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Abstract: The introduction of the new international accounting standard requires that assets and liabilities are valued at market value. Since pension liabilities are generally to some extent indexed according to inflation, this provides new challenges for both valuation and risk management. This paper discusses models that can be used to determine the fair value of inflation-sensitive pension rights. In addition, we show how these models can be used in risk management applications and we emphasize the discrepancy between currently popular methodologies, like duration analysis. We show that in case of pension schemes where indexation is conditional on the state of the pension fund, the value of the liabilities is determined by an interplay between economic variables, like interest rates and inflation, and the asset mix. A thorough understanding of such dependencies is crucial in order to come to a proper assessment of the risks to which the liabilities are exposed.

Keywords: inflation sensitive liabilities, pricing kernel, valuation

JEL codes: E31, G13, G23

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1 Introduction

Pension rights can easily be eroded by inflation. For this reason, many pension schemes offer indexation of retirement benefits to annual changes in a consumer price index (CPI). Retirees are therefore insulated from inflation risk, at least as long as their preferred consumption bundle is not too different from the consumption bundle used to compute the CPI. Some pension schemes also aim at indexation to the overall welfare level and index with respect to wage rather than price inflation. Full (price-)indexation is offered by real annuities as they are widely available in the UK and of which examples can be found in other markets as well (see e.g. Brown, Mitchell, and Poterba (2001)). Full indexation has also been offered for many years by the U.S. social security system as well as by many defined benefit schemes world-wide (e.g. in the U.S. and in the Netherlands). Besides financial products and pension schemes that offer full indexation, recently products and pension schemes have also been designed that offer indexation to inflation in certain scenarios only, e.g. if the funding ratio of the defined benefit system is sufficiently high. Such schemes will be referred to in the sequel as conditionally indexed pension schemes.

Nominal pension rights are relatively easy to value using data on the market for nominal bonds. Assuming that longevity risk can be diversified, the market risk to which the nominal pension rights are exposed are the same as the risks in a corresponding portfolio of nominal bonds. Apart from the fact that the maturity structure will probably have to be extrapolated somewhat, the market price of such pension rights is easy to construct. Valuation of inflation sensitive pension rights however is non-trivially harder than valuation of nominal pension rights, unless a sufficiently deep and liquid market for bonds indexed to the appropriate price index is available. Bonds that are indexed to price inflation are traded in many countries such as the U.K., Sweden, Israel and nowadays also in France and the U.S. (see Deacon, Derry, and Mirfendereski (2004) for an extensive overview). However, traded bonds are available for a limited number of maturities only, no bonds related to wage inflation or local price inflation within e.g. the U.S. are available, and the markets do not even exist for many other countries or are highly illiquid (including e.g. Germany and the Netherlands). Clearly also the value of the conditional indexation schemes that were referred to cannot be derived directly from market prices of traded assets. In this chapter we discuss how pricing models can be used to value inflation sensitive pension rights, even if no perfectly equivalent assets are traded in the market.

Valuation of pension rights is important for many reasons. First of all, market valuation of the pension rights increases the transparency of a pension scheme and helps the beneficiary to choose between alternatives, such as nominal or real annuities and / or voluntary participation in a collective pension scheme. Valuation of the pension rights is also required to compute a fair premium for new entrants to be included in a pension scheme as well as for transfers out of the scheme. Supervisory authorities will need the value of the liabilities of a pension fund in order to be able to judge its solvency. Adequate risk management of that fund (e.g. to protect indexation rights) moreover requires that one realizes how the value of assets and liabilities will fluctuate if underlying fundamentals fluctuate, which requires once more valuation. Moreover, valuation of the liabilities can be a powerful tool in labor agreement negotiations to confront parties with the monetary value of elements of the pension deal, and is required to determine generational accounts, which illustrates solidarity issues within the pension scheme, see also Kortleve and Ponds (2005) and Exley (2005). Finally, valuation of inflation sensitive assets is of course also relevant for topics far from the applications to pensions and retirement on which we focus, e.g. in valuing an inflation sensitive firm.

The plan of this chapter is as follows. In Section 2, we discuss how valuation of pension liabilities and models of the term structure are related. In Section 3, we will first of all sketch an overly simple term structure model along the lines of duration analysis that can be used to value nominal or fully indexed liabilities, assuming that certain strong additional assumptions have been satisfied. This model, which is often used in risk management for "back of the envelope" calculations, is shown to be severely misspecified. In Section 4, we introduce the use of models of the real and nominal term structure to value inflation sensitive assets. In line with the recent literature (Evans (1998), Campbell and Viceira (2001), Brennan and Xia (2002), Ang and Bekaert (2004)) we discuss how pricing kernels can be specified which can be used to price nominal, real as well as conditionally indexed liabilities. Moreover, we will indicate briefly how valuation along these lines is related to pricing through a replicating strategy, to risk neutral valuation, and to state contingent discounting, see also Hibbert (2005). Section 5 will present a number of numerical results, where we focus on valuation of the conditional indexation contracts that have been implemented in the Netherlands as of the year 2004. In Section 6, the same numerical example is used in order to illustrate the use of models for the purpose of risk management. Section 7 concludes and provides references to recent literature to extend the models to be more aligned with observed market data. Moreover, we will discuss several practical limitations of the approach.

2 Valuation of nominal and real pension rights

Valuation of straight real (nominal) pension rights requires knowledge of the real (nominal) term structure. In order to be able discuss term structure models and to link them to valuation, we will first of all introduce some notation. A nominal discount bond maturing in period $t+n$ is a bond that pays the principal (normalized at 1) in period $t+n$ and never pays any coupon. The price of such a nominal bond in period t will be denoted as $P_t^{(n)}$. The yield to maturity (i.e. the nominal interest rate) of a discount bond is defined as the $R_t^{(n)}$ that solves

$$P_t^{(n)} = \left(1 + R_t^{(n)}\right)^{-n}. \tag{1}$$

Since $\log(1+x) \approx x$ for x close to zero this definition is approximately equivalent to

$$P_t^{(n)} = e^{-nR_t^{(n)}}. \tag{2}$$

In the sequel we will use (2) for mathematical convenience.

The set of interest rates $\left(R_t^{(n)}\right)$ at a specific point in time (t) with varying maturities (n) is known as the nominal term structure of interest rates. Likewise, a real or indexed bond is a bond that pays one real unit at maturity. If the underlying price index is denoted as Π_t the pay-off at maturity will be $\frac{\Pi_{t+n}}{\Pi_t}$ in nominal terms. The payoff of a real bond is therefore scaled proportionally to the increase in the price index. Fully analogous to the nominal case, the yield to maturity (i.e. the real interest rate) of a real discount bond is defined as the $R_t^{R(n)}$ that solves

$$P_t^{R(n)} = e^{-nR_t^{R(n)}}. \tag{3}$$

Now consider the market value of a pension scheme that generates cash flows F_{t+n} in period $t+n$. These cash flows are in principle random variables, and might depend e.g. on future inflation, on survival of the participants etc. By definition of the expected return $\mu_t^{(n)}$, the market value at time t of the cash flow at time $t+n$, $V_t^{(n)}$, can be written as

$$V_t^{(n)} = \mathbb{E}_t\left(F_{t+n}e^{-n\mu_t^{(n)}}\right), \tag{4}$$

where $\mathbb{E}_t$ denotes the conditional expectation operator, where the conditioning takes place on all information available at time t. Consequently, the value of the pension scheme can be written as

$$V_t = \sum_{n=1}^{N} V_t^{(n)} = \sum_{n=1}^{N} \mathbb{E}_t\left(F_{t+n}e^{-n\mu_t^{(n)}}\right). \tag{5}$$

Equations (4) and (5) are not very informative, unless the expected returns, $\mu_t^{(n)}$, are known. This is the case if the cash flows are non-random, but also, for example, if micro longevity risk is the only source of randomness[1]. We assume throughout that the size of the pension fund is sufficiently large to diversify longevity risk. Since all randomness in the cash flow is fully diversifiable, standard finance theory imposes that it will not be priced, i.e. the expected return equals the corresponding nominal interest rate, implying that

$$V_t = \sum_{n=1}^{N} \mathbb{E}_t\left(F_{t+n}e^{-nR_t^{(n)}}\right) = \sum_{n=1}^{N} P_t^{(n)}\mathbb{E}_t\left(F_{t+n}\right). \tag{6}$$

Valuation of indexed pension schemes is possible using the real term structure along the same lines. First of all, denote the one period inflation rate corresponding to the relevant price index Π_t as

$$\pi_{t+1} = \log \Pi_{t+1} - \log \Pi_t, \tag{7}$$

and define average inflation over n periods as

$$\pi_{t+n}^{(n)} = \frac{1}{n}\sum_{i=1}^{n} \pi_{t+i} = \frac{1}{n}\left(\log \Pi_{t+n} - \log \Pi_t\right).$$

We denote the risk premium that alters the nominal interest rate in order account for inflation risk by $\lambda_t^{(n)}$. Using this notation the market value of the indexed pension scheme can be written as

$$\begin{aligned} V_t^R &= \sum_{n=1}^{N} \mathbb{E}_t\left(\frac{\Pi_{t+n}}{\Pi_t}F_{t+n}e^{-n\left(R_t^{(n)}+\lambda_t^{(n)}\right)}\right) \\ &= \sum_{n=1}^{N} \mathbb{E}_t\left(F_{t+n}e^{-n\left(R_t^{(n)}+\lambda_t^{(n)}-\pi_{t+n}^{(n)}\right)}\right) \\ &= \sum_{n=1}^{N} \mathbb{E}_t\left(F_{t+n}e^{-n\left(R_t^{R(n)}\right)}\right) = \sum_{n=1}^{N} P_t^{R(n)}\mathbb{E}_t\left(F_{t+n}\right), \end{aligned} \tag{8}$$

where we define the real interest rate of an n period bond as

$$R_t^{R(n)} = R_t^{(n)} - \mathbb{E}_t\left(\pi_{t+n}^{(n)}\right) + \lambda_t^{(n)}. \tag{9}$$

Many models assume that inflation risk is not priced, i.e. the expected return on nominal bonds coincides with that on indexed bonds, in which case $\lambda_t^{(n)}$ vanishes. If the assumption is valid the real term structure can be obtained through knowledge of the nominal term structure and inflation expectations only.

[1] Micro longevity risk is caused by the randomness of deaths for given survival probabilities. Macro longevity risk, which is due to changes in the survival probabilities due to e.g. improvements in health care, is beyond the scope of this chapter.

If the relevant term structure (either the nominal or the real) is observed for every maturity, term structure models are not required to value the pension scheme if the pension scheme either fully nominal or real. For most developed countries, data on the nominal term structure are readily accessible, but typically only for maturities up to a limited number of years. In many countries, the swap curve generates yields for maturities, say, up to fifty years. Valuation of pension schemes requires assumptions on the interest rate for even longer maturities, which can be obtained through modelling. More importantly, term structure data for indexed bonds are available for a limited number of countries, for specific choices of the index used for indexation (e.g. price rather than wage inflation), and for very particular maturities only. Term structure models can be used to estimate interest rates that are not directly observed. As a consequence, the valuation of either nominal or real pension schemes will typically rely on the specification of the model that also implies a specific term structure.

Modeling is even more crucial if no assets are traded that are closely related to the liabilities to be valued. This represents the current case of pension funds in countries where no inflation-linked products are traded, or markets are highly illiquid. In Section 5, we will consider the example of conditional indexation schemes as they have been recently introduced in the Netherlands. Given this particular pension scheme, we will show that the value of the liabilities is dependent on the asset mix, the current state of the economy, and the current state of the pension fund.

3 Risk management of nominal and real pension rights using duration analysis

As discussed in Section 2, term structure models and assumptions regarding parameters such as the inflation risk premium are sometimes required for valuation of nominal or real pension rights, when no indexed securities are traded within the economy. Even if the relevant term structure is fully observed, i.e. no modeling assumptions are required for valuation, the models are essential for risk managing and risk budgeting of trustees and participants of pension schemes. Through the use of pricing models "what if" questions can be answered, which give the value of the scheme if the interest rate drops, the inflation rate rises etc. and which enable portfolio strategies that would hedge the surplus of the fund. Duration analysis is currently one of the most popular tools for this aim.

Duration analysis is based on assumptions on the sensitivity of the market value of assets with respect to fluctuations in the interest rates. Duration analysis assumes that the fluctuations in interest rates with arbitrary maturities can be adequately described by fluctuations in one rate (say the short rate) only. The relative interest rate exposure of a nominal zero coupon bond with arbitrary maturity n with respect to the short rate can be written as

$$\varepsilon = \frac{1}{P_t^{(n)}} \frac{\partial P_t^{(n)}}{\partial R_t^{(1)}} = -n \frac{\partial R_t^{(n)}}{\partial R_t^{(1)}}, \tag{10}$$

where the second equality follows from (2).

According to duration analysis, a 1% increase in the short rate implies a 1% increase the interest rates with all other maturities, i.e. duration analysis allows for parallel shift in the term structure only. This is illustrated in Figure 1.

Insert here Figure 1

This assumption implies that the interest exposure of a zero coupon bond duration n (i.e. with n periods to maturity) simplifies to

$$\varepsilon = -n, \tag{11}$$

Equation (11) states the important result that, within the assumptions made, the interest exposure of a zero coupon bond happens to coincide with the duration of the bond. This explains the terminology "duration analysis".

In an analogous way the relative interest rate exposure of a portfolio as in (6) can be obtained as

$$\varepsilon_V = \frac{1}{V_t}\frac{\partial V_t}{\partial R_t^{(1)}} = -\frac{\sum_{n=1}^{N} nV_t^{(n)}}{V_t}\frac{\partial R_t^{(n)}}{\partial R_t^{(1)}}, \tag{12}$$

which simplifies under the assumptions of duration analysis to

$$\varepsilon_V = -\frac{\sum_{n=1}^{N} nV_t^{(n)}}{V_t}, \tag{13}$$

i.e. the duration of the portfolio is simply the weighted duration of each of the assets or liabilities. The duration of the liabilities of a pension scheme is often in the range $15-20$ years. According to duration analysis this implies that the market value of the liabilities will rise by $15-20\%$ if the interest rate drops 1%. For a more detailed textbook treatment of duration analysis and the use of convexity corrections to obtain second order approximations to the sensitivity of market values to interest rate changes, see for instance Campbell, Lo, and MacKinlay (1997) and Jarrow and Turnbull (2000).

The argument underlying duration analysis can be extended in a straightforward way to the analysis of real rather than nominal bonds by referring to real rather than nominal interest rates. Both the nominal and the real term structure will shift in parallel fashion if, on top of the assumptions on the nominal curve, shifts in the inflation risk premium $\lambda_t^{(n)}$ and the expected average annual inflation $\mathbb{E}_t\left(\pi_{t+n}^{(n)}\right)$ are identical for all maturities.

Duration analysis is heavily used in applications and is an important tool for "back of the envelope" calculations. It should be realized however that the model underlying duration analysis is severely misspecified for at least two reasons

1. The model allows for parallel shifts in the term structure only;
2. The model implies that long rates are as volatile as short rates.

Both assumptions are in contrast with the stylized facts in the data, see e.g. Brandt and Chapman (2002). In the next section we will introduce a model that relaxes the assumption underlying duration analysis, i.e. $\frac{\partial R_t^{(n)}}{\partial R_t^{(1)}}$ is not necessarily equal to 1. This model illustrates that duration analysis based on the fluctuations of short rates overemphasizes the exposure to interest rate risk in the market value of pension schemes, due to the implicit assumption that long rates are as volatile as short rates.

4 The use of pricing kernels to value inflation sensitive assets

In Section 2, we discussed how nominal or real pension schemes can be valued at market value using observations or assumptions on the nominal and real term structure respectively. Valuation of more general inflation sensitive assets and liabilities, such as the conditionally indexed pension schemes, as referred to in the introduction, requires additional modelling. In this section, we will introduce the notion of a pricing kernel and show how it can be used to derive term structure models and to price interest and inflation sensitive assets. Moreover, we will explain how the model can be used for risk management and how the model avoids the limitations of duration analysis that were referred to in Section 3.

To motivate the nature of the pricing kernel, consider a two period economy, where an agent maximizes expected utility and the utility function of the agent is time-separable, i.e. can be written as

$$\mathbb{E}_t\left(U(C_t, C_{t+1})\right) = u(C_t) + \varphi \mathbb{E}_t\left(u(C_{t+1})\right),$$

where C_t denotes consumption at time t. The parameter φ is the subjective discount factor which captures the impatience of the agent, i.e. consumption today is preferred over consumption next period ($\varphi \in (0,1)$). The income that the agent receives from an exogenous source, say labor income, will be denoted L_t. Next period's income, L_{t+1}, is a random variable. Now assume that a number of financial assets is traded in this economy with corresponding price vector P_t at time t. Next period's price vector, P_{t+1}, is a vector of random variables. The agent has to decide on his portfolio choice, i.e. will have to choose the number of units of each of the financial assets that will be held, which is reflected in the vector θ. Formally, the agent solves the problem[2]

$$\begin{aligned}&\max_{\theta} u(C_t) + \varphi \mathbb{E}_t\left(u(C_{t+1})\right)\\ &s.t.\ C_t = L_t - \theta^{\intercal} P_t,\ C_{t+1} = L_{t+1} + \theta^{\intercal} P_{t+1}\end{aligned}$$

Substituting the constraints into the objective and setting the derivative with respect to θ equal to zero yields the first order condition for an optimal consumption and portfolio choice[3]

$$P_t = \mathbb{E}_t\left(M_{t+1} P_{t+1}\right), \tag{14}$$

where

$$M_{t+1} = \varphi \frac{u_c(C_{t+1})}{u_c(C_t)}, \tag{15}$$

where u_c represents the marginal utility of consumption.

Because of (15), the pricing kernel is often referred to as the marginal rate of substitution. It is also known as the stochastic discount factor, since M_{t+1} discounts future payoffs to their current value. Assuming that satiation does not occur, M_{t+1} is positive since the marginal utility of consumption will be positive. In general, when arbitrage opportunities are excluded in financial markets, the existence of a positive pricing kernel is ensured.

Although (14) has here been derived for simplicity in a two period economy it can be shown that the relation is equally valid in a multi-period setting. Moreover, it can be shown that the assumption that the agent maximizes an expected time separable utility function is not needed to obtain (14); direct use of the law of one price and exclusion of arbitrage opportunities yields the same result (see e.g. Cochrane (2001)).

[2] In this illustration we do not consider portfolio constraints like short-sale constraints or borrowing constraints.

[3] Conditions on the utility function ensure that the necessary first order condition is as well sufficient for the optimal portfolio allocation.

The pricing kernel is a very powerful tool for valuation of all assets for which the joint distribution of the kernel and the future pay-off of the asset has to be modelled. Many influential papers in the recent literature on modelling nominal and real interest rates rely on specifications of the pricing kernel (see e.g. Campbell and Viceira (2001), Ang and Bekaert (2004), Ang and Piazzesi (2003), and Brennan and Xia (2002)). The use of the pricing kernel is probably best explained by considering a specific example. Assume that the prices of nominal and real bonds of all maturities as well as the stock market index in a specific economy are determined by a single state vector only. This state vector, x_t, could consist e.g. of the nominal annual interest rate, the annual inflation rate, and the return on the stock market index. Assume that the state vector is generated by a first order vector autoregressive process

$$x_{t+1} = \mu + \Gamma(x_t - \mu) + \varepsilon_{t+1}, \tag{16}$$

where $\varepsilon_{t+1} \overset{i.i.d.}{\sim} N(0, \Sigma)$. Assume in addition that the logarithm of the pricing kernel can be written as

$$-\log M_{t+1} = \alpha + \delta^{\intercal} x_t + \beta^{\intercal} \varepsilon_{t+1} + \eta_{t+1}, \tag{17}$$

where $\eta_{t+1} \overset{i.i.d.}{\sim} N\left(0, \sigma_\eta^2\right)$ and ε_s and η_t are mutually independent for all t and s. See Campbell, Lo, and MacKinlay (1997) for a motivation of this pricing kernel specification. Given this representation of the pricing kernel, modeling is tantamount to specification of the prices of risk, β.

Equation (14) can be used to price all assets that do not depend on η_{t+1}, the unspecified component of the pricing kernel. The reason is that σ_η^2 is not identified on the basis of the assets traded, since the innovations in the traded assets are solely determined by ε_{t+1}, which is by construction orthogonal to η_{t+1}.

To price the assets, one way is to compute the expectation in (14) through Monte Carlo simulation of (16) and (17). Consider an asset with a single pay-off at time $t+n$, e.g. a European option that matures at time $t+n$. Let the function $P_{t+n}(x_{t+1}, .., x_{t+n})$ denote how the pay-off depends on the future states of the economy. Substitution of (14) in itself and applying the law of iterated expectations yields

$$P_t = \mathbb{E}_t\left(P_{t+n}(x_{t+1}, .., x_{t+n}) \prod_{i=1}^{n} M_{t+i}\right) \tag{18}$$

The asset can be priced by approximating the expectation in (18) through Monte Carlo simulation of the state vector as well as the kernel simultaneously. Let $x_{t+s}^{(k)}$ and $M_{t+s}^{(k)}$ denote simulation result for period $t+s$ $(s = 1, .., n)$ in replication k $(k = 1, .., K)$. As K tends to infinity, the empirical mean

$$\hat{P}_t = \frac{1}{K} \sum_{k=1}^{K} \left[P_{t+n}(x_{t+1}^{(k)}, .., x_{t+n}^{(k)}) \prod_{i=1}^{n} M_{t+i}^{(k)}\right], \tag{19}$$

will converge in probability to the true price of the asset on the basis of the weak law of large numbers. In order to limit the computational effort, it may turn out to be useful to employ variance reduction techniques in order to reduce the sampling error. Examples are importance sampling, antithetic variables, and control variates, see for instance Glasserman (2003). Along the same lines, the value of a portfolio with cash flows at several points in time can be determined by adding the value of each of the components.

Monte Carlo simulation will be in used in Section 5 to value inflation sensitive liabilities of a particular pension scheme. The model structure that is chosen is such that analytical expressions can be obtained for the price of nominal as well as real bonds. It is well-known (see for instance Campbell, Lo, and MacKinlay

(1997), Campbell and Viceira (2001)) that the specific model structure in (16) and (17) implies, if the prices of risk, β, are affine in the state variables, for the nominal term structure of interest rates

$$\log P_t^{(n)} = -na_{x,n} - nb_{x,n}^{\intercal} x_t, \tag{20}$$

and for the real term structure of interest rates

$$\log P_t^{R(n)} = -na_{x,n}^R - nb_{x,n}^{R\intercal} x_t, \tag{21}$$

where the coefficients $a_{x,n}$, $b_{x,n}$, $a_{x,n}^R$, and $b_{x,n}^R$ are known functions of the parameters in (16) and (17). The model in (20) and (21) is typically referred to as an affine term structure model, because (2) and (3) imply that the nominal and real term yields are affine in the state vector. The appendix provides more details on the derivation of (20) and (21) and specifies explicitly how the coefficients in these equations can be determined from those in (16) and (17).

Note that (20) and (21) can also be used as tool for hedging and risk management in order to generalize duration analysis. Equations (20) and (21) describe how the value of nominal and real bonds with various maturities will depend on the vector of state variables. In duration analysis, only one state variable is considered (the level of the term structure) and the interest exposure of all bonds is set equal to the duration, n, of the bond (i.e. $b_{x,n} = 1$ is imposed for all n). An application of use of (20) and (21) in risk management will be discussed in Section 6.

As is well known in the option pricing literature, the use of pricing kernels to value assets is equivalent to several other approaches to valuation (see e.g. Cochrane (2001)). In case of complete markets, the use of the pricing kernel as outlined above is equivalent to pricing by replication. In case of incomplete markets, it is no longer possible to replicate any contingent claim and the pricing kernel is not uniquely defined. Consequently, infinitely many valid pricing kernels exist that price all traded assets correctly. Apart from modeling the asset dynamics, which is required both in complete and incomplete markets, this requires additional modeling of the pricing kernel. In this chapter, we have assumed a particular specification of the pricing kernel that is often used in finance theory. We assume that the prices of risk are constant, in line with e.g. Campbell and Viceira (2001). Alternatively, the prices of risk may be assumed to be affine in the factors, leading to essentially affine models that have been introduced by Duffee (2002) and applied in discrete time by Ang and Piazzesi (2003).

For the sake of completeness, we remark that the use of the pricing kernel is also equivalent to the use of state contingent discounting or risk neutral valuation, see Hibbert (2005). A specific model to compute discount factors implies a specific pricing kernel. Explicit specification of this kernel might be preferable to ensure consistency of the model parameters, the discount factors, and the true state probabilities.

5 Numerical example of the valuation of inflation sensitive liabilities

In this section, we illustrate the use of the model that was developed in Section 4 for the valuation of the liabilities of a stylized pension scheme. We distinguish between valuation at actuarial value and valuation at market value. In case of valuation at market values, i.e. fair value valuation, we distinguish between nominal, fully indexed, and conditionally indexed schemes. The results will indicate that in case of fair value valuation, the value of the liabilities can depend on the current level of state vector, such as interest and inflation rates, and, in case of the conditional indexation schemes, also on the funding ratio of the scheme and even on the asset mix. Throughout, we consider that the model structure that is used in valuation as given. Model-based valuation as proposed here clearly implies the risk that models or parameters are misspecified and leads to new questions related to adequate use of models and supervision. These issues will be discussed in Section 7.

The pension scheme that we consider in this example has linearly decreasing expected cash flows over the subsequent 60 years. The actuarial value of the liabilities (i.e. the value obtained by discounting all future cash flows at 4%) is set to 1000. The duration of the liabilities equals 13 years. These two assumptions fully determine the liability structure which is illustrated in Figure 2.

Insert here Figure 2

Note that we take a discontinuity perspective, i.e. we abstract from new premium inflows. Actuarial risks in the nominal cash flows are assumed to be fully diversifiable, which implies that this risk can be ignored for valuation purposes. In case of indexed or conditionally index pension schemes, the cash flows will obviously depend on future inflation.

The model that is used in valuation is a special case of the model introduced in Section 4. We assume for the dynamics of the state variables

$$R_{t+1}^{R(1)} = \mu_R + \varphi_R(R_t^{R(1)} - \mu_R) + \varepsilon_{t+1}^R; \quad (22)$$

$$\pi_{t+1} = \mu_\pi + \varphi_\pi(\pi_t - \mu_\pi) + \varepsilon_{t+1}^\pi; \quad (23)$$

$$r_{t+1} = \mu_r + R_t^{(1)} + \varepsilon_{t+1}^r, \quad (24)$$

where R^R, π, and r denote respectively the real short rate, one period inflation, and stock returns. For notational convenience, we define

$$\varepsilon_{t+1} = \left[\begin{array}{ccc} \varepsilon_{t+1}^R & \varepsilon_{t+1}^\pi & \varepsilon_{t+1}^r \end{array} \right]^\intercal,$$

where $\varepsilon_{t+1} \overset{i.i.d.}{\sim} N(0_{3\times1}, \Sigma)$. For the specification of the real pricing kernel[4] (M_{t+1}), we postulate

$$-\log M_{t+1} = R_t^{R(1)} + \alpha + \beta_R \varepsilon_{t+1}^R + \beta_r \varepsilon_{t+1}^r + \eta_{t+1}. \quad (25)$$

In principle, the parameters in (22) - (25) can be estimated from observed data, as discussed in e.g. Ang and Piazzesi (2003), Campbell and Viceira (2001), and Brennan and Xia (2002). Commonly used estimation procedures are maximum likelihood and GMM. For this illustration, we did not estimate the parameters along these lines, but rather selected, in an admittedly somewhat ad hoc way, a set of parameters that reflects the main stylized facts in the data. The data show that inflation and interest changes are highly persistent, and that long rates are less volatile than short rates . The choice of the parameters is based on the assumption that the real interest rates and inflation move independently. If we in addition assume that stock returns

[4] The relation between the nominal and real pricing kernel is given by

$$M_{t+1}^{\$} \frac{\Pi_{t+1}}{\Pi_t} = M_{t+1},$$

where $M_{t+1}^{\$}$ denotes the nominal pricing kernel and M_{t+1} indicates the real pricing kernel.

are independent of inflation and the process for the real short rate, the implied covariance matrix of the innovations Σ has a diagonal structure. The diagonal elements are selected so that the standard deviation of the innovations in real rates, inflation rates, and stock returns is 1.1%, 0.8%, and 15.5% respectively, which is roughly in line with historical data. Regarding the unconditional expectation of the state variables, we assumed that the unconditionally expected real rate (μ_R) is set to 4.0%, the corresponding figure for the inflation rate (μ_π) is 2.0%, while the equity risk premium (μ_r) is assumed to be 3.0%, which determines together with the stock return volatility β_r. The persistence parameters for the real rate and the inflation rate (φ_R and φ_π) are set to 0.94 and 0.90 respectively. The long-term bond risk premium, i.e. the premium for holding nominal bonds with fifty rather than one year to maturity, is set to 2%, which rejects the pure expectation hypothesis. This assumption determines β_R, which reflects the price of interest rate risk. Note that in (25) we have already assumed that inflation risk is not priced, i.e. the expected return of nominal and real bonds with the same maturity differs only due to an exposition of Jensen's inequality. Formally, this is reflected into the fact that $\beta_\pi = 0$, which has been incorporated already into the above-mentioned pricing kernel specification. We assume in this example that all risk premia are constant, but the model can be extended to exhibit time-varying risk premia, without adding much complexity. From a practical perspective, it may be important to note that these models are generally not able to fit the initial term structure exactly. Models in the same spirit that do allow for an exact fit of the current term structure are proposed by Hull and White (1996).

It should be noted that when pensions are indexed against wage rather than price inflation, this can be incorporated as well. A convenient way to accomplish this is to model wage inflation as price inflation plus an additional innovation. This risk factor, which determines the discrepancy between price and wage inflation, can be priced in the pricing kernel which leads to a real term structure on the basis of wage rather than price inflation.

As indicated in Section 4, an important property of the model in (22) - (25) is that it generates affine term structures. Rewriting (20) and (21), we find

$$R_t^{(n)} = a_{y,n} + b_{y,n}^{\intercal} y_t, \tag{26}$$

for the nominal term structure, whereas for the real term structure, we obtain

$$R_t^{R(n)} = a_{y,n}^R + b_{y,n}^{R\intercal} y_t, \tag{27}$$

i.e. both the nominal and real term structure are affine in the state vector, which contains in this case $y_t = \begin{bmatrix} R_t^{R(1)} & \pi_t & r_t^e \end{bmatrix}^{\intercal}$. We denote by r_t^e the return on stocks in excess of the nominal short rate. In (22) - (24) we have chosen the state vector to reflect the real rate with one year to maturity and the current inflation rate. Using (26) and (27), the selection of the state vector can be easily adjusted through rotation without affecting any of the results. Because the real rate is not directly observed in some countries, like e.g. the Netherlands, we will present several numerical results assuming that the nominal rate with one year to maturity and the current inflation rate constitute the state vector, which we denote by x_t once we add as a third element the excess return on stocks. The coefficients $a_{x,n}$, $b_{x,n}$, $a_{x,n}^R$, and $b_{x,n}^R$ with respect to the transformed state vector reflect the exposure of nominal and real rates with maturity n with respect to the the annual nominal rate and current inflation. The coefficients are fully determined by the parameters in (22) - (25), see the appendix.

The numerical values for the coefficients $a_{y,n}$, $b_{y,n}$, $a^R_{y,n}$, and $b^R_{y,n}$, as implied by the parameter assumptions referred to above are given in Table *1*. Note that due to the independence of the process for the excess stock returns of the interest rate and inflation processes, excess stock returns do not have an impact on either the nominal or real term structure, i.e. $(b_{y,n})_3 = \left(b^R_{y,n}\right)_3 = 0$ for every n.

Insert here Table 1

We provided the coefficients that correspond to the state vector y_t that contains the annual real rate and inflation, which are by assumption independent, to highlight the impact of changes in either of the factors. Note that long rates are much less sensitive too fluctuations in the annual nominal rate than short rates. Note also that inflation affects nominal rates positively, through the assumption that the real rate and inflation are independent. In valuing nominal and real liabilities, we assume for simplicity that the model in (22) - (25) perfectly fits the current nominal and real structure. In Section 6, we will see that, in case of market valuation, the exposure coefficients, $b_{y,n}$ and $b^R_{y,n}$, can play moreover a vital role in risk management of portfolios with interest and inflation sensitive liabilities.

As a first illustration of the use of the model in (22) - (25), we determine the fair value of nominal and fully indexed pension schemes and compare it to the actuarial value. In case of valuation at market value, the value of the scheme will depend on the current state vector, i.e. on the current nominal annual rate and inflation. We consider four cases for the current state of the economy, i.e. the nominal rate is either 5% or 7% and inflation equals either 2% or 4%. The nominal and real term structures implied by the model (and the parameter assumptions that have been made) are presented in Figure 3 and 4.

Insert here Figure 3

Insert here Figure 4

Using (6) and (8) these term structures determine the market value of a nominal and real scheme.

Insert here Table 2

Table *2* compares the impact of the selection of the discount factors on the value of the liabilities. The actuarial value obviously does not depend on the states and equals 1000 by construction. Figures 3 and 4 indicate that in all four cases the nominal term structure exceeds 4% for all maturities. It is therefore not surprising that the market value of a nominal pension with these expected cash flow is substantially less than 1000. As a first approximation to the value of these cash flows one could use duration analysis as discussed in Section 3. E.g. for the case where the nominal annual rate equals 7% and inflation 2%, the nominal term structure is approximately flat at the level of 7%. Duration analysis suggests that the value of the scheme is approximately 39% less than 1000 (i.e. from the actuarial 4% to the 'constant' 7%) since the duration of the liabilities is 13 years. As indicated before, the current model is somewhat more general than the one underlying duration analysis and not surprisingly the value obtained from the current model (644.1) does not fully coincide with the prediction of duration analysis (610).

The market value of fully indexed schemes is presented in the last column of Table 2. Discounting the expected nominal cash flows against the real rather than the nominal term structure implies that the market value of the liabilities increases substantially. The cost of indexation are estimated to be in the order of magnitude of $20\% - 40\%$, depending on the current state vector. The parameter assumptions made in this chapter imply that the market value of indexed liabilities is not too different from the actuarial value of the liabilities. Clearly, this is not the case though if the nominal interest rates are low and current inflation is high or the current nominal interest rate is high and the current inflation rate is low. Currently, funding ratios of pension funds are typically based on actuarial valuation of the liabilities. Comparison of the actuarial and market values of the liabilities is of course crucial to determine whether the use of market valuation will reduce the funding ratios further or will increase them.

Unlike valuation of the liabilities of straight nominal or indexed schemes as in (6) and (8), consistent valuation of liabilities that include derivative components of some sort can not easily be based on discounting expected future cash flows, since these discount rates are generally unknown. Equation (18) is equally valid for liabilities with derivative elements though and Monte Carlo analysis can be used for valuation using (19), and therefore enables consistent valuation. Many types of derivative elements can occur in pension schemes. Some schemes offer stock market exposure but with a fixed minimum return. Many pension schemes in the UK are characterized by indexation up to a maximum inflation of, say, 5%. In the Netherlands, many pension funds have introduced conditional indexation schemes of some form. These schemes offer only indexation if the funding ratio of the pension fund is sufficiently high. One example is a scheme that offers indexation if the nominal funding ratio is above 125% and none if this is not the case, see also Kortleve and Ponds (2005).

To illustrate valuation of liability schemes with conditional indexation and the value transfer that is implicit in a decision to switch from a fully indexed to a conditionally indexed scheme, we consider a specific conditional indexation scheme that roughly corresponds to the scheme adopted by ABP and PGGM. ABP and PGGM are by far the largest pension funds in the Netherlands and among the largest world-wide. ABP and PGGM have adopted a so-called policy ladder ("indexatie-staffel" in Dutch), which offers indexation if the nominal funding ratio of the fund is above an upper threshold, no indexation if the nominal funding ratio is below a lower threshold, and fractional indexation if it is between the two thresholds. In the numerical results, we set the upper threshold equal to 136% and the lower threshold to 105% a scheme. Intuitively, it is clear that such a scheme is close to a nominal scheme if the current funding ratio is low and the probability that it will ever recover from the low funding ratio is negligible. Equivalently, the scheme will be close to a fully indexed scheme if the funding ratio is high and the asset allocation strategy is such that the funding ratio is unlikely to fall below the upper threshold. This reasoning indicates that the value of the liabilities in such a scheme will depend on the current funding ratio as well as on the asset allocation. Dependence of the value of the liabilities on the asset allocation is at first sight surprising, but is a natural consequence of the fact that in case of low funding ratios the participants of the scheme will prefer risk seeking since it can increase their cash flows while the nominal cash flows are guaranteed anyway[5]. This is not unlike the preference for participants of an underfunded pension scheme for a risky asset allocation if the sponsor is obliged to pay for the down-side risk.

We allow for three different choices of the asset mix that includes nominal bonds with a maturity of 10 years as well as stocks. The initial nominal funding ratio takes the values 1 and 1.4. The numerical results on conditional indexation are presented in Table *3*.

Insert here Table 3

Not surprisingly, Table *3* shows that the value of conditionally indexed schemes is bounded from below

[5] This paper abstracts from additional complications when there is positive probability that the sponsor defaults.

by the value of nominal schemes and from above by the indexed schemes. When the initial funding ratio is low, the value of conditional indexation is close to the value of the nominal scheme. After all, the likelihood of ending up in a state in which the liabilities are indexed is small. The opposite is true when the initial funding ratio is relatively high, since in that case the conditional indexation scheme in similar to a fully indexed scheme.

Furthermore, we find for the dependence on the state vector that the value of conditional indexation is generally increasing in the current level of inflation and decreasing in the current nominal short rate. The intuition is that when the current inflation is higher, the liabilities are indexed against a higher inflation rate in case of indexation, implying that the value of the liabilities increases. If the nominal short rate increases, the whole nominal and real term structure increases, which implies that the liabilities are discounted at a higher rate. This leads to a decrease in the value of the liabilities.

Finally, the value of the liabilities is dependent on the choice of the asset mix. The rationale is that when the initial funding ratio is low, the probability of obtaining indexation is low and the conditional indexation scheme resembles a nominal scheme. Since the value of the liabilities is not lower in bad states of the world, i.e. when the asset returns are low, but higher in good states of the world, there is incentive for risk-seeking. The argument is analogous to standard option pricing theory. In case of a high initial funding ratio, the value of conditional indexation is lower in bad states of the world, since the liabilities will not be indexed. In good states of the world, the value of the value of the liabilities does not change that much, since indexation occurs already. Consequently, the value of the liabilities is higher when a less risky asset mix is implemented by the pension fund. This implies that the preferred asset mix can be influenced by the liabilities structure of the pension fund.

6 Numerical example of the risk management of inflation sensitive liabilities

In Section 3, we have discussed risk management of nominal and indexed liabilities using duration analysis. We emphasized that duration analysis is based on strong assumptions, which are easily shown to be at odds with stylized facts in the data. In particular, duration analysis assumes that long rates are as volatile as short rates and therefore tends to overestimate the interest rate risk of pension schemes. In this section, we indicate how duration analysis can be generalized to the models that are based on less restrictive assumptions as well as to risk management of liabilities that contain derivative components, such as the conditional indexation schemes that have been discussed in Section 5. The approach to risk management that is developed in this section can be used e.g. to hedge all risks of specific liability schemes, such as the conditional indexation scheme discussed in Section 5. In continuous time settings where markets are complete, this approach leads to replication of the stream of liabilities.

The relative exposure of the market value of an asset or liability with respect to changes of the short term interest rate was the corner stone of the discussion of risk management in Section 3. This concept can easily be generalized to the much more general models in (16) and (17) by considering the relative exposure of the market value of assets with respect to each of the elements of the state vector y_t. Using (26) the exposure of the market value of a zero-coupon bond with n periods to maturity with respect to the the k^{th} element of the state vector can be written as

$$\varepsilon_k^{(n)} = \frac{1}{P_t^{(n)}} \frac{\partial P_t^{(n)}}{\partial y_{k,t}} = -n \left(b_y^{(n)} \right)_k, \tag{28}$$

and analogously (27) implies for a real bond

$$\varepsilon_k^{R(n)} = \frac{1}{P_t^{R(n)}} \frac{\partial P_t^{R(n)}}{\partial y_{k,t}} = -n \left(b_y^{R(n)} \right)_k. \tag{29}$$

In order to illustrate the use of (28) and (29) to generate intuition on the price impact of changes in the state vector note that Table 1 implies $\left(b_y^{(10)} \right)_1 = 0.77$, i.e. market value of a nominal zero coupon bond with 10 years to maturity will increase approximately $10 \cdot 0.77\% = 7.70\%$ if the annual real interest rate drops by 1% and the inflation rate is unchanged. The corresponding number for duration analysis is 10%, as duration analysis is simply the special case of (28), where the state vector coincides with the annual nominal interest rate and $b_y^{(n)} = 1$ for all n. If both the inflation rate and the annual real interest rate drop by 1% the implied increase in the nominal price of the nominal bond with 5 years to maturity is $5 \cdot (0.89\% + 0.74\%) = 8.15\%$. Likewise, Table 2 implies the negative price impact on the equivalent indexed bond will be $5 \cdot 0.89\% = 4.45\%$ if both the nominal rate and inflation increase by 1%.

Like duration analysis equation (28) and (29) can not only be used to estimate the price impact of changes in the state vector, but can in principle also be used to derive hedging strategies. Consider e.g. an indexed zero-coupon bond with 10 years to maturity. According to Table 2, the price of an indexed bond has only an exposure (per unit invested) to the first factor, i.e. the annual real interest rate, of 7.7. The exposure of a portfolio with $w_1 \cdot 100\%$ of wealth invested in 1 year nominal bonds, $w_5 \cdot 100\%$ of wealth invested in 5 year nominal bonds and $(1 - w_1 - w_5) \cdot 100\%$ of wealth invested in 10 year nominal bonds can easily be verified to equal $w_1 \cdot b_y^{(1)} + 5 \cdot w_5 \cdot b_y^{(5)} + 10 \cdot (1 - w_1 - w_5) \cdot b_y^{(10)}$, with respect to both factors. Substitution of the numerical values from Table 1 shows that the exposure of the indexed bond coincides with that of the portfolio of nominal bonds if

$$w_1 \cdot \begin{bmatrix} 1.00 \\ 0.90 \end{bmatrix} + 5 \cdot w_5 \cdot \begin{bmatrix} 0.89 \\ 0.74 \end{bmatrix} + 10 \cdot (1 - w_1 - w_5) \cdot \begin{bmatrix} 0.77 \\ 0.59 \end{bmatrix} = 10 \cdot \begin{bmatrix} 0.77 \\ 0 \end{bmatrix}. \tag{30}$$

Equation (30) can easily be solved for w_1 and w_5 which yields $w_1 = 1269.9\%$ and $w_5 = -2617.9\%$. The asset allocation locally hedges all risk factors in the indexed bond under consideration. Since the maturities of the assets and their market values vary over time, a fully replicating strategy would require that the local hedge is adjusted very frequently and that the market is dynamically complete. It should be noted that the hedge portfolio requires rather extreme long and short positions, which can hamper the practical implementation of such a hedge strategy.

In Section 5, we have used pricing kernels to derive the market values of liabilities. Once the market is dynamically complete, the valuation on the basis of pricing kernels corresponds to cost of constructing a replicating portfolio. Although our market is not dynamically complete in the discrete time setting, we can determine the composition of the replicating portfolio, pretending that the market is complete. Knowing the market value of each of the components of the indexed scheme, the portfolio that initializes the replicating strategy for the indexed pension scheme can readily be obtained using (8) and (29), i.e.

$$\begin{aligned} \varepsilon_k^{V^R} &= \frac{1}{V_t^R} \frac{\sum_{n=1}^{N} \mathbb{E}_t \left(F_{t+n}\right) \partial P_t^{R(n)}}{\partial y_{k,t}} \\ &= -\frac{1}{V_t^R} \sum_{n=1}^{N} \mathbb{E}_t \left(F_{t+n}\right) n P_t^{R(n)} \left(b_y^{R(n)}\right)_k \end{aligned} \tag{31}$$

The appendix describes how the $\left(b_y^{R(n)}\right)_k$ coefficients can be obtained from the structural coefficients of the model in Section 4. For some specific maturities these parameters have already been reported in Table 2. For the cash flow scheme in Figure 1, which underlies the numerical example in Section 5, the interest and inflation exposure of the indexed scheme can easily be calculated to be $\varepsilon_1^{V^R} = -6107.9$ and $\varepsilon_2^{V^R} = 0$, when the state vector equals its long-term mean. The relative exposures, i.e. the exposures normalized by the value of the indexed liabilities, equals -7.2 for the first factor and 0 for the second factor. By solving a system of two equations in two unknowns like the one in (30), one finds that if the true price of the indexed liabilities equals $V_t^R = 848.1$, the hedge portfolio that invests in nominal bonds with maturities of 1 year, 5 year, and 10 years has the composition $w_1 = 1197.0\%$, $w_5 = -2452.3\%$, and $w_{10} = 1355.3\%$.

It should be noted that this hedge portfolio is dependent on the current value of the state vector. For example, when the initial state vector equals $(7\%, 4\%, 3\%)$ for the nominal short rate, the annual inflation and excess return in stocks instead of the long-term mean $(6\%, 2\%, 3\%)$, the hedge portfolio changes to $w_1 = 1216.8\%$, $w_5 = -2497.4\%$, and $w_{10} = 1380.5\%$, whereas in case of $(5\%, 2\%, 3\%)$, the hedge portfolio turns out to be $w_1 = 1221.7\%$, $w_5 = -2508.5\%$, and $w_{10} = 1386.8\%$. These portfolios show that the composition of the hedge portfolio is not too volatile given our setting of the parameters and structure of the liabilities. Whether this result carries over to other situations, remains an empirical question.

Until now we restricted ourselves in this section to the case of a fully indexed scheme. In case of non-linear elements in the liability structure, such as the conditional indexation scheme considered in Section 5, analytical expressions like (31) are no longer available. Nevertheless, the exposures of the market value of conditional indexation schemes to changes in the state vector can easily be determined numerically. Along the lines sketched above, one can therefore easily find the composition of the hedge portfolio that corresponds to such a liabilities structure. However, since the value of the liabilities is now dependent on the funding ratio, the asset mix, and the state vector, the hedge portfolio will exhibit these dependencies as well.

Note however that these numbers indicate that significant short sales and substantial trading may be required in the replicating strategy. The implicit assumption that these replicating strategies can be readily and costlessly implemented underlies a large part of the recent literature on modelling the real term structure, on valuation of indexed bonds, and on the utility gains of having access to indexed bonds. Analysis of the possible impact that transaction costs and short sell constraints can have on the replication argument is an important topic for future research.

7 Conclusions

In this chapter, we have considered market valuation and risk management of inflation sensitive pension rights. This topic deserves attention due to the introduction of the new international accounting standards, which requires fair value valuation of both assets and liabilities. In case of nominal or real liabilities, valuation is relatively straightforward. However, indexation schemes have been put in place that provide indexation of pension benefits only in particular states of the world. Valuation of such liabilities is considerably more complicated and introduces new challenges for risk management.

This chapter proposes a framework to value inflation sensitive liabilities by modeling the pricing kernel in the economy. The specification proposed is in line with the recent affine term structure literature. We discuss how to incorporate different assets, like stocks and nominal bonds, in the menu of assets. Using a stylized model, we illustrate the use of the model in valuing inflation sensitive pension rights. The model provides interesting insights in the determinants of the fair value of pension liabilities. We find that the asset mix, the initial funding ratio, and the current state of the economy, as expressed by the current term structure and inflation rate, can influence the value of the liabilities substantially. Especially the interplay between these factors is highlighted.

Apart from valuation, the model turns out to be particularly useful in risk management applications. First of all, we highlight the assumptions that underlie duration analysis, which is an important risk management tool nowadays in finance practice. By relaxing the assumptions justifying duration analysis, we illustrate the potential errors made by this rule. We discuss in detail how the model proposed can be used in order to come to a better assessment of the risk exposure of inflation sensitive liabilities.

Several topics are left for future research. In line with the literature, we have assumed throughout that there are no portfolio constraints, nor transaction costs. Although practically relevant, incorporation of market frictions is beyond the scope of this chapter. For ease of exposition, we have moreover restricted ourselves to simple models to illustrate the argument. It is well-known in the literature that market behavior can be more adequately described by higher order factor models for the term structure of interest rates, more general processes for inflation, as well as time-varying risk premia, see Dai and Singleton (2000, 2003), Duffee (2002), Ang and Piazzesi (2003), Campbell and Viceira (2001), and Sangvinatsos and Wachter (2005). In addition, Ang and Bekaert (2004) argue that regime-switching models are required to model interest rate data properly. Implementation issues of this particular class of models are addressed as well by Ang and Bekaert (2004).

Practical implementation of the model-based valuation and risk management procedures that are advocated in this chapter requires consensus on both the models and parameter values in order to satisfy the requirements of accountants and supervisors. Moreover, a better understanding of model risk, which is inherent to misspecification of aspects of the model, is required. All this is beyond the scope of this chapter.

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A Appendix

A.1 Derivation of the nominal term structure

In this appendix, we show how to derive the parameters of the term structure of interest rates, provided the model outlined in (16) and (17). The factors are left unspecified in what follows and can be specialized to the cases discussed in the main text. We assume that the factors that drive the term structure obey a VAR(1)-model

$$x_{t+1} = \mu + \Gamma(x_t - \mu) + \varepsilon_{t+1}, \tag{32}$$

where $\varepsilon_{t+1} \overset{i.i.d.}{\sim} N(0, \Sigma)$. We assume that both the nominal short rate and inflation are affine in the factors

$$R_t^{(1)} = \delta^\intercal x_t, \ \ \pi_t = \zeta^\intercal x_t. \tag{33}$$

Subsequently, we specify the nominal pricing kernel[6]

$$-\log M_{t+1}^{\$} = \delta^\intercal x_t + \frac{1}{2}\beta^\intercal \Sigma \beta + \beta^\intercal \varepsilon_{t+1}, \tag{34}$$

where $\eta_{t+1} \overset{i.i.d.}{\sim} N\left(0, \sigma_\eta^2\right)$ and ε_s and η_t are mutually independent for all t and s. The term $\frac{1}{2}\beta^\intercal \Sigma \beta$ is a Jensen's correction term to ensure that relation (14) is valid, i.e.

$$\mathbb{E}_t\left(M_{t+1}^{\$} \cdot 1\right) = P_t^{(1)}, \tag{35}$$

which excludes arbitrage opportunities in the economy.

We derive first of all an expression for the risk premium of holding an n period bond for 1 period. The risk premium is defined by

$$c^{(n)} = \mathbb{E}_t\left(\log P_{t+1}^{(n-1)} - \log P_t^{(n)}\right) - R_t^{(1)}, \tag{36}$$

i.e. the expected 1 period holding return in excess of the risk free rate. It follows from the structure of the model that bond prices are exponentially affine in the factors[7], i.e.

$$P_t^{(n)} = e^{-A_{x,n} - B_{x,n}^\intercal x_t}. \tag{37}$$

As a consequence, bond prices and the pricing kernel in (34) are jointly log-normally distributed. This implies for (36)

$$c^{(n)} = -\frac{1}{2} Var_t\left(\log P_{t+1}^{(n-1)}\right) - Cov_t\left(\log P_{t+1}^{(n-1)}, \log M_{t+1}^{\$}\right), \tag{38}$$

where we made use of the distributional properties and relation (14). Var_t and Cov_t denote respectively the conditional variance and covariance operators. Using the specification in (34) and (37), we find

$$c^{(n)} = -\frac{1}{2} B_{x,n}^\intercal \Sigma B_{x,n} - B_{x,n}^\intercal \Sigma \beta, \tag{39}$$

where the first term is a Jensen's correction term.

[6] In pricing kernel specifications (17) and (25), the Jensen term $\frac{1}{2}\beta^\intercal \Sigma \beta$ is denoted by α.

[7] Note that this is indeed a model property. The proof follows by induction and iterating the pricing relation in (14).

To determine the coefficients $A_{x,n}$ and $B_{x,n}$, we use the relation in (14), where we use the properties of the log-normal distribution. The argument is based on induction and delivers difference equations for both coefficient. Suppose (37) is valid up to maturity $n-1$, then

$$\begin{aligned} \log P_t^{(n)} &= \mathbb{E}_t\left(\log P_{t+1}^{(n-1)} + \log M_{t+1}^{\$}\right) + \frac{1}{2} Var_t\left(\log P_{t+1}^{(n-1)} + \log M_{t+1}^{\$}\right) \\ &= -A_{x,n-1} - B_{x,n-1}^{\intercal}\left(\mu + \Gamma(x_t-\mu)\right) - \delta^{\intercal} x_t - c^{(n)} \\ &= -A_{x,n} - B_{x,n}^{\intercal} x_t. \end{aligned} \tag{40}$$

Hence, $A_{x,n}$ and $B_{x,n}$ satisfy

$$\begin{aligned} A_{x,n} &= A_{x,n-1} + B_{x,n-1}^{\intercal}\left(I-\Gamma\right)\mu + c^{(n)}; \\ B_{x,n} &= \Gamma^{\intercal} B_{x,n-1} + \delta, \end{aligned} \tag{41}$$

where the difference equations are subject to the initial conditions

$$A_{x,1} = 0,\ B_{x,1} = \delta. \tag{42}$$

Note that the difference equation for $B_{x,n}$ is solved by

$$B_{x,n} = \left(I - \Gamma^{\intercal}\right)^{-1}\left(I - \Gamma^{\intercal n}\right)\delta. \tag{43}$$

Obviously, the coefficients $a_{x,n}$ and $b_{x,n}$ relate to $A_{x,n}$ and $B_{x,n}$ via

$$a_{x,n} = n^{-1} A_{x,n},\ b_{x,n} = n^{-1} B_{x,n}. \tag{44}$$

A.2 Derivation of the real term structure

The derivation of the real term structure resembles to a large extent the derivation of the nominal term structure. The main difference is the indexation that takes place every period. Formally, the price of a real bond at time t with maturity n is given by

$$P_t^{R(n)} = e^{-A_{x,n}^R - B_{x,n}^{R\intercal} x_t}, \tag{45}$$

whereas the payoff is given by

$$P_{t+1}^{R(n-1)} \frac{\Pi_{t+1}}{\Pi_t}. \tag{46}$$

We define the risk premium on an n period real bond as

$$c^{R(n)} = \mathbb{E}_t\left(\log P_{t+1}^{R(n-1)} + \pi_{t+1} - \log P_t^{R(n)}\right) - R_t^{(1)}, \tag{47}$$

which can be rewritten along the lines of (38) to

$$\begin{aligned} c^{R(n)} &= -\frac{1}{2} Var_t\left(\log P_{t+1}^{(n-1)} + \pi_{t+1}\right) - Cov_t\left(\log P_{t+1}^{(n-1)} + \pi_{t+1}, \log M_{t+1}^{\$}\right) \\ &= -\frac{1}{2}\left(B_{x,n-1}^R - \zeta\right)^{\intercal} \Sigma \left(B_{x,n-1}^R - \zeta\right) - \left(B_{x,n-1}^R - \zeta\right)^{\intercal} \Sigma\beta. \end{aligned} \tag{48}$$

In order to determine the coefficients $A_{x,n}^R$ and $B_{x,n}^R$, the same induction argument can be applied, i.e.

$$\begin{aligned} \log P_t^{(n)} &= \mathbb{E}_t\left(\log P_{t+1}^{(n-1)} + \pi_{t+1} + \log M_{t+1}^{\$}\right) + \frac{1}{2} Var_t\left(\log P_{t+1}^{(n-1)} + \pi_{t+1} + \log M_{t+1}^{\$}\right) \\ &= -A_{x,n-1}^R - \left(B_{x,n}^R - \zeta\right)^{\intercal}\left(\mu + \Gamma(x_t - \mu)\right) - \delta^{\intercal} x_t - c^{R(n)} \\ &= -A_{x,n}^R - B_{x,n}^{R\intercal} x_t. \end{aligned} \tag{49}$$

As a result, we obtain the following difference equations

$$\begin{aligned} A^R_{x,n} &= A^R_{x,n-1} + \left(B^R_{x,n-1} - \zeta\right)^\intercal (I - \Gamma)\,\mu + c^{R(n)}; \\ B^R_{x,n} &= \Gamma^\intercal \left(B^R_{x,n-1} - \zeta\right) + \delta. \end{aligned} \tag{50}$$

The boundary conditions are given by

$$A^R_{x,0} = 0,\ B^R_{x,0} = 0. \tag{51}$$

The difference equation for $B^R_{x,n}$ can again be solved in closed form, i.e.

$$B^R_{x,n} = \left(I - \Gamma^\intercal\right)^{-1} \left(I - \Gamma^{\intercal n}\right)\left(\delta - \Gamma^\intercal \zeta\right). \tag{52}$$

Finally, we remark that $a^R_{x,n}$ and $b^R_{x,n}$ equal

$$a^R_{x,n} = n^{-1} A^R_{x,n},\ b^R_{x,n} = n^{-1} B^R_{x,n}. \tag{53}$$

B Figures

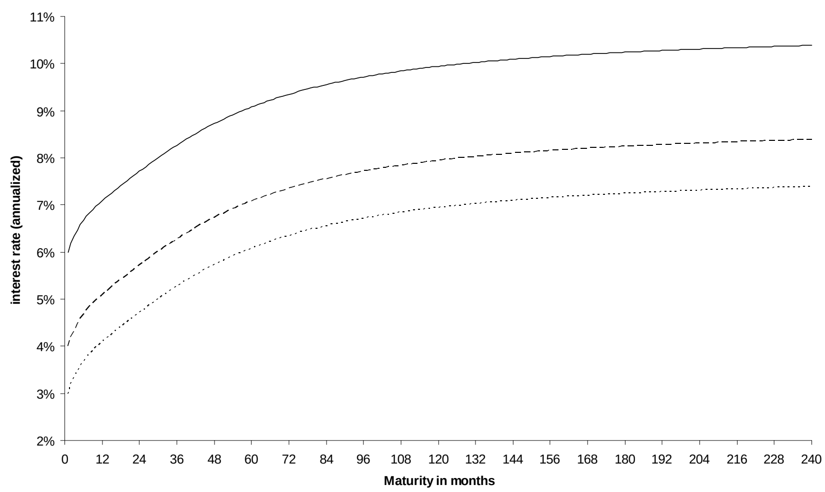

Figure 1: Impact of changes in the interest rate on the term structure if the assumptions underlying duration analysis are satisfied. Only shifts in a parallel fashion can occur according to duration analysis.

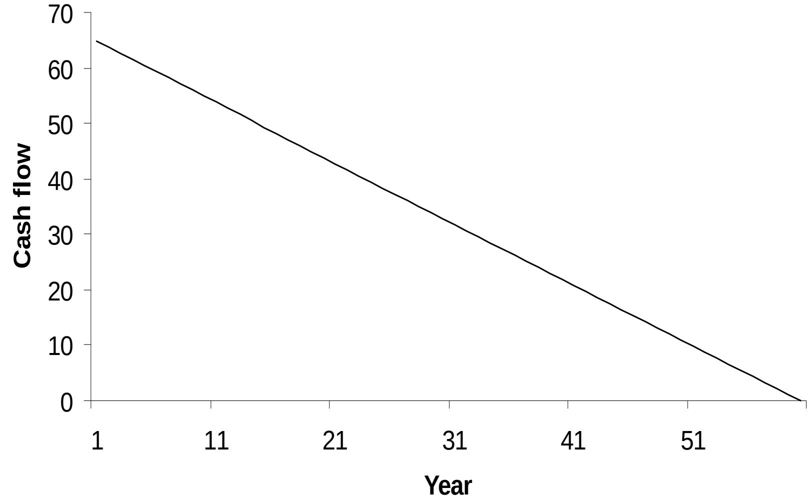

Figure 2: Liabilities structure of the artificial pension fund considered in Section 5. Dependent on the assumptions regarding indexation scheme that has been put in place, these cash flows are indexed annually. The linearly declining scheme implies that we do not consider new premium inflows.

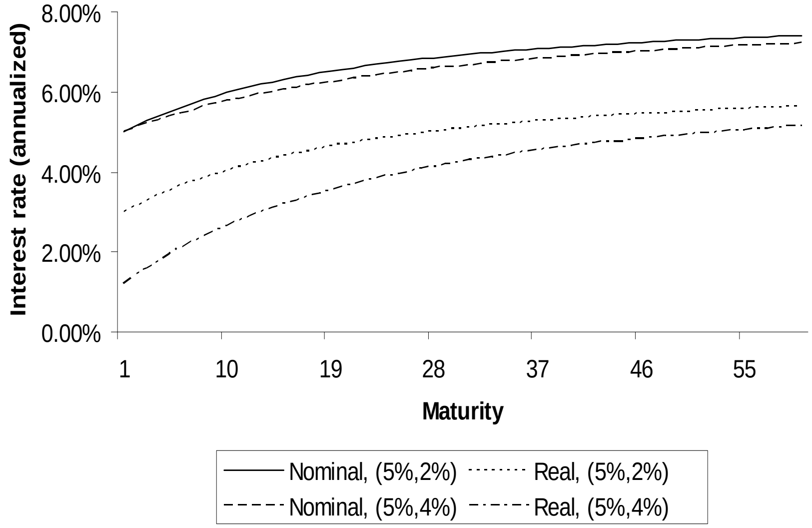

Figure 3: Nominal and real term structures for different values of the initial state vector. The first element refers to the current nominal short rate, whereas the second element corresponds to the current 1 year inflation rate.

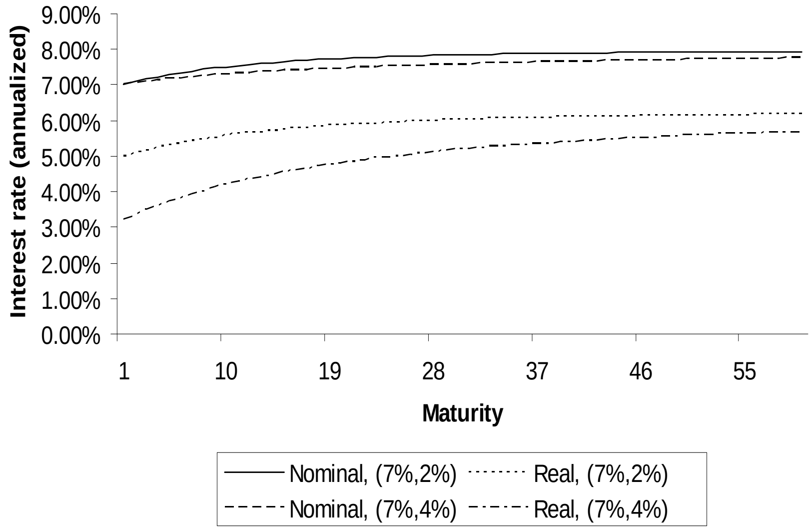

Figure 4: Nominal and real term structures for different values of the initial state vector. The first element refers to the current nominal short rate, whereas the second element corresponds to the current 1 year inflation rate.

C Tables

Maturity	$a_{y,n}$	$(b_{y,n})_1$	$(b_{y,n})_2$	Risk premium
1	0.20%	1.00	0.90	0.00%
2	0.52%	0.97	0.86	0.23%
3	0.83%	0.94	0.81	0.42%
4	1.11%	0.91	0.77	0.59%
5	1.38%	0.89	0.74	0.75%
10	2.49%	0.77	0.59	1.27%
20	4.00%	0.59	0.40	1.73%
30	4.93%	0.47	0.29	1.89%
50	5.98%	0.32	0.18	1.99%
Maturity	$a^R_{y,n}$	$\left(b^R_{y,n}\right)_1$	$\left(b^R_{y,n}\right)_2$	Risk premium
1	0.00%	1.00	0	0.00%
2	0.24%	0.97	0	0.24%
3	0.46%	0.94	0	0.44%
4	0.67%	0.91	0	0.63%
5	0.87%	0.89	0	0.80%
10	1.73%	0.77	0	1.40%
20	2.91%	0.59	0	1.96%
30	3.68%	0.47	0	2.17%
50	4.55%	0.32	0	2.29%

Table 1: Implied coefficients of both the nominal and the real term structure of interest rates. In addition, the risk premium of holding a n period bond for one holding period is repeated in column 5.

Current 1Y nominal rate	Current inflation	Actuarial	Nominal	Real
5%	2%	1000	736.9	914.0
5%	4%	1000	755.2	1050.4
7%	2%	1000	644.1	788.3
7%	4%	1000	658.8	900.3

Table 2: The nominal, real, and actuarial value of the liabilities has been determined for different initial states of the economy. The nominal short rate equals either 5% or 7%, whereas the current inflation equals either 2% or 4%.

Current nom. short rate	Current inflation	Current funding ratio	Fraction stocks	Value conditional indexation	Nom. value liabilities	Real value liabilities
Panel A						
5%	2%	1	0%	740.4		
5%	2%	1.4	0%	895.7		
5%	2%	1	50%	768.1	736.9	914.0
5%	2%	1.4	50%	868.7		
5%	2%	1	100%	780.1		
5%	2%	1.4	100%	840.9		
Panel B						
5%	4%	1	0%	759.1		
5%	4%	1.4	0%	980.5		
5%	4%	1	50%	796.7	755.2	1050.4
5%	4%	1.4	50%	949.3		
5%	4%	1	100%	817.4		
5%	4%	1.4	100%	914.0		
Panel C						
7%	2%	1	0%	647.8		
7%	2%	1.4	0%	776.2		
7%	2%	1	50%	669.4	644.1	788.3
7%	2%	1.4	50%	754.7		
7%	2%	1	100%	679.4		
7%	2%	1.4	100%	731.1		
Panel D						
7%	4%	1	0%	663.1		
7%	4%	1.4	0%	850.9		
7%	4%	1	50%	692.7	658.8	900.3
7%	4%	1.4	50%	823.4		
7%	4%	1	100%	709.9		
7%	4%	1.4	100%	792.5		

Table 3: Value of liabilities when the liabilities are conditionally indexed. Full indexation occurs when the (nominal) funding ratio is higher than 1.36, whereas no indexation is granted whenever the (nominal) funding ratio is lower than 1.05. In between, indexation is provided proportionally. The value of conditionally indexed liabilities is compared to the value of nominal and real liabilities. The actuarial value of the liabilities is equal to 1000. We determine these values for different states of the economy, different initial funding ratios, and different choices of the asset mix. The asset mix selects between stocks and 10Y nominal bonds.