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Cooperative Investment in Incomplete Markets Under Financial Fairness

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Abstract

Groups of agents, such as participants in a collective pension fund, can decide to undertake a joint investment and to define, ex ante, a rule for the division of proceeds. The collective investment decision and the allocation rule together form a risk sharing scheme. Such a scheme defines a contingent claim for each participant. Given a proposed risk sharing scheme and an agreed-upon pricing functional, the values of these claims can be determined and can be compared to the values of the contributions made by the agents. A risk sharing scheme is said to be financially fair if, for each agent, the value of the agent's claim as defined by the scheme is equal to the value of the agent's contribution. The paper provides conditions under which there exists a unique risk sharing scheme that is both Pareto efficient (in the sense of expected utility) and financially fair. Furthermore, an iterative algorithm is presented by which this scheme can be computed. The theory is illustrated by a simple example which shows that, in an incomplete financial market, agents may benefit substantially from forming a collective.

1 Introduction

In a widely cited paper published in 1968, Wilson [29] defined a "syndicate" as a group of individuals who must make a common decision under uncertainty, resulting in a monetary payoff that will be shared among them. The problem that was addressed by Wilson has lost nothing of its

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relevance. One possible application is the design of collective pension schemes. In recent years, so called collective defined-contribution (CDC) schemes have been drawing increased attention. In these schemes, the contribution by the employer is fixed and all risks are shared between active and retired participants only. The fund's asset allocation as well as the rules used for indexation of benefits imply particular forms of risk sharing between generations. As stressed, for instance, by Cui et al. [15] and Gollier [22], the design of a collective pension system calls for a joint decision on the asset allocation strategy and on the benefit policy. This is a decision of the type studied in the theory of syndicates.

In this paper, we consider group investment decisions under the assumption that agents agree on a pricing functional for financial contracts, so that it is possible to define a notion of "financial fairness". The payoffs of the agents are contingent claims, with payoff distributions that depend on the collective decision as well as on the allocation rule. On the basis of an agreed-upon pricing functional, the ex-ante value of the claim held by each agent can be determined when a risk sharing scheme (i.e. a combination of a collective decision and an allocation rule) is specified. The scheme is said to be *financially fair* if, for each agent, this claim value is equal to the value of the agent's contribution, which is available ex ante as well. Our principal aims are to provide conditions under which the imposition of financial fairness solves the bargaining problem, in the sense that it defines a unique solution within the set of all Pareto efficient solutions, and to provide an algorithm by which this solution can be effectively computed.

The redistribution feature distinguishes the problem discussed here from the group decision framework as developed by Arrow [3]. We assume no information asymmetry between the participants in the collective, in contrast to the theory of risk sharing within the principal-agent setting as discussed by Grossman and Hart [23] and many others. The point of view of the present paper is also different from the literature on investment games where the condition of financial fairness is typically not imposed (see Xia [30] and the references cited therein). The risk exchange that is effectively established by a financially fair risk sharing scheme can alternatively be viewed as an exchange of derivative contracts, and in this way the paper is related to the literature on security design [25, 7].

The prototypical example of a risk sharing problem of the type that we consider in this paper is the situation of a number of insurance companies who want to pool their risks. In the actuarial literature, the problem of optimal risk sharing within reciprocal reinsurance treaties was formulated already in the 1940's by de Finetti [16] (as cited by Seal [28]). The first modern treatment of this problem was given in 1962 by Borch [8], who characterized the collection of Pareto optimal solutions and explored the relations to general equilibrium theory; cf. Aase [1] for a survey and further references. In the work of Borch, the risk faced by the collective is given exogenously, as would be a natural assumption in the case of insurance pools. There are many situations, however, in which the collective can decide to a certain extent how much risk it wants to take; in particular this is true when agents form a pool to undertake a joint investment. The element of collective decision that is introduced in this way leads to the framework of syndicates as formulated by Wilson in the paper already cited above [29].

Financial fairness with respect to a given pricing functional is not addressed in the paper by Wilson, which was written at a time when contingent claim pricing was still in its infancy. Generally speaking however, the idea that ownership rights should constrain allocation rules is as old as the world. A simple way to respect the ownership rights is to divide the proceeds proportionally ex post, but as noted by Gale in 1977 [18], such a division rule is often not Pareto efficient. For instance, in cases where one participant in a group is much more risk tolerant than the others, it may be reasonable to let this participant carry the bulk of the risk, and to ensure balance with respect to ownership rights by means of deterministic side payments. Divisions that do not follow the simple proportional rule can in fact be found already in the Talmud [4].

Gale calls a risk sharing scheme *proportional* (in an ex-ante sense) if the expected values of the payoffs of agents are proportional to their ownership shares. The main purpose of the papers by Gale and Sobel [20, 21] is to show that, under various conditions, there is only one risk sharing scheme that is both Pareto efficient and proportional in this sense. A similar problem formulation was given in the actuarial literature by Bühlmann and Jewell [10, 11]. These authors used valuation on the basis of expectation under a risk-neutral measure, as is standard in modern financial economics for the purpose of pricing, in contrast to Gale and Sobel who worked with expectations under the statistical measure. Bühlmann and Jewell speak of a FAIRPOREX (fair and Pareto optimal risk exchange). The proof of uniqueness that appears in [10] is based on the original idea by Gale [18]. Another closely related problem formulation has been studied by Balasko [5] from the point of view of general equilibrium theory.

The papers by Gale and Sobel [20, 21], Bühlmann and Jewell [10, 11], and Balasko [5] cited above are all concerned with the situation of an exogenously given risk, as in Borch's work. The case of endogenous risk, as in Wilson's paper, has been addressed by Gale and Machado [19]. They prove that also in this case, under certain conditions, there is a unique Pareto optimal and proportional risk sharing scheme, which now consists of the combination of a joint decision and a rule for the allocation of proceeds to participants. While the formulation they use is stated in the terms of production economies, a reinterpretation in terms of risk exchanges is possible. Their conditions for uniqueness include separability of the production set; in the financial interpretation of their model, this condition comes down to market completeness.

In the present paper, we consider essentially the model of Gale and Machado, with the following main modifications: for the purpose of valuation we work with a risk-neutral measure, as in the work of Bühlmann and Jewell [10, 11]; and our main emphasis is on incomplete markets. The problem of cooperative investment in incomplete markets was discussed previously in a paper by Xia [30]. Xia however does not consider financial fairness, and relies instead on traditional game-theoretic criteria (coalitional stability) to narrow down the set of possible solutions. Our approach is in a sense opposite: we do impose financial fairness, and we do not make use of individual rationality as a criterion for admissibility of solutions. We argue that, while in an incomplete market there is no unique pricing functional determined by the market, the agents in a collective may still reach an agreement on which pricing functional to choose from the ones that are admitted by the market; support for this decision may come for instance from the theory of good-deal bounds [13]. There is no guarantee that a financially fair and Pareto efficient solution is utility-improving for every agent in a given collective; in such a case, our conclusion is simply that the collective may be unsuitable for risk sharing, and the composition of the collective should be modified accordingly.

The present paper can be viewed in the light of showing that cooperative investment under financial fairness market helps to reduce the detrimental effects of market incompleteness. As illustrated in a simple example below, agents in an incomplete market may derive considerable utility gains from participation in a collective, even if the collective does not have access to a larger asset space than is available to the individual agents. The reason is that trading within the collective effectively lifts some of the restrictions faced by agents, while financial fairness can still be maintained. The fact that in incomplete markets several constraints play a role, rather than just one as in the case of a complete market, is what makes it possible to relax restrictions while preserving financial fairness. For an example of internal trading within the context of a collective pension fund, see for instance Cui and Ponds [14]. Of course, in practice there can be additional benefits to collective investing, resulting for instance from economies of scale, which are not addressed in the stylized setting of this paper.

The main aims of the paper are: (i) to give conditions under which there exists exactly one combination of investment decision and allocation rule that is Pareto efficient as well as financially fair, and (ii) to prove convergence of an iterative method by which this risk sharing scheme can be computed. The paper is a continuation of [27] in which we addressed the construction of Pareto efficient and financially fair risk sharing rules in the case of exogenously given risk. As in the earlier paper, we employ the mathematical tools of nonlinear Perron-Frobenius theory. We work in the simple setting of a one-period model and a finite probability space. We consider collectives consisting of agents whose preferences can be described by expected utility, and we suppose that the agents have common beliefs in the sense that they all assign the same probabilities to events. The utility functions, which may be different for different agents, will be assumed to satisfy a number of standard conditions. We assume that investments take place in a liquid financial market so that the possible investment decisions are parametrized by a subspace of the payoff space.

The paper is organized as follows. Assumptions and notations are specified in Section 2. Section 3 presents the main results of the paper. Examples are provided in Section 5, and conclusions follow in Section 6. Most proofs are given in the appendix.

2 Notation, assumptions, and definitions

2.1 Mathematical conventions

Notation and terminology relating to vector inequalities will be as follows. Inequalities are interpreted componentwise; in other words, the expression $a \geq b$ for $a, b \in \mathbb{R}^n$ means that $a_i \geq b_i$ for all $i = 1, \ldots, n$, and $a > b$ means $a_i > b_i$ for all i. The expression $a \gneq b$ indicates that $a \geq b$ and $a \neq b$. The same notation will be used in the case of function spaces; when f and g are functions from a set D to $\mathbb{R}$, we write $f \geq g$ when $f(x) \geq g(x)$ for all $x \in D$, $f > g$ when $f(x) > g(x)$ for all $x \in D$, and $f \gneq g$ when $f \geq g$ while $f \neq g$.

A function f from (a subset of) $\mathbb{R}^n$ into $\mathbb{R}^m$ will be said to be *strictly increasing* if, for x and y in the domain of f, the inequality $x \gneq y$ implies that $f(x) \gneq f(y)$, and *strongly increasing* when $x \gneq y$ implies that $f(x) > f(y)$. A sequence of vectors $(a^k)_{k=1,2,\ldots}$ in $\mathbb{R}^n$ is said to *converge to infinity* if all scalar sequences $(a_i^k)_{k=1,2,\ldots}$ $(i = 1, \ldots, n)$ tend to infinity.

The *nonnegative orthant* and the *positive orthant* in $\mathbb{R}^n$ are defined by $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n \mid x > 0\}$, respectively. A *cone* in a real vector space V is a subset of V that is closed under multiplication by nonnegative scalars. The term *positive cone* refers to the set $\mathbb{R}^n_{++} \cup \{0\}$. The *closed unit simplex* (or simply *unit simplex*) in $\mathbb{R}^n$ is the set $\{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$. The restriction of this set to the positive orthant is called the *open unit simplex*. We use the term "homogeneous" as shorthand for "positively homogeneous of degree 1"; that is, a function f defined on a cone in $\mathbb{R}^n$ is *homogeneous* if the equality $f(\alpha x) = \alpha f(x)$ holds for all $\alpha > 0$ and all x in the domain of f.

2.2 Economic assumptions

The risks in the economy that we consider are modeled as random variables on a finite outcome space Ω. These random variables can be identified with vectors in $\mathbb{R}^m$, where m is the cardinality of Ω. The probability of outcome $\omega_j \in \Omega$ $(j = 1, \ldots, m)$ is given by $p_j > 0$, and the measure that is defined in this way is denoted by $\mathbb{P}$. The expectation operator under this measure is denoted by E, as opposed to $E^{\mathbb{Q}}$ which is the expectation under the pricing measure to be introduced below. Elements of $\mathbb{R}^m$ will also be referred to as *assets* or *contingent claims.*

We consider a group of $n < \infty$ agents in the economy. The agents are endowed with utility functions $u_i : (0, \infty) \to \mathbb{R}$ which are strictly increasing, strictly concave, and twice continuously differentiable, and which moreover satisfy the Inada conditions

$$\lim_{x \downarrow 0} u_i'(x) = \infty, \qquad \lim_{x \to \infty} u_i'(x) = 0. \tag{2.1}$$

The utility functions give rise to preference functions defined on the space of risks $\mathbb{R}^m$ by

$$U_i(X) = E[u_i(X)] = \sum_{j=1}^{m} p_j u_i(X_j), \qquad X \in \mathbb{R}^m_{++}. \tag{2.2}$$

The inverse of the marginal utility function u_i' is denoted by I_i; this is a strictly decreasing and continuous function defined on $(0, \infty)$ with

$$\lim_{z \downarrow 0} I_i(z) = \infty, \qquad \lim_{z \to \infty} I_i(z) = 0.$$

A pricing functional is defined on all of the asset space $\mathbb{R}^m$ by expectation under a measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$; we set the discount factor equal to unity without loss of generality. The $\mathbb{Q}$-probability of an outcome $\omega_j \in \Omega$ is denoted by q_j, so that the expression

$$E^{\mathbb{Q}} X = \sum_{j=1}^{m} q_j X_j$$

gives the price of an asset $X \in \mathbb{R}^m$.

We assume furthermore that a vector $X^{\text{in}} \in \mathbb{R}^m$ and a subspace $\mathcal{X}_0 \subset \{X \in \mathbb{R}^m \mid E^{\mathbb{Q}} X = 0\}$ are given, which are interpreted respectively as the *initial endowment of the group of agents* (collective risk) and the *space of zero-cost trades that are accessible to the group.* Under the assumption that the collective as a whole is only trading assets that have well-determined market prices, the space $\mathcal{X}_0$ is unambiguously defined even in an incomplete market setting. By its investment decision, the collective modifies its position from X^{in} to $X^{\text{in}} + X$, with $X \in \mathcal{X}_0$. We write

$$\mathcal{X} := \{X^{\text{in}} + X \mid X \in \mathcal{X}_0\}. \tag{2.3}$$

The space of accessible trades $\mathcal{X}_0$ is a subspace of the space of zero-price assets $\{X \mid E^{\mathbb{Q}} X = 0\}$; it is allowed, and actually it is the main situation of interest below, that the inclusion is strict. A standing assumption is that the initial endowment of the collective X^{in} and the space of accessible trades $\mathcal{X}_0$ are such that the space $\mathcal{X} = X^{\text{in}} + \mathcal{X}_0$ has nonempty intersection with the positive orthant. In other words, on the basis of its initial endowment it is possible for the collective to take positions that produce positive payoffs in every state of nature. This condition is required because the utility functions that we use are defined only on the positive half line.

Finally we assume that for each agent i a number w_i is given, which is referred to as the *initial wealth* of the agent or the *value of the contribution* of that agent to the group. These numbers will be used to define the notion of financial fairness for risk sharing rules. The numbers w_i must satisfy the consistency requirement

$$\sum_{i=1}^{n} w_i = E^{\mathbb{Q}} X^{\mathrm{in}}. \tag{2.4}$$

Moreover, the numbers w_i should be positive for the allocation problem to be meaningful.

For the purpose of the design of risk sharing systems along the lines of this paper, no further information concerning the agents is required. However, to be able to compare the outcomes of collective schemes to individual schemes, one needs to define initial endowments as well as accessible trading spaces for the individual agents. If $X_i^{\mathrm{in}} \in \mathbb{R}^m$ denotes the initial endowment of agent i ($i = 1, \dots, n$), then one may define

$$X^{\mathrm{in}} = \sum_{i=1}^{n} X_i^{\mathrm{in}} \quad \text{and} \quad w_i = E^{\mathbb{Q}} X_i^{\mathrm{in}}$$

so that the consistency requirement (2.4) is satisfied automatically.

2.3 Risk sharing schemes

A risk sharing scheme is a combination of a collective investment decision and a rule for the allocation of investment returns to the participants in the collective. In more formal terms, a risk sharing scheme is a tuple $(X; y)$ where $X \in \mathbb{R}^m$ and $y = (y_1, \dots, y_n)$ is a collection of functions from $(0, \infty)$ to $(0, \infty)$. The risk sharing scheme is said to be *feasible* if the vector X belongs to the constraint set $\mathcal{X}$ and is moreover positive, and the allocation functions $y_1, \dots, y_n$ satisfy the redistribution property

$$\sum_{i=1}^{n} y_i(x) = x \quad \text{for all } x > 0. \tag{2.5}$$

Positivity of the position X taken by the collective is needed, because the agent's utilities are defined only on the positive halfline. If there would be a state of nature in which the payoff of the collective would be nonpositive, then no feasible allocation assigns positive payoffs to all agents.

Above, we have introduced the values w_i of agent's contributions. These are used below to define the notion of financial fairness. The definition below follows Bühlmann and Jewell [11].

Definition 2.1. Let a pricing measure $\mathbb{Q}$, a collective risk X, and values of agent's contributions w_i be given, with $E^{\mathbb{Q}} X = \sum_{i=1}^{n} w_i$. An allocation rule $(y_1, \dots, y_n)$ is *financially fair* if $E^{\mathbb{Q}}[y_i(X)] = w_i$ for all $i = 1, \dots, n$.

We work with a notion of Pareto efficiency that is constrained only by feasibility, not by financial fairness.

Definition 2.2. A feasible risk sharing scheme $(X; y)$ for n agents with preference functions U_i is *Pareto efficient* if there does not exist another feasible scheme $(\tilde{X}; \tilde{y})$ such that

$$\big[U_1\big(\tilde{y}_1(\tilde{X})\big), \dots, U_n\big(\tilde{y}_n(\tilde{X})\big)\big] \gneq \big[U_1\big(y_1(X)\big), \dots, U_n\big(y_n(X)\big)\big].$$

In this paper we consider preferences to be given by expected utility as in (2.2). In (2.2), expectations are taken with respect to the objective probability measure $\mathbb{P}$, whereas the financial fairness constraint involves the pricing measure $\mathbb{Q}$.

Remark 2.3. Uniqueness of risk sharing schemes will always be understood in the sense of uniqueness with probability 1. In other words, two risk sharing schemes (X, y) and (X', y') (both for n agents) are considered to be the same if the random variables $Y_i := y_i(X)$ and $Y_i' := y_i'(X')$ are equal with probability 1 for all $i = 1, \ldots, n$.

3 Main result

In this section we establish, under some conditions, the existence of a unique Pareto efficient and financially fair (PEFF) risk sharing scheme. The result applies to situations in which the collective is to a certain extent free in choosing its risk exposure. The classical work of Borch [8] is concerned with the situation in which a collective faces a *given* risk, such as when a number of insurance companies form a pool with respect to the claims that may be received in a given year. It was shown by Borch that, in this case, the collection of all Pareto optimal solutions for risk sharing between n agents can be parametrized in terms of positive parameters $c_1, \ldots, c_n$, as follows:

$$y_i(x) = I_i\big(J(x)/c_i\big) \tag{3.1}$$

where the function $J(x) = J(x; c_1, \ldots, c_n)$ is defined implicitly by the requirement

$$\sum_{i=1}^{n} I_i\big(J(x)/c_i\big) = x \qquad (x > 0). \tag{3.2}$$

Below we also use the notation c to refer to the vector of parameters $(c_1, \ldots, c_n)$. As is easily verified, for fixed $c \in \mathbb{R}^n_{++}$, the equation (3.2) indeed uniquely defines $J(x)$ for each given $x > 0$. The proof of Borch's result (cf. for instance [17, 6] for details) is based on the fact that, under convexity assumptions, all points in the Pareto efficient set can be found by solving weighted-sum optimization problems of the form

$$\sum_{i=1}^{n} c_i E\big[u_i\big(y_i(X)\big)\big] \to \max \quad \text{subject to } \sum_{i=1}^{n} y_i(X) = X \tag{3.3}$$

where X is the given total risk. The restriction to positive (rather than nonnegative) coefficients c_i can be motivated by noting that the weighted-sum optimization problem that results from setting one or more (but not all) of the c_i's equal to zero does not give rise to boundary points of the set of achievable joint preference levels of the agents, since the levels of agents who are taken into account in the weighted sum can always be improved when the allocations to the agents with zero weight are made smaller.

When the total risk X is not fixed but can be chosen to a given extent by the collective, the idea of weighted-sum optimization still applies, but this time the optimization is carried out not only with respect to the allocation functions but also with respect to X. The optimal allocation functions for a given risk are provided by Borch's result; in fact, as noted by Borch, these functions do not depend on the distribution of the risk X because the problem (3.3) can be solved for each realization of X separately. As a consequence, the weighted-sum optimization problem can be formulated directly in terms of the risk X that is to be chosen by the collective:

$$\sum_{i=1}^{n} c_i E\big[u_i\big(I_i(J(X)/c_i)\big)\big] \to \max \quad \text{subject to } X \in \mathcal{X} \tag{3.4}$$

where the set $\mathcal{X} \subset \mathbb{R}^m$ represents the positions that can be taken. Given the above, it is convenient to introduce the function

$$u(x) = \sum_{i=1}^{n} c_i u_i\big(I_i(J(x)/c_i)\big) \tag{3.5}$$

which may also be written as $u(x; c)$ or $u(x; c_1, \dots, c_n)$ when it is desired to emphasize the dependence on the parameters c_i. Calculation shows (cf. [30]) that in fact

$$u'(x; c) = J(x; c) \tag{3.6}$$

so that, if $u(x; c)$ is interpreted as a weighted group utility, then $J(x; c)$ is the corresponding marginal group utility. It can be verified that the function $u(x)$ inherits all properties that were assumed for the utility functions u_i in Section 2.2. It then follows that the optimization problem in (3.4) has a unique solution. Consequently [30, Thm. 3.1], a risk sharing scheme $(\hat{X}; y)$ is Pareto efficient if and only if there exists a set of positive parameters $(c_1, \dots, c_n)$ such that

$$y_i(x) = I_i\big(J(x; c)/c_i\big) \quad (i = 1, \dots, n) \tag{3.7a}$$

$$\hat{X} = \arg\max_{X \in \mathcal{X}} E[u(X; c)] \tag{3.7b}$$

where the weighted group utility $u(\cdot)$ is defined in (3.5) and the constraint set $\mathcal{X}$ is given by (2.3). Financial fairness for the scheme above holds if

$$E^{\mathbb{Q}}[y_i(\hat{X})] = w_i. \tag{3.7c}$$

It can be verified immediately that the solutions of (3.7a–3.7b) corresponding to $(c_1, \dots, c_n)$ and $(\alpha c_1, \dots \alpha c_n)$ are identical for any positive α, so that we are free to impose the constraint

$$\sum_{i=1}^{n} c_i = 1 \tag{3.8}$$

on parameter vectors. Our parameter set then becomes the open unit simplex.

In the analysis below, an important role is played by the mapping from the open unit simplex to $\mathbb{R}^m$ that assigns to a vector $c = (c_1, \dots, c_n)$ the random variable $J(\hat{X})$, where $\hat{X}$ is defined by the optimization problem (3.7b) and J is defined in (3.2). For a fixed random variable X, the mapping $c \mapsto J(X;c)$ is strictly (even strongly) increasing; this follows from the fact that the function $I(z;c)$ is increasing in c, which in turn is a consequence of the fact that the marginal utilities u_i' and hence also their inverses I_i are decreasing functions of their arguments. However, the same may not hold for the mapping $c \mapsto J(\hat{X})$, since the location of the optimum $\hat{X}$ depends on the parameter vector c. An example of nonmonotonicity is shown in the appendix. Whether or not monotonicity holds depends on the specifications of the preferences of the agents, as well as on the restrictions that are imposed on the positions that may be taken by the collective. We state the monotonicity below as an assumption.

Assumption 3.1. The mapping $c \mapsto J(\hat{X};c)$, where J is defined by (3.2) and $\hat{X}$ by (3.7b), is strictly increasing.

One sufficient condition for this assumption to be satisfied is that the agents form a so called equicautious HARA collective; see the discussion in Section 4.3 below. While that is a condition in terms of the agents' utilities, the proposition below (which is proved in the appendix) gives a sufficient condition that is independent of the utilities and instead is stated in terms of the constraint set.

Proposition 3.2. *Assumption 3.1 is satisfied if there exists a random variable G defined on Ω such that the space of accessible zero-cost trades $\mathcal{X}_0$ is given by*

$$\mathcal{X}_0 = \{X \in \mathbb{R}^m \mid E^{\mathbb{Q}}[X \mid G] = 0\}. \tag{3.9}$$

The condition appearing in (3.9) may be rewritten in a more explicit form as

$$\sum_{j \in S(g)} q_j X_j = 0 \quad \text{for all } g \text{ such that } \mathbb{P}(G = g) > 0$$

where

$$S(g) = \{j \mid G(\omega_j) = g\}.$$

The random variable G can be thought of as a device by which the outcomes ω_j are grouped into non-overlapping categories. Situations in which there indeed exists such a categorizer include the case of no choice ($\mathcal{X}_0 = \{0\}$; take $G = X$) and the complete-market case (take $G =$ constant). An example of a situation that does not correspond to one of these extremes is the following.

Example 3.3. Let Ω consist of four points, and assume that the space $\mathcal{X}_0$ is spanned by the vectors $[1, 1, -1, -1]$ and $[1, -1, 1, -1]$. This situation is obtained when the accessible assets are bets on the results of two Bernoulli trials, both with equal $\mathbb{Q}$-probabilities of the two outcomes "heads" and "tails". The space Ω may then be described as $\{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$. Introduce a new random variable G by $G(\mathtt{HH}) = G(\mathtt{TT}) = 1$ and $G(\mathtt{HT}) = G(\mathtt{TH}) = 2$. The condition (3.9) is equivalent to the two conditions $X(\mathtt{HH}) + X(\mathtt{TT}) = 0$ and $X(\mathtt{HT}) + X(\mathtt{TH}) = 0$, which together characterize the space $\mathcal{X}_0$.

At this point we can state the main result of this paper and give an outline of its proof. Details of the proof are provided in the appendix. For the reader's convenience, we recapitulate the setting. Our context is a single-period financial market, given by a finite probability space with objective probability measure $\mathbb{P}$ and pricing measure $\mathbb{Q}$. A collective is formed by n agents whose characteristics are given by their utility functions u_i (of the type described in Section 2.2) and by the market values $w_i > 0$ of their contributions. A risk sharing scheme consists of a combination of a collective investment decision and a rule for allocation of the investment returns to the participants. The position that can be taken by the collective is constrained to lie in a given affine subset $\mathcal{X}$ of the asset space $\mathbb{R}^m$ such that $E^{\mathbb{Q}}X = w := \sum_{i=1}^n w_i$ for all $X \in \mathcal{X}$. We are looking for risk sharing schemes that are both financially fair in the sense of Def. 2.1 and unconstrained Pareto efficient in the sense of Def. 2.2 (i.e. Pareto efficiency should hold even among schemes that are not financially fair). Our main result is as follows.

Theorem 3.4. *In the setting as described above, if Assumption 3.1 holds, then there exists a unique risk sharing scheme that is both Pareto efficient and financially fair.*

The proof is based on the parametrization of Pareto efficient solutions in terms of the unit simplex in $\mathbb{R}^n$. Each parameter vector $c = (c_1, \dots, c_n) \in \mathbb{R}^n_{++}$ with $\sum_{i=1}^n c_i = 1$ gives rise to a Pareto efficient risk sharing scheme through the specification (3.7). The resulting individual risks of the agents are given by $Y_i = I_i(\hat{Z}(c)/c_i)$ where $\hat{Z}(c) := J(\hat{X})$ represents the group marginal utility corresponding to the collective decision $\hat{X}$ defined in (3.7b). On the other hand, by the fact that the individual inverse marginal utilities I_i are strictly decreasing functions traversing all positive values, any given positive random variable Z defined on Ω gives rise to a uniquely determined vector $\hat{c} \in \mathbb{R}^n_{++}$ through the equations

$$E^{\mathbb{Q}}[I_i(Z/\hat{c}_i)] = w_i \qquad (i = 1, \dots, n). \tag{3.10}$$

The Pareto efficient risk sharing scheme associated to a given parameter vector c is financially fair if and only if the mapping $Z \mapsto \hat{c}(Z)$ defined by (3.10) takes $\hat{Z}(c)$ to c. The problem of finding a Pareto efficient and financially fair risk sharing scheme can therefore be phrased as a fixed-point problem for the composite mapping

$$\varphi : c \mapsto \hat{Z} \mapsto \hat{c}. \tag{3.11}$$

This fixed-point problem is addressed in the appendix. For convenience, the mapping $c \mapsto \hat{Z}(c)$ will be formulated as the composition of two separate mappings, so that the mapping φ is actually written as a composition of three mappings. The required fixed-point theorem will be obtained as a consequence of an eigenvalue theorem in nonlinear Perron-Frobenius theory due to Oshime

[26]. To make application of this theorem possible, a number of properties of the mapping φ must be established; among other things, it needs to be shown that φ can be extended to a continuous mapping defined on the closed unit simplex, and that positive eigenvectors of the mapping φ can only occur with eigenvalue 1, so that a positive eigenvector of φ gives rise to a fixed point. These properties are shown in a series of lemmas which are relegated to the appendix.

The reformulation of the problem of finding a PEFF solution as a fixed-point problem naturally suggests the use of iteration as a numerical procedure. The mapping φ as defined above does not map the unit simplex into itself. However, the mapping always produces a strictly positive vector, so that we can construct a related mapping ψ that does map the unit simplex into itself by

$$\psi(c) = \frac{\varphi(c)}{\sum_{i=1}^{n}(\varphi(c))_i}\,. \tag{3.12}$$

A vector c in the open unit simplex is a fixed point of ψ if and only if it is an eigenvector of the mapping φ.

Theorem 3.5. *In the same setting as in Thm. 3.4, if Assumption 3.1 holds, then the sequence $(c^k)_{k=1,2,\dots}$ defined by $c^{k+1} = \psi(c^k)$ converges to the unique fixed point in the open unit simplex of the mapping φ defined in (3.11), for any choice of the initial point $c^0 \in \mathbb{R}^n_{++}$.*

The theorem can be proved in the same way as Thm. 3.10 in [27], on the basis of the properties of the mapping φ as established in the appendix.

4 Special cases

This section is devoted to a few special cases in which explicit solutions exist to the problem of finding a Pareto efficient and financially fair risk sharing scheme.

4.1 Identical preferences and identical contributions

One situation in which existence and uniqueness of the PEFF solution can be proved directly, without a monotonicity assumption, is the case of agents with identical preferences and identical contributions.

Proposition 4.1. *If a collective consists of n agents who all employ the same utility function $u_0(x)$ and whose contributions to the collective all have the same economic value w_0, then the risk sharing problem allows a unique Pareto efficient and financially fair solution. This solution is given by $(\hat{X}; y_1, \dots, y_n)$ with*

$$y_i(x) = x/n \quad (i = 1, \dots, n) \tag{4.1a}$$

$$\hat{X} = \arg\max_{X \in \mathcal{X}} E[u_0(X/n)] \tag{4.1b}$$

where $\mathcal{X}$ represents the set of positions available to the collective.

Proof. The parameter vector $c = [1, \dots, 1]$ leads to the solution (4.1) which is financially fair. To prove uniqueness, suppose that a parameter vector $(c_1, \dots, c_n) > 0$ gives rise to a risk sharing scheme $(X; y)$ that is Pareto efficient and financially fair. Write $Z = J(X; c)$, and let $I_0(\cdot)$ denote the inverse marginal utility corresponding to the utility function $u_0(x)$. The Pareto optimal allocation functions are given by (3.1). From the financial fairness constraint, we then obtain that $E^{\mathbb{Q}}[I_0(J(Z)/c_i)] = w_0$ for all $i = 1, \dots, n$, so that in particular for all i and j we have

$$E^{\mathbb{Q}}[I_0(J(Z)/c_i)] = E^{\mathbb{Q}}[I_0(J(Z)/c_j)].$$

Since the inverse marginal utility $I_0(\cdot)$ is strictly decreasing, the function $\alpha \mapsto E^{\mathbb{Q}}[I_0(J(Z)/\alpha)]$ is strictly monotonic. The equality above therefore implies that $c_i = c_j$, and we find that all entries of c must be equal. $\square$

In the situation of the proposition above, the agents may still differ in their initial endowments; also, they might face different trading constraints. As a consequence, the utility gains (or losses) of the agents in a Pareto efficient and financially fair risk sharing scheme may be different as well.

In an incomplete market, an agent may profit from risk sharing even if the collective does not have access to a larger space of trades than is available to individual agents. When agents enter a PEFF risk sharing scheme under the conditions of equal preferences and equal contributions, they effectively replace their initial endowment X_i^{in} by the average initial endowment $\frac{1}{n}\sum_{i=1}^n X_i^{\text{in}}$. This replacement can be beneficial to all participants when the collective is of suitable composition, as illustrated in the example in Section 5. When agents are heterogeneous in terms of preferences and/or contributions as well, the benefits of cooperation in a collective may even be larger.

4.2 Complete market

In case the collective faces a complete market, the PEFF solution can be implemented as follows: all agents turn over their wealth w_i to the collective and communicate their individual optimal investment plans; the collective then implements these plans, and returns to the agents the proceeds that they would have received if they would have implemented their plans on their own. The following proposition shows that this is indeed the only PEFF solution. We use X_i^* to indicate the solution of the individual optimization problem of agent i in a complete market:

$$\text{maximize } E[u_i(X)] \quad \text{subject to } E^{\mathbb{Q}}X = w_i. \tag{4.2}$$

Due to the assumptions we have imposed on the utility functions u_i, the above problem indeed has a unique solution $X_i^* > 0$. Moreover, the Lagrange multiplier associated to the budget constraint is positive, so that the optimal utility $E[u_i(X_i^*)]$ is a strictly increasing function of the available wealth w_i.

Proposition 4.2. *In a complete market, there is a unique Pareto efficient and financially fair risk sharing scheme, which is given by the collective investment decision*

$$X = \sum_{i=1}^{n} X_i^*, \quad \textit{with } X_i^* = \operatorname*{arg\,max}_{X:\, E^{\mathbb{Q}} X = w_i} E[u_i(X)] \tag{4.3}$$

and the allocation functions

$$y_i(X) = X_i^*. \tag{4.4}$$

Proof. The position X given by (4.3) satisfies $E^{\mathbb{Q}} X = \sum_{i=1}^{n} E^{\mathbb{Q}} X_i = \sum_{i=1}^{n} w_i = w$, and therefore, since the market is complete, this is a feasible position. The allocation rules are also obviously financially fair. To show Pareto efficiency, suppose that a not necessarily financially fair risk sharing scheme (X', y') is such that

$$\big(E[u_1(y_1'(X_1'))], \ldots, E[u_n(y_n'(X_n'))]\big) \geq \big(E[u_1(X_1^*)], \ldots, E[u_n(X_n^*)]\big). \tag{4.5}$$

Define $Y_i' = y_i'(X')$. Suppose first that there is an index k such that $E^{\mathbb{Q}} Y_k' \neq E^{\mathbb{Q}} X_k^*$. Because $\sum_{i=1}^{n} E^{\mathbb{Q}} Y_i' = w = \sum_{i=1}^{n} E^{\mathbb{Q}} X_i^*$, this means that there also exists an index (still denoted by k) such that $E^{\mathbb{Q}} Y_k' < E^{\mathbb{Q}} X_k^*$. Since X_k^* optimizes the utility of agent k subject only to the budget constraint, and since optimal utility is a strictly increasing function of available wealth, it follows that $E[u_k(Y_k')] < E[u_k(X_k^*)]$, which contradicts the inequality (4.5). Therefore, we must have $E^{\mathbb{Q}} Y_i' = E^{\mathbb{Q}} X_i^*$ for all i. In particular, $E^{\mathbb{Q}} Y_i' = w_i$ for all i. From the two relations $E^{\mathbb{Q}} Y_i' = w_i$ and $E[u_i(Y_i')] \geq E[u_i(X_i^*)]$ it follows that actually $Y_i' = X_i^*$. This shows that the proposed scheme is Pareto efficient. Finally, to show the uniqueness of the PEFF solution, consider any risk sharing scheme (X', y') that is Pareto efficient and financially fair, and again write $Y_i' = y_i'(X_i')$. Due to the financial fairness constraint, we must have $E[u_i(Y_i')] \leq E[u_i(X_i^*)]$ for all i. Because the scheme (X', y') is assumed to be Pareto efficient, in fact equality must hold, and this implies that $Y_i' = X_i^*$ for all i. □

If there are no access restrictions for agents, so that the agents operate in the same complete market as the collective does, then the role of the collective is merely administrative. The collective can provide gains to the participants only by effects that are not modeled in this paper, such as economies of scale.

4.3 Equicautious HARA collectives

A utility function $u(x)$ is said to belong to the *HARA class* (hyperbolic absolute risk aversion) if its coefficient of risk tolerance is affine in x; that is, there exist constants $a > 0$ and b such that $-u'(x)/u''(x) = ax + b$. The derivative of the risk tolerance with respect to its argument x has been called *cautiousness* by Wilson [29]; in the case of a HARA utility, the cautiousness is given by the parameter a. A collective all of whose members have the same cautiousness is called *equicautious* [2]. It is well known that equicautious HARA collectives enjoy special properties. The formulation of the following lemma is taken from [27].

Lemma 4.3. *Suppose that $u_1, \ldots, u_n$ are functions of the HARA class, and write $-u_i'(x)/u_i''(x) = a_i x + b_i$. If all coefficients a_i are the same, say $a_i = a$ for all i, then, for any set of weights $(c_1, \ldots, c_n) \in \mathbb{R}^n_{++}$, the group utility function $u(x)$ defined in (3.5) is in the HARA class as well. Specifically, we have $-u'(x)/u''(x) = ax + b$ where $b = \sum_{i=1}^n b_i$.*

In particular the lemma shows that, in the case of an equicautious HARA collective, the group risk tolerance does not depend on the weights c_i. Since the group risk tolerance defines the group utility up to equivalence of utility functions, it follows that the optimal decision $\hat{X}$ given by (3.7b) does not depend on $(c_1, \ldots, c_n)$. From the point of view of reallocation, the situation is then the same as when the risk is given exogenously. In this situation, the mapping $(c_1, \ldots, c_n) \mapsto J(\hat{X})$ is strictly increasing, as already noted in the paragraph preceding the statement of Assumption 3.1. It follows that Assumption 3.1 is satisfied for equicautious HARA collectives, so that uniqueness holds in this case.

A utility function that is defined on the positive halfline, as required in this paper, belongs to the HARA class if and only if it belongs to the CRRA class (constant relative risk aversion); these are the power utilities and logarithmic utility. An equicautious CRRA collective consists of agents who all have the same constant coefficient of risk aversion, so essentially all members have the same utility function. The group utility is also the same. The allocation rule in the unique Pareto efficient and financially fair risk sharing scheme for an equicautious CRRA collective is

$$y_i(x) = \frac{w_i}{E^Q X} X.$$

Therefore this is a case in which the combination of Pareto efficiency and financial fairness leads to the ex-post proportional division rule (cf. the discussion in the introduction). The investment decision that is taken by the collective is the same as the one that would be taken by an individual with the same coefficient of relative risk aversion as all participants have, and with a capital that is equal to the sum of the contributions of all participants.

5 Example

We present a simple example, which illustrates the notions of Pareto efficiency and financially fair risk sharing in complete and incomplete markets in the context of a coin tossing experiment. We compare the levels of utility that can be reached by the agents when they operate on their own and when they form a collective, under various assumptions on the restrictions to which their bets are subject. The example may be viewed as a highly stylized representation of the situation of two generations that take part in a collective pension fund.

Consider a situation in which two Bernoulli trials will take place, and in which there are two agents who are allowed to place bets. The two possible outcomes of the trials are denoted by $\mathtt{H}$ (heads) and $\mathtt{T}$ (tails), so that the outcome space is $\{\mathtt{HH}, \mathtt{HT}, \mathtt{TH}, \mathtt{TT}\}$. It will be assumed that all outcomes have equal $\mathbb{Q}$-probabilities, or in other words, the state prices of all outcomes are the same. We shall consider a number of different settings, which are distinguished by the constraints that are placed on the positions that can be taken by the agents. First consider the "autarky" settings, in which the agents operate on their own.

A1 Agent A can only bet on the outcome of Coin 1, and agent B can only bet on the outcome of Coin 2. In this setting, both agents have only one degree of freedom. Specifically, agent A has access to the zero-price payoff given by the vector $[1,\ 1,\ -1,\ -1]$, and agent B has access to the zero-price payoff given by $[1,\ -1,\ 1,\ -1]$.

A2 Both agents can bet on both coins. The agents now each have access to the two-dimensional space of zero-price payoff vectors generated by $[1,\ 1,\ -1,\ -1]$ and $[1,\ -1,\ 1,\ -1]$.

A3 Both agents have access to the full (three-dimensional) set of payoffs that have zero price. For instance, this means that they may place bets on "first coin heads and second coin tails" with payoff vector $[-1,\ 3,\ -1,\ -1]$, which would not be possible in setting A2.

Next we proceed to the settings in which the agents may form a collective. We consider situations in which the agents are subject to financial fairness, as well as situations in which this constraint is not imposed.

C2F The collective has access to both coins, as in setting A2 for the individual agents, and can implement any financially fair division rule.

C2U The collective has access to both coins and can use any division rule.

C3F The collective has access to the full space of zero-price payoffs, as in setting A3, and can use any financially fair division rule.

C3U The collective has access to the full space of zero-price payoffs and can use any division rule.

All of these settings correspond to certain subspaces of the eight-dimensional space of vectors along which the initial endowments of the two agents can be modified. In the autarky settings, these subspaces are just the products of the corresponding subspaces for the agents individually, and consequently the subspaces corresponding to settings A1, A2, and A3 have dimensions 2, 4, and 6 respectively. The subspaces associated to the setting A3 and the setting C3F coincide; this reflects the fact that when agents individually have access to a complete market, they can by themselves

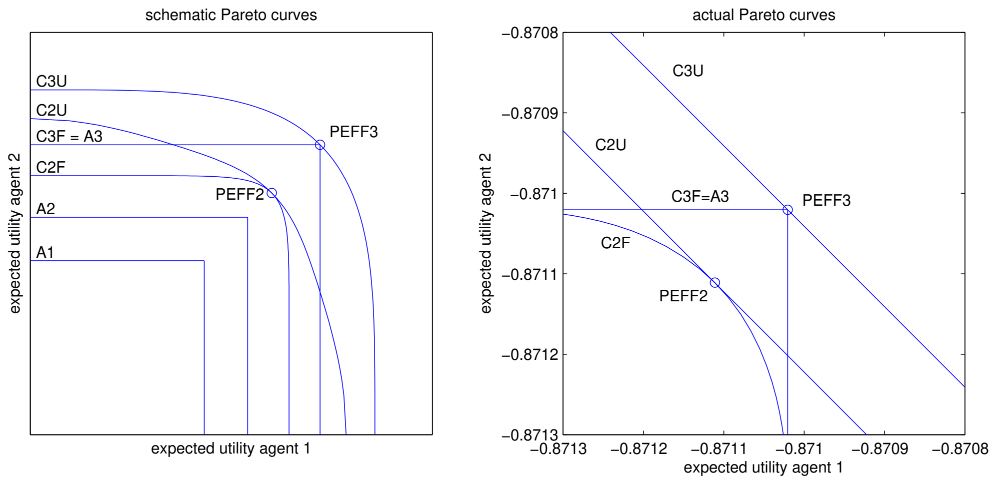

Figure 1: Schematic and actual Pareto curves for the coin toss example. PEFF solutions are indicated by circles. In the right panel, the scale is chosen such that the distinctions between the C2U and C3U curves and between the C2F and C3F/A3 curves are visible; as a consequence, only small parts of these curves are shown, and the Pareto curves A1 and A2 are outside the domain of the plot.

construct any position that can be obtained from a collective implementing financially fair risk sharing (cf. Section 4.2). The subspace associated to setting C2F however has dimension 5 and is strictly larger than the space associated to A2. The subspace associated to C2U has dimension 6 just like the subspace associated to A3 and C3F, but these subspaces do not coincide. Finally, the subspace associated to C3U is subject only to the overall budget constraint; it has dimension 7 and contains all lower-dimensional subspaces mentioned above.

Given preference functions of the agents, any point in the product of the payoff spaces of the two agents gives rise to a point in the two-dimensional space of preference levels of the two agents. The collection of these points for any of the settings specified above generates a Pareto set (set of points for which no strict Pareto improvement exists within the specified setting). The inclusion relations discussed above give rise to certain relations among these sets which are indicated schematically in the left panel of Fig. 1. If the agents' preferences are of the form as assumed in Section 2.2, it follows from our main result and Example 3.3 that there exist unique PEFF solutions both in the incomplete-market setting C2 and in the complete-market setting C3. In the diagram, these solutions are indicated by circles marked PEFF2 and PEFF3. The autarky settings give rise to autarky solutions, in which both agents select their optimal policy; the solutions appear as corner points of the corresponding Pareto sets. The autarky solution A3 coincides with PEFF3.

In order to get an impression of the actual location of various Pareto curves in a specific case, we assume the following. The real-world probability of outcome `H` is taken to be equal to 0.6 in

	agent	HH	HT	TH	TT	premium
initial	1	2.0000	1.0000	0.5000	1.0000	–
	2	0.5000	1.0000	2.0000	1.0000	–
A1	1	1.7559	0.7559	0.7442	1.2441	6.52
	2	0.7435	0.7565	2.2435	0.7565	6.87
A2	1	1.8000	0.7500	0.7500	1.2000	6.53
	2	0.9000	1.5000	1.5000	0.6000	15.78
PEFF2	1	1.3500	1.1250	1.1250	0.9000	17.74
	2	1.3500	1.1250	1.1250	0.9000	29.48
PEFF3/A3	1	1.3638	1.1135	1.1135	0.9092	17.75
	2	1.3638	1.1135	1.1135	0.9092	29.49

Table 1: The final holdings of the agents in the coin toss example with different spaces of accessible trades. The term "premium" refers to the percentage of initial wealth that an agent would be maximally willing to give up in order to take part in a given risk sharing scheme. The solutions given in the autarky settings are the ones that are individually optimal for both agents.

both experiments, so that there is an incentive for both agents to place bets even when they are risk averse, since the risk-neutral probabilities of outcome H is 0.5. The agents' preferences are described by power utility $u_i(x) = x^{1-\gamma_i}/(1-\gamma_i)$, $i = 1, 2$, with relative risk aversion coefficients $\gamma_1 = \gamma_2 = 2$. The initial allocations of the agents are given by $X_1^{\text{in}} = [2, 1, 0.5, 1]$ and $X_2^{\text{in}} = [0.5, 1, 2, 1]$, so that the values of the contributions of the two agents are the same: $w_1 = w_2 = 1.125$. Table 1 shows the optimal allocations for both agents in the three autarky settings as well as the two PEFF solutions corresponding to the settings C2F and C3F. In both PEFF solutions, the proceeds are divided equally between the two agents in all states of the world, as it should be the case according to Prop. 4.1. In the example, it turns out that both the allocations and the utility gains for both agents in the PEFF solution under incomplete accessibility are only slightly below the ones obtained from the individually optimal complete-market solutions. The proximity of the incomplete-market PEFF solution to the complete-market autarky solution is also demonstrated in the right panel of Fig. 1. By taking part in a collective that follows the PEFF rule, both agents in the C2 setting realize a utility gain that is almost the same as the one they would realize in a complete-market situation.

6 Conclusion

In collective decision theory, there is no absolute notion of optimality, and any specific solution of the bargaining problem is to some extent arbitrary. In this paper, we have used the notion of unconstrained Pareto efficient and financially fair solutions, and we have proved that, under certain conditions, there is a unique solution. Given also the natural interpretations of both Pareto efficiency and financial fairness, this appears to make the PEFF rule a natural candidate as a decision tool in situations of risk sharing. Of course, it should be noted that the model requires specification of the agents' preferences in terms of expected utilities, which may not be straightforward, and it also requires agreement of the agents on a pricing functional, even in an incomplete market. There is no guarantee that the PEFF solution is utility-improving for all participants; in other words, the result of a PEFF analysis for a given collective could be that the composition of the group is unsuitable for risk sharing.

The analysis in this paper is restricted to discrete probability spaces and single-period models; a further step would be to formulate and solve problems combining Pareto efficiency and financial fairness for continuous and possible unbounded random variables in a multiperiod context, or in continuous time. Further investigations may also address the question as to which characteristics of collectives can ensure that PEFF solutions satisfy individual rationality. We have worked in this paper with agents who satisfy the von Neumann-Morgenstern axioms, so that their preferences can be described in terms of expected utility with respect to an objective probability measure. Of course this is a limitation, and it would be interesting to consider collectives of agents whose preferences are not necessarily of this type. The relation under the PEFF decision rule of individual preferences to group preferences, to the extent that these can be defined, is another topic of interest.

Assumption 3.1 (monotonicity) presents a challenge. The assumption seems quite far from being necessary for existence and uniqueness of PEFF solutions, but on the other hand it appears that a rather different mathematical approach would be needed to avoid this assumption altogether.

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Appendix

A.1 Proof of Prop. 3.2

Since the random variable G is defined on a finite probability space, it can only take finitely many values, say $g_1, \dots, g_\ell$. For $k = 1, \dots, \ell$, write $S_k = \{j \mid G(\omega_j) = g_k\}$. The sets $S_1, \dots, S_\ell$ form a partition of the index set $\{1, \dots, m\}$. The Lagrangian function associated to the constrained optimization problem (3.4) can be written as

$$L(X_1,\dots,X_m,\lambda_1,\dots,\lambda_\ell) = \sum_{j=1}^{m} p_j u(X_j) + \sum_{k=1}^{\ell} \lambda_k \sum_{j\in S_k} q_j (X_j^{\mathrm{in}} - X_j).$$

Differentiation with respect to X_j leads to the conditions $p_j u'(X_j) = \lambda_k q_j$ ($j \in S_k$, $k = 1,\dots,\ell$). It follows that the location of the optimum is given by

$$\hat{X}_j = I(\lambda_k q_j/p_j) \qquad (j \in S_k,\ k = 1,\dots,\ell) \tag{A.1}$$

where the Lagrange multipliers λ_k are determined by the constraints $\sum_{j\in S_k} q_j(X_j^{\mathrm{in}} - \hat{X}_j) = 0$ ($k = 1,\dots,\ell$). These constraints may be written, with explicit indication of dependence on the weights c_i, as

$$\sum_{j\in S_k} q_j I(\lambda_k q_j/p_j\,; c_1,\dots,c_n) = \sum_{j\in S_k} q_j X_j^{\mathrm{in}} \qquad (k = 1,\dots,\ell). \tag{A.2}$$

The expression that appears on the left-hand side of (A.2) is strictly decreasing as a function of λ_k and strictly increasing as a function of any of the weights c_i. Since the right-hand side of (A.2) does not depend on either the variable λ_k or the variables c_i, it follows that an increase of any of the variables c_i must be accompanied by an increase of λ_k in order to maintain equality in (A.2). In other words, the function $(c_1,\dots,c_n) \mapsto \lambda_k$ defined implicitly by (A.2) is strictly increasing. Since $(J(\hat{X}))_j = J(\hat{X}_j) = \lambda_k q_j/p_j$ by (A.1), the statement of the proposition follows.

A.2 Mapping used in fixed-point theorem

The proof of our main result, Theorem 3.4, is based on showing existence and uniqueness of the fixed point of a particular mapping φ from the nonnegative cone into itself. This mapping is constructed as the composition of three mappings, as follows:

$$\mathbb{R}_+^n \xrightarrow{\varphi_1} \mathcal{L} \xrightarrow{\varphi_2} \mathbb{R}_{++}^m \cup \{0\} \xrightarrow{\varphi_3} \mathbb{R}_+^n.$$

Here, $\mathcal{L}$ is the cone defined by $\mathcal{L} = \mathcal{L}' \cup \{0\}$, where $\mathcal{L}'$ denotes the set of all continuous and strictly decreasing functions $J : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that

$$\lim_{x\downarrow 0} J(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} J(x) = 0.$$

Below we make use of $\mathcal{L}$ equipped with the topology of pointwise convergence.

Our aim is to show that the composite mapping φ satisfies the conditions for the existence of a unique positive eigenvector as formulated by Oshime [26], and that the corresponding eigenvalue is equal to 1. In the following three subsections, the three mappings are defined and it is proved that they satisfy certain continuity and monotonicity properties. On the basis of this, we then show that the mapping φ indeed has the required properties.

A.2.1 First component φ_1

Lemma A.1. *For any $c \in \mathbb{R}^n_+ \setminus \{0\}$, the function I defined on $\mathbb{R}_{++}$ by*

$$I(z) := \sum_{i:c_i \neq 0} I_i(z/c_i) \qquad (z > 0) \tag{A.3}$$

is invertible. The inverse $J = I^{-1}$ belongs to $\mathcal{L}'$.

Proof. For given $c \in \mathbb{R}^n_+ \setminus \{0\}$, the functions $I_i(\cdot/c_i)$ in (A.3) are strictly decreasing and continuous on $(0, \infty)$. Therefore, their sum I is strictly decreasing and continuous as well, so that its inverse J exists. Clearly, J is strictly decreasing and continuous. The Inada conditions (2.1) imply that $\lim_{z\to\infty} I(z) = 0$ and $\lim_{z\downarrow 0} I(z) = \infty$. It follows that $\lim_{x\downarrow 0} J(x) = \infty$ and $\lim_{x\to\infty} J(x) = 0$, i.e., $J \in \mathcal{L}'$. ◻

On the basis of the lemma, we can define a mapping φ_1 from $\mathbb{R}_+$ to $\mathcal{L}$ by

$$\varphi_1(c) = \begin{cases} J \text{ as defined in Lemma A.1} & \text{if } c \neq 0 \\ 0 & \text{if } c = 0. \end{cases} \tag{A.4}$$

Given that J is defined as the inverse function of the mapping I which appears in A.3, and also given that $\lim_{c\to 0} I(z) = 0$ for all $z > 0$, the definition $\varphi_1(0) = 0$ in (A.4) can be viewed as corresponding to the convention that the inverse function of 0 is 0. While this might not seem intuitive at first, the convention becomes natural if we think of functions in $\mathcal{L}'$ in terms of their graphs as subsets of the extended nonnegative quadrant $[0, \infty]^2$. In this point of view, the function denoted by 0 in fact represents the union of the two nonnegative semi-axes $\big(\{0\} \times [0, \infty]\big) \cup \big([0, \infty] \times \{0\}\big)$, and the customary generalization of the notion of inverse function to multivalued mappings leads to the statement $0^{-1} = 0$. We now note some properties of the mapping φ_1 that will be needed below.

Lemma A.2. *The mapping φ_1 is a continuous and homogeneous mapping from $\mathbb{R}^n_+$ to $\mathcal{L}$.*

Proof. For the first statement, see Lemma A.4 in [27]. Homogeneity is immediate. ◻

Lemma A.3. *If $(c^k)_{k=1,2,\ldots}$ is a sequence in $\mathbb{R}^n_+$ such that there is an index i' with $c^k_{i'} \to \infty$, then $\varphi_1(c^k) \to \infty$.*

Proof. Let $(c^k)_k$ be a sequence as in the statement of the lemma, and write $J^k = \varphi_1(c^k)$. Take $x > 0$. By definition, we have

$$\sum_{i=1}^{n} I_i(J^k(x)/c^k_i) = x. \tag{A.5}$$

If the sequence $(J^k(x))_k$ would have a bounded subsequence, then the corresponding subsequence of $(I_{i'}(J^k(x)/c^k_{i'})$ would tend to infinity. Since all terms on the left-hand side of (A.5) are positive, this leads to a contradiction. It follows that $J^k(x) \to \infty$, and since this holds for all $x > 0$, we obtain $J^k \to \infty$. ◻

A.2.2 Second component φ_2

When $u : (0, \infty) \to \mathbb{R}$ is a utility function of the type described in Section 2.2, then its derivative u' belongs to $\mathcal{L}'$. Given also the subspace $\mathcal{X}_0$ of accessible zero-cost trades and an initial endowment X^{in} such that $\big(X^{\text{in}} + \mathcal{X}_0\big) \cap \mathbb{R}^m_{++} \neq \emptyset$, we can formulate the optimization problem

$$\text{maximize } E[u(X)] \quad \text{subject to } X \in \mathcal{X} \cap \mathbb{R}^m_{++} \tag{A.6}$$

where $\mathcal{X} := X^{\text{in}} + \mathcal{X}_0$. A vector $\hat{X} \in \mathbb{R}^m_{++}$ is a solution of this optimization problem if and only if the conditions

$$\hat{X} \in \mathcal{X}, \quad J(\hat{X}) \in \mathcal{X}_0^{\perp} \tag{A.7}$$

hold, where $J := u'$ is the marginal utility and where orthogonality is taken in the sense of the inner product defined by expectation under $\mathbb{P}$, i.e.

$$\mathcal{X}_0^{\perp} = \{Z \mid E[X_0 Z] = 0 \text{ for all } X_0 \in \mathcal{X}_0\}.$$

Lemma A.4. *For $J \in \mathcal{L}'$, the equations (A.7) admit a unique solution $\hat{X} \in \mathbb{R}^m_{++}$.*

Proof. Let $u : (0, \infty) \to \mathbb{R}$ be such that $u' = J$. The function u is increasing, strictly concave, and can be extended continuously to a function from $[0, \infty)$ to $\mathbb{R} \cup \{-\infty\}$ which will still be denoted by u. The function $X \mapsto E[u(X)]$ is continuous, bounded from above on the compact set $\mathcal{X} \cap \mathbb{R}^m_+$, and strictly concave. Therefore there exists a unique point $\hat{X}$ in $\mathcal{X} \cap \mathbb{R}^m_+$ at which the function takes its maximum. The set $\mathcal{X} \cap \mathbb{R}^m_+$ is convex and has nonempty relative interior by assumption. Take $X^1 \in \mathcal{X} \cap \mathbb{R}^m_{++}$, and define the function $\tilde{u} : [0, 1] \to [-\infty, \infty)$ by

$$\tilde{u}(\lambda) = E[u\big((1 - \lambda)\hat{X} + \lambda X^1\big)].$$

Given that the function $E[u(X)]$ is maximized at $X = \hat{X}$, the function $\tilde{u}(\lambda)$ takes its maximum at $\lambda = 0$. The derivative of $\tilde{u}(\lambda)$ on the open interval $(0, 1)$ is given by

$$\tilde{u}'(\lambda) = \sum_{j=1}^{m} p_j J\big((1 - \lambda)\hat{X}_j + \lambda X^1_j\big)(X^1_j - \hat{X}_j).$$

For indices j such that $\hat{X}_j = 0$, we have $X^1_j - \hat{X}_j = X^1_j > 0$, so that the corresponding term in the summation at the right-hand side of the equation above tends to ∞ as λ tends to 0. In the case of indices j such that $\hat{X}_j \neq 0$, the corresponding term in the summation tends to a finite limit. Therefore, if there are any indices j such that $\hat{X}_j = 0$, then $\lim_{\lambda \downarrow 0} \tilde{u}'(\lambda) = \infty$, which is not possible given that $\tilde{u}$ takes its maximum at 0. It follows that all entries of $\hat{X}$ are strictly positive, and consequently the vector $\hat{X}$ satisfies the necessary conditions (A.7). From the strict concavity of the mapping $X \mapsto E[u(X)]$ it follows that these conditions are sufficient as well, so that the solution is unique. $\square$

On the basis of the lemma above, we can define a mapping $\varphi_2 : \mathcal{L} \to \mathbb{R}^m_{++} \cup \{0\}$ as follows:

$$\varphi_2(J) = \begin{cases} J(\hat{X}) \text{ where } \hat{X} \text{ is defined by (A.7)} & \text{if } J \in \mathcal{L}' \\ 0 & \text{if } J = 0. \end{cases} \tag{A.8}$$

The question of continuity of the mapping φ_2 leads to the problem of showing that the solution of a portfolio optimization problem depends continuously on the preference specification. This problem has been studied in the literature in various contexts; a discrete-time formulation has been used by Carassus and Rasonyi [12], while Jouini and Napp [24] have obtained results in continuous time. Here we cannot follow the work of Carassus and Rasonyi entirely since they work with utility functions that are defined on the whole real line, but part of the proof of Lemma A.6 below makes use of ideas in their paper. First we show an auxiliary result.

Lemma A.5. *If $(J^k)_{k=1,2,\ldots}$ is a sequence of functions in $\mathcal{L}'$ that converges pointwise to a function $J \in \mathcal{L}$, then the sequence $(J^k(\hat{X}^k))_{k=1,2,\ldots}$ is bounded, where $\hat{X}^k$ denotes the element of $\mathbb{R}^m_{++}$ associated to J^k via Lemma A.4.*

Proof. All vectors $\hat{X}^k$ belong to the bounded set $\mathcal{X}_0 \cap \mathbb{R}^m_{++}$, so that it is sufficient to conduct the proof under the assumption that the sequence $\hat{X}^k$ converges to a limit, say $\hat{X}^\infty$. Then $\hat{X}^\infty \in \mathcal{X}_0 \cap \mathbb{R}^m_+$. Because $J^k(\hat{X}^k) \in \mathcal{X}_0^\perp$ for all k, we have in particular

$$\sum_{j=1}^m J^k(\hat{X}^k_j)(\hat{X}^k_j - X^{\text{in}}_j) = 0 \tag{A.9}$$

for all k. Moreover, we may assume (shifting X^{in} by an element of $\mathcal{X}_0$ if necessary) that all components of X^{in} are positive. Define index sets S_+ and S_- by

$$S_+ = \{j \in \{1, \ldots, m\} \mid \hat{X}^\infty_j \geq X^{\text{in}}_j\}, \quad S_- = \{j \in \{1, \ldots, m\} \mid \hat{X}^\infty_j < X^{\text{in}}_j\}.$$

From (A.9) we have

$$\sum_{j \in S_+} J^k(\hat{X}^k_j)(\hat{X}^k_j - X^{\text{in}}_j) = \sum_{j \in S_-} J^k(\hat{X}^k_j)(X^{\text{in}}_j - \hat{X}^k_j). \tag{A.10}$$

For $j \in S_+$, the inequality $\hat{X}^k_j \geq \frac{1}{2} X^{\text{in}}_j$ holds for all sufficiently large k, so that

$$J^k(\hat{X}^k_j) \leq J^k(\tfrac{1}{2} X^{\text{in}}_j) \xrightarrow{k \to \infty} J(\tfrac{1}{2} X^{\text{in}}_j).$$

This shows that the sequences $\big(J^k(\hat{X}^k_j)\big)_{k=1,2,\ldots}$ are bounded for $j \in S_+$. Given that $\lim_{k\to\infty} \hat{X}^k_j = \hat{X}^\infty_j$, it follows that the left-hand side of (A.10) is bounded as k tends to infinity. Consequently the right-hand side is bounded as well. Since the sequences $\big(X^{\text{in}}_j - \hat{X}^k_j\big)_{k=1,2,\ldots}$ for $j \in S_-$ converge to limits that are strictly positive, this implies the boundedness of $\big(J^k(\hat{X}^k_j)\big)_{k=1,2,\ldots}$ for $j \in S_-$. $\square$

Lemma A.6. *Let $(J^k)_{k=1,2,\dots}$ be a sequence of functions in $\mathcal{L}'$ that converges pointwise to a function $J \in \mathcal{L}'$, and let $\hat{X}^k$ $(k = 1, 2, \dots)$ and $\hat{X}$ be the elements of $\mathbb{R}^m_{++}$ that are associated to J^k and J respectively via Lemma A.4. Under these conditions, we have $\lim_{k\to\infty} J^k(\hat{X}^k) = J(\hat{X})$.*

Proof. We first show that all limit points of the sequence $(\hat{X}^k)_k$ belong to the positive orthant $\mathbb{R}^m_{++}$. Let $\hat{X}^\infty$ denote such a limit point. Then there is a subsequence that converges to $\hat{X}^\infty$, which for convenience of notation will still be denoted by $(\hat{X}^k)_k$. All vectors $\hat{X}^k$ belong to $\mathbb{R}^m_{++}$, and therefore $\hat{X}^\infty \in \mathbb{R}^m_{+}$. Suppose that j' is such that $\hat{X}^\infty_{j'} = 0$. It follows from the previous lemma that the sequence $J^k(\hat{X}^k_{j'})$ must remain bounded as k tends to infinity. Let M be such that $J^k(\hat{X}^k_{j'}) \leq M$ for all k. Define $\bar{x} > 0$ by $J(2\bar{x}) = M$. Because the function J is strictly decreasing, we have $J(\bar{x}) > M$. The inequality $\hat{X}^k_{j'} < \bar{x}$ holds for all sufficiently large k, so that $J^k(\bar{x}) < J^k(X^k_{j'}) \leq M$ for all such k. But this implies that $\lim_{k\to\infty} J^k(\bar{x})$ is strictly less than $J(\bar{x})$, which contradicts the assumption that the sequence $(J^k)_k$ converges pointwise to J. Therefore the limit point $\hat{X}^\infty$ must be strictly positive.

We next show that the sequence $\hat{X}^k$ converges to $\hat{X}$. Since the set $\mathcal{X} \cap \mathbb{R}^m_{++}$ is bounded, it is enough to show that any limit point of the sequence $\hat{X}^k$ is equal to $\hat{X}$. Let $\hat{X}^\infty$ denote such a limit point. Passing to a subsequence if necessary, we can assume that $\lim_{k\to\infty} \hat{X}^k = \hat{X}^\infty$. A classical theorem of analysis [9] states that pointwise convergence of the monotonically decreasing functions J^k to J implies uniform convergence of the sequence J^k to J on compacts. Let u and u^k be primitive functions of J and J^k respectively; we may and will assume that $u(1) = u^k(1) = 0$ for all k. Under this condition, it is an easy exercise to show that the uniform convergence of J^k to J on compacts implies the uniform convergence of u^k to u on compacts. By definition, we have

$$E[u^k(\hat{X}^k)] \geq E[u^k(\hat{X})] \tag{A.11}$$

for all k. The estimate

$$\big|E[u^k(\hat{X}^k)] - E[u(\hat{X}^\infty)]\big| \leq \big|E[u^k(\hat{X}^k)] - E[u(\hat{X}^k)]\big| + \big|E[u(\hat{X}^k)] - E[u(\hat{X}^\infty)]\big|$$

shows, in combination with the fact that $\hat{X}^\infty \in \mathbb{R}^m_{++}$ and the uniform convergence of (u^k) on compacts, that $\lim_{k\to\infty} E[u^k(\hat{X}^k)] = E[u(\hat{X}^\infty)]$. Since the right-hand side of the inequality (A.11) converges to $E[u(\hat{X})]$ as k tends to infinity, it follows that $E[u(\hat{X}^\infty)] = E[u(\hat{X})]$ and hence that $\hat{X}^\infty = \hat{X}$, by the uniqueness of the maximizer in the optimization problem defined by J.

Given that the sequence $\hat{X}^k$ converges to $\hat{X} \in \mathbb{R}^m_{++}$, the convergence of $J^k(\hat{X}^k)$ to $J(\hat{X})$ is now immediate from the uniform convergence of J^k to J on compacts. $\square$

To complete the proof of sequential continuity of φ_2 on $\mathcal{L}$, we show the continuity at 0.

Lemma A.7. *Let $(J^k)_{k=1,2,\dots}$ be a sequence of functions in $\mathcal{L}'$ converging pointwise to 0, and let $\hat{X}^k$ $(k = 1, 2, \dots)$ be the elements of $\mathbb{R}^m_{++}$ that are associated to J^k via Lemma A.4. Then $J^k(\hat{X}^k) \to 0$.*

Proof. Write $Z^k := J^k(\hat{X}^k)$. It follows from Lemma A.5 that the sequence $(Z^k)_k$ is bounded, so that without loss of generality we may assume that the sequence tends to a limit. By restricting to a further subsequence if necessary, we may furthermore assume that the sequence $(\hat{X}^k)_k$ tends to a limit as well; let the limit be denoted by $\hat{X}^\infty$. Since $Z^k \in \mathcal{X}_0^\perp$ for all k, we have in particular

$$\sum_{j=1}^{m} p_j Z_j^k X_j^{\text{in}} = \sum_{j=1}^{m} p_j Z_j^k \hat{X}_j^k \qquad (k = 1, 2, \dots) \tag{A.12}$$

where, as before, we may assume that $X_j^{\text{in}} > 0$ for all j. For j such that $\hat{X}_j^\infty > 0$, we have $0 \leq Z_j^k = J^k(\hat{X}_j^k) \leq J^k(\frac{1}{2}\hat{X}_j^\infty)$ for all sufficiently large k, and since $\lim_{k\to\infty} J^k(\frac{1}{2}X_j^\infty) = 0$ it follows that the term $Z_j^k \hat{X}_j^k$ tends to 0 as k tends to infinity. For j such that $\hat{X}_j^\infty = 0$, the term $Z_j^k \hat{X}_j^k$ tends to 0 as well, since the sequence (Z^k) is bounded as shown in Lemma A.5. Therefore the right-hand side of (A.12) tends to 0 as k tends to infinity. The left-hand side therefore converges to 0 as well; given that $p_j > 0$ and $X_j^{\text{in}} > 0$ for all j, this implies that $Z^k \to 0$. □

Finally we establish two simple properties of the mapping φ_2 that will be required below.

Lemma A.8. *The mapping φ_2 is homogeneous.*

Proof. Multiplication of $J \in \mathcal{L}'$ by a positive scalar has no effect on the vector $\hat{X}$ defined by (A.7). Hence the mapping $J \mapsto J(\hat{X})$ is homogeneous for all $J \in \mathcal{L}'$. For $J = 0$, the statement holds trivially. □

Lemma A.9. *Let $(J^k)_{k=1,2,\dots}$ be a sequence of functions in $\mathcal{L}$ with vectors $\hat{X}^k$ $(k = 1, 2, \dots)$ associated via Lemma A.4. If $J^k(x) \to \infty$ for all $x > 0$, then $J^k(\hat{X}^k) \to \infty$.*

Proof. Let M be such that $\hat{X}_j^k \leq M$ for all k and all j. Then, for all j, $J^k(\hat{X}_j^k) \geq J^k(M) \to \infty$ as $k \to \infty$. □

A.2.3 Third component φ_3

Lemma A.10. *Let $Z \in \mathbb{R}^m_{++}$ and let $w_i > 0$, for $i = 1, \dots, n$, be given. Then there exists a unique vector $c = (c_1, \dots, c_n) \in \mathbb{R}^n_{++}$ such that*

$$E^{\mathbb{Q}}\big[I_i(Z/c_i)\big] = w_i \qquad (i = 1, \dots, n). \tag{A.13}$$

Proof. For each i, the mapping $c_i \mapsto E^{\mathbb{Q}} I_i(Z/c_i)$ is strictly monotonic and continuous. From the Inada conditions (2.1), we also have $\lim_{c_i\to\infty} E^{\mathbb{Q}}[I_i(Z/c_i)] = \infty$ and $\lim_{c_i\downarrow 0} E^{\mathbb{Q}} I_i(Z/c_i) = 0$. Therefore, there exists a unique positive vector $c = (c_1, \dots, c_n)$ that solves (A.13). □

The above lemma enables us to define a mapping φ_3 from $\mathbb{R}^m_{++} \cup \{0\}$ to $\mathbb{R}^n_+$ in the following way:

$$\varphi_3(Z) = \begin{cases} c \text{ as defined by (A.13)} & \text{if } Z > 0 \\ 0 \in \mathbb{R}^n & \text{if } Z = 0. \end{cases} \tag{A.14}$$

Lemma A.11. *The mapping φ_3 defined by (A.14) is a continuous mapping from $\mathbb{R}^m_{++} \cup \{0\}$ to R^n_+.*

Proof. Let $(Z^k)_{k=1,2,\dots}$ be a sequence in $\mathbb{R}^m_{++} \cup \{0\}$ converging to $\bar{Z} \in \mathbb{R}^m_{++} \cup \{0\}$. Write $c^k = (c^k_1, \dots, c^k_n) = \varphi_3(Z^k)$, and $\bar{c} = (\bar{c}_1, \dots, \bar{c}_n) = \varphi_3(\bar{Z})$. First consider the case in which $\bar{Z} = 0$, and take $i \in \{1, \dots, n\}$. To avoid trivialities, we can assume that $Z^k > 0$ for all k. If there would be a subsequence of $(c^k_i)_k$ that would be bounded away from 0, then the corresponding quantities $E^{\mathbb{Q}}[I_i(Z^k/c^k_i)]$ would tend to infinity given that $Z^k \to 0$, contradicting the requirement $E^{\mathbb{Q}}[I_i(Z^k/c^k_i)] = w_i$. It follows that $\lim_{k\to\infty} c^k_i = 0$ for all i.

Now, consider the case in which $Z > 0$. There is no loss in generality if we assume that $Z^k > 0$ for all k. Fix $i \in \{1, \dots, n\}$. The numbers c^k_i and $\bar{c}_i$ are defined uniquely by the equations $f(c^k_i, Z^k) = w_i$ and $f(\bar{c}_i, \bar{Z}) = w_i$ respectively, where f is the mapping from $\mathbb{R}_{++} \times \mathbb{R}^m_{++}$ defined by $f(c, Z) = E^{\mathbb{Q}}[I_i(Z/c)]$. The mapping f is continuous. Moreover, because the sequence (Z^k) is bounded, the sequence (c^k_i) is bounded as well. Indeed, if there would be a subsequence converging to infinity, then the corresponding quantities $E^{\mathbb{Q}}[I_i(Z^k/c^k_i)]$ would converge to infinity also, which is not possible as already noted. We can therefore conclude (cf. [27, Lemma A.3]) that $\lim_{k\to\infty} c^k_i = \bar{c}_i$. Since this holds for each i, it follows that $\lim_{k\to\infty} c^k = \bar{c}$. □

Lemma A.12. *The mapping φ_3 is strongly increasing.*

Proof. Take Z^1 and Z^2 in $\mathbb{R}^m_{++} \cup \{0\}$ with $Z^2 \gneq Z_1$, and write $c^i = \varphi_3(Z^i)$ $(i = 1, 2)$. If $Z^1 = 0$, then $c^1 = 0 < c^2$. Assume now that $Z^1 > 0$, and take $i \in \{1, \dots, n\}$. Since the mapping $Z \mapsto E^{\mathbb{Q}}[I_i(Z)]$ is strongly decreasing, it would follow from the supposition $c^2_i \le c^1_i$ that $Z^2/c^2_i \gneq Z^1/c^1_i$ and hence

$$w_i = E^{\mathbb{Q}}\big[I_i(Z^2/c^2_i)\big] < E^{\mathbb{Q}}\big[I_i(Z^1/c^1_i)\big] = w_i$$

which is a contradiction. It follows that $c^2_i > c^1_i$ for all i. □

Lemma A.13. *If the sequence $(Z^k)_{k=1,2,\dots}$ in $\mathbb{R}^m_{++} \cup \{0\}$ tends to infinity, then the sequence $(c^k)_{k=1,2,\dots}$ defined by $c^k = \varphi_3(Z^k)$ tends to infinity as well.*

Proof. Suppose to the contrary that for some $i \in \{1, \dots, n\}$ there are a subsequence $(c^{k_j}_i)_{j=1,\dots}$ and a constant M such that $c^{k_j}_i \le M$ for all j. Then

$$w_i = E^{\mathbb{Q}}[I_i(Z^{k_j}/c^{k_j}_i)] < E^{\mathbb{Q}}[I_i(Z^{k_j}/M)].$$

The right-hand side tends to 0 because Z^{k_j} tends to infinity, and so we arrive at a contradiction since $w_i > 0$. □

Lemma A.14. *The mapping φ_3 is homogeneous.*

Proof. This is immediate from the definition. □

A.3 Proof of Thm. 3.4

We now define a mapping φ from the nonnegative cone to itself by

$$\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1 \tag{A.15}$$

where the components φ_1, φ_2, and φ_3 are given by (A.4), (A.8), and (A.14) respectively.

Lemma A.15. *The mapping φ defined by (A.15) is continuous and homogeneous. Under Assumption 3.1, the mapping φ is strongly increasing.*

Proof. Continuity follows from lemmas A.2, A.6, A.7, and A.11; homogeneity results from lemmas A.2, A.8, and A.14. Given Assumption 3.1, strong monotonicity is a consequence of Lemma A.12. □

To allow the application of an appropriate fixed-point theorem, we verify an additional condition that acts as a substitute for the indecomposability condition of linear Perron-Frobenius theory. The definition below follows Oshime [26].

Definition A.16. A mapping φ from $\mathbb{R}^n_+$ into itself is *nonsectional* if, for every partition of the index set $\{1, \ldots, n\}$ into two nonempty subsets R and S, there exists $s \in S$ such that

(i) $(\varphi(x))_s > (\varphi(y))_s$ for all x, $y \in \mathbb{R}^n_+$ such that $x_R > y_R$ and $x_S = y_S > 0$

(ii) $(\varphi(x^k))_s \to \infty$ for all sequences $(x^k)_{k=1,2,\ldots}$ in $\mathbb{R}^n_+$ such that $x^k_R \to \infty$ while x^k_S is fixed and positive.

Lemma A.17. *Under Assumption 3.1, the mapping φ defined by (A.15) is nonsectional.*

Proof. Let a nontrivial partition of the index set $\{1, \ldots, n\}$ into complementary subsets R and S be given, and take any $s \in S$. Item (i) follows from the fact that φ is strongly increasing (Lemma A.15). Item (ii) follows from lemmas A.3, A.9, and A.13. □

We will use the following nonlinear Perron-Frobenius theorem, due to Oshime [26]; cf. also [27] for the statement as given below.

Theorem A.18 (Oshime)**.** *If a mapping φ from $\mathbb{R}^n_+$ into itself is continuous, monotonic, homogeneous of degree 1, and nonsectional, then the mapping φ has a positive eigenvector, which is unique up to scalar multiplication. In other words, there exist a constant $\alpha^* > 0$ and a vector $x^* \in \mathbb{R}^n_{++}$ such that $\varphi(x^*) = \alpha^* x^*$, and if $\alpha > 0$ and $x \in \mathbb{R}^n_{++}$ are such that $\varphi(x) = \alpha x$, then x is a scalar multiple of x^*.*

The theorem by itself does not prove that the mapping φ has a fixed point. For this, it needs to be shown that the corresponding eigenvalue is equal to 1. This is established in the lemma below.

Lemma A.19. *Let φ be defined by (A.15). If $\lambda \in \mathbb{R}_{++}$ is such that $\varphi(c) = \lambda c$ for some $c \in \mathbb{R}^n_{++}$, then $\lambda = 1$.*

Proof. Let $c \in \mathbb{R}^n_{++}$ and $\lambda > 0$ be such that $\varphi(c) = \lambda c$. Write $J = \varphi_1(c)$. Because $\varphi_3(J(\hat{X})) = \lambda c$, we have

$$\sum_{i=1}^{n} E^{\mathbb{Q}}[I_i(J(\hat{X})/(\lambda c_i))] = \sum_{i=1}^{n} w_i =: w. \tag{A.16}$$

On the other hand, from the relation $\sum_{i=1}^{n} I_i(J(x)/c_i) = x$ it follows that

$$\sum_{i=1}^{n} E^{\mathbb{Q}}[I_i(J(\hat{X})/c_i)] = E^{\mathbb{Q}}\hat{X} = w. \tag{A.17}$$

Since the mapping $\lambda \mapsto E^{\mathbb{Q}}[I_i(J(\hat{X})/(\lambda c_i))]$ is strictly increasing, a comparison of (A.16) and (A.17) shows that $\lambda = 1$. $\square$

This leads to the proof of our main result.

Proof of Theorem 3.4. By [30, Thm. 3.1], the collection of all Pareto efficient risk sharing schemes $\big(\hat{X}; y_1(\cdot), \dots, y_n(\cdot)\big)$ is parametrized completely by the open unit simplex in $\mathbb{R}^n$ through the prescription (3.7). It was already argued in the main text, following the statement of Theorem 3.4, that PEFF schemes correspond exactly to those parameter vectors that are fixed points under the mapping φ. By Lemma A.15 and Lemma A.17, if Assumption 3.1 holds, then the mapping φ satisfies all conditions of Oshime's theorem, and consequently a unique positive eigenvector exists. Lemma A.19 shows that the corresponding eigenvalue is equal to 1, and consequently the eigenvector gives rise to a fixed point of the mapping φ on the open unit simplex. Since conversely any fixed point of φ on the open simplex gives rise to an eigenvector, the uniqueness of the eigenvector implies the uniqueness of the fixed point. The statement of the theorem follows. $\square$

A.4 On the necessity of the monotonicity assumption

The following example demonstrates that the condition of Assumption 3.1 is not always satisfied. Let two agents be given with identical marginal utilities defined by

$$u_i'(x) = \begin{cases} \frac{1}{x} & \text{for } x \in (0, z_1), \\ \frac{z_2 - x}{z_2 - z_1}\frac{1}{z_1} + \frac{x - z_1}{z_2 - z_1}\frac{1}{z_1 + d} & \text{for } x \in [z_1, z_2), \\ \frac{1}{x - z_2 + z_1 + d} & \text{for } x \in [z_2, \infty) \end{cases} \tag{A.18}$$

$(i = 1, 2)$, with $z_1 = 0.05$, $z_2 = 2$, and $d = 0.01$; actually a small modification of the definition above would be needed to ensure continuity of the second derivative as required by our assumptions, but we will skirt this issue. The agents also have identical initial endowments given by [1, 1, 1]. The objective probability measure is given by $[p_1,\ p_2,\ p_3] = [0.1,\ 0.1,\ 0.8]$, whereas the pricing

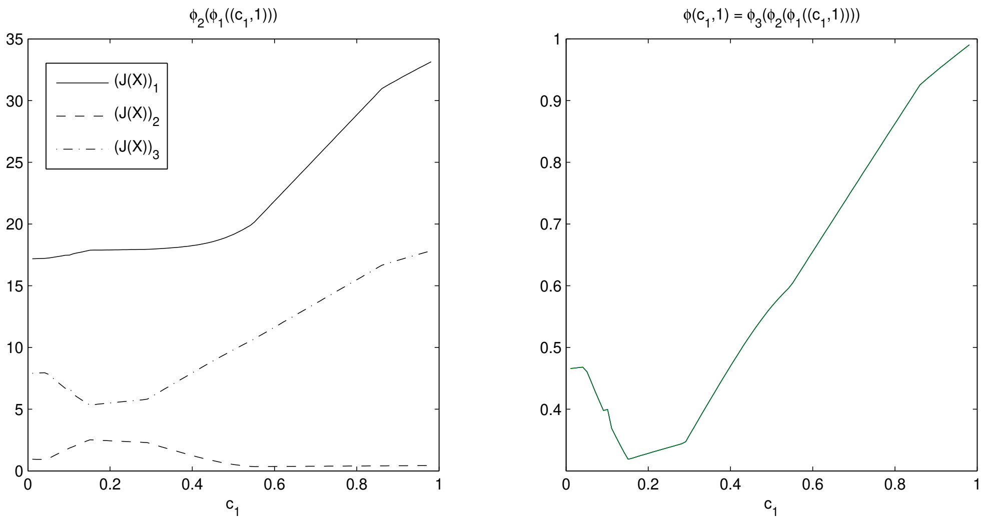

Figure 2: The panels show the three components of $J(\hat{X};c) = \varphi_2(\varphi_1(c))$ (left) and the two components of $\varphi(c)$ (right) as a function of c_1, while c_2 is kept fixed at the value 1. The two components of $\varphi(c)$ are identical to each other.

measure is defined by $[q_1,\ q_2,\ q_3] = [0.5,\ 0.125,\ 0.375]$. The space of accessible zero-cost trades is defined by

$$\mathcal{X}_0 = \ker \begin{bmatrix} 4 & 1 & 3 \\ 10 & 2 & 15 \end{bmatrix}.$$

The left panel of Figure 2 shows a plot of the three components of the vector $J(\hat{X};c)$ as a function of c_1, when c_2 is kept constant at the value 1. Monotonicity does not hold. The right panel shows the behavior of the two components of $\varphi(c_1, 1)$; since the agents have identical utility functions and identical wealths, these components are equal to each other. It is seen that the mapping φ is in this case also not monotonic. However, the symmetry between the agents implies that there is a unique Pareto efficient and financially fair solution, as shown in Prop. 4.1. It is therefore apparent that the monotonicity in Assumption 3.1 is only a sufficient and not a necessary condition.

The example shown above was obtained by experimentation, and seems too complicated for a clear economic interpretation. A plot of the marginal utility defined by (A.18) is shown in Fig. 3. The particular shape of the utility is likely to play a role in the violation of monotonicity.

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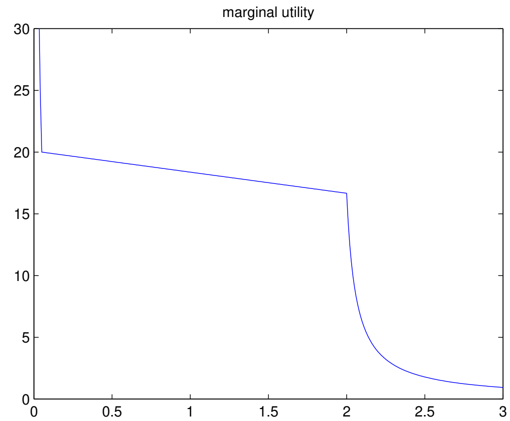

Figure 3: Marginal utility as defined by (A.18).

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