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JEL subject codes: G11, D91, D14

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Abstract:

In a calibrated consumption-portfolio model with stock, housing, and labor income predictability, we evaluate the welfare effects of predictability on life-cycle consumption-portfolio choice. We compare skilled investors who are able to take advantage of all sources of predictability with unskilled investors ignoring predictability. For an unskilled investor the certainty equivalent of wealth is 0.3-6.8% lower than for a skilled investor, depending on the market entry date. We also determine the effect of luck to enter the market at a favorable time. Across market entry dates, skilled but unlucky investors can lose up to 15.4% compared to unskilled but lucky investors. Simulation studies confirm the relative importance of luck and document that, if anything, housing predictability is more important than stock predictability.

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1 Introduction

This paper studies the importance of predictability in stock prices, house prices, and labor income in a rich life-cycle consumption-portfolio choice problem. We compare the welfare of skilled investors who are able to time the market and take all sources of predictability into account to unskilled investors ignoring predictability. A framework with predictable returns is also apt to analyze welfare effects along a second dimension. Whereas in models with iid returns the expected returns are always the same and consequently an investor's market entry date does not matter, the values of the predictor variables indicate whether a particular economic scenario is more or less favorable. Therefore, investors can enter the market under quite different economic conditions. An investor's market entry date is however predetermined by his birthday so a favorable entry date is pure luck.

We can thus compare investors along two dimensions: their skills and their luck to live under favorable economic conditions. We determine the welfare effects of living under the same economic conditions, but having different skills, and the welfare effects of living under different economic conditions. For 16 cohorts of investors entering the market between 1961 and 1976 and retiring 35 years later the certainty-equivalent welfare loss of being unskilled is between 0.3% and 6.8% with an average of 4.1%.¹ The three unluckiest but skilled investors entering the market in 1973, 1974, and 1976 realized average welfare losses of 7.1%, 11.2%, and 13.0% compared to lucky but unskilled investors entering the market about ten years earlier (1963, 1964, and 1965). In fact, the losses can be as high as 15.5% (skilled entering in 1974 vs. unskilled entering in 1964). Our findings are supported by a simulation study. In particular, we find that in 25% of the cases a skilled investor would rather trade in skill for luck, i.e. he would rather give up all his skills to live under more favorable economic conditions.²

To model skills, we estimate the joint dynamics of stock prices, house prices, and labor income based on aggregate, annual U.S. data for the CRSP value-weighted stock market portfolio, the national Case-Shiller home price index, and the disposable income per capita (all series are inflation adjusted). In our setting the net corporate payout yield predicts both the stock market index and house prices, whereas the excess growth of the log rent-price ratio over inflation predicts both house prices and labor income.³ The predictive power of the net payout yield on stock returns is

¹ Our setting allows investors to exploit predictability more than is typically possible in practice. The numbers reported here are conservative upper bounds and the effect of predictability is presumably smaller. This strengthens our conclusion that luck tends to be more important than market timing skills (see Section 6 for a detailed discussion).

²Bach, Calvet, and Sodini (2015) study a related issue empirically.

³The net payout of a company in a given period equals the dividends plus equity repurchases less equity issuances.

known from Boudoukh, Michaely, Richardson, and Roberts (2007), but its relation to house price growth rates has not been established before. The rent-price ratio is known to predict house prices (Himmelberg, Mayer, and Sinai (2005); Plazzi, Torous, and Valkanov (2010)).

We embed the estimated dynamics in a rich model of household decisions involving consumption of perishable goods and housing services, unspanned labor income, stochastic house prices, home renting and owning, stock investments, and portfolio constraints. Within this model we study the effect of predictability on welfare and portfolio performance using both a simulation-based and a historical approach. Our study is the first to analyze the effects of the skill to implement portfolio decisions that are based on signals predicting the returns of stock prices, house prices, and income. This joint view is essential for household decisions. Due to the contemporaneous correlation between the prices and income on one hand and the predictors on the other hand, the model produces a rich longer-run correlation structure between stock prices, house prices, and labor income that, for example, allows expected stock returns to be correlated with house prices or labor income.⁴

We do not only consider skilled and unskilled investors who either take all sources of predictability into account or disregard predictability all together. We also study the performance of what we refer to as semi-skilled investors who either ignore stock or housing or labor income predictability. This allows us to quantify the relevance of the skill to predict returns of a particular asset separately. Our paper thus also complements findings that quantify the impact of return predictability on stock-bond asset allocation decisions (see e.g. Campbell and Viceira (1999), Barberis (2000), and the references given below). Most papers focus on a particular type of predictability (in particular stock return predictability) and do not take the perspective of a household. For households, however, portfolio decisions should be seen in a life-cycle perspective incorporating human capital and real estate, the dominant assets for many households (Campbell (2006)). The ability to predict house prices and labor income is potentially as important for households as stock market predictability.

Our paper builds on the large literature on stock return predictability which reports that expected stock returns vary with such variables as the price-earnings ratio (Campbell and Shiller (1988)), the net payout yield (Boudoukh, Michaely, Richardson, and Roberts (2007)), past stock returns (Fama and French (1988); Moskowitz, Ooi, and Pedersen (2012)), or short-term inter-

⁴The housing collateral ratio and the ratio of aggregate labor income to aggregate consumption are also reported to help predicting stock returns (Lustig and van Nieuwerburgh (2005); Santos and Veronesi (2006)), which supports our assumption that the expected stock return can be correlated with house prices and labor income.

est rates (Ang and Bekaert (2007)). Kojien and van Nieuwerburgh (2011) survey this literature. The implications for stock-bond asset allocation has been explored by Kim and Omberg (1996), Campbell and Viceira (1999), Barberis (2000), and Wachter (2002) in stylized models disregarding housing and income. Since expected stock returns vary counter cyclically in these models, intertemporal hedging considerations lead to an increased demand for stocks. However, it is well known by now that optimal portfolios change substantially with the inclusion of labor income or housing, so the impact of stock return predictability on household portfolios should be explored in richer models.

Cocco, Gomes, and Maenhout (2005) show that a labor income process calibrated to life-cycle data is more bond-like than stock-like and thus induces agents to invest a large share of financial wealth in stocks, in particular early in life where human capital dominates. Two papers extend their study by allowing the expected income growth to depend on a business cycle variable, either the level of the short-term interest rate level (Munk and Sørensen (2010)) or the stock market dividend yield (Lynch and Tan (2011)), but these papers ignore housing aspects.

Only few papers derive life-cycle consumption and investment strategies in settings capturing both human capital, housing, and investment risk. Assuming for tractability a perfect correlation between house prices and aggregate income shocks, Cocco (2005) concludes that house price risk crowds out stock holdings and can therefore help in explaining limited stock market participation. Yao and Zhang (2005) generalize Cocco's setting to an imperfect house-income correlation and endogenize the renting/owning decision. They find that home-owners invest less in stocks than home-renters, which confirms that housing risk crowds out stock market risk.⁵ In our more general setting, we also find that the optimal stock investment is zero or low for many young households. Our paper documents that this result dampens the welfare effect of stock return predictability.

While several papers find evidence of predictability in real estate prices (Case and Shiller (1990), Poterba (1991), Malpezzi (1999), Ghysels, Plazzi, Valkanov, and Torous (2013)), only few papers discuss the implications for household decisions. Fischer and Stamos (2013) solve a life-cycle utility maximization problem assuming that expected housing returns depend on realized past returns. We add time-varying drift rates in stocks and income, and we allow the drift rates to be correlated with the levels of stock, house price, and income, which can substantially affect the magnitude and risk characteristics of human capital and therefore optimal investment and consumption decisions. In a

⁵van Hemert (2010) generalizes further by allowing for stochastic variations in interest rates and thereby introducing a role for bonds, but his focus is on the interest rate exposure and mortgage choice over the life cycle.

model without stock or income predictability, Corradin, Fillat, and Vergara-Alert (2014) show that predictability in house prices causes house [stock] investments to be increasing [decreasing] in the current expected house price growth. As in our paper, they also find cross effects of predictability in house prices on stock investments.

The remainder of the paper is organized as follows. Section 2 sets up and estimates the dynamic model of stock prices, house prices, and labor income. In Section 3 we formulate the life-cycle utility maximization problem of an individual consumer-investor and explain how we solve it. Section 4 presents the welfare effects that emerge in a simulation study. We also discuss the portfolio performances of the various investors. Section 5 determines the welfare effects for 16 cohorts of investors entering the market between 1961 and 1976. Section 6 argues that relaxing some of our model assumptions potentially reduces the welfare losses of being unskilled. Finally, Section 7 concludes.

2 The joint dynamics of stock prices, house prices, and income

This section presents our model for the joint dynamics of labor income, stock prices, and house prices, and calibrates it to U.S. data. We use a continuous-time formulation since this facilitates the derivation of the optimal consumption and investment decisions in subsequent sections.

2.1 Our main model

The time t level of the stock market index is denoted by S_t , the unit house price by H_t , and the labor income rate by L_t . In our framework, the dynamics of these variables (in real terms) are

$$\frac{dS_t}{S_t} = (r + \mu_S + \chi_S x_t) dt + \sigma_S dB_{St}, \quad (1)$$

$$\frac{dH_t}{H_t} = (r + \mu_H + \chi_{Hx} x_t + \chi_{Hy} y_t) dt + \sigma_H (\rho_{HS} dB_{St} + \hat{\rho}_H dB_{Ht}), \quad (2)$$

$$\frac{dL_t}{L_t} = (\mu_L(t) + \chi_L y_t) dt + \sigma_L (\rho_{LS} dB_{St} + \hat{\rho}_{LH} dB_{Ht} + \hat{\rho}_L dB_{Lt}). \quad (3)$$

Here x and y are the predictors with x_t represented by the corporate net payout yield and y_t by the excess growth of the log rent-price ratio over inflation (more details will be given below). Both

predictors are detrended and demeaned following

$$dx_t = -\kappa_x x_t dt + \sigma_x (\rho_{xS} dB_{St} + \hat{\rho}_{xH} dB_{Ht} + \hat{\rho}_{xL} dB_{Lt} + \hat{\rho}_x dB_{xt}), \quad (4)$$

$$dy_t = -\kappa_y y_t dt + \sigma_y (\rho_{yS} dB_{St} + \hat{\rho}_{yH} dB_{Ht} + \hat{\rho}_{yL} dB_{Lt} + \hat{\rho}_{yx} dB_{xt} + \hat{\rho}_y dB_{yt}), \quad (5)$$

where B_S, B_H, B_L, B_x, B_y are independent standard Brownian motions. All instantaneous correlations are constant. We let $\rho_{HS} = \rho_{SH}$ denote the instantaneous correlation between stock price and the house price and use similar notation for the other pairs of processes. In addition, define

$$\begin{aligned} \hat{\rho}_H &= \sqrt{1 - \rho_{HS}^2}, & \hat{\rho}_{LH} &= \frac{\rho_{LH} - \rho_{LS}\rho_{HS}}{\hat{\rho}_H}, & \hat{\rho}_L &= \sqrt{1 - \rho_{LS}^2 - \hat{\rho}_{LH}^2}, \\ \hat{\rho}_{xH} &= \frac{\rho_{xH} - \rho_{xS}\rho_{HS}}{\hat{\rho}_H}, & \hat{\rho}_{xL} &= \frac{\rho_{xL} - \rho_{xS}\rho_{LS} - \hat{\rho}_{xH}\hat{\rho}_{LH}}{\hat{\rho}_L}, & \hat{\rho}_x &= \sqrt{1 - \rho_{xS}^2 - \hat{\rho}_{xH}^2 - \hat{\rho}_{xL}^2}, \\ \hat{\rho}_{yH} &= \frac{\rho_{yH} - \rho_{yS}\rho_{HS}}{\hat{\rho}_H}, & \hat{\rho}_{yL} &= \frac{\rho_{yL} - \rho_{yS}\rho_{LS} - \hat{\rho}_{yH}\hat{\rho}_{LH}}{\hat{\rho}_L}, & \hat{\rho}_y &= \sqrt{1 - \rho_{yS}^2 - \hat{\rho}_{yH}^2 - \hat{\rho}_{yL}^2 - \hat{\rho}_{yx}^2}, \end{aligned}$$

and

$$\hat{\rho}_{yx} = \frac{\rho_{yx} - \rho_{yS}\rho_{xS} - \hat{\rho}_{yH}\hat{\rho}_{xH} - \hat{\rho}_{yL}\hat{\rho}_{xL}}{\hat{\rho}_x}.$$

By construction μ_S and μ_H are the long-term average expected growth rates of the stock and house prices per year in excess of the real risk-free interest rate r , which is assumed constant. The stock price and house price volatilities σ_S and σ_H are also constant. The stock index pays a constant dividend yield of \bar{D} so that the total dividends paid out over a short interval $[t, t + dt]$ are $\bar{D}S_t dt$. The parameters κ_x and σ_x denote the speed of mean reversion and the diffusion parameter of the net payout yield. Similarly for κ_y and σ_y . Increments to x and y are correlated with increments to S, H , and L , which implies that the longer-term relations between S, H , and L can be markedly different from the short-term relations.

2.2 Estimation and calibration

We estimate the above dynamics on time series of the stock market index, the national Case-Shiller home price index, and aggregate labor income. In this estimation we assume μ_L, χ_L , and σ_L are constant. When embedding the dynamics in the life-cycle decision problem in subsequent sections, we adjust the estimates of the house price and income volatilities to be more representative of individual house prices and labor income.

We use annual U.S. data for stock prices, house prices, and aggregate labor income from the beginning of 1960 (where available data on the home rent-price ratio begins) until the end of 2010 (where available data on net payout yield ends). As stock market data, we use returns on the CRSP value-weighted market portfolio inclusive of the NYSE, AMEX, and NASDAQ markets (cum dividend). The risk-free asset is estimated from the Treasury bill yield provided by the Risk Free File on CRSP Bond tape. The house price is represented by the national Case-Shiller home price index with data taken from Robert Shiller’s homepage.⁶ From the National Income and Product Accounts (NIPA) tables published by the Bureau of Economic Analysis of the U.S. Department of Commerce, we obtain quarterly U.S. data for aggregated disposable personal income (per capita). The annual returns are computed from quarterly data. To obtain real values, all time-series are deflated using the consumer price index (CPI) taken from the website of the Bureau of Labor Statistics.⁷

As suggested by Boudoukh, Michaely, Richardson, and Roberts (2007) we use the log of the sum of 0.1 and the net payout yield as our x -variable (for simplicity, we refer to x as the net payout yield in the following), and Professor Michael Roberts supplies the data until 2010 on his homepage.⁸ We use the net payout yield for nonfinancials, but obtain very similar results when including all firms. The data for the rent-price ratio is described in Davis, Lehnert, and Martin (2008) and is downloaded from the homepage of the Lincoln Institute of Land Policy.⁹ Since the log rent-price ratio is nonstationary but integrated of order one, we use the difference in the log rent-price ratio. This is normalized by subtracting the growth of CPI. Hence, the y -predictor is the excess growth of the log rent-price ratio over inflation. In other words, we benchmark rent increases against inflation.¹⁰ We will refer to y as the (normalized) change of the log rent-price ratio or change of the log rent-price ratio (benchmarked against inflation). To avoid that extreme outliers significantly affect the estimation, we winsorize x at the 4% level and y at the 2% level. This dampens the biases resulting from two outliers of x and one outlier of y . Figure 1 depicts the time series of the detrended predictors before winsorizing.

⁶<http://www.econ.yale.edu/~simshiller/data.htm>

⁷<https://www.bls.gov/cpi/>

⁸<http://finance.wharton.upenn.edu/~mrrobert/>

⁹<http://www.lincolninst.edu/resources/>

¹⁰Defining y as the difference of the two variables is a pragmatic way to avoid a third predictor and thus an additional state variable in our portfolio optimization. We could also use a model with predictors (x, \tilde{y}, z) where x is payout-ratio, \tilde{y} is change in the log rent-price ratio, and z is inflation. Then x predicts stock and housing returns, \tilde{y} predicts housing returns, and z predicts income. The VAR results are almost equivalent, since \tilde{y} and z have a correlation of virtually zero. Consequently, our findings on welfare and performance of skilled vs. unskilled or semi-skilled investors are hardly affected, but the computational complexity is reduced significantly.

In our sample, the net payout yield is a statistically better stock predictor than various alternatives suggested in the existing literature, namely the dividend yield, the log price-earnings ratio, the cyclically adjusted log price-earnings ratio, and the GDP growth rate, and none of these predictor candidates notably improve the prediction when added along with the net payout yield. However, the net payout yield has no predictive power for income growth, and the (normalized) rent-price ratio has no predictive power for stock returns. Therefore, we arrive at the model (1)-(5).

[INSERT FIGURE 1 ABOUT HERE]

We estimate a VAR(1) system which is a discretization of our continuous-time model and transform the VAR parameter estimates into estimates of our model parameters. Details are given in Appendix A. We estimate both the main model described above as well as four special cases. For these cases, we remove the predictors from one or all dynamics of S , H , and L , since we are going to explore the isolated and joint effects of predictability in stock prices, house prices, and labor income.

Table 1 lists the parameter values used as the benchmark in the following. These values equal the empirical estimates with a few exceptions. First, we reduce the equity premium from the estimated 5.6% to 4% to account for survivorship bias (Brown, Goetzmann, and Ross (1995)) as well as the decline in discount rates and the implied unexpected capital gains over the sample period (Fama and French (2002)). Moreover, a 4% equity premium is used in related papers such as Cocco, Gomes, and Maenhout (2005) and Yao and Zhang (2005). Secondly, the use of a house price index and aggregate income underestimates the volatilities of an individual house price and the labor income of a typical worker. We increase the volatility of house prices from the estimated value of 6.1% to 12%, which is identical to the value assumed by Flavin and Yamashita (2002) and Yao and Zhang (2005) and in the range estimated by Case and Shiller (1989) and Bourassa, Haurin, Haurin, Hoesli, and Sun (2009). Furthermore, we increase the income volatility from the estimate of 2.1% to 10%, in line with the estimate in Cocco, Gomes, and Maenhout (2005). We do not change the volatility in retirement for several reasons: (i) some retirees continue to earn income from proprietary businesses or other non-traded assets; (ii) uncertainty about medical expenses implies that the disposable income is risky (De Nardi, French, and Jones (2010)); (iii) because of mortality risk, the individual may miss retirement payments and, while we do not model mortality formally, retirement income risk captures this effect parsimoniously.

The average growth rate of the aggregate income series is 1.7% per year, but this is not reflecting

the income growth an individual can expect. As our benchmark we assume an expected income growth rate of 1% throughout the working life. Over the 35-year working period the income is then expected to grow by a factor $\exp(0.01 \times 35) \approx 1.42$, which seems reasonable and is close to the 38% reported as the median individual's income growth by Guvenen, Karahan, Ozkan, and Song (2015). In retirement, we set the growth rate to 0%.

[INSERT TABLE 1 ABOUT HERE]

For our sample, the average excess house price growth μ_H is estimated to be -0.5%. We adjust this number to -1% so that the average house price growth $r + \mu_H$ is equal to zero, which is for instance also used in Yao and Zhang (2005). This is in line with the long-term average reported in Shiller (2005). The net payout yield has a mean reversion speed of 0.234 (expected half-life of $(\ln 2)/\kappa_x \approx 3.0$ years) and a long-run standard deviation of $\sigma_x/\sqrt{2\kappa_x} \approx 0.127$. The normalized change of the rent-price ratio has a mean reversion speed of 0.298 (expected half-life of 2.3 years) and a long-run standard deviation of 0.048.

The pairwise contemporaneous correlations between stock prices, house prices, and labor income are all positive. The stock price is positively related to the net payout yield, $\chi_S > 0.329$, and since the two variables are negatively correlated, the model captures to mean reversion in stock returns. In contrast, since the house price index is both negatively related to and negatively correlated with its predictors, the model captures momentum in house prices, referred to as housing cycles by Fischer and Stamos (2013). The labor income is virtually uncorrelated with its predictor. Note that $\hat{\rho}_L^2 = 90.8\%$ of the variance of income shocks is unspanned.

The correlations of prices and income with the predictors generate interesting lagged effects. For example, a positive shock to stock prices this period tends to be accompanied by a negative shock to x (since $\rho_{xS} < 0$), which increases the expected house price growth next period (since $\chi_{Hx} < 0$).

3 The decision problem of a consumer-investor

We embed the estimated model (1)–(5) for the dynamics of stock price S_t , house price H_t , and labor income L_t in the life-cycle consumption and investment choice problem of an individual agent (consumer-investor or household). We assume the individual retires at a known time \tilde{T} and lives

on until a known time T . At retirement, the income rate drops to by a fixed proportion $1 - \Upsilon$,

$$L_{\tilde{T}+} = \Upsilon L_{\tilde{T}-},$$

where $\Upsilon \in (0, 1)$ is the so-called replacement rate. This is consistent with the wide-spread final-salary pension schemes and a common assumption in the literature (e.g., Cocco, Gomes, and Maenhout (2005); Lynch and Tan (2011)). We assume $\mu_L(t) = \mu_L = 0.01$ before retirement (active phase) and $\mu_L(t) = 0$ in retirement and that both the sensitivity χ_L towards the predictor and the income volatility are not changing at retirement, i.e. we allow for retirement income risk (see also our discussion in the previous section).

The agent consumes a perishable good and housing services from living in a house (we let “house” represent any type of residential real estate). The perishable good serves as the numéraire. The agent can invest in a bank account with a constant interest rate r and in the stock index with value S_t . The agent can invest in and rent houses. A house is characterized by a number of housing units, where a “unit” is a one-dimensional representation of the size, quality, and location. Prices of all houses are assumed to move in parallel. The purchase of a units of housing costs aH_t . The unit rental cost of houses is assumed proportional to their market prices so that the total costs of renting ϕ housing units over a short period $[t, t + dt]$ are $\phi RH_t dt$. These assumptions are standard in the consumption and investment literature involving housing (e.g. Yao and Zhang (2005); Fischer and Stamos (2013)). Following Kraft and Munk (2011), simultaneous owning and renting is possible. The agent derives utility from the number of housing units occupied, whether rented or owned.

To facilitate the solution of the agent’s utility maximization problem, we assume the agent can continuously adjust both the number of units rented and the number of units owned without transaction costs. Observed changes in the physical ownership of housing units seem rare and costly, but the remodeling or the extension of a house would also count as an increase in the number of housing units owned due to the higher quality or increased space. Moreover, real-life agents can also invest in housing units by purchasing shares in residential REITs (Real Estate Investment Trusts), exchange-traded funds tracking the REIT market, or other financial assets closely linked to house prices such as the Case-Shiller derivatives.¹¹ Homeowners can thus implement short-term variations to their desired housing investment position by investing in REITs, whereas they might

¹¹Well-developed REIT markets exist in many countries. Cotter and Roll (2015) study the risk and return characteristics of U.S. REITs. Tsai, Chen, and Sing (2007) report that REITs behave more and more like real estate and less and less like ordinary stocks.

prefer implementing larger changes in both desired housing consumption and investment through (rare) physical transactions of housing units. Note that the welfare gains of skilled investors reported in this paper rely on their ability to time the housing market by trading REITs. If we assumed that the agent has no access to this asset class, then the welfare gains of taking (housing) predictability into account were significantly lower (see also Section 6).

A housing investment can be seen as a physical purchase of a housing unit (by either the agent himself or by a REIT he holds shares in) which is then rented out. Ownership entails maintenance costs (including property taxes) equal to a constant fraction $m \geq 0$ of the property value. The rate of return on a housing investment over a period of length dt is therefore

$$(R - m) dt + \frac{dH_t}{H_t} = (r + \mu'_H + \chi_{Hx}x_t + \chi_{Hy}y_t) dt + \sigma_H (\rho_{HS} dB_{St} + \hat{\rho}_H dB_{Ht}) ,$$

where $\mu'_H = \mu_H + R - m$ is the average excess expected return on housing investments. Let ϕ_{ot} and ϕ_{rt} denote the number of housing units owned and rented, respectively, at time t , and let ϕ_{ft} denote the housing units owned via financial assets like REITs. What matters for the agent are the total units of houses occupied, ϕ_{Ct} , which provides utility from housing services, and the total units of housing invested in, ϕ_{It} , either physically owned or through REITs, where

$$\phi_{Ct} \equiv \phi_{ot} + \phi_{rt}, \quad \phi_{It} \equiv \phi_{ot} + \phi_{ft}. \quad (6)$$

Hence, we have a degree of freedom. Physical ownership and REITs investments complement each other, but we do not distinguish them in the model.

Let W_t denote the tangible wealth of the agent at time t , which includes the positions in the bank account, the stock index, REITs, and physically owned housing units, but not the agent's human wealth, i.e., the present value of her future labor income. Let Π_{St} and $\Pi_{Ht} = \phi_{It}H_t/W_t$ denote the fractions of tangible wealth invested in the stock and in housing units, respectively, at time t . The wealth invested in the bank account is residually determined as $W_t(1 - \Pi_{St} - \Pi_{Ht})$. The rate of perishable consumption at time t is represented by c_t . The wealth dynamics is then

$$dW_t = W_t \left[(r + \Pi_{St}(\mu'_S + \chi_{Sx}x_t) + \Pi_{Ht}(\mu'_H + \chi_{Hx}x_t + \chi_{Hy}y_t)) dt + (\Pi_{St}\sigma_S + \Pi_{Ht}\sigma_H\rho_{HS}) dB_{St} + \Pi_{Ht}\sigma_H\hat{\rho}_H dB_{Ht} \right] + (L_t - c_t - \phi_{Ct}RH_t) dt, \quad (7)$$

where $\mu'_S = \mu_S + \bar{D}$.

The objective of the investor is to maximize life-time expected utility from perishable consumption and consumption of housing services. The indirect utility function is

$$J(t, W, H, L, x, y) = \sup_{(c, \phi_C, \Pi_S, \Pi_H) \in \mathcal{A}_t} \mathbb{E}_t \left[\int_t^T e^{-\delta(u-t)} U(c_u, \phi_{C_u}) du \right], \quad (8)$$

where W , H , L , x , and y denote time t values of wealth, house price, labor income, and the two predictors, and where U is a Cobb-Douglas-power utility function

$$U(c, \phi_C) = \frac{1}{1-\gamma} \left(c^a \phi_C^{1-a} \right)^{1-\gamma}. \quad (9)$$

Here $\gamma > 1$ is the relative risk aversion, and $a \in (0, 1)$ the relative utility weight of the two goods.¹² Similar preferences are assumed in other recent papers, such as Cocco (2005), Yao and Zhang (2005), and van Hemert (2010). The set \mathcal{A}_t contains all admissible control processes over the time interval $[t, T]$. Constraints on the controls are thus reflected by \mathcal{A}_t . We shall impose the constraints

$$\Pi_S \geq 0, \quad \Pi_H \geq 0, \quad \Pi_S + q\Pi_H \leq 1, \quad (10)$$

which rule out short-selling and limits borrowing to a fraction $(1 - q)$ of the current value of the housing investment.

Because of incomplete markets (shocks to labor income and the predictors are not spanned by traded assets) and portfolio constraints, we are unable to solve the problem in closed form, but apply the approach introduced by Bick, Kraft, and Munk (2013). The method exploits that we can derive an explicit expression for the optimal strategy in each of various *artificial* markets.¹³ In any of the artificial markets the agent is unconstrained, has access to the same assets (with identical or higher returns) as in the true market plus additional assets completing the market, so the agent can obtain at least as high an expected utility as in the true market. Cvitanić and Karatzas (1992) show theoretically that the solution to the true, constrained and incomplete market problem is identical to the solution in the worst of all the artificial markets. We can solve the utility maximization problems in some artificial markets in closed form. The fractions of wealth optimally invested in

¹²We disregard utility of bequests which is known to have a negligible impact on portfolio decisions except maybe in the final few years of life. In an empirical study, Hurd (1989) concludes that bequest motives in various countries are close to zero.

¹³Appendix B explains the construction of the computable artificial markets and the solution to the corresponding utility maximization problem in more detail.

stocks and housing units are of the form

$$\begin{aligned} \Pi_S = & \frac{1}{\gamma \hat{\rho}_H^2 \sigma_S^2} \left(\mu'_S(t, x, y) + \chi_S x - \frac{\rho_{HS} \sigma_S}{\sigma_H} (\mu'_H(t, x, y) + \chi_{Hx} x + \chi_{Hy} y) \right) \frac{W + LF}{W} \\ & + \left(M_{xS} \frac{B_x}{B} + M_{yS} \frac{B_y}{B} \right) \frac{W + LF}{W} - \left(M_{LS}(t) + M_{xS} \frac{F_x}{F} + M_{yS} \frac{F_y}{F} \right) \frac{LF}{W}, \end{aligned} \quad (11)$$

$$\begin{aligned} \Pi_H = & \frac{1}{\gamma \hat{\rho}_H^2 \sigma_H^2} \left(\mu'_H(t, x, y) + \chi_{Hx} x + \chi_{Hy} y - \frac{\rho_{HS} \sigma_H}{\sigma_S} (\mu'_S(t, x, y) + \chi_S x) \right) \frac{W + LF}{W} \\ & + k \frac{W + LF}{W} + \left(M_{xH} \frac{B_x}{B} + M_{yH} \frac{B_y}{B} \right) \frac{W + LF}{W} - \left(M_{LH}(t) + M_{xH} \frac{F_x}{F} + M_{yH} \frac{F_y}{F} \right) \frac{LF}{W}. \end{aligned} \quad (12)$$

Here μ'_S and μ'_H are the adjusted expected excess stock and house returns (including dividends and rents). The functions $F = F(t, x, y)$ and $B = B(t, x, y)$ are found by solving simple partial differential equations which involve the Sharpe ratios on the fictitious assets completing the market. The product LF is the human capital, which is uniquely determined in any artificial market. The stock investment consists of the speculative demand, a term hedging the variations in expected stock returns, and an adjustment for the extent to which the human capital replaces a direct stock investment. The housing investment consists of three similar terms plus the term $k \frac{W+LF}{W}$, where $k = (1 - a)(\gamma - 1)/\gamma$, that hedges against increases in housing consumption costs.

The explicit, optimal strategy in any of the artificial markets is infeasible in the true market, but following Cvitanić and Karatzas (1992), Ex. 14.9, we can transform it into a feasible strategy in the true market—and evaluate the expected utility it generates in the true market by standard Monte Carlo simulation. We then maximize over these feasilized strategies. The corresponding maximizer is used in our analysis.

Just as with other numerical methods, the suggested strategy is unlikely to be identical to the unknown, truly optimal strategy. However, in the examples studied by Bick, Kraft, and Munk (2013) the relative welfare difference between the suggested strategy and a strategy coming from a grid-based model, which is supposed to deliver the optimal solution, are typically negligible. In any case, the number of state variables (five plus time) prevents us from implementing grid-based methods with reasonable grid sizes that bring us near the continuous-time solution.

4 Simulation results

This section studies the effects of predictability on portfolio decisions and quantifies the welfare effects of skill and luck in a simulation study. Unless otherwise noted, the estimation-based parameter values in Table 1 are used. Furthermore, we assume a relative risk aversion coefficient of $\gamma = 5$. We set the relative utility weight of the goods to $a = 0.7$, implying that total consumption expenditures consist of 70% on perishable goods and 30% on housing consumption, which seems consistent with observed household expenditure, cf. a report by the U.S. Department of Labor (2003). The subjective time preference rate is $\delta = 0.05$. All agents are initially of age $t = 30$, retires at age $\tilde{T} = 65$, and live on until age $T = 80$. We assume an income replacement rate of $\Upsilon = 0.6$.¹⁴ We set the proportional rental rate to $R = 0.067$ as motivated by Fischer and Stamos (2013) and assume maintenance costs of $m = 0.035$ (includes property taxes that constitute 1-2% in many U.S. states). Finally, we assume a 60% maximal loan-to-value ratio corresponding to $q = 0.4$.¹⁵

To be specific, we think of a housing unit as 1,000 square feet of average quality and location. Using a monetary unit of a thousand U.S. dollars, we set the initial unit house price to $H = 250$, which implies an initial annual rent of \$16,750 for a housing unit. Furthermore, the initial tangible wealth is set to $W = \$20,000$ and the initial annual income to $L = \$20,000$ which are roughly equal to the median values for individuals of age 30-40 in the 2007 Survey of Consumer Finances (see Kraft and Munk (2011)). Finally, we set the initial values of the predictors to $x = y = 0$.

Therefore, all simulations start at the average values of the predictors. However, since the predictors are rather volatile, there is already quite some dispersion of the predictor values after a few years so that the investors live under diverse economic circumstances along the corresponding predictor paths. This allows us to study welfare effects of different economic conditions described by the values of the predictors.

4.1 Portfolio decisions

The portfolio strategies (11) and (12) depend on time and four state variables and are thus difficult to depict graphically. To get a first impression of the determinants of the portfolio strategies, we

¹⁴The reduction from the 68%-93% estimate of Cocco, Gomes, and Maenhout (2005) is a way to implicitly incorporate the higher medical expenses in retirement as well as the increased mortality risk that lowers expected future income.

¹⁵Robustness checks for loan-to-value ratios of 80% are available upon request. The portfolio strategies are similar, but the house becomes even more attractive.

calculate them for about 1,000,000 grid points (see Table 2) and run three regressions of π_S , π_H , and c/W on the state variables and some interaction terms.¹⁶ The regression outputs are shown in Table 3 that reports standardized regression coefficients so that we can interpret the magnitude of the loadings. The high R^2 values indicate that our specification approximates the true relation between the decision variables and the explanatory variables well.¹⁷

[INSERT TABLES 2 and 3 ABOUT HERE]

The house investment tends to be decreasing in age. This is because the hedging motive for future housing consumption becomes smaller over time. Besides, housing is more important for young agents, as it can be used as collateral and gives access to credit. In turn, the stock investment tends to be increasing in age. Furthermore, the consumption wealth ratio increases over time, which is a standard result given the size of the agent's time preference rate.

The predictors affect the decisions in several ways: The house investment decreases in both predictors because their loadings χ_{Hx} and χ_{Hy} in the expected house price return are negative leading to a momentum effect. This tilts investments towards stocks if the predictor values are positive. Since a high payout ratio x additionally predicts high stock returns in the future (a well-known result since x generates mean reversion in stock returns), whereas y has no direct impact on stock returns, the effect is stronger for x . Notice that a high value of y signals more future income. Since the agent tries to smooth consumption, the consumption-wealth ratio thus increases with both x and y .

Furthermore, a high income-wealth ratio indicates high current and future income. The agent's motive to smooth consumption thus drives him to increase both perishable and durable consumption. To hedge his future housing consumption, he also increases his housing investment, which in turn leads to a smaller stock investment (see fourth row of Table 3).

Finally, we study the effect of interaction variables between the income-wealth ratio and time t or the predictors x and y . For the optimal investments the signs do not change compared to loadings on t , x , and y alone. Notice however that the consumption-wealth ratio loads negatively on the payout ratio interacted with the income-wealth ratio. In this case, the agent cuts down on

¹⁶Since the optimal spending on housing consumption relative to perishable consumption is equal to a constant we do not consider ϕ_C separately.

¹⁷Of course, this is only true for the grid values. For outliers the relation is weaker. This is why one cannot simply use the approximation in the portfolio choice problem. Notice also that this is not a linear approximation. If we interpret our approximation as Taylor expansion, then the interaction variables are second-order terms.

both perishable and durable consumption and invests heavily in stocks, since x signals high stock returns in the future. In turn, he reduces his housing investment.

To assess *average* life-cycle patterns, we perform the following analysis: We simulate 10,000 paths of exogenous state variables and calculate the optimal consumption, housing, and investment strategies as well as the resulting wealth along these paths. Finally, we calculate expectations of consumption, wealth, investments, and portfolio weights by averaging over the simulations.

[INSERT FIGURE 2 ABOUT HERE]

Figure 2 illustrates the average optimal investment strategy over the life cycle with baseline parameter values. The horizontal axes depicts the time passed after the initial date where the agent is assumed to be of age 30. The left panel shows the amounts invested in the housing asset, the stock index, and the risk-free asset (bond), whereas the right panel depicts the portfolio weights relative to tangible wealth. The agent builds up wealth in the active phase to finance consumption in retirement where income is markedly lower. The portfolio is dominated by housing, especially early in life, where the investment is maximally leveraged. Later in life, borrowing is less than the allowed 60% of the house value. The stock weight is around 25% early in life, but increases rather quickly to around 50%, where it remains relatively stable. In settings ignoring housing, labor income typically leads to a full stock investment being optimal (leveraged if possible), cf. Cocco, Gomes, and Maenhout (2005), even for relatively high levels of risk aversion. Our results confirm the findings of Cocco (2005), among others, that housing crowds out stocks. Not only is housing a decent investment in itself (especially considering the rents), it also provides access to leverage, and constitutes a hedge against increases in housing consumption costs.

Now, we compare the investment profiles of agents accounting for predictability and agents disregarding predictability. We refer to these agents as *skilled* and *unskilled* investors. The latter agent assumes that the dynamics of stock prices, house prices, and labor income are given by (1)–(3), but without the terms involving x and y , i.e., effectively imposing $\chi_S = \chi_{Hx} = \chi_{Hy} = \chi_L = 0$, and thus arriving at the parameter estimates in the column labeled “Not at all” of Table 1. The solid lines in Figure 3 show the two agents’ expected portfolio weights over the life cycle. Predictability materializes as mean reversion in stock returns and thus leads to a larger *average* portfolio share of the stock as found in simpler settings by Kim and Omberg (1996) and others. In contrast, predictability in house price growth emerges as momentum and thus lowers the *average* share of housing in the portfolio.

[INSERT FIGURE 3 ABOUT HERE]

Looking at *unconditional* averages is however too simplistic if asset returns are predictable. A skilled investor tries to take advantage of the information contained in the values of the predictors. We thus perform a scenario-based analysis of the portfolio decisions. Since the distributions of both predictors x and y are symmetric around zero, we study scenarios where the predictors are positive or negative.

As can be seen on the right-hand side of Figure 3, not surprisingly the decisions of the unskilled agent are hardly affected by the values of the predictors. The small deviations of the strategies are solely driven by wealth effects. This is in sharp contrast to the decisions of the skilled agent. He times the market and for instance significantly reduces his stock investment if the payout ratio x is low. In this case the optimal stock investment is even lower than the one of the agent ignoring predictability (except for the first years). Put differently, although the average is higher it turns out that the optimal stock demand is in 50% of the cases smaller. In that sense, statements like “stock predictability increases the stock demand” are only half-true.

Furthermore, as in Corradin, Fillat, and Vergara-Alert (2014) there are cross-effects of the house return predictor y on stock investments, although the stock returns are not predicted by y . Since low values of y predict high house returns, the optimal housing investment is in this case larger than the average. Therefore, the agent must reduce his stock exposure due to borrowing constraints. It turns out that the stock investment for both agents is almost of the same order of magnitude after 20 years if y is low.

4.2 Welfare effects and portfolio performance

By taking predictability into account, the skilled agent can time the market and thus generate higher investment returns, which leads to a higher average consumption level. Table 4 reports the welfare effects of disregarding predictability. We compare the optimal decisions with situations where the agent disregards stock, housing or income predictability or predictability all together. Whereas the latter investor is referred to as unskilled, we refer to all agents that disregard only one type of predictability as *semi-skilled*. Our main measure to assess the welfare loss is a utility-based

measure (RWEL) that is defined as follows:

$$1 - \left(\frac{J^i}{J^{skilled}} \right)^{\frac{1}{1-\gamma}}, \quad (13)$$

where $J^{skilled}$ is the indirect utility of a skilled investor, while J^i is the expected utility of a semi- or unskilled investor operating in the market described by our full model (see equation (58) in the Appendix). Additionally, we also report two present-value-based measures that we refer to as relative gains (RGmean or RGmedian). For the relative gains, we compute the present values of consumption (perishable and housing) per path for the optimal and a suboptimal strategy. Then we either calculate the mean or the median and divide by the mean or median realized by the skilled investor. Formally,¹⁸

$$\frac{A(PV_n^{skilled}) - A(PV_n^i)}{A(PV_n^{skilled})} \quad (14)$$

where $A(\cdot)$ is either the mean or the median and $PV_n^{skilled}$ and PV_n^i are the present values of total consumption over the n -th path calculated using an interest rate of 1% (see Table 1).

[INSERT TABLE 4 ABOUT HERE]

Table 4 reports several interesting results: First, an unskilled investor disregarding predictability all together suffers a welfare loss of about 5.73% (measured in RWEL). For an semi-skilled investor ignoring either stock or housing predictability, the loss reduces to 1.41% or 1.86%. In our setting, disregarding housing predicability is thus slightly worse than disregarding stock predictability. Finally, ignoring income predictability leads to a smaller loss of 0.74%.

The welfare losses are higher if measured in terms of relative gains. This indicates that the strategies of the semi- and unskilled investors perform relatively well in bad states and are particularly outperformed in good states. In fact, the distribution of the present value of consumption realized by the skilled investor is more positively skewed than the corresponding distributions of the semi-skilled or the unskilled investors.¹⁹ This also leads to higher average than median gains. Consequently, taking predictability into account generates more asymmetric distributions since the skilled investor tries to time the market. If this gamble works out well, then he realizes high gains,

¹⁸Alternatively, we could also calculate pathwise gains $(PV_n^{skilled} - PV_n^i)/PV_n^{skilled}$ and then average over these gains. By Jensen's inequality, the resulting losses would be smaller.

¹⁹The skewness is 4.1 for the skilled investor, 3.1 (1.9) for the investor disregarding stock (housing) predictability, and 1.7 for the unskilled investor.

whereas the losses are limited due to borrowing constraints.²⁰

Table 4 also reports the welfare losses for paths with positive and/or negative average values of the predictors. For instance, for the case $\bar{x} > 0$ ($\bar{x} < 0$) we consider only paths where the average value of x along the path is positive (negative). One can interpret a path with $\bar{x} > 0$ as a scenario where the agent lives in times with a comparably high payout ratio. Analogously, we also report the numbers for scenarios where we jointly condition on both averages. Notice that up to sampling errors all one-dimensional scenarios are equally likely, i.e. for instance that about half of the paths fall in the category $\bar{x} > 0$ and the other half into $\bar{x} < 0$. The same is true for the four two-dimensional scenarios where about 25% of all paths fall into one of the four scenarios.

If the agent disregards stock predictability only, then RWEL is high if $\bar{x} > 0$ since he would benefit from taking predictability into account and increasing his stock exposure. In the opposite scenario, $\bar{x} \leq 0$, the skilled investor reduces his stock position or even wants to short stocks which is not possible due to short-sale constraints. Therefore, RWEL is low. The worst scenario for this particular semi-skilled investor is $\bar{x} > 0$ and $\bar{y} > 0$. In this case, a skilled investor would overweight his stock position since $\bar{x} > 0$ and underweight his housing exposure since $\bar{x} > 0$ and $\bar{y} > 0$. However, the semi-skilled investor does not increase his stock position enough and retains too much housing exposure which is on average performing poorly. Notice that the comparably high loss of 1.78% for $\bar{y} > 0$ is slightly misleading since it consists of the two-dimensional scenarios $\bar{x} > 0, \bar{y} > 0$ and $\bar{x} \leq 0, \bar{y} > 0$, which are quite different. In the first scenario, the agent ignores a buy signal leading to a bigger loss of 2.33%, whereas in the second scenario he disregards a sell signal, which leads to a smaller loss of 1.18%. The average of the two losses is about 1.76%, which is close to 1.78%. This asymmetry of the losses in the two-dimensional scenarios is driven by binding short-sale constraints that prevent skilled investors from fully exploiting sell signals.

If the agent disregards housing predictability only, then the situation is straightforward since house price returns are predicted by both x and y . Therefore, RWEL is the biggest if the agent ignores buy signals to invest in housing ($\bar{x} < 0$ or $\bar{y} < 0$). The worst case for the semi-skilled investor is a combination of both ($\bar{x} < 0$ and $\bar{y} < 0$). In this scenario, investing in housing is very beneficial, but the semi-skilled investor misses out on this opportunity.

If the agent is unskilled and disregards predictability all together, then RWEL is between 5.02% and 6.42% across the one-dimensional scenarios where the payout ratio predicting stock returns has

²⁰Welfare effects can well be slightly negative in specific scenarios. However, by construction, the welfare effect must be positive when averaging over all possible paths since the strategies are optimized unconditionally at time 0.

a more pronounced effect. To understand the relative importance of the predictors, consider the two-dimensional scenarios generating more dispersion of RWEL. The scenarios involving a positive payout ratio, $\bar{x} > 0$, lead to the largest losses, i.e. conditionally x matters more than y . The worst scenario is $\bar{x} > 0$, $\bar{y} > 0$ where it is beneficial to buy stocks and reduce the housing exposure. The skilled investor thus tilts his portfolio heavily towards stocks, whereas the unskilled investor misses out on this opportunity and keeps a more balanced portfolio. Notice that the skilled investor cannot benefit from negative payout ratios in the same way due to short sale constraints. If we however compare the two scenarios with $\bar{x} \leq 0$, then the loss is bigger when y is on average negative as well, $\bar{y} \leq 0$. This is because jointly negative values of the predictors generate a clear signal to sell stocks and increase the housing exposure. By contrast, the unskilled investor is not aware of this opportunity.

We have also sorted on the realized volatility of x or y along a path. We do not report the results here, since the findings are straightforward: A skilled investor benefits the most if one or both volatilities of the predictors are high. In these scenarios market timing pays off more frequently leading to a better performance of the skilled investor. The semi-skilled or unskilled investors implement more static strategies and (partly) miss out on these market timing opportunities.

[INSERT TABLE 5 ABOUT HERE]

To support our previous statements, we now study the portfolio returns and holdings of the skilled, semi-skilled, and unskilled investors in more detail. Table 5 reports several conditional and unconditional averages over the time points of the simulated paths. The row labeled “All points” provides the unconditional averages over all time points of all simulated paths.²¹ The rows below report the conditional averages if we condition on the predictor values of the previous time point on the particular path. This allows us to study how portfolio decisions at the beginning of a trading interval, time $t - 1$, generate returns at the end of that interval, time t .²² The results for the semi-skilled investor are omitted since they are similar to the findings for the skilled investor.

We first focus on the unconditional results labeled by “All points”. The average annualized excess portfolio return before consumption (pfret) at time t is 4.1% for the unskilled investor,

²¹In total, we have 10,000,000 time points (= 10,000 paths \times 50 years \times 20 steps/year).

²²As a robustness check, we have also performed this exercise where we do not sort pointwise on the realizations of the predictors, but on the average realizations of the predictors along a path. Intuitively, this means that one considers portfolio performances of investors living in times with for instance on average high or low values of the payout ratio x . The results are similar and thus not reported here.

but 2.8% higher for the skilled investor. Notice that mean and median are almost identical for the unskilled agent, but the median of the skilled agent is about 1% smaller than his average portfolio return. This confirms that the market timing of the skilled investor generates a more positively skewed return distribution, which in turn leads to a more positively skewed consumption distribution.

The average portfolio fractions invested in the two asset classes (stock, house) at the beginning of every trading interval are reported in the columns labeled π_S and π_H . Unconditionally, the skilled agent invests much more in stocks than the unskilled one (on average 40.9% compared to 24.5%), which is reported in several other papers (see, e.g., Barberis (2000)). One reason is that stocks are less risky due to the mean reversion in returns, which leads to higher stock positions also in settings without housing. Additionally, in our setting housing is more risky due to the momentum effect. Therefore, even a semi-skilled investor disregarding stock predictability tilts his portfolio towards stocks (on average 33.3% compared to 24.5% for an unskilled investor).

Notice also that the portfolio strategies of the skilled and the semi-skilled investors are more volatile than the strategies of the unskilled agent, since the latter does not try to time the market. In any case, for all investors the housing investment is at least two and a half times bigger than the stock investment and thus dominates the portfolio holdings. The reason is twofold: First, the house is a durable good and all agents must hedge their future housing consumption. Second, agents can borrow against housing, which makes it comparably more attractive than stocks.

The unconditional volatility of the portfolio returns is similar across all investor types and between 19.2% and 20.5%. This is higher than the individual asset volatilities of stock and house price since the investors take leveraged positions. The differences in volatility are driven by the size of the housing position because only the house can be used as collateral. Since the unskilled investor has the biggest position, his portfolio-return volatility is the biggest. The semi-skilled investor disregarding house-price predictability has the smallest housing position and thus the smallest volatility. Only the semi-skilled investor disregarding stock predictability is kind of an outlier compared to the skilled investor. He has a slightly higher housing position, but the same return volatility. This is because he is less exploiting his opportunity to lever up and invest in the stock market. Notice that the unconditional Sharpe ratios are mostly determined by the average portfolio returns since there is comparably few variation in the unconditional volatilities.

To obtain a more balanced picture of the portfolio performances, we study the portfolios in four two-dimensional scenarios where we report returns and holdings conditional on whether the predictors are positive or negative at the beginning of the particular trading period, $t - 1$. It turns out that the out-performance realized by the skilled investor (relatively to the unskilled investor) is driven by his returns in three out of four scenarios where one scenario, $x_{t-1} > 0$ and $y_{t-1} > 0$, stands out in particular.

The best scenario for the portfolio performance of all investors is a situation where both predictors are negative, $x_{t-1} \leq 0$ and $y_{t-1} \leq 0$. This is because all investors have high housing exposures and expected house price appreciation is high in this scenario. So the average returns in this scenario are 9.5% and 6.9% for the skilled and unskilled investor. The large return differential of 2.6% comes from the fact that the median housing position of the skilled investor is about 50% bigger than the position of the unskilled investor. In fact, this is the only two-dimensional scenario where the housing exposure of the skilled investor is bigger than the one of the unskilled investor. As a result, the skilled investor holds virtually no stocks (median is zero, mean is 3.5%) and is thus not diversified. He interprets the negative value of x_{t-1} (low payout ratio) as a signal for low stock returns. Additionally, he perceives the combination of negative values of x_{t-1} and y_{t-1} as a strong signal that house prices will appreciate in the future. His high housing position also generates the highest portfolio-return volatility of 24.6% across all investor types and scenarios. This explains why his conditional Sharpe ratio is of similar magnitude as the Sharpe ratios of the other investors although he realizes significant average return differentials compared to some of them.

The worst scenario for the unskilled investor is a situation where both predictors are positive ($x_{t-1} > 0$ and $y_{t-1} > 0$). The skilled investor realizes that in this scenario house prices on average decrease significantly and stock prices increase since $x_{t-1} > 0$. Therefore, his median portfolio positions are again extreme with 100% in stocks and 0% in housing. Notice that all the other investors hold on to housing positions. Of course, in our simulations the assumptions about predictability hold and the estimated parameters are also the ones that are used to simulate the paths. Therefore, the skilled investor performs well despite his extreme strategy. His return differential of 5.7% with respect to the unskilled investor is the highest across scenarios.

The worst situation for the skilled investor is a scenario with $x_{t-1} \leq 0$ and $y_{t-1} > 0$, where profitable market-timing is hardly possible and thus the median returns of all investors are similar. Negative values of x_{t-1} are a sell signal for stocks. Therefore, the median stock position of the

skilled investor is zero, whereas his median housing position is above average and close to the one of the unskilled investor. Notice that a situation with $x_{t-1} \leq 0$ and $y_{t-1} > 0$ does not give a clear signal whether to increase or decrease the housing investment.

Finally, in the scenario $x_{t-1} > 0$ and $y_{t-1} \leq 0$ the skilled investor tilts his portfolio significantly towards stocks, since a positive value of x_{t-1} (high payout ratio) is a buy signal for stocks and the combination of $x_{t-1} > 0$ and $y_{t-1} \leq 0$ does not provide a clear signal for housing investments. In fact, the skilled investor implements the most balanced strategy across all scenarios and realizes the second highest return. He outperforms the unskilled investor by 2.2%. On the other hand, this is the only scenario where his conditional Sharpe ratio is smaller than the Sharpe ratio of a lesser skilled investor. More precisely, the semi-skilled investor disregarding housing predictability has a much smaller housing position and thus realizes a smaller return volatility, but a similar average excess return.

To summarize, the out-performance is primarily generated in three out of four scenarios. These are the scenarios involving a buy signal for stocks, $x_{t-1} > 0$, or a strong buy signal for housing investments, $x_{t-1} \leq 0$, $y_{t-1} \leq 0$. However, the attempt of the skilled investor to time the market comes at the cost of very undiversified portfolio positions. In three out of four scenarios, he invests in stocks or housing only (measured by the median). From a practical point of view, this could be an issue since it relies on the assumption that predictability is correctly specified and estimated. In fact, our analysis puts the skilled investor in the best position possible since our model is simulated with the estimated parameters. If this is not the case (as most likely in practice), then the reported welfare gains and return differentials of a skilled investor are smaller. In particular, it might become an issue that the optimal portfolio strategy can lead to very undiversified portfolio holdings that involve significant transaction costs when the skilled investor has to adjust his portfolio if the scenario changes.

[INSERT TABLE 6 ABOUT HERE]

Finally, we compare the welfare effects across scenarios for four different investor types. Since the welfare effects for a semi-skilled investor ignoring income predictability are moderate, we disregard this particular semi-skilled investor. Not surprisingly, the skilled investor realizes the highest utility in every scenario, whereas the unskilled investor realizes the lowest utility. This is however not the case across scenarios. For instance, the unskilled investor has a higher utility for $\bar{x} \leq 0$, $\bar{y} > 0$ than the skilled investor for $\bar{x} > 0$, $\bar{y} \leq 0$. This is because housing is more affordable in the first case and

thus the unskilled investor can occupy a large house. Intuitively, we can think of such a comparison as a comparison between two generations of investors that live in different times. Following this idea we could then ask the following question: In how many cases would a skilled investor rather give up (some of) his skills and live under better conditions? In other words, when would he be willing to trade in skill for luck? Comparing the skilled and unskilled investor this is never the case for the one-dimensional scenarios since all conditional indirect utilities of the skilled investor are larger than the ones of the unskilled investor. For the two-dimensional cases we obtain a different result: In 25% of the cases ($\bar{x} > 0, \bar{y} \leq 0$) the skilled investor would rather give up his skills and live in the scenario $\bar{x} \leq 0, \bar{y} > 0$, where his perishable consumption would be slightly smaller, but his housing consumption much bigger. If we compare the skilled with a semi-skilled investor, the willingness to trade in skill for luck is even more pronounced. In 50% of the one-dimensional cases, the skilled investor would like to give up either his abilities to time the stock or housing market if he could choose another scenario to live in. For the two-dimensional scenarios, it is even so that in three out of four scenarios, i.e. in 75% of the cases ($\bar{x} > 0, \bar{y} > 0$ or $\bar{x} > 0, \bar{y} \leq 0$ or $\bar{x} \leq 0, \bar{y} \leq 0$), he would like to trade in parts of his skills to live in the scenario $\bar{x} \leq 0, \bar{y} > 0$ with fewer skills.

5 Historical perspective

Figure 1 shows the realized paths of the two predictors over our sample period 1960-2010. For 16 different cohorts of agents entering the market between 1961 and 1976, we now study their performances over their corresponding working period of 35 years.²³ For each entry date, we have five types of agents: a skilled investor taking all kinds of predictability into account, three different types of semi-skilled investors disregarding either stock, housing, or labor income predictability and an unskilled investor disregarding predictability all together. For simplicity, we focus on the skilled investors, the semi-skilled investors disregarding stock predictability, and the unskilled investors.²⁴ This leaves us with 48 cohorts of investors. Agents starting in the same year are exposed to the same excess stock returns, excess housing returns, and labor income shocks that we extract from the historical time series. We keep the stock returns as they are, but scale the idiosyncratic shocks

²³We take the final wealth at time $t = 35$ into account via a fictitious bequest motive. The particular assumption however does not affect our results reported in Figure 7 significantly, since we follow the usual practice to report the welfare effects at time 0.

²⁴The semi-skilled investor disregarding labor income predictability performs very similar as the skilled investor. So we do not get much additional insights from this type. The results for the semi-skilled investor disregarding housing predictability are qualitatively similar to the semi-skilled investor disregarding stock predictability. All omitted results are available upon request.

of housing and labor income such that the total volatilities are 12% and 10% as in the baseline calibration of Table 1. We fix the real risk-free rate at 1%, which is about the sample average. Besides, we assume that all agents start with an initial tangible wealth of $W = \$20,000$ and an initial annual income of $L = \$20,000$. Investors revise their decisions annually. This seems to be a fair assumption, since we are disregarding transaction costs. Furthermore, we use the in-sample parameter estimates for all processes (stock, house, labor, predictors) in sample, i.e. skilled investors and to some extent semi-skilled investors have a great advantage if predictability matters.

Now, two interesting questions arise: How relevant is it for a particular market entry date that investors take predictability into account? In other words, how much do skills matter historically? Secondly, how big is the effect of being lucky, i.e. of being born at a favorable time? Put differently, how much do investors lose if they enter the market at a bad time? Table 7 summarizes our findings.

[INSERT TABLE 7 ABOUT HERE]

This table provides several pieces of information: The first column reports the ranking of a particular cohort which is characterized by its entry year (second column labeled “Start”) and skill level (third column labeled “Pred”). Skilled investors are marked yellow (label “1”), semi-skilled investors green (label “2”),²⁵ and unskilled investors red (label “5”). We rank the cohorts according to the investor’s utility of consumption over the working period and wealth at retirement ($t = 35$). For instance, a skilled investor who enters the market in the year 1964 is ranked number 1 among all 45 cohorts. The fourth column then reports the relative welfare losses of all other cohorts compared to this cohort (1964, skilled). For instance, a skilled agent who enters the market only two years later (1966, skilled) loses 7.1% and is ranked number 16. On the other hand, an unskilled investor entering the market in the same year (1964, unskilled) loses only 3.3% and is ranked number 5. All other welfare losses can also be found in the lower left triangle of Table 7.

The upper right triangle speaks to the question of how less skilled investors perform compared to more skilled investors across cohorts. Every filled box indicates that a less skilled investor beats a more skilled investor simply because the less skilled investor was lucky to enter the market at a more favorable time. A green box refers to a case where a semi-skilled investor beats a skilled investor, a red box to a case where an unskilled investor beats a semi-skilled investor, and a black box to a case where an unskilled investor even beats a skilled investor. For instance, consider the

²⁵Since we are only considering semi-skilled investors that disregard stock predictability, there should be no confusion in this section if we refer to them as semi-skilled.

row with rank 13 that contains the results for an unskilled investor who enters the market in 1965. This investor beats skilled agents entering in 1966 and 1972 (column with rank 14 and 16) and a semi-skilled investor entering in 1971 (column with rank 15), among others. To put this into perspective, about 20% of all boxes are filled, i.e. in 20% of the cases a less skilled investor beats a more skilled investor simply because of a more favorable market entry date.

[INSERT TABLES 8, 9 ABOUT HERE]

Let us now take a closer look at the welfare losses to get an overall picture on how these losses are affected by skill and luck. If we compare skilled and unskilled investors for a particular entry date, then Table 8 summarizes the relevant results from Table 7 and reports the welfare losses for skilled vs. unskilled, skilled vs. semi-skilled and semi-skilled vs. unskilled investors. Skilled investors entering the market in 1970 were able to realize the highest overall welfare gain of 6.8% (compared to an unskilled investor). On average, their gain is 4.1%, but it could be as low as 0.3%. The average numbers for skilled vs. semi-skilled and semi-skilled vs. unskilled are 2.9% and 1.3%. Notice that these losses are almost additive (sum of the last two columns yields first column).²⁶ To summarize, the welfare losses of being unskilled are historically at most 6.8%.

To put these results into perspective, we compare them to intergenerational losses reported in Table 7. We focus on the three highest-ranked unskilled investors entering the market in 1963, 1964 or 1965 (columns with rank 5, 13 or 17) and the three lowest-ranked skilled investors entering the market in 1973, 1974 or 1976 (rows with rank 39, 43 or 44). The corresponding losses can also be found in Table 9. For instance, the skilled investor entering the market in 1974 loses 15.4% (12.3%, 11.2%) compared to an unskilled investor entering the market in 1964 (1965, 1963), i.e. about 10 years earlier. For a skilled investor entering the market in 1973 (row with rank 43) the average loss compared to these unskilled investors (1963, 1964, 1965) is 11.2%. Besides, the corresponding average is 7.1% for an investor entering the market in 1976 (row with rank 39). These numbers are all bigger than the largest loss of 6.8% from the previous paragraph. Finally, notice that the luckiest investors in our sample enter the market in 1964. They rank number 1, 2 and 5. The loss of the most unlucky investor (1974, semi-skilled) compared to these agents is 21.1%, 20% and 18.5%. Therefore, the effect of skills is again moderate compared to the large effect of being lucky.

²⁶Negative losses can occur, since we consider just one path only.

6 Robustness to model limitations

Although we study a rich life-cycle model, we make some simplifying assumptions to obtain a tractable framework. This section argues that relaxing some of these assumptions would strengthen our conclusion that luck is often more important than market timing skills.

Estimation and parameter uncertainty The predictors that we used were the best that we could find. For instance, the payout ratio has a better predictive power in our sample than the dividend yield. Besides, we estimate the model in sample. Our skilled investor has access to this estimate over his full life-time. In real life, the best he can do is to learn about predictability (see, e.g., Xia (2001)). This reduces the benefits from predictability.

Transaction costs A market timer like our skilled investor has to trade frequently and substantially, since he is often fully invested either in stocks or in house. In practice, this can generate substantial transaction costs that reduce the benefits from predictability since an unskilled investor does not trade as frequently.

REITs Our investors have access to a liquid market for REITs so that they can hedge their housing demands. However, this also allows the skilled investor to benefit from housing predictability by actively timing the market. If REITs are not available, then he cannot benefit from house price predictability, which reduces the overall effect of predictability.

Partial vs. general equilibrium We use a partial equilibrium framework to study the effect of predictability. Of course, we cannot all time the market. If a majority of investors tries to exploit the benefits of predictability, then they might compete these benefits away. This however suggests that the welfare effects are smaller if we impose market clearing conditions.

Simulation vs. real life Real investors do not live along 10,000 simulated paths, but just along one real path. This makes the underdiversified positions of a market timer very risky (in particular with the above mentioned parameter uncertainty). It can well happen that on a particular path an unskilled investor performs better than a skilled investor. For instance, in our simulation study the unskilled investor realizes a higher indirect utility along 12% of the 10,000 paths. One can thus

argue that already common sense dictates to implement less extreme positions than those of the skilled investor.

7 Conclusion

This paper evaluates the welfare effects of skill and luck in a life-cycle consumption-portfolio choice problem with stock, housing, and labor income. In a setting where stock, house, and income returns are predictable, skills are modeled by the ability to time the market, i.e. by the ability to “read” the signals of the predictors. First, we find that, if anything, house predictability is more relevant than stock predictability. Second, and more importantly, in our framework the welfare effect of being skilled is moderate compared to the effect of being born in favorable times. In fact, the latter effect is about 2-3 times bigger.

However, we consider highly skilled investors who have access to the true parameters that are estimated in sample. We also abstract from transaction costs and give the investors access to REITs. The latter opportunity allows the skilled investor to not only time the stock but also the housing market. If we relax these assumptions, the benefits of being skilled are smaller and the component of being lucky becomes even more dominant.

What are the practical implications of our findings? It can well be that neighbors of different, but similar age, might have very distinct lifestyles simply because one of them was so lucky to enter the asset markets under more favorable conditions. Even if the unlucky agent is skilled, he might not be able to compensate his bad luck of being born at the wrong time by using his skills. Although we are only considering a partial equilibrium model, this finding suggests that achieving inter-generational fairness is aggravated if different generations face different investment opportunities.

References

- Ang, Andrew, and Geert Bekaert, 2007, Stock Return Predictability: Is It There?, *Review of Financial Studies* 20, 651–707.
- Bach, Laurent, Laurent E. Calvet, and Paolo Sodini, 2015, Rich Pickings? Risk, Return, and Skill in the Portfolios of the Wealthy, *Working Paper*, Stockholm School of Economics and HEC Paris.
- Barberis, Nicholas, 2000, Investing for the Long Run when Returns are Predictable, *Journal of Finance* 55, 225–264.
- Bick, Björn, Holger Kraft, and Claus Munk, 2013, Solving Constrained Consumption-Investment Problems by Simulation of Artificial Market Strategies, *Management Science* 59, 485–503.
- Boudoukh, Jacob, Roni Michaely, Matthew Richardson, and Michael R. Roberts, 2007, On the Importance of Measuring Payout Yield: Implications for Empirical Asset Pricing, *Journal of Finance* 62, 877–916.
- Bourassa, Steven C., Donald R. Haurin, Jessica L. Haurin, Martin Hoesli, and Jian Sun, 2009, House Price Changes and Idiosyncratic Risk: The Impact of Property Characteristics, *Real Estate Economics* 37, 259 – 278.
- Brown, S., W. Goetzmann, and S. A. Ross, 1995, Survival, *Journal of Finance* 50, 853–873.
- Campbell, John Y., 2006, Household Finance, *Journal of Finance* 61, 1553–1604.
- Campbell, John Y., and Robert J. Shiller, 1988, The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors, *Review of Financial Studies* 1, 195–227.
- Campbell, John Y., and Luis M. Viceira, 1999, Consumption and Portfolio Decisions when Expected Returns are Time Varying, *Quarterly Journal of Economics* 114, 433–495.
- Case, Karl E., and Robert J. Shiller, 1989, The Efficiency of the Market for Single-Family Homes, *American Economic Review* 79, 125–137.
- Case, Karl E., and Robert J. Shiller, 1990, Forecasting Prices and Excess Returns in the Housing Market, *Real Estate Economics* 18, 253–273.
- Cocco, Joao F., 2005, Portfolio Choice in the Presence of Housing, *Review of Financial Studies* 18, 535–567.

- Cocco, Joao F., Francisco J. Gomes, and Pascal J. Maenhout, 2005, Consumption and Portfolio Choice over the Life Cycle, *Review of Financial Studies* 18, 491–533.
- Corradin, Stefano, José L. Fillat, and Carles Vergara-Alert, 2014, Optimal Portfolio Choice with Predictability in House Prices and Transaction Costs, *Review of Financial Studies* 27, 823–880.
- Cotter, John, and Richard Roll, 2015, A Comparative Anatomy of Residential REITs and Private Real Estate Markets: Returns, Risks and Distributional Characteristics, *Real Estate Economics* 43, 209–240.
- Cvitanić, Jakša, and Ioannis Karatzas, 1992, Convex Duality in Constrained Portfolio Optimization, *Annals of Applied Probability* 2, 767–818.
- Cvitanić, Jakša, Walter Schachermayer, and Hui Wang, 2001, Utility Maximization in Incomplete Markets with Random Endowment, *Finance and Stochastics* 5, 259–272.
- Davis, Morris A., Andreas Lehnert, and Robert F. Martin, 2008, The Rent-Price Ratio for the Aggregate Stock of Owner-Occupied Housing, *Review of Income and Wealth* 54, 279–284.
- De Nardi, Mariacristina, Eric French, and John B. Jones, 2010, Why do the Elderly Save? The Role of Medical Expenses, *Journal of Political Economy* 118, 39–75.
- Fama, Eugene F., and Kenneth R. French, 1988, Permanent and Temporary Components of Stock Prices, *Journal of Political Economy* 96, 246–273.
- Fama, Eugene F., and Kenneth R. French, 2002, The Equity Premium, *Journal of Finance* 57, 637–659.
- Fischer, Marcel, and Michael Stamos, 2013, Optimal Life Cycle Portfolio Choice with Housing Market Cycles, *Review of Financial Studies* 26, 2311–2352.
- Flavin, Marjorie, and Takashi Yamashita, 2002, Owner-Occupied Housing and the Composition of the Household Portfolio, *American Economic Review* 91, 345–362.
- Ghysels, Eric, Alberto Plazzi, Rossen Valkanov, and Walter Torous, 2013, Forecasting Real Estate Prices, in Graham Elliott, and Allan Timmermann, eds.: *Handbook of Economic Forecasting* (Elsevier,).

- Guvenen, Fatih, Fatih Karahan, Serdar Ozkan, and Jae Song, 2015, What Do Data on Millions of U.S. Workers Reveal About Life-Cycle Earnings Risk, Available at SSRN: <http://ssrn.com/abstract=2563279>.
- van Hemert, Otto, 2010, Household Interest Rate Risk Management, *Real Estate Economics* 38, 467–505.
- Himmelberg, C., C. Mayer, and T. Sinai, 2005, Assessing High House Prices: Bubbles, Fundamentals, and Misperceptions, *Journal of Economic Perspectives* 19, 67–92.
- Hurd, Michael D., 1989, Mortality Risk and Bequest, *Econometrica* 57, 779–813.
- Karatzas, I., J. Lehoczky, S. Shreve, and G. Xu, 1991, Martingale and Duality Methods for Utility Maximization in an Incomplete Market, *SIAM Journal on Control and Optimization* 29, 702–730.
- Kim, Tong Suk, and Edward Omberg, 1996, Dynamic Nonmyopic Portfolio Behavior, *Review of Financial Studies* 9, 141–161.
- Koijen, Ralph S. J., and Stijn van Nieuwerburgh, 2011, Predictability of Returns and Cash Flows, *Annual Review of Financial Economics* 3, 467–491.
- Kraft, Holger, and Claus Munk, 2011, Optimal Housing, Consumption, and Investment Decisions over the Life-Cycle, *Management Science* 57, 1025–1041.
- Lustig, Hanno, and Stijn van Nieuwerburgh, 2005, Housing Collateral, Consumption Insurance, and Risk Premia: An Empirical Perspective, *Journal of Finance* 60, 1167–1219.
- Lynch, Anthony W., and Sinan Tan, 2011, Labor Income Dynamics at Business-cycle Frequencies: Implications for Portfolio Choice, *Journal of Financial Economics* 101, 333–359.
- Malpezzi, Stephen, 1999, A Simple Error Correction Model of House Prices, *Journal of Housing Economics* 8, 27–62.
- Moskowitz, Tobias, Yao Hua Ooi, and Lasse Heje Pedersen, 2012, Time Series Momentum, *Journal of Financial Economics* 104, 228–250.
- Munk, Claus, and Carsten Sørensen, 2010, Dynamic Asset Allocation with Stochastic Income and Interest Rates, *Journal of Financial Economics* 96, 433–462.

- Plazzi, Alberto, Walter Torous, and Rossen Valkanov, 2010, Expected Returns and Expected Growth in Rents of Commercial Real Estate, *Review of Financial Studies* 23, 3469–3519.
- Poterba, James M., 1991, House Price Dynamics: The Role of Tax Policy and Demography, *Brookings Papers on Economic Activity* 22, 143–203.
- Santos, Tano, and Pietro Veronesi, 2006, Labor Income and Predictable Stock Returns, *Review of Financial Studies* 19, 1–44.
- Shiller, Robert J., 2005, Irrational Exuberance, *Princeton University Press*, 2nd edition.
- Tsai, I-Chun, Ming-Chi Chen, and Tien-Foo Sing, 2007, Do REITs behave More like Real Estate Now?, Available at <http://ssrn.com/abstract=1079590>.
- U.S. Department of Labor, Bureau of Labor Statistics, 2003, Consumer Expenditures in 2001, Report 966.
- Wachter, Jessica A., 2002, Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets, *Journal of Financial and Quantitative Analysis* 37, 63–91.
- Xia, Yihong, 2001, Learning about Predictability: The Effects of Parameter Uncertainty on Dynamic Asset Allocation, *Journal of Finance* 56, 205–246.
- Yao, Rui, and Harold H. Zhang, 2005, Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints, *Review of Financial Studies* 18, 197–239.

A Details of the calibration

We estimate the following VAR(1)

$$\begin{pmatrix} r_{S,t+1} - r_{t+1} \\ r_{H,t+1} - r_{t+1} \\ r_{L,t+1} \\ x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} \beta_S \\ \beta_H \\ \beta_L \\ \beta_x \\ \beta_y \end{pmatrix} + \begin{pmatrix} \beta_{x,S} & \beta_{y,S} \\ \beta_{x,H} & \beta_{y,H} \\ \beta_{x,L} & \beta_{y,L} \\ \beta_{x,x} & 0 \\ 0 & \beta_{y,y} \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{S,t+1} \\ \varepsilon_{H,t+1} \\ \varepsilon_{L,t+1} \\ \varepsilon_{x,t+1} \\ \varepsilon_{y,t+1} \end{pmatrix} \quad (15)$$

where $(\varepsilon_S, \varepsilon_H, \varepsilon_L, \varepsilon_x, \varepsilon_y, \varepsilon_z) \sim N(0, \Sigma)$. The Stata regression output is shown in Table 10.

[INSERT TABLE 10 ABOUT HERE]

At first we have estimated the model (15) where excess stock returns, excess housing returns, and income returns are potentially predicted by x and y . It turns out that y is not significant for the excess stock returns and x is not significant for income returns. Therefore, in our benchmark model, we set $\beta_{y,S} = 0$ and $\beta_{x,L} = 0$, which also helps to simplify the numerical calculations later on. The estimation results are in the column labeled “Full” of Table 10. This model is used to generate the data in Section 4 and a skilled investor applies it to calculate his optimal portfolio. The net payout yield is borderline significant in predicting excess stock returns (p -value of 5.1%) and highly significant in predicting house price growth (p -value of 1.0%). The positive estimate of χ_S shows that the net payout yield positively predicts stock prices as found by Boudoukh, Michaely, Richardson, and Roberts (2007), although the slope coefficient of 0.329 is smaller than they report for the univariate regression using 1926-2003 data. The net payout yield negatively predicts house price growth as indicated by the estimate -0.106 of χ_{Hx} . The normalized rent-price ratio is a significant (p -value 0.4%) negative predictor of house price growth with a χ_{Hy} coefficient of -0.398 and a significant (p -value 1.7%) positive predictor of income growth with a χ_L coefficient of 0.126.

An unskilled investor disregards predictability all together and thus estimates (15) with $\beta_{x,S} = \beta_{y,S} = \beta_{x,H} = \beta_{y,H} = \beta_{x,L} = \beta_{y,L} = 0$. The estimation results are in the column labeled “Not at all” of Table 10. The semi-skilled investors disregarding stock, housing, or income predictability set $\beta_{x,S} = \beta_{y,S} = 0$, $\beta_{x,H} = \beta_{y,H} = 0$, or $\beta_{x,L} = \beta_{y,L} = 0$ and then estimate (15). The estimation results are reported in the columns labeled “No stock”, “No house”, and “No income” of Table 10. Finally, Table 1 reports the resulting parameter values for the different model specifications of (1)-(5) with some estimates adjusted as explained in the main text.

B Details on the numerical method

B.1 Artificial markets

Building on the idea of Cvitanić and Karatzas (1992), the constrained, incomplete market problem is embedded in a family of artificial, unconstrained, complete market problems for which we can derive exact closed-form solutions. In order to handle the constraints (10), we modify the risk-free rate as well as the drift rates of the stock and the house as follows

$$\mu_{St} = \mu_S + \nu_{St}, \quad \mu_{Ht} = \mu_H + \nu_{Ht}, \quad r_t = r + \max\left(\nu_{St}^-, \frac{1}{q} \nu_{Ht}^-\right), \quad (16)$$

where $\nu^- = \max(-\nu, 0)$; see Cvitanić and Karatzas (1992) and Bick, Kraft, and Munk (2013), Sec. 8. Note that for any values of ν_{St} and ν_{Ht} , we have $r_t \geq r$ and $r_t + \mu_{St} \geq r + \mu_S$ as well as $r_t + \mu_{Ht} \geq r + \mu_H$. Intuitively, if the unconstrained Π_S or Π_H is above 1, we increase the risk-free rate to make investing in the bank account relatively more attractive and bring down the risky investment. Conversely, if the unconstrained Π_S or Π_H is negative, we increase the drift rate to boost the investment in the asset. To complete the market in our case, we introduce an artificial asset for the idiosyncratic labor income risk, for the x risk, and for the y risk with price processes given by

$$dV_{xt} = V_{xt} [(r_t + \lambda_{xt}) dt + dB_{xt}], \quad (17)$$

$$dV_{yt} = V_{yt} [(r_t + \lambda_{yt}) dt + dB_{yt}], \quad (18)$$

$$dV_{Lt} = V_{Lt} [(r_t + \lambda_{Lt}) dt + dB_{Lt}], \quad (19)$$

where λ_{xt} , λ_{yt} , λ_{Lt} denote the market prices of risk associated with x , y , and L shocks, respectively. The assumption of a unit volatility is without loss of generality. An artificial market is characterized by the “modifiers” ν_S , ν_H , λ_L , λ_x , and λ_y .

The modifiers could be quite general stochastic processes, but for tractability we focus on the family of artificial markets in which the modifiers are polynomials of time (age), x , and y . We therefore denote write $\nu_{St} = \nu_S(t, x, y)$ and similar for other quantities, and specify

$$\begin{aligned} \nu_S(t, x, y) = & \nu_{S,0,\text{act}} + \nu_{S,0,\text{ret}} + \nu_{S,1,\text{act}} t + \nu_{S,1,\text{ret}} t + \nu_{S,2} x + \nu_{S,3} y + \nu_{S,4} x^2 + \nu_{S,5} y^2 + \nu_{S,6} xy \\ & + \nu_{S,7} t^2 + \nu_{S,8} xt + \nu_{S,9} yt, \end{aligned} \quad (20)$$

$$\begin{aligned} \nu_H(t, x, y) = & \nu_{H,0,\text{act}} + \nu_{H,0,\text{ret}} + \nu_{H,1,\text{act}} t + \nu_{H,1,\text{ret}} t + \nu_{H,2} x + \nu_{H,3} y + \nu_{H,4} x^2 + \nu_{H,5} y^2 + \nu_{H,6} xy \\ & + \nu_{H,7} t^2 + \nu_{H,8} xt + \nu_{H,9} yt, \end{aligned} \quad (21)$$

$$\begin{aligned} \lambda_x(t, x, y) = & \Lambda_{x,0,\text{act}} + \Lambda_{x,0,\text{ret}} + \Lambda_{x,1,\text{act}} t + \Lambda_{x,1,\text{ret}} t + \Lambda_{x,2} x + \Lambda_{x,3} y + \Lambda_{x,4} x^2 + \Lambda_{x,5} y^2 + \Lambda_{x,6} xy \\ & + \Lambda_{x,7} t^2 + \Lambda_{x,8} xt + \Lambda_{x,9} yt, \end{aligned} \quad (22)$$

$$\begin{aligned}\lambda_y(t, x, y) &= \Lambda_{y,0,\text{act}} + \Lambda_{y,0,\text{ret}} + \Lambda_{y,1,\text{act}} t + \Lambda_{y,1,\text{ret}} t + \Lambda_{y,2} x + \Lambda_{y,3} y + \Lambda_{y,4} x^2 + \Lambda_{y,5} y^2 + \Lambda_{y,6} xy \\ &\quad + \Lambda_{y,7} t^2 + \Lambda_{y,8} xt + \Lambda_{y,9} yt,\end{aligned}\tag{23}$$

$$\begin{aligned}\lambda_L(t, x, y) &= \Lambda_{L,0,\text{act}} + \Lambda_{L,0,\text{ret}} + \Lambda_{L,1,\text{act}} t + \Lambda_{L,1,\text{ret}} t + \Lambda_{L,2} x + \Lambda_{L,3} y + \Lambda_{L,4} x^2 + \Lambda_{L,5} y^2 + \Lambda_{L,6} xy \\ &\quad + \Lambda_{L,7} t^2 + \Lambda_{L,8} xt + \Lambda_{L,9} yt.\end{aligned}\tag{24}$$

We refer to such markets as the *computable artificial markets* since in these markets we can solve the agent's utility maximization problem in closed form as shown below.

For notational convenience, we define

$$\mu'_S(t, x, y) = \mu_S(t, x, y) + \bar{D}, \quad \mu'_H(t, x, y) = \mu_H(t, x, y) + R - m,$$

where \bar{D} is the dividend yield, R is the rental rate, and m the maintenance cost rate. By Π_{it} we denote the fraction of wealth invested in asset i at time t with $i \in \{S, H, L, x, y\}$.

B.2 Human capital in computable artificial markets

In any of the computable artificial markets, the labor income is spanned and the agent can borrow against future income. By combining these features with the assumed income dynamics, we can compute the human capital—the present value of all future income—by solving a relatively simple partial differential equation (PDE).

Lemma 1 *In a computable artificial market, the human capital at time t equals $L_t F(t, x_t, y_t)$, where F solves the PDE (25) stated in the proof below, with the discrete adjustment $F(\tilde{T}^-, x, y) = \Upsilon F(\tilde{T}^+, x, y)$ at the retirement date.*

Proof: In a complete, unconstrained market we can represent the human capital by the risk-neutral expectation of the future labor income L_s discounted by (the integral of) the short-term interest rate $r(u, x_u, y_u)$. Let \mathbb{Q} denote the unique risk-neutral probability measure in a given artificial market. To compute the human capital we must therefore identify the \mathbb{Q} -dynamics of L , x , and y . For that purpose we have to identify the market prices of risk associated with the Brownian shocks B_S, B_H, B_L, B_x, B_y . While the market prices of risk associated with B_L, B_x, B_y are $\lambda_L(t, x, y), \lambda_x(t, x, y), \lambda_y(t, x, y)$ by assumption, we identify the market prices of risk m_{St}, m_{Ht} associated with B_S, B_H by using the fact that the excess expected return on an asset is the product of its sensitivities towards the shocks and the market prices of risks associated with the shocks. For the stock, this means

$$\mu'_S(t, x, y) + \chi_S x_t = \sigma_S m_{St} \quad \Rightarrow \quad m_{St} = \frac{\mu'_S(t, x, y) + \chi_S x_t}{\sigma_S}.$$

For an investment in housing units, this implies

$$\begin{aligned} \mu'_H(t, x, y) + \chi_{Hx}x_t + \chi_{Hy}y_t &= \sigma_H \rho_{HS} m_{St} + \sigma_H \hat{\rho}_H m_{Ht} \Rightarrow \\ m_{Ht} &= \frac{\mu'_H(t, x, y) + \chi_{Hx}x_t + \chi_{Hy}y_t}{\sigma_H \hat{\rho}_H} - \frac{\rho_{HS}(\mu'_S(t, x, y) + \chi_S x_t)}{\hat{\rho}_H \sigma_S}. \end{aligned}$$

The risk-neutral income dynamics is therefore

$$\begin{aligned} \frac{dL_t}{L_t} &= \left[\mu_L(t) + \bar{\chi}_L(t)y_t - \sigma_L(t) (\rho_{LS} m_{St} + \hat{\rho}_{LH} m_{Ht} + \hat{\rho}_L \lambda_L(t, x, y)) \right] dt \\ &\quad + \sigma_L(t, x, y) \left(\rho_{LS} dB_{St}^{\mathbb{Q}} + \hat{\rho}_{LH} dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_L dB_{Lt}^{\mathbb{Q}} \right) \\ &= \left[- (M_{LS}(t)\chi_S + M_{LH}(t)\chi_{Hx}) x_t - (M_{LH}(t)\chi_{Hy} - \bar{\chi}_L(t)) y_t + \mu_L(t) \right. \\ &\quad \left. - M_{LS}(t)\mu'_S(t, x, y) - M_{LH}(t)\mu'_H(t, x, y) - \sigma_L(t)\hat{\rho}_L \lambda_L(t, x, y) \right] dt \\ &\quad + \sigma_L \left(\rho_{LS} dB_{St}^{\mathbb{Q}} + \hat{\rho}_{LH} dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_L dB_{Lt}^{\mathbb{Q}} \right), \end{aligned}$$

where

$$M_{LS}(t) = \frac{\sigma_L(t)}{\sigma_S} \left(\rho_{LS} - \frac{\rho_{HS} \hat{\rho}_{LH}}{\hat{\rho}_H} \right), \quad M_{LH}(t) = \frac{\sigma_L(t) \hat{\rho}_{LH}}{\sigma_H \hat{\rho}_H}.$$

For x , we obtain

$$\begin{aligned} dx_t &= \left[-\kappa_x x_t - \sigma_x (\rho_{xS} m_{St} + \hat{\rho}_{xH} m_{Ht} + \hat{\rho}_{xL} \lambda_L(t, x, y) + \hat{\rho}_x \lambda_x(t, x, y)) \right] dt \\ &\quad + \sigma_x \left(\rho_{xS} dB_{St}^{\mathbb{Q}} + \hat{\rho}_{xH} dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_{xL} dB_{Lt}^{\mathbb{Q}} + \hat{\rho}_x dB_{xt}^{\mathbb{Q}} \right) \\ &= \left[-(\kappa_x + M_{xS}\chi_S + M_{xH}\chi_{Hx}) x_t - M_{xH}\chi_{Hy} y_t \right. \\ &\quad \left. - M_{xS}\mu'_S(t, x, y) - M_{xH}\mu'_H(t, x, y) - \sigma_x (\hat{\rho}_{xL} \lambda_L(t, x, y) + \hat{\rho}_x \lambda_x(t, x, y)) \right] dt \\ &\quad + \sigma_x \left(\rho_{xS} dB_{St}^{\mathbb{Q}} + \hat{\rho}_{xH} dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_{xL} dB_{Lt}^{\mathbb{Q}} + \hat{\rho}_x dB_{xt}^{\mathbb{Q}} \right), \end{aligned}$$

where

$$M_{xS} = \frac{\sigma_x}{\sigma_S} \left(\rho_{xS} - \frac{\rho_{HS} \hat{\rho}_{xH}}{\hat{\rho}_H} \right), \quad M_{xH} = \frac{\sigma_x \hat{\rho}_{xH}}{\sigma_H \hat{\rho}_H}.$$

For y , we obtain

$$\begin{aligned} dy_t &= \left[-\kappa_y y_t - \sigma_y (\rho_{yS} m_{St} + \hat{\rho}_{yH} m_{Ht} + \hat{\rho}_{yL} \lambda_L(t, x, y) + \hat{\rho}_{yx} \lambda_x(t, x, y) + \hat{\rho}_y \lambda_y(t, x, y)) \right] dt \\ &\quad + \sigma_y \left(\rho_{yS} dB_{St}^{\mathbb{Q}} + \hat{\rho}_{yH} dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_{yL} dB_{Lt}^{\mathbb{Q}} + \hat{\rho}_{yx} dB_{xt}^{\mathbb{Q}} + \hat{\rho}_y dB_{yt}^{\mathbb{Q}} \right) \\ &= \left[- (M_{yS}\chi_S + M_{yH}\chi_{Hx}) x_t - (\kappa_y + M_{yH}\chi_{Hy}) y_t \right. \\ &\quad \left. - M_{yS}\mu'_S(t, x, y) - M_{yH}\mu'_H(t, x, y) - \sigma_y (\hat{\rho}_{yL} \lambda_L(t, x, y) + \hat{\rho}_{yx} \lambda_x(t, x, y) + \hat{\rho}_y \lambda_y(t, x, y)) \right] dt \\ &\quad + \sigma_y \left(\rho_{yS} dB_{St}^{\mathbb{Q}} + \hat{\rho}_{yH} dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_{yL} dB_{Lt}^{\mathbb{Q}} + \hat{\rho}_{yx} dB_{xt}^{\mathbb{Q}} + \hat{\rho}_y dB_{yt}^{\mathbb{Q}} \right), \end{aligned}$$

$$\begin{aligned}
& - M_{yS}\mu'_S(t, x, y) - M_{yH}\mu'_H(t, x, y) - \sigma_y \left(\hat{\rho}_{yL}\lambda_L(t, x, y) + \hat{\rho}_{yx}\lambda_x(t, x, y) + \hat{\rho}_y\lambda_y(t, x, y) \right) \Big] dt \\
& + \sigma_y \left(\rho_{yS} dB_{St}^{\mathbb{Q}} + \hat{\rho}_{yH} dB_{Ht}^{\mathbb{Q}} + \hat{\rho}_{yL} dB_{Lt}^{\mathbb{Q}} + \hat{\rho}_{yx} dB_{xt}^{\mathbb{Q}} + \hat{\rho}_y dB_{yt}^{\mathbb{Q}} \right),
\end{aligned}$$

where

$$M_{yS} = \frac{\sigma_y}{\sigma_S} \left(\rho_{yS} - \frac{\rho_{HS}\hat{\rho}_{yH}}{\hat{\rho}_H} \right), \quad M_{yH} = \frac{\sigma_y\hat{\rho}_{yH}}{\sigma_H\hat{\rho}_H}.$$

It follows that the human capital in retirement $E_t^{\mathbb{Q}} \left[\int_t^T e^{-\int_t^s r(u, x_u, y_u) du} L_s ds \right]$ is a function $\mathcal{P}(t, L, x, y)$ and that we can separate it as $\mathcal{P}(t, L, x, y) = LF(t, x, y)$. From general derivatives pricing results we know that if $z = (L, x, y)^\top$ and

$$dz_t = \mu_z(t, z_t) dt + \Sigma_z(t, z_t) dB_t^{\mathbb{Q}},$$

the function $\mathcal{P}(t, z)$ satisfies the partial differential equation (PDE)

$$\frac{\partial \mathcal{P}}{\partial t} + \frac{\partial \mathcal{P}}{\partial z} \cdot \mu_z + \frac{1}{2} \text{tr} \left(\frac{\partial^2 \mathcal{P}}{\partial z^2} \Sigma_z \Sigma_z^\top \right) + L = r\mathcal{P}$$

and the terminal condition $\mathcal{P}(T, z) = 0$. Given the separation $\mathcal{P} = LF$, it follows that $F(t, x, y)$ satisfies the PDE

$$\begin{aligned}
0 = 1 + \frac{\partial F}{\partial t} & + \left[- (M_{LS}(t)\chi_S + M_{LH}(t)\chi_{Hx}) x + (\bar{\chi}_L(t) - M_{LH}(t)\chi_{Hy}) y + \mu_L(t) \right. \\
& \left. - M_{LS}(t)\mu'_S(t, x, y) - M_{LH}(t)\mu'_H(t, x, y) - \sigma_L(t)\hat{\rho}_L\lambda_L(t, x, y) - r(t, x, y) \right] F \\
& + \left[- (\kappa_x + M_{xS}\chi_S + M_{xH}\chi_{Hx}) x - M_{xH}\chi_{Hy} y + \sigma_{xL}(t) \right. \\
& \left. - M_{xS}\mu'_S(t, x, y) - M_{xH}\mu'_H(t, x, y) - \sigma_x (\hat{\rho}_{xL}\lambda_L(t, x, y) + \hat{\rho}_x\lambda_x(t, x, y)) \right] F_x \\
& + \left[- (M_{yS}\chi_S + M_{yH}\chi_{Hx}) x - (\kappa_y + M_{yH}\chi_{Hy}) y + \sigma_{yL}(t) \right. \\
& \left. - M_{yS}\mu'_S(t, x, y) - M_{yH}\mu'_H(t, x, y) - \sigma_y (\hat{\rho}_{yL}\lambda_L(t, x, y) + \hat{\rho}_{yx}\lambda_x(t, x, y) + \hat{\rho}_y\lambda_y(t, x, y)) \right] F_y \\
& + \frac{1}{2} \sigma_x^2 F_{xx} + \frac{1}{2} \sigma_y^2 F_{yy} + \sigma_{xy} F_{xy},
\end{aligned} \tag{25}$$

where subscripts on F denote partial derivatives, and where $\sigma_{xL}(t) = \rho_{xL}\sigma_x\sigma_L(t)$, $\sigma_{yL}(t) = \rho_{yL}\sigma_y\sigma_L(t)$, and $\sigma_{xy} = \rho_{xy}\sigma_x\sigma_y$. Given the specification of the interest rate $r(t, x, y)$ in (16), we cannot solve the PDE (25) in closed form, so we solve it backwards from the terminal date T where $F(T, x, y) = 0$ using standard finite difference methods.

Before retirement, the human capital is computed from

$$E_t^{\mathbb{Q}} \left[\int_t^{\tilde{T}} e^{-\int_t^s r(u, x_u, y_u) du} ds + \int_{\tilde{T}}^T e^{-\int_t^s r(u, x_u, y_u) du} L_s ds \right],$$

where in the second integral we have to incorporate the drop in income at the retirement time \tilde{T} . We can handle that in the finite difference solution by multiplying the values $F(\tilde{T}^+, x, y)$ immediately after retirement by Υ to get the values immediately before retirement. \square

Given the structure of the PDE (25), it follows that F can be written in the form

$$F(t, x, y) = \begin{cases} \int_t^T A(t, s, x, y) ds, & t \in (\tilde{T}, T], \\ \int_t^{\tilde{T}} A(t, s, x, y) ds + \Upsilon \int_{\tilde{T}}^T A(t, s, x, y) ds, & t \in [0, \tilde{T}], \end{cases}$$

and that we can derive some PDE for $A(t, s, x, y)$, but this would still have to be solved numerically and then integrated up numerically.

B.3 Optimality in computable artificial markets

The agent's total time t wealth is the sum of tangible wealth and human capital, i.e., $W_t + L_t F(t, x_t, y_t)$. Due to power utility, the indirect utility function is conjectured to have the form $\frac{1}{1-\gamma} G(\cdot)^\gamma (W_t + L_t F(t, x_t, y_t))^{1-\gamma}$, where G depends on time and on variables driving shifts in investment opportunities (risk-free rate and risk premia) as well as changes in relative prices of consumer goods. In our case, G therefore depends on x_t, y_t , and the house price H_t . The relative good price H_t is expected to enter proportionally with a power (Kraft and Munk (2011)) so that G can be separated as $\tilde{k} H^k B(t, x, y)$ for appropriate constants k and \tilde{k} . These considerations motivate the form of the indirect utility function given below. The optimal strategies then follow from the first-order conditions to the associated Hamilton-Jacobi-Bellman (HJB) equation.

Theorem 1 *In a computable artificial market the indirect utility is*

$$J(t, W, H, L, x, y) = \frac{1}{1-\gamma} a^{1-\gamma} \left(\frac{aRH}{1-a} \right)^{(1-a)(1-\gamma)} B(t, x, y)^\gamma (W + LF(t, x, y))^{1-\gamma}, \quad (26)$$

where F solves the PDE (25) and B solves the PDE (52) stated in the proof below. The optimal portfolio weights are

$$\begin{aligned} \Pi_S &= \frac{1}{\gamma \hat{\rho}_H^2 \sigma_S^2} \left(\mu'_S(t, x, y) + \chi_S x - \frac{\rho_{HS} \sigma_S}{\sigma_H} (\mu'_H(t, x, y) + \chi_{Hx} x + \chi_{Hy} y) \right) \frac{W + LF}{W} \\ &\quad + \left(M_{xS} \frac{B_x}{B} + M_{yS} \frac{B_y}{B} \right) \frac{W + LF}{W} - \left(M_{LS}(t) + M_{xS} \frac{F_x}{F} + M_{yS} \frac{F_y}{F} \right) \frac{LF}{W}, \end{aligned} \quad (27)$$

$$\begin{aligned} \Pi_H &= \frac{1}{\gamma \hat{\rho}_H^2 \sigma_H^2} \left(\mu'_H(t, x, y) + \chi_{Hx} x + \chi_{Hy} y - \frac{\rho_{HS} \sigma_H}{\sigma_S} (\mu'_S(t, x, y) + \chi_S x) \right) \frac{W + LF}{W} + k \frac{W + LF}{W} \\ &\quad + \left(M_{xH} \frac{B_x}{B} + M_{yH} \frac{B_y}{B} \right) \frac{W + LF}{W} - \left(M_{LH}(t) + M_{xH} \frac{F_x}{F} + M_{yH} \frac{F_y}{F} \right) \frac{LF}{W}, \end{aligned} \quad (28)$$

$$\begin{aligned}\Pi_L &= \frac{1}{\gamma} \lambda_L(t, x, y) \frac{W + LF}{W} + \sigma_L(t) \left(M_{xL} \frac{B_x}{B} + M_{yL} \frac{B_y}{B} \right) \frac{W + LF}{W} \\ &\quad - \sigma_L(t) \left(\hat{\rho}_L + M_{xL} \frac{F_x}{F} + M_{yL} \frac{F_y}{F} \right) \frac{LF}{W},\end{aligned}\tag{29}$$

$$\Pi_x = \frac{1}{\gamma} \lambda_x(t, x, y) \frac{W + LF}{W} + \sigma_x \left(\hat{\rho}_x \frac{B_x}{B} + M_{yx} \frac{B_y}{B} \right) \frac{W + LF}{W} - \sigma_x \left(\hat{\rho}_x \frac{F_x}{F} + M_{yx} \frac{F_y}{F} \right) \frac{LF}{W},\tag{30}$$

$$\Pi_y = \frac{1}{\gamma} \lambda_y(t, x, y) \frac{W + LF}{W} + \sigma_y \hat{\rho}_y \frac{B_y}{B} \frac{W + LF}{W} - \sigma_y \hat{\rho}_y \frac{F_y}{F} \frac{LF}{W},\tag{31}$$

where $k = \frac{(1-a)(\gamma-1)}{\gamma}$ and where the functions and constants M_{ij} were defined in the proof of Lemma 1. The optimal consumption is

$$c = a \frac{W + LF}{B},\tag{32}$$

$$\phi_C = (1-a) \frac{W + LF}{RHB}.\tag{33}$$

Proof: First we set up the HJB equation, then we conjecture and verify a solution to it.

Setting up the HJB equation. The wealth dynamics in the artificial market is similar to (7), but adjusted because of the possibility to invest in the artificial asset with price dynamics (17) as well as the modification of r , μ'_S , and μ'_H :

$$\begin{aligned}dW_t &= \left(W_t \left[r(t, x, y) + \Pi_{St}(\mu'_S(t, x, y) + \chi_S x_t) + \Pi_{Ht}(\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t) \right. \right. \\ &\quad \left. \left. + \Pi_{Lt} \lambda_L(t, x, y) + \Pi_{xt} \lambda_x(t, x, y) + \Pi_{yt} \lambda_y(t, x, y) \right] + (L_t - c_t - \phi_{Ct} RH_t) \right) dt \\ &\quad + W_t \left[(\Pi_{St} \sigma_S + \Pi_{Ht} \sigma_H \rho_{HS}) dB_{St} + \Pi_{Ht} \sigma_H \hat{\rho}_H dB_{Ht} + \Pi_{Lt} dB_{Lt} + \Pi_{xt} dB_{xt} + \Pi_{yt} dB_{yt} \right] \\ &= (r(t, x, y) W_t + \alpha_t^\top \lambda_t + L_t - \phi_{Ct} RH_t - c_t) dt + \alpha_t^\top \Sigma dB_t,\end{aligned}$$

where

$$\alpha_t = \begin{pmatrix} \alpha_{St} \\ \alpha_{Ht} \\ \alpha_{Lt} \end{pmatrix} = \begin{pmatrix} \Pi_{St} \sigma_S W_t \\ \Pi_{Ht} \sigma_H W_t \\ \Pi_{Lt} W_t \\ \Pi_{xt} W_t \\ \Pi_{yt} W_t \end{pmatrix}, \quad \lambda_t = \begin{pmatrix} (\mu'_S(t, x, y) + \chi_S x_t) / \sigma_S \\ (\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t) / \sigma_H \\ \lambda_L(t, x, y) \\ \lambda_x(t, x, y) \\ \lambda_y(t, x, y) \end{pmatrix},$$

$$B_t = \begin{pmatrix} B_{St} \\ B_{Ht} \\ B_{Lt} \\ B_{xt} \\ B_{yt} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \rho_{HS} & \hat{\rho}_H & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $Z = (H, L, x, y)^\top$ be the vector of state variables, which has the dynamics

$$dZ_t = \mu_Z(t, Z_t) dt + \Sigma_Z(t, Z_t) dB_t,$$

where

$$\mu_Z(t, Z_t) = \begin{pmatrix} H_t[r(t, x, y) + \mu_H(t, x, y) + \chi_{Hx}x_t + \chi_{Hy}y_t] \\ L_t[\mu_L(t) + \bar{\chi}_L(t)y_t] \\ -\kappa_x x_t \\ -\kappa_y y_t \end{pmatrix},$$

$$\Sigma_Z(Z_t) = \begin{pmatrix} H_t\sigma_H\rho_{HS} & H_t\sigma_H\hat{\rho}_H & 0 & 0 & 0 \\ L_t\sigma_L(t)\rho_{LS} & L_t\sigma_L(t)\hat{\rho}_{LH} & L_t\sigma_L(t)\hat{\rho}_L & 0 & 0 \\ \sigma_x\rho_{xS} & \sigma_x\hat{\rho}_{xH} & \sigma_x\hat{\rho}_{xL} & \sigma_x\hat{\rho}_x & 0 \\ \sigma_y\rho_{yS} & \sigma_y\hat{\rho}_{yH} & \sigma_y\hat{\rho}_{yL} & \sigma_y\hat{\rho}_{yx} & \sigma_y\hat{\rho}_y \end{pmatrix}.$$

The Hamilton-Jacobi-Bellman equation (HJB) associated with the problem can be written as

$$\delta J = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J, \tag{34}$$

where

$$\begin{aligned} \mathcal{L}_1 J &= \max_{c, \phi_C} \{U(c, \phi_C) - J_W(c + HR\phi_C)\}, \\ \mathcal{L}_2 J &= \max_{\alpha} \left\{ J_W \alpha^\top \lambda + \frac{1}{2} J_{WW} \alpha^\top \Sigma \Sigma^\top \alpha + \alpha^\top \Sigma \Sigma_Z^\top J_{WZ} \right\}, \\ \mathcal{L}_3 J &= \frac{\partial J}{\partial t} + J_W(rW + L) + J_Z^\top \mu_Z + \frac{1}{2} \text{trace}(J_{ZZ} \Sigma_Z \Sigma_Z^\top). \end{aligned}$$

Recall that $J = J(t, W, H, L, x, y) = J(t, W, Z)$ so that

$$J_Z = \begin{pmatrix} J_H \\ J_L \\ J_x \\ J_y \end{pmatrix}, \quad J_{ZZ} = \begin{pmatrix} J_{HH} & J_{HL} & J_{Hx} & J_{Hy} \\ J_{HL} & J_{LL} & J_{Lx} & J_{Ly} \\ J_{Hx} & J_{Lx} & J_{xx} & J_{xy} \\ J_{Hy} & J_{Ly} & J_{xy} & J_{yy} \end{pmatrix}, \quad J_{WZ} = \begin{pmatrix} J_{WH} \\ J_{WL} \\ J_{Wx} \\ J_{Wy} \end{pmatrix}.$$

First, consider $\mathcal{L}_1 J$. The first-order conditions are $U_c(c^*, \phi_C^*) = J_W$ and $U_\phi(c^*, \phi_C^*) = RHJ_W$. These imply $U_\phi(c^*, \phi_C^*)/U_c(c^*, \phi_C^*) = RH$ so that

$$\phi_C^* = c^* \left(\frac{aRH}{1-a} \right)^{-1}.$$

We substitute that relation into $U_c = J_W$ and find

$$c^* = J_W^{-1/\gamma} a^{1/\gamma} \left(\frac{aRH}{1-a} \right)^k, \quad (35)$$

and hence

$$\phi_C^* = J_W^{-1/\gamma} a^{1/\gamma} \left(\frac{aRH}{1-a} \right)^{k-1}, \quad (36)$$

where $k = (1-a)(\gamma-1)/\gamma$. These maximizers lead to

$$\mathcal{L}_1 J = \frac{\gamma}{1-\gamma} J_W^{\frac{\gamma-1}{\gamma}} a^{\frac{1-\gamma}{\gamma}} \left(\frac{aRH}{1-a} \right)^k. \quad (37)$$

Next, consider $\mathcal{L}_2 J$. The first-order condition for α reads $J_W \lambda + J_{WW} \Sigma \Sigma^\top \alpha + \Sigma \Sigma_Z^\top J_{WZ} = 0$ or

$$\alpha = -\frac{J_W}{J_{WW}} (\Sigma \Sigma^\top)^{-1} \lambda - \frac{1}{J_{WW}} (\Sigma \Sigma^\top)^{-1} \Sigma \Sigma_Z^\top J_{WZ} = -\frac{J_W}{J_{WW}} (\Sigma \Sigma^\top)^{-1} \lambda - \frac{1}{J_{WW}} (\Sigma_Z \Sigma_Z^{-1})^\top J_{WZ}. \quad (38)$$

Substituting the optimal α back into $\mathcal{L}_2 J$ leads to

$$\mathcal{L}_2 J = -\frac{1}{2} \frac{J_W^2}{J_{WW}} \lambda^\top (\Sigma \Sigma^\top)^{-1} \lambda - \frac{J_W}{J_{WW}} J_{WZ}^\top \Sigma_Z \Sigma_Z^{-1} \lambda - \frac{1}{2} \frac{1}{J_{WW}} J_{WZ}^\top \Sigma_Z \Sigma_Z^\top J_{WZ}. \quad (39)$$

The matrix products are

$$\Sigma \Sigma^\top = \begin{pmatrix} 1 & \rho_{HS} & 0 & 0 & 0 \\ \rho_{HS} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow (\Sigma \Sigma^\top)^{-1} = \frac{1}{\hat{\rho}_H^2} \begin{pmatrix} 1 & -\rho_{HS} & 0 & 0 & 0 \\ -\rho_{HS} & 1 & 0 & 0 & 0 \\ 0 & 0 & \hat{\rho}_H^2 & 0 & 0 \\ 0 & 0 & 0 & \hat{\rho}_H^2 & 0 \\ 0 & 0 & 0 & 0 & \hat{\rho}_H^2 \end{pmatrix},$$

$$\Sigma_Z \Sigma_Z^{-1} = \begin{pmatrix} 0 & H\sigma_H & 0 & 0 & 0 \\ LM_{LS}\sigma_S & LM_{LH}\sigma_H & L\hat{\rho}_L\sigma_L & 0 & 0 \\ M_{xS}\sigma_S & M_{xH}\sigma_H & \hat{\rho}_x L\sigma_x & \hat{\rho}_x \sigma_x & 0 \\ M_{yS}\sigma_S & M_{yH}\sigma_H & \hat{\rho}_y L\sigma_y & \hat{\rho}_y \sigma_y & \hat{\rho}_y \sigma_y \end{pmatrix}, \quad \Sigma_Z \Sigma_Z^\top = \begin{pmatrix} H^2\sigma_H^2 & HL\sigma_{HL} & H\sigma_{Hx} & H\sigma_{Hy} \\ HL\sigma_{HL} & L^2\sigma_L^2 & L\sigma_{Lx} & L\sigma_{Ly} \\ H\sigma_{Hx} & L\sigma_{Lx} & \sigma_x^2 & \sigma_{xy} \\ H\sigma_{Hy} & L\sigma_{Ly} & \sigma_{xy} & \sigma_y^2 \end{pmatrix},$$

where we have applied the covariance notation $\sigma_{ab} = \rho_{ab}\sigma_a\sigma_b$ and constants defined in Appendix B.2.

Substitution of these matrix products into (38) gives

$$\begin{aligned}\alpha_S = & -\frac{J_W}{J_{WW}} \frac{1}{\sigma_S \hat{\rho}_H^2} \left(\mu'_S(t, x, y) + \chi_S x - \frac{\rho_{HS} \sigma_S}{\sigma_H} [\mu'_H(t, x, y) + \chi_{Hx} x + \chi_{Hy} y] \right) \\ & - \frac{J_{WL}}{J_{WW}} LM_{LS}(t) \sigma_S - \frac{J_{Wx}}{J_{WW}} M_{xS} \sigma_S - \frac{J_{Wy}}{J_{WW}} M_{yS} \sigma_S,\end{aligned}\quad (40)$$

$$\begin{aligned}\alpha_H = & -\frac{J_W}{J_{WW}} \frac{1}{\sigma_H \hat{\rho}_H^2} \left(\mu'_H(t, x, y) + \chi_{Hx} x + \chi_{Hy} y - \frac{\rho_{HS} \sigma_H}{\sigma_S} [\mu'_S(t, x, y) + \chi_S x] \right) \\ & - \frac{J_{WH}}{J_{WW}} H \sigma_H - \frac{J_{WL}}{J_{WW}} LM_{LH}(t) \sigma_H - \frac{J_{Wx}}{J_{WW}} M_{xH} \sigma_H - \frac{J_{Wy}}{J_{WW}} M_{yH} \sigma_H,\end{aligned}\quad (41)$$

$$\alpha_L = -\frac{J_W}{J_{WW}} \lambda_L(t, x, y) - \frac{J_{WL}}{J_{WW}} L \hat{\rho}_L \sigma_L(t) - \frac{J_{Wx}}{J_{WW}} \hat{\rho}_{xL} \sigma_x - \frac{J_{Wy}}{J_{WW}} \hat{\rho}_{yL} \sigma_y,\quad (42)$$

$$\alpha_x = -\frac{J_W}{J_{WW}} \lambda_x(t, x, y) - \frac{J_{Wx}}{J_{WW}} \hat{\rho}_x \sigma_x - \frac{J_{Wy}}{J_{WW}} \hat{\rho}_{yx} \sigma_y,\quad (43)$$

$$\alpha_y = -\frac{J_W}{J_{WW}} \lambda_y(t, x, y) - \frac{J_{Wy}}{J_{WW}} \hat{\rho}_y \sigma_y.\quad (44)$$

Conjecture of the solution to the HJB equation. We conjecture

$$J(t, W, H, L, x, y) = \frac{1}{1-\gamma} G(t, H, x, y)^\gamma (W + LF(t, x, y))^{1-\gamma}.\quad (45)$$

It turns out to be useful to express the derivatives of J in terms of J itself:

$$\begin{aligned}J_W &= \frac{(1-\gamma)J}{W+LF}, & J_{WW} &= -\frac{\gamma(1-\gamma)J}{(W+LF)^2}, \\ J_L &= (1-\gamma)J \frac{F}{W+LF}, & J_{LL} &= -\gamma(1-\gamma)J \frac{F^2}{(W+LF)^2}, \\ J_H &= \gamma J \frac{G_H}{G}, & J_{HH} &= \gamma(1-\gamma)J \left[\frac{1}{1-\gamma} \frac{G_{HH}}{G} - \left(\frac{G_H}{G} \right)^2 \right], \\ J_{WL} &= -\gamma(1-\gamma)J \frac{F}{(W+LF)^2}, & J_{WH} &= \gamma(1-\gamma)J \frac{1}{W+LF} \frac{G_H}{G}, \\ J_{HL} &= \gamma(1-\gamma)J \frac{G_H}{G} \frac{F}{W+LF}, & J_x &= (1-\gamma)J \left[\frac{\gamma}{1-\gamma} \frac{G_x}{G} + \frac{LF_x}{W+LF} \right], \\ J_y &= (1-\gamma)J \left[\frac{\gamma}{1-\gamma} \frac{G_y}{G} + \frac{LF_y}{W+LF} \right], & J_{Wx} &= \gamma(1-\gamma)J \left[\frac{G_x}{G} \frac{1}{W+LF} - \frac{LF_x}{(W+LF)^2} \right], \\ J_{Wy} &= \gamma(1-\gamma)J \left[\frac{G_y}{G} \frac{1}{W+LF} - \frac{LF_y}{(W+LF)^2} \right], & J_t &= \frac{(1-\gamma)J}{W+LF} L \frac{\partial F}{\partial t} + \gamma J \frac{1}{G} \frac{\partial G}{\partial t}, \\ J_{xx} &= (1-\gamma)J \left[\frac{\gamma}{1-\gamma} \frac{G_{xx}}{G} - \gamma \left(\frac{G_x}{G} \right)^2 + 2\gamma \frac{G_x}{G} \frac{LF_x}{W+LF} - \gamma \left(\frac{LF_x}{W+LF} \right)^2 + \frac{LF_{xx}}{W+LF} \right], \\ J_{yy} &= (1-\gamma)J \left[\frac{\gamma}{1-\gamma} \frac{G_{yy}}{G} - \gamma \left(\frac{G_y}{G} \right)^2 + 2\gamma \frac{G_y}{G} \frac{LF_y}{W+LF} - \gamma \left(\frac{LF_y}{W+LF} \right)^2 + \frac{LF_{yy}}{W+LF} \right], \\ J_{Hx} &= \gamma(1-\gamma)J \left[\frac{1}{1-\gamma} \frac{G_{Hx}}{G} - \frac{G_x G_H}{G^2} + \frac{LF_x}{W+LF} \frac{G_H}{G} \right],\end{aligned}$$

$$\begin{aligned}
J_{Hy} &= \gamma(1-\gamma)J \left[\frac{1}{1-\gamma} \frac{G_{Hy}}{G} - \frac{G_y G_H}{G^2} + \frac{L F_y}{W+LF} \frac{G_H}{G} \right], \\
J_{Lx} &= (1-\gamma)J \left[\gamma \frac{G_x}{G} \frac{F}{W+LF} + \frac{F_x}{W+LF} - \gamma \frac{L F F_x}{(W+LF)^2} \right], \\
J_{Ly} &= (1-\gamma)J \left[\gamma \frac{G_y}{G} \frac{F}{W+LF} + \frac{F_y}{W+LF} - \gamma \frac{L F F_y}{(W+LF)^2} \right], \\
J_{xy} &= (1-\gamma)J \left[\frac{\gamma}{1-\gamma} \frac{G_{xy}}{G} - \gamma \frac{G_x G_y}{G^2} + \gamma \frac{G_x}{G} \frac{L F_y}{W+LF} + \gamma \frac{G_y}{G} \frac{L F_x}{W+LF} + \frac{L F_{xy}}{W+LF} - \gamma \frac{L^2 F_x F_y}{(W+LF)^2} \right].
\end{aligned}$$

Next, we rewrite the terms $\mathcal{L}_1 J$, $\mathcal{L}_2 J$, $\mathcal{L}_3 J$ exploiting the conjecture for J . First, since

$$J_w^{\frac{\gamma-1}{\gamma}} = J_w J_w^{-1/\gamma} = \frac{(1-\gamma)J}{W+LF} \left\{ G^\gamma [W+LF]^{-\gamma} \right\}^{-1/\gamma} = (1-\gamma)J \frac{1}{G},$$

we get from (37) that

$$\mathcal{L}_1 J = \gamma J \frac{1}{G} a^{\frac{1-\gamma}{\gamma}} \left(\frac{aRH}{1-a} \right)^k.$$

Next, we have from (39) that

$$\mathcal{L}_2 J = \mathcal{L}_{2,1} J + \mathcal{L}_{2,2} J + \mathcal{L}_{2,3} J,$$

where

$$\begin{aligned}
\mathcal{L}_{2,1} J &= -\frac{1}{2} \frac{J_W^2}{J_{WW}} \lambda^\top (\Sigma \Sigma^\top)^{-1} \lambda = \frac{1-\gamma}{2\gamma} J \lambda^\top (\Sigma \Sigma^\top)^{-1} \lambda \\
&= \frac{1-\gamma}{2\gamma} J \left\{ \left(\frac{\mu'_S(t, x, y) + \chi_S x_t}{\sigma_S \hat{\rho}_H} \right)^2 + \left(\frac{\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t}{\sigma_H \hat{\rho}_H} \right)^2 \right. \\
&\quad \left. - 2\rho_{HS} \frac{\mu'_S(t, x, y) + \chi_S x_t}{\sigma_S \hat{\rho}_H} \frac{\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t}{\sigma_H \hat{\rho}_H} + \lambda_L(t, x, y)^2 + \lambda_x(t, x, y)^2 + \lambda_y(t, x, y)^2 \right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{2,2} J &= -\frac{J_W}{J_{WW}} J_{WZ}^\top \Sigma_Z \Sigma^{-1} \lambda = \frac{1}{\gamma} (W+LF) J_{WZ}^\top \Sigma_Z \Sigma^{-1} \lambda \\
&= (1-\gamma) J \begin{pmatrix} \frac{G_H}{G} \\ -\frac{F}{W+LF} \\ \frac{G_x}{G} - \frac{L F_x}{W+LF} \\ \frac{G_y}{G} - \frac{L F_y}{W+LF} \end{pmatrix}^\top \begin{pmatrix} 0 & H\sigma_H & 0 & 0 & 0 \\ L\ell(t)\sigma_S & LM_{LH}(t)\sigma_H & L\hat{\rho}_L\sigma_L(t) & 0 & 0 \\ M_{xS}\sigma_S & M_{xH}\sigma_H & \hat{\rho}_{xL}\sigma_x & \hat{\rho}_x\sigma_x & 0 \\ M_{yS}\sigma_S & M_{yH}\sigma_H & \hat{\rho}_{yL}\sigma_y & \hat{\rho}_{yx}\sigma_y & \hat{\rho}_y\sigma_y \end{pmatrix} \begin{pmatrix} \frac{\mu'_S(t, x, y) + \chi_S x_t}{\sigma_S} \\ \frac{\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t}{\sigma_H} \\ \lambda_L(t, x, y) \\ \lambda_x(t, x, y) \\ \lambda_y(t, x, y) \end{pmatrix} \\
&= (1-\gamma) J \left\{ \frac{HG_H}{G} (\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t) + \frac{G_x}{G} \left(-(M_{xS}\chi_S + M_{xH}\chi_{Hx}) x_t - M_{xH}\chi_{Hy} y_t \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - M_{xS}\mu'_S(t, x, y) - M_{xH}\mu'_H(t, x, y) - \sigma_x (\hat{\rho}_{xL}\lambda_L(t, x, y) + \hat{\rho}_x\lambda_x(t, x, y)) \\
& + \frac{G_y}{G} \left(- (M_{yS}\chi_S + M_{yH}\chi_{Hx}) x_t - M_{yH}\chi_{Hy}y_t \right. \\
& \quad \left. - M_{yS}\mu'_S(t, x, y) - M_{yH}\mu'_H(t, x, y) - \sigma_y (\hat{\rho}_{yL}\lambda_L(t, x, y) + \hat{\rho}_{yx}\lambda_x(t, x, y) + \hat{\rho}_y\lambda_y(t, x, y)) \right) \\
& - \frac{L}{W + LF} \left[F \left(- (M_{LS}(t)\chi_S + M_{LH}(t)\chi_{Hx}) x_t + (\text{bar}\chi_{yL}(t) - M_{LH}(t)\chi_{Hy}) y_t + \mu_L(t) - r(t) \right. \right. \\
& \quad \left. \left. - M_{LS}(t)\mu'_S(t, x, y) + M_{LH}(t)\mu'_H(t, x, y) - \sigma_L(t)\hat{\rho}_L\lambda_L(t, x, y) \right) \right. \\
& \quad \left. + F_x \left(- (M_{xS}\chi_S + M_{xH}\chi_{Hx}) x_t - M_{xH}\chi_{Hy}y_t \right. \right. \\
& \quad \left. \left. - M_{xS}\mu'_S(t, x, y) - M_{xH}\mu'_H(t, x, y) - \sigma_x (\hat{\rho}_{xL}\lambda_L(t, x, y) + \hat{\rho}_x\lambda_x(t, x, y)) \right) \right. \\
& \quad \left. + F_y \left(- (M_{yS}\chi_S + M_{yH}\chi_{Hx}) x_t - M_{yH}\chi_{Hy}y_t - M_{yS}\mu'_S(t, x, y) - M_{yH}\mu'_H(t, x, y) \right. \right. \\
& \quad \left. \left. - \sigma_y (\hat{\rho}_{yL}\lambda_L(t, x, y) + \hat{\rho}_{yx}\lambda_x(t, x, y) + \hat{\rho}_y\lambda_y(t, x, y)) \right) \right] \Bigg\},
\end{aligned}$$

and, furthermore,

$$\begin{aligned}
\mathcal{L}_{2,3}J &= -\frac{1}{2} \frac{1}{J_{WW}} J_{WZ}^\top \Sigma_Z \Sigma_Z^\top J_{WZ} \\
&= \frac{1}{2} \gamma(1-\gamma) J \begin{pmatrix} \frac{G_H}{G} \\ -\frac{F}{W+LF} \\ \frac{G_x}{G} - \frac{LF_x}{W+LF} \\ \frac{G_y}{G} - \frac{LF_y}{W+LF} \end{pmatrix}^\top \begin{pmatrix} H^2\sigma_H^2 & HL\sigma_{HL} & H\sigma_{Hx} & H\sigma_{Hy} \\ HL\sigma_{HL} & L^2\sigma_L^2 & L\sigma_{Lx} & L\sigma_{Ly} \\ H\sigma_{Hx} & L\sigma_{Lx} & \sigma_x^2 & \sigma_{xy} \\ H\sigma_{Hy} & L\sigma_{Ly} & \sigma_{xy} & \sigma_y^2 \end{pmatrix} \begin{pmatrix} \frac{G_H}{G} \\ -\frac{F}{W+LF} \\ \frac{G_x}{G} - \frac{LF_x}{W+LF} \\ \frac{G_y}{G} - \frac{LF_y}{W+LF} \end{pmatrix} \\
&= \gamma(1-\gamma) J \left\{ \frac{1}{2} \sigma_H^2 \frac{H^2 G_H^2}{G^2} + \frac{1}{2} \sigma_x^2 \frac{G_x^2}{G^2} + \frac{1}{2} \sigma_y^2 \frac{G_y^2}{G^2} + \sigma_{Hx} \frac{HG_H G_x}{G^2} + \sigma_{Hy} \frac{HG_H G_y}{G^2} + \sigma_{xy} \frac{G_x G_y}{G^2} \right. \\
& \quad \left. - \frac{L}{W + LF} \left[\frac{HG_H}{G} (\sigma_{HL}F + \sigma_{Hx}F_x + \sigma_{Hy}F_y) \right. \right. \\
& \quad \left. \left. + \frac{G_x}{G} (\sigma_{Lx}F + \sigma_x^2 F_x + \sigma_{xy}F_y) + \frac{G_y}{G} (\sigma_{Ly}F + \sigma_{xy}F_x + \sigma_y^2 F_y) \right] \right. \\
& \quad \left. + \frac{L^2}{(W + LF)^2} \left[\frac{1}{2} \sigma_L^2 F^2 + \frac{1}{2} \sigma_x^2 F_x^2 + \frac{1}{2} \sigma_y^2 F_y^2 + \sigma_{Lx} F F_x + \sigma_{Ly} F F_y + \sigma_{xy} F_x F_y \right] \right\}.
\end{aligned}$$

Finally, we can rewrite \mathcal{L}_3J as

$$\mathcal{L}_3J = \mathcal{L}_{3,1}J + \mathcal{L}_{3,2}J,$$

where

$$\mathcal{L}_{3,1}J = \frac{\partial J}{\partial t} + J_W(r(t, x, y)W + L) + J_Z^\top \mu_Z$$

$$\begin{aligned}
&= (1-\gamma)J \left\{ r(t, x, y) + \frac{\gamma}{1-\gamma} \left[\frac{1}{G} \frac{\partial G}{\partial t} + \frac{HG_H}{G} (r(t, x, y) + \mu_H(t, x, y) + \chi_{Hx}x_t + \chi_{Hy}y_t) - \kappa_x x_t \frac{G_x}{G} - \kappa_y y_t \frac{G_y}{G} \right] \right. \\
&\quad \left. + \frac{L}{W+LF} \left[\frac{\partial F}{\partial t} + 1 + (\mu_L(t) + \bar{\chi}_L(t)y_t - r(t, x, y))F - \kappa_x x_t F_x - \kappa_y y_t F_y \right] \right\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{3,2}J &= \frac{1}{2} \text{trace} (J_{ZZ} \Sigma_Z \Sigma_Z^\top) \\
&= \frac{1}{2} \sigma_H^2 H^2 J_{HH} + \frac{1}{2} \sigma_L^2 L^2 J_{LL} + \frac{1}{2} \sigma_x^2 J_{xx} + \frac{1}{2} \sigma_y^2 J_{yy} + \sigma_{HL} H L J_{HL} + \sigma_{Hx} H J_{Hx} \\
&\quad + \sigma_{Hy} H J_{Hy} + \sigma_{Lx} L J_{Lx} + \sigma_{Ly} L J_{Ly} + \sigma_{xy} J_{xy} + \sigma_{xz} J_{xz} + \sigma_{yz} J_{yz} \\
&= (1-\gamma)J \left\{ \frac{\gamma}{1-\gamma} \left[\frac{1}{2} \sigma_H^2 H^2 \frac{G_{HH}}{G} + \frac{1}{2} \sigma_x^2 \frac{G_{xx}}{G} + \frac{1}{2} \sigma_y^2 \frac{G_{yy}}{G} - \frac{1-\gamma}{2} \sigma_H^2 H^2 \frac{G_H^2}{G^2} - \frac{1-\gamma}{2} \sigma_x^2 \frac{G_x^2}{G^2} \right. \right. \\
&\quad - \frac{1-\gamma}{2} \sigma_y^2 \frac{G_y^2}{G^2} + \sigma_{Hx} H \frac{G_{Hx}}{G} + \sigma_{Hy} H \frac{G_{Hy}}{G} + \sigma_{xy} \frac{G_{xy}}{G} \\
&\quad \left. \left. - (1-\gamma) \sigma_{Hx} H \frac{G_H G_x}{G^2} - (1-\gamma) \sigma_{Hy} H \frac{G_H G_y}{G^2} - (1-\gamma) \sigma_{xy} \frac{G_x G_y}{G^2} \right] \right. \\
&\quad + \frac{L}{W+LF} \left[\frac{1}{2} \sigma_x^2 F_{xx} + \frac{1}{2} \sigma_y^2 F_{yy} + \sigma_{xy} F_{xy} + \sigma_{Lx} F_x + \sigma_{Ly} F_y \right. \\
&\quad \left. + \gamma \left[\frac{HG_H}{G} (\sigma_{HL} F + \sigma_{Hx} F_x + \sigma_{Hy} F_y) \right. \right. \\
&\quad \left. \left. + \frac{G_x}{G} (\sigma_{Lx} F + \sigma_x^2 F_x + \sigma_{xy} F_y) + \frac{G_y}{G} (\sigma_{Ly} F + \sigma_{xy} F_x + \sigma_y^2 F_y) \right] \right] \\
&\quad \left. - \frac{L^2}{(W+LF)^2} \gamma \left[\frac{1}{2} \sigma_L^2 F^2 + \frac{1}{2} \sigma_x^2 F_x^2 + \frac{1}{2} \sigma_y^2 F_y^2 + \sigma_{Lx} F F_x + \sigma_{Ly} F F_y + \sigma_{xy} F_x F_y \right] \right\}
\end{aligned}$$

By adding $\mathcal{L}_{2,3}J$ and $\mathcal{L}_{3,2}J$ numerous terms cancel so that we are left with

$$\begin{aligned}
\mathcal{L}_{2,3}J + \mathcal{L}_{3,2}J &= (1-\gamma)J \left\{ \frac{\gamma}{1-\gamma} \left[\frac{1}{2} \sigma_H^2 H^2 \frac{G_{HH}}{G} + \frac{1}{2} \sigma_x^2 \frac{G_{xx}}{G} + \frac{1}{2} \sigma_y^2 \frac{G_{yy}}{G} + \sigma_{Hx} H \frac{G_{Hx}}{G} + \sigma_{Hy} H \frac{G_{Hy}}{G} + \sigma_{xy} \frac{G_{xy}}{G} \right] \right. \\
&\quad \left. + \frac{L}{W+LF} \left[\frac{1}{2} \sigma_x^2 F_{xx} + \frac{1}{2} \sigma_y^2 F_{yy} + \sigma_{xy} F_{xy} + \sigma_{Lx} F_x + \sigma_{Ly} F_y \right] \right\}.
\end{aligned}$$

If we further add $\mathcal{L}_{2,2}J$ and $\mathcal{L}_{3,1}J$ to this, all the terms multiplying $L/(W+LF)$ cancel because F satisfies the PDE (25). In sum, we get

$$\begin{aligned}
\delta J &= \mathcal{L}_1 J + \mathcal{L}_{2,1} J + \mathcal{L}_{2,2} J + \mathcal{L}_{2,3} J + \mathcal{L}_{3,1} J + \mathcal{L}_{3,2} J \\
&= \gamma J \frac{1}{G} a^{\frac{1-\gamma}{\gamma}} \left(\frac{aRH}{1-a} \right)^k
\end{aligned}$$

$$\begin{aligned}
& + J \frac{1-\gamma}{2\gamma} \left[\left(\frac{\mu'_S(t, x, y) + \chi_S x_t}{\sigma_S \hat{\rho}_H} \right)^2 + \left(\frac{\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t}{\sigma_H \hat{\rho}_H} \right)^2 \right. \\
& \quad \left. - 2\rho_{HS} \frac{\mu'_S(t, x, y) + \chi_S x_t}{\sigma_S \hat{\rho}_H} \frac{\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t}{\sigma_H \hat{\rho}_H} + \lambda_L(t, x, y)^2 + \lambda_x(t, x, y)^2 + \lambda_y(t, x, y)^2 \right] \\
& + (1-\gamma) J \frac{1}{G} \left\{ r(t, x, y) + HG_H (\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t) + \frac{G_x}{G} \left(- (M_{xS} \chi_S + M_{xH} \chi_{Hx}) x_t - M_{xH} \chi_{Hy} y_t \right. \right. \\
& \quad \left. \left. - M_{xS} \mu'_S(t, x, y) - M_{xH} \mu'_H(t, x, y) - \sigma_x (\hat{\rho}_{xL} \lambda_L(t, x, y) + \hat{\rho}_x \lambda_x(t, x, y)) \right) \right. \\
& \quad + \frac{G_y}{G} \left(- (M_{yS} \chi_S + M_{yH} \chi_{Hx}) x_t - M_{yH} \chi_{Hy} y_t \right. \\
& \quad \left. - M_{yS} \mu'_S(t, x, y) - M_{yH} \mu'_H(t, x, y) - \sigma_y (\hat{\rho}_{yL} \lambda_L(t, x, y) + \hat{\rho}_{yx} \lambda_x(t, x, y) + \hat{\rho}_y \lambda_y(t, x, y)) \right) \\
& \quad + \frac{\gamma}{1-\gamma} \left[\frac{1}{2} \sigma_H^2 H^2 G_{HH} + \frac{1}{2} \sigma_x^2 G_{xx} + \frac{1}{2} \sigma_y^2 G_{yy} + \sigma_{Hx} H G_{Hx} + \sigma_{Hy} H G_{Hy} + \sigma_{xy} G_{xy} \right. \\
& \quad \left. + \frac{\partial G}{\partial t} + HG_H (r(t, x, y) + \mu_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t) - \kappa_x x_t G_x - \kappa_y y_t G_y \right] \left. \right\}.
\end{aligned}$$

Therefore, it follows that the HJB equation is satisfied if the G function solves the PDE

$$\begin{aligned}
0 = & \tilde{k} H^k + \frac{\partial G}{\partial t} + \frac{1}{2} \sigma_H^2 H^2 G_{HH} + \frac{1}{2} \sigma_x^2 G_{xx} + \frac{1}{2} \sigma_y^2 G_{yy} + \sigma_{Hx} H G_{Hx} + \sigma_{Hy} H G_{Hy} + \sigma_{xy} G_{xy} \\
& + HG_H \bar{\mu}_H(t, x, y) + G_x \bar{\mu}_x(t, x, y) + G_y \bar{\mu}_y(t, x, y) - \bar{r}_G(t, x, y) G,
\end{aligned} \tag{46}$$

where $\tilde{k} = a^{\frac{1-\gamma}{\gamma}} \left(\frac{aR}{1-a} \right)^k$, and

$$\begin{aligned}
-\bar{r}_G(t, x, y) = & \frac{\gamma-1}{\gamma} r(t, x, y) + \frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2} \left[\left(\frac{\mu'_S(t, x, y) + \chi_S x_t}{\sigma_S \hat{\rho}_H} \right)^2 + \left(\frac{\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t}{\sigma_H \hat{\rho}_H} \right)^2 \right. \\
& \left. - 2\rho_{HS} \frac{\mu'_S(t, x, y) + \chi_S x_t}{\sigma_S \hat{\rho}_H} \frac{\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t}{\sigma_H \hat{\rho}_H} + \lambda_L(t, x, y)^2 + \lambda_x(t, x, y)^2 + \lambda_y(t, x, y)^2 \right], \\
\bar{\mu}_H(t, x, y) = & r(t, x, y) - \frac{\gamma-1}{\gamma} [R - m] + \frac{1}{\gamma} (\mu_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t), \\
\bar{\mu}_x(t, x, y) = & -(\kappa_x + M_{xS} \chi_S + M_{xH} \chi_{Hx}) x - M_{xH} \chi_{Hy} y - M_{xS} \mu'_S(t, x, y) - M_{xH} \mu'_H(t, x, y) \\
& - \sigma_x (\hat{\rho}_{xL} \lambda_L(t, x, y) + \hat{\rho}_x \lambda_x(t, x, y)), \\
\bar{\mu}_y(t, x, y) = & -(M_{yS} \chi_S + M_{yH} \chi_{Hx}) x - (\kappa_y + M_{yH} \chi_{Hy}) y - M_{yS} \mu'_S(t, x, y) - M_{yH} \mu'_H(t, x, y) \\
& - \sigma_y (\hat{\rho}_{yL} \lambda_L(t, x, y) + \hat{\rho}_{yx} \lambda_x(t, x, y) + \hat{\rho}_y \lambda_y(t, x, y)).
\end{aligned}$$

The terminal condition is $G(T, H, x, y) = 0$ because of the no-bequest assumption.

Coming back to the optimal investment strategy, we first note that

$$\begin{aligned}
-\frac{J_W}{J_{WW}} &= \frac{1}{\gamma} (W + LF), & -\frac{J_{Wx}}{J_{WW}} &= \frac{G_x}{G} (W + LF) - LF_x, \\
-\frac{J_{WL}}{J_{WW}} &= -F, & -\frac{J_{Wy}}{J_{WW}} &= \frac{G_y}{G} (W + LF) - LF_y,
\end{aligned}$$

$$-\frac{J_{WH}}{J_{WW}} = \frac{G_H}{G}(W + LF).$$

By substituting these expressions into (40)-(44), we obtain

$$\begin{aligned} \alpha_S &= \frac{1}{\gamma}(W + LF) \frac{1}{\sigma_S \hat{\rho}_H^2} \left(\mu'_S(t, x, y) + \chi_S x - \frac{\rho_{HS} \sigma_S}{\sigma_H} [\mu'_H(t, x, y) + \chi_{Hx} x + \chi_{Hy} y] \right) - FLM_{LS}(t) \sigma_S \\ &\quad + \left[\frac{G_x}{G}(W + LF) - LF_x \right] M_{xS} \sigma_S + \left[\frac{G_y}{G}(W + LF) - LF_y \right] M_{yS} \sigma_S, \end{aligned} \quad (47)$$

$$\begin{aligned} \alpha_H &= \frac{1}{\gamma}(W + LF) \frac{1}{\sigma_H \hat{\rho}_H^2} \left(\mu'_H(t, x, y) + \chi_{Hx} x + \chi_{Hy} y - \frac{\rho_{HS} \sigma_H}{\sigma_S} [\mu'_S(t, x, y) + \chi_S x] \right) + \frac{G_H}{G}(W + LF) H \sigma_H \\ &\quad - FLM_{LH}(t) \sigma_H + \left[\frac{G_x}{G}(W + LF) - LF_x \right] M_{xH} \sigma_H + \left[\frac{G_y}{G}(W + LF) - LF_y \right] M_{yH} \sigma_H, \end{aligned} \quad (48)$$

$$\begin{aligned} \alpha_L &= \frac{1}{\gamma}(W + LF) \lambda_L(t, x, y) - FL \hat{\rho}_L \sigma_L(t) + \left[\frac{G_x}{G}(W + LF) - LF_x \right] \hat{\rho}_{xL} \sigma_x \\ &\quad + \left[\frac{G_y}{G}(W + LF) - LF_y \right] \hat{\rho}_{yL} \sigma_y, \end{aligned} \quad (49)$$

$$\alpha_x = \frac{1}{\gamma}(W + LF) \lambda_x(t, x, y) + \left[\frac{G_x}{G}(W + LF) - LF_x \right] \hat{\rho}_x \sigma_x + \left[\frac{G_y}{G}(W + LF) - LF_y \right] \hat{\rho}_{yx} \sigma_y, \quad (50)$$

$$\alpha_y = \frac{1}{\gamma}(W + LF) \lambda_y(t, x, y) + \left[\frac{G_y}{G}(W + LF) - LF_y \right] \hat{\rho}_y \sigma_y. \quad (51)$$

Below we show that $G(t, H, x, y) = \tilde{k} H^k B(t, x, y)$ so that

$$\frac{G_x}{G} = \frac{B_x}{B}, \quad \frac{G_y}{G} = \frac{B_y}{B}, \quad \frac{HG_H}{G} = k.$$

Then (27)–(31) in the theorem follows since $\Pi_S = \alpha_S / (\sigma_S W)$, $\Pi_H = \alpha_H / (\sigma_H W)$, $\Pi_L = \alpha_L / W$, $\Pi_x = \alpha_x / W$, and $\Pi_y = \alpha_y / W$.

If we substitute the conjecture $G(t, H, x, y) = \tilde{k} H^k B(t, x, y)$ into the PDE (46) and dividing by $\tilde{k} H^k$, we obtain

$$0 = 1 + \frac{\partial B}{\partial t} + \frac{1}{2} \sigma_x^2 B_{xx} + \frac{1}{2} \sigma_y^2 B_{yy} + \sigma_{xy} B_{xy} + \bar{\mu}_x B_x + \bar{\mu}_y B_y - \bar{r}_G B \quad (52)$$

with terminal condition $B(T, x, y) = 0$. Because of the complicated form of the coefficient functions (in particular $r(t, x, y)$), we solve the PDE (52) using standard finite difference techniques. \square

B.4 Upper bound on obtainable utility

The returns on the stock and house and the risk-free rate are at least as high in the artificial markets as in the true market. Therefore, a feasible strategy in the true market leads to at least the same expected utility in any of the artificial markets as in the true market. Since many other strategies are feasible in the artificial market, the indirect utility there is always greater or equal the indirect utility in the true market. Karatzas, Lehoczky, Shreve, and Xu (1991), Cvitanic and Karatzas (1992), and Cvitanic, Schachermayer, and Wang

(2001) show that, under certain technical conditions, the solution in the true market is equal to the solution in the worst of all the artificial markets but, in complex models as our, it seems impossible to identify the worst market.

Theorem 1 provides a closed-form solution in any “computable” artificial market. The worst among these artificial markets defines an upper bound on the maximum expected utility in the true market. Each computable artificial market is parameterized by the constants appearing in (20)-(24). For easy reference, let Θ denote a set of such constants. We find the worst of the corresponding artificial markets by a standard unconstrained numerical optimization over Θ . Let $\bar{\Theta}$ denote the parameter set for which the minimum is obtained. Hence,

$$\bar{J}(t, W, H, L, x, y) = J(t, W, H, L, x, y; \bar{\Theta})$$

is the upper bound on the obtainable indirect utility in the true market.

B.5 Feasible strategies for the true problem

We derive a feasible strategy in the true market from the optimal strategies in the parameterized family of artificial markets in the following way. For each parameter set Θ , we take the optimal strategy in the corresponding artificial market and *feasibilize* it, i.e., transform it into a strategy which is feasible in the true market. Obviously, we disregard the investment in the artificial assets and focus on Π_S , Π_H and the consumption processes c , ϕ_C .

In the artificial markets labor income is fully spanned and tangible wealth can be allowed to be temporarily negative if balanced by human capital. We require tangible wealth to stay non-negative because of the unhedgeable shocks that may bring income close to zero. We follow Bick, Kraft, and Munk (2013) and multiply the human capital by a factor $(1 - e^{-\eta W_t})$, where $\eta > 0$ is a constant to be determined. This is consistent with the intuition that future income has a smaller present value when current wealth W_t is small. Define

$$\tilde{F}_t = (1 - e^{-\eta W_t})F(t, x_t, y_t).$$

Furthermore, we prune the optimal portfolios to make sure the constraints (10) are met. To sum up, the feasible strategy derived from the artificial market with parameters Θ is determined from

$$\begin{aligned} \Pi_{St} &= \frac{1}{\gamma \hat{\rho}_H^2 \sigma_S^2} \left(\mu'_S(t, x, y) + \chi_S x_t - \frac{\rho_{HS} \sigma_S}{\sigma_H} [\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t] \right) \frac{W_t + L_t \tilde{F}_t}{W_t} \\ &\quad + \left(M_{xS} \frac{B_x}{B} + M_{yS} \frac{B_y}{B} \right) \frac{W_t + L_t \tilde{F}_t}{W_t} - \left(M_{LS}(t) + M_{xS} \frac{F_x}{F} + M_{xH} \frac{F_y}{F} \right) \frac{L_t \tilde{F}_t}{W}, \end{aligned} \quad (53)$$

$$\begin{aligned} \Pi_{Ht} &= \frac{1}{\gamma \hat{\rho}_H^2 \sigma_H^2} \left(\mu'_H(t, x, y) + \chi_{Hx} x_t + \chi_{Hy} y_t - \frac{\rho_{HS} \sigma_H}{\sigma_S} [\mu'_S(t, x, y) + \chi_S x_t] \right) \frac{W_t + L_t \tilde{F}_t}{W_t} \\ &\quad + k \frac{W_t + L_t \tilde{F}_t}{W_t} + \left(M_{xH} \frac{B_x}{B} + M_{yH} \frac{B_y}{B} \right) \frac{W_t + L_t \tilde{F}_t}{W_t} - \left(M_{LH}(t) + M_{xH} \frac{F_x}{F} + M_{yH} \frac{F_y}{F} \right) \frac{L_t \tilde{F}_t}{W}, \end{aligned} \quad (54)$$

and

$$c_t = a \frac{W_t + L_t \tilde{F}_t}{B(t, x_t, y_t)}, \quad (55)$$

$$\phi_{Ct} = (1 - a) \frac{W_t + L_t \tilde{F}_t}{RH_t B(t, x_t, y_t)}, \quad (56)$$

where we suppress the dependence of F , F_x , F_y , B , B_x , and B_y on t, x, y and the parameter set Θ . If necessary to satisfy the portfolio constraints (10), we prune the portfolio weights Π_S, Π_H following Cvitanić and Karatzas (1992). After these potential transformations, the residual wealth (positive or negative) constitutes the position in the bank account. If financial wealth should equal zero at any point in time, the investment in the risky assets is restricted to zero and consumption is set to fraction of current income, $c_t = \omega Y_t$ and $\phi_{Ct} = \omega Y_t / (RH_t)$, where $\omega \in (0, 1/2)$. This ensures that the liquidity constraint is respected.

For any (Θ, η) , we can approximate the expected utility $J(t, W, H, L, x, y; \Theta, \eta)$ generated with the above strategy by Monte Carlo simulation of the wealth W_t and state variables H_t, L_t, x_t, y_t . Searching over (Θ, η) , we find the best of the feasible strategies. This is our candidate for a near-optimal consumption-investment strategy in the true market. Again, this search can be implemented by a standard unconstrained numerical optimization algorithm.

We can evaluate the performance of any admissible strategy $(c, \phi_C, \Pi_S, \Pi_H)$ —including our candidate defined above—in the following way. We compare the expected utility generated by the strategy, $J^{c, \phi_C, \Pi_S, \Pi_H}(t, W, H, L, x, y)$, to the upper bound $\bar{J}(t, W, H, L, x, y)$ on the maximum utility. If the distance is small, the strategy is indeed near-optimal. More precisely, we can compute an upper bound $\text{Loss} = \text{Loss}^{c, \phi_C, \Pi_S, \Pi_H}(t, W, H, L, x, y)$ on the welfare loss suffered when following the specific strategy $(c, \phi_C, \Pi_S, \Pi_H)$ by solving the equation

$$J^{c, \phi_C, \Pi_S, \Pi_H}(t, W, H, L, x, y) = \bar{J}(t, W[1 - \text{Loss}], H, L[1 - \text{Loss}], x, y). \quad (57)$$

We can interpret Loss as an upper bound on the fraction of total wealth (current wealth plus current and future income) that the individual is willing to sacrifice to get access to the unknown optimal strategy, instead of following the strategy $(c, \phi_C, \Pi_S, \Pi_H)$. Theorem 1 implies

$$\begin{aligned} \bar{J}(t, W[1 - \text{Loss}], H, L[1 - \text{Loss}], x, y) &= J(t, W[1 - \text{Loss}], H, L[1 - \text{Loss}], x, y; \bar{\Theta}) \\ &= (1 - \text{Loss})^{1-\gamma} J(t, W, H, L, x, y; \bar{\Theta}), \end{aligned}$$

so that the upper bound on the welfare loss becomes

$$\text{Loss}^{c, \phi_C, \Pi_S, \Pi_H}(t, W, H, L, x, y) = 1 - \left(\frac{J^{c, \phi_C, \Pi_S, \Pi_H}(t, W, H, L, x, y)}{J(t, W, H, L, x, y; \bar{\Theta})} \right)^{\frac{1}{1-\gamma}}. \quad (58)$$

Figure 1: The time series of detrended predictors. The figure depicts the annual net payout yield (left panel) and the annual change of the normalized log rent-price ratio (right panel), both series are shown before winsorizing.

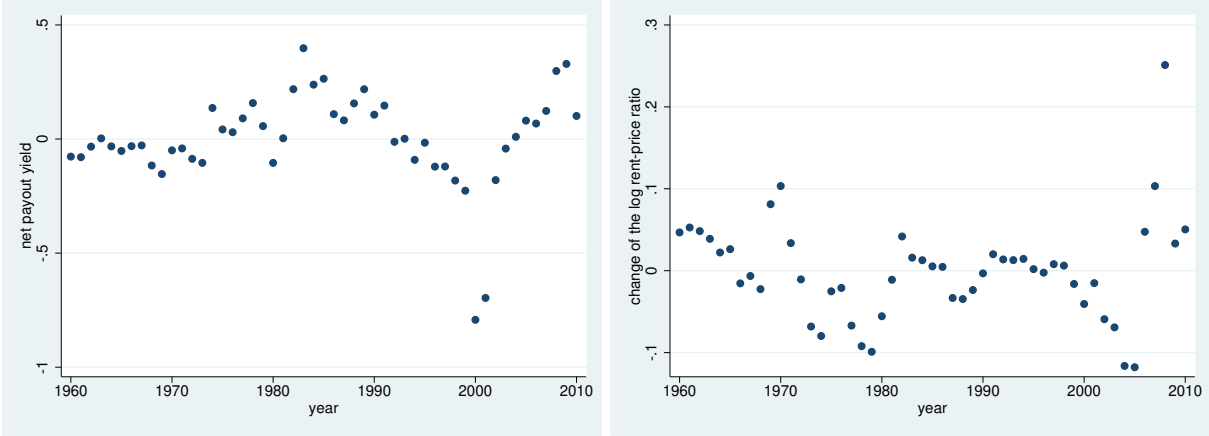


Table 1: Baseline parameter values. This table reports the estimates of the model parameters based on 1960-2010 US data. Here, x refers to the net payout ratio and y to the normalized change of the log home rent-price ratio. Some parameter estimates have been adjusted as explained in the text. (N.A. means ‘not available’ as the parameter is not included in that model.)

Parameter		Predictability taken into account				
Symbol	Explanation	Full	Not at all	No stock	No house	No income
<i>Drift and volatility</i>						
r	Interest rate	0.01	0.01	0.01	0.01	0.01
$\mu_S + \bar{D}$	Equity premium	0.04	0.04	0.04	0.04	0.04
χ_S	Stock predictor coefficient	0.329	N.A.	N.A.	0.483	0.313
σ_S	Stock volatility	0.17	0.175	0.175	0.172	0.17
μ_H	Excess expected house price growth	-0.01	-0.01	-0.01	-0.01	-0.01
χ_{Hx}	House predictor coefficient	-0.106	N.A.	-0.133	N.A.	-0.108
χ_{Hy}	House predictor coefficient	-0.398	N.A.	-0.4	N.A.	-0.443
σ_H	House price volatility	0.12	0.12	0.12	0.12	0.12
μ_L	Expected income growth (active)	0.01	0.01	0.01	0.01	0.01
μ_L	Expected income growth (retire)	0.00	0.00	0.00	0.00	0.00
χ_L	Income predictor coefficient	0.126	N.A.	0.123	0.15	N.A.
σ_L	Income volatility	0.10	0.10	0.10	0.10	0.10
κ_x	Mean reversion speed	0.234	N.A.	0.200	0.266	0.227
σ_x	Volatility of net payout yield	0.087	N.A.	0.087	0.087	0.087
κ_y	Mean reversion speed	0.298	N.A.	0.297	0.491	0.295
σ_y	Volatility of rent-price ratio	0.037	N.A.	0.037	0.039	0.037
<i>Correlations</i>						
ρ_{HS}	House, stock	0.300	0.194	0.291	0.332	0.297
ρ_{LS}	Income, stock	0.268	0.211	0.232	0.283	0.235
ρ_{LH}	Income, house	0.212	0.092	0.205	0.244	0.208
ρ_{xS}	Net payout yield, stock	-0.249	N.A.	-0.249	-0.256	-0.248
ρ_{yS}	Rent-price ratio, stock	0.007	N.A.	0.028	-0.025	0.008
ρ_{xH}	Net payout yield, house	-0.121	N.A.	-0.120	-0.154	-0.117
ρ_{yH}	Rent-price ratio, house	-0.619	N.A.	-0.614	-0.640	-0.619
ρ_{xL}	Net payout yield, income	-0.228	N.A.	-0.222	-0.237	-0.196
ρ_{yL}	Rent-price ratio, income	-0.003	N.A.	-0.003	-0.025	-0.004
ρ_{yx}	Rent-price ratio, net payout yield	-0.027	N.A.	-0.031	-0.001	-0.028
<i>Derived correlation parameters</i>						
$\hat{\rho}_H$		0.954	0.981	0.957	0.943	0.955
$\hat{\rho}_{LH}$		0.138	0.052	0.144	0.159	0.145
$\hat{\rho}_L$		0.953	0.976	0.962	0.946	0.961
$\hat{\rho}_{xH}$		-0.049	N.A.	-0.050	-0.073	-0.045
$\hat{\rho}_{xL}$		-0.162	N.A.	-0.163	-0.162	-0.136
$\hat{\rho}_x$		0.954	N.A.	0.953	0.950	0.958
$\hat{\rho}_{yH}$		-0.651	N.A.	-0.650	-0.670	-0.651
$\hat{\rho}_{yL}$		0.089	N.A.	0.087	0.094	0.092
$\hat{\rho}_{yx}$		-0.044	N.A.	-0.044	-0.043	-0.045
$\hat{\rho}_y$		0.752	N.A.	0.753	0.735	0.752

Table 2: Output grid for regression of optimal strategies. This table reports the values of our state variables for which we compute the optimal strategies. These values are used for the regressions reported in Table 3.

Variable	Steps	Min. value	Max. value
t	8	0	35
H	5	200	300
L/W	200	0.01	2.00
x	11	-0.85	0.85
y	11	-0.25	0.25

Table 3: First-order sensitivities of optimal strategies. The table shows the regression results based on the state variable values shown in Table 2. The table reports standardized betas obtained by first standardizing all variables to have a mean of 0 and a standard deviation of 1. All parameters are significant at the 0.1% level. All regressions are based on 968,000 grid points that are defined according to Table 2.

	π_S	π_H	c/W
t	0.0033	-0.0028	0.0122
x	0.0615	-0.0465	0.0022
y	0.0125	-0.0401	0.0019
L/W	-0.0291	0.2051	0.9415
$t \times L/W$	0.0663	-0.0876	0.0160
$x \times L/W$	0.9162	-0.7156	-0.1086
$y \times L/W$	0.1868	-0.5853	0.1028
R^2	0.983	0.985	0.931

Figure 2: Optimal investments over the life cycle: baseline case. The left panel depicts the expected tangible wealth and its decomposition into stock investment, housing investment, and risk-free (bond) investment (the unit is thousands of U.S. dollars). The right panel shows the expected percentage investments of tangible wealth into stocks, housing, and bonds. Baseline parameter values are used.

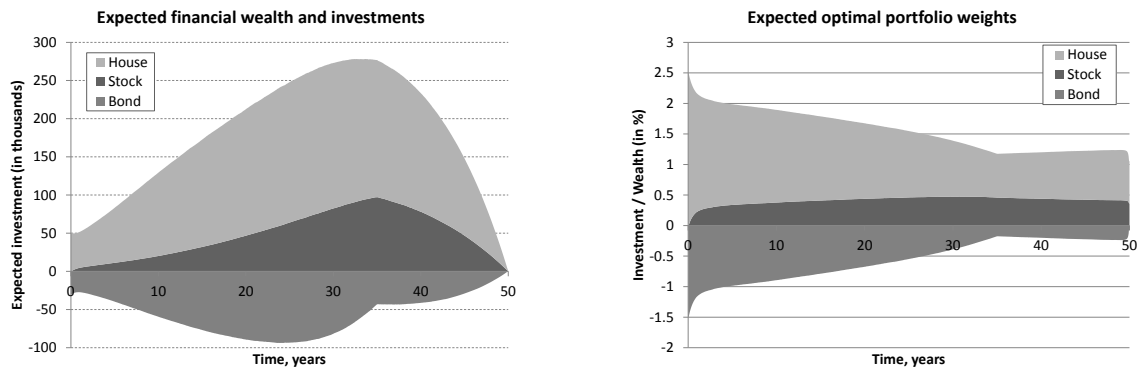


Figure 3: Effect of predictability on the portfolio weights. The upper [lower] panel shows the effect of predictability on the expected percentage investments of financial wealth in stocks [housing assets]. The graphs on the left-hand side depict the optimal demands (skilled investor), whereas the graphs on the right-hand side show the demands of an agent ignoring predictability (unskilled investor). The solid lines depict averages. The dotted lines show averages conditional on whether the current values of the predictors that determine the investors' decisions are positive or negative. We run 10,000 simulations. Baseline values are used for other parameters.

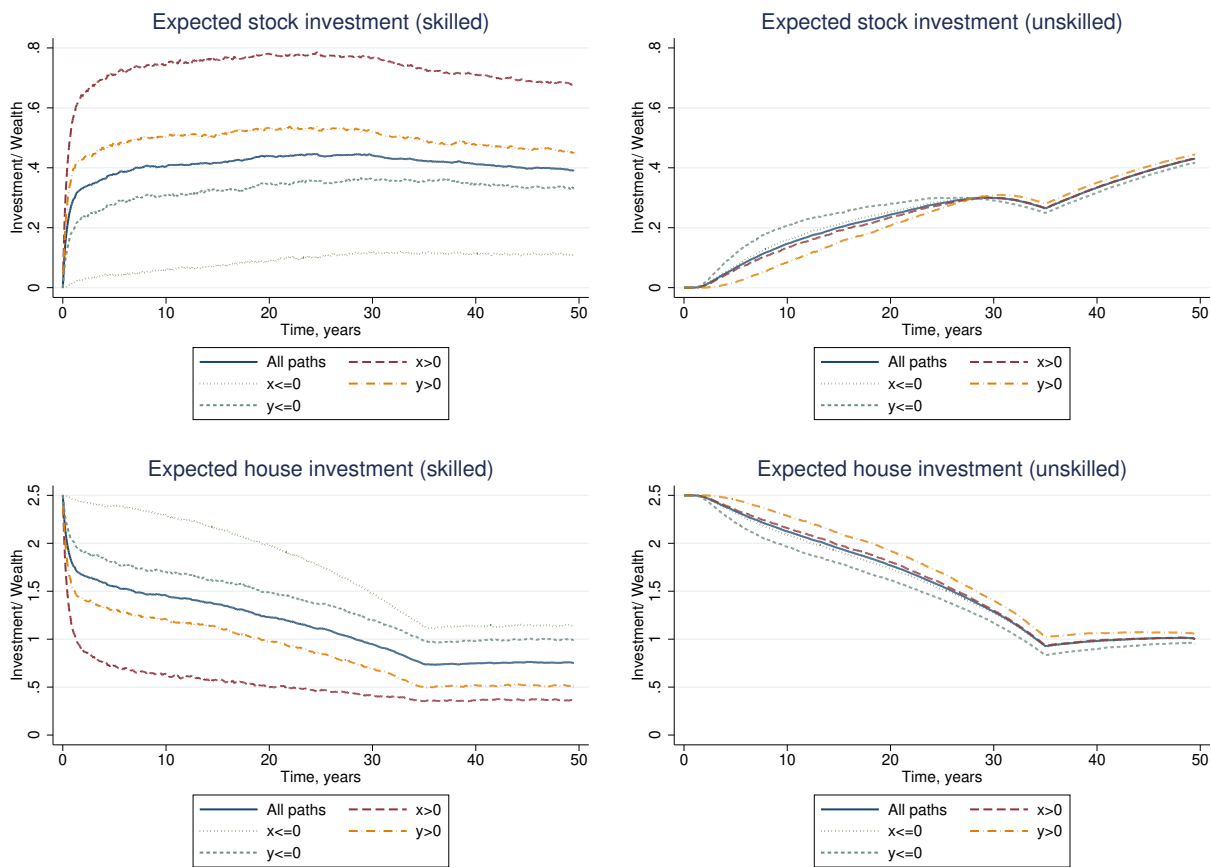


Table 4: Welfare effect of predictability: simulation results. This table reports the welfare effects if an agent is only semi-skilled or unskilled. The case of an agent ignoring predictability all together is referred to as unskilled. The cases of agents ignoring stock, house or income predictability (semi-skilled) are referred to as no stock, no house or no income. *RWEL* is the the welfare loss defined in (13). *RGmean* and *RGmedian* are relative gains of a skilled investor based on normalized present-value differentials of consumption that are defined in (14). We distinguish several cases: First we report the results for all 10,000 paths. Then we report results conditional on whether the average value of the predictors x or y along a path are positive or negative. Baseline parameter values are used.

Variable	Unskilled			Semi-skilled (no stock)			Semi-skilled (no house)			Semi-skilled (no income)		
	RGmean	RGmed	RWEL	RGmean	RGmed	RWEL	RGmean	RGmed	RWEL	RGmean	RGmed	RWEL
All paths	12.49	10.57	5.73	4.51	3.71	1.41	3.02	1.20	1.86	4.24	3.22	0.74
$\bar{x} > 0$	13.38	11.86	6.42	5.85	4.81	1.69	1.84	0.44	1.58	4.59	3.54	0.94
$\bar{x} \leq 0$	11.59	9.26	5.02	3.13	2.58	1.13	4.22	1.97	2.14	3.89	2.89	0.53
$\bar{y} > 0$	10.83	9.34	6.30	4.09	3.29	1.78	0.89	0.35	1.36	3.00	2.18	0.39
$\bar{y} \leq 0$	14.16	12.18	5.37	4.92	4.12	1.15	5.15	2.83	2.21	5.48	4.53	1.04
$\bar{x} > 0, \bar{y} > 0$	13.24	11.82	7.65	5.62	4.55	2.33	0.28	-0.10	1.33	3.72	2.76	0.77
$\bar{x} > 0, \bar{y} \leq 0$	13.51	11.89	5.46	6.07	5.18	1.21	3.31	1.39	1.76	5.41	4.42	1.06
$\bar{x} \leq 0, \bar{y} > 0$	8.50	7.19	4.77	2.62	2.15	1.18	1.49	0.76	1.39	2.31	1.62	-0.01
$\bar{x} \leq 0, \bar{y} \leq 0$	14.88	12.53	5.25	3.68	3.16	1.08	7.14	4.69	2.81	5.56	4.69	1.02

Table 5: Portfolio performance and holdings: simulation results. This table reports the conditional and unconditional summary statistics of the portfolio returns and holdings. $pfret$ is the annualized excess portfolio return and π_S and π_H are the corresponding optimal strategies. The unconditional statistics can be found in the row labeled “All points”. Additionally, we also report the statistics of four two-dimensional scenarios where we condition on lagged predictor values. SD is the standard deviation. SR is the Sharpe ratio ($=\text{Mean}(pfret)/SD$). Baseline parameter values are used.

Investor Scenarios	Skilled				Unskilled				Semi-skilled (no stock)				Semi-skilled (no house)			
	pfret	π_S	π_H		pfret	π_S	π_H		pfret	π_S	π_H		pfret	π_S	π_H	
All points	Mean	0.069	0.409	1.121	0.041	0.245	1.573		0.060	0.333	1.282		0.061	0.413	1.038	
	SD	0.205	0.397	0.936	0.218	0.151	0.632		0.205	0.274	0.819		0.192	0.387	0.844	
	Median	0.060	0.345	0.994	0.040	0.300	1.616		0.052	0.307	1.206		0.056	0.358	0.859	
	SR	0.337			0.188				0.293				0.317			
$x_{t-1} > 0, y_{t-1} > 0$	Mean	0.067	0.851	0.206	0.010	0.225	1.703		0.051	0.607	0.551		0.061	0.770	0.411	
	SD	0.163	0.207	0.403	0.232	0.169	0.643		0.160	0.264	0.629		0.164	0.229	0.465	
	Median	0.064	1.000	0.000	0.013	0.299	1.666		0.046	0.579	0.365		0.059	0.792	0.316	
	SR	0.412			0.043				0.319				0.372			
$x_{t-1} > 0, y_{t-1} \leq 0$	Mean	0.078	0.606	0.820	0.056	0.255	1.482		0.065	0.323	1.291		0.073	0.737	0.453	
	SD	0.183	0.292	0.693	0.209	0.136	0.617		0.197	0.204	0.688		0.161	0.230	0.445	
	Median	0.076	0.585	0.765	0.054	0.299	1.483		0.059	0.327	1.181		0.070	0.742	0.414	
	SR	0.426			0.268				0.329				0.452			
$x_{t-1} \leq 0, y_{t-1} > 0$	Mean	0.036	0.141	1.523	0.029	0.235	1.664		0.035	0.282	1.438		0.033	0.070	1.741	
	SD	0.218	0.223	0.862	0.228	0.164	0.630		0.213	0.228	0.764		0.229	0.124	0.706	
	Median	0.025	0.000	1.479	0.026	0.302	1.651		0.027	0.287	1.340		0.027	0.000	1.769	
	SR	0.165			0.127				0.164				0.144			
$x_{t-1} \leq 0, y_{t-1} \leq 0$	Mean	0.095	0.035	1.938	0.069	0.263	1.444		0.091	0.122	1.847		0.076	0.073	1.547	
	SD	0.246	0.078	0.606	0.204	0.128	0.597		0.239	0.123	0.597		0.206	0.117	0.665	
	Median	0.084	0.000	2.148	0.060	0.300	1.408		0.081	0.105	1.859		0.064	0.000	1.414	
	SR	0.387			0.338				0.380				0.369			

Table 6: Consumption and indirect utility: simulation results. This table reports the conditional and unconditional means of the following variables: present value of total consumption (PVtot) per path, present value of perishable consumption (PVC) per path, average housing units occupied (ϕ) over a particular path, indirect utility (J) of a particular path. The unconditional means are calculated across all paths. We also report results conditional on whether the average values of the predictors x or y along a path are positive or negative. We consider four investor types: skilled, unskilled, semi-skilled (disregarding stock predictability) and semi-skilled (disregarding housing predictability). The indirect utility is multiplied by 100 to avoid too many decimal places. Baseline parameter values are used.

Investor	Skilled			Unskilled			Semi-skilled (no stock)			Semi-skilled (no house)						
	PVtot	PVC	ϕ	J	PVtot	PVC	ϕ	J	PVtot	PVC	ϕ	J				
All paths	1254	878	0.85	-1.21	1063	744	0.72	-1.54	1188	832	0.80	-1.28	1189	832	0.84	-1.30
$\bar{x} > 0$	1152	806	0.94	-1.31	973	681	0.76	-1.70	1073	751	0.86	-1.40	1114	780	0.94	-1.39
$\bar{x} \leq 0$	1359	951	0.76	-1.11	1156	809	0.68	-1.37	1306	914	0.74	-1.16	1266	886	0.74	-1.21
$\bar{y} > 0$	1142	799	1.16	-1.08	1006	704	0.98	-1.41	1091	764	1.09	-1.17	1125	788	1.16	-1.15
$\bar{y} \leq 0$	1367	957	0.54	-1.34	1120	784	0.46	-1.67	1285	900	0.51	-1.40	1253	877	0.53	-1.46
$\bar{x} > 0, \bar{y} > 0$	1066	746	1.31	-1.13	912	638	1.05	-1.56	999	699	1.20	-1.24	1058	741	1.32	-1.19
$\bar{x} > 0, \bar{y} \leq 0$	1233	863	0.59	-1.47	1031	722	0.49	-1.84	1143	800	0.54	-1.54	1166	816	0.58	-1.58
$\bar{x} \leq 0, \bar{y} > 0$	1215	851	1.01	-1.04	1098	768	0.91	-1.26	1180	826	0.98	-1.09	1190	833	1.00	-1.10
$\bar{x} \leq 0, \bar{y} \leq 0$	1512	1058	0.49	-1.19	1217	852	0.43	-1.48	1440	1008	0.47	-1.24	1347	943	0.47	-1.33

Table 8: Historical welfare effects of skills for all market entry dates. This table extracts some information from Table 7 and reports the welfare losses of being less skilled for a given market entry. This date is given in the first column labeled “Start”. The second column shows the losses of an unskilled investor compared to a skilled investor. The third column reports the losses of a semi-skilled investor disregarding stock predictability compared to a skilled investor. The fourth column provides the losses of an unskilled investor compared to the semi-skilled investor.

Start	Skilled vs. unsk.	Skilled vs. semi-sk.	Semi-sk. vs. unsk.
61	4.0	2.0	2.0
62	4.2	1.7	2.6
63	3.5	1.3	2.2
64	3.3	1.4	1.9
65	3.7	2.4	1.3
66	4.6	3.2	1.4
67	4.5	3.1	1.4
68	5.4	3.7	1.8
69	6.2	4.3	1.9
70	6.8	4.2	2.7
71	5.9	3.8	2.1
72	5.0	4.2	0.8
73	4.6	4.3	0.4
74	3.0	3.6	-0.7
75	1.4	1.8	-0.4
76	0.3	1.0	-0.7
Max	6.8	4.3	2.7
Mean	4.1	2.9	1.3
Min	0.3	1.0	-0.7

Table 9: Historical welfare effects of luck for specific entry dates. This table extracts some information from Table 7. It compares the best performing cohorts of unskilled investors with the worst performing cohorts of skilled investors.

		Rank	5	13	17	
	Start	Start	64	65	63	
Rank	Start	Pred	5	5	5	Average
39	76	1	9.6	6.3	5.2	7.1
43	73	1	13.7	10.5	9.4	11.2
44	74	1	15.4	12.3	11.2	13.0

Table 10: Regression results for model with cum returns for the stock. The table shows the regression results based on annual data from the beginning of 1960 to the end of 2010. The house market uses the national Case/Shiller home price index, for stock market data we use returns on the CRSP value-weighted market portfolio inclusive of the NYSE, AMEX, and NASDAQ markets. The risk-free asset is the Treasury bill yield from the Risk Free File on CRSP Bond tape. From NIPA tables, we obtain U.S. data for aggregated disposable personal income (per capita). To obtain real values, all time-series are deflated using the consumer price index (CPI) taken from the website of the Bureau of Labor Statistics. For the predictor variables we use for x the net payout ratio, and for y the the normalized change of the log home rent-price. Both are demeaned, i.e., by construction, their constants are zero. In parentheses are the p-values. $*p < 0.05$, $**p < 0.01$, $***p < 0.001$

	Predictability taken into account				
	Full	Not at all	No stock	No house	No income
Excess stock return					
Lag x	0.329 (0.051)			0.483** (0.004)	0.313 (0.064)
Constant	0.042 (0.072)	0.041 (0.100)	0.042 (0.082)	0.042 (0.074)	0.042 (0.072)
Excess house return					
Lag x	-0.106** (0.010)		-0.133*** (0.001)		-0.108** (0.009)
Lag y	-0.398** (0.004)		-0.400** (0.004)		-0.443** (0.002)
Constant	-0.012* (0.040)	-0.011 (0.195)	-0.012* (0.040)	-0.011 (0.077)	-0.012* (0.039)
Income return					
Lag y	0.126* (0.017)		0.123* (0.022)	0.150** (0.008)	
Constant	0.018*** (0.000)	0.017*** (0.000)	0.018*** (0.000)	0.018*** (0.000)	0.018*** (0.000)
Predictor x					
Lag x	0.766*** (0.000)		0.800*** (0.000)	0.734*** (0.000)	0.773*** (0.000)
Constant	0.000		0.000	0.000	0.000
Predictor y					
Lag y	0.702*** (0.000)		0.703*** (0.000)	0.509*** (0.000)	0.705*** (0.000)
Constant	0.000		0.000	0.000	0.000