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STOCHASTIC PORTFOLIO SPECIFIC MORTALITY AND THE QUANTIFICATION OF MORTALITY BASIS RISK

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Abstract

The last decennium a vast literature on stochastic mortality models has been developed. However, these models are often not directly applicable to insurance portfolios because:

a) For insurers and pension funds it is more relevant to model mortality rates measured in insured amounts instead of measured in number of policies.
b) Often there is not enough insurance portfolio specific mortality data available to fit such stochastic mortality models reliably.

In practice, these issues are often solved by applying a (deterministic) portfolio experience factor to projected (stochastic) mortality rates of the whole country population. This factor is usually based on historical portfolio mortality rates, measured in amounts. However, it is reasonable to assume that this portfolio experience factor is also a stochastic variable. Therefore, in this paper a stochastic model is proposed for portfolio mortality experience. Adding this stochastic process to a stochastic country population mortality process leads to stochastic portfolio specific mortality rates, measured in insured amounts. The proposed stochastic process is applied to two insurance portfolios, and the impact on the Value at Risk for longevity risk is quantified. Furthermore, the model can be used to quantify the basis risk that remains when hedging portfolio specific mortality risk with instruments of which the payoff depends on population mortality rates. The conclusion is that adding this stochastic process can have a significant impact on the Value at Risk of an insurance portfolio and the hedge efficiency of a possible hedge, depending on the size of this portfolio.

Keywords: mortality risk, longevity risk, stochastic mortality models, portfolio specific mortality, Value at Risk, Economic Capital, Solvency 2, mortality basis risk, hedge efficiency, longevity hedging, Monte Carlo simulation, insurance companies, pension funds.

1. Introduction

In recent years there has been an increasing amount of attention of the insurance industry for the quantification of the risks that insurers are exposed to. Important drivers of this development are the increasing internal focus on risk measurement and risk management and the introduction of Solvency 2 (expected to be implemented around 2011).

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Solvency 2 will lead to a change in the regulatory required solvency capital for insurers. At this moment this capital requirement is a fixed percentage of the mathematical reserve or the risk capital. Under Solvency 2 the so-called Solvency Capital Requirement (SCR) will be risk-based, and market values of assets and liabilities will be the basis for these calculations.

Also for pension funds, a new solvency framework will be developed, either as part of Solvency 2 or as a separate project (usually named IORP 2). It is expected that the general principles will be similar as Solvency 2, meaning market valuation of assets and liabilities and risk-based solvency requirements.

An important risk to be quantified is mortality or longevity risk. Not only is this an important risk for most (life) insurers, the resulting solvency margin will also be part of the fair value reserve. Reason for this is that it is becoming best practice for the quantification of the Market Value Margin to apply a Cost of Capital rate to the solvency capital necessary to cover for unhedgeable risks.

There is a vast literature on stochastic modeling of mortality rates. Often used models are for example those of Lee and Carter (1992), Brouhns et al (2002), Renshaw and Haberman (2006), Cairns et al (2006a), Currie et al (2004) and Currie (2006). These models are generally tested on a long history of mortality rates for large country populations, such as the United Kingdom or the United States. However, the ultimate goal is to quantify the risks for specific insurance portfolios. And in practice there is often not enough insurance portfolio specific mortality data to fit such stochastic mortality models reliably, because:

- The historical period for which observed mortality rates for the insurance portfolio are available is usually shorter, often in a range of only 5 to 15 years.
- The number of people in an insurance portfolio is much less than the country population.

Also, for insurers it is more relevant to model mortality rates measured in insured amounts instead of measured in number of people, because in the end the insured amounts have to be paid by the insurer. Measuring mortality rates in insured amounts has two effects:

- Policyholders with higher insured amounts tend to have lower mortality rates[2]. So measuring mortality rates in insured amounts will generally lead to lower mortality rates.
- The standard deviation of the observations will increase. For example, the risk of an insurance portfolio with 100 males with average salaries will be lower then that of a portfolio with 99 males with average salaries and 1 billionaire.

So fitting the before mentioned stochastic mortality models to the limited mortality data of insurers, measured in insured amounts, will in many cases not lead to reliable results. In practice, this issue is often solved by applying a (deterministic) portfolio experience factor to projected (stochastic) mortality rates of the whole country population. However, it is reasonable to assume that this portfolio experience factor is also a stochastic variable.

In this paper a stochastic model is proposed for portfolio specific mortality experience. This stochastic process can be added to the stochastic country population mortality process, leading to

[2] See for example CMI (2004).

stochastic portfolio specific mortality rates. The process is, amongst others, based on historical mortality rates measured in insured amounts, but can also be used when only historical mortality rates measured in number of policies are available.

The model can be used to quantify portfolio specific mortality or longevity risks for the purpose of determining the Value at Risk (VaR) or SCR, which is often also the basis for the quantification of the Market Value Margin. Also, it gives more insight in the basis risk when hedging portfolio mortality or longevity risks with hedge instruments of which the payoff depends on country population mortality. The market for mortality or longevity derivatives is emerging (see J.P. Morgan (2007a)) and one of the characteristics of these derivatives is that the payoff depends on country population mortality. While this certainly has advantages regarding transparency and market efficiency, the impact of the basis risk is unclear.

Measurement of mortality rates in insured amounts is already used for a long time, starting with CMI (1962) and more recently for example in Verbond van Verzekeraars (2008) and CMI (2008). However, apart from a sub-paragraph in Van Broekhoven (2002) we are not aware of any literature on stochastic modeling of portfolio specific experience.

The remainder of the paper is organized as follows. First, in section 2 the general model for stochastic portfolio specific experience mortality is defined. In section 3 a 1-factor version of this model is applied to two insurance portfolios. Then in section 4 the impact on the VaR and on the hedge effectiveness is quantified and conclusions are given in section 5.

2. General model for stochastic portfolio specific mortality experience

The first step in the process is determining the historical portfolio mortality rates, measured by insured amounts. There are different kinds of definitions for mortality rates which are calculated in a slightly different manner (see J.P. Morgan (2007b)), for example the initial mortality rate or the central mortality rate. Regardless which definition is used, it is important that the same mortality rate definition is used for setting the country population mortality rates and the portfolio specific mortality rates. In the remaining part of this paper, we determine the portfolio mortality rate, measured by insured amounts, as follows:

$$(2.1) \qquad q^{A}_{x,t} = \frac{A^{D}_{x,t}}{\frac{1}{2}(A^{P}_{x,t} + A^{U}_{x,t} + A^{D}_{x,t})}$$

where $A^{P}_{x,t}$ and $A^{U}_{x,t}$ are the insured amounts primo and ultimo for the total portfolio and $A^{D}_{x,t}$ the insured amount of the deaths, for age x and year t.

Now the aim is to define a stochastic mortality model for the so-called portfolio experience mortality factor $P_{x,t}$ for age x and year t:

(2.2) $$P_{x,t} = \frac{q_{x,t}^{A}}{q_{x,t}^{Pop}}$$

where $q_{x,t}^{Pop}$ is the specific country population mortality rate for age x and year t.

Given that the model will often be based on a limited amount of data and given the specific nature of $P_{x,t}$, a few requirements can be set for he stochastic model:

a) The results of the model should be biologically reasonable
b) The model should be as parsimonious as possible
c) $P_{x,t}$ should approach 1 near the closing age (normally 120 years).

Biologically reasonableness is a standard requirement for mortality models (see for example Cairns et al (2006b)). Furthermore, the model should be as parsimonious as possible because it is not very useful to fit a very complex model to a limited amount of data.
One reason that $P_{x,t}$ should approach 1 is to avoid mortality rates larger than 1, assuming that the mortality rates for the country population also approach 1 for higher ages. Another reason is that the difference between portfolio mortality and country population mortality is expected to be less at higher ages, since the country population at higher ages is expected to have a relatively high percentage of people with higher salaries.

2.1 The basic model

We propose to model the mortality experience factor $P_{x,t}$ as:

(2.3) $$P_{x,t} = 1 + \sum_{i=1}^{n} X^{i}(x)\beta_t^{i} + \varepsilon_{x,t}$$

where n is the number of factors of the model, X^{i} (x) is the element for age x in the i^{th} column of design matrix X, β_t^{i} is the i^{th} element of a vector with factors for year t and $\varepsilon_{x,t}$ the error term. Another way to define the model is in matrix notation:

(2.4) $$P_t = \mathbf{1} + X\beta_t + \varepsilon_t$$

where P_t is the vector of mortality experience factors, β_t the vector with factors and ε_t the vector of error terms for time t. Furthermore, to ensure $P_{x,t}$ approaches 1, we require:

(2.5) $$\sum_{i=1}^{n} X_t^{i}(\omega)\beta_t^{i} = 0$$

where ω is the closing age of the mortality table (usually 120).

Now given a design matrix X, the vector β_t has to be estimated for each year. The structure of X and the corresponding β's can be set in different ways, depending on what fits best with the data and the problem at hand. One could use for example:

1) principal components analysis to derive the preferred shape of the columns X^i.
2) a similar structure as the multi-factor model proposed by Nelson and Siegel (1987) for modeling of yield curve dynamics[3].
3) a more simple structure, for example using 1 factor where the vector X is a linear function in age.

To avoid multicollinearity the factors should be set in such a way that they are as independent as possible.

For very large portfolios, structure 1) and 2) could the most appropriate solutions. However, for the insurance portfolios considered in this paper, with 14 years of history and respectively about 100.000 policies and about 45.000 policies, principle components analysis didn't lead to usable results, and structure 2) did not fit the data better than structure 3). Since structure 3) also uses less parameters, the Bayesian Information Criterion (*BIC*) was more favorable for this structure.

2.2 Fitting the basic model

The structure of the model is such that it could be fitted with Ordinary Least Squares (OLS). However, the observations $P_{x,t}$ are all based on different exposures to death and observed deaths, so there is significant heteroskedasticity. Therefore Generalized Least Squares (GLS) should be used (Verbeek (2008)). This means that a weight matrix W_t is applied to (2.4):

$$(2.6) \quad W_t\left(P_t - \mathbf{1}\right) = W_t X \beta_t + W_t \varepsilon_t \quad or \quad \left(P_t - \mathbf{1}\right)^* = X^* \beta_t + \varepsilon_t^*$$

Where the vectors or matrices labeled with an * are weighted with W_t. This weight matrix is a diagonal matrix that ideally is based on the known or estimated error covariance matrix for year t. However, due to data limitations this error covariance matrix can often not be estimated and therefore usually a practical alternative is used. Examples are the use of the number of deaths as weights (Wilmoth (1993)) or the square root of the number of deaths (Tornij (2006), Verbeek (2008)). Now applying OLS to (2.6) gives the GLS estimator for β_t:

$$(2.7) \quad \hat{\beta}_t = \left(X^{*\prime} X^*\right)^{-1} X^{*\prime}\left(P_t - \mathbf{1}\right)^* = \left(X' W' W X\right)^{-1} X' W' W \left(P_t - \mathbf{1}\right)$$

This procedure can be repeated for each historical observation year, leading to a time series of vector $\hat{\beta}_t$.

2.3 Adding stochastic behavior

Now using the time series of the fitted β_t's, a Box-Jenkins analysis can be performed to determine which stochastic process fit the historical data best[4]. However, an important requirement in this case is biologically reasonableness. For example, when assuming a non-

[3] An example of a possible 2-factor structure is given in Appendix 1.
[4] This is possible under the assumption that the historical fitted parameters are certain. Another possible approach would be to fit the parameters and the stochastic process at once, for example using a state space method combined with the Kalman filtering technique.

stationary process such as a Random Walk for the β_t's, in certain scenario's the $P_{x,t}$'s could be 0 for all ages for some time, which is not biologically reasonable. Since the difference between country population mortality and portfolio mortality is dependent on factors that are normally relatively stable (size, composition and relative welfare of the portfolio), it doesn't seem reasonable to assume that this difference can increase unlimitedly. Therefore, a stationary process seems the most appropriate in this case. Given the often limited historical period of observations and the requirement of parsimoniousness, in most cases the most appropriate model will then be a set of correlated first order autoregressive (*AR*(1)) processes or equivalently, a restricted first order Vector Autoregressive (*VAR*) model:

(2.8) $\beta_t = \delta + \Theta_1 \beta_{t-1} + \varepsilon_t$

where Θ_1 is a n x n diagonal matrix, δ is a n-dimensional vector and ε_t is a n-dimensional vector of white noise processes with covariance matrix Σ.

Possible alternatives are an unrestricted *VAR*(1) model or a first order (restricted) Vector Moving Average (*VMA*) model.

Model (2.8) can be fitted using OLS equation by equation. From the residuals e of the n equations the elements (i,j) of Σ can be estimated as:

(2.9) $\hat{\sigma}_{ij} = \frac{1}{(T-n)} \sum_{t=1}^{T} e_{it} e_{jt}$

An alternative is to estimate this simultaneously with the stochastic processes of the country population mortality model, which is the subject of the next paragraph.

When the insurance portfolio has developed significantly over the years, the fitted parameters over time are subject to heteroskedasticity. In this case GLS could be used, using either the results from table A2.1 in Appendix 2 or one of the practical alternatives mentioned in paragraph 2.2 as weights. When the portfolio has grown significantly and the current size of the portfolio is believed to be more representative for the future, the *relative* weights can also be applied to the residuals, weighting the earlier residuals less than the more recent ones.

2.4 Combine the process with the stochastic country population model

To end up with simulated portfolio specific mortality rates, the country population mortality rates and the portfolio mortality experience factors have to be simulated, taking into account correlations between the different stochastic processes. Let's assume that the country population mortality is driven by m factors of which the processes can be written as:

(2.10) $\alpha_t^k = X_k^{\alpha} \eta_k^{\alpha} + \varepsilon_k^{\alpha} \qquad k = 1, ..., m$

Now when the historical observation period is equal for the country mortality rates and the portfolio mortality experience factors, Seemingly Unrelated Regression (SUR, see Zellner (1963)) can be applied to fit all processes simultaneously. The processes don't have to be similar, so *AR*(1), Random Walk or other *ARIMA* models can be combined.

Re-writing (2.8) for each element *i* in a more general form as $\beta_t^i = X_i^\beta \eta_i^\beta + \varepsilon_i^\beta$ and combining all processes gives:

$$(2.11)\quad \begin{bmatrix} \beta^1 \\ \vdots \\ \beta^n \\ \alpha^1 \\ \vdots \\ \alpha^m \end{bmatrix} = \begin{bmatrix} X_1^\beta & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & 0 \\ \vdots & 0 & X_n^\beta & & & \vdots \\ \vdots & \vdots & & X_1^\alpha & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & X_m^\alpha \end{bmatrix} \begin{bmatrix} \eta_1^\beta \\ \vdots \\ \eta_n^\beta \\ \eta_1^\alpha \\ \vdots \\ \eta_m^\alpha \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \varepsilon_{n+m} \end{bmatrix}$$

which can be written more compactly as:

$$(2.12)\quad Y = X^{\alpha,\beta}\eta + \varepsilon$$

Now these processes can be fitted with SUR using the following steps:

1) Fit equation by equation using OLS
2) Use the residuals to estimate the total covariance matrix $\hat{\Omega}$ with (2.9)
3) Estimate $\hat{\eta}$ using GLS

To be more specific, the resulting estimator in step 3) is determined as:

$$(2.13)\quad \hat{\eta} = \left(X^{\alpha,\beta\prime}\, \hat{\Omega}^{-1} X^{\alpha,\beta} \right)^{-1} \left(X^{\alpha,\beta\prime}\, \hat{\Omega}^{-1} Y \right)$$

As mentioned earlier, in most cases the historical data period for portfolio mortality will be shorter than of country population mortality. In this case and alternative is only to do steps 1) and 2). In step 1) all available historical observations can be used for the different processes. In step 2) for the country population mortality the same historical data period should be used as is available for the portfolio mortality.

3. Application to example insurance portfolios

In this section the general model described in section 2 is applied to two insurance portfolios[5]. The portfolios are respectively large and medium sized, and only data for males and from age 65

[5] The author thanks the Centrum voor Verzekeringsstatistiek (CVS) and Erik Tornij for the data of the large portfolio, and Femke Nawijn and Christel Donkers for the data of the medium portfolio.

on is taken into account. The large portfolio is a collection of collective pension portfolios of the Dutch insurers and contains about 100.000 male policyholders aged 65 or higher. The medium portfolio is an annuity portfolio with about 45.000 male policyholders age 65 or higher[6]. For both portfolios 14 years of historical mortality data is available.

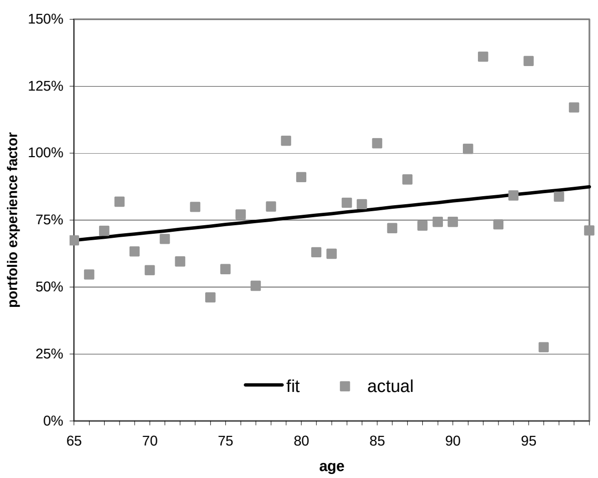
For both portfolios, we examined a collection of 1-, 2-, and 3-factor models and concluded that the 2- and 3- factor models did not fit the data much better than a 1-factor model. Since the 1-factor model uses less parameters, the Bayesian Information Criterion (*BIC*) is more favorable for this structure. Therefore, the model we use is model (2.3) with n = 1 and:

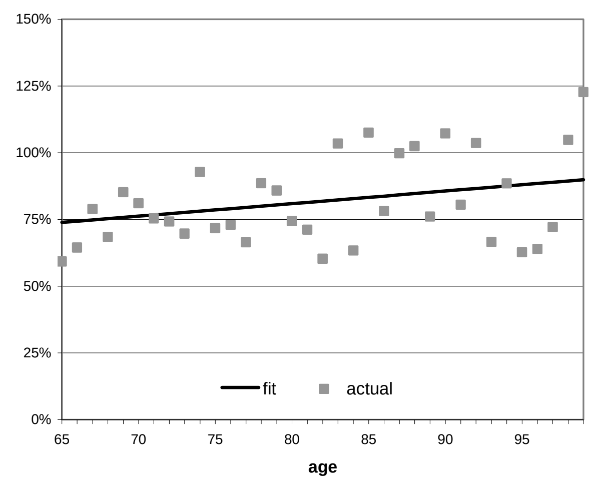
$$(3.1) \qquad X^1(x) = 1 - \frac{x-\xi}{\omega-\xi} \qquad\qquad \xi \le x \le \omega$$

where ξ is the start age (in this case 65) and ω is the end age (120). So in this formulation of model (2.3), the vector X is a linear function in age and, as required, $X^1(\omega) = 0$.

The reason why the 1-factor model fits the data as well as 2- or 3-factor models is that the data shows an upward slope for increasing ages, but the pattern along the ages is very volatile. For example, figure 1 shows two fits for the years 2006 and 2000. Fitting a more complex model through this data will not reduce the residuals significantly. Of course, this observation depends on the characteristics of the specific portfolio to which the model is fitted. For larger portfolios a 2- or 3-factor could give better results, since such a model is able to capture more shapes of the portfolio experience mortality factor curve.

Figure 1: example fit of model to actual observations for years 2006 and 2000





The model is fitted using the procedure described in paragraph 2.2, where we have used the square root of the number of deaths as weights. The fitted β's are shown in figure 2[7]. Further results are given in table A2.1 in Appendix 2.

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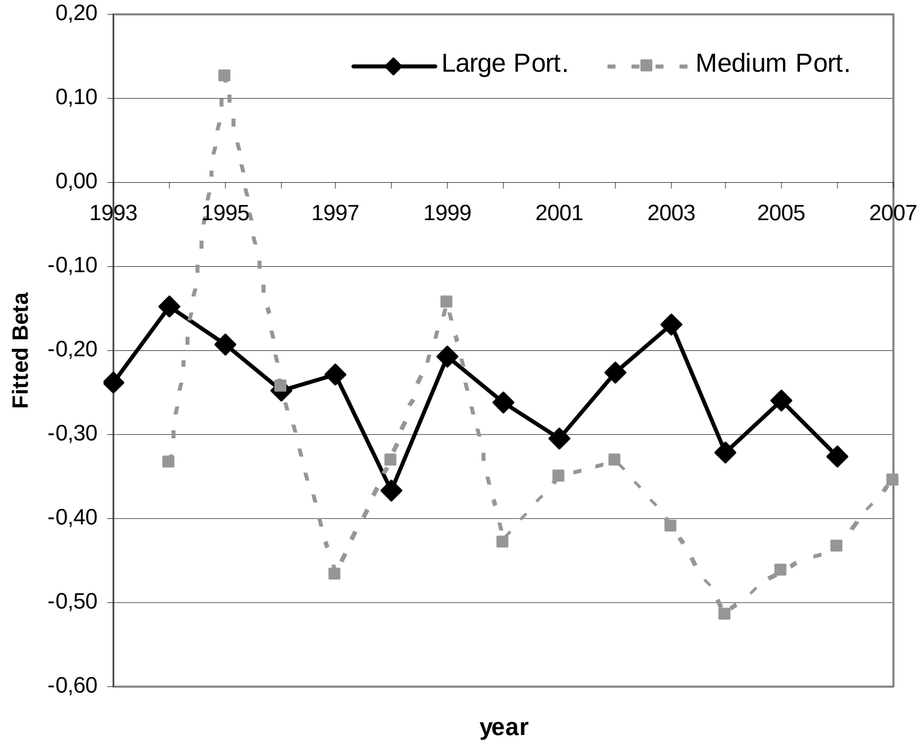
[6] Note that this portfolio has developed over time, so the annuity portfolio had less than 45.000 policyholders in earlier years.

[7] Note that although we have 14 years of data for both portfolios, the periods are slightly different, having data from 1993-2006 for the large portfolio and from 1994-2007 for the medium portfolio.

Figure 2: fitted β's for historical years 2003-2007



For both portfolios the results show an autoregressive pattern for the β's. Now a stochastic process for the future β's has to be selected. As mentioned in paragraph 2.3, a stationary process will be most appropriate. Also, since the historical data period is limited, the model should be as parsimonious as possible. Therefore an *AR*(1) model is assumed for both portfolios. Also an *AR*(2) process is fitted to the data, but this led to a less favorable *BIC* compared to the *AR*(1) process.

Because of the significant development of the medium sized portfolio over the historical years, GLS is used for fitting the *AR*(1) process. The square roots of the *relative* number of deaths in a year are used as weights. Relative means relative to the average number of deaths. These weights are also applied to the residuals, giving less weight to years where the portfolio was relatively small. Since the large portfolio was relatively stable over time, OLS is used for fitting the *AR*(1) process for this portfolio.

The fitted processes for the portfolios are:

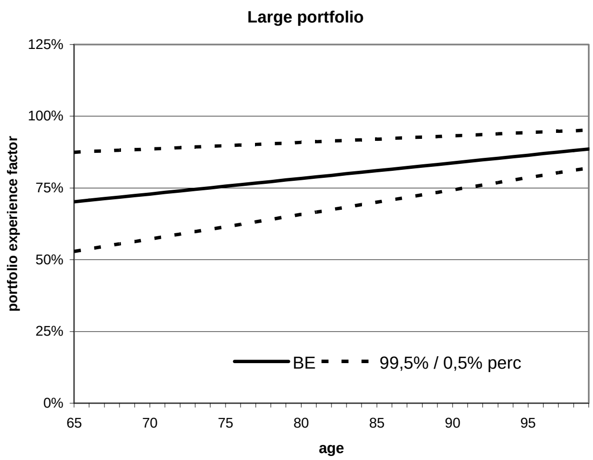
(3.2) Large portfolio: $\beta_t = -0,2731 - 0,0924\,\beta_{t-1} + \varepsilon_t\,, \qquad \hat{\sigma} = 0,0676$

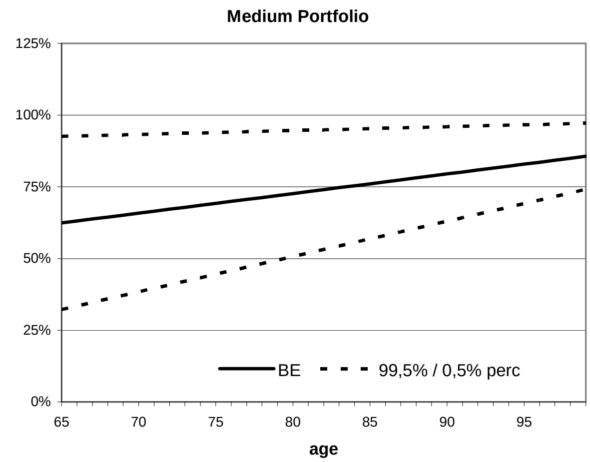
(3.3) Medium portfolio: $\beta_t = -0,2906 + 0,2339\,\beta_{t-1} + \varepsilon_t\,, \qquad \hat{\sigma} = 0,1150$

The estimated error standard deviation $\hat{\sigma}$ is significantly larger for the medium sized portfolio, which is mainly the result of having less policyholders. The result of this is shown in figure 3, where the best estimates and the 99,5% / 0,5% percentiles are given for the portfolio experience mortality factors in the year 2016[8]. These specific percentiles are shown because the SCR of Solvency 2 is based on a 99,5% percentile.

The figure shows that for the large portfolio the difference between the best estimate and the percentile(s) is in the range 10% - 20% for ages 65-80. So taking this stochastic behavior of the portfolio experience mortality factor into account can have a reasonable impact on for example the Value at Risk. As expected, the impact is more significant for the medium portfolio, where the difference between the best estimate and the percentile(s) is about 30% at its maximum.

Figure 3: best estimates and 99,5% / 0,5% percentiles for both portfolios





4. Numerical example 1: Value at Risk

An important application of the presented model is the quantification of the Value at Risk (VaR) or SCR for longevity or mortality risk. In this paragraph the VaR is determined for the two portfolios, for different definitions / horizons of the VaR. First the model has to be combined with a model for country population mortality risk.

4.1 Stochastic country population mortality model

For the stochastic country population model we use the model of Cairns et al (2006a):

(4.1) $\text{logit}\ q_{x,t}^{Pop} = \kappa_t^1 + \kappa_t^2 \left(x - \overline{x} \right)$

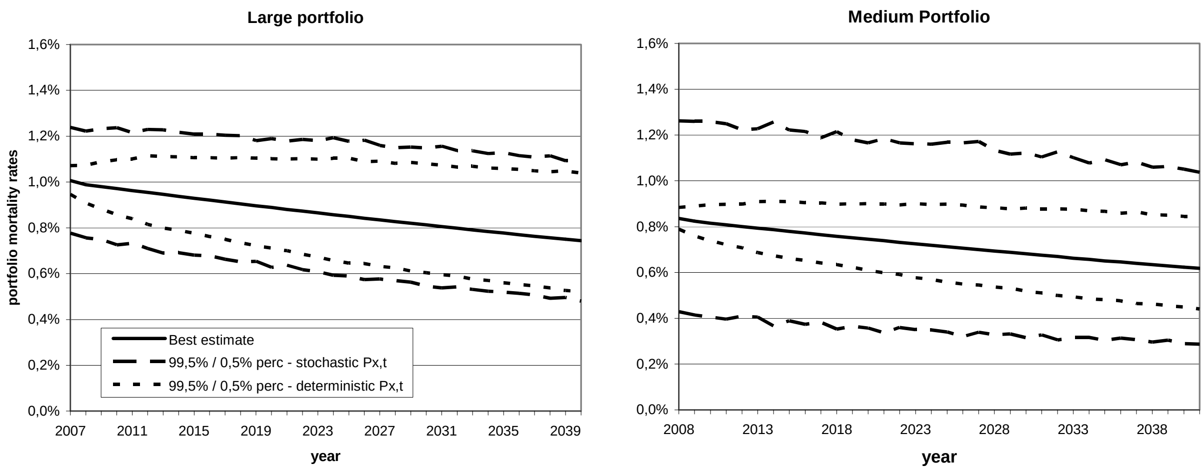
Where $\overline{x}$ is the mean age in the sample range and κ_t^1 and κ_t^2 the two (stochastic) factors. We fitted this model to data of the Dutch population for the years 1950 – 2007. Using the resulting time series of parameter estimates, a 2-dimensional random walk process is fitted for the factors.

[8] Since a stationary process is assumed, the figure will be similar for other projection years.

The fitted parameters and the covariance matrices, including the covariance's with the portfolio experience mortality process of both portfolios, are given in Appendix 2.

Now combining the stochastic process above and the process described in section 3 leads to stochastic portfolio specific mortality rates. Figure 4 gives the best estimate mortality rates and percentiles for age 65. The percentiles are based on respectively deterministic and stochastic $P_{x,t}$'s.

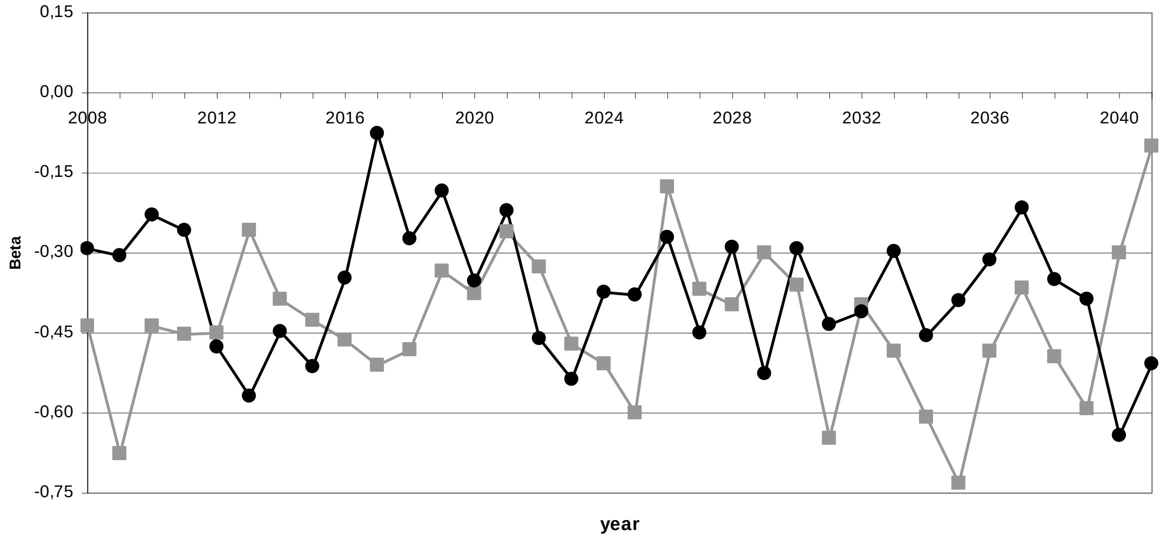
Figure 4: best estimates and percentiles, with stochastic or deterministic $P_{x,t}$



The figure shows that the additional risk of including stochastic $P_{x,t}$'s is highest at the start of the projection and decreases slowly in time. The reason for this is that the country population mortality rate is gradually increasing over time, resulting in a higher diversification effect between country population mortality rates and the $P_{x,t}$'s over time.

The percentiles for the medium portfolio seem quite dramatic. However, note that the shown percentiles are a result of picking the particular percentile every year, and not picking 1 scenario that represents the x%-percentile for the whole projection. Because of the assumed $AR(1)$ process the extremely low outliers will normally be (partially) compensated somewhere in time by high outliers. This is shown in figure 5, where two random (simulated) scenarios of the β's are given as an example.

Figure 5: two random (simulated) scenarios for β



4.2 Impact on Value at Risk

Now using the described stochastic processes the impact on the VaR of stochastic (instead of deterministic) $P_{x,t}$'s is determined for both portfolios. The (present) value of liabilities is calculated for all simulated mortality rate scenarios[9]. The VaR is then defined as the difference between the *x*%-percentile and the average value of the liabilities. The impact is determined for three different definitions / horizons, which are all being used in practice:

1) 1-year horizon, 99,5% percentile, including effect on best estimate after 1 year
2) 10-year horizon, 95% percentile, including effect on best estimate after 10 years
3) Run-off of the liabilities, 90% percentile

So for definitions 1) and 2), at the 1-year or 10-year horizon all parameters are re-estimated using the (simulated) observations in the first 1 or 10 years, for each simulated scenario. The impact of the new parameterization on the best estimate of liabilities (for each scenario) is taken into account in the VaR. The results for the large and medium portfolio are given in respectively table 1 and table 2.

Table 1: impact of stochastic $P_{x,t}$ on VaR – large portfolio

VAR definition	Deterministic $P_{x,t}$	Stochastic $P_{x,t}$	% difference
1-year, 99,5%	102.380.661	121.250.929	+ 18,4%
10-year, 95%	139.480.321	154.661.171	+ 10,9%
Run off, 90%	109.837.423	118.480.837	+ 7,9%

[9] For convenience we assumed that the portfolios only contain pension or annuity payments, so no spouse pension or annuities on a second life.

Table 2: impact of stochastic $P_{x,t}$ on VaR – medium portfolio

VAR definition	Deterministic $P_{x,t}$	Stochastic $P_{x,t}$	% difference
1-year, 99,5%	48.380.106	81.483.079	+ 68,4%
10-year, 95%	72.242.177	112.385.203	+ 55,6%
Run off, 90%	58.270.036	86.285.812	+ 48,1%

Table 1 shows that for the large portfolio the impact of stochastic $P_{x,t}$'s on the VaR can be significant when the horizon is short. For longer horizons, the impact is less significant (but not negligible). The reason for this is that on a longer horizon the impact of stochastic $P_{x,t}$'s levels out because of the assumed autoregressive process.

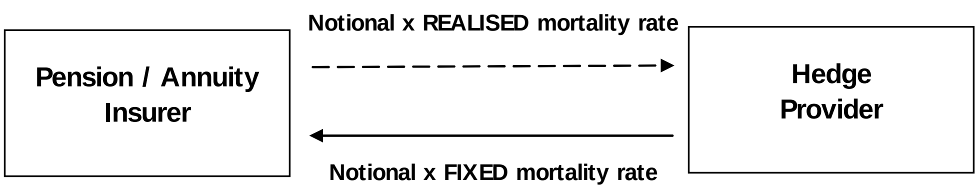
Table 2 shows that the impact for the medium portfolio is very significant. The increase in VaR is between 48% and 68%, depending on the definition for VaR used. The reason for this is the significant increase in volatility due to the addition of the stochastic $P_{x,t}$'s, which is mainly related to the size of the portfolio. Since a large part of the insurance portfolios in practice are of this size or smaller, this should be a point of attention when developing or reviewing internal models for mortality and longevity.

5. Numerical example 2: hedge effectiveness / basis risk

Because of the increasing external requirements and focus on risk measurement and risk management, the interest in hedging mortality or longevity risk is also increasing. A result of this is that a market for mortality and longevity derivatives is emerging (see J.P. Morgan (2007a)). One of the main characteristics of these derivatives is that the payoff depends on country population mortality. While this certainly has advantages regarding transparency and market efficiency, the impact of the basis risk is unclear. Basis risk is the risk arising from difference between the underlying of the derivative and the actual risk in the liability portfolio. The model presented in this paper can be used to quantify this basis risk. In the example below the basis risk will be quantified for the two portfolios, where the longevity risk is (partly) hedged with the so-called q-forwards.

A q-forward is a simple capital market instrument with similar characteristics as an interest rate swap. The instrument exchanges a realized mortality rate in a future period for a pre-agreed fixed mortality rate. This is shown in figure 1. The pre-agreed fixed mortality rate is based on a projection of mortality rates, using a freely available and well documented projection tool[10].

Figure 6: working of a q-forward



[10] For more information, see http://www.jpmorgan.com/pages/jpmorgan/investbk/solutions/lifemetrics

For example, when the realized mortality rate is lower than expected, the pension / annuity insurer will receive a payment which (partly) compensates the increase of the expected value of the insurance liabilities (caused by the decreasing mortality rates).

The basis for the instrument is the (projected) mortality of a country population, not the mortality of a specific company or portfolio. This makes the product and the pricing very transparent compared to traditional reinsurance.

For both insurance portfolios we determined a minimum variance hedge, based on deterministic $P_{x,t}$'s. The hedge is determined for a horizon of 10 years, but including the effect on the best estimate after 10 years. The hedge is determined for age-buckets of 5 years. For every bucket *i*, the impact of small shocks of the two factors of the country population model on the value of the liabilities and the value of an appropriate q-forward contract are calculated. The required nominal a_i^* for the q-forward of bucket *i* is then determined as:

$$(5.1) \qquad a_i^* = \frac{l_1 h_1 + l_2 h_2}{h_1^2 + h_2^2}$$

where l_i and h_i are the impact of the shock of the i^{th} factor on respectively the liabilities (*l*) and the hedge instrument (*h*).

The resulting hedge portfolio consists of 5 q-forwards for age-buckets of 5, from age 65 till age 89. The payoff of such a q-forward depends on the average mortality rate for the 5 ages in the bucket. The exact composition of both the hedge portfolios is given in Appendix 3.

Table 3 and 4 show the impact on the hedge effectiveness when the $P_{x,t}$'s are assumed to follow the stochastic process described in section 3.

Table 3: impact of stochastic $P_{x,t}$ on hedge effectiveness – large portfolio

	VAR unhedged	VAR hedged	% reduction
Deterministic $P_{x,t}$	139.480.321	52.416.247	62,4%
Stochastic $P_{x,t}$	154.661.171	70.367.427	54,5%

Table 4: impact of stochastic $P_{x,t}$ on hedge effectiveness – medium portfolio

	VAR unhedged	VAR hedged	% reduction
Deterministic $P_{x,t}$	72.242.177	25.672.404	64,5%
Stochastic $P_{x,t}$	112.385.203	64.843.202	42,3%

The tables show that given deterministic $P_{x,t}$'s, the hedge reduces the VaR with 62%-65%. The risk is not fully hedged, because the hedge is based on small shocks of the two country population factors, while the factors in the tails of the distributions (which are relevant for VaR) are often more extreme.

For the large portfolio, table 3 shows that the hedge quality is decreasing, but is still reasonable. The basis risk for this portfolio is therefore limited. The reason for this is (again) that on a longer horizon the impact of stochastic $P_{x,t}$'s levels out because of the assumed autoregressive process.

For the medium portfolio the hedge effectiveness is reduced more significantly. The effectiveness of the hedge can be improved by periodically adjusting the hedge portfolio. For smaller portfolios then this, it is probably questionable whether it is sensible to set up such hedge constructions.

6. Conclusions

In this paper a stochastic model is proposed for stochastic portfolio experience. Adding this stochastic process to a stochastic country population mortality model leads to stochastic portfolio specific mortality rates, measured in insured amounts. The proposed stochastic process is applied to two insurance portfolios. The results show that the uncertainty for the portfolio experience factor $P_{x,t}$ can be significant, mostly depending on the size of the portfolio.

The impact of the VaR for longevity risk is quantified. Depending on the definition used, the impact for the large portfolio is an 8%-18% increase of VaR. The impact for the medium portfolio is very significant. The increase in VaR is 70%-80% for the longer horizons and the VaR doubles for the 1-year horizon. The reason for this is the significant increase in volatility due to the addition of the stochastic $P_{x,t}$'s. Since a large part of the insurance portfolios in practice are of this size or smaller, this should be a point of attention when developing or reviewing internal models for mortality and longevity.

Furthermore, the basis risk is quantified when hedging portfolio specific mortality risk with q-forwards, of which the payoff depends on country population mortality rates. For the large portfolio the hedge quality is decreasing, but is still reasonable. The reason for this is that on a longer horizon the impact of stochastic $P_{x,t}$'s levels out because of the assumed autoregressive process. For the medium portfolio the additional risks from the stochastic $P_{x,t}$'s is so significant that the hedge effectiveness is reduced significantly. Although the effectiveness of the hedge can be improved by periodically adjusting the hedge portfolio, it is questionable whether it is sensible to set up such hedge constructions for portfolios that are small or medium sized.

Appendix 1: example 2-factor model based on Nelson & Siegel

Nelson and Siegel (1987) proposed a parsimonious model for yield curves, which allows for different shapes of the curve. The Nelson-Siegel forward curve can be viewed as a constant plus a Laquerre function, which is a polynomial times an exponential decay term. It has three elements, respectively for the short, medium and long term. The model is very often used for yield curves and could serve as a basis for thinking for the P_t curves that are the subject of this paper. However, the Nelson-Siegel curve cannot directly be used for the P_t curves because $P_{x,t}$

should approach 1 near the closing age. Also, another requirement mentioned in section 2 is that the model is as parsimonious as possible, so a 2-factor model might be more appropriate in most cases.

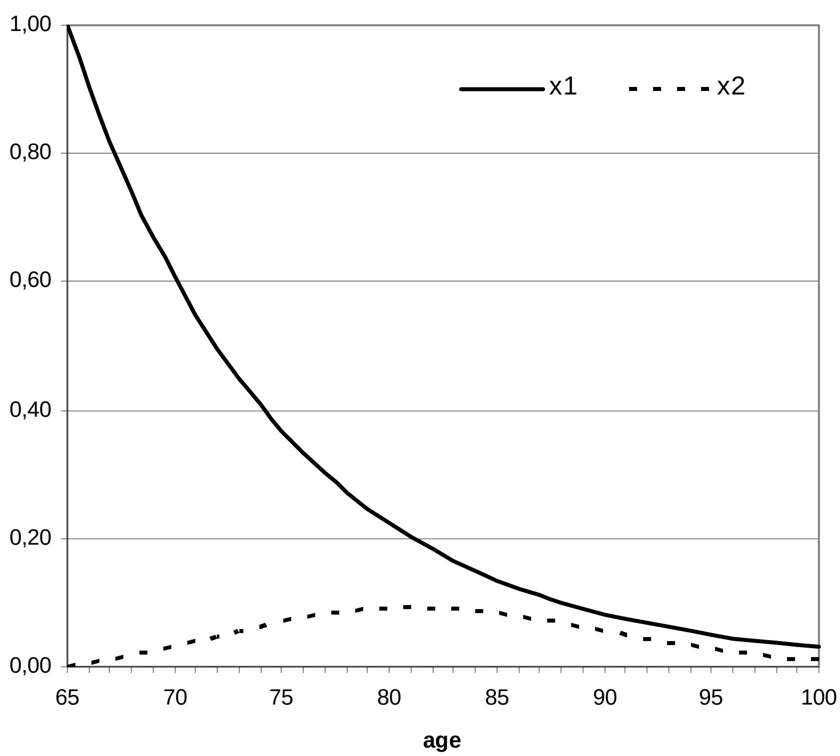
Many variations on the Nelson-Siegel curve are possible. An example of such a model is the following model:

(A.1) $P_t(\tau) = 1 + \beta_{1t} e^{-\lambda_1 \tau} + \beta_{2t} w_\tau \left(e^{-\lambda_1 \tau} - e^{-\lambda_2 \tau} \right)$

$$where\ w_\tau = \varphi\left(\alpha \left[\frac{\tau - \tau_m}{\tau_m} \right] \right) \phi$$

The variable τ is 0 for the starting age of the data (in this case 65 years), τ_m is a strategically set middle point of the age interval (in this case 20, representing age 85), φ is the density of a standard normal distributed variable, α is a variable that arranges the shape of w_τ and can be set at 2 for example, and ϕ is a scale variable. The variable λ_1 can be solved in such a way that the second term of (A.1) approaches 0 for the closing age. The variable λ_2 can be solved in such a way that the third term of (A.1) is at its maximum somewhere between $\tau = 0$ and τ_m (in this case 75 years). The factors are shown in figure A.1, where x_1 represents the second term and x_2 the third term of (A.1).

Figure A.1: factors for model (A.1)



As can be seen from the figure and (A.1), the curve starts at age 65 at 1 + β_{1t} (where β_{1t} will be negative in general) and ends at 1 at higher ages. With the model (A.1) different shapes of the curve can be fitted, and the requirements in section 2 are met. A disadvantage of the model is the large number of parameters, of which some are set more or less arbitrary.

Appendix 2: further results

Table A.2.1 shows the fitting results for the β's in each year, for the large and medium sized portfolio.

Table A2.1: yearly fitting results for β's

Results large portfolio			
Year	β	**s.e.**	**t-ratio**
1993	-0,239	0,036	-6,55
1994	-0,149	0,041	-3,67
1995	-0,194	0,030	-6,55
1996	-0,246	0,033	-7,43
1997	-0,228	0,032	-7,20
1998	-0,368	0,023	-16,12
1999	-0,208	0,036	-5,77
2000	-0,261	0,029	-8,91
2001	-0,304	0,032	-9,46
2002	-0,226	0,033	-6,88
2003	-0,168	0,046	-3,62
2004	-0,321	0,048	-6,71
2005	-0,259	0,042	-6,11
2006	-0,325	0,040	-8,18

Results medium portfolio			
Year	β	**s.e.**	**t-ratio**
1994	-0,333	0,103	-3,23
1995	0,127	0,201	0,63
1996	-0,243	0,127	-1,92
1997	-0,467	0,091	-5,14
1998	-0,330	0,056	-5,92
1999	-0,143	0,065	-2,21
2000	-0,427	0,057	-7,56
2001	-0,349	0,089	-3,94
2002	-0,331	0,052	-6,32
2003	-0,408	0,050	-8,11
2004	-0,515	0,035	-14,77
2005	-0,462	0,047	-9,81
2006	-0,434	0,046	-9,52
2007	-0,355	0,079	-4,52

Table A.2.2 shows the fitted parameters for the 2-dimensional random walk model of section 4, and the covariance matrix including the covariance's with the process of section 3. Note that the country population parameter estimates slightly differ for the large and medium portfolio, because for the medium portfolio the year 2007 is also taken into account.

Table A.2.2: fit of country population model and covariance matrices

Fit - large portfolio		
	μ	σ
κ^1	-0,006206	0,02475844
κ^2	0,000182	0,00145148

Fit - medium portfolio		
	μ	σ
κ^1	-0,006722	0,023468
κ^2	0,000175	0,001494

Covariance matrix - large portfolio			
	β	κ^1	κ^2
β	0,004575	0,000408	0,000042
κ^1	0,000408	0,000613	0,000020
κ^2	0,000042	0,000020	0,000002

Covariance matrix - large portfolio			
	β	κ^1	κ^2
β	0,030651	0,003140	0,000089
κ^1	0,003140	0,000551	0,000021
κ^2	0,000089	0,000021	0,000002

Appendix 3: hedge portfolios

Table A.3.1: hedge portfolios for large and medium insurance portfolio

Charateristics hedge portfolio - large portfolio				
q-forward	**Start age**	**End age**	**Nominal**	**Tick Size**
1	65	69	117.717.992	100
2	70	74	34.530.493	100
3	75	79	10.635.390	100
4	80	84	2.290.751	100
5	85	89	331.753	100

Characteristics hedge portfolio - medium portfolio				
q-forward	**Start age**	**End age**	**Nominal**	**Tick Size**
1	65	69	74.636.291	100
2	70	74	23.996.948	100
3	75	79	3.126.779	100
4	80	84	137.166	100
5	85	89	7.733	100

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