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# Mean-Coherent Risk and Mean-Variance Approaches in Portfolio Selection: an Empirical Comparison

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# Mean-Coherent Risk and Mean-Variance Approaches in Portfolio Selection: an Empirical Comparison

## **Abstract**

We empirically analyze the implementation of coherent risk measures in portfolio selection. First, we compare optimal portfolios obtained through mean-coherent risk optimization with corresponding mean-variance portfolios. We find that, even for a typical portfolio of equities, the outcomes can be statistically and economically different. Furthermore, we apply spanning tests for the mean-coherent risk efficient frontiers, which we compare to their equivalents in the mean-variance framework. For portfolios of common stocks the outcomes of the spanning tests seem to be statistically the same.

*Keywords:* portfolio choice, mean variance, mean coherent risk, comparison.

*JEL Classification:* G11.

# I Introduction

There is an ongoing debate in the financial literature on which risk measure to use in risk management and portfolio choice. As some risk measures are more theoretically appealing, others are easier to implement practically. For a long time, the standard deviation has been the predominant measure of risk in asset management. Mean-variance portfolio selection via quadratic optimization, introduced by Markowitz (1952), used to be the industry standard (see, for instance, Tucker et al. (1994)). Two justifications for using the standard deviation in portfolio choice can be given. First, an institution can view the standard deviation as a measure of risk, which needs to be minimized to limit the risk exposure. Second, a mean-variance portfolio maximizes expected utility of an investor if the utility index is quadratic or asset returns jointly follow an elliptically symmetric distribution.<sup>1</sup>

Despite the computational advantages, the variance is not a satisfactory risk measure from the risk measurement perspective. First, mean-variance portfolios are not consistent with second-order stochastic dominance (SDD) and, thus, with the benchmark expected utility approach for portfolio selection. Second, but not independently, as a symmetric risk measure, the variance penalizes gains and losses in the same way.

Artzner et al. (1999) give an axiomatic foundation for so-called coherent risk measures. They propose that a "rational" risk measure related to capital requirements<sup>2</sup> should be monotonic, subadditive, linearly homogeneous, and translation invariant. Tasche (2002) and Kusuoka (2001) demonstrate that a Choquet expectation with a concave distortion function represents a general class of coherent risk measures. Moreover, with some additional regularity restrictions, as imposed by Kusuoka (2001), the class of coherent risk measures becomes consistent with the second order stochastic dominance principle and thus generates portfolios consistent with the expected utility paradigm, see, for example, Ogryczak and Ruszczyński (2002) and De Giorgi (2005).

The class of coherent risk measures generalizes expected shortfall, a co-

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<sup>1</sup>See, for instance, Ingersoll (1987).

<sup>2</sup>The capital requirements are relevant for asset management since they are directly applied to financial institutions, see the Basel Accord (1999).

herent risk measure which received a lot of attention in the recent literature due to its easy practical implementability and tractability. Tasche (2002) discusses theoretical properties of expected shortfall and its generalizations. He suggests a general method how to calculate expected shortfall risk contributions of individual assets in a portfolio. At the same time, a literature on how to apply expected shortfall in portfolio optimization appeared. Rockafellar and Uryasev (2000) provide an algorithmic solution to the expected shortfall-based portfolio optimization and hedging. Bertsimas et al. (2004) report theoretical properties of expected shortfall and show that the mean-expected shortfall optimization problem can be solved efficiently as a convex optimization problem. They also provide some empirical evidence on asset allocation and index tracking applications.

There is also a broad empirical literature on expected shortfall. Bassett et al. (2004) show that a sample portfolio choice problem based on expected shortfall is equivalent to a quantile regression. Focusing mainly on the quantitative economic effect, they demonstrate that for certain asymmetric distributions of asset returns the difference between mean-variance and mean-expected shortfall efficient portfolio weights can be substantial. Kerkhof and Melenberg (2004) develop a framework for backtesting expected shortfall using the functional delta method. They show in a simulation study that tests for expected shortfall have better performance than tests for value-at-risk with acceptably low probability thresholds. Bertsimas et al. (2004) discuss various properties of expected shortfall. They provide empirical evidence based on asset allocation and tracking index examples that the mean-expected shortfall approach might have advantages over the mean-variance approach. Similarly to Bassett et al. (2004), the authors focus mainly on examples with simulated returns.

Even though the literature on coherent risk measures emphasizes the importance of the difference between these and conventional risk measures in asset allocation and risk management, there still seems to be lack of evidence on the statistical and economic significance of this difference in practical applications. The aim of this paper is to analyze the degree of statistical and economic relevance of the switch from the traditional standard deviation to a coherent risk measure in a typical asset allocation problem, which consists of

determining the optimal portfolio weights or of deciding whether particular assets have to be additionally included into the portfolio. Our contribution is twofold. First, we compare portfolios obtained by mean-coherent risk and mean-variance optimizations both statistically and economically. We do this for simulated asset returns as well as for actually traded securities. If the distribution of asset returns and liabilities were elliptically symmetric then any coherent regular risk measure of a portfolio would be proportional to its standard deviation, and, as a result, would lead to the same implications in risk management. In reality, asset returns are likely to be skewed and fat tailed. It is, however, an empirical question whether skewness and excess kurtosis alone are sufficient to generate statistically and economically different efficient portfolios if the variance is replaced by a coherent risk measure in a portfolio optimization problem. Here, we address this question by first deriving the asymptotic distribution of the mean-coherent risk portfolio weights and using these to statistically and economically compare the mean-coherent risk and mean-variance efficient portfolio weights. Additionally, we explain how to reformulate the point mass approximated mean-coherent risk problem as a linear program, which can be efficiently solved by numerical algorithms. The results obtained for simulated and actual portfolios suggest that portfolios based on coherent risk measures are often statistically and economically different from the portfolios based on standard deviation for a typical portfolio of equities. Our simulation study confirms that for portfolios with asymmetric distributions of returns, such as portfolios of derivatives or credit instruments, an optimization based on a coherent risk measure behaves differently as it accounts mostly for negative returns<sup>3</sup>. As second contribution, we implement spanning tests for the mean-coherent risk efficient frontiers as developed by Polbennikov and Melenberg (2005). These tests can be regarded as an analog for the usual mean-variance spanning tests, see DeRoos and Nijman (2001) for a survey of the mean-variance tests. The test statistics are compared to their counterparts in the mean-variance framework. Our mean-variance and mean-coherent risk spanning tests for portfolios of common equities give statistically and economically similar results.

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<sup>3</sup>We do not study actual portfolios with derivatives due to related problems with stationarity.

The remainder of the paper is structured as follows. Section II describes the methodology, including the statistical comparison of mean-variance and mean-coherent risk efficient portfolio weights and spanning tests for coherent risk measures. Empirical results on the comparison of the efficient portfolio weights and applications of the coherent risk-spanning test are described in section III. Section V discusses effects of estimation error in expected asset returns. Finally, section VI concludes.

## II Methodology

### II.A Coherent risk measures and portfolio choice

Consider a probability space  $(\Omega, \mathcal{F}, P)$ ,<sup>4</sup> and let  $L_0(\Omega, \mathcal{F}, P)$  be the space of all equivalence classes of real valued random variables  $X : \Omega \rightarrow \mathbb{R}$ . A random variable  $X \in L_0(\Omega, \mathcal{F}, P)$  can be seen as a risky financial position (profit or loss) and we call it a *risk*. If we consider the set  $\mathcal{X} := L_0(\Omega, \mathcal{F}, P, \mathbb{R})$  of all risks then a *risk measure*  $\rho$  defined on  $\mathcal{X}$  is a map from  $\mathcal{X}$  to  $\mathbb{R} \cup \{+\infty\}$ , see Delbaen (2000).<sup>5</sup> Intuitively, one can consider a risk measure as measuring the riskiness of the position or cost of risk. The concept of the cost of risk can be formalized by defining the *capital requirement* or amount of reserved capital ("sweetener") as a function of the risk measure  $\rho$ . We consider risk measures defined on general probability spaces  $L_0(\Omega, \mathcal{F}, P)$ , and probability spaces of bounded random variables  $L_\infty(\Omega, \mathcal{F}, P) = \{X \in L_0(\Omega, \mathcal{F}, P) : P[|X| < \infty] = 1\}$ . Denote

$$\rho_\infty : L_\infty(\Omega, \mathcal{F}, P, \mathbb{R}) \rightarrow \mathbb{R}, \quad (1)$$

$$\rho_0 : L_0(\Omega, \mathcal{F}, P, \mathbb{R}) \rightarrow \mathbb{R} \cup \{\infty\}. \quad (2)$$

For a long time, the standard deviation has served as the common risk measure.<sup>6</sup> Since it measures the "degree of the deviation" of a random variable from its mean it was perceived as a good measure of risk. Moreover, it

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<sup>4</sup> $\Omega$  is the set of states,  $\mathcal{F}$  is the  $\sigma$ -algebra, and  $P$  is the probability measure.

<sup>5</sup>The range includes  $\infty$  to make *coherent* risk measures on  $L_0(\Omega, \mathcal{F}, P)$  possible.

<sup>6</sup>Well defined on the space  $L_2(\Omega, \mathcal{F}, P)$  and set equal to  $+\infty$  on  $L_0(\Omega, \mathcal{F}, P) \setminus L_2(\Omega, \mathcal{F}, P)$ , where  $L_k(\Omega, \mathcal{F}, P) = \left\{X \in L_0 : \int |X|^k dP < \infty\right\}$  for  $k > 0$ .



has some very attractive properties. In particular, the standard deviation is closely related to the measure concept of square integrable random variables. This property leads to some nice theoretical results in mean-variance analysis. The standard deviation is also attractive for its analytical and numerical tractability. Indeed, it is easy to model, estimate, and implement in empirical problems of asset management. The main criticism regarding the standard deviation is related to the fact that it symmetrically measures losses and profits as contributions to riskiness of a financial position. Many different alternatives that concentrate on the downside part of the risk distribution have been proposed. The paper by Pedersen and Satchell (1998) illustrates this effort by providing an overview and classifying common measures of risk.

Artzner et al. (1999) follow the axiomatic approach to define a risk measure coherent from a regulator's point of view. They relate a risk measure to the regulatory capital requirement and deduce four axioms which should be satisfied by a "rational" risk measure. Delbaen (2000) extends the definition to general probability spaces  $L_0(\Omega, \mathcal{F}, P)$ .

**Definition 1** *A mapping  $\rho = \rho_0 : \mathcal{X} \rightarrow \mathbb{R} \cup \{+\infty\}$  is called a coherent measure of risk if it satisfies the following conditions for all  $X, Y \in \mathcal{X}$ .*

- *Monotonicity: if  $X \leq Y$ , then  $\rho(X) \geq \rho(Y)$ .*
- *Translation Invariance: if  $m \in \mathbb{R}$ , then  $\rho(X + m) = \rho(X) - m$ .*
- *Positive Homogeneity: if  $\lambda \geq 0$ , then  $\rho(\lambda X) = \lambda \rho(X)$ .*
- *Subadditivity:  $\rho(X + Y) \leq \rho(X) + \rho(Y)$ .*

The financial meaning of monotonicity is clear: The downside risk of a position is reduced if the payoff profile is increased. Translation invariance is motivated by the interpretation of the risk measure  $\rho(X)$  as a capital requirement, i.e.,  $\rho(X)$  is the amount of capital which should be added to the position to make  $X$  acceptable from the point of view of the regulator. Thus, if the amount  $m$  is added to the position, the capital requirement is reduced by the same amount. Positive homogeneity says that riskiness of a financial position grows in a linear way as the size of the position increases. This

assumption is not always realistic. Withdrawing the positive homogeneity axiom leads to a family of convex risk measures, see Föllmer and Schied (2002).<sup>7</sup> The subadditivity property allows one to decentralize the task of managing the risk arising from a collection of different positions: If separate risk limits are given to different desks, then the risk of the aggregate position is bounded by the sum of the individual risk limits. The subadditivity is also closely related to the concept of risk diversification in a portfolio of risky positions.

These axioms rule out many of the conventional measures of risk traditionally used in finance. For instance, the standard deviation is ruled out by the monotonicity requirement.

Kusuoka (2001) adds another two axioms that further constraint the set of coherent risk measures

- *Law Invariance:* if  $P[X \leq t] = P[Y \leq t] \forall t$ , then  $\rho(X) = \rho(Y)$ .
- *Comonotonic Additivity:* if  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  are measurable and non-decreasing, then  $\rho(f \circ X + g \circ X) = \rho(f \circ X) + \rho(g \circ X)$ .

The intuition of the two axioms is simple: the Law of Invariance means that financial positions with the same distribution should have the same risk. It allows empirical identification of the risk measure. The second condition on Comonotonic Additivity refines slightly the subadditivity property: subadditivity becomes additivity when two positions are comonotonic. By comonotonicity we understand that the random variables are monotonic transformations of the same random variable. Suppose that we are given two non-decreasing functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  and a random variable  $X \in L_0(\Omega, \mathcal{F}, P)$ . Then the random variables  $Z = f(X)$  and  $Y = g(X)$  are called comonotonic. The following result was shown by Kusuoka (2001), Tasche (2002), and Denneberg (1990):

*A risk measure  $\rho = \rho_\infty$  defined on  $L_\infty(\Omega, \mathcal{F}, P)$ , with  $P$  non-atomic, is coherent, law invariant, and comonotonic additive if and only if for any random variable  $X$  with cumulative distribution function  $F_X(\cdot)$  it can be*

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<sup>7</sup>See, however, De Giorgi (2005) on homogenizing risk measures.

represented as

$$\rho(X) = \int_0^1 s_\alpha(X) d\phi(\alpha), \quad (3)$$

where  $\phi$  is a probability measure defined on the interval  $[0, 1]$ , and  $s_\alpha$  is the expected shortfall of  $X$

$$s_\alpha(X) = -\alpha^{-1} \int_0^\alpha F_X^{-1}(t) dt.$$

This risk measure defined on the general probability space  $L_0(\Omega, \mathcal{F}, P)$  for non-positive random variables  $X$  stays coherent, law invariant, and comonotonic additive, see Delbaen (2000). We call a coherent, law invariant, and comonotonic additive measure of risk represented by equation (3) a *coherent regular risk* (CRR) measure.

*Expected Shortfall.* A CRR risk measure that gained a lot of attention in the recent literature is the expected shortfall, given by

$$s_\tau(X) = -\tau^{-1} \int_0^\tau F_X^{-1}(t) dt,$$

which corresponds to  $\phi(\alpha) = I(\alpha \geq \tau)$ . Being a coherent regular risk measure, it satisfies comonotonic additivity, law invariance and all axioms of a coherent risk measure. Many useful properties of expected shortfall are established, for example, in Tasche (2002) and Bertsimas et al. (2004).

*Point Mass Approximation (PMA) of CRR measure.* Bassett et al. (2004) suggested to approximate a CRR measure by a weighted sum of Dirac's point mass functions.<sup>8</sup> This approximation corresponds to the probability measure  $\phi'(\alpha) = \sum_{k=1}^m \phi_k \delta_{\tau_k}(\alpha)$  in expression (3), with  $\phi_k \geq 0$  and  $\sum \phi_k = 1$ . The PMA CRR measure can be written as

$$\rho(X) = \sum_{k=1}^m \phi_k s_{\tau_k}(X). \quad (4)$$

Notice, that the PMA CRR measure is itself a CRR measure, and the term PMA refers to the fact that the integral in expression (3) is replaced by a

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<sup>8</sup>The point mass function  $\delta_\tau(\alpha)$  is defined through the integral  $\int_{-\infty}^x \delta_\tau(\alpha) d\alpha = I(x \geq \tau)$ .

finite weighted sum in (4). From the form of the PMA CRR measure it is clear that the expected shortfall is a particular case of this approximation.

A nice property of these two examples is that in both cases the in-sample mean-CRR optimization problem can be reformulated as a linear program, which can be solved efficiently. The mean-expected shortfall optimization is considered among others by Rockafellar and Uryasev (2000), Bertsimas et al. (2004), and Bassett et al. (2004). The mean-PMA CRR optimization is discussed in subsection II.D. Additionally, as special cases of the mean-CRR portfolio selection problem, mean-expected shortfall and mean-PMA CRR optimizations are consistent with second-order stochastic dominance and, thus, fall in the reward-risk theoretical framework developed by De Giorgi (2005).

For a fixed set of random returns  $\{R_0, \dots, R_p\}$ , a risk measure  $\rho = \rho(\sum_{i=0}^p w_i R_i)$  can be considered as a function of portfolio weights,

$$\rho(w_0, \dots, w_p) : \left\{ (w_0, \dots, w_p) \in \mathbb{R}^p : \sum_i w_i = 1 \right\} \rightarrow \mathbb{R}.$$

Denote by  $\mu_i = E[R_i]$  the expected return of asset  $i$  (which we assume to exist). Given the portfolio expected return  $\nu = \sum_i \mu_i$  we try to find portfolio weights  $\{w_i\}$  that minimize the chosen risk measure. The corresponding optimization problem can be formulated as follows:

$$\min_{\{w_0, \dots, w_p\}} \rho(w_0, \dots, w_p) \quad s.t. \quad \sum_{i=0}^p w_i = 1, \quad \sum_{i=0}^p w_i \mu_i = \nu. \quad (5)$$

When solving this problem, we assume that  $\rho(w_1, \dots, w_p) < \infty$ . It is straightforward that the first equality constraint can be eliminated by passing it to the objective function. Denote by  $y = R_0$  the return on the benchmark asset  $R_0$ . Define by  $x = (R_1 - R_0, \dots, R_p - R_0)'$  the vector of excess returns of the other assets. The mean-risk optimization problem (5) can be rewritten as

$$\min_{\theta \in \mathbb{R}^p} \rho(y + x'\theta) \quad s.t. \quad E[y + x'\theta] = \nu, \quad (6)$$

where  $\theta$  is the  $p \times 1$  vector of portfolio weights of assets  $1, \dots, p$ . When one chooses the standard deviation as the risk measure  $\rho$  in optimization (6) the

standard mean-variance portfolio problem is obtained. Alternatively, when a CRR measure is chosen, the solution to (6) is the vector of mean-CRR portfolio weights. The standard deviation has an advantage over other risk measures in empirical applications since the estimation and optimization parts can be separated from each other. In this case the random returns  $(R_0, \dots, R_p)$  should be square integrable. The expected shortfall portfolio optimization problem is an example of the mean-CRR portfolio that can be solved by convex programming methods as, for example, suggested by Bertsimas et al. (2004) and Rockafellar and Uryasev (2000). Bassett et al. (2004) show that the mean-expected shortfall efficient portfolio problem is equivalent to a quantile regression with linear constraints. As a result the problem can be solved by well developed standard methods.<sup>9</sup>

## II.B Comparison of Portfolio Weights

The question of the comparison of the efficient portfolio weights for the standard deviation and a CRR risk measure arises naturally. For elliptically symmetric distributions the standard deviation and a CRR measure give the same portfolio weights in the mean-risk optimization.<sup>10</sup> For other distributions the efficient portfolio weights will, in general, alter. But the question then is whether this difference is significant, either economically or statistically, or both.

To statistically compare the mean-variance and mean-CRR portfolio weights we need to derive their joint asymptotic distribution. Then, standard statistical procedures can be applied. The asymptotic results on portfolio weights as well as the equality test for mean-CRR and mean-variance portfolio weights are given in Appendixes A, B, and C.

It is well known that portfolio weights are very sensitive to estimation inaccuracy in asset expected returns, see, for example, Chopra and Ziemba (1993). This often leads to insignificance of estimated portfolio weights due to high standard errors and potentially can yield insignificant comparison

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<sup>9</sup>See Portnoy and Koenker (1997).

<sup>10</sup>This fact is a straightforward generalization of proposition 1 in Bertsimas et al. (2004) for expected shortfall.

results for portfolio weights in practical sample sizes. Therefore, we consider two situations. First, we ignore the estimation inaccuracy in asset expected returns, taking the viewpoint of Markowitz (1952) who suggests existence of *a priori* beliefs about the future expected returns. Then we include the asset expected return estimation inaccuracy into the portfolio weight comparison test.

## II.C Mean-variance and mean-CRR spanning tests

By analogy with the mean-variance spanning test, which tests whether two mean-variance frontiers generated by different sets of assets coincide, it is possible to develop a similar test for a CRR measure, see Polbennikov and Melenberg (2005). The standard question to be answered is whether the introduction of a new asset to a set of assets forming the optimal portfolio shifts the mean-CRR efficient frontier in a statistical sense.

In the literature spanning tests are usually considered in the mean-variance context. A conventional procedure for such a spanning test is suggested by Huberman and Kandel (1987). It is based on the notion that the restrictions on the tangent portfolio weights can be expressed as moment restrictions on excess returns of assets in the portfolio. These moment restrictions can be reformulated in terms of restrictions in an OLS regression, see, for example also, DeRoos and Nijman (2001). Polbennikov and Melenberg (2005) develop a test similar to Huberman and Kandel (1987) for mean-CRR spanning, expressed in terms of restrictions on an IV regression coefficients.

An alternative approach to the spanning test is followed by Britten-Jones (1999), who formulates the spanning hypothesis in the mean-variance framework in terms of restrictions on the tangent portfolio weights. These weights can be found as OLS regression coefficient. Results from the previous subsection can be used to implement this approach in the mean-CRR setup with the restrictions on the OLS regression coefficients in Britten-Jones (1999) replaced by restrictions on the corresponding mean-CRR portfolio weights.

In this paper we follow the approach developed by Huberman and Kandel (1987) for the mean-variance spanning and by Polbennikov and Melenberg (2005) for the mean-CRR spanning. The mean-variance spanning test is

based on the notion that the restrictions on the tangent portfolio weights can be expressed as moment restrictions on excess returns of assets in the portfolio. These moment restrictions can be reformulated in terms of restrictions on OLS regression coefficients. In particular, let  $Y^e$  be a random return excess of the risk-free rate of an asset for which we want to perform a spanning test. Let  $Z^e$  be the excess return of the mean-variance optimal market portfolio. Consider the OLS regression

$$\begin{aligned} Y_i^e &= \alpha + \beta Z_i^e + \epsilon_i, \\ E[\epsilon_i] &= 0, \\ E[Z_i^e \epsilon_i] &= 0. \end{aligned}$$

The spanning hypothesis can be reformulated in terms of the restrictions on parameters  $\alpha$  and  $\beta$ :

$$\alpha = 0, \tag{7}$$

$$\beta \text{Var}(Z^e) - \text{Cov}(Y^e, Z^e) = 0. \tag{8}$$

Restriction (8) shows that the coefficient  $\beta$  can be consistently estimated by an OLS regression, while restriction (7) states that the constant term in the regression (Jensen's  $\alpha$ ) should be equal to 0.

Polbennikov and Melenberg (2005) show that the test for mean-CRR spanning can be reformulated in terms of restrictions on the instrumental variable (IV) regression

$$\begin{aligned} Y_i^e &= \alpha + \beta Z_i^e + \epsilon_i, \\ E[\epsilon_i] &= 0, \\ E[V_i \epsilon_i] &= 0, \end{aligned}$$

where  $V$  is the instrumental variable<sup>11</sup>

$$V = \int_{F_z(Z)}^1 \alpha^{-1} d\phi(\alpha).$$

This instrumental variable defines a monotonic transformation of the original cumulative probability function  $F_z$  of portfolio returns. As a result more

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<sup>11</sup>Notice, that in an empirical application the instrumental variable  $V$  has to be non-parametrically estimated.

probability is assigned to the least favorable outcomes. We call this instrumental variable the *risk instrument* as it also defines the CRR measure. The restrictions imposed by the spanning hypothesis are

$$\alpha = 0, \tag{9}$$

$$\beta \text{Cov}(Z^e, V) - \text{Cov}(Y^e, V) = 0. \tag{10}$$

It follows from relation (10) that under the spanning hypothesis coefficient  $\beta$  can be consistently estimated by the IV regression with the risk instrument  $V$ . restriction (9) can then be checked as a zero-intercept test. Thus, the spanning test in case of the mean-variance portfolio is equivalent to the significance test of the intercept  $\alpha$  in OLS regression,<sup>12</sup> and the mean-CRR spanning test is equivalent to the significance test of the intercept  $\alpha$  in the IV regression. The asymptotic properties of the IV intercept coefficient are discussed in Polbennikov and Melenberg (2005).

## II.D Sample mean-CRR optimization

In this section we discuss algorithmic solutions to the sample mean-CRR optimization. A CRR measure can be viewed as a weighted combination of expected shortfalls for the whole range of probability thresholds, see (3). In practical applications, however, one would deal with the PMA version of a CRR measure, given in (4). Numerical solutions to an in-sample mean-expected shortfall optimization were proposed, among others, by Rockafellar and Uryasev (2000), Bertsimas et al. (2004), and Bassett et al. (2004). Generally, a sample analog of the mean-expected shortfall optimization can be reformulated as a linear program and solved efficiently with existing numerical algorithms, see Barrodale and Roberts (1974), Koenker and D'Orey (1987), and Portnoy and Koenker (1997). The method can be generalized to

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<sup>12</sup>The spanning tests discussed in this subsection takes into account the estimation inaccuracy in the asset expected returns. Alternatively, one can ignore the estimation error in the asset expected returns by following the approach of Britten-Jones (1999). The mean-variance and the mean-CRR spanning tests can be straightforwardly performed by testing the significance of the new asset tangent portfolio weight, using the results derived in the Appendix.



a PMA CRR measure, which uses Dirac's point mass functions to approximate an arbitrary CRR measure. This also corresponds to a piecewise linear approximation of the concave cumulative probability function  $\phi(\alpha)$  in (3).

Suppose that a PMA approximation of a CRR measure is given by the piecewise linear cumulative distribution function  $\phi$ :

$$\phi(\alpha) = \sum_{k=1}^m \phi_k I(\alpha \geq \tau_k).$$

Then the population mean-CRR portfolio problem is

$$\min_{\theta} \sum_{k=1}^m \phi_k s_{\tau_k}(v) \quad s.t. \quad E[v] = \nu, \quad (11)$$

where  $s_{\tau_k}(\cdot)$  is the expected shortfall with the probability threshold  $\tau_k$ ,  $v = y + x'\theta$  is the return of the portfolio, and  $\nu$  is the required expected return of the portfolio. As noticed by Bassett et al. (2004) expected shortfall can be equivalently expressed as

$$s_{\tau_k}(v) = \tau_k^{-1} \min_{\vartheta \in \mathbb{R}} E \varrho_{\tau_k}(v - \vartheta) - \nu,$$

where  $\varrho_{\alpha}(u) = u(\alpha - I(u < 0))$  and  $\nu$  is the expected return of portfolio  $v$ . Using this expression for expected shortfall the mean-CRR problem (11) can be reformulated as

$$\min_{\theta \in \mathbb{R}^p, \vartheta \in \mathbb{R}^m} \sum_{k=1}^m \phi_k \tau_k^{-1} E [\varrho_{\tau_k}(v - \vartheta_k)] \quad s.t. \quad E[v] = \nu. \quad (12)$$

Introduce auxiliary variables  $u_k^+ \in \mathbb{R}_+^p$  and  $u_k^- \in \mathbb{R}_+^p$  for  $k = 1, 2, \dots, m$ . Denote by  $\mu$  a vector of asset expected excess returns  $E[x]$  and by  $\mu_y$  the expected excess return of the asset  $y$ . Then the sample analog of the problem (12) can be formulated as the linear program

$$\begin{aligned} & \min \sum_{k=1}^m \phi_k \tau_k^{-1} (\tau_k e' u_k^+ + (1 - \tau_k) e' u_k^-) \\ & s.t. \\ & Y + X\theta - u_k^+ + u_k^- - e\vartheta_k = 0, \\ & \mu'\theta = \nu - \mu_y, \\ & (u_k^+, u_k^-, \theta, \vartheta_k) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}^p \times \mathbb{R} \text{ for } k = 1, 2, \dots, m, \end{aligned}$$

where  $Y$  is the  $(n \times 1)$ -vector of sample returns of the asset  $y$ ,  $X$  is the  $(n \times p)$ -matrix of sample excess returns of assets  $x$ , and  $e$  is the  $(n \times 1)$ -vector of ones. This linear program can be solved very efficiently by classical simplex and interior point methods, see Barrodale and Roberts (1974) and Portnoy and Koenker (1997).

### III Statistical Comparison of Portfolio Weights

#### III.A Simulated Returns

First, we compare the mean-variance and the mean-CRR efficient portfolio weights for simulated returns. We focus our attention on the expected shortfall and PMA CRR measure. This exercise emphasizes the fact that the variance and a CRR measure in the portfolio optimization context give different outcomes only in the case when the distribution of returns substantially deviates from the elliptically symmetric case. For expected shortfall similar examples with simulated returns were considered in Bertsimas et al. (2004) and Bassett et al. (2004). However, Bassett et al. (2004) do not perform a statistical comparison of the mean-variance and mean-expected shortfall efficient portfolio weights, while Bertsimas et al. (2004) use Monte-Carlo simulations instead of asymptotic theory.

As a benchmark we consider a sample of returns drawn from a three-variate normal distribution with population vector of means  $[0.06, 0.08, 0.08]^T$  and covariance matrix:

$$\begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}.$$

This simple example corresponds to a portfolio of assets with normally and independently distributed returns with an annual standard deviation of 20%. The independence of the returns makes the diversification motive very simple, so that it is easy to see which outcome in a portfolio optimization to anticipate. We expect the efficient mean-variance and mean-CRR portfolio weights to be equal in this case, because the considered risk measures are pro-

portional under normality. We shall refer to this case with the abbreviation "NORM".

Next, we simulate returns from a three-variate Student distribution with the same vector of expected returns and covariance matrix as in the normal case. This example might be more realistic than the multivariate normal one since observable market returns usually have fat distributional tails. Nevertheless, from a theoretical point of view the standard deviation and the expected shortfall are equivalent in the case of a Student  $t$ -distribution from a portfolio optimization perspective. This is so because the Student  $t$ -distribution belongs to the class of elliptically symmetric distributions. We shall refer the simulation from the Student distribution with an abbreviation "t".

To illustrate the difference between a CRR measure and the standard deviation in a portfolio choice framework, we consider a sample of returns drawn from a three-variate asymmetric distribution, "ASYM", using returns on the following independent assets. Asset  $A$  has a lognormal distribution such that its log return is normally distributed with mean 0.06 and variance 0.04. Asset  $B$  consists of a long position in an equity and an at-the-money European call option written on this equity. We assume normally distributed equity log-returns and use the Black-Scholes formula to calculate the price of the option. We normalize the distribution of log-returns on asset  $B$  to have mean 0.08 and variance 0.04. Its distribution is significantly skewed to the left. Asset  $C$  consists of a long position in an equity and the money market account and a short position in the European call option on the equity. We normalize the distribution of the log-returns on asset  $C$  to have mean 0.08 and variance 0.04. This distribution is skewed to the right. Figure 1 shows kernel density estimates of the simulated log-return distributions for the assets  $A$ ,  $B$ , and  $C$ .<sup>13</sup>

FIGURE 1 HERE

Summary statistical information on all considered assets is provided in

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<sup>13</sup>We use the Gaussian kernel density with the bandwidth chosen according to the Silverman's rule of thumb, see Silverman (1986).

Table 1. It can be seen that for the returns simulated from the three-variate normal distribution, NORM, the values of skewness and kurtosis are close to the theoretical ones, i.e., 0 and 3, respectively. For the returns simulated from the three-variate Student  $t$ -distribution we observe significantly higher sample kurtosis than for the normal case. As the returns are generated from a  $t$ -distribution with 6 degrees of freedom, the sample kurtosis is close to 6, the theoretical result for a  $t$ -distribution with six degrees of freedom. Finally, for the case of asymmetric returns, we observe a substantial positive sample skewness for asset  $B$  and a negative sample skewness for asset  $C$ , while the kurtosis of all assets in the portfolio is close to 3, i.e., not very different from the normal case.

TABLE 1 HERE

For the three simulated classes of returns we first perform a statistical comparison of the efficient portfolio weights resulting from the mean-variance and mean-expected shortfall portfolio optimization problems. We apply the asymptotic test for equality of the portfolio weights developed in Section II to all three cases of the simulated returns, NORM,  $t$  and ASYM. Since we want to make sure that a particular test result is not due to a specific portfolio expected return or shortfall probability threshold, we apply this test for different expected returns on the risk-efficient portfolio and different probability thresholds for the expected shortfall. The expected returns of the efficient portfolios are chosen to guarantee that the resulting portfolio belongs to the upper part of the efficient frontier. In particular, annual returns of 10%, 12%, 14% and 16% were chosen as portfolio target returns. Table 2 contains the corresponding  $p$ -values of the test.

TABLE 2 HERE

The results indicate that there is no statistical difference in the mean-variance and expected shortfall efficient portfolio weights for the multivariate normal and  $t$ -distribution of the asset returns. In fact, this result aligns well

with the theoretical predictions for elliptically symmetric distributions, see Bertsimas et al. (2004), and Embrechts et al. (1999). For the ASYM case, when the asset returns are simulated from a three-variate asymmetric distribution, we generally see a statistically significant difference between the variance and expected shortfall based portfolio weights. For the probability threshold of 2.5% the result holds in the whole range of the required portfolio expected returns at the 5% significance level. For required portfolio expected returns 14% and 16% and probability thresholds in the range of 2.5%-10% there is a difference between mean-variance and mean-expected shortfall portfolio weights significant at the 10% significance level. The test statistics become insignificant for the probability threshold of 12.5% and required portfolio expected returns of 14% and 16%. Usually, as can be noticed, the  $p$ -values of the test increase with the threshold probability and the required portfolio expected return. This means that the sensitivity of the expected shortfall to changes in the portfolio weights differs from the sensitivity of the standard deviation mostly in the tail area. The two risk measures become closer to each other as we increase the tail probability or portfolio expected return.

The expected shortfall gives the value of expected loss in the portfolio provided that the loss exceeds a certain quantile. For an investor such a measure of risk might not be the best reflection of riskiness of the position because for different quantiles the expected loss can behave differently with respect to portfolio weights. Therefore, a more general coherent risk measure can be a better choice. Here we consider the case of the point mass approximation (PMA) of a CRR measure described in section II. In particular, we choose an equally weighted PMA CRR with probability thresholds of 2.5%, 5%, 7.5%, 10%, and 12.5%, which aggregates the expected shortfalls used for portfolio weight comparison before. Table 3 shows  $p$ -values for the comparison test between the mean-PMA CRR portfolio weights and the mean-variance portfolio weights. Similar to the results for expected shortfall reported in Table 2, the equality hypothesis is strongly rejected only for portfolios of asymmetric returns. The rejection holds for all required expected portfolio returns.

TABLE 3 HERE

In addition, we investigate the economic effect of the differences between the mean-variance and the mean-shortfall portfolio weights. In Table 4 we report the decrease in the expected shortfall, which results from shifting from the standard deviation to the expected shortfall in a portfolio allocation decision. These numbers can be interpreted as a decrease in the expected loss in the portfolio for a given loss probability threshold.

TABLE 4 HERE

As can be seen from Table 4, the results support our statistical conclusions. The economic significance of the difference between the mean-variance and the mean-shortfall efficient portfolios is economically negligible for the returns simulated from the multivariate normal and the multivariate Student  $t$ -distributions. The effect from using the expected shortfall instead of the standard deviation is substantially less than a one-percent decrease in the expected conditional loss. In the case of the asymmetric returns the situation is different. We can observe a significant reduction in the expected loss for small probability thresholds and medium expected portfolio returns. In this example the effect decays as the probability threshold and the expected portfolio return increase. Overall, we observe more pronounced results in the tail of the portfolio return distribution.

In summary, the example in this section indicates that the portfolio allocations based on the mean-shortfall optimization can significantly differ from those based on the mean-variance approach. Furthermore, this difference depends on the choice of the risk level for the expected shortfall risk measure. This suggests that for portfolios of assets with asymmetric distributions of returns, such as equity and credit derivatives, an investor can benefit from using the expected shortfall risk measure when making an allocation decision. By doing so, he can better avoid the risk exposure from the extreme tail events while taking advantage on a positive skewness of the returns,

i.e., extreme events from the positive side. Clearly, the standard deviation, which treats positive and negative returns symmetrically, cannot do the job of distinguishing the positively skewed returns from the negatively skewed ones.

### III.B Market Returns

It is well known that returns observed in the market usually substantially deviate from the normal distribution. Generally, asset returns have fat tails and negative or positive skewness. These empirical facts potentially make the CRR measure an attractive alternative to the standard deviation. However, in reality, asset allocation decisions involve work with empirical data, including estimation procedures, so that there is always a level of uncertainty in the obtained result. As a consequence, the question of statistical and economic significance of the difference between CRR and variance based allocation decisions arises. In this section we compare the mean-variance and mean-CRR efficient portfolio weights for portfolios of market returns. We consider three cases: the daily exchange rates for the British pound, the Canadian dollar, the German mark, and the Japanese yen ("ER") with respect to the US dollar; the daily returns on the Fama-French value/book-to-market portfolios ("Fama-French"); the daily returns on S&P 500 index, US government bond JPM index, and Small Caps S&P 600 index ("Index"). The sample statistics for these portfolios are shown in Table 5.

TABLE 5 HERE

It follows from the table that for most of these portfolios the deviation from the normal distribution is very substantial. In particular, all exchange rates in the ER portfolio have excess kurtosis, with the Japanese yen being the most fat tailed. It is also the case for the Japanese yen exchange rate that its empirical distribution is substantially positively skewed. The deviation

from the normal distribution for the Fama-French and Index portfolios is even more pronounced. In particular, we observe large negative skewness for all returns in the Fama-French portfolio. For the indexes, we see that the S&P 500 and the Small Cap returns are negatively skewed. All reported returns have a large excess kurtosis with the S&P 500 being the most fat tailed. As the deviation of the reported returns from the normal distribution is so striking, we could expect substantially different weights for the variance and CRR based efficient portfolios as well.

Table 6 shows the outcomes of the equality test between the mean-variance and mean-expected shortfall efficient portfolios for different required portfolio expected returns and probability thresholds. These results ignore the estimation inaccuracy of the expected returns, see section V.

TABLE 6 HERE

Surprisingly, the results from Table 6 indicate that the variance and the shortfall-based efficient portfolio weights are not always significantly different. The weight-equality hypothesis cannot be rejected at standard significance levels for the portfolios of exchange rates. For the Fama-French efficient portfolios the equality hypothesis is strongly rejected for the low probability threshold of 2.5%, but cannot be rejected at the 5% significance level for higher thresholds. Significance levels of the test are especially high for the probability thresholds higher than 5%, where the equality hypothesis is generally accepted. For the Index portfolios the situation is reversed. The equality hypothesis is accepted at conventional significance levels for the low probability threshold of 2.5%, while for higher thresholds the equality hypothesis is usually rejected. These results indicate that mean-expected shortfall portfolio weights depend on the tail behavior of the return distribution function. If the sensitivities of the expected shortfall with respect to portfolio weights are proportional to those of the standard deviation, then the resulting portfolio weights are similar. Otherwise, they are different. One interesting point is that even though the market returns are usually fat tailed and negatively skewed, the portfolio weights produced by the expected shortfall and the



standard deviation are not necessarily statistically different. As we have already seen in the example of the multivariate  $t$ -distribution, fat tails do not always mean a difference in allocation between the mean-variance and the mean-shortfall portfolios, because distributions of the returns can still be close to elliptically symmetric. Now, we discover that skewness per se might not matter as well. There are two overlapping factors which determine the test outcomes. First, the test results are driven by the covariance matrix of the portfolio weights, which depends on the sample variance of the returns. Thus, the test outcome is dependent on the relation between skewness and variance in the return distributions. Second, the difference between the mean-expected shortfall and mean-variance portfolios is due to the asymptotic tail behavior of the return distributions. Skewness and kurtosis are only partial measures of this behavior and cannot completely reflect the sensitivity of the risk measures with respect to the portfolio weights. Table 7 illustrates the change of the difference between the mean-expected shortfall and mean-variance portfolio weights with the probability threshold for the Fama-French and Index portfolios with a required annualized expected portfolio return of 10%.

TABLE 7 HERE

The results confirm the conclusions of the tests in Table 6. In particular, the difference between the three first mean-expected shortfall and mean-variance portfolio weights is relatively large and statistically significant for the probability threshold 2.5% in the Fama-French portfolio. These outcomes suggest that the rejection of the equality hypothesis in the Fama-French portfolio for the probability threshold of 2.5% in Table 6 was caused by differences between the mean-expected shortfall and mean-variance portfolio weights of Big/Med, Big/High, and Small/Low size-book-to-market factors. As we increase the probability threshold to 7.5%, the behavior of the expected shortfall risk measure becomes similar to the behavior of the standard deviation. As a result, the differences between the mean-expected shortfall and mean-variance portfolio weights become small and insignificant. The same effect is

observed in Table 6. For the Index portfolio we observe a reverse situation: the increase of the probability threshold leads to significant difference between the mean-expected shortfall and mean-variance portfolio weights. As Table 7 suggests, the rejection of the null-hypothesis in Table 6 for higher probability thresholds is caused by the difference between the mean-expected shortfall and mean-variance portfolio weights of the Small Cap index. For the low probability threshold of 2.5% this difference is insignificant, and so is the test statistic in Table 6.

Additionally, as in the case of simulated returns, we perform a statistical comparison of the mean-PMA CRR and mean-variance portfolio weights. The equally weighted probability thresholds for the point mass approximation are chosen to be 2.5%, 5%, 7.5%, 10%, and 12.5%. Table 8 reports  $p$ -values of the test for different required expected portfolio returns.

TABLE 8 HERE

Even though the results of this table align well with the results for the expected shortfall reported in Table 6, they indicate the statistical difference between the mean-variance and mean-CRR portfolios better. In particular,  $p$ -values of the Fama-French and Index portfolios are relatively small, which can be attributed to the contribution of the corresponding expected shortfalls with small significant test statistics.

Finally, Table 9 shows the economic size of the difference between the variance and the shortfall-based portfolio allocations.

TABLE 9 HERE

For the Fama-French and Index portfolios the results support our statistical conclusions as we observe higher economic effect for those required portfolio expected returns and probability thresholds for which we also had smaller  $p$ -values of the equality test. The smaller economic effect is observed for the required portfolio expected returns and probability thresholds for which the equality hypothesis was not rejected.

Surprisingly, we observe high economic effect for portfolios of exchange rates (ER), where the decrease in expected loss with a given probability is up to 9%. At the same time the equality hypothesis is not rejected for these portfolios, see Table 5. The explanation for this phenomenon is high volatility of the exchange rates. The standard errors for the economic effects of the ER portfolios are relatively high, so that we can attribute the high  $p$ -values of the test statistics in Table 5 to the high volatility of the ER portfolio weights.

We conclude that for a typical portfolio of equities the expected shortfall and the standard deviation might produce statistically and economically different results. However, in certain cases the difference in portfolio weights is offset by the estimation error. When portfolios with asymmetric returns are considered, the portfolio weights for shortfall and standard deviation are significantly different, as in the ASYM case. In this situation it might be beneficial to use Choquet risk measures which account for downside returns.<sup>14</sup>

## IV Spanning tests

Comparison of the mean-variance and mean-CRR approach is not confined to the comparison of the portfolio weights. Additionally, one might ask the question whether the introduction of a new asset that shifts the mean-variance frontier has the same effect on the mean-CRR efficient frontier or conversely. Statistically, shifts in efficient frontiers can be characterized by spanning tests. In this section we are going to apply tests for mean-variance and mean-CRR spanning to several sets of assets, including the simulated returns from the previous section, the Fama-French value-book-to-market portfolios, and the S&P500 industry index returns. The results for the mean-variance and mean-CRR spanning tests are compared. In principle, as described in Polbennikov and Melenberg (2005), we can perform the spanning test for an arbitrary CRR measure. However, to make our analysis concise we focus on the mean-expected shortfall and mean-PMA CRR cases.

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<sup>14</sup>A natural extension of this study would be to investigate asymmetric portfolios that include options or credit derivatives. However, due to non-stationarity problems, caused by the maturity of derivative contracts, the methodology would have to be significantly adjusted. We postpone this for a separate study.

## IV.A Simulated returns

In this subsection we apply the mean-variance and the mean-CRR spanning tests to the sets of returns simulated in the previous section. First, for the three sets of assets, NORM,  $t$ , and ASYM, we perform market efficiency tests with respect to the first asset, which we denote by  $R_1$ . The null hypothesis is that the asset  $R_1$  is market efficient, so that the remaining assets, which we respectively denote by  $R_2$  and  $R_3$  are redundant. We perform three spanning tests. First, as a benchmark, the test for mean-variance spanning is performed. Then, the mean-expected shortfall efficiency for probability thresholds 2.5%, 7.5%, and 12.5% is tested. Finally, we implement the mean-PMA CRR spanning test, with equally weighted probability thresholds of 2.5%, 5%, 7.5%, 10%, and 12.5%. The risk-free interest rate is assumed to be 2.5%. The test  $p$ -values are reported in Table 10.

TABLE 10 HERE

It can be clearly seen that the spanning hypothesis is strongly rejected for all risk measures, which means that the remaining assets  $R_2$  and  $R_3$  are not redundant. We do not report significance levels for asset  $R_1$  as it should be, of course, redundant.

The inclusion of the assets  $R_2$  and  $R_3$  in a mean-risk portfolio improves diversification from both the mean-variance and mean-CRR perspectives. The difference between the mean-variance and mean-CRR spanning tests can be shown by testing the spanning hypothesis for a mean-variance market efficient portfolio. We form this portfolio from the three available assets  $R_1$ ,  $R_2$ , and  $R_3$ . Table 11 reports  $p$ -values of the spanning tests with respect to the mean-variance portfolio of the available assets. The null hypothesis is that assets  $R_2$  and  $R_3$  are redundant.

TABLE 11 HERE

As could be anticipated, the mean-variance hypothesis cannot be rejected at the conventional significance levels for all sets of assets. The mean-CRR

spanning hypothesis cannot be rejected<sup>15</sup> for the returns simulated from the normal (NORM) and multivariate  $t$ -distributions. At the same time the spanning hypothesis is strongly rejected for the portfolio of asymmetric returns ASYM. This demonstrates that the difference between the outcomes of mean-variance and mean-CRR spanning tests should be expected for portfolios of non-standard instruments with asymmetric return distributions. Such instruments could include equity derivatives or pooled credit securities.

The spanning tests in Tables 10 and 11 can be interpreted from the point of view of an investor who considers the given 3 equities as an investment possibility set. The fact that the spanning hypothesis is accepted for an individual equity indicates the redundancy of this equity with respect to the market portfolio (or the set of other equities from which the "market portfolio" is formed). Rejection of the spanning test for the asset  $R_1$  in Table 10 means that from the investor's perspective this asset cannot be viewed as a market portfolio, neither from the mean-variance nor from the mean-CRR perspective. The mentioned redundancy is related to a risk measure that is used by the investor for allocation purposes. Suppose that the mean-CRR investor forms a portfolio based on the mean-variance principle. In this case she invests her wealth in the combination of the risk-free asset and the mean-variance market portfolio. The results in Table 11 for asymmetrically distributed returns show that assets  $R_2$  and  $R_3$  are not redundant to such an allocation, i.e., the portfolio can still be improved from the mean-CRR perspective. On the other hand, an investor, who uses the mean-variance instead of the mean-CRR analysis gets almost the same diversification in the case of elliptically symmetric returns NORM or  $t$ .

## IV.B Market returns

Skewness and excess kurtosis of empirical distributions of asset returns is a frequent phenomenon observed in the market. In this subsection we apply spanning tests to the set of Fama-French portfolios based on the size and book-to-market factors as well as to the set of the S&P 500 sector indexes to check whether the mean-CRR spanning test produces significantly different

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<sup>15</sup>The same results are obtained if the estimation error in mean returns is ignored.

conclusions from the mean-variance one. Sample statistics of the observed returns are reported in Table 12. The sample returns demonstrate substantial excess kurtosis and, in most of the cases, negative skewness.

TABLE 12 HERE

Table 13 reports the results of the spanning tests. For the Fama-French set we perform the spanning tests with respect to the Fama-French market portfolio. For the set of sector indexes the tests are performed with respect to the S&P 500 composite index.

TABLE 13 HERE

The results indicate that for the portfolio of small companies with high and medium book-to-market ratio as well as for the portfolio of big companies with high book-to-market ratio the spanning hypothesis is strongly rejected in all tests. At the same time it can be seen that for the portfolio of small companies with low book-to-market ratio the  $p$ -value of the mean-variance spanning test is almost twice as high as the  $p$ -values of the mean-CRR tests, which could possibly indicate a difference between the two tests. Generally, the market portfolio is not optimal both from the mean-variance and mean-CRR perspectives. Its risk can be further diversified by inclusion of Small/High, Small/Medium, and Big/High Fama-French portfolios.

Testing the S&P 500 composite index for market efficiency with respect to the S&P 500 sector indexes shows that no test can reject the spanning hypothesis at the conventional significance levels. The mean-variance and mean-CRR spanning tests produce the same conclusions and similar  $p$ -values.

Since both mean-variance and mean-CRR spanning tests lead to the same conclusion in both the Fama-French and the S&P 500 examples, one could wonder whether these spanning tests can be distinguished at all for sets of common assets, such as stocks, stock indexes, etc. To check this we form the optimal mean-variance portfolios in both the Fama-French and S&P 500

sector index sets. For these portfolios we perform the mean-expected shortfall and mean-CRR PMA spanning tests. The results are reported in Table 14.

TABLE 14 HERE

The spanning hypothesis cannot be rejected by any of the tests at the conventional significance levels,<sup>16</sup> which means that the mean-CRR and mean-variance optimal portfolios are statistically similar. Thus, for portfolios of common equities mean-variance and mean-CRR spanning tests can be used interchangeably.

## V Estimation inaccuracy in expected returns

The results on the portfolio weight equality tests discussed in section III are considered from the viewpoint of Markowitz (1952) who suggests that there are *a priori* beliefs about the future expected returns. Given these beliefs an investor compares two alternative approaches in portfolio allocation decision: mean-variance or mean-CRR. In this section we investigate the effect of estimation inaccuracy in expected returns when these are also estimated using sample averages. It is known that the portfolio weights in the mean-variance analysis are very sensitive to errors in expected returns, see, for example, Chopra and Ziemba (1993). The same is the case for the mean-CRR portfolios. The asymptotic variance of the equality tests would typically increase due to the estimation inaccuracy, so that the test statistics yield insignificant results in practical sample sizes. In this section we use the portfolio weight equality tests to illustrate this. Table 15 shows the  $p$ -values of the portfolio comparison tests for the ASYM, the Fama-French, and the Index portfolios when the estimation inaccuracy in expected returns is taken into account. Comparing these results to the results in Tables 2 and 6, we see the increase in significance levels of the tests due to the estimation inaccuracy in expected returns. As a result, the majority of test statistics become

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<sup>16</sup>Ignoring estimation errors in mean returns lead to the same conclusions.

insignificant at the standard significance levels, confirming the findings of the sensitivity analysis by Chopra and Ziemba (1993).

## VI Conclusion

In this paper we empirically investigated the statistical implications of coherent risk measures, advocated in the literature, to the portfolio selection problem. We showed that efficient portfolio weights generated by mean-variance and mean-CRR optimizations can be statistically different for various portfolios of stocks if the estimation error in the mean returns can be ignored. Our results suggest that a CRR measure can better account for the downside risk in the case when one can include derivatives or other assets with asymmetric returns in the portfolio. In this case mean-variance and mean-CRR portfolio weights are likely to be statistically different. Economic differences between the mean-variance and the mean-CRR approaches align well with the statistical ones. The differences in expected loss between mean-variance and mean-expected shortfall portfolios are high for portfolios of asymmetric returns and relatively low for portfolios of common equities.

Secondly, we applied the mean-CRR spanning test to simulated returns, the Fama-French portfolios, and a number of sector indexes included in the S&P500. We showed that the difference between the mean-variance and the mean-CRR tests is especially pronounced for portfolios of asymmetric returns. For elliptically symmetric distributions of returns, as well as for portfolios of common equities, the mean-variance and mean-CRR tests lead to the same statistical conclusions. Both tests strongly reject the hypothesis that the market portfolio spans the set of Fama-French size-book-to-market portfolios. At the same time, both mean-variance and mean-CRR tests cannot reject market efficiency of the S&P 500 composite index. This means that the S&P500 composite index fulfills the role of market portfolio both for mean-variance investors, as well as for mean-CRR investors. Our results demonstrate that the mean-variance and the mean-CRR approaches are often statistically and economically similar for the equity asset classes considered.

Finally, we considered the sample mean estimation inaccuracy effect on the mean-variance and mean-CRR portfolio weight equality tests. In line



with the existing literature on the sensitivity of the mean-variance analysis to the sampling error, the test statistics become insignificant.

## A Limit distribution of a constrained extremum estimator

Our optimal portfolio problem can be expressed as a constrained extremum estimator problem

$$\min_{\theta \in \mathbb{R}^p} Ef(\theta) \text{ s.t. } Eg(\theta) = 0, \quad (13)$$

The first order conditions of this problem are

$$\begin{aligned} E[\nabla_{\theta} f] - \lambda E[\nabla_{\theta} g] &= 0, \\ Eg(\theta) &= 0, \end{aligned}$$

where  $\lambda$  is the Lagrange multiplier of the equality constraint. Denote by  $\psi_{\nabla f}(\theta)$  and  $\psi_{\nabla g}(\theta)$  the influence functions of the gradient functionals  $E[\nabla_{\theta} f]$  and  $E[\nabla_{\theta} g]$  respectively. Let the influence function of the constraint functional  $E[g(\theta)]$  be  $\psi_g(\theta)$ . Then, using the first order Taylor expansion of the FOC, we obtain:

$$\begin{bmatrix} H & -G \\ -G' & 0 \end{bmatrix} \sqrt{n} \begin{bmatrix} \hat{\theta} - \theta \\ \hat{\lambda} - \lambda \end{bmatrix} = \frac{1}{\sqrt{n}} \begin{bmatrix} \sum_{i=1}^n (\lambda \psi_{\nabla g i} - \psi_{\nabla f i}) \\ \sum_{i=1}^n \psi_{g i} \end{bmatrix} + \begin{bmatrix} r_{1n} \\ r_{2n} \end{bmatrix},$$

where

$$H = E[\nabla_{\theta}^2 f] - \lambda E[\nabla_{\theta}^2 g], \quad (14)$$

$$G = E[\nabla_{\theta} g], \quad (15)$$

and  $r_{1n}$  and  $r_{2n}$  are the residual terms converging in probability to zero. Solving this system of linear equations for  $\sqrt{n}(\hat{\theta} - \theta)$ , we obtain the result for the asymptotic distribution of the constrained extremum estimator  $\hat{\theta}$  expressed in terms of the influence functions  $\psi_f(\theta)$  and  $\psi_g(\theta)$

$$\sqrt{n}(\hat{\theta} - \theta) = H^{-1} \begin{bmatrix} bGG'H^{-1} - I_p \\ -bG \end{bmatrix}' \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{bmatrix} \psi_{\nabla f i} - \lambda \psi_{\nabla g i} \\ \psi_{g i} \end{bmatrix} + o_p(1), \quad (16)$$

where  $b = (G'H^{-1}G)^{-1}$ . Notice, that the Lagrange multiplier  $\lambda$  for a given optimal  $\theta$  can be found from the first order condition, for example,

$$\lambda = (i'G)^{-1}i'E[\nabla_{\theta}f], \quad (17)$$

where  $i$  stands for a  $p \times 1$  vector of ones.

Finally, for the case when the constraint and gradient functionals  $E[g(\theta)]$ ,  $E[\nabla_{\theta}f]$  and  $E[\nabla_{\theta}g]$  do not involve a non-parametric estimation of population distribution functions, their influence functions can be found in a usual way, i.e.,  $\psi_g = g$ ,  $\psi_{\nabla f} = \nabla_{\theta}f$  and  $\psi_{\nabla g} = \nabla_{\theta}g$ .

Suppose now that one wants to eliminate the estimation uncertainty from the constraint in (13). In this case the problem can be reformulated as

$$\min_{\theta \in \mathbb{R}^p} Ef(\theta) \text{ s.t. } g(\theta) = 0.$$

It is straightforward to see that as a result all the constraint related terms in (16) disappear so that the limit distribution of the constrained extremum estimator is given by

$$\sqrt{n}(\hat{\theta} - \theta) = H^{-1}(bGG'H^{-1} - I_p) \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi_{\nabla f i} + o_p(1).$$

## B Limit distribution of portfolio weights

The Mean-CRR portfolio problem is obtained from the mean-risk problem (6) when a CRR measure (3) is used as an objective function

$$\min_{\theta \in \mathbb{R}^p} \int_0^1 s_{\alpha}(y + x'\theta) d\phi(\alpha) \text{ s.t. } E[y + x'\theta] = \nu.$$

This mean-CRR portfolio problem can be reformulated as an extremum estimation problem as discussed in Appendix A, since a CRR measure can be expressed as an expectation. To simplify the exposition we use the notation  $v = y + x'\theta$  for the portfolio return and  $F_v$  for its cumulative distribution function. Both  $v$  and  $F_v$  depend on the portfolio weights  $\theta$ . First, we express the expected shortfall  $s_{\alpha}(v)$  as an expectation

$$s_{\alpha}(v) = -\alpha^{-1}E[vI(F_v(v) \leq \alpha)].$$

Substituting this expression into equation (3) we obtain a CRR measure as an expectation

$$\begin{aligned}\rho(v) &= - \int_0^1 \alpha^{-1} E[vI(F_v(v) \leq \alpha)] d\phi(\alpha) \\ &= -E \left[ v \int_0^1 \alpha^{-1} I(F_v(v) \leq \alpha) d\phi(\alpha) \right] \\ &= -E \left[ v \int_{F_v(v)}^1 \alpha^{-1} d\phi(\alpha) \right].\end{aligned}$$

The mean-CRR optimal portfolio problem becomes

$$\min_{\theta} E \left[ -v \int_{F_v(v)}^1 \alpha^{-1} d\phi(\alpha) \right] \text{ s.t. } E[v] = \nu. \quad (18)$$

Problem (18) is a constrained extremum estimator problem, so the asymptotic results derived in Appendix A apply. The asymptotic distribution of the mean-CRR portfolio weights can be expressed through the influence function  $\xi(x, v)$  of the estimated portfolio weights<sup>17</sup>

$$\sqrt{n} (\hat{\theta} - \theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi(x_i, v_i) + o_p(1) \rightarrow_d N(0, E[\xi\xi']),$$

where the index  $i$  identifies a particular observation in the sample. The influence function of the portfolio weights that ignores constraint estimation inaccuracy is

$$\xi(x_i, v_i) = H^{-1} (bGG'H^{-1} - I_p) \psi_{\nabla fi}. \quad (19)$$

The influence function of the mean-CRR portfolio weights that takes into account the estimation inaccuracy in asset expected returns is

$$\xi(x_i, v_i) = H^{-1} \begin{bmatrix} bGG'H^{-1} - I_p \\ -bG \end{bmatrix}' \begin{bmatrix} \psi_{\nabla fi} - \lambda\psi_{\nabla gi} \\ \psi_{gi} \end{bmatrix}. \quad (20)$$

The vector  $G$  is the gradient of the constraint function with respect to portfolio weights  $G = E[x]$ , and  $\lambda$  is the Lagrange multiplier

$$\lambda = -e' \frac{\partial}{\partial \theta} E \left[ v \int_{F_v(v)}^1 \alpha^{-1} d\phi(\alpha) \right] E[e'x]^{-1},$$

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<sup>17</sup>Notice, that we assumed the asset sample returns to be identically and independently distributed. Our results, however, can be straightforwardly extended to the case of stationary sample returns, see Newey and West (1987).

matrix  $H$  is the Hessian of the objective function with respect to portfolio weights

$$H = -\frac{\partial^2}{\partial\theta\partial\theta'} E \left[ v \int_{F_v(v)}^1 \alpha^{-1} d\phi(\alpha) \right],$$

functions  $\psi_{\nabla f}$  and  $\psi_{\nabla g}$  are the influence functions of the the objective and constraint function gradient functionals correspondingly, function  $\psi_g$  is the influence function of the constraint functional, and scalar  $b$  is the notation

$$b = (G' H^{-1} G)^{-1}.$$

The exact expressions for the Lagrange multiplier  $\lambda$ , the Hessian  $H$ , and the influence function  $\psi_{\nabla f}$  in case of a mean-CRR portfolio are as follows

$$\begin{aligned} \lambda &= -E \left[ e'x \int_{F_v(v)}^1 \alpha^{-1} d\phi(\alpha) \right] E[e'x]^{-1}, \\ H &= E \left[ \frac{\phi'(F_v(v))f(v)}{F(v)} \text{Cov}(x|v) \right], \\ \psi_{\nabla f} &= \chi_{\nabla f} - E[\chi_{\nabla f}], \\ \chi_{\nabla f} &= - \int_{F_v(v)}^1 (x - E[x|v = F_v^{-1}(\alpha)]) \alpha^{-1} d\phi(\alpha). \end{aligned}$$

The derivation details can be found in the companion paper by Polbennikov and Melenberg (2005). Finally, the influence functions  $\psi_{\nabla g}$  and  $\psi_g$  are

$$\begin{aligned} \psi_{\nabla g} &= x - E[x], \\ \psi_g &= v - \nu. \end{aligned}$$

## B.1 Expected shortfall

In the case of expected shortfall the probability function  $\phi(\alpha)$  is

$$\phi(\alpha) = I(\alpha \geq \tau),$$

so that the influence function of the mean-expected shortfall portfolio weights is given by (19) or (20) with

$$\begin{aligned} \lambda &= -\tau^{-1} E[e'x I(F_v(v) \leq \tau)] E[e'x]^{-1}, \\ H &= \tau^{-1} f(F_v^{-1}(\tau)) \text{Cov}(x|v = F_v^{-1}(\tau)), \\ \chi_{\nabla f} &= \tau^{-1} I(F_v(v) \leq \tau) (x - E[x|v = F_v^{-1}(\tau)]). \end{aligned}$$

## B.2 Point Mass Approximation (PMA) of a CRR measure

In the case of PMA CRR measure the probability function  $\phi(\alpha)$  is a stepwise function

$$\phi(\alpha) = \sum_{k=1}^m \phi_k I(\alpha \geq \tau_k),$$

so that the influence function of the mean-expected shortfall portfolio weights is given by (19), if one wants to ignore the estimation error in the asset expected returns, or (20), if one wants to take into account the estimation constraint uncertainty, with

$$\begin{aligned} \lambda &= - \sum_{k=1}^m \phi_k \tau_k^{-1} E[e'x I(F_v(v) \leq \tau_k)] E[e'x]^{-1}, \\ H &= \sum_{k=1}^m \phi_k \tau_k^{-1} f(F_v^{-1}(\tau_k)) \text{Cov}(x|v = F_v^{-1}(\tau_k)), \\ \chi_{\nabla f} &= - \sum_{k=1}^m \phi_k \tau_k^{-1} I(F_v(v) \leq \tau_k) (x - E[x|v = F_v^{-1}(\tau_k)]). \end{aligned}$$

## B.3 Mean-variance portfolio weights

Using the same notations as in (6) we write the mean-variance portfolio problem

$$\min_{\theta} E[(y + x'\theta)^2] \quad \text{s.t.} \quad E[y + x'\theta] = \nu.$$

This problem can also be viewed as a constrained extremum estimator problem, so, again, the limit distribution results of the Appendix A apply. The influence function of the mean-variance portfolio weights is given by expression (19) or (20) with the Lagrange multiplier  $\lambda$  given by

$$\lambda = e' \frac{\partial}{\partial \theta} E[(y + x'\theta)] E[e'x]^{-1} = E[(y + x'\theta)e'x] E[e'x]^{-1},$$

and the Hessian  $H$  of the objective function given by

$$H = \frac{\partial^2}{\partial \theta \partial \theta'} E[(y + x'\theta)] = E[xx'].$$

Finally, the influence functions of the gradient and constraint functionals are

$$\begin{aligned}\psi_{\nabla f} &= (y + x'\theta)x - E[(y + x'\theta)x], \\ \psi_{\nabla g} &= x - E[x], \\ \psi_g &= y + x'\theta - \nu.\end{aligned}$$

## C Statistical comparison of portfolio weights

Let  $\beta$  be the vector of mean-variance portfolio weights, and  $\theta$  be the vector of mean-CRR portfolio weights. Denote by  $\eta(x, v)$  the influence function of the mean-variance portfolio weights, and by  $\xi(x, v)$  the influence function of the mean-CRR portfolio weights. The exact expressions for these influence functions are provided in Appendix B. The joint asymptotic distribution of the mean-variance and the mean-CRR weights is

$$\sqrt{n}(\hat{\gamma} - \gamma) \equiv \sqrt{n} \begin{pmatrix} \hat{\beta} - \beta \\ \hat{\theta} - \theta \end{pmatrix} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} \eta(x_i, v_i) \\ \xi(x_i, v_i) \end{pmatrix} + o_p(1) \rightarrow_d N(0, \Omega),$$

where

$$\Omega = E \begin{bmatrix} \eta\eta' & \eta\xi' \\ \xi\eta' & \xi\xi' \end{bmatrix}.$$

The hypothesis  $H_0 : \beta = \theta$  vs.  $H_1 : \beta \neq \theta$  can be tested in a standard way. Introduce the restriction matrix  $R = [I_p, -I_p]$ , then

$$\hat{\gamma}' R' (R \hat{\Omega} R')^{-1} R \hat{\gamma} \rightarrow_d \chi_p^2$$

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| Portfolios | Assets  | N Obs | Avg. Return | Skewness | Kurtosis | Covariance |      |      |
|------------|---------|-------|-------------|----------|----------|------------|------|------|
| NORM       | Asset 1 | 3000  | 0.06        | -0.04    | 3.09     | 0.04       | 0.00 | 0.00 |
|            | Asset 2 |       | 0.08        | 0.00     | 3.00     | *          | 0.04 | 0.00 |
|            | Asset 3 |       | 0.08        | -0.01    | 2.99     | *          | *    | 0.04 |
| t          | Asset 1 | 3000  | 0.06        | 0.22     | 5.40     | 0.04       | 0.00 | 0.00 |
|            | Asset 2 |       | 0.08        | -0.12    | 6.45     | *          | 0.04 | 0.00 |
|            | Asset 3 |       | 0.08        | -0.13    | 5.10     | *          | *    | 0.04 |
| ASYM       | Asset A | 3000  | 0.06        | -0.04    | 3.01     | 0.04       | 0.00 | 0.00 |
|            | Asset B |       | 0.08        | 0.66     | 3.16     | *          | 0.04 | 0.00 |
|            | Asset C |       | 0.08        | -0.79    | 4.03     | *          | *    | 0.04 |

Table 1: Sample statistics for simulated asset returns. NORM - returns from the three-variate normal distribution,  $t$  - returns from the three-variate  $t$ -distribution, ASYM - returns from the three-variate asymmetric distribution.

| Portfolios | Probability Threshold | Expected Portfolio Return |       |       |       |
|------------|-----------------------|---------------------------|-------|-------|-------|
|            |                       | 10%                       | 12%   | 14%   | 16%   |
| NORM       | 2.5%                  | 76.0%                     | 86.8% | 81.7% | 87.1% |
|            | 5%                    | 63.6%                     | 63.5% | 56.1% | 74.7% |
|            | 7.5%                  | 39.7%                     | 50.2% | 63.0% | 64.3% |
|            | 10%                   | 54.4%                     | 42.4% | 47.0% | 41.1% |
|            | 12.5%                 | 60.6%                     | 68.1% | 68.1% | 56.0% |
| t          | 2.5%                  | 94.4%                     | 87.6% | 89.0% | 90.4% |
|            | 5%                    | 80.2%                     | 95.1% | 96.1% | 96.8% |
|            | 7.5%                  | 67.7%                     | 82.5% | 95.7% | 95.2% |
|            | 10%                   | 99.3%                     | 92.4% | 96.4% | 94.2% |
|            | 12.5%                 | 98.5%                     | 92.6% | 88.0% | 99.4% |
| ASYM       | 2.5%                  | 0.0%                      | 0.0%  | 0.0%  | 0.1%  |
|            | 5%                    | 0.0%                      | 1.3%  | 3.5%  | 7.8%  |
|            | 7.5%                  | 0.0%                      | 0.9%  | 5.3%  | 7.1%  |
|            | 10%                   | 0.0%                      | 0.9%  | 6.3%  | 9.9%  |
|            | 12.5%                 | 0.0%                      | 3.9%  | 13.3% | 22.5% |

Table 2:  $p$ -values of the test for equality of the mean-variance and the mean-shortfall portfolio weights in portfolios of simulated returns.

|            | Expected Portfolio Return |       |       |       |
|------------|---------------------------|-------|-------|-------|
| Portfolios | 10%                       | 12%   | 14%   | 16%   |
| NORM       | 86.1%                     | 68.3% | 60.8% | 63.6% |
| t          | 85.5%                     | 85.0% | 92.9% | 93.8% |
| ASYM       | 0.0%                      | 0.1%  | 0.9%  | 2.3%  |

Table 3:  $p$ -values of the test for equality of the mean-variance and the mean-PMA CRR portfolio weights in portfolios of simulated returns. The probability thresholds for the PMA CRR measure are 2.5%, 5%, 7.5%, 10%, and 12.5%.

|            |                       | Expected Portfolio Return |             |             |             |
|------------|-----------------------|---------------------------|-------------|-------------|-------------|
| Portfolios | Probability Threshold | 10%                       | 12%         | 14%         | 16%         |
| NORM       | 2.5%                  | 0.03%                     | 0.01%       | 0.01%       | 0.03%       |
|            |                       | <i>0.08</i>               | <i>0.10</i> | <i>0.19</i> | <i>0.16</i> |
|            | 5%                    | 0.05%                     | 0.10%       | 0.09%       | 0.08%       |
|            |                       | <i>0.09</i>               | <i>0.19</i> | <i>0.35</i> | <i>0.28</i> |
|            | 7.5%                  | 0.04%                     | 0.10%       | 0.21%       | 0.32%       |
|            | <i>0.12</i>           | <i>0.21</i>               | <i>0.24</i> | <i>0.31</i> |             |
|            | 10%                   | 0.03%                     | 0.11%       | 0.13%       | 0.14%       |
|            | <i>0.06</i>           | <i>0.16</i>               | <i>0.24</i> | <i>0.35</i> |             |
|            | 12.5%                 | 0.01%                     | 0.02%       | 0.05%       | 0.11%       |
|            | <i>0.04</i>           | <i>0.07</i>               | <i>0.10</i> | <i>0.18</i> |             |
| t          | 2.5%                  | 0.01%                     | 0.29%       | 0.32%       | 0.30%       |
|            |                       | <i>0.14</i>               | <i>0.53</i> | <i>0.83</i> | <i>0.99</i> |
|            | 5%                    | 0.04%                     | 0.02%       | 0.02%       | 0.01%       |
|            |                       | <i>0.11</i>               | <i>0.06</i> | <i>0.07</i> | <i>0.06</i> |
|            | 7.5%                  | 0.01%                     | 0.01%       | 0.00%       | 0.01%       |
|            | <i>0.12</i>           | <i>0.13</i>               | <i>0.05</i> | <i>0.09</i> |             |
|            | 10%                   | 0.00%                     | 0.00%       | 0.00%       | 0.00%       |
|            | <i>0.00</i>           | <i>0.04</i>               | <i>0.03</i> | <i>0.05</i> |             |
|            | 12.5%                 | 0.00%                     | 0.00%       | 0.01%       | 0.00%       |
|            | <i>0.00</i>           | <i>0.02</i>               | <i>0.06</i> | <i>0.00</i> |             |
| ASYM       | 2.5%                  | 3.88%                     | 4.62%       | 5.48%       | 6.46%       |
|            |                       | <i>1.10</i>               | <i>1.50</i> | <i>2.04</i> | <i>2.45</i> |
|            | 5%                    | 2.14%                     | 2.15%       | 2.34%       | 2.59%       |
|            |                       | <i>0.63</i>               | <i>0.84</i> | <i>0.99</i> | <i>1.10</i> |
|            | 7.5%                  | 1.52%                     | 1.13%       | 1.16%       | 1.32%       |
|            | <i>0.42</i>           | <i>0.58</i>               | <i>0.74</i> | <i>0.93</i> |             |
|            | 10%                   | 1.16%                     | 0.90%       | 0.77%       | 0.72%       |
|            | <i>0.35</i>           | <i>0.45</i>               | <i>0.49</i> | <i>0.60</i> |             |
|            | 12.5%                 | 0.88%                     | 0.67%       | 0.53%       | 0.48%       |
|            | <i>0.33</i>           | <i>0.33</i>               | <i>0.37</i> | <i>0.39</i> |             |

Table 4: Economic size of the difference between the mean-shortfall and mean-variance simulated efficient portfolios. The effect is measured as a decrease in the expected shortfall when switching from the standard deviation to the expected shortfall risk measure in portfolio optimization. The standard errors are given in italics.

| Portfolios   | Assets     | N Obs | Avg. Return | Skewness | Kurtosis | Covariance |       |       |       |       |       |
|--------------|------------|-------|-------------|----------|----------|------------|-------|-------|-------|-------|-------|
| ER           | BP         | 3913  | 0.34%       | -0.21    | 6.61     | 0.009      | 0.001 | 0.007 | 0.004 |       |       |
|              | CAN        |       | -0.70%      | 0.00     | 5.39     | *          | 0.003 | 0.001 | 0.000 |       |       |
|              | DM         |       | 1.03%       | 0.03     | 4.72     | *          | *     | 0.012 | 0.005 |       |       |
|              | JAP        |       | 1.15%       | 0.77     | 10.70    | *          | *     | *     | 0.013 |       |       |
| Fama- French | Small/Low  | 10448 | 9.6%        | -0.67    | 11.66    | 0.028      | 0.019 | 0.018 | 0.022 | 0.017 | 0.017 |
|              | Small/Med  |       | 14.9%       | -0.86    | 13.78    | *          | 0.015 | 0.014 | 0.015 | 0.013 | 0.013 |
|              | Small/High |       | 16.9%       | -0.88    | 14.75    | *          | *     | 0.014 | 0.014 | 0.012 | 0.013 |
|              | Big/Low    |       | 10.7%       | -0.47    | 17.25    | *          | *     | *     | 0.026 | 0.019 | 0.018 |
|              | Big/Med    |       | 11.8%       | -1.10    | 31.06    | *          | *     | *     | *     | 0.018 | 0.016 |
|              | Big/High   |       | 13.6%       | -0.89    | 24.21    | *          | *     | *     | *     | *     | 0.018 |
| Index        | S&P 500    | 4797  | 8.6%        | -2.08    | 46.41    | 0.030      | 0.022 | 0.001 |       |       |       |
|              | Small Caps |       | 7.6%        | -0.94    | 16.61    | *          | 0.025 | 0.000 |       |       |       |
|              | Gov. Bonds |       | 2.6%        | -0.04    | 7.48     | *          | *     | 0.002 |       |       |       |

Table 5: Sample statistics for market returns. ER - exchange rates, Fama-French - returns on the Fama-French portfolios, Index - returns on market indexes.

| Portfolios  | Probability Threshold | Expected Portfolio Return |       |       |       |
|-------------|-----------------------|---------------------------|-------|-------|-------|
|             |                       | 10%                       | 12%   | 14%   | 16%   |
| ER          | 2.5%                  | 39.8%                     | 29.7% | 28.6% | 25.2% |
|             | 5%                    | 50.1%                     | 55.9% | 56.3% | 50.9% |
|             | 7.5%                  | 39.2%                     | 41.4% | 45.8% | 48.3% |
|             | 10%                   | 16.2%                     | 14.8% | 15.3% | 13.9% |
|             | 12.5%                 | 28.7%                     | 22.7% | 28.0% | 24.4% |
| Fama-French | 2.5%                  | 0.0%                      | 0.0%  | 0.0%  | 0.0%  |
|             | 5%                    | 7.1%                      | 6.3%  | 10.3% | 5.1%  |
|             | 7.5%                  | 60.1%                     | 87.3% | 64.2% | 26.2% |
|             | 10%                   | 53.0%                     | 51.3% | 69.6% | 61.1% |
|             | 12.5%                 | 46.1%                     | 88.2% | 80.1% | 52.6% |
| Index       | 2.5%                  | 67.5%                     | 71.2% | 80.4% | 63.2% |
|             | 5%                    | 0.6%                      | 2.0%  | 2.0%  | 1.4%  |
|             | 7.5%                  | 6.7%                      | 5.3%  | 4.0%  | 3.5%  |
|             | 10%                   | 4.7%                      | 5.2%  | 5.9%  | 5.5%  |
|             | 12.5%                 | 5.0%                      | 5.2%  | 4.1%  | 4.5%  |

Table 6:  $p$ -values of the test for equality of the mean-variance and mean-shortfall portfolio weights in portfolios of market returns.

| Comparison of Portfolio Weights |            |       |       |       |           |
|---------------------------------|------------|-------|-------|-------|-----------|
|                                 |            | M-ShF | M-V   | Diff  | Std. Err. |
| FF, 2.5%                        | Big/Med    | 1.55  | 1.36  | 0.20  | 0.055     |
|                                 | Big/High   | 1.13  | 1.28  | -0.16 | 0.050     |
|                                 | Small/Low  | 0.45  | 0.27  | 0.19  | 0.074     |
|                                 | Small/Med  | -0.19 | -0.17 | -0.02 | 0.064     |
|                                 | Small/High | -0.22 | -0.18 | -0.03 | 0.045     |
| FF, 7.5%                        | Big/Med    | 1.38  | 1.36  | 0.02  | 0.052     |
|                                 | Big/High   | 1.24  | 1.28  | -0.04 | 0.040     |
|                                 | Small/Low  | 0.32  | 0.27  | 0.05  | 0.035     |
|                                 | Small/Med  | -0.14 | -0.17 | 0.03  | 0.037     |
|                                 | Small/High | -0.17 | -0.18 | 0.02  | 0.028     |
| Index, 2.5%                     | Small Caps | 0.47  | 0.52  | -0.05 | 0.114     |
|                                 | Gov. Bonds | -0.23 | -0.24 | 0.01  | 0.015     |
| Index, 5%                       | Small Caps | 0.29  | 0.52  | -0.24 | 0.086     |
|                                 | Gov. Bonds | -0.21 | -0.24 | 0.03  | 0.011     |

Table 7: Effect of the probability threshold on the difference between mean-expected shortfall and mean-variance portfolio weights. Portfolio weights are reported for the required expected portfolio return of 10%. Portfolio names and probability thresholds are given in the left column.

|             | Expected Portfolio Return |       |       |       |
|-------------|---------------------------|-------|-------|-------|
| Portfolios  | 10%                       | 12%   | 14%   | 16%   |
| ER          | 25.6%                     | 21.8% | 22.3% | 28.6% |
| Fama-French | 2.1%                      | 5.1%  | 13.2% | 4.3%  |
| Index       | 4.2%                      | 4.9%  | 4.5%  | 4.5%  |

Table 8:  $p$ -values of the test for equality of the mean-variance and mean-shortfall portfolio weights in portfolios of market returns. The probability thresholds for the PMA CRR measure are 2.5%, 5%, 7.5%, 10%, and 12.5%.

| Portfolios  | Probability Threshold | Expected Portfolio Return |              |              |              |
|-------------|-----------------------|---------------------------|--------------|--------------|--------------|
|             |                       | 10%                       | 12%          | 14%          | 16%          |
| ER          | 2.5%                  | 5.83%                     | 7.16%        | 8.23%        | 9.36%        |
|             |                       | <i>11.54</i>              | <i>14.33</i> | <i>16.69</i> | <i>19.76</i> |
|             | 5%                    | 2.60%                     | 3.53%        | 4.51%        | 5.44%        |
|             |                       | <i>4.39</i>               | <i>5.10</i>  | <i>6.02</i>  | <i>7.08</i>  |
|             | 7.5%                  | 2.63%                     | 3.12%        | 3.83%        | 4.58%        |
|             |                       | <i>4.46</i>               | <i>5.21</i>  | <i>5.89</i>  | <i>6.60</i>  |
|             | 10%                   | 4.19%                     | 5.07%        | 5.89%        | 6.54%        |
|             |                       | <i>3.88</i>               | <i>4.83</i>  | <i>5.80</i>  | <i>6.71</i>  |
|             | 12.5%                 | 2.40%                     | 2.79%        | 3.21%        | 3.71%        |
|             |                       | <i>2.48</i>               | <i>3.42</i>  | <i>3.81</i>  | <i>4.39</i>  |
| Fama-French | 2.5%                  | 2.74%                     | 0.96%        | 0.97%        | 0.88%        |
|             |                       | <i>1.41</i>               | <i>0.87</i>  | <i>0.80</i>  | <i>1.05</i>  |
|             | 5%                    | 0.67%                     | 0.26%        | 0.28%        | 0.62%        |
|             |                       | <i>0.55</i>               | <i>0.35</i>  | <i>0.35</i>  | <i>0.54</i>  |
|             | 7.5%                  | 0.30%                     | 0.22%        | 0.31%        | 0.25%        |
|             |                       | <i>0.32</i>               | <i>0.28</i>  | <i>0.31</i>  | <i>0.26</i>  |
|             | 10%                   | 0.16%                     | 0.17%        | 0.14%        | 0.23%        |
|             |                       | <i>0.20</i>               | <i>0.20</i>  | <i>0.17</i>  | <i>0.28</i>  |
|             | 12.5%                 | 0.16%                     | 0.07%        | 0.11%        | 0.21%        |
|             |                       | <i>0.15</i>               | <i>0.11</i>  | <i>0.17</i>  | <i>0.25</i>  |
| Index       | 2.5%                  | 0.40%                     | 0.17%        | 0.06%        | 0.22%        |
|             |                       | <i>1.13</i>               | <i>1.15</i>  | <i>0.90</i>  | <i>2.01</i>  |
|             | 5%                    | 3.78%                     | 4.74%        | 5.47%        | 5.99%        |
|             |                       | <i>3.23</i>               | <i>3.19</i>  | <i>3.81</i>  | <i>4.85</i>  |
|             | 7.5%                  | 2.35%                     | 2.95%        | 3.91%        | 4.48%        |
|             |                       | <i>1.94</i>               | <i>2.62</i>  | <i>3.04</i>  | <i>3.48</i>  |
|             | 10%                   | 1.91%                     | 2.30%        | 2.75%        | 3.12%        |
|             |                       | <i>1.44</i>               | <i>1.71</i>  | <i>1.97</i>  | <i>2.33</i>  |
|             | 12.5%                 | 1.46%                     | 1.90%        | 2.35%        | 2.79%        |
|             |                       | <i>1.17</i>               | <i>1.43</i>  | <i>1.78</i>  | <i>2.03</i>  |

Table 9: Economic size of the difference between the mean-shortfall and mean-variance market efficient portfolios. The effect is measured as a decrease in the expected shortfall when switching from the standard deviation to the expected shortfall risk measure in portfolio optimization. The standard errors are given in italics.



|       | <b>M-V</b>                    | <b>Mean-Expected Shortfall</b> |             |              | <b>M-PMA</b> |
|-------|-------------------------------|--------------------------------|-------------|--------------|--------------|
|       |                               | <b>2.5%</b>                    | <b>7.5%</b> | <b>12.5%</b> |              |
|       | Simulated NORM Returns vs. R1 |                                |             |              |              |
| $R_2$ | 0.00%                         | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |
| $R_3$ | 0.00%                         | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |
|       | Simulated t Returns vs. R1    |                                |             |              |              |
| $R_2$ | 0.00%                         | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |
| $R_3$ | 0.00%                         | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |
|       | Simulated ASYM Returns vs. R1 |                                |             |              |              |
| $R_2$ | 0.00%                         | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |
| $R_3$ | 0.00%                         | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |

Table 10:  $p$ -values of the spanning tests for simulated returns with respect to the asset  $R_1$ . The reported results are for the mean-variance (M-V), mean-expected shortfall, and mean-PMA CRR (M-PMA) spanning. The PMA probability thresholds are 2.5%, 5%, 7.5%, 10%, and 12.5% with equal weights of 20%.

|       | <b>M-V</b>                    | <b>Mean-Expected Shortfall</b> |             |              | <b>M-PMA</b> |
|-------|-------------------------------|--------------------------------|-------------|--------------|--------------|
|       |                               | <b>2.5%</b>                    | <b>7.5%</b> | <b>12.5%</b> |              |
|       | Simulated NORM Returns vs. MV |                                |             |              |              |
| $R_2$ | 100.00%                       | 63.02%                         | 65.61%      | 56.34%       | 68.99%       |
| $R_3$ | 100.00%                       | 47.48%                         | 99.94%      | 59.33%       | 76.13%       |
|       | Simulated t Returns vs. MV    |                                |             |              |              |
| $R_2$ | 100.00%                       | 76.39%                         | 94.53%      | 97.04%       | 86.91%       |
| $R_3$ | 100.00%                       | 74.38%                         | 75.91%      | 58.06%       | 64.14%       |
|       | Simulated ASYM Returns vs. MV |                                |             |              |              |
| $R_2$ | 100.00%                       | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |
| $R_3$ | 100.00%                       | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |

Table 11:  $p$ -values of the spanning tests for simulated returns with respect to the optimal mean-variance portfolio. The reported results are for the mean-variance (M-V), mean-expected shortfall, and mean-PMA CRR (M-PMA) spanning. The PMA probability thresholds are 2.5%, 5%, 7.5%, 10%, and 12.5% with equal weights of 20%.

| Portfolios   | Assets     | N Obs | Avg. Return | Skewness | Kurtosis | Volatility |
|--------------|------------|-------|-------------|----------|----------|------------|
| Fama- French | Mkt        | 10448 | 11.13%      | -0.75    | 21.17    | 14.2%      |
|              | Small/Low  |       | 9.59%       | -0.67    | 11.66    | 16.8%      |
|              | Small/Med  |       | 14.91%      | -0.86    | 13.78    | 12.1%      |
|              | Small/High |       | 16.89%      | -0.88    | 14.75    | 11.7%      |
|              | Big/Low    |       | 10.74%      | -0.47    | 17.25    | 16.1%      |
|              | Big/Med    |       | 11.77%      | -1.10    | 31.06    | 13.5%      |
|              | Big/High   |       | 13.63%      | -0.89    | 24.21    | 13.6%      |
| S&P 500 Ind. | SP         | 2609  | 9.5%        | -0.01    | 6.13     | 18.0%      |
|              | COD        |       | 10.6%       | 0.00     | 7.84     | 20.8%      |
|              | CST        |       | 8.8%        | -0.09    | 9.09     | 16.4%      |
|              | ENE        |       | 12.0%       | 0.09     | 5.27     | 21.8%      |
|              | FIN        |       | 14.0%       | 0.20     | 5.82     | 23.4%      |
|              | HCR        |       | 13.5%       | -0.06    | 6.42     | 20.7%      |
|              | IND        |       | 10.6%       | -0.10    | 7.02     | 19.7%      |
|              | INT        |       | 14.1%       | 0.37     | 6.55     | 34.4%      |
|              | MAT        |       | 6.4%        | 0.22     | 6.16     | 21.3%      |
|              | TEL        |       | 4.1%        | 0.05     | 6.27     | 24.4%      |
| UTL          | 4.8%       | -0.22 | 10.03       | 18.3%    |          |            |

Table 12: Sample statistics of the market returns used for spanning tests. Fama-French are the returns on the Fama-French size-book-to-market portfolios with MKT being the market portfolio. S&P500 Ind. are returns on the S&P500 industrial indexes. GICS sectors: consumer discretionary (COD), consumer staples (CST), energy (ENE), financials (FIN), health care (HCR), industrials (IND), information technology (INT), materials (MAT), telecommunications services (TEL), and utilities (UTL). SP is the S&P500 composite index.

|              | <b>M-V</b>                                       | <b>Mean-Expected Shortfall</b> |             |              | <b>M-PMA</b> |
|--------------|--|--------------------------------|-------------|--------------|--------------|
|              |  | <b>2.5%</b>                    | <b>7.5%</b> | <b>12.5%</b> |              |
|              | Fama-French Size/Book-to-Mkt. Portfolios vs. MKT |                                |             |              |              |
| Small/Low    | 20.42%   | 10.61%                         | 10.36%      | 10.69%       | 10.03%       |
| Small/Medium | 0.00%  | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |
| Small/High   | 0.00%  | 0.00%                          | 0.00%       | 0.00%        | 0.00%        |
| Big/Low      | 4.91%  | 13.32%                         | 8.79%       | 7.50%        | 10.16%       |
| Big/Medium   | 8.45%  | 6.94%                          | 6.44%       | 6.59%        | 6.59%        |
| Big/High     | 0.07%  | 0.06%                          | 0.06%       | 0.07%        | 0.06%        |
|              | S&P 500 Sector Indexes vs. S&P 500 Composite     |                                |             |              |              |
| COD          | 74.67%   | 78.19%                         | 74.84%      | 75.24%       | 76.02%       |
| CST          | 68.49%   | 72.71%                         | 66.66%      | 64.80%       | 67.54%       |
| ENE          | 43.00%   | 45.04%                         | 45.79%      | 44.56%       | 44.73%       |
| FIN          | 30.68%   | 31.10%                         | 27.89%      | 27.37%       | 28.70%       |
| HCR          | 27.04%   | 26.87%                         | 28.53%      | 28.93%       | 28.18%       |
| IND          | 66.05%   | 80.24%                         | 68.21%      | 65.47%       | 71.24%       |
| INT          | 78.81%   | 69.60%                         | 75.60%      | 79.44%       | 74.74%       |
| MAT          | 70.15%   | 62.75%                         | 68.82%      | 71.04%       | 67.87%       |
| TEL          | 36.49%   | 33.41%                         | 36.05%      | 35.66%       | 35.28%       |
| UTL          | 70.17%   | 59.93%                         | 64.10%      | 67.58%       | 63.91%       |

Table 13:  $p$ -values of the spanning tests for the Fama-French size-book-to-market portfolios with respect to the market portfolio and S&P 500 sector indexes with respect to the S&P 500 composite index. The reported results are for the mean-variance (M-V), mean-expected shortfall, and mean-PMA CRR (M-PMA) spanning. The PMA probability thresholds are 2.5%, 5%, 7.5%, 10%, and 12.5% with equal weights of 20%. GICS sectors: consumer discretionary (COD), consumer staples (CST), energy (ENE), financials (FIN), health care (HCR), industrials (IND), information technology (INT), materials (MAT), telecommunications services (TEL), and utilities (UTL).

|              | M-V   | Mean-Expected Shortfall |        |        | M-PMA  |
|--------------|---|-------------------------|--------|--------|--------|
|              |   | 2.5%                    | 7.5%   | 12.5%  |        |
|              | Fama-French Size/Book-to-Mkt. Portfolios vs. MV |                         |        |        |        |
| Small/Low    | 99.35%  | 41.53%                  | 66.74% | 73.00% | 57.98% |
| Small/Medium | 99.40%  | 42.92%                  | 68.67% | 74.80% | 59.54% |
| Small/High   | 99.35%  | 34.99%                  | 66.35% | 74.39% | 55.39% |
| Big/Low      | 99.62%  | 89.45%                  | 95.54% | 93.35% | 94.40% |
| Big/Medium   | 99.68%  | 61.39%                  | 94.41% | 96.74% | 83.82% |
| Big/High     | 99.82%  | 67.45%                  | 93.98% | 99.66% | 86.65% |
|              | S&P 500 Sector Indexes vs. MV                   |                         |        |        |        |
| COD          | 99.99%  | 93.59%                  | 95.06% | 91.27% | 93.97% |
| CST          | 100.00%   | 92.04%                  | 88.98% | 94.99% | 92.67% |
| ENE          | 99.98%  | 90.40%                  | 96.18% | 96.86% | 95.36% |
| FIN          | 100.00%   | 79.76%                  | 85.73% | 89.68% | 85.87% |
| HCR          | 99.98%  | 97.23%                  | 96.33% | 95.22% | 98.60% |
| IND          | 100.00%   | 87.71%                  | 93.34% | 93.05% | 92.48% |
| INT          | 100.00%   | 97.49%                  | 99.65% | 97.21% | 98.00% |
| MAT          | 99.98%  | 86.69%                  | 81.94% | 91.49% | 86.92% |
| TEL          | 99.99%  | 81.44%                  | 90.92% | 90.35% | 88.20% |
| UTL          | 99.98%  | 94.35%                  | 91.95% | 99.66% | 94.02% |

Table 14:  $p$ -values of the spanning tests for the Fama-French size-book-to-market portfolios and S&P 500 sector indexes with respect to the optimal mean-variance portfolio. The reported results are for the mean-variance (M-V), mean-expected shortfall, and mean-PMA CRR (M-PMA) spanning. The PMA probability thresholds are 2.5%, 5%, 7.5%, 10%, and 12.5% with equal weights of 20%. GICS sectors: consumer discretionary (COD), consumer staples (CST), energy (ENE), financials (FIN), health care (HCR), industrials (IND), information technology (INT), materials (MAT), telecommunications services (TEL), and utilities (UTL).

| Portfolios  | Probability Threshold | Expected Portfolio Return |       |       |       |
|-------------|-----------------------|---------------------------|-------|-------|-------|
|             |                       | 10%                       | 12%   | 14%   | 16%   |
| ASYM        | 2.5%                  | 2.4%                      | 8.6%  | 8.9%  | 13.2% |
|             | 5%                    | 10.5%                     | 41.1% | 50.9% | 58.5% |
|             | 7.5%                  | 7.7%                      | 38.4% | 56.4% | 60.4% |
|             | 10%                   | 9.2%                      | 43.6% | 60.9% | 66.5% |
|             | 12.5%                 | 16.1%                     | 59.8% | 71.2% | 77.1% |
| Fama-French | 2.5%                  | 21.3%                     | 93.1% | 95.9% | 91.4% |
|             | 5%                    | 84.5%                     | 99.7% | 99.7% | 98.8% |
|             | 7.5%                  | 94.4%                     | 99.8% | 99.9% | 99.9% |
|             | 10%                   | 94.6%                     | 99.7% | 99.9% | 99.8% |
|             | 12.5%                 | 95.8%                     | 99.9% | 99.9% | 99.9% |
| Index       | 2.5%                  | 94.7%                     | 95.1% | 96.5% | 93.8% |
|             | 5%                    | 78.5%                     | 79.7% | 81.2% | 82.5% |
|             | 7.5%                  | 87.3%                     | 87.3% | 85.6% | 85.2% |
|             | 10%                   | 81.5%                     | 80.8% | 81.0% | 81.3% |
|             | 12.5%                 | 73.0%                     | 74.1% | 73.1% | 72.6% |

Table 15:  $p$ -values of the test for equality of the mean-variance and mean-shortfall portfolio weights for ASYM, Fama-French and, Index portfolios with inaccuracy in the mean returns taken into account.

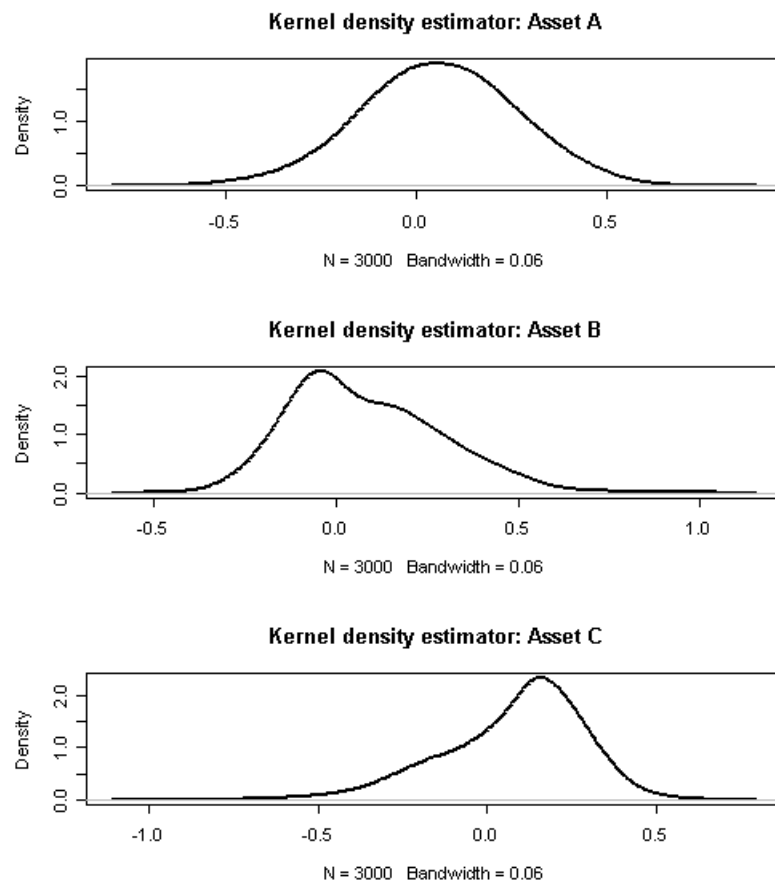


Figure 1: Kernel density of the returns simulated from ASYM distribution.