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Jan Budek

Peter Schotman

Rolf Tschernig

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Jan Budek[a,c] Peter C. Schotman[a,b,c] Rolf Tschernig[a,d]

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Abstract

This paper focuses on the implications of asset return predictability on long-term portfolio choice when return forecasting variables exhibit long memory. Recent research in empirical finance argues that expected asset returns are time-varying and relates them to various predicting variables that historically reveal very gradual movements in time; hence, we aim at careful modelling of their persistence properties. For that purpose, we exploit the class of fractionally integrated processes. Our theoretical derivations indicate profound impact of the long-memory component on optimal long-term portfolio weights. We illustrate our approach to the modelling of asset return dynamics on post-war US data for equities, Treasury bonds, and cash.

Keywords: Long-term portfolio choice; Term structure of risk; Linear processes with fractional integration

JEL classification: C32, G11

Affiliations: [a] Maastricht University, [b] CEPR, [c] NETSPAR, [d] University of Regensburg

Correspondence address: Jan Budek
Department of Finance / LIFE
Maastricht University
PO BOX 616
6200 MD Maastricht
The Netherlands
Fax: 0031 433884875
Email: `j.budek@finance.unimaas.nl`

1 Introduction

Since the seminal work of Merton (1973), it has been well known that asset return predictability introduces changes into the investment opportunity set of long-term investors. A vast body of recent research in empirical finance has indeed argued that expected asset returns are time-varying: short-term interest rates are used to forecast both stock and bond returns, valuation ratios such as the dividend yield or the price-earnings ratio appear to predict stock returns, and bond returns are, in addition, related to the yield spread.[1] These findings indicate that optimal investment strategies depend on the investment horizon.

Furthermore, observations based on various historical data suggest that many of the return forecasting variables are highly persistent. Along with the predictive power of these variables, this property may, to a great extent, influence the long-run behaviour of expected asset returns. In particular, a modest unexpected shock to the dividend yield would substantially affect the attractiveness of stocks for many periods ahead. If changes in expected returns are persistent, the expected return itself becomes an important source of risk as argued by Barsky and DeLong (1993) and Bansal and Yaron (2004). The effects of persistence in expected returns on asset prices and realized returns are analyzed in detail in Campbell, Lo, and MacKinlay (1997, ch 7) and Cochrane (2001, ch 20). In their models, expected returns follow a stationary AR(1) process with a high first-order autocorrelation parameter. Even a small shock to expected returns can have a big effect on asset prices and realized returns when expected returns are close to a random walk.

In the light of recent developments on asset return predictability, a growing literature focusing on recommendations on optimal portfolio choice for long-term investors has emerged. Nevertheless, these studies hardly consider any explicit specification of persistence and ignore, therefore, an important source of long-term risk. Campbell and Viceira (2002) and Campbell, Chan, and Viceira (2003), in line with other authors, assume that both asset returns and return forecasting variables follow a first-order vector autoregressive process. They demonstrate the important impact of asset return predictability on variances of long-horizon returns on stocks, bonds, and T-Bills. Nevertheless, a stationary VAR process may seriously underestimate the risk since all impulse responses decline exponentially, and persistence is restricted to zero.

[1] A brief and incomplete survey on asset return predictability includes Campbell (1987), Campbell and Shiller (1988, 1991), Fama (1984), Fama and French (1988, 1989), Lamont (1998), Campbell and Vuolteenaho (2004), and Campbell and Yogo (2005).

Being successfully exploited in various applications in economics and finance, linear stochastic processes with fractional integration facilitate a thorough econometric analysis of time series exhibiting persistent shifts, also referred to as time series with long memory. Fractional time series models possess the properties of a covariance stationary process yet allow for hyperbolically declining autocorrelations. In other words, the rate of decay associated with the impulse response coefficients of a fractionally integrated process is considerably more gradual compared to the exponential rate imposed by representations from the ARMA class of models. Instead of relying solely on the first-order autocorrelations as in the Campbell and Viceira (2002) vector autoregression, modelling persistence requires the consideration of autocorrelations of much higher order.

Nonlinear models with infrequent regime switches are another rich class of time series models that are able to capture long memory, e.g., see Guidolin and Timmermann (2005). For the purposes of our application, however, regime switching suffers from some drawbacks relative to fractional integration. Nonlinear models rely on intensive numerical simulations and do not provide analytical insights into long-run effects; moreover, persistence depends on the entire model structure rather than relates to a single parameter. Hence, fractionally integrated processes are able to tackle the underlying problem in a more parsimonious and better interpretable and tractable way.

In this paper, we pursue an approach that explicitly considers implications of persistence for the variability of long-horizon asset returns. We regress excess stock and bond log-returns on a set of state variables. Contrary to the usual approach, we use a multivariate fractionally integrated process to describe the dynamic behaviour of asset return predictors. We derive explicit formulas for the term structure of the risk-return trade-off in our framework and illustrate our approach empirically.

The rest of the paper is organized as follows. We describe our approach to the modelling of asset return dynamics in Section 2 and obtain the long-term buy-and-hold optimal portfolio weights in Section 3. Theoretical formulas for the term structure of the risk-return trade-off implied by persistent risk factors are derived and analyzed in Section 4. We discuss the dataset and model estimation procedure in Sections 5 and 6, respectively. Implications for long-term portfolio choice are provided in Section 7, and Section 8 concludes.

2 Model

As Campbell and Viceira (2005), we consider time-varying expected returns that are linearly related to a set of state variables. Let y_t be the vector of excess log-returns on stocks and bonds over the 3-month T-Bill rate. We obtain the state m-vector x_t as the difference between the vector of forecasting variables and a vector of constants μ, and let ϵ_t denote the m-vector of innovations to the vector x_t. We model asset log-returns as

$$y_{t+1} = \mu_0 + Bx_t + G\epsilon_{t+1} + \eta_{t+1} \tag{1}$$

with μ_0 denoting the 2-vector of intercepts, B and G $(2 \times m)$ matrices of coefficients, and η_{t+1} the 2-dimensional risk of stocks and bonds that is uncorrelated with innovations to x_t. We denote the covariance matrix $\mathsf{E}(\eta_t \eta_t')$ by Σ_η, and the $(m \times m)$ covariance matrix of ϵ_t by Σ.

The first element of the state variables vector is the real log-return on 3-month T-Bills, $r_t = h'(x_t + \mu)$, with $h = \begin{pmatrix} 1 & \mathbf{0}' \end{pmatrix}'$. Other predictive variables are the log nominal T-Bill rate, the log price-earnings ratio, the credit spread, and the yield spread. It is well-known from empirical time series literature that each of these variables has strong positive autocorrelations pointing to possible nonstationarity. Stambaugh (1999) and Campbell and Yogo (2005) study issues regarding the estimation of the B matrix if x_t is (close to) $I(1)$. The effects of nonstationary x_t on the term structure of risk and on long-term portfolio decisions have not been considered so far.

We consider two alternative representations for the dynamic behaviour of the state variables. Under the first assumption, they are generated by a stationary first-order VAR process as in Campbell and Viceira (2005),

$$x_t = Ax_{t-1} + \epsilon_t \tag{2}$$

with A an $(m \times m)$ matrix. The joint model for both y_t and x_t is a restricted first-order VAR, in which lagged asset returns, y_{t-1}, predict neither level returns nor state variables. From the empirical results in Campbell and Viceira (2005) and others, this restriction is not important. We impose the restriction in order to facilitate the analysis for the fractionally integrated model in a framework where we can compare results.

As indicated above, the stationarity assumption underlying (2) may be too restrictive. This paper pursues a more general assumption that all elements of x_t are fractionally integrated. We describe the dynamics of the state vector with

$$(I - AL)\Delta(L)x_t = \epsilon_t \tag{3}$$

where the diagonal matrix $\Delta(L)$ controls the order of integration,

$$\Delta(L) = \begin{pmatrix} (1-L)^{d_1} & & \\ & \ddots & \\ & & (1-L)^{d_m} \end{pmatrix}$$

In this model, each state variable may have a different order of fractional integration d_i, and state variables do not fractionally cointegrate among themselves. We make the usual assumption that $0 \leq d_i \leq 1$. After applying the fractional filter $\Delta(L)$, the remaining short term dynamics is described by a first-order VAR process with parameter matrix A. In case that all d_i's are equal to zero, the model reduces to the stationary first-order VAR in (2).

By construction, expected asset returns must cointegrate with the predicting variables. If $d_{\max}$ is the highest order of fractional integration among the state variables predicting the j^{th} element of y_t, then the j^{th} element of y_t will be $I(d_{max})$. Expected returns are as persistent as the state variables in this model. In the empirical section, we estimate d_i's using both semiparametric methods and maximum likelihood in low-order ARFIMA models.

Inspired by the fractional VECM representation in Davidson (2002), we can rewrite equations (1) and (3) as the following system

$$\left\{ I - \begin{pmatrix} \mathbf{0}_{(2\times 2)} & B \\ \mathbf{0}_{(m\times 2)} & A \end{pmatrix} L + \alpha\beta'(L) \left[\widetilde{\Delta}(L)^{-1} - I \right] \right\} \widetilde{\Delta}(L) \begin{pmatrix} y_t - \mu_0 \\ x_t \end{pmatrix} = \begin{pmatrix} G\epsilon_t + \eta_t \\ \epsilon_t \end{pmatrix} \quad (4)$$

where

$$\widetilde{\Delta}(L) = \begin{pmatrix} (1-L)^{d_{y1}} & 0 & \mathbf{0}' \\ 0 & (1-L)^{d_{y2}} & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & \Delta(L) \end{pmatrix}$$

Matrices

$$\begin{aligned} \alpha &= \begin{pmatrix} I_{(2\times 2)} \\ \mathbf{0}_{(m\times 2)} \end{pmatrix} \\ \beta'(L) &= \begin{pmatrix} I_{(2\times 2)} & -BL \end{pmatrix} \end{aligned}$$

represent the error correction and cointegrating relations, respectively. Note that the representation in (4) deviates from the fractional VECM representation in Davidson (2002). The cointegration relation in (1) is a noncontemporaneous one as is also reflected by the lag operator in the cointegrating matrix $\beta(L)$.

3 Portfolio Implications

In this section, we present the optimal mean-variance portfolio choice for a buy-and-hold investor with equities, bonds, and cash as her financial instruments. Campbell and Viceira (2002) formulate the mean-variance problem for an investor with an investment horizon of k periods as

$$\max_{\alpha_t^{(k)}} \ \ln \mathsf{E}_t \left[1 + R_{p,t+k}^{(k)} \right] - \tfrac{1}{2}\gamma\sigma_p^2(k) \tag{5}$$

where $R_{p,t+k}^{(k)}$ is the real cumulative return on the asset portfolio from t to $t+k$, $\sigma_p^2(k)$ is the conditional variance of k-period cumulative log-returns, and $\alpha_t^{(k)}$ denotes the vector of portfolio weights of equities and bonds. This formulation of the mean-variance problem is equivalent to maximizing power utility of real wealth over a k-period horizon. The investor chooses her optimal portfolio at the beginning of the first period and holds it for k periods without rebalancing. Although this is a static framework, it enables the investor to benefit from time diversification properties of the assets.

To obtain a closed-form solution for the optimal portfolio choice, Campbell and Viceira (2002) approximate the expectation of the simple return in (5) and obtain an equivalent maximization problem

$$\max_{\alpha_t^{(k)}} \ \mathsf{E}_t \left[r_{p,t+k}^{(k)} \right] + \tfrac{1}{2}\sigma_p^2(k) - \tfrac{1}{2}\gamma\sigma_p^2(k) \tag{6}$$

where $r \equiv \ln(1+R)$, and $r_{p,t+k}^{(k)} = \sum_{\ell=1}^{k} r_{p,t+\ell}$ is the cumulative logarithmic return on the portfolio. Using log-linear approximations, the cumulative portfolio log-return is rewritten as a function of the cumulative log-return on the real T-Bill, $r_{t+k}^{(k)}$, and the cumulative excess log-return on stocks and bonds, $y_{t+k}^{(k)}$,

$$\begin{aligned} r_{p,t+k}^{(k)} &= \ln\left(\exp\left(r_{t+k}^{(k)} \right) + \left(\alpha_t^{(k)} \right)' \exp\left(y_{t+k}^{(k)} \right) \right) \\ &\approx r_{t+k}^{(k)} + \left(\alpha_t^{(k)} \right)' y_{t+k}^{(k)} + \tfrac{1}{2} \left(\alpha_t^{(k)} \right)' \sigma_y^2(k) - \tfrac{1}{2} \left(\alpha_t^{(k)} \right)' \Sigma_{yy}(k)\, \alpha_t^{(k)} \end{aligned} \tag{7}$$

where

$$\begin{aligned} \Sigma_{yy}(k) &= \mathrm{Var}_t \left(y_{t+k}^{(k)} \right) \\ \sigma_y^2(k) &= \mathrm{diag}\left\{ \Sigma_{yy}(k) \right\} \end{aligned}$$

Noting that the real T-bill rate, r_t, is not a riskless return for a long-term investor, the conditional portfolio variance follows as

$$\sigma_p^2(k) = \sigma_r^2(k) + \left(\alpha_t^{(k)} \right)' \Sigma_{yy}(k)\, \alpha_t^{(k)} + 2 \left(\alpha_t^{(k)} \right)' \sigma_{yr}(k) \tag{8}$$

where $\sigma_r^2(k)$ is the conditional variance of $r_{t+k}^{(k)}$, and $\sigma_{yr}(k)$ is the conditional covariance vector of excess log-returns with the return on real 3-month T-Bills. Substituting (7) and (8) in the mean-variance problem (6) leads to a quadratic optimization problem with the solution

$$\alpha_t^{(k)} = \frac{1}{\gamma}\, \Sigma_{yy}^{-1}(k) \left(\mu_t(k) + \tfrac{1}{2}\sigma_y^2(k)\right) - \left(1 - \frac{1}{\gamma}\right)\, \Sigma_{yy}^{-1}(k)\, \sigma_{yr}(k) \tag{9}$$

where $\mu_t(k)$ is a vector of expected excess returns over a k-period horizon. Explicit expressions for the terms $\mu_t(k)$, $\Sigma_{yy}(k)$, and $\sigma_{yr}(k)$ are derived in the next section.

As is well-known, the optimal solution has two components: the speculative demand

$$\pi_t^{(k)} = \frac{1}{\gamma}\, \Sigma_{yy}^{-1}(k) \left(\mu_t(k) + \tfrac{1}{2}\sigma_y^2(k)\right)$$

and the hedging demand

$$\left(1 - \frac{1}{\gamma}\right) \rho^{(k)}$$

where

$$\rho^{(k)} = -\Sigma_{yy}^{-1}(k)\, \sigma_{yr}(k) \tag{10}$$

refers to the so-called global minimum variance portfolio. It denotes the hedging demand for an infinitely risk-averse investor with $\gamma \to \infty$. Note that the hedging demand is independent of the data realization vector up to time t while the opposite holds true for the speculative demand $\pi_t^{(k)}$ that contains the vector of expected excess returns, $\mu_t(k)$. An unconditional version of π_t^k, $\pi^{(k)}$, that one may call long-term speculative demand, can be obtained if the first unconditional moments of the expected returns exist, which is generally not the case for nonstationary variables.

In order to avoid data sample dependency in our results, we analyze solely the global minimum variance portfolio $\rho^{(k)}$ in the empirical section.

4 Term Structure of Risk and Return

To derive conditional moments of the k-period cumulative returns, we rewrite the model for x_t in (3) as an infinite MA process[2]

$$x_t = \sum_{j=0}^{\infty} \Theta_j \epsilon_{t-j} \tag{11}$$

with $\Theta_0 = I$,

$$\Theta_{j+1} = \Theta_j A + \Delta_{j+1}, \qquad j \geq 0$$

and $\Delta_0 = I$,

$$\Delta_{j+1} = \begin{pmatrix} \frac{j+d_1}{j+1} & & \\ & \ddots & \\ & & \frac{j+d_m}{j+1} \end{pmatrix} \Delta_j, \qquad j \geq 0$$

Note that all Δ_j terms for $j > 0$ vanish under the VAR(1) representation with all d_i's being equal to zero. By substituting (11) into (1), we obtain the implied process for excess returns as

$$y_t = \mu_0 + B \sum_{j=0}^{\infty} \Theta_j \epsilon_{t-j-1} + G\epsilon_t + \eta_t \tag{12}$$

Combining (11) and (12), the joint process for $z_t = \begin{pmatrix} y_t & x_t' \end{pmatrix}'$ follows as

$$z_t = c + \sum_{j=0}^{\infty} C_j \nu_{t-j}$$

where $c = \begin{pmatrix} \mu_0' & \mathbf{0}' \end{pmatrix}'$ is a vector of length $m+2$, $\nu_t = \begin{pmatrix} \eta_t' & \epsilon_t' \end{pmatrix}'$ has covariance matrix

$$\Omega = \begin{pmatrix} \Sigma_\eta & \mathbf{0}' \\ \mathbf{0} & \Sigma \end{pmatrix}$$

[2]Strictly speaking, $(1-L)^{d_i}$ is invertible only for $d_i < 0.5,\ i = 1, \ldots, m$. For $0.5 \leq d_i \leq 1$, we can, nevertheless, always write $(1-L)^{d_i} = (1-L)^{\tilde{d}_i}(1-L)$ where $\tilde{d}_i \leq 0$; thus, the i^{th} element of x_t in (11) can be adjusted as

$$x_{it} = x_{i,t-1} + \sum_{j=0}^{\infty} \Theta_j^*(i)\, \epsilon_{i,t-j} = \cdots = x_{i0} + \sum_{j=0}^{\infty} \widetilde{\Theta}_j(i)\, \epsilon_{i,t-j}$$

with $\Theta_j(i)$ denoting the i^{th} column of Θ_j. Without loss of generality, we therefore assume in the following that $0 \leq d_i < 0.5,\ i = 1, \ldots, m$. Further note that the actual construction of the term structure of risk exploits only a finite number of the impulse responses Θ_j.

and coefficient matrices C_j are given by

$$C_0 = \begin{pmatrix} I_{(2\times 2)} & G \\ \mathbf{0}_{(m\times 2)} & I \end{pmatrix}, \qquad C_j = \begin{pmatrix} \mathbf{0}_{(2\times 2)} & B\Theta_{j-1} \\ \mathbf{0}_{(m\times 2)} & \Theta_j \end{pmatrix} \text{ for } j > 0 \tag{13}$$

We are interested in the mean and covariance properties of the cumulative process

$$z_{t+k}^{(k)} = \sum_{\ell=1}^{k} z_{t+\ell} \tag{14}$$

for various fixed values of k. The first two elements of $z_{t+k}^{(k)}$ contain average expected excess returns, so that

$$\mu_t(k) = \frac{1}{k}\sum_{\ell=1}^{k} \mathsf{E}_t[y_{t+\ell}] \tag{15}$$

The ℓ-period ahead forecast of the joint process z_t is given by

$$\mathsf{E}_t[z_{t+\ell}] = c + \sum_{i=\ell}^{\infty} C_i \nu_{t+\ell-i} = c + \sum_{i=0}^{\infty} C_{i+\ell}\nu_{t-i}$$

and the ℓ-period innovation follows as

$$z_{t+\ell} - \mathsf{E}_t[z_{t+\ell}] = \sum_{i=0}^{\ell-1} C_i \nu_{t+\ell-i} = \sum_{i=1}^{\ell} C_{\ell-i}\nu_{t+i}$$

Consequently, we obtain the innovation in the k-period cumulative process

$$\begin{aligned} z_{t+k}^{(k)} - \mathsf{E}_t\left[z_{t+k}^{(k)}\right] &= \sum_{\ell=1}^{k}\left(z_{t+\ell} - \mathsf{E}_t[z_{t+\ell}]\right) \\ &= \sum_{\ell=1}^{k}\sum_{i=1}^{\ell} C_{\ell-i}\nu_{t+i} \\ &= \sum_{\ell=1}^{k}\left(\sum_{i=0}^{k-\ell} C_i\right)\nu_{t+\ell} \end{aligned}$$

Finally, we scale the k-period conditional covariance matrix

$$\begin{aligned} V_k &= \frac{1}{k}\sum_{\ell=1}^{k}\left(\sum_{i=0}^{k-\ell} C_i\right)\Omega\left(\sum_{j=0}^{k-\ell} C_j'\right) \\ &= \frac{1}{k}\sum_{\ell=1}^{k}\left(\sum_{i=0}^{\ell-1} C_i\right)\Omega\left(\sum_{j=0}^{\ell-1} C_j'\right) \end{aligned} \tag{16}$$

by the horizon k to compute per period variances and covariances.

In order to pursue the long-term portfolio implications of persistent risk factors, we explore in more detail the top-left element of V_k, namely terms $\Sigma_{yy}(k)$ and $\sigma_{yr}(k)$. We define partial sums

$$\Xi_j = \sum_{i=0}^{j} \Theta_i \tag{17}$$

and substitute them into the cumulative sums $\sum C_i$ in (16). We end up with the following insightful formulas

$$\begin{aligned} \Sigma_{yy}(k) &= \Sigma_\eta + G\Sigma G' + G\Sigma \left(\frac{1}{k} \sum_{\ell=0}^{k-2} \Xi'_\ell \right) B' + B \left(\frac{1}{k} \sum_{\ell=0}^{k-2} \Xi_\ell \right) \Sigma G' + B \left(\frac{1}{k} \sum_{\ell=0}^{k-2} \Xi_\ell \Sigma \Xi'_\ell \right) B' \\ \sigma_{yr}(k) &= \Sigma G \left(\frac{1}{k} \sum_{\ell=0}^{k-1} \Xi_\ell \right) h + B \left(\frac{1}{k} \sum_{\ell=0}^{k-2} \Xi_\ell \Sigma \Xi'_{\ell+1} \right) h \end{aligned} \tag{18}$$

The conditional long-term covariance matrix of excess asset returns consists of three components. $\Sigma_\eta + G\Sigma G'$ represents the risk of unpredictable excess returns that is independent of the time horizon. The second component includes all terms with $\left(\frac{1}{k} \sum_{\ell=0}^{k-2} \Xi_\ell \right)$ and captures the contemporaneous covariances with state variables. These covariances are usually small and negative: an increase in expected returns raises the discount factor and reduces, therefore, the current price in the present value formula. As a result, the contemporaneous correlation between realized returns and shocks to expected returns is negative in most cases. Finally, the third component explains the long-term risk of expected returns caused by the variability in the return forecasting variables. As an always positive quadratic form, the covariance matrix of expected returns increases the long-term variance relative to the one-period component $\Sigma_\eta + G\Sigma G'$. The magnitude depends on the behaviour of $\left(\frac{1}{k} \sum_{\ell=0}^{k-2} \Xi_\ell \Sigma \Xi'_\ell \right)$ with increasing k. Since the k-period conditional covariance of excess log-returns with the real T-Bill return, $\sigma_{yr}(k)$, depends on the behaviour of $\left(\frac{1}{k} \sum_{\ell=0}^{k-2} \Xi_\ell \Sigma \Xi'_\ell \right)$ as well, the behaviour of the global minimum variance portfolio $\rho^{(k)}$ with respect to an increasing time horizon k crucially depends on the matrix $\left(\frac{1}{k} \sum_{\ell=0}^{k-2} \Xi_\ell \Sigma \Xi'_\ell \right)$.

In case of a stationary VAR process, the impulse responses Θ_j converge to zero at an exponential rate; hence, the average cumulative impulse responses $\frac{1}{k} \sum \Xi_j$ converge to a constant as $k \to \infty$. For the same reason, if $\Xi_j \sim O(1)$, then $\frac{1}{k} \sum \Xi_j \Sigma \Xi'_j$ converges to a constant as well. On the contrary, if the i^{th} element of the state vector, x_{it}, is fractionally integrated with $0 < d_i \leq 1$, the $(i,i)^{th}$ element of the impulse responses Θ_j will be of order $j^{d_i - 1}$ for large j, and the corresponding element of the partial sums Ξ_j will be of order j^{d_i}. Since $\Xi_j \leq \Xi_k,\ j \leq k$, the $(i,l)^{th}$ element in $\frac{1}{k} \sum \Xi_j$ will be $O(k^{d_i})$ if $d_l \leq d_i$ whereas the $(i,l)^{th}$ element in $\frac{1}{k} \sum \Xi_j \Sigma \Xi_j$ will be $O(k^{d_i + d_l})$. We conclude that the long-term portfolio decisions are dominated by the long memory properties of the state variable that has the highest order of integration from all state variables entering the prediction equation.

5 Data

To calibrate our modelling approach to empirical dynamics of asset returns, we examine quarterly US post-war data on three asset classes - equities, Treasury bonds, and cash - for the period 1952 Q1 – 2004 Q2. The equity price series for S&P 500 Composite Index is obtained from the Yahoo!Finance monthly data;[3] annual dividends and earnings on the S&P 500 are constructed from the monthly dataset of Robert Shiller;[4] furthermore, we download the 3-month T-Bill, 10-year Constant Maturity Yield, Moody's Seasoned AAA Corporate Bond Yield, and non-seasonally adjusted Consumer Price Index series from the Economic Data - FRED website.[5]

We consider asset return series in logarithms (or continuously compounded) and following usual practice, cash as the benchmark asset. We approximate the real return on cash by the real return on 3-month T-Bills. Excess stock returns (including dividends) are computed with respect to the return on 3-month T-Bills, a measure of the short-term interest rate. Finally, we use the log-linear approximation technique described in Campbell, Lo, and MacKinlay (1997, ch 10) and construct the 10-year bond return series as

$$r_{n,t+1} = \frac{1}{4}\, \mathcal{Y}_{n-1,t+1} - D_{n,t}(\mathcal{Y}_{n-1,t+1} - \mathcal{Y}_{n,t}) \tag{19}$$

where $n = 10$ is bond maturity, the log bond yield $\mathcal{Y}_{nt} = \log(1 + Y_{nt})$, and D_{nt} is bond duration given by

$$D_{nt} = \frac{1 - (1 + Y_{nt})^{-n}}{1 - (1 + Y_{nt})^{-1}}$$

We approximate $\mathcal{Y}_{n-1,t+1}$ by $\mathcal{Y}_{n,t+1}$, and the 1/4 in (19) adjusts the formula for quarterly data. Again, excess bond returns are computed with respect to the return on 3-month T-Bills.

We consider four return forecasting variables identified by previous empirical research - the log nominal short-term interest rate, the log price-earnings ratio, the credit spread, and the yield spread. Log price-earnings ratio is the difference between log equity price index and log quarterly earnings on the index; credit spread is the difference between log

[3] http://finance.yahoo.com/

[4] http://www.econ.yale.edu/~shiller/data.htm

[5] http://research.stlouisfed.org/fred2/

corporate AAA yield and log long-term nominal interest rate; yield spread is the difference between log long-term and log short-term nominal interest rates.

Table 1 shows post-war sample means and standard deviations of variables included in the model. Except for the log price-earnings ratio, the sample statistics are annualized percentages; moreover, mean log-returns are adjusted by adding one half of their variance so that they reflect mean gross returns. US Treasury bills offer a low average real return of 1.20% per year along with low volatility. Average annual stock excess return of 7.14% is very high compared to 10-year Treasury bonds with 1.08% per year. Since stock return volatility is about two times higher than bond return volatility (15.81% vs. 7.81%, respectively), the resulting Sharpe ratio is about 3.25 times higher for stocks than for bonds. In nominal terms, US Treasury bills pay on average 4.95% annually; average credit spread and yield spread are 0.7% and 1.32%, respectively.

6 Model Estimation

Noting that the fractional first-order VAR model in (3) can be rewritten as

$$(I - AL)u_t = \epsilon_t \tag{20}$$

with the fractionally differenced vector

$$u_t = \Delta(L)x_t \tag{21}$$

we estimate the dynamics of state variables in two steps. First, we univariately estimate d_i for each state variable. Second, the estimate of the parameter matrix A is based on $\hat{u}_t = \widehat{\Delta}(L)\hat{x}_t$. To obtain $\hat{x}_t$, we subtract the vector of sample averages from the state variables vector.

As discussed in the previous section, long memory properties of state variables have substantial impact on long-term portfolio choice. We exploit both parametric and semi-parametric estimation techniques in order to determine d_i's.[6] The parametric exact maximum likelihood estimator requires a complete specification of both the short and the long memory component in the univariate model. For this purpose, we consider the class of ARFIMA(p, d, q) models[7]

$$\alpha(L)(1 - L)^{d_i}x_{it} = \beta(L)e_t \tag{22}$$

[6] All results are generated with the Arfima package (version 1.01) for Ox. See Doornik (2002) for general reference; computational and technical details of estimation procedures for models with fractional integration can be found in Doornik and Ooms (2001, 2003).

[7] For notational convenience, we suppress subscript i in the lag polynomials.

where $\alpha(L) = 1 - \alpha_1 L - \cdots - \alpha_p L^p$ and $\beta(L) = 1 + \beta_1 L + \cdots + \beta_q L^q$ denote the AR and MA polynomials of order p and q, respectively. It is well-known that the accuracy of the estimator of d_i is highly sensitive to the selection of appropriate AR and MA orders. Therefore, we employ the Akaike (AIC), Schwarz (SC), and Hannan-Quin (HQ) order selection criteria to compare all possible representations with $0 \leq p \leq 3,\ 0 \leq q \leq 3$, and $p + q < 6$, including submodels. The random walk type ARIMA$(p, 1, q)$ models are taken into account as well.[8]

Despite its merits, the exact maximum likelihood estimation of the long memory parameter d has several drawbacks. First, it is highly sensitive to the estimation of the mean, which is complicated already in the stationary long memory framework. Second, it is derived under the assumption of homoskedastic errors that is likely to be violated in the present case. Third, a misspecification of the short memory order may increase estimation bias. Therefore, we also consider the Gaussian semiparametric estimator introduced in Künsch (1987) and intensively studied in Robinson and Henry (1999). The estimator is derived from the maximum likelihood estimator as an approximation of the likelihood function in frequency domain. The long memory parameter estimate depends only on low frequency periodogram estimates and spectral density behaviour. Hence, only $m < n/2$ lowest Fourier frequencies are taken into account, and any impact of short memory dynamics is asymptotically eliminated. Robinson and Henry (1999) show that the Gaussian semiparametric estimator is asymptotically normally distributed even under conditional heteroskedasticity that itself exhibits long range dependence. In finite samples, the number of Fourier frequencies, m, may matter. In order to obtain asymptotically valid inference, m has to grow with the sample size at an appropriate rate, see Robinson and Henry (1999), equations (3.8) and (3.9).

Tables 2 to 6 report semiparametric estimates for each state variable with $m = n^{0.45}, n^{0.5}, n^{0.55}$ implying 11, 14, and 19 Fourier frequencies, respectively,[9] and three best fitting parametric ARFIMA representations identified by the Hannan-Quinn (HQ) criterion. Not surprisingly, the estimates of d for each series vary substantially; nevertheless, nonstationarity is indicated for both nominal and real T-Bill rate and the price-earnings ratio series. Since most of selected ARFIMA models include higher order AR or MA poly-

[8] Since the estimation methods require stationary series, we difference each element of x_t prior to estimation. All parameter values for d considered in the tables and graphs below refer to the level series, however.

[9] Note that the last two exponent values have to be modified for larger samples to fulfil the asymptotic requirements.

nomials, the fractional first-order VAR in (3) may not be able to capture the short memory dynamics appropriately. However, given our theoretical considerations in Section 4, this drawback does not matter for long-term implications.

The estimates in Tables 2 to 6 suggest a range for the degree of long memory in each state variable, and we pursue a rather pragmatic approach to pick a value for each series. For the real 3-month T-Bill rate, we set $\hat{d}_1 = 0.7$. Our results suggest that the nominal interest rate is slightly more persistent, and we set $\hat{d}_2 = 0.8$ for the nominal 3-month T-Bill rate. For the log price-earnings ratio, we adhere to the tight range estimated semi-parametricaly and choose $\hat{d}_3 = 0.75$. An estimate of 0.5 seems to be a reasonable choice for the credit spread. Given the broad range obtained for the yield spread, we assign $\hat{d}_5 = 0.3$ and, finally, arrive at an estimate of the vector of mutually unequal long memory parameters $\hat{d}_u = \begin{pmatrix} 0.7 & 0.8 & 0.75 & 0.5 & 0.3 \end{pmatrix}'$. Nevertheless, Sun and Phillips (2004) recently argue that short-term nominal and real interest rates are fractionally integrated of the same order. Furthermore, the highest order of fractional integration among state variables seems to have essential impact on the long-term portfolio choice as discussed in Section 4. Given the uncertainty about the estimates of individual d_i's, we consider an alternative set of long memory parameter values $\hat{d}_e = \begin{pmatrix} 0.75 & 0.75 & 0.75 & 0.5 & 0.3 \end{pmatrix}'$ where the order of fractional integration for the three most persistent state variables equals.

Table 7 reports regression estimates for the prediction equation in (1) where we impose zero coefficient restrictions on those state variables that do not significantly forecast excess asset returns, i.e., yield spread in the excess stock return equation and nominal return on T-Bills, price-earnings ratio, and credit spread in the excess bond return equation.

Fractional differences $\hat{u}_t$ in (21) are obtained by summing over the first t elements in the infinite VAR representation of the multivariate fractional operator $\widehat{\Delta}(L)$. In order to reduce the start-up problem, we extend the vector of state variables x_t by about two years of presample observations. Tables 8 to 10 contain the stationary first-order VAR estimation results for $\hat{u}_t$ computed with $\hat{d}_u$, $\hat{d}_e$, and zero d_i's, respectively. The latter case is included as a benchmark for comparison. The upper panel of each table provides an estimate of the parameter matrix A. Since persistence is filtered out by $\widehat{\Delta}(L)$, we observe much smaller diagonal elements for the two fractional VAR representations compared to Table 10. Lower panels describe residual covariance structure for the joint models of excess returns and state variables. More precisely, quarterly percent standard deviations are reported on the diagonal while residual cross-correlations are on the off-diagonal.

7 Term Structure Results

The term structure of the risk-return trade-off over 25 years implied by the model estimates from Tables 7 to 10 is illustrated in Figures 1 to 4 for real returns on stocks, 10-year bonds, and 3-month T-Bills. Figure 1 illustrates that the risk of stocks and T-Bills relative to long-term bonds, as measured by the ratios of annualized standard deviations, is affected by long memory. In the two-year horizon, stocks are about 1.6 times riskier than long-term bonds. In the long-run, the fractional model with the same highest order of fractional integration for the most persistent state variables implies that stocks become more than twice as risky as bonds in the long-run whereas stocks become more than three times as risky as bonds over the same time horizon under the stationarity assumption. On the other hand, T-Bills relative to long-term bonds bear indeed a very low risk in the short-run whereas, in the long-run, the risk level approaches 50% for fractional representations and reaches as high as 80% in the stationary case. Figure 2 points to a striking difference between the implications of the fractional and stationary framework for the correlation term structure between real returns on 10-year bonds and on 3-month T-Bills. Starting above 0.3, the correlation structure resulting from the fractional VAR estimates drops to as low as zero within the first year but subsequently grows to more than 0.9 over the 25-year horizon whereas the pattern generated by the stationary VAR estimates is considerably flatter, rising from 0.3 to 0.65 with the investment horizon. The strong positive correlation within the fractional framework suggests a one-factor term structure model in the long-run. Similarly, Figure 3 illustrates that the correlation structure between real returns on stocks and on 3-month T-Bills is affected by long memory as well. Fractional representations imply patterns that stay positive; stationary VAR estimates generate negative long-term correlations. Finally, correlation structures between stocks and bonds begin at 20% but quickly rise to 80% in the third year. The long-term gradual decline is about two times deeper for the stationary representation compared to the fractional ones.

To illustrate the long-term portfolio choice, we consider in Figures 5 to 7 compositions of the global minimum variance portfolio, i.e., the portfolio with the smallest variance in the efficient set. Given the double or triple risk of stocks compared to long-term bonds, stock portfolio weight in Figure 5 is negligible, ranging from about -10% in the short-run to about 7% in the long-run, regardless of the underlying model specification. Portfolio weights for 10-year bonds and 3-month T-Bills in the fractional framework result from the strong positive correlation between real returns on these securities. Since long-term bonds are more than twice as risky as T-Bills, the long-term investor wants to short-sell bonds up to 60% of her financial wealth and invest into T-Bills, see Figures 6 and 7. The long-term correlation between bonds and T-bills implied by the stationary VAR estimates is considerably lower so that respective portfolio weights are hardly affected. The long-term bond portfolio weight varies between -8% and slightly more than 10% over the 25-year horizon; after 10 years, the portfolio weight on T-Bills gradually declines, ending up at about 80% of investor's wealth at the end of the 25-year horizon.

8 Conclusion

Exploiting the class of stochastic processes with fractional integration, we have examined the effects of persistence in risk factors for long-term portfolio choice. Our theoretical results indicate that the long-run risk highly depends on the integration order of the most persistent series. Empirical findings suggest that due to the risk of persistent shocks to the return forecasting variables that may influence asset returns for many periods ahead, the attractiveness of risky assets for long-term investors decreases.

There are many issues left for further research. Precise estimation of long memory parameters, in particular, in multivariate framework, is both ambitious and important for correct inference on optimal portfolio choice. This problem points to the uncertainty about parameter values. A related topic is the reliability of our forecasts for long time horizons, i.e., how broad is the prediction interval for optimal portfolio weights over the 25-year investment horizon. Last but not least, our empirical results may be considerably dependent on the data sample. We expect to consider these issues in our future research.

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Summary Statistics

Table 1: Mean and standard deviation of returns and return forecasting variables

1952 Q1 – 2004 Q2	
Mean real return on 3-month T-Bills	1.197
Standard deviation of real return on 3-month T-Bills	0.978
Mean excess return on equity	7.144
Standard deviation of excess return on equity	15.809
Sharpe ratio of equity	0.452
Mean excess return on 10-year Treasury bonds	1.075
Standard deviation of excess return on 10-year Treasury bonds	7.814
Sharpe ratio of 10-year Treasury bonds	0.138
Mean nominal return on 3-month T-Bills	4.949
Standard deviation of nominal return on 3-month T-Bills	1.331
Mean log price-earnings ratio	2.729
Standard deviation of log price-earnings ratio	0.397
Mean credit spread	0.700
Standard deviation of credit spread	0.252
Mean yield spread	1.323
Standard deviation of yield spread	0.535

Notes: All variables except for log price-earnings ratio are annualized percentages.

Estimation of the Long Memory Parameter

Table 2: Real log-return on 3-month T-Bills

	AR-1	AR-2	AR-3	d	MA-1	MA-2	MA-3	$\log(L)$	AIC	SC	HQ
1952 Q1 – 2004 Q2											
GSP $n^{.45}$				0.653							
				(0.151)							
GSP $n^{.50}$				0.783							
				(0.134)							
GSP $n^{.55}$				0.729							
				(0.115)							
$(0, d, 3)$	-	-	-	0.681	-0.457	-	0.204	829.000	-7.857	-7.793	-7.831
				(0.122)	(0.122)		(0.069)				
$(3, 1, 1)$	-	-	0.188	1	-0.716	-	-	826.868	-7.846	-7.799	-7.827
			(0.078)		(0.054)						
$(2, d, 0)$	-0.593	-0.295	-	0.819	-	-	-	828.224	-7.850	-7.786	-7.824
	(0.099)	(0.088)		(0.088)							

Notes: Two estimators are used to determine the long memory parameter d: The Gaussian semi-parametric estimator (GSP) discussed in Robinson and Henry (1999) and the exact maximum likelihood estimator. The former employs Fourier frequencies corresponding to the floor of n^{α}, $\alpha \in \{0.45, 0.50, 0.55\}$. The latter requires full specification of the short memory dynamics; triples (p, d, q) denote the AR and the MA order, respectively. We consider full subset selection for $0 \leq p \leq 3$, $0 \leq q \leq 3$, including the random walk type ARIMA$(p, 1, q)$ specifications. Three best fitting representations selected by the Hannan-Quin (HQ) criterion are considered. The Akaike (AIC) and Schwarz (SC) criterions are mentioned for completeness. Standard errors stated in parentheses result from estimation for the stationary series of first differences. Reported d values for the level series are obtained by adding one.

Table 3: Nominal log-return on 3-month T-Bills

	AR-1	AR-2	AR-3	d	MA-1	MA-2	MA-3	$\log(L)$	AIC	SC	HQ
1952 Q1 – 2004 Q2											
GSP $n^{.45}$				0.642							
				(0.151)							
GSP $n^{.50}$				0.762							
				(0.134)							
GSP $n^{.55}$				0.877							
				(0.115)							
$(3, d, 0)$	-	-	0.313	0.814	-	-	-	999.5326	-9.4908	-9.4430	-9.4715
			(0.066)	(0.062)							
$(3, d, 1)$	-	-	0.356	0.675	0.208	-	-	1001.1949	-9.4971	-9.4333	-9.4713
			(0.069)	(0.084)	(0.097)						
$(0, d, 3)$	-	-	-	0.809	-	-	0.331	999.5129	-9.4906	-9.4428	-9.4713
				(0.060)			(0.070)				

For notes see Table 2.

Table 4: Log price-earnings ratio

	AR-1	AR-2	AR-3	d	MA-1	MA-2	MA-3	$\log(L)$	AIC	SC	HQ
1952 Q1 – 2004 Q2											
GSP $n^{.45}$				0.746							
				(0.151)							
GSP $n^{.50}$				0.765							
				(0.134)							
GSP $n^{.55}$				0.771							
				(0.115)							
$(1, d, 0)$	0.563	-	-	0.653	-	-	-	198.728	-1.864	-1.816	-1.845
	(0.170)			(0.169)							
$(1, 1, 0)$	0.244	-	-	1	-	-	-	196.243	-1.850	-1.818	-1.837
	(0.067)										
$(0, 1, 1)$	-	-	-	1	0.238	-	-	196.130	-1.849	-1.817	-1.836
					(0.064)						

For notes see Table 2.

Table 5: Credit spread

	AR-1	AR-2	AR-3	d	MA-1	MA-2	MA-3	$\log(L)$	AIC	SC	HQ
1952 Q1 – 2004 Q2											
GSP $n^{.45}$				0.467							
				(0.151)							
GSP $n^{.50}$				0.520							
				(0.134)							
GSP $n^{.55}$				0.406							
				(0.115)							
$(2, 1, 2)$	-0.268	0.695	-	1	-	-0.887	-	1262.069	-11.982	-11.918	-11.956
	(0.052)	(0.060)				(0.048)					
$(1, 1, 1)$	0.733	-	-	1	-0.973	-	-	1259.660	-11.968	-11.920	-11.949
	(0.071)				(0.036)						
$(1, d, 2)$	-0.960	-	-	0.488	1.248	0.324	-	1262.985	-11.981	-11.901	-11.949
	(0.039)			(0.084)	(0.100)	(0.095)					

For notes see Table 2.

Table 6: Yield spread

	AR-1	AR-2	AR-3	d	MA-1	MA-2	MA-3	$\log(L)$	AIC	SC	HQ
1952 Q1 – 2004 Q2											
GSP $n^{.45}$				0.157							
				(0.151)							
GSP $n^{.50}$				0.310							
				(0.134)							
GSP $n^{.55}$				0.492							
				(0.115)							
$(1, d, 3)$	0.688	-	-	0.071	-	-	0.319	1064.401	-10.099	-10.035	-10.073
	(0.232)			(0.265)			(0.082)				
$(3, d, 2)$	-	-	0.513	0.121	0.680	0.389	-	1064.212	-10.087	-10.008	-10.056
			(0.151)	(0.167)	(0.156)	(0.389)					
$(3, d, 1)$	-	-	0.269	0.491	0.320	-	-	1062.433	-10.080	-10.017	-10.055
			(0.269)	(0.089)	(0.088)						

For notes see Table 2.

Estimation of Excess Return Dynamics

Table 7: Prediction equation for excess asset log-returns

1952 Q1 – 2004 Q2							
	intercept	real log-return on T-Bills	nominal log-return on T-Bills	log PE ratio	credit spread	yield spread	R^2
excess stock	0.016	2.807	-4.245	-0.089	14.265	0	0.106
log-return	(0.005)	(1.182)	(1.009)	(0.020)	(5.196)	-	
excess bond	0.002	1.273	0	0	0	2.888	0.072
log-return	(0.003)	(0.543)	-	-	-	(0.993)	

Note: Standard errors in parenthesis.

Table 8: Fractional VAR(1) dynamics: Unequal d's

1952 Q1 – 2004 Q2						
	(1)	(2)	(3)	(4)	(5)	R^2
(1) real log-return on T-Bills ($\hat{d}_1$=0.70)	0.327	1.125	0.005	0.560	0.429	0.551
	(0.049)	(0.090)	(0.001)	(0.221)	(0.089)	
(2) nominal log-return on T-Bills ($\hat{d}_2$=0.80)	-0.136	-0.276	-0.001	-1.115	-0.226	0.076
	(0.065)	(0.120)	(0.001)	(0.295)	(0.118)	
(3) log PE ratio ($\hat{d}_3$=0.75)	-1.951	-8.764	0.429	23.695	-3.426	0.279
	(2.791)	(5.170)	(0.061)	(12.671)	(5.071)	
(4) credit spread ($\hat{d}_4$=0.50)	-0.010	-0.007	0.000	0.258	0.008	0.065
	(0.019)	(0.035)	(0.001)	(0.085)	(0.034)	
(5) yield spread ($\hat{d}_5$=0.30)	0.104	0.371	0.002	0.934	0.790	0.361
	(0.047)	(0.086)	(0.001)	(0.211)	(0.085)	

Quarterly percent standard deviations and cross-correlations of residuals							
	(a)	(b)	(1)	(2)	(3)	(4)	(5)
(a) excess stock log-return	7.490	0.140	0.274	-0.098	0.835	-0.084	-0.020
(b) excess bond log-return	0.140	3.764	0.292	-0.644	0.255	0.702	0.017
(1) real log-return on T-Bills ($\hat{d}_1$=0.70)	0.274	0.292	0.158	-0.286	0.352	0.134	0.130
(2) nominal log-return on T-Bills ($\hat{d}_2$=0.80)	-0.098	-0.644	-0.286	0.211	-0.248	-0.454	-0.749
(3) log PE ratio ($\hat{d}_3$=0.75)	0.834	0.255	0.352	-0.248	9.056	0.015	0.071
(4) credit spread ($\hat{d}_4$=0.50)	-0.084	0.702	0.134	-0.454	0.015	0.061	-0.009
(5) yield spread ($\hat{d}_5$=0.30)	-0.020	0.017	0.130	-0.749	0.071	-0.009	0.151

Note: Standard errors in parenthesis; orders of fractional differentiation indicated in parenthesis for each state variable.

Table 9: Fractional VAR(1) dynamics: Equal d's

1952 Q1 – 2004 Q2						
	(1)	(2)	(3)	(4)	(5)	R^2
(1) real log-return on T-Bills ($\hat{d}_1$=0.75)	0.244	1.144	0.006	0.595	0.471	0.542
	(0.049)	(0.093)	(0.001)	(0.225)	(0.092)	
(2) nominal log-return on T-Bills ($\hat{d}_2$=0.75)	-0.118	-0.239	-0.001	-1.141	-0.256	0.084
	(0.064)	(0.124)	(0.001)	(0.299)	(0.122)	
(3) log PE ratio ($\hat{d}_3$=0.75)	-2.017	-9.378	0.423	22.642	-4.100	0.280
	(2.757)	(5.297)	(0.062)	(12.816)	(5.238)	
(4) credit spread ($\hat{d}_4$=0.50)	-0.010	-0.005	0.000	0.261	0.009	0.065
	(0.019)	(0.036)	(0.001)	(0.086)	(0.035)	
(5) yield spread ($\hat{d}_5$=0.30)	0.100	0.383	0.002	0.957	0.808	0.362
	(0.047)	(0.088)	(0.001)	(0.214)	(0.087)	

Quarterly percent standard deviations and cross-correlations of residuals							
	(a)	(b)	(1)	(2)	(3)	(4)	(5)
(a) excess stock log-return	7.490	0.140	0.276	-0.098	0.835	-0.084	-0.020
(b) excess bond log-return	0.140	3.764	0.288	-0.641	0.257	0.701	0.015
(1) real log-return on T-Bills ($\hat{d}_1$=0.75)	0.276	0.288	0.158	-0.289	0.350	0.127	0.134
(2) nominal log-return on T-Bills ($\hat{d}_2$=0.75)	-0.098	-0.641	-0.289	0.211	-0.249	-0.450	-0.749
(3) log PE ratio ($\hat{d}_3$=0.75)	0.835	0.257	0.350	-0.249	9.049	0.015	0.074
(4) credit spread ($\hat{d}_4$=0.50)	-0.084	0.701	0.127	-0.450	0.015	0.061	-0.010
(5) yield spread ($\hat{d}_5$=0.30)	-0.020	0.015	0.134	-0.749	0.074	-0.010	0.151

Note: Standard errors in parenthesis; orders of fractional differentiation indicated in parenthesis for each state variable.

Table 10: Stationary VAR(1) dynamics

1952 Q1 – 2004 Q2						
	(1)	(2)	(3)	(4)	(5)	R^2
(1) real log-return on T-Bills	0.816	0.144	0.002	-0.601	0.014	0.807
	(0.035)	(0.031)	(0.001)	(0.161)	(0.065)	
(2) nominal log-return on T-Bills	-0.050	0.955	-0.000	-0.252	0.130	0.895
	(0.035)	(0.031)	(0.001)	(0.161)	(0.065)	
(3) log PE ratio	5.185	-3.536	0.882	27.774	-4.692	0.946
	(1.508)	(1.314)	(0.025)	(6.939)	(2.780)	
(4) credit spread	-0.006	0.027	0.001	0.780	0.026	0.777
	(0.010)	(0.008)	(0.001)	(0.045)	(0.018)	
(5) yield spread	0.011	0.025	-0.000	0.338	0.773	0.673
	(0.025)	(0.022)	(0.001)	(0.115)	(0.046)	

Quarterly percent standard deviations and cross-correlations of residuals							
	(a)	(b)	(1)	(2)	(3)	(4)	(5)
(a) excess stock log-return	7.490	0.140	0.243	-0.084	0.832	-0.095	-0.023
(b) excess bond log-return	0.140	3.764	0.272	-0.669	0.226	0.700	0.047
(1) real log-return on T-Bills	0.243	0.272	0.215	-0.351	0.192	0.046	0.239
(2) nominal log-return on T-Bills	-0.084	-0.669	-0.351	0.215	-0.182	-0.407	-0.766
(3) log PE ratio	0.832	0.226	0.192	-0.182	9.258	0.007	0.036
(4) credit spread	-0.095	0.700	0.046	-0.407	0.007	0.060	-0.046
(5) yield spread	-0.023	0.047	0.239	-0.766	0.036	-0.046	0.153

Note: Standard errors in parenthesis.

Implied Term Structure of Risk

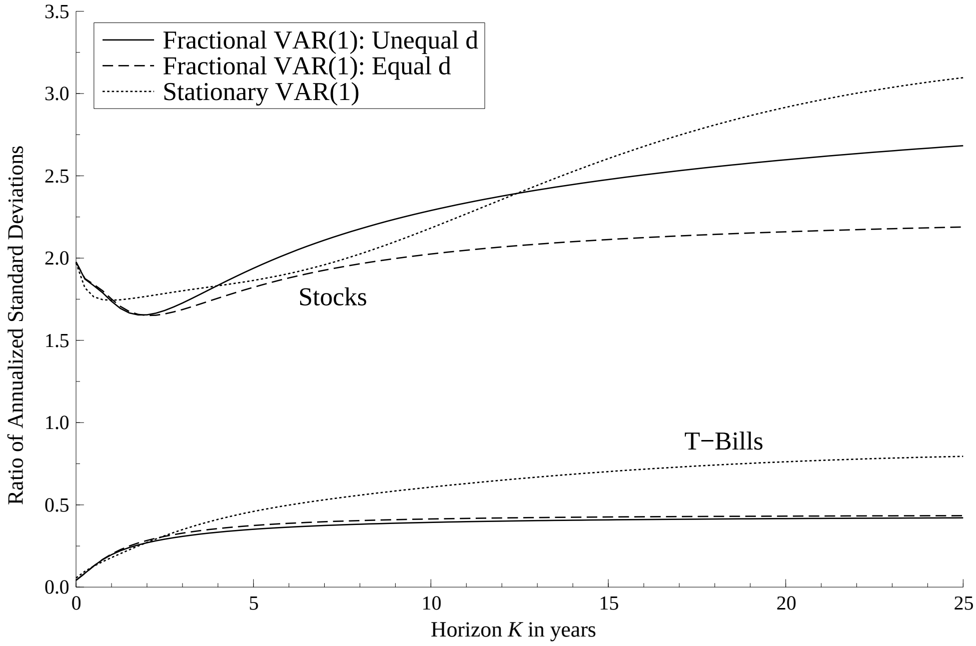

Figure 1: Term structure of risk for stocks and T-Bills relative to 10-year bonds

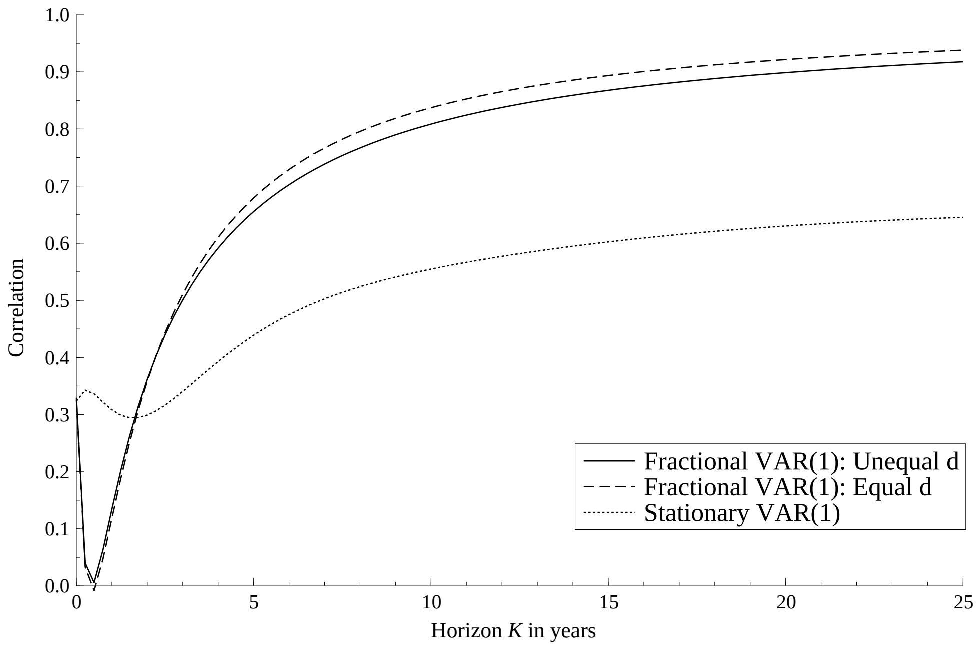

Figure 2: Term structure of correlation between real returns on 10-year bonds and on T-Bills

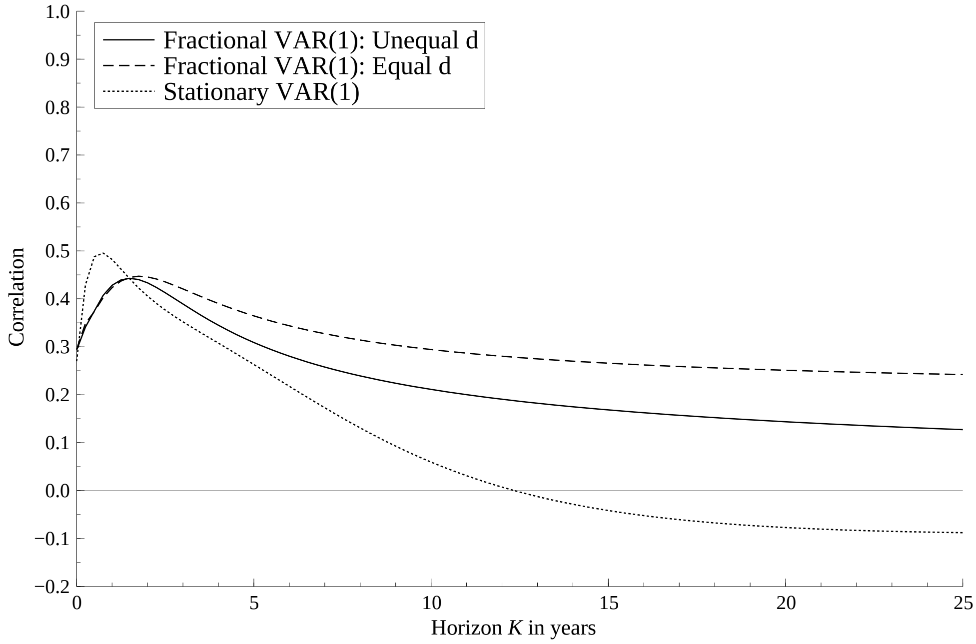

Figure 3: Term structure of correlation between real returns on stocks and on T-Bills

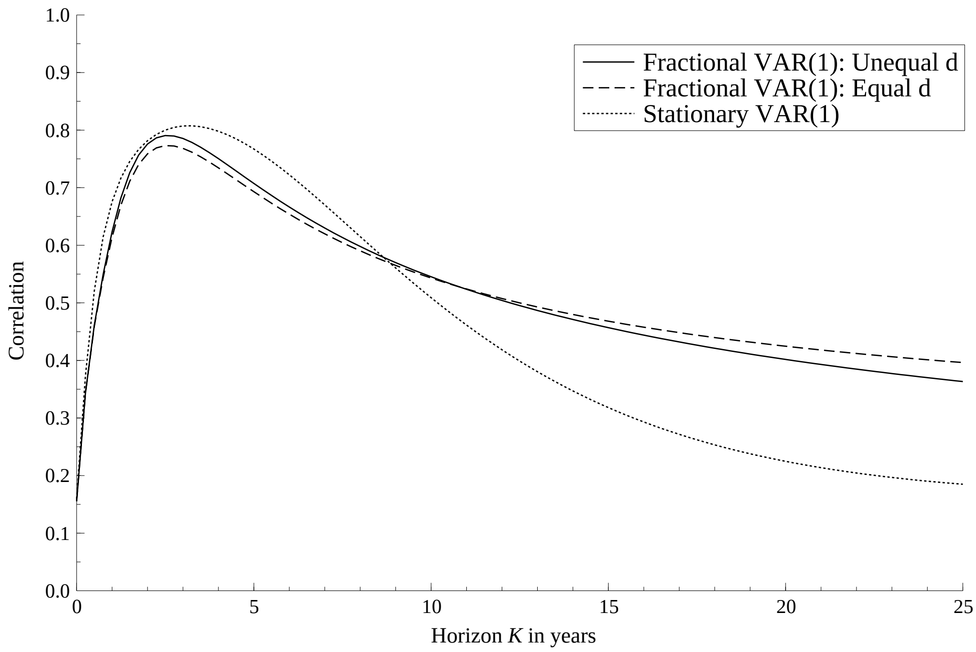

Figure 4: Term structure of correlation between real returns on stocks and on 10-year bonds

Global Minimum Variance Portfolio

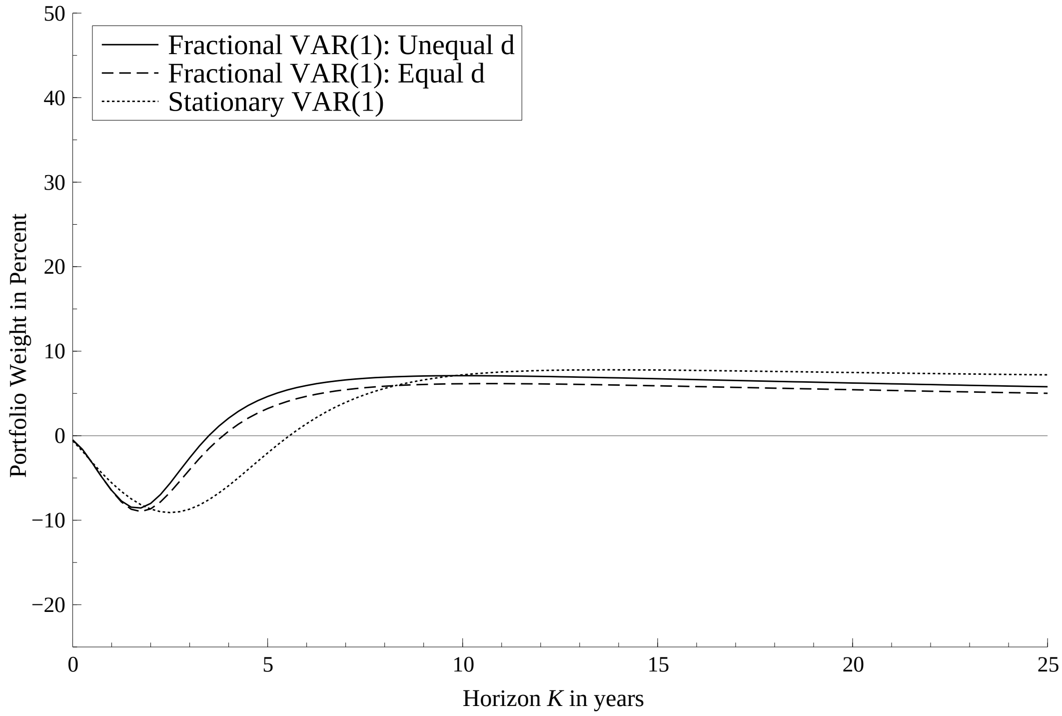

Figure 5: Portfolio weights for stocks

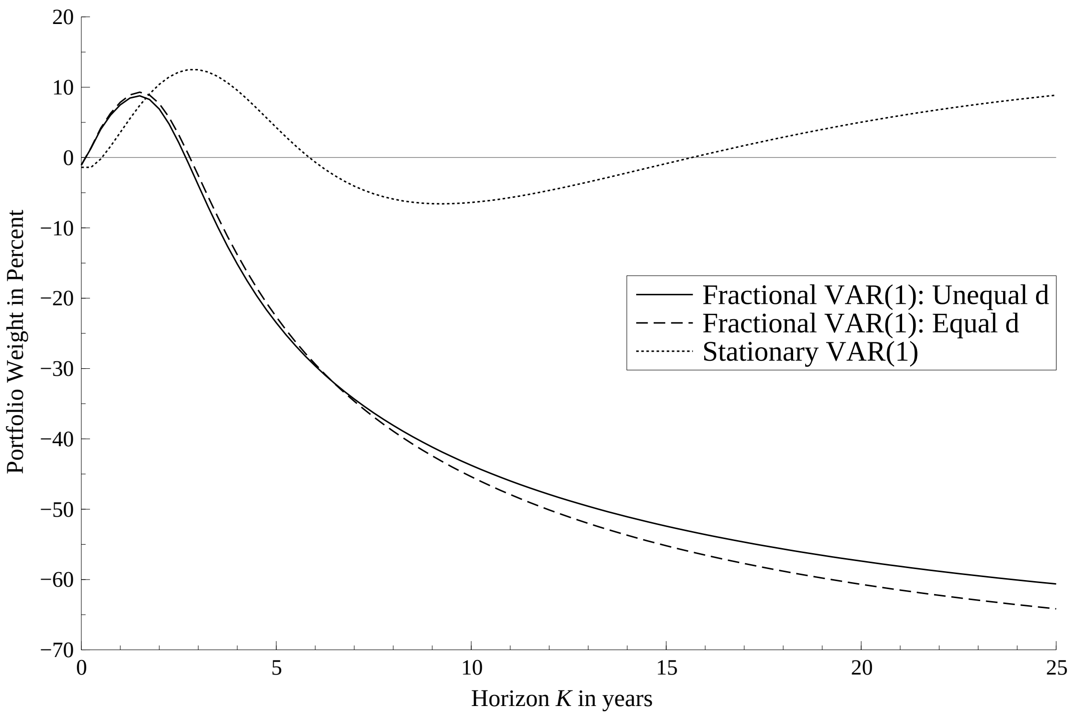

Figure 6: Portfolio weights for 10-year bonds

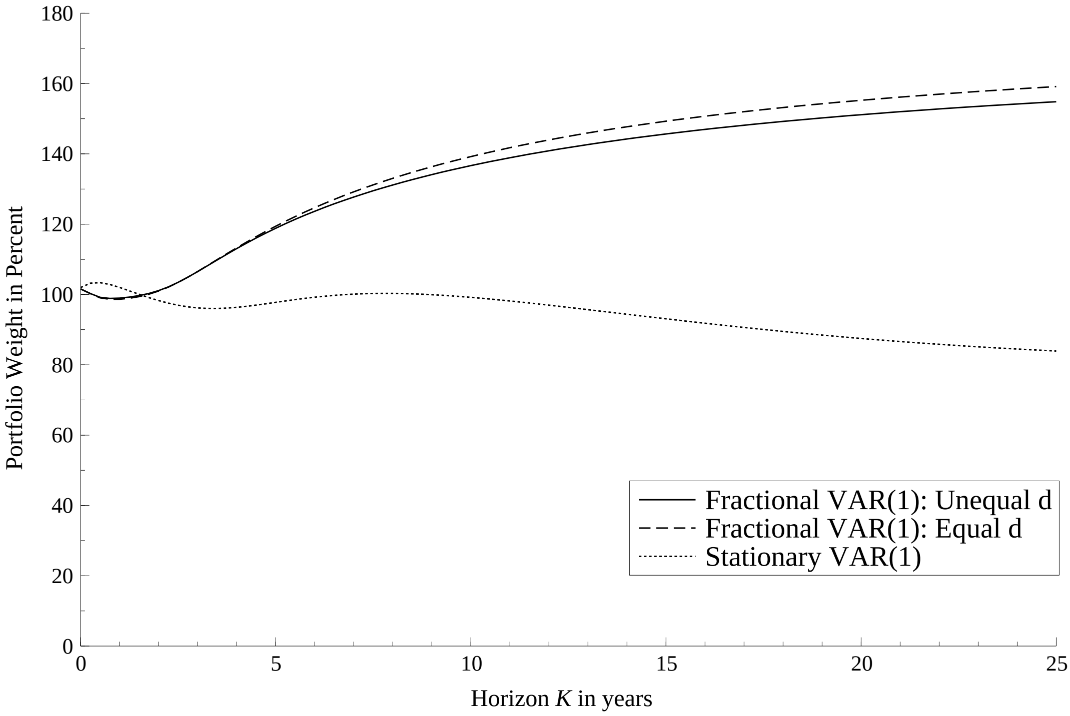

Figure 7: Portfolio weights for T-Bills