^*_{surv,f}. \tag{3.31}$$
$$\tilde{C}(mf), \tilde{C}(fm) \geq 0. \tag{3.32}$$

The first order conditions are displayed in Appendix 3.C.3. In case the couple has pre-accrued pension rights, the lifetime expected utility as given in (3.23) is maximized subject to the following constraints:

$$W(m) = C_{sl}(m)P^*_{sl,m} + C_{surv}(f)P^*_{surv,f}, \tag{3.33}$$
$$W(f) = C_{sl}(f)P^*_{sl,f} + C_{surv}(m)P^*_{surv,m}, \tag{3.34}$$

$$\tilde{C}(mf) + \tilde{C}(fm) \leq C_{sl}(m) + C_{sl}(f), \tag{3.35}$$
$$C(g) \leq C_{sl}(g) + C_{surv}(g), \text{ for } g \in \{m, f\}, \tag{3.36}$$
$$C_{sl}(m), C_{sl}(f), C_{surv}(m), C_{surv}(f) \geq 0, \tag{3.37}$$
$$\tilde{C}(mf), \tilde{C}(fm) \geq 0. \tag{3.38}$$

Note that in both cases because of the asymmetries in the utility function, constraint (3.32) or (3.38) may become binding. In both settings, it is also not optimal to save for the numerical examples we have considered.[14]

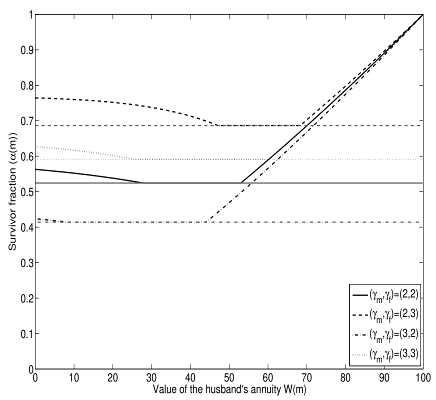
Figure 3.8 displays the optimal survivor fractions when the spouses have different degrees of risk aversion (3.8a and 3.8b) or joint consumption (3.8c and 3.8d) for the husband (3.8a and 3.8c) and wife (3.8b and 3.8d), respectively. From the first order conditions displayed in Appendix 3.C.3, we obtain that the following equality should hold in case of flat annuities:

$$C(m)^{-\gamma_m}\frac{P_{surv,m}}{P^*_{surv,m}} = C(f)^{-\gamma_f}\frac{P_{surv,f}}{P^*_{surv,f}}. \tag{3.40}$$
$$= \left[\left(\tilde{C}(mf) + \mu_m\tilde{C}(fm)\right)^{-\gamma_m} + \mu_f\left(\tilde{C}(fm) + \mu_f\tilde{C}(mf)\right)^{-\gamma_f}\right]\frac{P_{joint}}{P^*_{joint}} \tag{3.41}$$
$$= \left[\mu_m\left(\tilde{C}(mf) + \mu_m\tilde{C}(fm)\right)^{-\gamma_m} + \left(\tilde{C}(fm) + \mu_f\tilde{C}(mf)\right)^{-\gamma_f}\right]\frac{P_{joint}}{P^*_{joint}}. \tag{3.42}$$

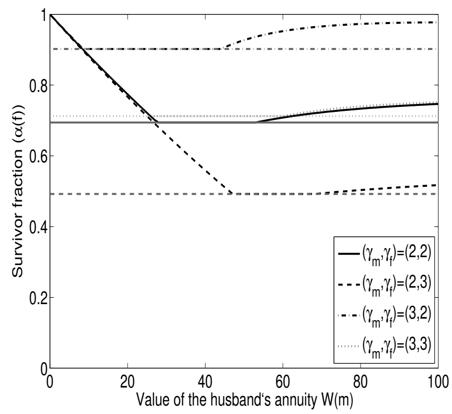
[14]Optimizing (3.30) subject to (3.31) and (3.32) yields the optimal consumption pattern under the assumption that the couple consumes all annuity income in each period, i.e., it ignores the possibility to change the consumption pattern through saving. However, if $r = \rho$, $\tilde{C}(mf), \tilde{C}(fm) > 0$, and condition (3.39) is satisfied, saving is not optimal. Let $\tilde{p}^{(g)}_{x+t}$ denote the probability that an individual with age $x + t$ and gender g survives at least one year. Let x be the husband's age at time $t = 0$ and y be the wife's age at time $t = 0$. Further, let $\tilde{C}(mf)$, $\tilde{C}(fm)$, $C(m)$, $C(f)$ be the solution of (3.30) subject to (3.31) and (3.32). Then, when the rate of time preference equals the interest rate and (3.32) is not binding, the couple will not save in period t when the when the following inequality holds:

$$\begin{aligned}&\left(\tilde{C}(mf) + \mu_m\tilde{C}(fm)\right)^{-\gamma_m} + \mu_f\left(\tilde{C}(fm) + \mu_f\tilde{C}(mf)\right)^{-\gamma_f} >\\ &\tilde{p}^{(m)}_{x+t}(1-\tilde{p}^{(f)}_{y+t})C(m)^{-\gamma_m} + \tilde{p}^{(f)}_{y+t}(1-\tilde{p}^{(m)}_{x+t})C(f)^{-\gamma_f}\\ &+\tilde{p}^{(m)}_{x+t}\tilde{p}^{(f)}_{y+t}\left(\left(\tilde{C}(mf) + \mu_m\tilde{C}(fm)\right)^{-\gamma_m} + \mu_f\left(\tilde{C}(fm) + \mu_f\tilde{C}(mf)\right)^{-\gamma_f}\right).\end{aligned} \tag{3.39}$$

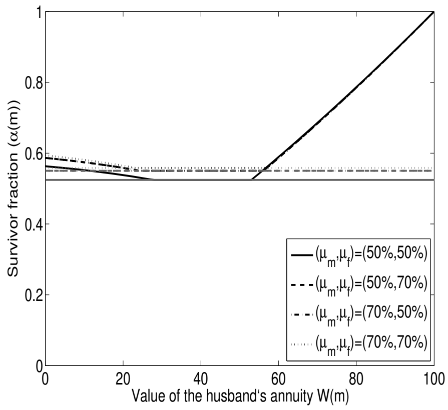
It can be verified that this condition is satisfied in all the numerical examples that we present for all $t \geq 0$. Equation (3.32) is not binding in all numerical examples considered for all $t \geq 0$. The derivation of the condition is presented in Appendix 3.C.3. We also present the more general conditions in case $r \neq \rho$. When (3.30) is maximized subject to (3.33) until (3.38), (3.39) also holds for all $t \geq 0$ in all numerical examples considered.



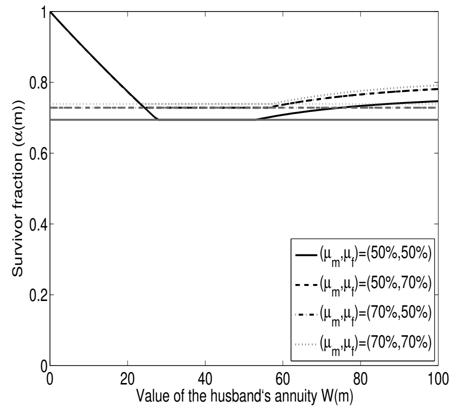
(a) husband



(b) wife



(c) husband



(d) wife

Figure 3.8 – The optimal survivor fraction as a function of the actuarial value of the husband's annuity portfolio ($W(m)$) for a man aged 65 and a wife aged 62 for the husband (a and c) and wife (b and d). In Figures (a) and (b) the degree of risk aversion differs and the degree of joint consumption is 50% for both the husband and the wife. Figures (c) and (d) the degree of joint consumption differs and the degree of risk aversion is 2 for both the husband and the wife. The actuarial value of the wife's annuity portfolio is given by $W(f) = 100 - W(m)$. The black lines are the survivor fractions under pre-accrued pension rights and the flat grey lines are the survivor fractions under flat-annuities. The interest rate is set at 3%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) with $\delta = 50\%$.

Let $\tilde{C}(mf), \tilde{C}(fm), C(m), C(f)$ be the optimal consumption for a couple. Then, suppose that only the husband's degree of risk aversion increases. Then, for the given $\tilde{C}(mf), \tilde{C}(fm), C(m), C(f)$, the husband's expected marginal utility and the couple's expected marginal utility decrease. In the optimum, all expected marginal utilities corrected for prices should be equal. So, in the optimum, either the wife's expected marginal utility should decrease, or both the husband's expected marginal utility and the couple's expected marginal utility should increase. Since the budget constraint should be satisfied in the optimum, $C(f)$ will increase, and $C(m)$ will decrease, and $\tilde{C}(mf)$ or $\tilde{C}(fm)$ will increase. Since the marginal utility with respect to $\tilde{C}(mf)$ should equal the marginal utility with respect to $\tilde{C}(fm)$, both will increase. When we compare the solid lines with the dashed (dashed-dotted) line in Figures 3.8a and 3.8b, we see indeed that when the wife's (husband's) degree of risk aversion increases, the husband's (wife's) optimal survivor fraction increases whereas the wife's (husband's) optimal survivor fraction decreases. Note that when the wife's degree of risk aversion is higher than the husband's degree of risk aversion (dashed lines), there is only a small region where the exchange restrictions are not binding due to the low optimal survivor fraction for the wife.

Equation (3.40) shows that the relation between the husband's and wife's marginal utility is not affected by a change in the degree of joint consumption of one of the spouses. When we compare the solid lines with the dashed lines in Figures 3.8c and 3.8d, we see indeed that when the wife's degree of joint consumption increases, both the husband's and the wife's optimal survivor fraction increase.

3.4 Welfare losses

In this section we analyze the welfare losses couples incur because of imperfections in the annuity markets. In Subsection 3.4.1 we quantify the relative importance of the different sources of market imperfection. In Subsection 3.4.2 we show how welfare losses are affected by the presence of a basic state pension. We also quantify the welfare losses in case the couple chooses the default option instead of exchanging pension rights optimally.

3.4.1 The effect of market incompleteness

Our benchmark for determining welfare losses is the case where a complete market with state- and time-dependent Arrow annuities is available, the annuity provider and the couple use the same survival probabilities, and all wealth is liquid. To quantify the relative contribution of different sources of market imperfections to the welfare losses that couples bear compared to the benchmark case, we first determine the welfare losses when the only market imperfection is that gender-neutral survival probabilities are used to price the annuities. Second, we determine the additional welfare losses due to the fact that, instead of state- and time-dependent annuities, only annuities with a flat benefit over time are available. Finally, we determine the additional welfare losses when couples have pre-accrued pension rights instead of liquid wealth.

To determine the welfare losses given market imperfections, we determine the couple's optimal life time utility L^* given the market imperfections, and given that total pension wealth is normalized at $W = 100$. We then determine the minimal wealth W^* needed to obtain the same lifetime expected utility L^* in the benchmark case where annuity markets are complete, $(p_t(m), p_t(f))$ equals $(p_t^*(m), p_t^*(f))$ for all t, and all wealth is liquid. We refer to W^* as the "annuity equivalent wealth" like in Brown and Poterba (2000). It is the minimal wealth required to be just as well off when there are no market restrictions as when market restrictions are imposed. If $W^* < W$ ($W^* > W$), less (more) wealth would be needed in absence of the restrictions to obtain the same lifetime expected utility, so the restrictions lead to welfare losses (gains). The following cases are analyzed:

- *Benchmark*: a complete market with state- and time-dependent Arrow annuities, the annuity provider and the couple use the same survival probabilities, and all wealth is liquid (solid line).

- *Gender neutral*: a complete market with state- and time-dependent Arrow annuities, gender-neutral survival probabilities with $\delta = 50\%$ are used for pricing, and all wealth is liquid (dashed line).

- *Flat annuities*: only annuities with a flat benefit level are available, gender-neutral survival probabilities with $\delta = 50\%$ are used for pricing, and all wealth is liquid (dotted line).

- *Pre-accrued pension rights*: only annuities with a flat benefit level are available, gender-neutral survival probabilities with $\delta = 50\%$ are used for pricing, and couples have pre-accrued pension wealth (dashed-dotted line).

Figure 3.9 displays the annuity equivalent wealth as a function of the actuarial value of annuity of the husband W. The annuity equivalent wealth of the *gender-neutral* case is higher than the annuity equivalent wealth in the *benchmark* case, indicating that couple's can profit from the gender-neutral pricing

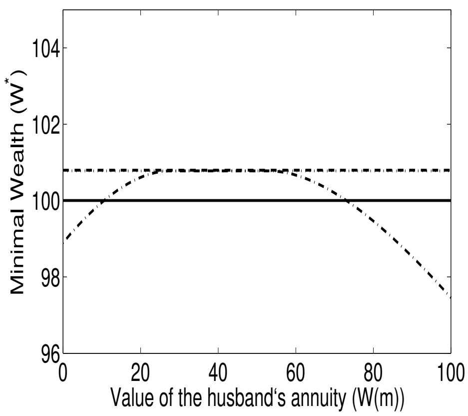


Figure 3.9 – The annuity equivalent wealth as a function of the actuarial value of the husband's annuity portfolio ($W(m)$), for a man aged 65 and a wife aged 62. The actuarial value of the wife's annuity portfolio is given by $W(f) = 100 - W(m)$. The discount rate is set at 3%. The degree of risk aversion γ equals 2 and the degree of joint consumption μ equals 50%. The solid line represents the *benchmark* case. The dashed line represents the *gender-neutral* case. The dotted line represents the *flat-annuities* case and the dashed-dotted line displays the *pre-accrued rights* case. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for $\delta = 50\%$.

through buying more annuities for the wife and less for the husband. In Section 3.2 we showed that when the annuities are priced gender-neutral, the optimal consumption pattern is not flat over time anymore. However, when we compare the minimal wealth of the *gender-neutral* case and the *flat annuities* case, we find that the annuity equivalent wealths are almost the same. So, although optimal consumption levels are not flat over time, the welfare loss of imposing the restriction that annuities should have a flat benefit level over time leads to only marginal welfare losses. When the couple has pre-accrued pension rights instead of liquid wealth, they bear additional welfare losses when one of the spouses has accrued significantly more pension wealth than the other spouse.

In previous sections we showed that the optimal consumption pattern depends strongly on the choice of the gender-neutral survival probabilities.

Figure 3.10 displays the annuity equivalent wealth for the pre-accrued rights case for different gender-neutral survival probabilities. The magnitude of the welfare losses depends strongly on the gender-neutral survival probabilities. For $\delta = 50\%$, the couple can gain from the favorable annuity prices for the

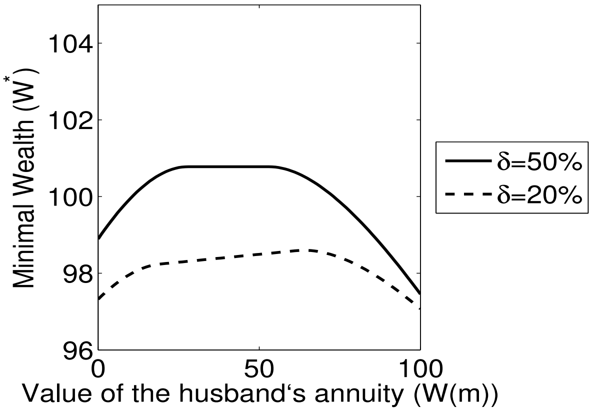


Figure 3.10 – The annuity equivalent wealth as a function of the actuarial value of the husband's annuity portfolio ($W(m)$), for a man aged 65 and a wife aged 62, in case of pre-accrued rights and gender-neutral pricing. The actuarial value of the wife's annuity portfolio is given by: $W(f) = 100 - W(m)$. The discount rate is set at 3%. The degree of risk aversion γ equals 2 and the degree of joint consumption μ equals 50%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for $\delta = 50\%$ (solid line), and for $\delta = 20\%$ (dashed line).

wife, whereas for $\delta = 20\%$, the couple bears a welfare loss compared to the *benchmark* case for all combinations of $(W(m), W(f))$. The difference in the annuity equivalent wealth for $\delta = 50\%$ and $\delta = 20\%$ ranges from 1.6 for $W(m) = 0$ to 2.5 for $W(m) = 26$ to 0.4 for $W(m) = 100$. When the husband has accrued all pension rights, the effect of the gender-neutral survival probabilities is small.

We have illustrated the welfare losses couples can incur because they cannot freely allocate their accrued pension rights over all types of annuities. When one of the spouses has accrued (almost) no pension rights, large welfare losses can be incurred. To avoid cases where one of the spouses has no after-retirement income, many countries have a basis state pension. In the next subsection, this state pension is introduced.

3.4.2 The effect of state pensions and default options

So far, the assumption was made that couples can exchange all their pre-accrued pension rights optimally in order to maximize their lifetime expected utility. In many countries, however, couples receive an illiquid basic state pension addition to their pre-accrued pension rights. So, they do not only have pre-accrued pension rights which can only be exchanged for a single-life or survivor annuity, but they also have accrued pension rights which cannot be exchanged at all.

Suppose that each spouse receives some state pension αSP annually when both spouses are alive, for some $\alpha \in [0,1]$, and SP annually when only one of the spouses is alive, independent of other sources of income. Now, a couple with pre-accrued pension wealth optimally exchanges their pension rights so as to maximize the lifetime expected utility, taking into account the presence of the state pension, i.e., they maximize:

$$\begin{aligned} L = 2\left[U(\frac{1}{2}C(mf) + \alpha SP, \frac{1}{2}C(mf) + \alpha SP)\right] \sum_{t=0}^{T} \frac{p_t(m)p_t(f)}{(1+\rho)^t} \\ + U(C(m) + SP, 0) \sum_{t=0}^{T} \frac{p_t(m)(1-p_t(f))}{(1+\rho)^t} \\ + U(C(f) + SP, 0) \sum_{t=0}^{T} \frac{(1-p_t(m))p_t(f)}{(1+\rho)^t}, \end{aligned} \tag{3.43}$$

subject to the budget constraints (3.25) until (3.29). Depending on the pre-accrued pension rights, the state pension may either be a large or only a small fraction of the after-retirement income of the couple. To reflect this, we choose values of the state pension SP such that the actuarial value of the state pension is x% of the actuarial value of the total pension wealth (actuarial value of their pre-accrued rights and of their state pension), which we normalize to 100. That is, for any given value of x, we determine SP such that

$$x = 2\alpha SP \times P^*_{joint} + SP \times P^*_{surv,m} + SP \times P^*_{surv,f}. \tag{3.44}$$

In the numerical results we set α equal to 0.75, and choose SP such that x is equal to 20 (small state pension), 50 (median state pension), or 80 (large state pension). The value of the husband's pre-accrued rights is $W(m) \in [0, 100 - x]$ and the value of the wife's pre-accrued right is $W(f) = 100 - x - W(m)$.

To determine the welfare losses/gains, we again determine the annuity equivalent wealth (W^*) which is needed in the *benchmark* case in order to obtain the same utility level as when the couple has pre-accrued pension rights and an illiquid basic state pension, and optimally exchanges the pre-accrued rights to maximize their lifetime expected utility (3.43). We display the annuity equivalent wealth (W^*) as function of the ratio of the husband's accrued pension rights ($W(m)$) over the couple's accrued pension rights ($W(m) + W(f)$).

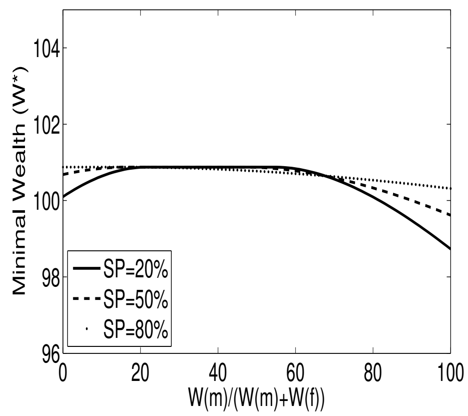


Figure 3.11 – The wealth equivalent as a function of the ratio of the husband's accrued pension rights ($W(m)$) over the couple's accrued pension rights ($W(m) + W(f)$) for a husband aged 65 and a wife aged 62. The results are displayed for various levels of the state pension. The actuarial value of the pre-accrued rights of the husband and wife are given by $W(m) \in [0, 100 - x]$, and $W(f) = 100 - x - W(m)$, respectively, where x is either 20 (solid line), 50 (dashed line), or 80 (dotted line). The discount rate is set at 3%. The degree of risk aversion γ equals 2 and the degree of joint consumption μ equals 70%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for $\delta = 50\%$.

In Figure 3.11 the welfare losses are displayed. Because of the state pension, the welfare losses have decreased, especially for those couples for who the state pension is an important source of their income. The reason for this is that because of the state pension, the extreme situations were one of the spouses had a much higher income than the other spouse have disappeared.

We have determined the welfare losses when the couple has an illiquid state pension next to the pre-accrued rights. So far, the assumption was made that the couple actively exchanges the pre-accrued rights optimally to maximize their lifetime expected utility. However, some research supports the idea that couples not actively exchange their single-life and survivor annuities but simply opt for the default. This may lead to additional welfare losses. We determine the wealth equivalent from opting for the default instead of annuity markets are complete, the couple's survival probabilities are used for pricing, and all wealth is liquid. In the U.S., the most common default option is a single-life annuity with at least a 50% survivor annuity for married couples. We set the default at a single-life annuity with a 50% survivor annuity and then determine welfare losses because of opting for the default instead of

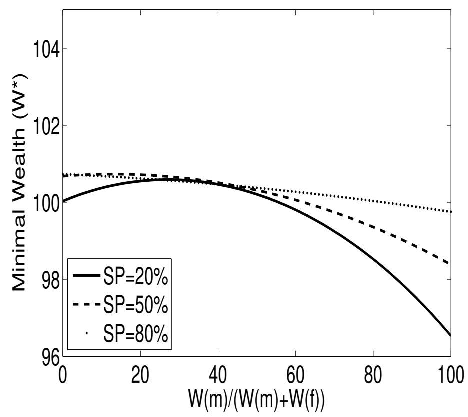


Figure 3.12 – The wealth equivalent of the default option of a single-life annuity with a 50% survivor annuity, as a function of the ratio of the husband's accrued pension rights ($W(m)$) over the couple's accrued pension rights ($W(m) + W(f)$) for a husband aged 65 and a wife aged 62. The results are displayed for various levels of the state pension. The actuarial value of the pre-accrued rights of the husband and wife are given by $W(m) \in [0, 100 - x]$, and $W(f) = 100 - x - W(m)$, respectively, where x is either 20 (solid line), 50 (dashed line), or 80 (dotted line). The discount rate is set at 3%. The degree of risk aversion γ equals 2 and the degree of joint consumption μ equals 70%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for $\delta = 50\%$.

exchanging optimally.

Figure 3.12 displays the wealth equivalent when the couple opts for the default instead of optimally exchanging. When couples opt for the default instead of exchanging optimally, they incur additional welfare losses. However, the size of the loss depends on the default option and on the importance of the state pension on the after retirement income. The welfare loss is the largest for those couples for which the state pension is a small part of their income, because the accrued pension rights form a big part of their after retirement income.

3.5 Conclusions

In this paper, we analyze the welfare losses couples bear due to imperfections in annuity markets. Our results suggest that welfare losses, due to the fact that only annuities with flat benefit levels over time are available, are relatively limited. In contrast, for couples who accrued pension rights in the form of a single-life annuity or a single-life with survivor annuity, restrictions on the exchange of these pension rights can induce significant welfare losses. The optimal annuity portfolio of couples with a relatively low level of joint consumption and relatively low degree of risk aversion are more likely to be affected by the exchange restrictions than the optimal annuity portfolio of couples with a higher degree of joint consumption or a higher degree of risk aversion. These welfare losses depend on the relative actuarial value of the pension rights accrued by the two spouses, on differences between survival probabilities used by the annuity provider and by the couple, and on whether the couple in addition holds illiquid pension wealth in the form of a state pension that cannot be exchanged for other types of longevity insurance. We find that welfare losses are significant for couples for which one of the spouses has accrued significantly more pension rights than the other spouse. We also find that the use of gender-neutral survival probabilities for pricing annuities has significant effect on these welfare losses. Finally, in case the couple is also entitled to receive an illiquid state pension, welfare losses become smaller for those couples for whom the state pension is an important source of their income after retirement.

3.A The degree of joint consumption

Consumption can be divided into two types: goods and services that can be publicly consumed by the household as a whole, and goods and services for which it is plausible that they are privately consumed by on or more of the household members. Public consumption contains: (1) expenditures on mortgages (rent and payment); (2) rent without expenditures on electricity and heating; (3) utilities (heating, electricity, water, telephone, internet, etc. but without insurances); (4) transportation costs; (5) insurances; (6) child care; (7) alimony and financial support to children who do not live at home; (8) expenditures to service debt; (9) trips and holidays with (part of) the family; (10) expenditures related to cleaning the house and gardening; and (12) other public expenditures not mentioned above. The private consumption contains: (1) food and drinks used at home; (2) food and drinks outside home; (3) cigarettes and other tobacco products; (4) clothing; (5) personal care and services; (6) medical expenditures not covered by an insurance; (7) leisure activities; (8) schooling; (9) gifts; and (10) other expenditures not mentioned above. All participants were asked how much they monthly spent on average on each of the above categories. Let C_{pri} denote the average monthly private consumption of a spouse and let C_{pub} denote the public consumption of the same household. We can than determine the degree of joint consumption of the household by solving the following equation:

$$(C_{pri} + \frac{1}{2}C_{pub})(1 + \mu) = C_{pri} + C_{pub}. \tag{3.45}$$

Each spouse pays only half of the public consumption of the household, but consumes all public consumption of the household.

3.B The cumulative survival probabilities of men and women

Throughout this paper, we use the one-year death rates differentiated with respect to age and gender from the Human Mortality Database of the U.S.A. for the years 2000 up to and including 2004.[15] Let $q_x^{(g)}$ denote the probability that an individual aged x with gender g dies within one year. The probability that an individual with gender g is alive over τ years conditional on being alive at age x is given by:

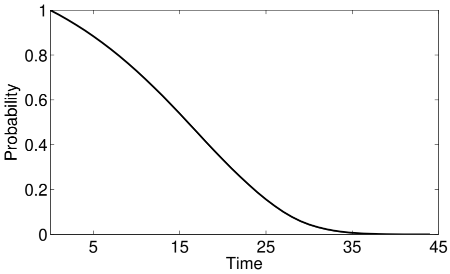
$$_\tau p_x^{(g)} = \prod_{v=1}^{\tau}(1 - q_{x+v-1}^{(g)})$$

Figure 3.13 displays the cumulative survival probabilities, conditional on being the husband being alive at age 65 , i.e., $_\tau p_{65}^{(m)}$, and the wife being alive at age 62, i.e., $_\tau p_{62}^{(f)}$, as a function of τ.

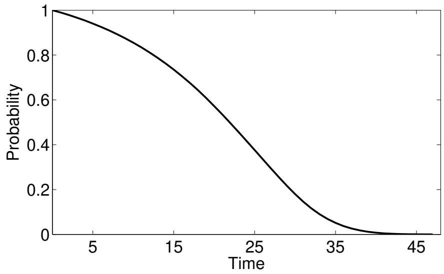
3.C The solutions of the maximization problems

In this Appendix, the maximization problems are solves analytically. In Subsection 3.C.1 we solve the maximization problem of Section 3.2 and in Subsection 3.C.2 we solve the maximization problem of Subsection 3.3.1. We solve the problems without taking into account that consumption should be non-negative in each state and in each period. Since the solutions yield non-negative consumption levels in each state, for each period, we do not have to take these constraints into account.

[15] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). Available at www.mortality.org or www.humanmortality.de (data downloaded on 05-01-2009).



(a) Men



(b) Women

Figure 3.13 – The cumulative survival probabilities, as a function of time for men (left) aged 65 and women (right) aged 62.

3.C.1 The maximization problem in case of complete annuity markets

In Section 3.2 (3.1) is maximized subject to the budget constraint (3.6) under the assumption that markets are complete, i.e., the couple can buy "Arrow annuities".

The objective function is given by:

$$
\begin{aligned}
L = & \sum_{t=0}^{T} 2U(\frac{1}{2}C_t(mf), \frac{1}{2}C_t(mf)) \frac{p_t(m)p_t(f)}{(1+\rho)^t} \\
& + \sum_{t=0}^{T} U(C_t(m), 0) \frac{p_t(m)(1-p_t(f))}{(1+\rho)^t} + \sum_{t=0}^{T} U(C_t(f), 0) \frac{(1-p_t(m))p_t(f)}{(1+\rho)^t} \\
& - \lambda \left(\sum_{t=0}^{T} \left(C_t(mf) \frac{p_t^*(m)p_t^*(f)}{(1+r)^t} + C_t(m) \frac{p_t^*(m)(1-p_t^*(f))}{(1+r)^t} \right.\right. \\
& \left.\left. + \ C_t(f) \frac{p_t^*(f)(1-p_t^*(m))}{(1+r)^t} \right) - \frac{W}{1+l} \right).
\end{aligned}
$$

The first order conditions are:

$$
\begin{aligned}
& \left[\frac{1}{2}(1+\mu)C_t(mf)\right]^{-\gamma} (1+\mu) \frac{p_t(m)p_t(f)}{(1+\rho)^t} = \lambda \frac{p_t^*(m)p_t^*(f)}{(1+r)^t}, \\
& C_t(g)^{-\gamma} \frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t} = \lambda \frac{p_t^*(g)(1-p_t^*(g'))}{(1+r)^t}, \text{ for } g, g' \in m, f, \\
& \frac{W}{1+l} = \sum_{t=0}^{T} \left(C_t(mf) \frac{p_t^*(m)p_t^*(f)}{(1+r)^t} + C_t(m) \frac{p_t^*(m)(1-p_t^*(f))}{(1+r)^t} \right. \\
& \left. + \ C_t(f) \frac{p_t^*(f)(1-p_t^*(m))}{(1+r)^t} \right),
\end{aligned}
$$

where λ is such that the budget constraint is satisfied. It can be verified that λ

is given by:

$$\lambda = \left(\frac{1+l}{W}\right)^{\gamma}$$
$$\left\{ \sum_t \left[\begin{array}{l} 2(1+\mu)^{\frac{1-\gamma}{\gamma}} \left(\frac{p_t(m)p_t(f)}{(1+\rho)^t}\right)^{\frac{1}{\gamma}} \left(\frac{p_t^*(m)p_t^*(f)}{(1+r)^t}\right)^{\frac{\gamma-1}{\gamma}} \\ + \left(\frac{p_t(m)(1-p_t(f))}{(1+\rho)^t}\right)^{\frac{1}{\gamma}} \left(\frac{p_t^*(m)(1-p_t^*(f))}{(1+r)^t}\right)^{\frac{\gamma-1}{\gamma}} \\ + \left(\frac{p_t(f)(1-p_t(m))}{(1+\rho)^t}\right)^{\frac{1}{\gamma}} \left(\frac{p_t^*(f)(1-p_t^*(m))}{(1+r)^t}\right)^{\frac{\gamma-1}{\gamma}} \end{array} \right] \right\}^{\gamma}.$$

When $\gamma = 1$, we have a log utility. This leads to the following first order conditions and optimal consumption:

$$\begin{aligned}
&\frac{2}{C_t(mf)}\frac{p_t(m)p_t(f)}{(1+\rho)^t} = \lambda\frac{p_t^*(m)p_t^*(f)}{(1+r)^t}, \\
&\frac{1}{C_t(g)}\frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t} = \lambda\frac{p_t^*(g)(1-p_t^*(g'))}{(1+r)^t}; \text{ for } g, g' \in m, f \\
&\frac{W}{1+l} = \sum_{t=0}^{T}\left(C_t(mf)\frac{p_t^*(m)p_t^*(f)}{(1+r)^t}\right. \\
&\left. + C_t(m)\frac{p_t^*(m)(1-p_t^*(f))}{(1+r)^t} + C_t(f)\frac{p_t^*(f)(1-p_t^*(m))}{(1+r)^t}\right), \\
&C_t(mf) = \frac{2}{\lambda}\frac{p_t(m)p_t(f)}{p_t^*(m)p_t^*(f)}\frac{(1+r)^t}{(1+\rho)^t}, \\
&C_t(g) = \frac{1}{\lambda}\frac{p_t(g)(1-p_t(g'))}{p_t^*(g)(1-p_t^*(g'))}\frac{(1+r)^t}{(1+\rho)^t}.
\end{aligned}$$

So, for a degree of risk aversion of 1, the optimal consumption pattern is independent of the degree of joint consumption. It can be verified that λ is given by:

$$\lambda = \frac{1+l}{W}\left[\sum_t\left(2\frac{p_t(m)p_t(f)}{(1+r)^t} + \frac{p_t(m)(1-p_t(f))}{(1+r)^t} + \frac{(1-p_t(m))p_t(f)}{(1+r)^t}\right)\right].$$

The maximization problem for general utility functions

In this subsection we solve the problem for general utility functions. We consider the case where, for all periods $t \in \{0, \cdots, T\}$, the utility that a spouse (husband or wife) derives from consuming C in period t, when the other spouse consumes C' in that period is given by:

$$U(C, C') = u\left(C + \mu C'\right),$$

where μ is the degree of joint consumption, and u is a utility function that satisfies $u'(x) > 0$ and $u''(x) < 0$. Further, $\lim_{x\to 0} u'(x) = \infty$ and $\lim_{x\to\infty} u'(x) = 0$. It can be verified that maximizing $U(C, C') + U(C', C)$ subject to $C + C' = C_t(mf)$ yields $C = C' = C_t(mf)/2$. Therefore, The couple's lifetime expected utility is given by:

$$L = \sum_{t=0}^{T} 2u(\frac{1}{2}(1+\mu)C_t(mf))\frac{p_t(m)p_t(f)}{(1+\rho)^t} + \sum_{t=0}^{T} u(C_t(m))\frac{p_t(m)(1-p_t(f))}{(1+\rho)^t}$$
$$+ \sum_{t=0}^{T} u(C_t(f))\frac{(1-p_t(m))p_t(f)}{(1+\rho)^t}.$$

Define $\tilde{C}_t = \frac{1}{2}(1+\mu)C_t(mf)$, i.e., $\tilde{C}_t$ is the consumption needed in case only one of the spouses is alive to obtain the same utility level per person as when both spouses are alive and consume $C_t(mf)$. The first order conditions are given by:

$$(1+\mu)u'(\tilde{C}_t)\frac{p_t(m)p_t(f)}{(1+\rho)^t} = \lambda\frac{p_t^*(m)p_t^*(f)}{(1+r)^t},$$
$$u'(C_t(g))\frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t} = \lambda\frac{p_t^*(g)(1-p_t^*(g'))}{(1+r)^t}, \text{ for } g, g' \in \{m, f\}.$$

The optimal consumption levels are given by:

$$C_t(mf) = \frac{2}{1+\mu}(u')^{-1}\left(\lambda\frac{P^*_{joint}(t)}{P_{joint}(t)}\frac{1}{1+\mu}\right),$$
$$C_t(g) = (u')^{-1}\left(\lambda\frac{P^*_{surv,g}(t)}{P_{surv,g}(t)}\right), \text{ for } g \in \{m, f\},$$

where λ is such that the budget constraint is satisfied. The optimal survivor fraction is given by:

$$\alpha_t(g) = \frac{(u')^{-1}\left(\lambda\frac{P^*_{surv,g}(t)}{P_{surv,g}(t)}\right)}{\frac{2}{1+\mu}(u')^{-1}\left(\lambda\frac{P^*_{joint}(t)}{P_{joint}(t)}\frac{1}{1+\mu}\right)}, \text{ for } g \in \{m, f\}.$$

3.C.2 The maximization problem in case of incomplete annuity markets

In Subsection 3.3.1 the restriction is imposed that only annuities with a flat benefit level are available. Consequently, the couple's lifetime expected utility is given by:

$$L = 2U(\frac{1}{2}C(mf), \frac{1}{2}C(mf)) \sum_{t=0}^{T} \frac{p_t(m)p_t(f)}{(1+\rho)^t} + U(C(m),0) \sum_{t=0}^{T} \frac{p_t(m)(1-p_t(f))}{(1+\rho)^t} + U(C(f),0) \sum_{t=0}^{T} \frac{(1-p_t(m))p_t(f)}{(1+\rho)^t}. \tag{3.46}$$

The lifetime expected utility is maximized subject to the budget constraint:

$$\frac{W}{1+l} = \sum_{t=0}^{T} \left(C(mf)\frac{p_t^*(m)p_t^*(f)}{(1+r)^t} + C(m)\frac{p_t^*(m)(1-p_t^*(f))}{(1+r)^t} + C(f)\frac{p_t^*(f)(1-p_t^*(m))}{(1+r)^t} \right). \tag{3.47}$$

The first order conditions are given by:

$$\left[\frac{1}{2}(1+\mu)C(mf)\right]^{-\gamma} (1+\mu) \sum_{t=0}^{T} \frac{p_t(m)p_t(f)}{(1+\rho)^t} = \lambda \sum_{t=0}^{T} \frac{p_t^*(m)p_t^*(f)}{(1+r)^t},$$

$$C(g)^{-\gamma} \sum_{t=0}^{T} \frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t} = \lambda \sum_{t=0}^{T} \frac{p_t^*(g)(1-p_t^*(g'))}{(1+r)^t}, \text{ for } g, g' \in m, f.$$

The solution is given by:

$$C(mf) = 2\,(1+\mu)^{\frac{1-\gamma}{\gamma}} \left[\frac{\sum_t \frac{p_t(m)p_t(f)}{(1+\rho)^t}}{\lambda \sum_t \frac{p_t^*(m)p_t^*(f)}{(1+r)^t}} \right]^{\frac{1}{\gamma}},$$

$$C(g) = \left[\frac{\sum_t \frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t}}{\lambda \sum_t \frac{p_t^*(g)(1-p_t^*(g'))}{(1+r)^t}} \right]^{\frac{1}{\gamma}},$$

$$\alpha(g) = \frac{1}{2}\,(1+\mu)^{\frac{\gamma-1}{\gamma}} \left[\frac{\sum_t \frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t}}{\sum_t \frac{p_t^*(g)(1-p_t^*(g'))}{(1+r)^t}} \frac{\sum \frac{p_t^*(g)p_t^*(g')}{(1+r)^t}}{\sum_t \frac{p_t(g)p_t(g')}{(1+\rho)^t}} \right]^{\frac{1}{\gamma}}, \tag{3.48}$$

where λ is such that the budget constraint is satisfied. It can be verified that λ is given by:

$$\lambda = \left(\frac{1+l}{W}\right)^{\gamma} \left\{ \begin{array}{l} 2(1+\mu)^{\frac{1-\gamma}{\gamma}} \left(\sum_t \frac{p_t(m)p_t(f)}{(1+\rho)^t}\right)^{\frac{1}{\gamma}} \left(\sum_t \frac{p_t^*(m)p_t^*(f)}{(1+r)^t}\right)^{\frac{\gamma-1}{\gamma}} \\ + \left(\sum_t \frac{p_t(m)(1-p_t(f))}{(1+\rho)^t}\right)^{\frac{1}{\gamma}} \left(\sum_t \frac{p_t^*(m)(1-p_t^*(f))}{(1+r)^t}\right)^{\frac{\gamma-1}{\gamma}} \\ + \left(\sum_t \frac{p_t(f)(1-p_t(m))}{(1+\rho)^t}\right)^{\frac{1}{\gamma}} \left(\sum_t \frac{p_t^*(f)(1-p_t^*(m))}{(1+r)^t}\right)^{\frac{\gamma-1}{\gamma}} \end{array} \right\}^{\gamma}.$$

The maximization problem for general utility functions

In this Appendix we solve the problem for general utility functions. We use the definition of $U(C, C')$ as in Appendix 3.C.1. The lifetime expected utility is given by:

$$L = 2u(\frac{1}{2}C(mf)) \sum_{t=0}^{T} \frac{p_t(m)p_t(f)}{(1+\rho)^t} + u(C(m)) \sum_{t=0}^{T} \frac{p_t(m)(1-p_t(f))}{(1+\rho)^t} + u(C(f)) \sum_{t=0}^{T} \frac{(1-p_t(m))p_t(f)}{(1+\rho)^t}. \quad (3.49)$$

Define $\tilde{C} = \frac{1}{2}(1+\mu)C(mf)$, i.e., $\tilde{C}$ is the consumption needed in case only one of the spouses is alive to obtain the same utility level per person as when both spouses are alive and consume $C(mf)$. The first order conditions are given by:

$$(1+\mu)u'(\tilde{C}) \sum_{t=0}^{T} \frac{p_t(m)p_t(f)}{(1+\rho)^t} = \lambda \sum_{t=0}^{T} \frac{p_t^*(m)p_t^*(f)}{(1+r)^t},$$
$$u'(C(g)) \sum_{t=0}^{T} \frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t} = \lambda \sum_{t=0}^{T} \frac{p_t^*(g)(1-p_t^*(g'))}{(1+r)^t}, \text{ for } g, g' \in \{m, f\}.$$

The optimal consumption levels are given by:

$$C(mf) = \frac{2}{1+\mu}(u')^{-1}\left(\lambda \frac{P^*_{joint}}{P_{joint}} \frac{1}{1+\mu}\right),$$
$$C(g) = (u')^{-1}(\lambda \frac{P^*_{surv,g}}{P_{surv,g}}), \text{ for } g \in \{m, f\}.$$

The optimal survivor fraction is given by:

$$\alpha(g) = \frac{(u')^{-1}(\lambda \frac{P^*_{surv,g}}{P_{surv,g}})}{\frac{2}{1+\mu}(u')^{-1}\left(\lambda \frac{P^*_{joint}}{P_{joint}} \frac{1}{1+\mu}\right)}, \text{ for } g \in \{m, f\}.$$

Conditions under which it is optimal not to save

In this Appendix we determine under which conditions the couple will not save in the complete market case. We assume that the couple fully annuitizes in period $t = 0$ and can consume and save in all periods. Optimizing (3.46) subject to (3.47) yields the optimal consumption pattern under the assumption that the couple consumes all annuity income in each period, i.e., it ignores the possibility to change the consumption pattern through saving. Let $u^{(m)}$,

$u^{(f)}$ and $u^{(c)}$ be the utility of the husband when only he is alive, the wife when only she is alive, and the couple when both spouses are alive. Then, the corresponding utilities are given by:

$$u^{(m)}(C) = U(C,0) = \frac{C^{1-\gamma}}{1-\gamma},$$
$$u^{(f)}(C) = U(C,0) = \frac{C^{1-\gamma}}{1-\gamma},$$
$$u^{(c)}(C) = 2U(\frac{1}{2}C, \frac{1}{2}C) = 2\frac{\left[\frac{1}{2}(1+\mu)C\right]^{1-\gamma}}{1-\gamma}.$$

Because the objective function is concave, and the constraint set is convex, the first order conditions are necessary and sufficient. For any given given annuity income $(A(mf), A(f), A(m))$, where $A(mf)$ denotes the annuity income when both spouses are alive, $A(f)$ denotes the annuity income when only the wife is alive, and $A(m)$ denotes the annuity income when only the husband is alive, the first order conditions for the optimal consumption pattern are given by:

$$\frac{\partial u^{(m)}}{\partial C}(C_t(m)) = \frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\frac{\partial u^{(m)}}{\partial C}(C_{t+1}(m)) + \nu_{s,t}^{(m)}, \tag{3.50}$$
$$\frac{\partial u^{(f)}}{\partial C}(C_t(f)) = \frac{1+r}{1+\rho}\frac{p_{t+1}(f)}{p_t(f)}\frac{\partial u^{(f)}}{\partial C}(C_{t+1}(f)) + \nu_{s,t}^{(f)}, \tag{3.51}$$
$$\frac{\partial u^{(c)}}{\partial C}(C_t(mf)) = \frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)}\frac{\partial u^{(c)}}{\partial C}(C_{t+1}(mf))$$
$$+ \frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)}\frac{\partial u^{(m)}}{\partial C}(C_{t+1}(m))$$
$$+ \frac{1+r}{1+\rho}\frac{p_{t+1}(f)}{p_t(f)}\frac{(1-p_{t+1}(m))}{p_t(m)}\frac{\partial u^{(f)}}{\partial C}(C_{t+1}(f)) + \nu_{s,t}^{(mf)}, \tag{3.52}$$
$$W_t(l_t) + A(l_t) - C_t(l_t) - S_t(l_t) \geq 0, \text{ for } l_t \in \{m, f, mf\}, \tag{3.53}$$
$$W_{t+1}(l_{t+1}) = (1+r)S_t(l_t), \text{ for } l_t \in \{m, f, mf\}, \tag{3.54}$$
$$\tag{3.55}$$

where $\nu_{s,t}^{(m)}$, $\nu_{s,t}^{(f)}$, and $\nu_{s,t}^{(mf)}$ are the corresponding Lagrange multipliers with respect to the nonnegativity constraint of savings. $\nu_{s,t}^{(l_t)}$ is positive when there are no savings in period t and zero else for $l_t \in \{m, f, mf\}$. $W_t(l_t)$ denotes the wealth at time t, and $S_t(l_t)$ are the savings at time t for $l_t \in \{m, f, mf\}$.

Because the objective function in strictly concave and the constraint set is convex, there is a unique optimum. Therefore, the consumption pattern

$(C(mf), C(f), C(m))$ that follows from optimizing (3.46) subject to (3.47) is optimal if saving is allowed if it satisfies the first order conditions with $(A(mf), A(f), A(m)) = (C(mf), C(f), C(m))$, $W_t(l_t) = S_t(l_t) = 0$ and $\nu_{s,t}^{(l_t)} = 0$ for all $t \geq 0$ and $l_t \in \{m, f, mf\}$. This yields the following conditions.

$$C(m)^{-\gamma} > \left(\frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\right) C(m)^{-\gamma}, \tag{3.56}$$

$$C(f)^{-\gamma} > \left(\frac{1+r}{1+\rho}\frac{p_{t+1}(f)}{p_t(f)}\right) C(f)^{-\gamma}, \tag{3.57}$$

$$\begin{aligned}(1+\mu)\left(\frac{1}{2}(1+\mu)C(mf)\right)^{-\gamma} > \left(\frac{1+r}{1+\rho}\right)\Bigg[&\frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)}C(m)^{-\gamma}\\ &+\frac{p_{t+1}(f)}{p_t(f)}\frac{(1-p_{t+1}(m))}{p_t(m)}C(f)^{-\gamma}\\ &+\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)}(1+\mu)\left(\frac{1}{2}(1+\mu)C(mf)\right)^{-\gamma}\Bigg].\end{aligned} \tag{3.58}$$

Inequalities (3.56) and (3.57) are fulfilled in all numerical examples we consider since $\rho = r$ and both $\frac{p_{t+1}(m)}{p_t(m)} < 1$ and $\frac{p_{t+1}(f)}{p_t(f)} < 1$ for all $t \geq 0$. We simplify condition (3.58) by dividing both sides by $C(mf)^{-\gamma}$ such that we obtain:

$$\begin{aligned}(1+\mu)\left(\frac{1}{2}(1+\mu)\right)^{-\gamma} > \left(\frac{1+r}{1+\rho}\right)\Bigg[&\frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)}\left(\frac{C(m)}{C(mf)}\right)^{-\gamma}\\ &+\frac{p_{t+1}(f)}{p_t(f)}\frac{(1-p_{t+1}(m))}{p_t(m)}\left(\frac{C(f)}{C(mf)}\right)^{-\gamma}\\ &+\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)}(1+\mu)\left(\frac{1}{2}(1+\mu)\right)^{-\gamma}\Bigg].\end{aligned} \tag{3.59}$$

Note that $\alpha(m) = \frac{C(m)}{C(mf)}$ and $\alpha(f) = \frac{C(f)}{C(mf)}$, such that we obtain:

$$\begin{aligned}(1+\mu)\left(\frac{1}{2}(1+\mu)\right)^{-\gamma} > \left(\frac{1+r}{1+\rho}\right)\Bigg[&\frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)}\alpha(m)^{-\gamma}\\ &+\frac{p_{t+1}(f)}{p_t(f)}\frac{(1-p_{t+1}(m))}{p_t(m)}\alpha(f)^{-\gamma}+\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)}(1+\mu)\left(\frac{1}{2}(1+\mu)\right)^{-\gamma}\Bigg].\end{aligned} \tag{3.60}$$

We can replace $\alpha(m)$ and $\alpha(f)$ using equation (3.48). In the numerical examples considered
$(1+\mu)\left(\frac{1}{2}(1+\mu)\right)^{-\gamma} > 1$, $\alpha(m) \leq 1$, and $\alpha(f) \leq 1$. Further $\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)} + \frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)} + \frac{(1-p_{t+1}(m))}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)} < 1$, such that for $\alpha(m) = 1$ and $\alpha(f) = 1$

the condition is fulfilled. In all our numerical examples, the conditions are fulfilled for all t (including $t = 0$).

3.C.3 The maximization problem for asymmetric utility functions

In this appendix we derive the first order conditions when the husband and wife have different degrees of risk aversion and joint consumption. Optimizing (3.30) subject to (3.31) yields the optimal consumption pattern under the assumption that the couple consumes all annuity income in each period, i.e., it ignores the possibility to change the consumption pattern through saving. The first order conditions are given by:

$$\left[\left(\tilde{C}(mf) + \mu_m \tilde{C}(fm)\right)^{-\gamma_m} + \mu_f \left(\tilde{C}(fm) + \mu_f \tilde{C}(mf)\right)^{-\gamma_f}\right] \sum_{t=0}^{T} \frac{p_t(m)p_t(f)}{(1+\rho)^t}$$
$$= \lambda P^*_{joint} + \nu_{\tilde{C}_{mf}},$$
$$\left[\mu_m \left(\tilde{C}(mf) + \mu_m \tilde{C}(fm)\right)^{-\gamma_m} + \left(\tilde{C}(fm) + \mu_f \tilde{C}(mf)\right)^{-\gamma_f}\right] \sum_{t=0}^{T} \frac{p_t(m)p_t(f)}{(1+\rho)^t}$$
$$= \lambda P^*_{joint} + \nu_{\tilde{C}_{fm}},$$
$$C(g)^{-\gamma_g} \sum_{t=0}^{T} \frac{p_t(g)(1-p_t(g'))}{(1+\rho)^t} = \lambda P^*_{surv,g}, \text{ for } g, g' \in m, f,$$

where $\nu_{\tilde{C}_{mf}}$ and $\nu_{\tilde{C}_{fm}}$ are the lagrange multipliers with respect to the non-negativity constraints of $\tilde{C}(mf)$ and $\tilde{C}(fm)$. In all numerical examples considered, $\nu_{\tilde{C}_{mf}}$ and $\nu_{\tilde{C}_{fm}}$ equal zero. We can again determine conditions under which it is optimal not to save. Let $u^{(m)}$, $u^{(f)}$ and $u^{(c)}$ be the utility of the husband when only he is alive, the wife when only she is alive, and the couple when both spouses are alive. Then, the corresponding utilities are given by:

$$u^{(m)}(C) = \frac{C^{1-\gamma_m}}{1-\gamma_m},$$
$$u^{(f)}(C) = \frac{C^{1-\gamma_f}}{1-\gamma_f},$$
$$u^{(c)}(C, C') = \frac{(C + \mu_m C')^{1-\gamma_m}}{1-\gamma_m} + \frac{(C + \mu_f C')^{1-\gamma_f}}{1-\gamma_f}.$$

For any given given annuity income $(A(mf)\ A(m)\ A(f))$, where $A(mf)$ denotes the annuity income used to consume $\tilde{C}(mf) + \tilde{C}(fm)$, the first order conditions for the optimal consumption pattern are given by:

$$\frac{\partial u^{(m)}}{\partial C}(C_t(m)) = \frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\frac{\partial u^{(m)}}{\partial C}(C_{t+1}(m)) + \nu_{s,t}^{(m)}, \tag{3.61}$$

$$\frac{\partial u^{(f)}}{\partial C}(C_t(f)) = \frac{1+r}{1+\rho}\frac{p_{t+1}(f)}{p_t(f)}\frac{\partial u^{(f)}}{\partial C}(C_{t+1}(f)) + \nu_{s,t}^{(f)}, \tag{3.62}$$

$$\begin{aligned}\frac{\partial u^{(c)}}{\partial C}(\tilde{C}_t(mf)) &= \frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)}\frac{\partial u^{(c)}}{\partial C}(\tilde{C}_{t+1}(mf))\\ &+ \frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)}\frac{\partial u^{(m)}}{\partial C}(C_{t+1}(m))\\ &+ \frac{1+r}{1+\rho}\frac{p_{t+1}(f)}{p_t(f)}\frac{(1-p_{t+1}(m))}{p_t(m)}\frac{\partial u^{(f)}}{\partial C}(C_{t+1}(f)) + \nu_{s,t}^{mf},\end{aligned} \tag{3.63}$$

$$\begin{aligned}\frac{\partial u^{(c)}}{\partial C}(\tilde{C}_t(fm)) &= \frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)}\frac{\partial u^{(c)}}{\partial C}(\tilde{C}_{t+1}(fm))\\ &+ \frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)}\frac{\partial u^{(m)}}{\partial C}(C_{t+1}(m))\\ &+ \frac{1+r}{1+\rho}\frac{p_{t+1}(f)}{p_t(f)}\frac{(1-p_{t+1}(m))}{p_t(m)}\frac{\partial u^{(f)}}{\partial C}(C_{t+1}(f)) + \nu_{s,t}^{(mf)},\end{aligned} \tag{3.64}$$

$$W_t(l_t) + A(l_t) - C_t(l_t) - S_t(l_t) \geq 0, \text{ for } l_t \in \{m, f, mf\}, \tag{3.65}$$

$$W_{t+1}(l_{t+1}) = (1+r)S_t(l_t), \text{ for } l_t \in \{m, f, mf\}, \tag{3.66}$$

where $\nu_{s,t}^{(m)}$, $\nu_{s,t}^{(f)}$, and $\nu_{s,t}^{(mf)}$ are the corresponding Lagrange multipliers with respect to the nonnegativity constraint of savings. $\nu_{s,t}^{(l_t)}$ is positive when the couple does not save in period t and zero else for $l_t \in \{m, f, mf\}$. $W_t(l_t)$ denotes the wealth at time t, and $S_t(l_t)$ are the savings at time t for $l_t \in \{m, f, mf\}$.

Because the objective function in strictly concave and the constraint set is convex, there is a unique optimum. Therefore, the consumption pattern $(\tilde{C}(mf), \tilde{C}(fm), C(f), C(m))$ that follows from optimizing (3.30) subject to (3.31) is optimal if saving is allowed if it satisfies the first order conditions with $(A(mf), A(f), A(m)) = (\tilde{C}(mf) + \tilde{C}(fm), C(f), C(m))$, $W_t(l_t) = S_t(l_t) = 0$ for $t \geq 0$, and $\nu_{s,t}^{(l_t)} = 0$ for all $t \geq 0$ and $l_t \in \{m, f, mf\}$. This yields the following conditions.

$$C(m)^{-\gamma_m} > \left(\frac{1+r}{1+\rho}\frac{p_{t+1}(m)}{p_t(m)}\right)C(m)^{-\gamma_m}, \tag{3.67}$$

$$C(f)^{-\gamma_f} > \left(\frac{1+r}{1+\rho}\frac{p_{t+1}(f)}{p_t(f)}\right)C(f)^{-\gamma_f}, \tag{3.68}$$

$$\left(\tilde{C}(mf) + \mu_m\tilde{C}(fm)\right)^{-\gamma_m} + \mu_f\left(\tilde{C}(fm) + \mu_f\tilde{C}(mf)\right)^{-\gamma_f} >$$

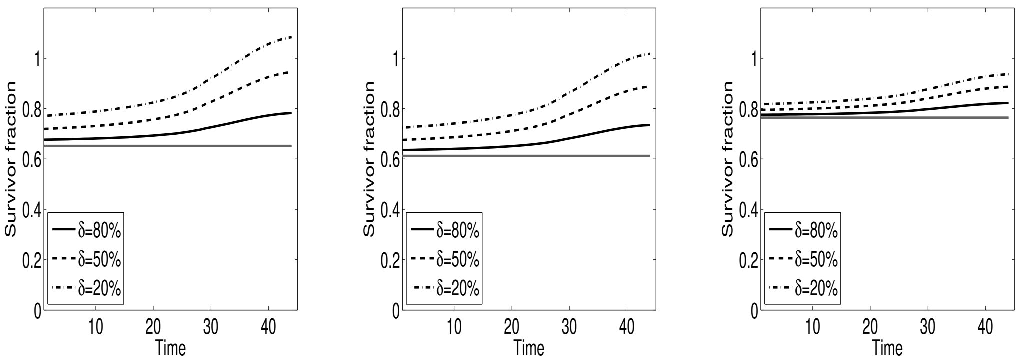
$$\left(\frac{1+r}{1+\rho}\right)\left[\frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)}C(m)^{-\gamma_m}+\frac{p_{t+1}(f)}{p_t(f)}\frac{(1-p_{t+1}(m))}{p_t(m)}C(f)^{-\gamma_f}\right.$$
$$\left.+\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)}\left(\left(\tilde{C}(mf)+\mu_m\tilde{C}(fm)\right)^{-\gamma_m}+\mu_f\left(\tilde{C}(fm)+\mu_f\tilde{C}(mf)\right)^{-\gamma_f}\right)\right], \tag{3.69}$$

$$\left(\tilde{C}(fm)+\mu_f\tilde{C}(mf)\right)^{-\gamma_f}+\mu_m\left(\tilde{C}(mf)+\mu_m\tilde{C}(fm)\right)^{-\gamma_m}>$$
$$\left(\frac{1+r}{1+\rho}\right)\left[\frac{p_{t+1}(m)}{p_t(m)}\frac{(1-p_{t+1}(f))}{p_t(f)}C(m)^{-\gamma_m}+\frac{p_{t+1}(f)}{p_t(f)}\frac{(1-p_{t+1}(m))}{p_t(m)}C(f)^{-\gamma_f}\right.$$
$$\left.+\frac{p_{t+1}(m)}{p_t(m)}\frac{p_{t+1}(f)}{p_t(f)}\left(\left(\tilde{C}(fm)+\mu_f\tilde{C}(mf)\right)^{-\gamma_f}+\mu_m\left(\tilde{C}(mf)+\mu_m\tilde{C}(fm)\right)^{-\gamma_m}\right)\right]. \tag{3.70}$$

Because the optimization problem is strict concave, the optimum is unique such that it is sufficient to show that these conditions are fulfilled in the optimums that we have found.

The left hand side and the right hand side of equations (3.69) and (3.70) are the same. Consequently, we only have to check one of the two equations. In all numerical examples considered, these conditions are satisfied for all $t \geq 0$.

3.D The wife's optimal survivor fraction



(a) $\gamma = 2, \mu = 70\%$ **(b)** $\gamma = 2, \mu = 50\%$ **(c)** $\gamma = 5, \mu = 70\%$

Figure 3.14 – The wife's optimal survivor fraction as a function of time, for a couple consisting of a husband and wife aged 65 and 62 respectively at the moment they buy the annuity, for three combinations of the degree of risk aversion γ and the degree of joint consumption μ: $\gamma = 2$ and $\mu = 70\%$ (Figure (a)), $\gamma = 2$ and $\mu = 50\%$ (Figure (b)), and $\gamma = 5$ and $\mu = 70\%$ (Figure (c)). The discount rate equals 3%. The survival probabilities $(p_t(m), p_t(f))$ used by the couple are age- and gender-specific population survival probabilities for U.S. individuals in the period $2000 - 2004$. The gender-neutral survival probabilities $(p_t^*(m), p_t^*(f))$ used by the annuity provider are determined from (3.13) and (3.14) for three values of δ. The black solid lines correspond to $\delta = 80\%$, the dashed lines correspond to $\delta = 50\%$, and the dotted lines correspond to $\delta = 20\%$. The grey lines represent the optimal consumption patterns for the case where $(p_t^*(m), p_t^*(f))$ equals $(p_t(m), p_t(f))$ for all t.

CHAPTER 4

ESTIMATING THE JOINT SURVIVAL PROBABILITIES OF MARRIED INDIVIDUALS

> "Some of my colleagues at the Department of Sociology in Helsinki wonder whether it is meaningfull to study mortality differences. After all, the death rate is the same for everyone: one death per person."
>
> Tapani Valkonem (1993)

4.1 Introduction

This chapter is based on Sanders and Melenberg (2011). In this paper we estimate the joint survival probabilities of married couples, taking into account the possible dependence between the remaining lifetimes of married individuals. The joint survival probabilities of spouses are important for determining the liabilities of pension providers, such as state retirement systems, pension and employee benefit funds, and insurance companies. There is a large literature emphasizing the importance of annuities (see for instance Yaari, 1965; Mitchell, Poterba, Warshawsky, and Brown, 1999; Davidoff, Brown, and Diamond, 2005; Horneff, Maurer, Mitchell, and Dus, 2006, to name just a few). For an overview of literature on annuities, see Cannon and Tonks (2008). Pension providers typically offer not only single-life annuities but also joint & survivor annuities. In the latter case, a periodic payment is made as long as at least one of the spouses is alive. Consequently, not only the marginal survival probabilities, but also the dependence between the remaining lifetimes of spouses may influence the demand for and the price of annuities. It is often claimed (see for instance Kotlikoff and Spivak, 1981; Brown and Poterba, 2000) that a couple may have less need for an annuity, because as a couple they can hedge part of the risk themselves, i.e., if one of the spouses becomes very old, he may outlive the other spouse and receive a bequest from the other spouse to finance (part of) his future consumption. However, when the remaining lifetimes of spouses are positively dependent, the hedging potential within a couple decreases, which may make an annuity more valuable for a couple. The price of an annuity is affected by the dependence between the marginal survival probabilities because it affects both the probability that both spouses are alive, and the probability that only one of the spouses is alive.

The joint survival probabilities of couples do not only affect the liabilities of pension providers, but also spatial planning, health care, and personal finances. As long as both spouses are alive they can provide informal care to each other, which may reduce the demand for formal care.[1] Spatial planning may also be affected because elderly couples may have different needs concerning housing than single elderly have. Further, elderly couples have to make decisions about saving, budgeting, and credit management. To plan savings adequately the joint survival probabilities of spouses are needed.

There are several reasons why the remaining lifetimes of spouses may be dependent. Youn and Shemyakin (1999) mention three of them. The first reason is the so-called "broken heart" syndrome, which is the impact of the death of one of the spouses on the mortality probability of the surviving spouse. Parkes, Benjamin, and Fitzgerald (1969) already found that mortality probabilities of widowers increase with 40% during the first six months of bereavement. They also found that mortality probabilities decreased relative to singles as of the fifth year of bereavement. So, they find positive dependence in the short run and negative dependence in the long run. Their results are based on a sample of 4486 widowers of 55 years and older who were followed for 10 years. The second reason why joint survival probabilities of spouses may be dependent is called "common disaster": two spouses are more likely to be involved in a disaster such as a car accident than two unrelated persons. Both reasons affect the time difference between the deaths of the spouses, but not directly the ages of death. To model the impact of the "common disaster"

[1] The availability of immediate family increases reliance on informal care and reduces reliance on formal care, see, e.g., Kemper (1992).

and "broken heart" syndrome, "common shock" models are used (see, for instance, Marshall and Olkin, 1988). The third reason is "common lifestyle": spouses are expected to have the same lifestyle (for instance both spouses may smoke) which influences their marginal survival probabilities in the same way. When spouses have the same lifestyle, their ages of death are expected to be correlated. The remaining lifetimes of spouses may be negatively dependent because of "complementariness of needs of husband and wife", i.e., individuals may select a mate with complementary needs, (see Winch, 1954). Opposed to the theory of heterogamy ("opposites attract each other") is the theory of homogamy, which states that "like attract likes" (see Hollingshead, 1950). In fact, as Burgess and Wallin (1943) already noted, both theories may be correct. Some couples may be attracted to each other by similarities and others by dissimilarities. Moreover, couples may be attracted to each other both by like and unlike characteristics, i.e., there may be similarities and dissimilarities between spouses.

Next to "common shock" models, copulas are also widely used to estimate joint survival probabilities. Different types of copulas have been used to estimate joint survival probabilities of spouses. For instance, Frees, Carriere, and Valdez (1996) focus on Frank's copula, whereas Carriere (2000) compares different marginal survival probabilities and different copulas to estimate bivariate survival probabilities. Youn and Shemyakin (1999) add an additional variable which captures the age difference between two spouses to estimate the joint survival probabilities. In a more recent paper, Luciano, Spreeuw, and Vigna (2008) estimate joint survival probabilities using copulas, finding that the best fit with the data is obtained with the so-called 4.2.20 Nelsen copula, which allows for positive dependence. A special feature of this copula is that the dependence is increasing with age. Not only copulas are used to estimate the dependence of the remaining lifetimes of couples, also Markovian models. Norberg (1989) and Wolthuis (1994) proposed a Markovian model with forces of mortality depending on marital status, see also, for instance, Denuit, Dhaene, Le Bailly de Tilleghem, and Teghem (2001). The latter authors use data from the Belgian National Institute of Statistics of the year 1991 to estimate the Markovian model. They also collected data from two cemeteries in Brussel. They found a weak positive correlation between the husband's and wife's remaining lifetimes. However, their data-set consists of only 533 couples. In a recent paper, Spreeuw and Wang (2008) extend the Markovian model by allowing the force of mortality to depend on the time of death of the spouse. Their extension of the model leads to more freedom which, in turn, may lead to a better fit of the data. However, a disadvantage of all these models remains that a specific dependence structure is imposed, leading to potential misspecification.

Frees, Carriere, and Valdez (1996), Carriere (2000), Youn and Shemyakin (1999), Luciano, Spreeuw, and Vigna (2008), and Spreeuw and Wang (2008) estimate the dependence based on a data-set of a Canadian insurer. This data-set contains information of around 15 thousand couples and the length of the observation period is limited to 5 years. Moreover, most couples enter the data-set at age 60 or later.[2] Additionally, only couples that have bought a joint & last survivor annuity enter this data-set. As a result, the data-set does not seem to be representative for the whole Canadian population. It is well-known that the marginal survival probabilities of annuitants tend to be higher than the population survival probabilities (see, for instance Mitchell, Poterba, Warshawsky, and Brown, 1999). It may be that not only the marginal survival probabilities differ, but also the dependence between the husband's and wife's remaining lifetime may differ. Couples where one of the spouses has deceased before retirement, or where one of the spouses is very ill, may buy a single-life annuity instead of a joint & survivor annuity or no annuity at all. As a consequence, these couples may be under-represented in the data-set of the Canadian insurer, leading to a potential overestimation of the dependence. Moreover, annuities are provided not only by insurance companies, but also (and mainly) by state retirement systems and pension funds. Although the Canadian data-set may be representative for couples who voluntarily buy annuities, it may not be representative for the couples who accrue annuities through the state or through pension funds. For these annuity providers, a data-set containing all individuals of the population may be more representative. Denuit, Dhaene, Le Bailly de Tilleghem, and Teghem (2001) use a different data-set. However, they have information of only one year from the Belgian National Institute of Statistics. The probability that both spouses die within the same calender year is very small. Consequently, they do not observe many couples where both spouses have died. The other data-set they use, contains information of only 533 couples.

Most of the models discussed above have a very limited number of parameters (also because only a very limited data-set was available). In contrast, we

[2]Chen, Cox, and Wen (2009) notice that in the data-set of the Canadian insurer, only very few people die above the age of 90. Therefore, they propose a model which uses extreme value theory to estimate the survival probabilities at high ages. They use the same data-set as Frees, Carriere, and Valdez (1996)

have a data-set containing the whole Dutch population. We have information on 3.9 million couples included in this data-set, who are followed for a period of 13 years. However, we base our results on a random sample of 50,000 couples. We use only a sub-sample because it takes too much computational time to estimate the likelihood on the whole census. We estimate the joint survival probabilities of spouses in a very flexible way, using the semi-nonparametric approach of Gallant and Nychka (1987) with two bivariate Weibull models serving as parametric benchmarks. The estimates are used to determine the period life expectancy. Because of the limited time period of the data available, we assume that the probability distribution does not change over time. However, our framework can be extended to allow for a cohort-dependent probability distribution.

The main results of the paper are the following. In the semi-nonparametric model, Spearman's rank correlation is well below the correlations found in for instance Frees, Carriere, and Valdez (1996), Carriere (2000),or Youn and Shemyakin (2001). We find that for all investigated ages, the remaining lifetimes of spouses are positively dependent. We prefer the more flexible approach because the dependence between the remaining lifetimes of spouses depends less on the structural form imposed. We find that the husband's and wife's life expectancy at birth is generally increasing with the other spouse's age of death. The different models are used to estimate actuarially fair annuity prices. We find that ignoring the possible dependence between the remaining lifetimes of spouses may lead to an underestimation of the value of a joint annuity for a couple of about five to ten percent, depending on the age as off which the annuity pays out. In addition, ignoring dependence may lead to an overestimation of the value of a survivor annuity on the husband's life 25 percent. When a joint & 70% survivor annuity is bought, these effects (partially) cancel out.

The remainder of this paper is organized as follows. In section 4.2 a description of the data-set is given. Section 4.3 describes the correlation between the remaining lifetimes we are interested in. Section 4.4 contains the methodology used to estimate the two bivariate Weibull models and their estimation results. In Section 4.5 the semi-nonparametric approach is explained and the results are shown. Section 4.6 determines the correlation between the remaining lifetimes of spouses. Section 4.7 estimates the value of joint & survivor annuities for the different models. Section 4.8 concludes.

4.2 Data Characteristics

In this section a description of the data used to estimate the joint survival probabilities of spouses is given. Data of the Dutch municipality registers (GBA) is used to estimate the joint survival probabilities of couples. The data-set we use is the census. Therefore, there is by construction no sample selection bias based on population choice when the whole population is of interest. In Subsection 4.2.1 we describe the data-set and in Subsection 4.2.2 we describe the sample actually used.

4.2.1 Description of the census

Our data-set consists of all individuals who are a member of the Dutch population from the period January 1995 until January 2008.[3] Of each individual we observe many characteristics like date of birth, gender, address, marital status, and ethnicity. At January 1995, the data-set has information of 7.6 million men and 7.8 million women (including children) of the 15.4 million Dutch citizens. For all these individuals we observe their marital status. In total, there are about 3.5 million married couples,[4] 157 thousand widowed men, and 720 thousand widowed women. The marital status of the remaining individuals is either "single" or "divorced". Individuals who live together as a couple but are not married to each other are not taken into account in the estimation. We restrict ourselves to the married couples of whom at least one of the spouses is alive at January 1, 1995. Consequently, we do not have any individuals younger than 16 in the data-set.[5]

Although the marital status of each individual is given, the data-set does not contain information about who is married to whom. Therefore, we couple the individuals ourselves. We assume that two individuals of opposite sex are married to each other when the marital status of both individuals is "married", they live at the same address at the same time, and no other married individual lives at that address at that time. This way, we match about 3.2 million couples. The following method is used to match some of the re-

[3]Individuals born in the Netherlands after January 1995 enter the data-set for in the year of their birth. Immigrants to the Netherlands enter after January 1995 enter the data-set in the year of their arrival.

[4]In this paper we refer to an individual as married either when the individual is married or when the individual has a civil partnership.

[5]In the Netherlands, the minimal age as of which an individual is allowed to marry is 18, unless there is a doctor's certificate which states that the wife is pregnant or has already had a baby. In that case, an individual is allowed to marry as of the age of 16.

maining couples: when more than two married individuals live at the same address at the same time, we assume that two married individuals of opposite sex are married to each other, when some of their personal characteristics change at the same day, for instance, if they move at the same moment in time and are also the only two who move, or when one dies and the other becomes widowed the same day, or when they get a child at the same day, and so on. About 74 thousand couples are matched using this additional method. In total we matched about 3.3 million couples. About 279 thousand couples of the 3.3 million are removed because they divorced during the observation period. One of the reasons we cannot couple all married individuals is that we only consider couples of opposite sex.

As stated before, some individuals enter the data-set with a marital status of "widowhood". Of these individuals we know that they were married before and that their spouse has deceased. However, we do not observe any characteristics of the former spouse. We assume that the widow(er) was married to an individual from opposite sex and died at age 18 or later. We add these 877 thousand widow(er)s with their former spouse to the data-set leading to a total of about 3.9 million couples.

All individuals enter the data-set at January 1995 and are observed for a maximum period of 13 years. Individuals mainly leave the data-set because of death, emigration, or because the end of the observation period is reached. Of the 3.9 million couples, 7.4 thousand men and 13.6 thousand women leave the data-set for an unspecified reason. Table 4.1 presents the frequency distribution of the couples by sex, entry age, and mortality status. Three mortality statuses are distinguished: individuals can die either *before* the observation period started (B), *during* the observation period (D), or *after* the observation period ended (A).[6]

[6]Note that we do not observe the entry age of individuals who died prior to the start of the observation period.

Mortality status		Entry age husband				Entry age Wife				
Husband	Wife	< 50	$50-65$	$65-80$	$80+$	< 50	$50-65$	$65-80$	$80+$	Total
D	D	1.9	18.6	104.7	45.8	3.3	26.8	110.6	30.3	170.9
		0.05	*0.47*	*2.67*	*1.17*	*0.08*	*0.68*	*2.82*	*0.77*	*4.35*
B	D	-	-	-	-	1.3	17.8	173.5	217.5	410.1
						0.03	*0.45*	*4.42*	*5.54*	*10.44*
D	B	0.5	7.1	48.9	48.5	-	-	-	-	105.0
		0.01	*0.18*	*1.24*	*1.23*					*2.67*
A	D	29.3	52.3	50.0	2.8	35.5	53.1	43.4	2.3	134.3
		0.75	*1.33*	*1.27*	*0.07*	*0.90*	*1.35*	*1.10*	*0.06*	*3.42*
D	A	43.3	130.4	185.0	19.0	64.8	158.7	149.3	5.0	377.7
		1.10	*3.32*	*4.70*	*0.48*	*1.65*	*4.04*	*3.80*	*0.13*	*9.61*
B	A	-	-	-	-	23.3	91.8	164.7	19.6	299.4
						0.59	*2.34*	*4.19*	*0.50*	*7.62*
A	B	5.5	16.3	19.0	1.9	-	-	-	-	42.7
		0.14	*0.41*	*0.48*	*0.05*					*1.09*
A	A	1528.8	689.6	167.5	1.9	1,691.7	587.8	107.8	0.6	2387.9
		38.92	*17.56*	*4.26*	*0.05*	*43.07*	*14.96*	*2.74*	*0.02*	*60.79*

Table 4.1 – Mortality status and entry age of all couples in the data-set by thousand couples. The Italic numbers represent the size of the group as percentage of all couples. Both husband and wife can die either *before* the observation period started (B), *during* the observation period (D) or *after* the observation period ended (A).

We find that of 171 thousand couples both spouses die within our observation period. Furthermore, of 134 (378) thousand couples the wife (husband) dies during the observation period, while the husband (wife) is still alive at the end of the observation period. Of most couples (2.4 million), both spouses survive the entire observation period. Of the individuals who enter the data-set as widow (widower), 410 (105) thousand die during the observation period and the remaining 299 (43) thousand survive the observation period. From these population statistics we can see that by far the largest part of our data is left truncated and right-censored.

We investigate in more detail the couples for whom we observed both the husband's and wife's age of death. In Table 4.2 the differences in time of deaths are displayed for this sub-population of couples.

Time difference between the two deaths	Number of couples	As percentage of couples for whom we observed both ages of death	As percentage of all couples
At most 1 day	340	0.18	0.01
At most 1 week	943	0.52	0.02
At most 1 month	3,482	1.93	0.09
At most 1 quarter	8,615	4.79	0.22
At most a half year	15,778	8.78	0.40
At most 1 year	29,129	16.20	0.74
At most 2 years	53,980	30.02	1.37
At most 5 years	114,750	63.82	2.92
More than 5 years	65,036	36.17	1.66

Table 4.2 – The time between the death of the two spouses of all couples of whom both spouses died during the observation period and the corresponding percentage of all couples in the data-set.

We find that of 340 couples (0.0087% of all couples in the data), both spouses die at the same day. In slightly more than 0.7% of the sample, both spouses die within one year. However, in contrast with this are the 342.1 thousand couples of whom one of the spouses died before the observation period started and the other spouse died or will die after the observation period ended. Of these couples the time of death is typically more than 13 years apart.[7] Note that the time between deaths of the two spouses of all couples who were still alive at the end of the observation period can also be very short.

4.2.2 Description of the sample actually used

Estimating the joint survival probabilities of spouses using the whole data-set requires much computational time. Therefore, we take a random sample of 50,000 couples and construct a confidence interval for, among others, the life expectancy of the two spouses at birth and the correlation between the lifetimes of spouses at birth. To construct the random sample we draw 50,000 couples with replacement from the data-set.

[7] Some of the individuals may leave the data-set before the 13-year observation period has ended because of, for instance, emigration. However, this is a very small part of the data-set.

Mortality status		Entry age husband				Entry age Wife				
Husband	Wife	< 50	$50-65$	$65-80$	$80+$	< 50	$50-65$	$65-80$	$80+$	Total
D	D	0.1	0.5	2.6	1.2	0.1	0.8	2.7	0.8	4.3
B	D	–	–	–	–	0.1	0.5	4.4	5.6	10.5
D	B	0.0	0.2	1.3	1.3	–	–	–	–	2.7
A	D	0.7	1.4	1.3	0.1	0.9	1.4	1.1	0.1	3.5
D	A	1.2	3.2	4.7	0.5	1.7	4.0	3.8	0.1	9.6
B	A	–	–	–	–	0.6	2.3	4.2	0.5	7.6
A	B	0.1	0.5	0.5	0.0	–	–	–	–	1.1
A	A	38.8	17.5	4.3	0.0	42.9	15.0	2.7	0.0	60.6
Total		41.3	25.3	22.4	11.1	46.4	24.7	20.9	8.0	100.0

Table 4.3 – Mortality status and entry age of a sample of 50, 000 couples in the dataset in percentages. Both husband and wife can die either *before* the observation period started (B), *during* the observation period (D) or *after* the observation period ended (A).

Table 4.3 displays a brief description of the sample, consisting of 1.3% of the couples of the the census used to estimate the joint survival probabilities of spouses. Because of the large sample size, there are many couples in most of the subgroups. If we compare the numbers in Table 4.3 with the Italic numbers in Table 4.1, we see that we the composition of our sample is comparable with the composition of the whole data-set.

4.3 Dependence between the remaining lifetimes of spouses

In this Section we introduce some notation. In Subsection 4.3.1 we discuss the left-truncation and right-censoring which occurs in our data-set. In Subsection 4.3.2 we discuss some notions of dependence. First we introduce some notation:

- $(X, Y)_i$ for the bivariate age-at-death random variable of couple i.
- $T(x)$ for the remaining lifetime of an individual aged x. Consequently, $T(x) = X - x$.
- min_{x_i} and min_{y_i} for the minimal age of death of the husband and wife, respectively.
- a_{x_i} for the entry age of the husband of couple i and a_{y_i} for the entry age of the corresponding wife in case the individual was alive at the beginning of the observation period. We set $a_{x_i} = min_{x_i}$ ($a_{y_i} = min_{y_i}$) in case the marital status of the wife (husband) of couple i is "widowhood" at the beginning of the observation period.
- b_{x_i} and b_{y_i} for the length of the observation period in the census of the husband and wife of couple i, respectively. We set $b_{x_i} = 0$ ($b_{y_i} = 0$) in case the marital status of the wife (husband) of couple i is "widowhood" at the beginning of the observation period.

Like in Frees, Carriere, and Valdez (1996), we consider models which assume that the remaining lifetimes of spouses are correlated at birth. Carriere (2000) criticized such models because it may be unrealistic that the lifetimes are dependent at birth since the two spouses have not even met. Therefore, Carriere (2000) proposes to couple the remaining lifetimes of spouses at the moment the individuals marry. However, in our data-set the moment at which individuals marry is typically not observed. Moreover, some couples may have a very long relationship before they actually marry whereas other couples may only have a short history together. Either way, survival probabilities are likely to be dependent before both individuals marry, meaning that the actual coupling should be done before the couple marries. Moreover, when "like marry likes" or "opposites attract each other", the survival probabilities are dependent before the spouses have met. Carriere (2000) does not observe when couples marry either and therefore chooses to couple the lives as soon as they enter the data-set. However, that is not possible in our case since we also observe individuals who have already lost their spouse. Therefore, we couple the husband's and wife's lifetimes at birth, like in Frees, Carriere, and Valdez (1996).

4.3.1 Left-truncation and right-censoring

The data is left-truncated because we only observe couples of whom both spouses are at least 18 years old and at least one of the spouses is alive at the beginning of the observation period. The data is right-censored because of most individuals we only observe a minimal age of death and not the actual age of death.

Let $z \in \{x, y\}$. When the age of death of an individual aged a_{z_i} at entry is right-censored, $T(a_{z_i}) \geq b_{z_i}$. Consequently, instead of the remaining lifetime $T(a_{z_i})$, we might only observe the possibly right-censored remaining lifetime $T^*(a_{z_i})$: when the individual died during the observation period $T^*(a_{z_i}) = T(a_{z_i})$, when the individual died before the observation period started or dies after the observation period ended $T^*(a_{z_i}) = b_{z_i}$. So, of each individual we observe:

$$T^*(a_{z_i}) = \min\left(T(a_{z_i}), b_{z_i}\right),\ z \in \{x, y\},$$

where in our case the maximum value of b_{z_i} is 13 years, which is the length of our observation period. We deal with the right-censoring and left-truncation similar as Frees, Carriere, and Valdez (1996).

Like Frees, Carriere, and Valdez (1996), we do not take future trends in mortality into account. This allows us to compare our results with previous research. Moreover, the length of our observation period is only 13 years, which is too short to estimate possible trends over time.

4.3.2 Some notions of dependence

In this subsection we discuss some qualifications of dependence. Different measures for dependence have been introduced in the literature. Let A and B be two sets in $\mathbb{R}^2$. We consider the following relation:

$$P(\{X,Y\} \in \{A \bigcap B\}) \geq P(\{X,Y\} \in A)P(\{X,Y\} \in B), \tag{4.1}$$

The random variables X and Y are *associated* if and only if inequality (4.1) holds for all open upper sets A and B, see Esary and Marshall (1976).

A special case of association is *positively quadrant dependent* (PQD). The random variables X and Y are *positively quadrant dependent* (PQD) if inequality (4.1) holds for $A_a = \{(x,y)|x > a, y \in \mathbb{R}\}$ and $B_b = \{(x,y)|x \in \mathbb{R}, y > b\}$ for all $\{a,b\} \in \mathbb{R}^2$, (see Lehmann, 1966). *Negative quadrant dependence* (NQD) is defined by reversing the inequality in the middle of (4.1). We use a parametric benchmark which can capture PQD and a parametric benchmark which can capture both PQD and NQD. A drawback of both association and quadrant dependence is that both concepts are relatively strong.

Shaked (1982) introduced some weaker notions of dependence by allowing the inequality above to hold only for some sets A and B. Define $\mathbb{A}$ and $\mathbb{B}$ as two collections of sets in $\mathbb{R}^2$. The random variables (X,Y) are *positively dependent relative to* $\mathbb{A}$ *and* $\mathbb{B}$, if inequality (4.1) holds whenever $A \in \mathbb{A}$ and $B \in \mathbb{B}$. Similarly, we define *negatively dependent relative to* $\mathbb{A}$ *and* $\mathbb{B}$ by reversing the inequality in the middle of (4.1). We consider positive and negative dependence relative to sets like, for instance, $A = \{(x,y)|x < 85, y \in \mathbb{R}\}$ and $B = \{(x,y)|x \in \mathbb{R}, y < 95\}$.

To compare our results with previous research, we also determine the linear correlation between the lifetimes of spouses as well as Spearman's ρ_s.

4.4 Two parametric benchmarks and their estimation

Two different bivariate Weibull models, are used as parametric benchmark to estimate the joint survival probabilities of spouses. One of the reasons for choosing these models is that previous research has found that the Weibull marginal survival probabilities of married individuals fit the data quite well (see Frees, Carriere, and Valdez, 1996). The benchmarks are described in Subsection 4.4.1. The first model captures only notions of positive dependence, whereas the second model can capture negative dependence as well.[8] In Subsection 4.4.2 the estimation results are presented.

4.4.1 Two bivariate Weibull models

Both benchmark models have Weibull marginals given by:

$$f_z(z) := f(z;\beta_z,\theta_z) = \frac{\beta_z}{\theta_z}\left(\frac{z}{\theta_z}\right)^{\beta_z-1}\exp\left[-\left(\frac{z}{\theta_z}\right)^{\beta_z}\right], \text{ for } z \in \{x,y\}. \quad (4.2)$$

For $z \in \{x,y\}$, θ_z, denotes the scale parameter and β_z denotes the shape parameter. We denote F_z for the cumulative distribution function of z and $\overline{F}_z$ for the survival function for $z \in \{x,y\}$. That is, $\overline{F}_z(z) = 1 - F_z(z)$ for $z \in \{x,y\}$. Let $\overline{F}(x,y)$ denote the cumulative survival function of the husband's and wife's ages of death. That is,

$$\overline{F}(x,y) = P(X \geq x, Y \geq y).$$

We refer to the first Weibull model, which is studied in Lu and Bhattacharyya (2003), as the positively dependent Weibull (PW) model. In copula-format, the model is parameterized as follows:

The positively dependent Weibull (PW) model

$$\overline{F}(x,y) = \exp\left\{-\left[\ln\left(\overline{F}_x(x)\right)^{\frac{1}{\delta}} + \ln\left(\overline{F}_y(y)\right)^{\frac{1}{\delta}}\right]^{\delta}\right\} \quad (4.3)$$
$$0 < \delta \leq 1, \;\; x,y \geq 0.$$

δ is the parameter which captures the dependence and independence corresponds to the boundary value $\delta = 1$. It is the same model as proposed by Hougaard (1986).

The second bivariate Weibull model, also proposed in Lu and Bhattacharyya (2003), allows for both PQD and NQD.[9] We refer to this model as the positively and negatively dependent Weibull (PNW) model. In copula-format, the model is parameterized as follows:

The positively and negatively dependent Weibull (PNW) model

$$\overline{F}(x,y) = \exp\left\{-\ln\left(\overline{F}_x(x)\right) - \ln\left(\overline{F}_y(y)\right) + \delta F_x(x)F_y(y)\right\}, \quad (4.4)$$

[8] A restriction of both models is that the models cannot be positively dependent for some combinations of the husband's and wife's age (x,y) and negatively dependent for some other combinations of the husband's and wife's age.

[9] The model cannot be positively dependent relative to some sets (A,B) and negatively dependent relative to some other sets (A',B').

$$-1 \leq \delta \leq 1,\ x, y \geq 0,\ \ 0 < \theta_i, \beta_i,\ \ i = x, y,$$

where independence corresponds to $\delta = 0$, positive dependence to $\delta > 0$, and negative dependence to $\delta < 0$. The linear correlation of this model is restricted to $(-0.20, 0.32)$, see Lu and Bhattacharyya (2003).

We estimate the two parametric Weibull models using the sample described in Subsection 4.2.2. The likelihood, similar to the one in Frees, Carriere, and Valdez (1996), is given in Appendix 4.A.

4.4.2 Estimation results of the two bivariate Weibull models

We estimate the parameters of the two parametric models (4.3) and (4.4) on the random sample of 50,000 couples as described in subsection 4.2.2.[10] The value of the likelihood, the Akaike Information Criterion (AIC)[11], and the Bayesian Information Criterion (BIC)[12] are also determined. Table 4.4 displays the re-

	β_x	θ_x	β_y	θ_y	δ
ML estimates *PW* (4.3)	8.82	82.70	9.28	86.83	1
ML estimates *IW* (4.3)	8.82	82.70	9.28	86.83	-
	(0.07)	(0.10)	(0.08)	(0.10)	
ML estimates *PNW* (4.4)	8.82	82.39	9.21	86.43	0.56
	(0.07)	(0.10)	(0.08)	(0.10)	(0.01)

	Likelihood	BIC	AIC
PW (4.3)	-69,315	138,684	138,640
IW (4.3)	-69,315	138,674	138,638
PNW (4.4)	-69,241	138,537	138,492

Table 4.4 – The ML estimates, the corresponding standard errors of the likelihood, the value of the likelihood, and the BIC and AIC based on the *PW* (4.3), the independent Weibull (IW) model ((4.3) with $\delta = 1$), and the positively and negatively dependent Weibull model *PNW* (4.4).

[10] Since in the PW model (4.3) the parameter for dependence (δ) should be between 0 and 1, the following transformation is used to estimate $\delta = \Phi(\overline{\delta})$, where $\overline{\delta} \in (-\infty, \infty)$. For the *PNW* model (4.4), the parameter for dependence δ should lie in the interval $[-1, 1]$. Therefore, the following transformation is used: $\delta = 2\Phi(\overline{\delta}) - 1$, where $\overline{\delta} \in (-\infty, \infty)$.

[11] The Akaike Information Criterion is given by $AIC = 2k - 2\ln(L)$, for k the number of parameters in the statistical model, and L the maximized value of the likelihood function for the estimated model.

[12] The Bayesian Information Criterion is given by $\text{BIC} = k\ln(n) - 2 \cdot \ln(L)$, for k the number of parameters in the statistical model, and L the maximized value of the likelihood function for the estimated model, and n the sample size.

sults of the maximimum likelihood estimation based. The ML estimate of dependence (δ) for the *PW* model indicates independence, which is a corner solution. Therefore, we estimate the *PW* model under the assumption of independence ($\delta = 1$). We refer to this model as the independent Weibull (IW) model. The results of that estimation are also displayed in Table 4.4. The *PNW* model finds positive dependence, which is in line with previous research. Note that the likelihood of the *PNW* model is well above the likelihood of the *IW* model. The Likelihood Ratio (LR) test statistic equals 148. Consequently, we reject the null-hypothesis that $\delta = 0$ at all conventional confidence levels. These results suggest that the *PW*-specification incorporates the correlation in an inappropriate way, since in this model we find no correlation while in the *PNW* model we find positive correlation.

The period life expectancy of the husband and wife, ignoring a possible trend over time, are calculated using the estimates in Table 4.4. All parametric benchmarks have Weibull marginals, leading to the following period life expectancy:

$$E(Z) = \theta_z \Gamma(1 + 1/\beta_z), \text{ for } z \in \{x, y\},$$

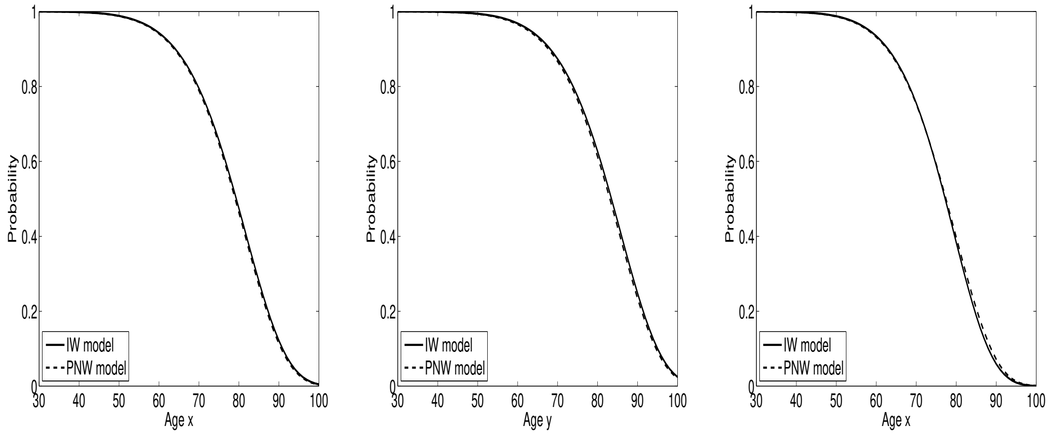
(see Lu and Bhattacharyya, 2003).

Gender	Men		
	Lower bound	Estimate	Upper bound
Model *IW* (4.3)	78.0	78.2	78.5
Model *PNW* (4.4)	77.7	78.0	78.2
	Women		
Gender	Lower bound	Estimate	Upper bound
Model *IW* (4.3)	82.1	82.3	82.5
Model *PNW* (4.4)	81.7	81.9	82.1

Table 4.5 – The period life expectancy at birth and the corresponding 95% confidence intervals for a man and woman based on the estimates of models (4.3) and (4.4) as described in Table 4.4.

Table 4.5 displays the period life expectancy of the husband and wife according to the parametric models. Both parametric models find similar life expectancies for the husband as well as the wife.

In Figure 4.1 the estimated cumulative survival probabilities are displayed for the two parametric benchmarks. Due to the large sample size, the 95% confidence intervals are quite small. Therefore, we plot only the estimated cumulative survival probabilities. Figure 4.1a and 4.1b display the husband's



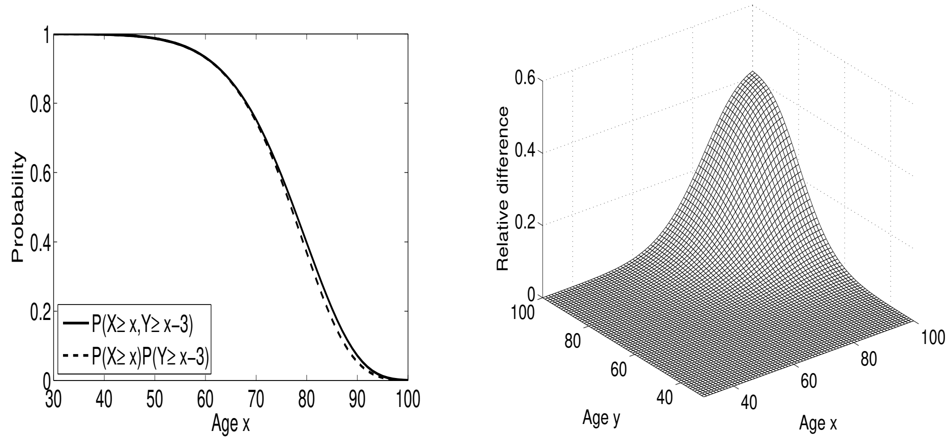
(a) Husband **(b)** Wife **(c)** Husband and wife

Figure 4.1 – The cumulative survival function for the husband ($P(X > x)$) (a), the wife ($P(Y > y)$) (b), and the joint survival function for $y = x - 3$ ($P(X > x, Y > x - 3)$) (c). The solid lines represent the cumulative survival probabilities under IW model. The dashed lines represent the cumulative survival probabilities when the PNW model (4.4) is used. Results are based on the ML estimates presented in Table 4.4.

and wife's cumulative survival probability, respectively. Figure 4.1c displays the probability that both spouses are alive. Figures 4.1a and 4.1b show that the cumulative survival probabilities for both the husband and wife are slightly lower based on the PNW model (dashed line) than the IW model (solid line). However, differences are only marginal. Because of the positive dependence, the probability that both spouses are alive under the PNW model is greater than the probability that both spouses are alive under the IW model as of age 75. However, these differences are also marginal.

Figure 4.2a displays the probability that both spouses are alive according to the estimates of the PNW model under dependence ($P(X \geq x, Y \geq x - 3)$, solid line) and independence ($P(X \geq x)P(Y \geq x - 3)$, dashed line). First, note that the difference between dependence and independence is negligible at young ages. As of age 65 the influence of dependence becomes more visible. However, for providers of annuities, these ages are the most important, influencing the value of their liabilities. Since the differences between independence and dependence are small, Figure 4.2b displays the relative difference between the probability that both spouses are alive under dependence and independence. The relative difference between the survival probabilities based on dependence and independence is strongly increasing with both the husband's and wife's age.

From the ML estimates, we know that the model is positively dependent



(a) Cumulative survival probability **(b)** Relative difference

Figure 4.2 – The probability that both the husband and wife are alive at a given age as a function of age (a) under dependence ($P(X \geq x, Y \geq x-3)$, solid line) and under independence ($P(X \geq x)P(Y \geq x-3)$, dashed line). and the relative difference in the probability that both spouses are live (b) under dependence and independence ($\frac{P(X \geq x, Y \geq y) - P(X \geq x)P(Y \geq y)}{P(X \geq x, Y \geq y)}$) as a function of the husband's and wife's age. Results are based on the ML estimates presented in Table 4.4 of the PNW model.

for each value of (x, y). We use the Metropolis-Hastings Algorithm to sample from the distribution to determine the correlation between the remaining lifetimes of the spouses in the PNW model.[13]

	Linear Correlation		
Gender	Lower bound	Estimate	Upper bound
Model PNW (4.4)	0.14	0.18	0.21
	Spearman's ρ		
Gender	Lower bound	Estimate	Upper bound
Model PNW (4.4)	0.15	0.19	0.23

Table 4.6 – The linear correlation and Spearman's ρ at birth and the corresponding 95% confidence intervals based on the estimates of models (4.3) and (4.4) as described in Table 4.4.

Table 4.6 displays the linear correlation and Spearman's ρ for the PNW model. Frees, Carriere, and Valdez (1996) find a 95% confidence interval of Spearman's rank correlation of $(0.28, 0.55)$. The dependence we find based on the estimates of the PNW model is much weaker than the dependence found in Frees, Carriere, and Valdez (1996). A reason may be that our estimation

[13]The Metropolis-Hastings Algorithm is explained in Appendix 4.B.

results are based on a random sample of the whole Dutch population whereas Frees, Carriere, and Valdez (1996) uses data of Canadian couples who voluntarily bought joint & survivor annuities. Previous research has indicated that the marginal survival probabilities of those who voluntarily buy annuities differ from the population survival probabilities (see for instance, Mitchell, Poterba, Warshawsky, and Brown, 1999). It may be that not only the marginal survival probabilities differ, but also the dependence between the marginal survival probabilities. Carriere (2000) estimates Spearman's rank correlation for different copula's. All these estimates of Spearman's rank correlation are well above our estimate.

Since our results so far are based on rather restrictive parametric models, we extend the flexibility of the model using the approach of Gallant and Nychka (1987). The extensions are described in Section 4.5.

4.5 The semi-nonparametric model

The parametric models considered in the previous section may be too restrictive models to capture the dependence between the remaining lifetimes of spouses in an appropriate way. Therefore, we study in this section flexible generalizations of the parametrice *IW* model. A natural approach is then to use semi-nonparametric estimation employing well-chosen sieves, (see Chen, 2007). As repeated from Chen (2007), for econometric applications where the only prior information on the unknown functions is their smoothness and supports, the choice of a sieve space is not important, as long as the chosen sieve space has the desired approximation error rate. We use as sieve a polynomial to increase flexibility of the models. In Subsection 4.5.1 we describe the extensions made. In Subsection 4.5.2, the estimation results are displayed based on the semi-nonparametric model. Furthermore, we compare the semi-nonparametric model with the two benchmark Weibull models.

4.5.1 The semi-nonparametric model

To obtain the semi-nonparametric model, we use (4.3) under the assumption of independence as a starting model. We take independence as a starting point because we do not want to force the outcome of either positive or negative dependence. Furthermore, the distribution function simplifies such that we can more easily trace the extensions analytically.

Following the ideas of Gallant and Nychka (1987), we extend the model by multiplying the probability density with a squared polynomial. The polynomial is then extended to increase flexibility. In this way, we can easily guarantee that the extension is again a density. First, because the squared polynomial is nonnegative. Second, because the squared polynomial allows a straightforward calculation of the normalizing constant. The reason is that we can use known expressions for the moments of the Weibull density. The resulting density turns out to have a very simple form. Moreover, given this specification, the likelihood can be obtained easily, as is shown in Appendix 4.A.

Under independence, the probability density is given by:

$$f(x,y) := f(x,y;\beta_x,\beta_y,\theta_x,\theta_y) = \frac{\beta_x}{\theta_x}\frac{\beta_y}{\theta_y}\widetilde{x}^{\beta_x-1}\widetilde{y}^{\beta_y-1}\exp\left(-\widetilde{x}^{\beta_x}\right)\exp\left(-\widetilde{y}^{\beta_y}\right), \tag{4.5}$$

where $\widetilde{x} := \frac{x}{\theta_x}$ and $\widetilde{y} := \frac{y}{\theta_y}$. The model will be enlarged using a polynomial. By construction, the new density becomes

$$g_\alpha(x,y) := g_\alpha(x,y;\beta_x,\beta_y,\theta_x,\theta_y) = \frac{1}{c}P_\alpha(x,y)^2 f(x,y),$$

where the constant c is a normalizing constant to ensure that $g_\alpha(x,y)$ integrates out to one. $P_\alpha(x,y)$ is the polynomial, which is squared to ensure that $g_\alpha(x,y)$ remains nonnegative. The factor $\frac{1}{c}P_\alpha(x,y)^2$ can be seen as the Radon-Nikodym derivative of $g_\alpha(x,y)$ w.r.t. $f(x,y)$. The polynomial $P_\alpha(x,y)$ is defined as

$$P_\alpha(x,y) := P_\alpha(x,y;\beta_x,\beta_y,\theta_x,\theta_y) = \sum_{i=0}^{n}\sum_{j=0}^{n}\alpha_{i,j}\left(\widetilde{x}^{\beta_x}\right)^i\left(\widetilde{y}^{\beta_y}\right)^j 1_{[i,j]}, \tag{4.6}$$

where $1_{[i,j]}$ represents an indicator function if the combination of (i,j) is included in the polynomial. The indicator function allows us to exclude some interaction terms between x and y. Define α as a vector containing the components $\alpha_{i,j}$ such that we can rewrite the squared polynomial $P_\alpha(x,y)^2$ as

$$P_\alpha(x,y)^2 = \alpha' P(x,y)\alpha \geq 0,$$

where $P(x,y)$ is the corresponding align. An example of a polynomial used is

$$P_\alpha(x,y) = \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y},$$

such that $\alpha = [\alpha_{0,0}\ \alpha_{1,0}\ \alpha_{0,1}]'$ and

$$P(x,y) = \begin{pmatrix} 1 & \widetilde{x}^{\beta_x} & \widetilde{y}^{\beta_y} \\ \widetilde{x}^{\beta_x} & \widetilde{x}^{2\beta_x} & \widetilde{x}^{\beta_x}\widetilde{y}^{\beta_y} \\ \widetilde{y}^{\beta_y} & \widetilde{x}^{\beta_x}\widetilde{x}^{\beta_y} & \widetilde{y}^{2\beta_y} \end{pmatrix}.$$

Now define Ω as follows:

$$\Omega := \int_0^\infty \int_0^\infty P(x,y)f(x,y)dxdy,$$

such that $c = \alpha'\Omega\alpha$, and the probability density is given by:

$$g_\alpha(x,y) = \frac{\alpha' P(x,y)^2 \alpha}{\alpha'\Omega\alpha} f(x,y) \tag{4.7}$$

The first element of α ($\alpha_{0,0}$) is set equal to one to avoid identification problems.

The polynomial is enlarged by adding either higher order or interaction terms. The polynomials we have estimated can be found in Appendix 4.C. The dependence for the semi-nonparametric model and the univariate survival function are described in Appendix 4.D.

4.5.2 Estimation results of the semi-nonparametric model

We can now estimate the different models described in Subsection 4.5.1. Table 4.7 displays the value of the likelihood, the BIC, and the AIC for different polynomials. Note that the value of the likelihood of the first polynomial model (i) is already well above the value of the PNW model although it has only one more degree of freedom. Because of the large sample-size, both the BIC and the AIC decrease when the polynomial is expanded for the first expansions, except when interaction terms are included. The computational time increases exponentially when we increase the flexibility of the model. The BIC reaches its minimum for polynomial (v), which is given by:

Polynomial	Likelihood	BIC	AIC	LR TS	DF
IW model	-69,315	138,674	138,638	-	-
(i)	-69,122	138,309	138,256	386	2
(ii)	-69,122	138,320	138,258	386	3
(iii)	-69,036	138,126	138,082	558	5
(iv)	-69,012	138,154	138,049	606	8
(v) (TOP)	-68,970	138,103	137,968	690	10
(vi)	-68,967	138,139	137,971	697	15
(vii)	-68,961	138,148	137,963	709	15

Table 4.7 – The value of the likelihood, the BIC and AIC for different polynomials and probability density (4.7). The LR Test Statistic ($LR\ TS$) is the likelihood ratio test statistic of the model with the IW model. DF are the corresponding degrees of freedom. The polynomials (i) until (vii) are displayed in Appendix 4.C.

Third Order Polynomial Weibull (TOP) model

$$
\begin{aligned}
P_\alpha &= \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y} + \alpha_{1,1}\widetilde{x}^{\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,0}\widetilde{x}^{2\beta_x} + \alpha_{0,2}\widetilde{y}^{2\beta_y} \\
&\quad + \alpha_{1,2}\widetilde{x}^{\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{2,1}\widetilde{x}^{2\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,2}\widetilde{x}^{2\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{3,0}\widetilde{x}^{3\beta_x} + \alpha_{0,3}\widetilde{y}^{3\beta_y} \qquad (4.8)
\end{aligned}
$$

To determine whether the semi-nonparametric model fits the data better than the *PNW* model, we perform a model selection test for overlapping models as in Vuong (1989). Note that the models overlap only for $\delta = 0$ in the *PNW* model and all components of α equal zero in the semi-nonparametric model, except the first component ($\alpha_{0,0}$) which was set to one. A brief summary of the test is displayed in Appendix 4.E. For more detailed information, see Vuong (1989). We perform the test only for the *TOP* model, see (4.8). The test consists of two sequential steps. First, we performed a variance test to test whether the two distributions differ at the ML estimates. This led to the conclusion that the *PNW* model and the *TOP* model indeed differ at the ML estimates at all conventional confidence levels. Second, we performed a likelihood ratio test, finding that we have to reject the null-hypotheses in favor of the hypotheses that the *TOP* model performs better than the *PNW* model, again at all conventional confidence levels.

ML estimates of the *TOP* model are displayed in Table 4.8 with the corresponding standard errors. We use the ML estimates to determine confidence intervals of the husband's and wife's period life expectancy. Table 4.9 displays the period life expectancy for a husband and wife at birth based on the *TOP* model. The life-expectancies based on the semi-nonparametric model are close to the life expectancy based on the parametric models for the husband as well as the wife. So, the restrictive parametric models seem to capture the firsts moments well.

The ML estimates as displayed in Table 4.8 are used to calculate the cumulative survival probabilities. Figure 4.3 displays the cumulative survival probabilities for the two parametric models and for the *TOP* model. Figure 4.3a shows that the husband's marginal survival probabilities are about the same for all models, in contrast with the wife's cumulative survival probabilities as showed in Figure 4.3b.The wife's cumulative survival probabilities are lower under the *TOP* model than under the *IW* and the *PNW* model for the ages 30 until 73 and 90 until 100 and higher for the ages 73 until 90.

Again the Metropolis-Hastings algorithm is used to determine the linear correlation and Spearman's rank correlation. Table 4.10 displays the linear correlation and Spearman's ρ based on the *TOP* model. We find that both the linear correlation and Spearman's rank correlation are slightly lower for the

Parameter	ML Estimate	St. Error
$\alpha_{1,0}$	1.46	(0.52)
$\alpha_{0,1}$	0.40	(0.34)
$\alpha_{1,1}$	0.01	(0.33)
$\alpha_{2,0}$	-0.34	(0.19)
$\alpha_{0,2}$	0.49	(0.14)
$\alpha_{1,2}$	0.53	(0.17)
$\alpha_{2,1}$	-0.02	(0.07)
$\alpha_{2,2}$	-0.06	(0.02)
$\alpha_{3,0}$	0.02	(0.02)
$\alpha_{0,3}$	-0.07	(0.02)
β_x	6.25	(0.21)
β_y	6.20	(0.16)
θ_x	75.00	(0.87)
θ_y	70.75	(0.78)

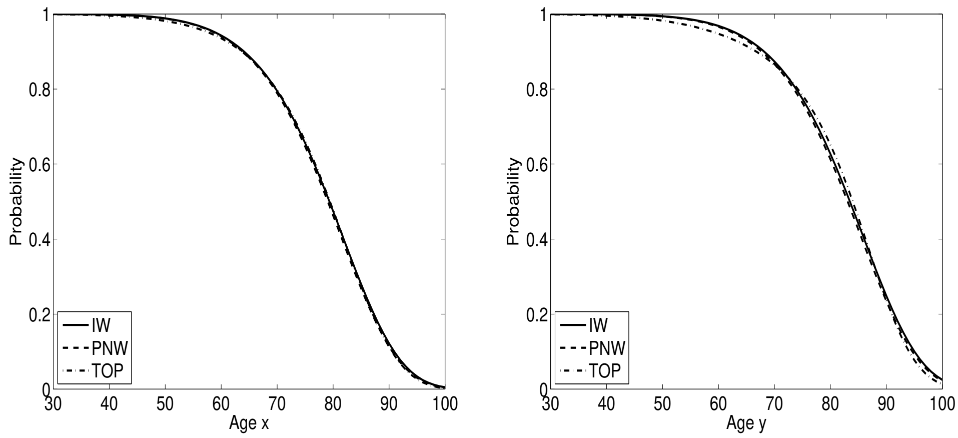
Table 4.8 – The ML estimates of the *TOP* model and the corresponding standard errors.

	Men		
Gender	Lower bound	Estimate	Upper bound
Model *TOP* (4.8)	77.5	77.9	78.3
	Women		
Gender	Lower bound	Estimate	Upper bound
Model *TOP* (4.8)	81.3	81.9	82.3

Table 4.9 – The period life expectancy at birth and the corresponding 95% confidence intervals for a man and woman based on the estimates of model *TOP* as described in Table 4.8.

TOP model than for the *PNW* model but higher compared with the *IW* model (which finds independence). However, the differences in terms of Spearman's ρ and the linear correlation between the *TOP* model and the *PNW* model are only marginal. Spearman's rank correlation is much lower than the Spearman's rank correlation found in Frees, Carriere, and Valdez (1996). The disadvantage of both the linear correlation and Spearman's ρ is that they capture the dependence in a single number, although the dependence may be stronger for some ages and weaker for some other ages.

The dependence can be observed from Figure 4.4. Figure 4.4a displays the



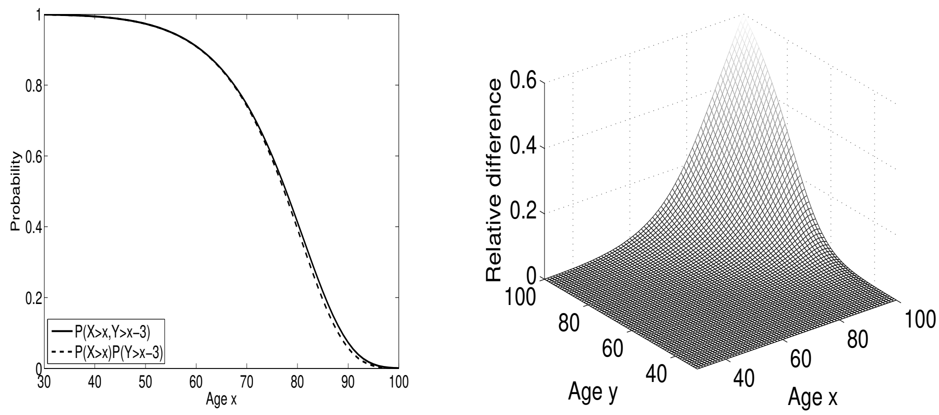
(a) Husband **(b)** Wife

Figure 4.3 – The probability that the husband (a) and the wife (b) are alive at a given age as a function of age. The cumulative survival probabilities are displayed for the *IW* model (solid lines), *PNW* model (dashed lines), *TOP* model (dashed-dotted lines).

	Linear Correlation		
Gender	Lower bound	Estimate	Upper bound
Model *TOP* (4.8)	0.09	0.14	0.19
	Spearman's ρ		
Gender	Lower bound	Estimate	Upper bound
Model *TOP* (4.8)	0.11	0.17	0.21

Table 4.10 – The linear correlation and Spearman's ρ at birth and the corresponding 95% confidence intervals based on the estimates of the *TOP* model as displayed in Table 4.7.

probability that both spouses are alive according to the estimates of the *TOP* model under dependence ($P(X > x, Y > y)$, solid line) and independence ($P(X > x)P(Y > y)$, dashed line). The remaining lifetimes of spouses are positively dependent. Figure 4.4b displays the relative difference between the probability that both spouses are alive under dependence ($P(X > x, y > y)$) and independence ($P(X > x)P(Y > y)$), as a function of both x and y. The relative difference in the cumulative survival probabilities under dependence and independence is strongly increasing with both the husband's and wife's age. When comparing Figure 4.4b with Figure 4.2b, we see that the increase based on the *TOP* model is much stronger than the increase found in the *PNW* model.



(a) Cumulative survival probability **(b)** Relative difference

Figure 4.4 – The probability that both the husband and wife are alive at a given (a) age as a function of age under dependence ($P(X > x, Y > y)$, solid line) under independence ($P(X \geq x)P(Y \geq y)$, dashed line) for $y = x - 3$. The relative difference in the probability that both spouses are live (b) under dependence and independence ($\frac{P(X \geq x, Y \geq y) - P(X \geq x)P(Y \geq y)}{P(X \geq x, Y \geq y)}$) as a function of x and y. Results are based on the ML estimates presented in Table 4.8.

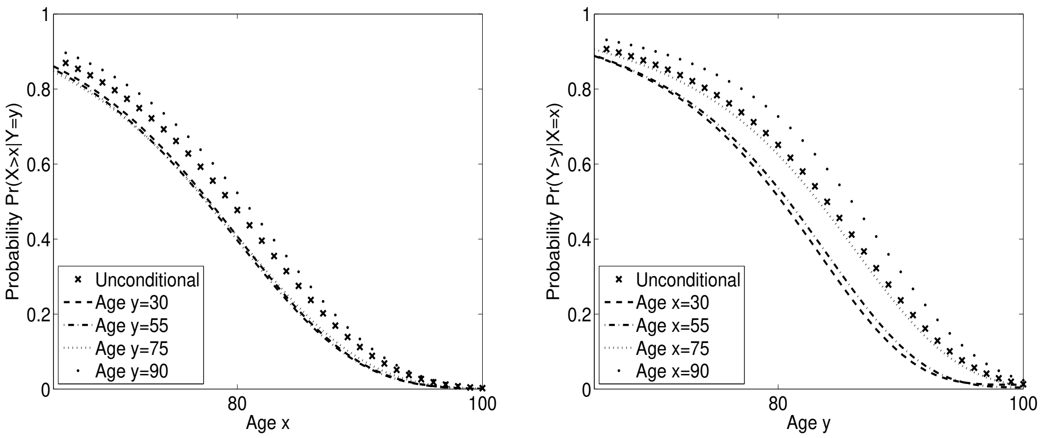
4.6 The impact of dependent remaining lifetimes

In this section, we use the estimates of the *TOP* model to determine the conditional survival probabilities and period life expectancy of the husband and wife.

We determine the cumulative survival probabilities of the husband (wife) given the wife's (husband's) age of death $P(X \geq x|Y = y)$ ($P(Y \geq y|X = x)$). Figure 4.5 displays the probability that the husband or wife is alive as a function of age given that the other spouse has died at some specific age.

When one of the spouses dies at a relatively young age (before age 75), the surviving spouse has relatively low survival probabilities at young ages compared to the unconditional survival probabilities. When one of the spouses dies at a relatively old age (dotted lines), the surviving spouse has relatively high survival probabilities compared to the unconditional survival probabilities. The wife's cumulative survival probabilities are more affected by the husband's death than the husband's cumulative survival probabilities are affected by the wife's death. Summarizing, a spouse's survival probabilities are relatively high when the other spouse died at a high age and low when the other spouse died at a low age, compared to the unconditional survival probabilities.

In Figure 4.6 the cumulative survival probabilities are displayed as a func-



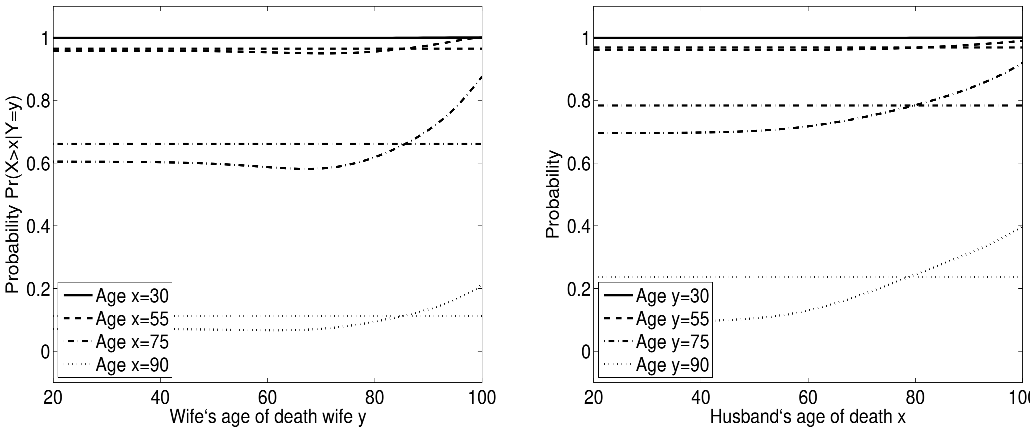
(a) Husband **(b)** Wife

Figure 4.5 – The probability that the husband is alive (a) as a function of age (x) given the age of death of the wife ($Y = y$) and the probability that wife is alive (b) as a function of age (y) given the age of death of the husband ($X = x$). The crosses represent the unconditional cumulative survival probabilities ($P(X \geq x)$ and $P(Y \geq y)$). The other lines represent the conditional cumulative survival probabilities ($P(X \geq x|Y = y)$ in (a) and $P(Y \geq y|X = x)$ in (b)) for y in (a) and x in (b) equal to 30 (dashed lines), 55 (dashed dotted lines), 75 (dotted lines), and 90 (dots). The *TOP* model (4.8) is used with as parameter values the estimates as displayed in Table 4.8.

tion of the age of death of the other spouse. That is, we display $P(X \geq x|Y = y)$ as a function of y for different values of x. Figure 4.6 displays the probability of being alive at a given age for the husband (4.6a) and wife (4.6b) as a function of the spouse's age of death. The probability of reaching age 30 is only marginally affected by the age of death of the other spouse, which is in line with the low dependence at young ages. The husband has lower survival probabilities when the wife dies before age 84 compared with the unconditional survival probabilities. The husband's survival probabilities are generally higher when the wife died after age 84 than when the wife died before age 84. The wife's survival probabilities are increasing with the husband's age of death. The probability that the wife is alive at a given age is more affected by the husband's age of death for high ages of the wife compared with low ages of the wife, which is in line with the increasing dependence with age.

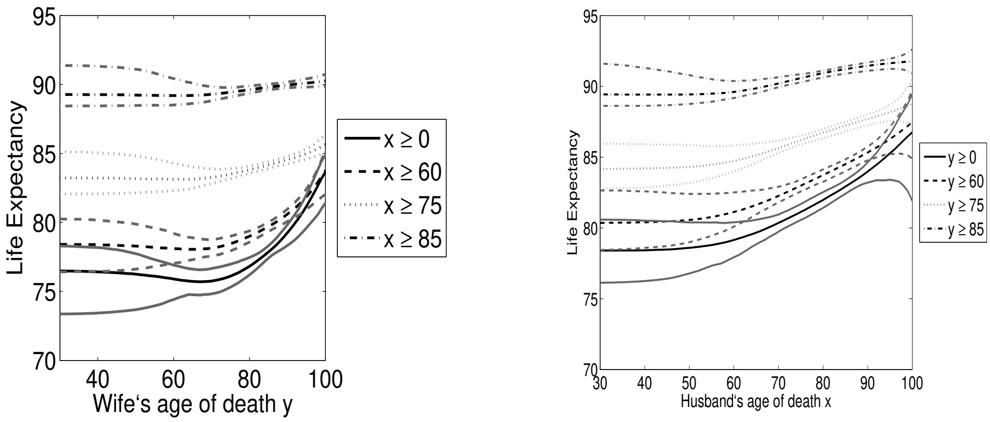
Guiaux (2010) finds that an old widower has a decreased survival probability after the death of the spouse whereas the survival probabilities of an old widow are not affected by the death of the spouse. We find that the effect depends on the age of death of the former spouse. Moreover, we find an effect for both spouses.

Figure 4.7 displays the husband's (4.7a) and wife's (4.7b) conditional peri-



(a) Husband **(b)** Wife

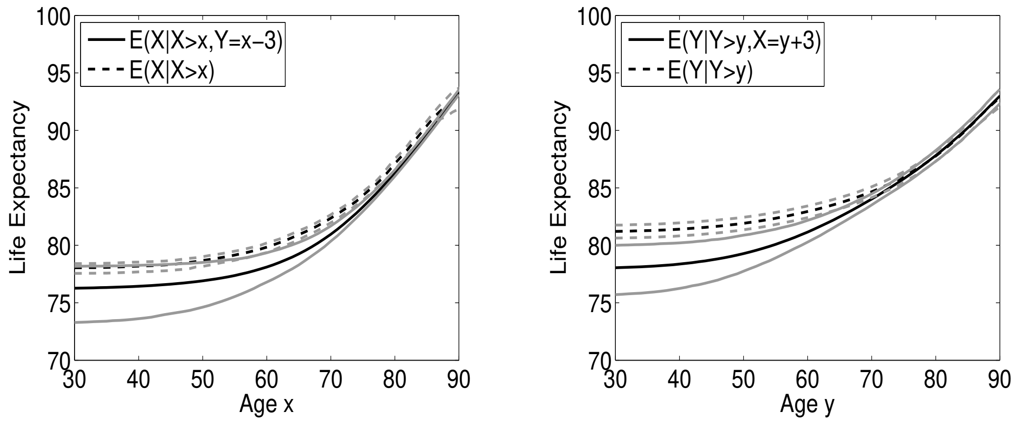
Figure 4.6 – The probability that the husband is alive (a) as a function of the wife's age of death y for different ages x. and the probability that the wife is alive (b) as a function of the husband's age of death x for different ages y. The lines represent the conditional cumulative survival probabilities for x in (a) and y in (b) equal to 30 (solid lines), 55 (dotted lines), 75 (dashed-dotted lines), and 90 (dotted-lines). The horizontal flat lines represent the unconditional cumulative survival probabilities ($P(X \geq x)$ and $P(Y \geq y)$) for the corresponding values of x and y. The TOP model (4.8) is used with as parameter values the estimates presented in Table 4.8.



(a) Husband **(b)** Wife

Figure 4.7 – The husband's life expectancy (a) as a function of the wife's age of death y given he is alive at a certain age ($E(X|X \geq x, Y = y)$) and the wife's life expectancy (b) as a function of the husband's age of death x given she is alive at a certain age ($E(Y|Y \geq y, X = x)$). The lines represent the conditional conditional life expectancies for x in (a) and y in (b) equal to 0 (solid line), 60 (dashed line), 75 (dotted line), and 85 (dashed-dotted line). The grey lines represent the corresponding 95% confidence interval.

odic life expectancy and corresponding 95% confidence interval as a function of the wife's and husband's age of death ($E(X|X \geq x, Y = y)$ $E(Y|Y \geq y, X = x)$).[14] We find that the husband's and wife's life expectancy are generally more volatile at young ages of death of the spouse than at high ages of death. The wife's life expectancy is more volatile than the husband's life expectancy at very high ages of death of the other spouse. This may because there are only a few husbands in the sub-sample who die at a young age. Generally, the remaining lifetimes of both the husband and wife are increasing with the age of death of the other spouse. The husband's remaining life expectancy is increasing steadily as of the wife's age of death of age 70. The wife's remaining life expectancy is increasing as of the husband's age of death of age 60.



(a) Husband **(b)** Wife

Figure 4.8 – The husband's (a) period life expectancy as a function of age x at the moment his wife died ($E(X|X \geq x, Y = x - 3)$) for the *TOP* model (solid line), and the husband's (a) period life expectancy unconditional on the wife's age of death ($E(X|X \geq x)$, dashed line). The wife's (b) period life expectancy as a function of age y at the moment her husband died ($E(Y|Y \geq y, X = y + 3)$) for the *TOP* model (solid line), and the wife's (b) period life expectancy unconditional on the husband's age of death ($E(Y|Y \geq y)$, dashed line). The husband is assumed to be three years older than the wife. The grey lines represent the corresponding 95% confidence interval.

Figure 4.8 displays the husband's (4.8a) and wife's (4.8b) period life expectancy as a function of age conditional on the spouse's age of death and unconditional with corresponding 95% confidence interval. When one of the spouses dies at a relatively young age, there is a large uncertainty in the sur-

[14] To determine the remaining lifetimes we simulate couples using scrambled Halton draws (see Halton, 1960; Train, 2002). Scrambled Halton draws are used to reduce the variance. To draw from the distribution $g_\alpha(x, y)$ as in Equation (4.7), we use $E(X) = \frac{1}{c} \int \int x P_\alpha(x, y) f(x, y) dy dx = \frac{1}{c} E_{f_{x,y}} x P_\alpha(x, y) dx dy$. See Appendix 4.D for more details.

viving spouse's life expectancy. At young ages the period life expectancy of the husband (wife) conditional on the wife's (husband's) age of death is well below the husband's (wife's) life expectancy unconditional on the wife's (husband's) death. The difference between the conditional and unconditional life expectancy at young ages is larger for the wife than for the husband. At high ages, the life expectancy conditional on the spouse's age of death and the unconditional life expectancy are similar.

4.7 Annuity valuation with dependent mortality

The marginal and joint survival probabilities differ for the different models. As a consequence, the actuarially fair value of annuities might also be different. The estimated survival probabilities are used to determine actuarially fair values of different kinds of annuities, ignoring possible trends in life-expectancy. We determine for each model the actuarially fair value of a single-life annuity, a joint annuity, a survivor annuity, and a joint & 70% survivor annuities. A single-life annuity makes an annual payment up to the death of the individual. The actuarially fair value of a single-life annuity for the husband with current age x is given by:

$$P_{sl,x} = \sum_{t=\tau}^{\infty} \frac{P(X \geq x+t | X \geq x)}{(1+r)^t},$$

where r is the interest rate and τ the start date of the first payment. For $\tau = 0$, we have an immediate annuity and for $\tau > 0$ we have a deferred annuity. The actuarially fair value of a single-life annuity for the wife with current age y is obtained by replacing X with Y and x with y. A joint annuity makes an annual payment as long as both the husband and wife are alive. The actuarially fair value of a joint annuity for a couple with current ages (x, y) for the husband and wife respectively is given by:

$$P_{joint} = \sum_{t=\tau}^{\infty} \frac{P(X \geq x+t, Y \geq y+t | X \geq x, Y \geq y)}{(1+r)^t}.$$

A survivor annuity on the husband's (wife's) life makes an annual payment as of the moment the wife (husband) has deceased until the death of the husband (wife). The actuarially fair value of a survivor annuity on that pays out when only the wife is alive for a husband aged x and wife aged y is given by:

$$P_{surv,y} = \sum_{t=\tau}^{\infty} \frac{P(X < x+t, Y \geq y+t | X \geq x, Y \geq y)}{(1+r)^t}.$$

We refer to the survivor annuity that pays out when only the husband (wife) is alive as the husband's (wife's) survivor annuity. The husband's survivor annuity, defined by $P_{surv,x}$ is obtained by replacing y with x and x with y. A joint & 70% survivor annuity pays the full benefit when both spouses are alive and a reduced benefit (70% of the full benefit) when only one of the spouse is alive. The actuarially fair value of a joint & 70% survivor annuity for a husband aged x and wife aged y is given by:

$$P_{j\&70surv} = P_{joint} + 0.7P_{surv,y} + 0.7P_{surv,x}.$$

The annual benefit levels are normalized to one. Results for immediate annuities for a husband aged 66 and wife aged 63 are displayed in Table 4.11, assuming an interest rate r of 3%.

Model	*IW*	*PNW*	*TOP*
Husband's single-life	12.07	11.93	12.09
	(11.99, 12.17)	(11.83, 12.02)	(12.03, 12.13)
Wife's single-life	15.30	15.12	15.57
	(15.22, 15.39)	(15.03, 15.21)	(15.54, 15.62)
Husband's survivor	1.60	1.31	1.19
	(1.55, 1.64)	(1.25, 1.38)	(1.16, 1.20)
Wife's survivor	4.83	4.63	4.80
	(4.74, 4.91)	(4.54, 4.72)	(4.78, 4.87)
Joint	10.48	10.69	10.97
	(10.41, 10.54)	(10.61, 10.77)	(10.91, 10.99)
Joint & 70% survivor	14.97	14.86	15.16
	(14.91, 15.04)	(14.79, 14.92)	(15.12, 15.19)

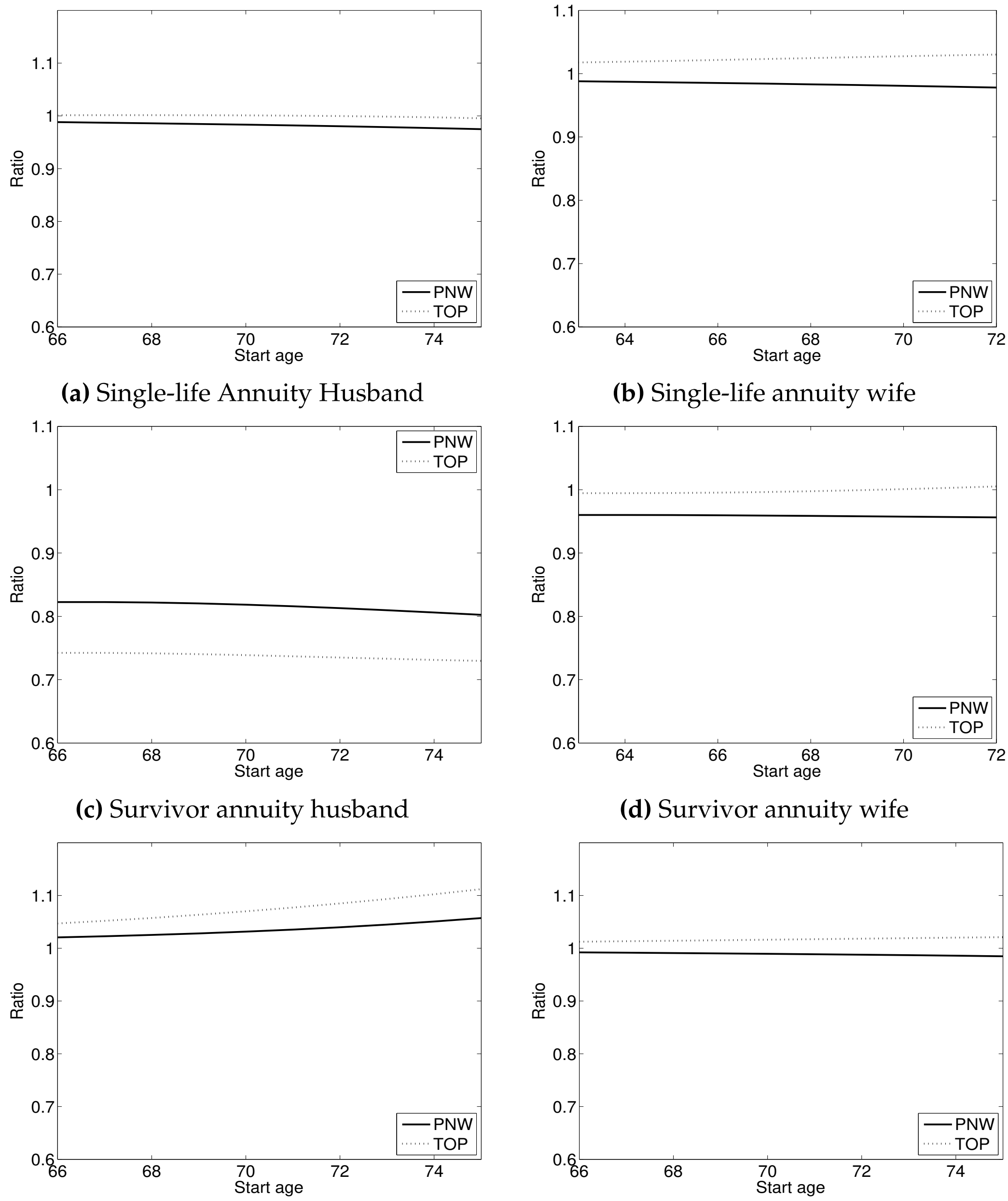
Table 4.11 – The actuarially fair values of different kind of immediate annuities using the estimates as displayed in Tables 4.4 and 4.8 and the corresponding 95% confidence intervals. Annual benefit levels are normalized to one. The interest rate (r) is assumed to be 3%, the husband is aged 66, and the wife is aged 63.

We find that the actuarially values of the single-life annuity for the wife, the joint annuity, and the joint & 70% survivor annuity are higher based on the semi-nonparametric model than on the parametric models. The actuarially fair value of the husband's survivor annuity is lower based on the semi-nonparametric model than the parametric models. Moreover, the accuracy of the values of the different annuities is much higher based on the *TOP* model than on the parametric models.

The wife's single-life annuity is the least expensive under the *PNW* model and the most expensive under the *TOP* model. The latter is because between the age of 73 and 90, the wife's cumulative survival probability is higher under the *TOP* model than under the *PNW* and *IW* model, increasing the value of the single-life annuity. Note that the wife's single-life annuity is more expensive than the husband's single-life annuity because the wife's single-life annuity starts at a younger age and because the wife's survival probabilities are higher than the husband's survival probabilities.

The survivor annuities are the most expensive under independence because positive dependence between the remaining lifetimes of spouses implies that both spouses are more likely to die either below their life-expectancy or above their life-expectancy compared with independence. As a consequence, the probability that only one of the spouses is alive decreases, decreasing the value of the survivor annuity. On the other hand, the joint survivor annuity is more expensive under positive dependence than under independence because the probability that both spouses are alive is greater under positive dependence than under independence. For the joint and survivor annuity, these effects partly cancel out. If we compare the mean values of the actuarially fair annuity prices, we find that the husband's survivor annuity is about 20% less expensive under the semi-nonparametric models than under independence. The joint annuity is 5% more expensive under the semi-nonparametric model than under independence and the joint & 70% survivor annuity is about 1% more expensive under the semi-nonparametric models than under independence.

The joint & 70% survivor annuity is the most expensive under the *TOP* model because of the high prices of the joint annuity and the wife's survivor annuity.



(a) Single-life Annuity Husband

(b) Single-life annuity wife

(c) Survivor annuity husband

(d) Survivor annuity wife

(e) Joint Annuity

(f) Joint & 70% survivor annuity

Figure 4.9 – The relative value of a single-life annuity based for the husband (a) and wife (b), respectively, the relative value of the survivor annuity for the husband (c) and wife (d), respectively, the relative value of the joint annuity (e), and the relative value of a joint & 70% survivor annuity (f). All figures are as a function of the start age of the annuity. We display the *PNW* (solid lines) and *TOP* (dotted lines) relative to the corresponding values of the *IW* models for a husband aged 66 and wife aged 63. An interest rate of 3% is assumed.

Figure 4.9 displays the ratio of single-life annuities, survivor annuities, and joint annuities for the different models with respect to the *IW* model. We investigate the ratios for both immediate and deferred annuities. Figures 4.9a and 4.9b display the value of immediate and deferred annuities for the husband (left) and wife (right), where the first annuity payment for the husband (wife) is made in the age range 66 (63) until 75 (72), conditional on being alive. Figure 4.9c and 4.9d display the actuarially fair prices of survivor annuities on the husband's life (left) or the wife's life (right). Figure 4.9e displays the actuarially fair price of a joint annuity for different start ages and Figure 4.9f displays the value of a joint & 70% survivor annuity for different start ages. For all results we assumed an interest rate of 3%.

We find that for a single-life annuity, the values of the annuity calculated with the best estimates of the IW model and the PNW model are very similar, i.e., the ratio is close to one. The semi-nonparametric models yield higher actuarially fair prices for the wife's single life annuity than the parametric models. Moreover, the difference with the value of the wife's single-life annuity under independence is slightly increasing with the start age. The increased flexibility of the semi-nonparametric models compared with the parametric benchmarks has not only led to more freedom to capture the dependence structure, but also to more freedom to capture the marginal survival probabilities.

The value of the survivor annuities for both husband and wife are typically lower for the models that take the dependence between the survival probabilities into account than for the IW model, especially for the husband. The value of the joint annuity under dependence is increasing with the start age relative to the value of the joint annuity under independence (because the dependence is increasing with age).

When couples buy a joint & 70% survivor annuity, these opposing effects cancel each other partly out, making the actuarially fair value of the joint & survivor annuity under independence almost equal to the actuarially fair value of the joint & survivor annuity under dependence. Because the actuarially fair value of the joint annuity is increasing relative to the actuarially fair value of the survivor annuity when the start age increases, the joint & survivor annuity is almost constant with the start age. Frees, Carriere, and Valdez (1996) finds that actuarially fair value joint & survivor annuities are reduced by approximately 5% when dependent mortality models are used compared to independence. We find that the effects of dependence on the actuarially fair annuity values is much lower. A reason for this may be that we find a much weaker dependence between the remaining lifetimes of spouses.

To conclude, ignoring the possible dependence may lead to an underestimation of the value of a joint annuity and an overestimation of the value of the survivor annuity. Imposing a restrictive dependence structure may lead to an overestimation of the value of the husband's survivor annuity and un underestimation of the value of the joint annuity and the wife's survivor annuity.

4.8 Conclusions

Different methods are used to estimate the joint survival probability of spouses using a random sample of the whole Dutch population. Previous literature has found that the remaining lifetimes of spouses are positively dependent, affecting the value of a joint & survivor annuity. We extend the literature by allowing more flexible models and by using a much richer data-set. The remaining lifetimes of spouses are estimated using two bivariate Weibull models. The first model, which allows only for positive dependence and independence, suggests that the remaining lifetimes are independent. The second model, which also allows for negative dependence, indicates that the remaining lifetimes of spouses are positively dependent. From this we see that the choice of parametric model might influence the results. Incorporating the correlation structure in an inappropriate way might yield invalid results. As a consequence, we have extended the bivariate Weibull model under the assumption of independence by enlarging it with a polynomial, based on the approach of Gallant and Nychka (1987), increasing the flexibility of the model, such that a better fit with the data can be found. The semi-nonparametric approach also led to a positive dependence between the remaining lifetimes of spouses. However, the dependence found with the semi-nonparametric model is much weaker than the dependence found in, for instance, Frees, Carriere, and Valdez (1996).

We find that the husband's life expectancy at birth is generally increasing with his wife's age of death and the wife's life expectancy at birth is generally increasing with the husband's age of death. The prices of annuities differ among the varying models. Ignoring the dependence between the remaining lifetimes of spouses may lead to an underestimation of the value of a joint annuity and an overestimation of the value of a single-life annuity. However, we find that the effects of ignoring the dependence between the remaining lifetimes of spouses has less impact on the actuarially fair value of annuities than previous literature has found.

Throughout the paper, we assumed that the marginal survival probabilities of individuals did not change over time, although life expectancy has increased significantly. Not only the marginal survival probabilities may change over time, but also the dependence between the remaining lifetimes of spouses may change over time. However, if the change in the bivariate age-of-death is deterministic as a function of age and time, the dependence may remain constant over time although life expectancy increases. We leave this for future research.

Of all individuals, we have much more data like for instance ethnicity. In the estimation we did however not use any covariates. Including covariates may lead to a better estimation of the dependence. We also leave this for future research.

4.A The likelihood

We develop the likelihood as in Frees, Carriere, and Valdez (1996). To set up the likelihood, we first create a dummy for each spouse to indicate whether the spouse's remaining lifetime is right-censored. Dummy D_{x_i} (D_{y_i}) is one in case the remaining lifetime of the husband (wife) of couple i is right-censored and zero else.

To deal with right-censoring, three cases have to be considered.

1. No censoring occurs. This represents the case where both spouses die during the observation period. That is $0 < T(a_x) < b_x$ and $0 < T(a_y) < b_y$.

2. X is right-censored and Y is uncensored. X survives the observation period or died prior to the start of the observation period while Y dies during the observation period. Consequently, $T(a_x) \geq b_x$ and $0 < T(a_y) < b_y$, where $b_x = 0$ in case the individual died prior to the start of the observation period.
   We also have this case for Y right-censored and X uncensored.

3. Both are right-censored, meaning that both spouses survive the observation period or one of the spouses died prior to the start of the observation period while the other spouse was still alive at the end of the observation period. In this case, $T(a_x) > b_x$ and $T(a_y) > b_y$, where b_x or b_y equals zero in case the individual died prior to the start of the observation period.

To ease notation, define $T_x = T(a_x)$ and $T_y = T(a_y)$. Because of the left-truncation, we estimate the conditional distribution of $T(a_x), T(a_y)$ defined as:

$$\begin{aligned} F_\tau(t_x, t_y) &= P(0 \leq T_x \leq t_x, 0 \leq T_y \leq t_y | T_x \text{ and } T_y \text{are observed}) \\ &= \frac{P(T_x \leq t_x, T_y \leq t_y) - P(T_x \leq t_x, T_y \leq 0) - P(T_x \leq 0, T_y \leq t_y)}{P(T_x \geq 0, T_y \geq 0)} \\ &+ \frac{P(T_x \leq 0, T_y \leq 0)}{P(T_x \geq 0, T_y \geq 0)}. \end{aligned}$$

If both lifetimes are uncensored, all dummies equal zero and $t_x < b_x$, $t_y < b_y$. The contribution to the likelihood for a couple with entry ages (a_{x_i}, a_{y_i}) and remaining lifetimes (t_{x_i}, t_{y_i}) equals:

$$\frac{\partial^2 F_\tau(a_{x_i} + t_{x_i}, a_{y_i} + t_{y_i})}{\partial t_{x_i} \partial t_{y_i}} = \frac{f(a_{x_i} + t_{x_i}, a_{y_i} + t_{y_i})}{\overline{F}(a_{x_i}, a_{y_i})}.$$

When the husband's lifetime is uncensored and the wife's lifetime is censored, we have $D_x = 0, D_y = 1$, such that for $0 < t_x < b_x, t_y \geq b_y$

$$\frac{P(0 < T_x < t_x, T_y = b_y)}{P(T_x > 0, t_y > 0)} = F_\tau(t_x, \infty) - F_\tau(t_x, b_y).$$

The contribution to the likelihood for a couple with entry ages (a_{x_i}, a_{y_i}) and remaining lifetimes (t_{x_i}, t_{y_i}) equals:

$$\frac{\partial}{\partial t_{x_i}} \left[F_\tau(t_{x_i}, \infty) - F_\tau(t_{x_i}, b_y) \right] = \frac{F_x(a_{x_i} + t_{x_i}, \infty) - F_x(a_{x_i} + t_{x_i}, a_{y_i} + b_{y_i})}{\overline{F}(a_{x_i}, a_{y_i} + b_{y_i})}.$$

The case where the husband's lifetime is censored and the wife's lifetime is uncensored is similar to the previous case. When both lifetimes are right-censored, $D_x = D_y = 1$, such that for $t_x \geq b_x$ and $t_y \geq b_y$:

$$\frac{P(T_x^* = b_x, T_y^* = b_y)}{P(T_x > 0, T_y > 0)} = \frac{P(T_x > b_x, T_y > b_y)}{P(T_x > 0, T_y > 0)} =$$

The contribution to the likelihood for a couple with entry age (a_{x_i}, a_{y_i}) and remaining lifetime (t_{x_i}, t_{y_i}) is given by:

$$\frac{\overline{F}(a_{x_i} + b_{x_i}, a_{y_i} + b_{y_i})}{\overline{F}(a_{x_i}, a_{y_i})}$$

The log-likelihood of a single observation is given by:

$$\ln L(a_x, a_y, t_x, t_y, D_x, D_y) = (1 - D_x)(1 - D_y) \ln f(a_x + t_x, a_y + t_y)$$

$$+ D_x)(1 - D_y) \ln \left(F_y(\infty, a_y + t_y) - F_y(a_x + b_x, a_y + t_y) \right)$$
$$+ (1 - D_x) D_y \ln \left(F_x(a_x + t_x, \infty) - F_x(a_x + t_x, a_y + b_y) \right)$$
$$+ D_x D_y \ln \left(\overline{F}(a_x + b_x, a_y + b_y) \right)$$
$$- \ln \left(\overline{F}(a_x, a_y) \right)$$

The log-likelihood for the whole data-set is given by:

$$\ln L = \sum_{i=1}^{n} \ln L(a_{x_i}, a_{y_i}, t_{x_i}, t_{y_i}, b_{x_i}, b_{y_i}, D_{x_i}, D_{y_i}). \tag{4.9}$$

The likelihood can now be maximized using the data-set described in Section 4.2. Standard function maximization routines yield the maximum likelihood estimates and the variance-covariance align. Estimation results are presented in Subsections 4.4.2 and 4.5.2.

Note that for the semi-nonparametric models, the derivation of the log-likelihood is easily to obtain. Recall that the density is given by:

$$g_\alpha(x, y) = \frac{1}{c} \left(\sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{i,j} \left(\tilde{x}^{\beta_x} \right)^i \left(\tilde{y}^{\beta_y} \right)^j 1_{[i,j]} \right)^2 f(x) f(y). \tag{4.10}$$

First, note that because of the truncation, the normalizing constant cancels out in the log-likelihood. Second, because we use the IW model, we can determine the integrals for x and y separately. All expressions are of the form:

$$\left(\frac{z}{\theta_z} \right)^{k\beta_z} \left(\frac{\beta_z}{\theta_z} \right) \left(\frac{z}{\theta_z} \right)^{\beta_z - 1} \exp \left[- \left(\frac{z}{\theta_z} \right)^{-\beta_z} \right] \text{ for } z \in x, y \text{ and } k \in \{0, .., 2n\}. \tag{4.11}$$

The primitive is given by:

$$- \sum_{i=0}^{k} \left(\frac{z}{\theta_z} \right)^{i\beta_z} \prod_{j=i}^{k} \left(I_{j=k} + (j+1) I_{j \neq k} \right) \exp \left[- \left(\frac{z}{\theta_z} \right)^{-\beta_z} \right], \tag{4.12}$$
$$\text{for } z \in x, y \text{ and } k \in \{0, .., 2n\}.$$

To obtain for instance the integrals needed for $\overline{F}(a_x + b_x, a_y + b_y)$, all needed integrals are of the form:

$$\sum_{i=0}^{k} \left(\frac{a_z}{\theta_z} \right)^{i\beta_z} \prod_{j=i}^{k} \left(I_{j=k} + (j+1) I_{j \neq k} \right) \exp \left[- \left(\frac{a_z}{\theta_z} \right)^{-\beta_z} \right], \tag{4.13}$$
$$\text{for } z \in x, y \text{ and } k \in \{0, .., 2n\}.$$

The integral needed for $\overline{F}(x, y)$ is displayed in Appendix 4.D.

4.B The Metropolis-Hastings Algorithm

To determine the linear correlation and Spearman's ρ_s we sample from the estimated distribution function using the Metropolis-Hastings Algorithm, initially developed by Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller (1953) and generalized by Hastings (1970). We follow the procedure as described in Train (2002).

We would like to draw a random sample of 50,000 of ages of death of a married couple from the estimated density function $f(x,y)$. The procedure for one draw is as followed:

1. We start with drawing the ML estimates $\tau = (\beta, \theta, \delta)$, where $\tau \sim N(\hat{\tau}, \hat{V}_\tau / n)$, with $\hat{\tau}$ the ML estimates of τ.

2. We draw a value of the vector ϵ, labeled ϵ^0, where $\epsilon = (\epsilon_x, \epsilon_y)$. The following holds:
$$\epsilon_j = \theta_j. \left(-\ln\left(1 - v_j\right)\right)^{1/\beta_j} \quad \text{for } j = x, y,$$
where v_j is uniform distributed on (0,1). By construction, ϵ_j follows a Weibull distribution.

3. The variance of a random age of death variable Z, which follows a Weibull distribution is given by $Var(Z) = \theta_z^2 * \Gamma(2/\beta_z + 1) - (\theta_z * \gamma(1/\beta_z + 1))^2$ for $z \in x, y$ (Lu and Bhattacharyya, 2003, see). We draw a trial value of ϵ^1 as $\tilde{\epsilon_j}^1 = \epsilon_j^0 + \eta_j$, where η_j follows a normal distribution with mean 0 and covariance align $\Sigma = diag(Var(X), Var(Y))$. [15]

4. The density at the trial value $\tilde{\epsilon}^1$ is compared with the density at the original value ϵ^0. If $f(\tilde{\epsilon}^1) > f(\epsilon^0)$, $\tilde{\epsilon}^1$ is accepted and labeled ϵ^1. We then move to step 5. If $f(\tilde{\epsilon}^1) \leq f(\epsilon^0)$, $\tilde{\epsilon}^1$ is accepted with probability $f(\tilde{\epsilon}^1)/f(\epsilon^0)$, and rejected with probability $1 - f(\tilde{\epsilon}^1)/f(\epsilon^0)$.

5. We choose a trial value of ϵ^2 as $\tilde{\epsilon}^2 = \epsilon^1 + \eta$, where η is drawn from $g(\eta)$ and apply the rule in step 4 to either accept or reject $\tilde{\epsilon}^2$.

The procedure described above is repeated until we have generated a sample of 50,000 couples. Each sample of 50,000 couples is used to estimate the linear correlation and Spearman's ρ. To obtain a confidence interval of the correlation and Spearman's ρ we have generated 400 samples of 50,000 couples.

[15] For the semi-nonparametric model we use the expression of $E(X^k Y^l)$ as given in Appendix 4.D to determine $Var(X)$ and $Var(Y)$.

4.C The different polynomials

(i)

$$P_\alpha = \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y};$$

(ii)

$$P_\alpha = \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y} + \alpha_{1,1}\widetilde{x}^{\beta_x}\widetilde{y}^{\beta_y};$$

(iii)

$$P_\alpha = \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y} + \alpha_{1,1}\widetilde{x}^{\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,0}\widetilde{x}^{2\beta_x} + \alpha_{0,2}\widetilde{y}^{2\beta_y};$$

(iv)

$$\begin{aligned} P_\alpha = {} & \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y} + \alpha_{1,1}\widetilde{x}^{\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,0}\widetilde{x}^{2\beta_x} + \alpha_{0,2}\widetilde{y}^{2\beta_y} \\ & + \alpha_{1,2}\widetilde{x}^{\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{2,1}\widetilde{x}^{2\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,2}\widetilde{x}^{2\beta_x}\widetilde{y}^{2\beta_y}; \end{aligned}$$

(v)

$$\begin{aligned} P_\alpha = {} & \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y} + \alpha_{1,1}\widetilde{x}^{\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,0}\widetilde{x}^{2\beta_x} + \alpha_{0,2}\widetilde{y}^{2\beta_y} \\ & + \alpha_{1,2}\widetilde{x}^{\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{2,1}\widetilde{x}^{2\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,2}\widetilde{x}^{2\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{3,0}\widetilde{x}^{3\beta_x} + \alpha_{0,3}\widetilde{y}^{3\beta_y}; \end{aligned}$$

(vi)

$$\begin{aligned} P_\alpha = {} & \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y} + \alpha_{1,1}\widetilde{x}^{\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,0}\widetilde{x}^{2\beta_x} + \alpha_{0,2}\widetilde{y}^{2\beta_y} \\ & + \alpha_{1,2}\widetilde{x}^{\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{2,1}\widetilde{x}^{2\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,2}\widetilde{x}^{2\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{3,0}\widetilde{x}^{3\beta_x} + \alpha_{0,3}\widetilde{y}^{3\beta_y} \\ & + \alpha_{1,3}\widetilde{x}^{\beta_x}\widetilde{y}^{3\beta_y} + \alpha_{3,1}\widetilde{x}^{3\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,3}\widetilde{x}^{2\beta_x}\widetilde{y}^{3\beta_y} + \alpha_{3,2}\widetilde{x}^{3\beta_x}\widetilde{y}^{2\beta_y} \\ & + \alpha_{3,3}\widetilde{x}^{3\beta_x}\widetilde{y}^{3\beta_y}; \end{aligned}$$

(vii)

$$\begin{aligned} P_\alpha = {} & \alpha_{0,0} + \alpha_{1,0}\widetilde{x}^{\beta_x} + \alpha_{0,1}\widetilde{y}^{\beta_y} + \alpha_{1,1}\widetilde{x}^{\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,0}\widetilde{x}^{2\beta_x} + \alpha_{0,2}\widetilde{y}^{2\beta_y} \\ & + \alpha_{1,2}\widetilde{x}^{\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{2,1}\widetilde{x}^{2\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,2}\widetilde{x}^{2\beta_x}\widetilde{y}^{2\beta_y} + \alpha_{3,0}\widetilde{x}^{3\beta_x} + \alpha_{0,3}\widetilde{y}^{3\beta_y} \\ & + \alpha_{1,3}\widetilde{x}^{\beta_x}\widetilde{y}^{3\beta_y} + \alpha_{3,1}\widetilde{x}^{3\beta_x}\widetilde{y}^{\beta_y} + \alpha_{2,3}\widetilde{x}^{2\beta_x}\widetilde{y}^{3\beta_y} + \alpha_{3,2}\widetilde{x}^{3\beta_x}\widetilde{y}^{2\beta_y} \\ & + \alpha_{3,3}\widetilde{x}^{3\beta_x}\widetilde{y}^{3\beta_y} + \alpha_{4,0}\widetilde{x}^{4\beta_x} + \alpha_{0,4}\widetilde{y}^{4\beta_y}; \end{aligned}$$

4.D Moments and dependence in the semi-nonparametric model

In the previous subsection, we have explained the semi-nonparametric model we use to estimate the joint survival probabilities of spouses. In this subsection, we determine when the semi-nonparametric model is positive dependence or negative dependence. We also determine the joint survival function and the moments. Moreover, we also determine the linear correlation ($Corr(X,Y)$) between the remaining lifetimes of spouses based on the semi-nonparametric model. To determine the dependence, we rewrite the semi-nonparametric model as

$$P(X \geq x, Y \geq y) = \frac{\alpha' A_{\theta_x,\beta_x}(x) \odot B_{\theta_y,\beta_y}(y)\alpha}{\alpha'\Omega\alpha} \exp\left[-\left(\frac{x}{\theta_x}\right)^{\beta_x}\right] \exp\left[-\left(\frac{y}{\theta_y}\right)^{\beta_y}\right] \tag{4.14}$$

where $A_{\theta_x,\beta_x}(x)$ and $B_{\theta_y,\beta_y}(y)$ are displayed at the end of this appendix for the TOP model. $A \odot B$ means that A and B should be multiplied componentwise (the Hadamadard product).

The model is PQD when

$$\frac{\alpha' A_{\theta_x,\beta_x}(x) \odot B_{\theta_y,\beta_y}(y)\alpha}{\alpha'\Omega\alpha} \geq \frac{\alpha' A_{\theta_x,\beta_x}(x) \odot B_{\theta_y,\beta_y}(0)\alpha}{\alpha'\Omega\alpha} \frac{\alpha' A_{\theta_x,\beta_x}(0) \odot B_{\theta_y,\beta_y}(y)\alpha}{\alpha'\Omega\alpha} \; \forall\, x, y \geq 0 \tag{4.15}$$

where $A_{\theta_x,\beta_x}(0)$ and $B_{\theta_y,\beta_y}(0)$ do not depend on $\theta_x, \beta_x, \theta_y$, and β_y.

To determine the linear correlation, we use the following:

Define

$$\tilde{G}(k,l,\theta_x,\theta_y) :\equiv \tilde{G}(k,l,\beta_x,\beta_y,\theta_x,\theta_y) \equiv \frac{1}{c}\int_0^\infty \int_0^\infty x^k y^l P(x,y) f(x,y)\, dx dy, \tag{4.16}$$

such that

$$E(X^k Y^l) \equiv \alpha' \tilde{G}(k,l,\theta_x,\theta_y)\alpha. \tag{4.17}$$

and define

$$G(k,l) \equiv \tilde{G}(k,l,1,1). \tag{4.18}$$

Since $\int_0^\infty x^k f(x)dx = \theta_x \Gamma(1 + k/\beta_x)$, (see Lu and Bhattacharyya, 2003) and since $f(x,y) = f(x)f(y)$, we can rewrite $E(X^k Y^l)$ as

$$E(X^k Y^l) \equiv \theta_x^k \theta_y^l \alpha' G(k,l)\alpha. \tag{4.19}$$

Therefore, the linear correlation between the remaining lifetimes of spouses at birth based on the semi-nonparametric model is given by:

$$Corr(X,Y) = \frac{\alpha' G(1,1)\alpha - (\alpha' G(1,0)\alpha)\,(\alpha' G(0,1)\alpha)}{\left[\alpha' G(2,0)\alpha - (\alpha' G(1,0)\alpha)^2\right]^{\frac{1}{2}} \left[\alpha' G(0,2)\alpha - (\alpha' G(0,1)\alpha)^2\right]^{\frac{1}{2}}}, \tag{4.20}$$

So, the correlation between the remaining lifetimes of spouses at birth does not depend on the scaling parameters (θ_x, θ_y).

To determine the matrices $A(x)$ and $B(y)$ for the TOP polynomial, note that in the TOP model

$\alpha = [\; \alpha_{0,0} \quad \alpha_{1,0} \quad \alpha_{0,1} \quad \alpha_{1,1} \quad \alpha_{2,0} \quad \alpha_{0,2} \quad \alpha_{1,2} \quad \alpha_{2,1} \quad \alpha_{2,2} \quad \alpha_{3,0} \quad \alpha_{0,3} \;]$. Define $\tilde{x} := \left(\frac{x}{\theta_x}\right)^{\beta x}$ and similar define $\tilde{y} := \left(\frac{y}{\theta_y}\right)^{\beta y}$ such that $P(x,y)^2$ is given by:

$$P(x,y)^2 = \begin{pmatrix}
1 & \tilde{x} & \tilde{y} & \tilde{x}\tilde{y} & \tilde{x}^2 & \tilde{y}^2 & \tilde{x}\tilde{y}^2 & \tilde{x}^2\tilde{y} & \tilde{x}^2\tilde{y}^2 & \tilde{x}^3 & \tilde{y}^3 \\
\tilde{x} & \tilde{x}^2 & \tilde{x}\tilde{y} & \tilde{x}^2\tilde{y} & \tilde{x}^3 & \tilde{x}\tilde{y}^2 & \tilde{x}^2\tilde{y}^2 & \tilde{x}^3\tilde{y} & \tilde{x}^3\tilde{y}^2 & \tilde{x}^4 & \tilde{x}\tilde{y}^3 \\
\tilde{y} & \tilde{x}\tilde{y} & \tilde{y}^2 & \tilde{x}\tilde{y}^2 & \tilde{x}^2\tilde{y} & \tilde{y}^3 & \tilde{x}\tilde{y}^3 & \tilde{x}^2\tilde{y}^2 & \tilde{x}^2\tilde{y}^3 & \tilde{x}^3\tilde{y} & \tilde{y}^4 \\
\tilde{x}\tilde{y} & \tilde{x}^2\tilde{y} & \tilde{x}\tilde{y}^2 & \tilde{x}^2\tilde{y}^2 & \tilde{x}^3\tilde{y} & \tilde{x}\tilde{y}^3 & \tilde{x}^2\tilde{y}^3 & \tilde{x}^3\tilde{y}^2 & \tilde{x}^3\tilde{y}^3 & \tilde{x}^4\tilde{y} & \tilde{x}\tilde{y}^4 \\
\tilde{x}^2 & \tilde{x}^3 & \tilde{x}^2\tilde{y} & \tilde{x}^3\tilde{y} & \tilde{x}^4 & \tilde{x}^2\tilde{y}^2 & \tilde{x}^3\tilde{y}^2 & \tilde{x}^4\tilde{y} & \tilde{x}^4\tilde{y}^2 & \tilde{x}^5 & \tilde{x}^2\tilde{y}^3 \\
\tilde{y}^2 & \tilde{x}\tilde{y}^2 & \tilde{y}^3 & \tilde{x}\tilde{y}^3 & \tilde{x}^2\tilde{y}^2 & \tilde{y}^4 & \tilde{x}\tilde{y}^4 & \tilde{x}^2\tilde{y}^3 & \tilde{x}^2\tilde{y}^4 & \tilde{x}^3\tilde{y}^2 & \tilde{y}^5 \\
\tilde{x}\tilde{y}^2 & \tilde{x}^2\tilde{y}^2 & \tilde{x}\tilde{y}^3 & \tilde{x}^2\tilde{y}^3 & \tilde{x}^3\tilde{y}^2 & \tilde{x}\tilde{y}^4 & \tilde{x}^2\tilde{y}^4 & \tilde{x}^3\tilde{y}^3 & \tilde{x}^3\tilde{y}^4 & \tilde{x}^4\tilde{y}^2 & \tilde{x}\tilde{y}^5 \\
\tilde{x}^2\tilde{y} & \tilde{x}^3\tilde{y} & \tilde{x}^2\tilde{y}^2 & \tilde{x}^3\tilde{y}^2 & \tilde{x}^4\tilde{y} & \tilde{x}^2\tilde{y}^3 & \tilde{x}^3\tilde{y}^3 & \tilde{x}^4\tilde{y}^2 & \tilde{x}^4\tilde{y}^3 & \tilde{x}^5\tilde{y} & \tilde{x}^2\tilde{y}^4 \\
\tilde{x}^2\tilde{y}^2 & \tilde{x}^3\tilde{y}^2 & \tilde{x}^2\tilde{y}^3 & \tilde{x}^3\tilde{y}^3 & \tilde{x}^4\tilde{y}^2 & \tilde{x}^2\tilde{y}^4 & \tilde{x}^3\tilde{y}^4 & \tilde{x}^4\tilde{y}^3 & \tilde{x}^4\tilde{y}^4 & \tilde{x}^5\tilde{y}^2 & \tilde{x}^2\tilde{y}^5 \\
\tilde{x}^3 & \tilde{x}^4 & \tilde{x}^3\tilde{y} & \tilde{x}^4\tilde{y} & \tilde{x}^5 & \tilde{x}^3\tilde{y}^2 & \tilde{x}^4\tilde{y}^2 & \tilde{x}^5\tilde{y} & \tilde{x}^5\tilde{y}^2 & \tilde{x}^6 & \tilde{x}^3\tilde{y}^3 \\
\tilde{y}^3 & \tilde{x}\tilde{y}^3 & \tilde{y}^4 & \tilde{x}\tilde{y}^4 & \tilde{x}^2\tilde{y}^3 & \tilde{y}^5 & \tilde{x}\tilde{y}^5 & \tilde{x}^2\tilde{y}^4 & \tilde{x}^2\tilde{y}^5 & \tilde{x}^3\tilde{y}^3 & \tilde{y}^6
\end{pmatrix}. \tag{4.21}$$

Define $A_{\theta_x,\beta_x}(x) = [A^1 A^2]$ with A^1 and A^2 respectively given by:

$$A^1 = \begin{pmatrix} 1 & \tilde{x}+1 & 1 & \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & 1 \\ \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}+1 \\ 1 & \tilde{x}+1 & 1 & \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & 1 \\ \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}+1 \\ \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^2+2\tilde{x}+2 \\ 1 & \tilde{x}+1 & 1 & \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & 1 \\ \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}+1 \\ \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^2+2\tilde{x}+2 \\ \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^2+2\tilde{x}+2 \\ \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^5+5\tilde{x}^4+20\tilde{x}^3+60\tilde{x}^2+120\tilde{x}+120 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 \\ 1 & \tilde{x}+1 & 1 & \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & 1 \end{pmatrix}, \tag{4.22}$$

$$A^2 = \begin{pmatrix} \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & 1 \\ \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}+1 \\ \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & 1 \\ \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}+1 \\ \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^5+5\tilde{x}^4+20\tilde{x}^3+60\tilde{x}^2+120\tilde{x}+120 & \tilde{x}^2+2\tilde{x}+2 \\ \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & 1 \\ \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}+1 \\ \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^5+5\tilde{x}^4+20\tilde{x}^3+60\tilde{x}^2+120\tilde{x}+120 & \tilde{x}^2+2\tilde{x}+2 \\ \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^5+5\tilde{x}^4+20\tilde{x}^3+60\tilde{x}^2+120\tilde{x}+120 & \tilde{x}^2+2\tilde{x}+2 \\ \tilde{x}^4+4\tilde{x}^3+12\tilde{x}^2+24\tilde{x}+24 & \tilde{x}^5+5\tilde{x}^4+20\tilde{x}^3+60\tilde{x}^2+120\tilde{x}+120 & \tilde{x}^5+5\tilde{x}^4+20\tilde{x}^3+60\tilde{x}^2+120\tilde{x}+120 & \tilde{x}^6+6\tilde{x}^5+30\tilde{x}^4+120\tilde{x}^3+360\tilde{x}^2+720\tilde{x}+720 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 \\ \tilde{x}+1 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^2+2\tilde{x}+2 & \tilde{x}^3+3\tilde{x}^2+6\tilde{x}+6 & 1 \end{pmatrix}. \tag{4.23}$$

Similarly, define and $B(y)_{\theta_y,\beta_y}(y) = [B^1 B^2]$ with B^1 and B^2 respectively given by:
is given by:

$$B^1 = \begin{pmatrix} 1 & 1 & \tilde{y}+1 & \tilde{y}+1 & 1 & \tilde{y}^2+2\tilde{y}+2 \\ 1 & 1 & \tilde{y}+1 & \tilde{y}+1 & 1 & \tilde{y}^2+2\tilde{y}+2 \\ \tilde{y}+1 & \tilde{y}+1 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}+1 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 \\ \tilde{y}+1 & \tilde{y}+1 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}+1 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 \\ 1 & 1 & \tilde{y}+1 & \tilde{y}+1 & 1 & \tilde{y}^2+2\tilde{y}+2 \\ \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 \\ \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 \\ \tilde{y}+1 & \tilde{y}+1 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}+1 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 \\ \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 \\ 1 & 1 & \tilde{y}+1 & \tilde{y}+1 & 1 & \tilde{y}^2+2\tilde{y}+2 \\ \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^5+5\tilde{y}^4+20\tilde{y}^3+60\tilde{y}^2+120\tilde{y}+120 \end{pmatrix}. \tag{4.24}$$

$$B^2 = \begin{pmatrix}
\tilde{y}^2+2\tilde{y}+2 & \tilde{y}+1 & \tilde{y}^2+2\tilde{y}+2 & 1 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 \\
\tilde{y}^2+2\tilde{y}+2 & \tilde{y}+1 & \tilde{y}^2+2\tilde{y}+2 & 1 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 \\
\tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}+1 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 \\
\tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}+1 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 \\
\tilde{y}^2+2\tilde{y}+2 & \tilde{y}+1 & \tilde{y}^2+2\tilde{y}+2 & 1 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 \\
\tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^5+5\tilde{y}^4+20\tilde{y}^3+60\tilde{y}^2+120\tilde{y}+120 \\
\tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^5+5\tilde{y}^4+20\tilde{y}^3+60\tilde{y}^2+120\tilde{y}+120 \\
\tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}+1 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 \\
\tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^2+2\tilde{y}+2 & \tilde{y}^5+5\tilde{y}^4+20\tilde{y}^3+60\tilde{y}^2+120\tilde{y}+120 \\
\tilde{y}^2+2\tilde{y}+2 & \tilde{y}+1 & \tilde{y}^2+2\tilde{y}+2 & 1 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 \\
\tilde{y}^5+5\tilde{y}^4+20\tilde{y}^3+60\tilde{y}^2+120\tilde{y}+120 & \tilde{y}^4+4\tilde{y}^3+12\tilde{y}^2+24\tilde{y}+24 & \tilde{y}^5+5\tilde{y}^4+20\tilde{y}^3+60\tilde{y}^2+120\tilde{y}+120 & \tilde{y}^3+3\tilde{y}^2+6\tilde{y}+6 & \tilde{y}^6+6\tilde{y}^5+30\tilde{y}^4+120\tilde{y}^3+360\tilde{y}^2+720\tilde{y}+720
\end{pmatrix}. \tag{4.25}$$

The joint survival probabilities are given by:

$$P(X \geq x, Y \geq y) = \frac{\alpha' A_{\theta_x,\beta_x}(x) \odot B_{\theta_y,\beta_y}(y)\alpha}{\alpha'\Omega\alpha} \exp\left[-\left(\frac{x}{\theta_x}\right)^{\beta_x}\right] \exp\left[-\left(\frac{y}{\theta_y}\right)^{\beta_y}\right]. \tag{4.26}$$

Note that the following holds:

$$P(X > x) = P(X > x, Y > 0) = \frac{\alpha' A_{\theta_x,\beta_x}(x) \odot B_{\theta_y,\beta_y}(0)\alpha}{\alpha'\Omega\alpha} \exp\left[-\left(\frac{x}{\theta_x}\right)^{\beta_x}\right]. \tag{4.27}$$

Consequently, we can write the condition for PQD as

$$\frac{\alpha' A_{\theta_x,\beta_x}(x) \odot B_{\theta_y,\beta_y}(y)\alpha}{\alpha'\Omega\alpha} \exp\left[-\left(\frac{x}{\theta_x}\right)^{\beta_x}\right] \exp\left[-\left(\frac{y}{\theta_y}\right)^{\beta_y}\right] \geq \frac{\alpha' A_{\theta_x,\beta_x}(x) \odot B_{\theta_y,\beta_y}(0)\alpha}{\alpha'\Omega\alpha} \frac{\alpha' A_{\theta_x,\beta_x}(0) \odot B_{\theta_y,\beta_y}(y)\alpha}{\alpha'\Omega\alpha} \exp\left[-\left(\frac{x}{\theta_x}\right)^{\beta_x}\right] \exp\left[-\left(\frac{y}{\theta_y}\right)^{\beta_y}\right]. \tag{4.28}$$

4.E A likelihood ratio test for overlapping models

To test whether the semi-nonparametric model is better than the PNW, we perform the LR test for overlapping models as in Vuong (1989). We perform the test only for the *TOP* model, see (4.8). The LR Test Statistic for the model F_η against the model G_γ is:

$$LR_n(\hat{\eta}_n, \hat{\gamma}_n) = \log L_n^f(\hat{\eta}_n) - \log L_n^g(\hat{\gamma}_n), \tag{4.29}$$

where $\hat{\eta}_n$ and $\hat{\gamma}_n$ are the ML estimates of η_* and γ_*.

The sequential procedure as given in Vuong (1989) is the following:

1. Test $H_0^\omega : \omega_*^2 = 0$ against $H_A^\omega : \omega_*^2 \neq 0$ using the variance test based on $n\tilde{\omega}_n^2$, where ω_*^2 denotes the variance of $\log\left(f(Z_i|\eta_*)/g(Z_i|\gamma_*)\right)$.

2. If H_0^ω is not rejected, conclude that both distributions cannot be discriminated given the data.

3. If H_0^ω is rejected, test $H_0 : E^0\left[\frac{f(Z_i|\eta_*)}{g(Z_i|\gamma_*)}\right] = 0$ against $H_f : E^0\left[\frac{f(Z_i|\eta_*)}{g(Z_i|\gamma_*)}\right] > 0$ or $H_g : E^0\left[\frac{f(Z_i|\eta_*)}{g(Z_i|\gamma_*)}\right] < 0$ using the normal model selection test based on $n^{-1/2}LR_n(\hat{\eta}_n, \hat{\gamma}_n)/\tilde{\omega}_n$, where in our case f is the probability distribution of the *PNW* model, g is the probability distribution of the *TOP* model, $\hat{\eta}_n$ represent the maximum likelihood estimates of the *PNW* model and $\hat{\gamma}_n$ represent the maximum likelihood estimate of the *TOP* model.

Under H_0^ω $n\tilde{\omega}_n^2$ converges to a chi-square distribution. Under H_0, $n^{-1/2}LR_n(\hat{\eta}_n, \hat{\gamma}_n)/\tilde{\omega}_n \xrightarrow{D} N(0,1)$, whereas under H_f, $n^{-1/2}LR_n(\hat{\eta}_n, \hat{\gamma}_n)/\tilde{\omega}_n \xrightarrow{D} \infty$ and under H_g, $n^{-1/2}LR_n(\hat{\eta}_n, \hat{\gamma}_n)/\tilde{\omega}_n \xrightarrow{D} -\infty$.

We find that $\tilde{\omega}_n^2 = 0.001202$ such that $n\tilde{\omega}_n^2 = 601$. Consequently, H_0^ω is rejected at all conventional confidence levels. We find that $n^{-1/2}LR_n(\hat{\eta}_n, \hat{\gamma}_n)/\hat{\omega}_n = -101$. We can reject H_0 in favor of H_g at all conventional confidence levels. So, the *TOP* model fits the data better than the *PNW* model.

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