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Pension Liability Valuation and Asset Allocation in the Presence of Funding Risk

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ABSTRACT

Defined benefit pension liabilities are usually computed by discounting future pension promises with yields of risk-free or AA-rated bonds. We argue that a pension plan in financial distress should use discount rates that reflect the inherent funding risk. We propose a new valuation approach that utilizes the term structure of funding-risk-adjusted discount rates. These discount rates depend on the current asset allocation of the pension plan which affects future funding ratios. We show that an optimal asset allocation which accounts for this dependency varies in a nonlinear way with the initial funding ratio of the pension plan.

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1. Introduction

The valuation and funding of corporate defined benefit (DB) pension liabilities has attracted considerable attention from financial economists ever since the U.S. Employee Retirement Income Security Act (ERISA) came into force in 1974.[1] ERISA requires that firms both pre-fund pension liabilities, which are defined as the present discounted value of expected future pension payments, and invest plan assets prudently. This paper proposes a new and integrated approach to both pension liability valuation and prudent plan asset allocation. The motivation for our paper is the observation that current methods of pension liability valuation ignore one of the fundamental insights of modern financial economics, namely that future cash flows should be discounted using (a term structure of) discount rates that appropriately reflect the risks underlying the cash flows. Petersen (1996) shares this view: "the correct discount rate should depend upon the type of risk inherent in the pension promise."[2] Despite this insight, no formal model for valuing pension liabilities using "correct" discount rates currently exists. Our purpose here is to correct this anomaly.

Before discussing our model in more detail, it is worth having a closer look at existing practice in pension liability valuation. We need to distinguish between practice before and after the release by the U.S. Financial Accounting Standards Board (FASB) of Statement No. 87 concerning "Employers' Accounting for Pensions", which came into force for the fiscal years after December 15, 1986. Prior to FAS 87, companies sponsoring a DB pension plan used a wide array of assumptions to determine the market value of plan liabilities. Feldstein and Mørck (1983) showed that the discount rates assumed for the valuation of pension liabilities ranged from 5 percent to 10.5 percent for a sample of large manufacturing firms in 1979. The authors discover evidence that the precise choice was determined by a sponsoring company's trade-off between the tax advantage of a low discount rate and the cosmetic benefit to the annual report arising from a high discount rate; high discount rates could also be used to escape a Department of Labor request for additional contributions to the pension plan. Bodie et al. (1987) found evidence that more profitable firms use lower discount rates to calculate pension liabilities in an attempt to smooth corporate earnings. Overall, before FAS 87, the choice of discount rate appeared to be guided more by strategic management considerations than by the exercise of fiduciary responsibility towards plan beneficiaries by plan sponsors.

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[1] Although corporate DB pension plans are increasingly being replaced by defined contribution plans, private sector DB plan assets are still substantial and amounted to $1.9 trillion in the U.S. at the end of 2003 (Buessing and Soto, 2006).

[2] Ippolito (2002) argues along the same lines.

FAS 87 reduced the discretion sponsoring companies had over the choice of the discount rate. The Statement requires that "assumed discount rates shall reflect the rates at which the pension benefits could be effectively settled." The discount rates regularly published by the Pension Benefit Guaranty Corporation (PBGC), which was also established by ERISA, and used to value the liabilities of terminated pension plans, satisfied this condition. However, FAS 87 also allows the company "to look to rates of return of high-quality fixed income investments currently available and expected to be available during the period to maturity of the pension benefits." In practice, sponsoring companies often use the average yield to maturity on long-term corporate bonds with a Moody's AA rating (Coronado and Sharpe, 2003).[3] However, this practice still leaves some degree of discretion and this can be exercised strategically to manipulate earnings. Bergstresser et al. (2006) offer a post-FAS 87 analysis of pension assumptions and show that the expected return on plan assets, another assumption required by FAS 87, tends to be used to manipulate reported earnings. Similarly, Cocco and Volpin (2007) show, for the U.K., that insider trustees, who are also executive directors of the sponsoring company, tend to act in favor of the shareholders of the sponsor rather than in the interests of the pension plan members.

We propose a method of liability valuation which will generate heterogeneous, pension-plan-specific, discount rates as in the pre-FAS 87 period, but in a systematic and standardized way as in the post-FAS 87 period. Our method will remove any discretionary freedom with respect to the choice of the discount rate and thus effectively prevents manipulations of the form detected by Bergstresser et al. (2006) and Cocco and Volpin (2007). The discount rates will be plan specific whenever the risks inherent in the pension promise are plan specific. The relevant plan specific risk for a corporate pension plan, which promises nominal pension benefits and is large enough to diversify away mortality risk, is funding risk, namely the risk that the future funding positions of the plan are insufficient to guarantee the promised pension benefits. This risk is plan specific because it depends on the current funding ratio of plan assets to plan liabilities, on the sponsor's ability to close funding gaps over time, and on all decisions of the sponsor that affect future funding ratios. These include decisions relating to the magnitude of contributions and the generosity of pension benefits, as well as the plan's asset allocation. Crucially, we find that the asset allocation and discount rate choice are interdependent. Petersen (1996) predicted this: "As the firm shifts the pension assets from low risk assets (cash) to higher risk assets (stocks), the discount rate will rise only if the pension liability does not remain risk free."

[3] This choice is also consistent with U.K. and international accounting standards FRS 17 and IAS 19, respectively.

We will derive a term structure of funding spreads which appropriately reflects the funding risk of the pension plan. The term structure of funding-risk-adjusted (FRA) discount rates results from combining the term structures of funding spreads and risk-free (RF) discount rates. We distinguish between the RF liability – which is the value of the promised pension found by discounting promised future pension payments using the term structure of RF discount rates – and the FRA liability – obtained by discounting the same promised payments using the term structure of FRA discount rates. The funding spreads are endogenous (along the lines of Duffie and Singleton, 1999) in the sense that they are dependent on expected future funding ratios and, thus, on the current asset allocation. When the asset allocation is based on an objective function in the FRA liability, discount rates and portfolio weights become completely interdependent and can be jointly determined in a single optimization step. We illustrate this by extending the asset-liability model proposed by Hoevenaars et al. (2005), which is itself an extension of Campbell and Viceira (2005), to the case of an institutional investor with liabilities. Hoevenaars et al. consider an objective function in the expected utility of the log return on the funding ratio of assets to RF liabilities. We modify the objective function by introducing FRA liabilities and show that optimizing this function with respect to the asset allocation will automatically generate the desired term structure of funding spreads and hence a value for the pension liability that appropriately reflects future funding risk.

Earlier asset-liability models for pension plans were proposed by Sundaresan and Zapatero (1997), Rudolf and Ziemba (2004), Boulier et al. (2005) and Van Binsbergen and Brandt (2007). None of these papers, however, considers funding risk and its implications for pension liability valuation and optimal asset allocation.

The plan sponsor modeled by Sundaresan and Zapatero (1997) maximizes a power utility function in final surplus by choosing an optimal asset allocation. In contrast with our framework, the authors assume that investment opportunities are constant.

Rudolf and Ziemba (2004) consider an intertemporal asset allocation framework for a pension plan endowed with HARA utility in the surplus. The authors describe liability returns by an Itô process and do not explicitly refer to an underlying term structure of interest rates which determines the current value of pension liabilities. Hence, the choice of discount rates for the valuation of pension liabilities is not discussed.

The plan sponsor investigated by Boulier at al. (2005) attempts to minimize the expected discounted value of future contributions over a given horizon by means of asset allocation and contribution policies subject to the constraint that the plan assets are always sufficient to cover pension payments. The authors do not discuss the valuation of pension liabilities. The investment opportunity set is assumed to be constant.

Van Binsbergen and Brandt (2007) consider a rebalancing institutional investor with pension liabilities in the presence of time-varying investment opportunities. The investor maximizes a power utility function in the terminal surplus which includes a penalty on additional financial contributions. The authors compare three different discount rates for valuing liabilities: the current yield on a RF long-term government bond, a moving average of past yields and the unconditionally expected yield. They show that significant economic costs emerge from not using current yields.

In our model, the plan itself does not have to make *any* assumptions about discount rates and this helps to prevent the gaming exercises mentioned above. Instead, we show that it is optimal for the plan to derive a term structure of endogenous funding spreads – based on current yields – that appropriately reflects the risks underlying future pension payments. This brings benefits to all stakeholders of the pension plan, including even the shareholders in the sponsoring company. Feldstein and Seligman (1981) argue that promised future pension benefits are a substitute for current wages.[4] Consequently, if the promise is not fully funded, then a company introducing a pension plan in exchange for lower current wages will create accounting profits. If the stock market correctly values the underfunded pension obligations, the company's share price will drop by the extent of underfunding. In this case, shareholders will not be fooled by temporary accounting profits and, hence, will leave their lifetime consumption plan unchanged. However, if the market incorrectly values underfunded pension liabilities, shareholders may interpret temporary accounting profits as an increase in permanent income and increase consumption accordingly.[5] Contrary to the findings of studies of the pre-FAS 87 period by Feldstein and Seligman (1981) and Bulow et al. (1987), recent studies of the post-FAS 87 period by Coronado and Sharpe (2003) and Franzoni and Marín (2006) find that the market does not correctly value firms with a DB pension plan. Coronado and Sharpe find evidence that all companies with a DB plan are overvalued, while Franzoni and Marín show that the market only overvalues companies with underfunded DB pension liabilities. Thus, the Feldstein and Seligman argument applies and shareholders are at risk of suboptimally cutting savings. A regulator adopting our proposed method of pension liability discounting with a FRA term structure of discount rates would help to increase shareholder awareness of possible future funding gaps, since the funding spreads that emerge from our method reflect both future underfunding probabilities and expected funding ratios conditional

[4] It is an empirical question as to whether there are perfect compensating wage differentials for pension benefits.

[5] This would reduce national savings as Feldstein and Seligman (1981) point out, since the combined consumption of shareholders and employees in the sponsoring company increases.

on underfunding. Reporting the term structure of funding spreads would increase transparency and help shareholders make optimal consumption decisions.

Pension plan members are a second group of stakeholders with a clear interest in the FRA value of their pension promise, since the future pension payments they expect to receive will also influence their lifetime savings and consumption choices. If the value of the RF liability exceeds the value of the FRA liability, then the member might decide to compensate through increased private pension savings at the cost of reduced consumption. Again, transparency would help plan beneficiaries make optimal choices.

Transparency with respect to future underfunding probabilities will benefit the third key stakeholder of the pension plan, namely the sponsoring company itself. Rauh (2006) shows that, for companies facing financial constraints, capital expenditures decline by the amount of mandatory contributions to their DB pension plans. The term structure of funding spreads immediately highlights possible future financial constraints arising from the current decisions of the plan sponsor in respect of the funding level, the magnitude of contributions, the generosity of benefits and the asset allocation. Henceforth, the plan sponsor will not be surprised by the need to make future contributions to the plan, nor by the consequential requirement to curtail corporate investment.

On the basis of applying our model to U.S. data, we conclude that a pension plan with an initial funding ratio that is not critically low and which chooses its asset allocation to maximize our proposed objective function in the log return of assets over FRA liabilities will exhibit lower underfunding probabilities, higher recovery fractions, and lower funding spreads than a pension plan which determines its asset allocation by maximizing a conventional objective function in the log return of assets over RF liabilities. The optimal allocation to stocks will also be lower, unless the initial funding ratio is severe (a state of underfunding which lies between critical and moderate). By contrast with conventional asset allocation approaches, the optimal asset allocation implied by our methodology will vary with the initial funding ratio of the pension plan.

The remainder of this paper is organized as follows. In Section 2, we derive the term structure of FRA discount rates which depends on the asset allocation of the pension fund. We follow Campbell et al. (2003), Campbell and Viceira (2005), Hoevenaars et al. (2005), and Van Binsbergen and Brandt (2007) among others and use a VAR(1) model for the return dynamics which will be discussed in Section 3. In Section 4, we estimate the term structure of funding spreads implied by the optimal asset allocation. We compare the funding spreads automatically generated by our methodology with those manually calculated from applying the optimal asset allocation procedure specified in Hoevenaars et al. (2005) to the funding spread formula. We compare the resulting asset allocations. Section 5 summarizes the results of the paper and concludes.

2. Asset-Liability Management with Funding Risk

This section considers the liability valuation and asset allocation problems that a pension plan has to solve as a part of an asset-liability management (ALM) exercise. We modify existing ALM approaches by considering FRA liabilities at the valuation stage and an objective function in the terminal funding ratio of assets to FRA liabilities at the asset allocation stage. The outcome of this exercise is a liability value which appropriately reflects the funding risk inherent in the pension promise and an asset allocation which is consistent with the liability valuation.

2.1 Assets and Risk-Free Pension Liabilities

Assume the pension plan has an investment horizon of k periods. Let A_t denote the value of the assets in the pension plan at time t and $A_{t+k} = A_t R^A_{t+k}$ the value of the assets at time $t+k$, where R^A_{t+k} is the cumulative portfolio gross return between t and $t+k$ defined as

$$R^A_{t+k} = w_t' R_{t+k} + (1 - w_t' 1_v) R^f_{t+k} = R^f_{t+k} + w_t' \left(R_{t+k} - R^f_{t+k} 1_v\right) = R^f_{t+k} + w_t' R^e_{t+k} \tag{1}$$

where R^f_t denotes the cumulative risk-free gross return and 1_v is the unit vector of appropriate length. R_{t+k} and $R^e_{t+k} = R_{t+k} - R^f_{t+k} 1_v$ are vectors of gross and excess returns, respectively, for a number of risky assets the pension plan considers for asset allocation purposes and w_t is the vector of portfolio weights in these assets. Using the now familiar approximation of the log portfolio return introduced by Campbell and Viceira (2002), the log of (1) can be written as[6]

$$r^A_{t+k} = \ln R^A_{t+k} = r^f_{t+k} + w_t' r^e_{t+k} + 0.5 w_t' \mathrm{diag}\left(V_t\left[r^e_{t+k}\right]\right) - 0.5 w_t' V_t\left[r^e_{t+k}\right] w_t \tag{2}$$

with $r^f_{t+k} = \ln R^f_{t+k}$ and $r^e_{t+k} = \ln R_{t+k} - \ln R^f_{t+k} 1_v$. Lower-case letters denote variables in logs (unless defined otherwise) and $V_t[\cdot]$ is the conditional variance-covariance matrix operator at

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[6] The approximation is based on a second-order Taylor series expansion and is needed because the log portfolio return is not equal to the portfolio-weighted average of log returns.

time t, with $\text{diag}(V_t[\cdot])$ as a vector of the main diagonal elements of $V_t[\cdot]$.

When valuing pension liabilities, we make a number of simplifying assumptions. First, we assume that all future promised pension benefit payments, B, are constant in nominal terms in order to abstract from inflation risk.[7] Second, we assume that the plan is sufficiently large that longevity risk is diversified away.[8] Third, in line with Hoevenaars et al. (2005), we assume that the maturity of the pension liability is constant which holds for a pension plan in a stationary state where the distribution of age cohorts and accrued benefit rights of plan members remains constant over time. Fourth, like Hoevenaars et al. (2005), we assume that new contributions to the plan exactly offset any increase in accrued pension rights. The overarching purpose of these assumptions is to allow us to focus on the change in liability value arising exclusively from changes in the yield curve.

The current value of the liability of the pension plan is given by the present discounted value of all future promised pension benefit payments to current plan members, which, using the first two assumptions above, is equal to

$$L_t^0 = \sum_{s=1}^{T} \frac{B}{\left(1+Y_t^s\right)^s} = B\sum_{s=1}^{T} P_t^s \ . \qquad (3)$$

T is the liability horizon and defines the maturity of the pension plan. In (3), we assume that all future pension payments are risk free or equivalently that the pension plan is always sufficiently funded to guarantee the payments. Correspondingly, we will refer to (3) as the RF value of pension liabilities. Since the discount rate should reflect the risks underlying future cash flows, it follows from this that Y_t^s is the time-t nominal yield on a default-free, zero-coupon bond with unit par value and maturity s and $P_t^s = \left(1+Y_t^s\right)^{-s}$ is the time-t price of this bond. The zero superscript in L_t^0 reflects the fact that the discount rates have a zero funding spread.

Under the third and fourth assumptions above, the $t+k$ value of the liabilities is

$$L_{t+k}^0 = \sum_{s=k+1}^{T+k} \frac{B}{\left(1+Y_{t+k}^{s-k}\right)^{s-k}} = \sum_{s=1}^{T} \frac{B}{\left(1+Y_{t+k}^{s}\right)^{s}} = B\sum_{s=1}^{T} P_{t+k}^s \qquad (4)$$

[7] This is a fairly realistic assumption. In the U.S., for example, companies are obliged to publish pension liabilities which are calculated on a nominal basis. In the Netherlands, nominal pensions are guaranteed. Indexation is subject to a sufficiently high funding ratio. See Nijman and Koijen (2006) and De Jong (2007) for the valuation of conditionally indexed pension liabilities.

[8] This is a common assumption in the literature. See e.g. Hoevenaars et al. (2005) and Nijman and Koijen (2006).

and the cumulative gross liability return between period t and period $t+k$ follows as

$$R_{t+k}^{L,0} = \frac{L_{t+k}^{0}}{L_t^{0}} = \frac{\sum_{s=1}^{T} P_{t+k}^{s}}{\sum_{s=1}^{T} P_t^{s}} = \sum_{s=1}^{T} \frac{P_t^{s}}{\sum_{r=1}^{T} P_t^{r}} R_{t+k}^{0,s} = \sum_{s=1}^{T} u_t^{s} R_{t+k}^{0,s} = u_t' R_{t+k}^{0}, \qquad \text{where } u_t^{s} = \frac{P_t^{s}}{\sum_{r=1}^{T} P_t^{r}} \tag{5}$$

is the time-t liability portfolio weight of a default-free zero-coupon bond with constant maturity s and $R_{t+k}^{0,s} = P_{t+k}^{s} / P_t^{s}$ denotes the cumulative gross return on this bond. The $T \times 1$ vectors u_t and R_{t+k}^{0} include the elements u_t^{s} and $R_{t+k}^{0,s}$ for $s = 1, \ldots, T$. We can rewrite the liability return with the risk-free rate as $R_{t+k}^{L,0} = u_t' R_{t+k}^{0} = R_{t+k}^{f} + u_t'\left(R_{t+k}^{0} - R_{t+k}^{f} 1_v\right)$ in order to apply again a log-linearization with $r_{t+k}^{0} = \ln R_{t+k}^{0}$

$$r_{t+k}^{L,0} = \ln R_{t+k}^{L,0} = u_t' r_{t+k}^{0} + 0.5 u_t' \operatorname{diag}\left(V_t\left[r_{t+k}^{0} - r_{t+k}^{f} 1_v\right]\right) - 0.5 u_t' V_t\left[r_{t+k}^{0} - r_{t+k}^{f} 1_v\right] u_t . \tag{6}$$

2.2 Funding-Risk-Adjusted Pension Liabilities

In the case where future pension payments are not risk free, the simple valuation framework (3) cannot be used. A stochastic discount factor (SDF) should be employed in order to discount risky future pension payments. Using the s-period SDF, M_{t+s}, the current value of a pension payment in period $t+s$, $s = 1, \ldots, T$, promised by a pension plan which is subject to funding risk equals

$$E_t\left[M_{t+s} \text{Payoff}_{t+s}\right] = \left[\left(1 - \pi_t^{s}\right) M_{t+s}^{o} + \pi_t^{s} M_{t+s}^{u} \lambda_t^{s}\right] B \tag{7}$$

where we decompose the expectation into the probability weighted realizations for two states of the world which we call “o” (overfunding) and “u” (underfunding). M_{t+s}^{o} and M_{t+s}^{u} are the SDF’s in the states of over- and underfunding, respectively. We will show below that it is reasonable to assume that $M_{t+s}^{o} < M_{t+s}^{u}$. π_t^{s} denotes the conditional probability of underfunding at time $t+s$. Correspondingly, $\left(1 - \pi_t^{s}\right)$ denotes the conditional probability of overfunding at time $t+s$. The pension benefit payoff equals B in the state of overfunding and $\lambda_t^{s} B$ in the state of underfunding where λ_t^{s}, $0 \le \lambda_t^{s} \le 1$, denotes the recovery fraction at time $t+s$, conditional on underfunding at $t+s$.[9] We assume that each future pension benefit is scaled by the

[9] Since we treat the pension promise B as fixed, we do not allow for any increase in benefits such as those that might be granted when the pension plan is overfunded. We are only concerned about shortfall funding risk.

conditionally expected funding ratio whenever the latter falls below a threshold, τ, which defines a situation of underfunding.[10] Thus, we formally define π_t^s and λ_t^s for $s = 1, \ldots, T$ as

$$\pi_t^s = \Pr_t\left(\frac{A_{t+s}}{L_{t+s}^0} < \tau\right) \tag{8}$$

$$\lambda_t^s = \frac{1}{\tau} E_t\left[\frac{A_{t+s}}{L_{t+s}^0} \mid \frac{A_{t+s}}{L_{t+s}^0} < \tau\right]. \tag{9}$$

Note that the funding ratio in these equations is expressed in terms of the liability value (4), which assumes that future pension payments are risk free. It is important to recognize that funding risk should be defined with respect to the promised liability (4) and not with respect to the funding-risk-adjusted liability which we derive in the current section. Because the recovery fraction (9) is scaled by the funding threshold, τ, it is possible to have threshold levels larger than unity without violating the inequality $\lambda_t^s \leq 1$ in the case where a regulatory regime requires the sponsoring company to maintain a funding buffer.[11]

If the pension plan is completely independent of the sponsoring company and the latter is under no obligation to close any funding gap, then $\tau = 1$ would seem to be a sensible choice for the plan trustees to make. If the regulatory authority obliges the sponsoring company to close any funding gap, the funding threshold could be modeled as

$$\tau = 1 - \frac{N_{t+s}}{L_{t+s}^0} \qquad \text{such that} \qquad \pi_t^s = \Pr_t\left(\frac{A_{t+s} + N_{t+s}}{L_{t+s}^0} < 1\right) \tag{10}$$

where N_{t+s} denotes the net worth of the sponsoring company at time $t+s$. In this case, underfunding only occurs whenever the sum of the pension plan assets and the net worth of the sponsoring company falls below the value of the pension liability.[12]

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[10] Other scaling mechanisms are possible. For example, nearby pension payments could receive a higher weight than distant pension payments. We adopt the simplest scaling scheme because it treats all retirees as equals.

[11] In the Netherlands, for example, the pension regulator requires pension plans to be 105% funded at all times, which implies $\tau = 1.05$.

[12] Similarly, N_{t+s} could be set at the level of liabilities that are covered by a pension guarantee fund such as the Pension Benefit Guaranty Corporation (PBGC) in the U.S. or the Pension Protection Fund (PPF) in the U.K. in case of default by the plan sponsor. However, because of large concentration risks and moral hazard affecting the behavior of the covered companies, the guarantee fund is itself subject to underfunding and default risk unless the government underwrites any funding gap (see McCarthy and Neuberger, 2005).

We find it useful to decompose (7) into the product of the risk-neutral value of the pension payment and a s-period funding-risk adjustment factor, Θ_t^s [13]

$$E_t\left[M_{t+s}\text{Payoff}_{t+s}\right] = E_t\left[M_{t+s}\right]\left[\left(1-\pi_t^s\right)+\pi_t^s\lambda_t^s\right]\Theta_t^s B \qquad \text{where} \tag{11}$$

$$E_t\left[M_{t+s}\right] = \left(1-\pi_t^s\right)M_{t+s}^o + \pi_t^s M_{t+s}^u = \left(1+Y_t^s\right)^{-s} \tag{12}$$

is the expected SDF (which has to equal to the inverse of the gross risk-free discount rate of appropriate maturity s). The risk-neutral value of the random pension benefit payoff is $E_t\left[M_{t+s}\right]\left[\left(1-\pi_t^s\right)+\pi_t^s\lambda_t^s\right]B$ in (11). Because the pension benefit payoff at time s is assumed to be risky, the pension plan members demand an additional funding-risk premium, θ_t^s, which is incorporated in the funding-risk adjustment factor, $\Theta_t^s \equiv \left(1+\theta_t^s\right)^{-s}$, in (12). This factor can be readily derived from equating equations (7) and (11)

$$\Theta_t^s = \frac{\left(1-\pi_t^s\right)}{\left[\left(1-\pi_t^s\right)+\pi_t^s\lambda_t^s\right]}\frac{M_{t+s}^o}{E_t\left[M_{t+s}\right]} + \frac{\pi_t^s\lambda_t^s}{\left[\left(1-\pi_t^s\right)+\pi_t^s\lambda_t^s\right]}\frac{M_{t+s}^u}{E_t\left[M_{t+s}\right]}. \tag{13}$$

We can see that $\Theta_t^s < 1$ ($\theta_t^s > 0$) if $M_{t+s}^u > M_{t+s}^o$.

We assume that the preferences of the pension plan members can be represented by a consumption-based asset pricing model (see Cochrane, 2001). Thus, the stochastic discount factors for the states of overfunding and underfunding take the form

$$M_{t+s}^o = \beta^s\left(\frac{C_{t+s}^o}{C_t}\right)^{-\gamma} = \beta^s\left(g_{t+s}^o\right)^{-\gamma} \quad \text{and} \quad M_{t+s}^u = \beta^s\left(\frac{C_{t+s}^u}{C_t}\right)^{-\gamma} = \beta^s\left(g_{t+s}^u\right)^{-\gamma} \tag{14}$$

where $g_{t+s}^o = C_{t+s}^o / C_t$ and $g_{t+s}^u = C_{t+s}^u / C_t$ denote consumption growth in the states of over- and underfunding, respectively, β denotes the subjective time-discount factor and γ the coefficient of relative risk aversion for the representative pension plan member. Using (14) we find that $E_t\left[M_{t+s}\right] = \left(1-\pi_t^s\right)\beta^s\left(g_{t+s}^o\right)^{-\gamma} + \pi_t^s\beta^s\left(g_{t+s}^u\right)^{-\gamma}$. $M_{t+s}^u > M_{t+s}^o$ whenever $g_{t+s}^o > g_{t+s}^u$, i.e., consumption growth in the state of overfunding exceeds consumption growth in the state of underfunding. This is likely to be the case when overfunding corresponds to a state of high as-

[13] $E_t[M_{t+s}\text{Payoff}_{t+s}]$ can be rewritten as $E_t[M_{t+s}]E_t[\text{Payoff}_{t+s}]+\text{cov}_t[M_{t+s},\text{Payoff}_{t+s}]$ where the first term denotes the risk-neutral value of the random payoff and the second term denotes an additive risk adjustment (Cochrane, 2001) term. We prefer to work a multiplicative risk adjustment factor as reflected in the term Θ^s_t. Of course, any additive risk adjustment term can be translated into a multiplicative one and vice versa.

set values and the representative pension plan member invests in the same asset classes as the pension plan and increases his consumption when his wealth is high. Given (14), we can determine the remaining unknown elements of the funding-risk adjustment factor on the right-hand-side of (13) as

$$\frac{M^o_{t+s}}{E_t[M_{t+s}]} = \frac{\left(g^o_{t+s}\right)^{-\gamma}}{\left(1-\pi^s_t\right)\left(g^o_{t+s}\right)^{-\gamma} + \pi^s_t\left(g^u_{t+s}\right)^{-\gamma}} = \left[\left(1-\pi^s_t\right) + \pi^s_t\left(\frac{g^o_{t+s}}{g^u_{t+s}}\right)^{\gamma}\right]^{-1} \tag{15}$$

$$\frac{M^u_{t+s}}{E_t[M_{t+s}]} = \frac{\left(g^u_{t+s}\right)^{-\gamma}}{\left(1-\pi^s_t\right)\left(g^o_{t+s}\right)^{-\gamma} + \pi^s_t\left(g^u_{t+s}\right)^{-\gamma}} = \left[\left(1-\pi^s_t\right)\left(\frac{g^o_{t+s}}{g^u_{t+s}}\right)^{-\gamma} + \pi^s_t\right]^{-1}. \tag{16}$$

For $g^o_{t+s} > g^u_{t+s}$, the ratio (15) will be smaller than unity, while the ratio (16) will be larger than unity. The funding-risk premium, $\theta^s_t = -1 + \left(\Theta^s_t\right)^{-1/s}$, is known, once g^o_{t+s} / g^u_{t+s}, the ratio of consumption growth in the state of overfunding to consumption growth in the state of under-funding, is known. We will assume this ratio, $\phi = g^o_{t+s} / g^u_{t+s}$, is constant, with $\phi > 1$. In the empirical section of the paper, ϕ will be varied in order to assess the responsiveness of the resulting funding-risk premia.

It is useful to derive some comparative statics results from (13) for the funding-risk premium, θ^s_t. First, we can see that θ^s_t is zero (Θ^s_t = 1), whenever $\pi^s_t = 0$ or $\pi^s_t = 1$. Thus, if one of the two possible states of the world occurs with certainty, the funding-risk premium is zero, whether or not this state is favorable or unfavorable for the pension plan member. For conditional underfunding probabilities between the two extreme outcomes, $0 < \pi^s_t < 1$, we can show (after some simple but tedious calculations) that the funding-risk premium increases with an increasing underfunding probability (for fixed s and λ^s_t), so long as $\phi^{-\gamma} > \lambda^s_t\left(\pi^s_t\right)^2\left(1-\pi^s_t\right)^{-2}$ is fulfilled. If this inequality is reversed, the funding-risk premium decreases with an increasing underfunding probability. The point at which the inequality switches corresponds with a point of maximum uncertainty about the future state of the world and so pension plan members will demand the highest risk premium here. The funding-risk premium decreases with increasing maturity, s, and recovery fraction, λ^s_t, and becomes zero for $\lambda^s_t = 1$. It also becomes zero for $\phi = 1$, is negative for $\phi < 1$ and positive for $\phi > 1$. All comparative statics results conform with a priori expectations.

We are now able to derive the FRA value of pension liabilities as

$$L^{\Delta}_t = \sum_{s=1}^{T} \frac{\left[\left(1-\pi^s_t\right) + \pi^s_t\lambda^s_t\right]\,\Theta^s_t\,B}{\left(1+Y^s_t\right)^s} = \sum_{s=1}^{T} \frac{B}{\left[\left(1+Y^s_t\right)\left(1+\Delta^s_t\right)\right]^s} = \sum_{s=1}^{T} P^s_t D^s_t \tag{17}$$

where the first term follows immediately from (11). For the second term, we express the FRA liability value in terms of a funding spread, Δ_t^s, over the risk-free yield, Y_t^s

$$\Delta_t^s = -1 + \left(1+\theta_t^s\right)\left(1-\pi_t^s + \pi_t^s\lambda_t^s\right)^{-1/s} \tag{18}$$

and use $D_t^s = \left(1+\Delta_t^s\right)^{-s}$ for the third term. We introduce the funding spread to emphasize that the discount rate for calculating the present value of all future pension payments should reflect the degree of funding risk in the case where future pension payments are not risk free.[14] The funding spread increases with an increasing funding-risk premium, θ_t^s. For a given funding-risk premium, the funding spread increases with an increasing conditional underfunding probability, π_t^s, and decreases with an increasing recovery fraction, λ_t^s, in the case of underfunding. The delta superscript in L_t^Δ reflects the fact that discount rates have a non-zero funding spread.

As before, we can derive the FRA liability value at time $t+k$

$$L_{t+k}^\Delta = \sum_{s=k+1}^{T+k} \frac{B}{\left[\left(1+Y_{t+k}^{s-k}\right)\left(1+\Delta_{t+k}^{s-k}\right)\right]^{s-k}} = \sum_{s=1}^{T} \frac{B}{\left[\left(1+Y_{t+k}^{s}\right)\left(1+\Delta_{t+k}^{s}\right)\right]^{s}} = B\sum_{s=1}^{T} P_{t+k}^s D_{t+k}^s \tag{19}$$

and the cumulative gross liability return between time t and time $t+k$ follows as

$$R_{t+k}^{L,\Delta} = \frac{L_{t+k}^\Delta}{L_t^\Delta} = \sum_{s=1}^{T} \frac{P_t^s D_t^s}{\sum_{r=1}^{T} P_t^r D_t^r} R_{t+k}^{\Delta,s} = \sum_{s=1}^{T} v_t^s R_{t+k}^{\Delta,s} = v_t' R_{t+k}^\Delta, \qquad \text{where } v_t^s = \frac{P_t^s D_t^s}{\sum_{r=1}^{T} P_t^r D_t^r} \tag{20}$$

is the time-t liability portfolio weight of a defaultable zero-coupon bond with constant maturity s and $R_{t+k}^{\Delta,s} = P_{t+k}^s D_{t+k}^s / P_t^s D_t^s = R_{t+k}^{0,s} D_{t+k}^s / D_t^s$ denotes the cumulative gross return on this bond. The $T \times 1$ vectors v_t and R_{t+k}^Δ include the elements v_t^s and $R_{t+k}^{\Delta,s}$ for $s = 1,\ldots,T$. We can rewrite the liability return in terms of the risk-free rate as $R_{t+k}^{L,\Delta} = v_t' R_{t+k}^\Delta = R_{t+k}^f + v_t'\left(R_{t+k}^\Delta - R_{t+k}^f 1_v\right)$ in order to make use of log-linearization with $r_{t+k}^\Delta = \ln R_{t+k}^\Delta$

[14] The reader will notice that our approach of deriving funding spreads resembles the derivation of credit spreads in structural credit risk models. The structural approach originates with Merton (1974) and models the value of the issuing firm: default occurs either at maturity (as in Merton, 1974) or when the firm's asset value falls below a critical threshold (as put forward by Black and Cox, 1976). Our model is related to the latter with τ denoting the critical threshold. By contrast, the reduced-form (or intensity-based) approach, introduced by Jarrow and Turnbull (1995) and extended by Duffee (1999), Duffie and Singleton (1999) and others, assumes an exogenous model for the default and recovery processes.

$$r^{L,\Delta}_{t+k} = \ln R^{L,\Delta}_{t+k} = v_t' r^{\Delta}_{t+k} + 0.5 v_t' \mathrm{diag}\left(V_t\left[r^{\Delta}_{t+k} - r^{f}_{t+k} 1_v\right]\right) - 0.5 v_t' V_t\left[r^{\Delta}_{t+k} - r^{f}_{t+k} 1_v\right] v_t \,.^{15} \qquad (21)$$

Since (18) holds for any s, we can immediately derive a term structure of funding spreads as soon as we have estimates of the respective underfunding probabilities and recovery fractions for all s. In order to obtain the latter quantities from an econometric model with lognormally distributed returns, which we discuss below, we express (8) and (9) in terms of the log funding ratio return.[16] We denote the funding ratios for RF liabilities as $F^0_{t+s} = A_{t+s} / L^0_{t+s}$ and for FRA liabilities as $F^{\Delta}_{t+s} = A_{t+s} / L^{\Delta}_{t+s}$ for any s. Let us assume the following distribution of the log funding ratio at time $t+s$

$$\ln\left(F^0_{t+s}\right) = \ln\left(F^0_t \frac{R^A_{t+s}}{R^{L,0}_{t+s}}\right) = f^0_t + s^0_{t+s} \sim N\left(f^0_t + E_t\left[s^0_{t+s}\right], V_t\left[s^0_{t+s}\right]\right) \qquad (22)$$

where $f^0_t = \ln\left(F^0_t\right)$ is the log of the initial funding ratio and $s^0_{t+s} = r^A_{t+s} - r^{L,0}_{t+s}$ is the log funding ratio return with respect to F^0_{t+s}. The corresponding log funding ratio return with respect to F^{Δ}_{t+s} is $s^{\Delta}_{t+s} = r^A_{t+s} - r^{L,\Delta}_{t+s}$. From (22), we derive the following expressions

$$\pi^s_t = \Pr_t\left(\frac{A_{t+s}}{L^0_{t+s}} < \tau\right) = \Pr_t\left(\ln\left(\frac{A_{t+s}}{L^0_{t+s}}\right) < \ln(\tau)\right) = \Pr_t\left(f^0_t + s^0_{t+s} < \ln(\tau)\right) = \Phi\left(\frac{\ln(\tau) - \left(f^0_t + E_t\left[s^0_{t+s}\right]\right)}{V^{0.5}_t\left[s^0_{t+s}\right]}\right) \qquad (23)$$

$$\begin{aligned}\lambda^s_t &= \frac{1}{\tau} E_t\left[\frac{A_{t+s}}{L^0_{t+s}} \mid \frac{A_{t+s}}{L^0_{t+s}} < \tau\right] \\ &= \frac{1}{\tau\pi^s_t} \exp\left(f^0_t + E_t\left[s^0_{t+s}\right] + 0.5 V_t\left[s^0_{t+s}\right]\right) \cdot \Phi\left(\frac{\ln(\tau) - \left(f^0_t + E_t\left[s^0_{t+s}\right]\right) - V_t\left[s^0_{t+s}\right]}{V^{0.5}_t\left[s^0_{t+s}\right]}\right)\end{aligned} \qquad (24)$$

for (8) and (9) where $\Phi(\cdot)$ denotes the c.d.f. of the standard normal distribution. The second line of (24) exploits properties of the truncated lognormal distribution (see Lien, 1985). It is evident from (23) and (24) that only the ratio of the initial funding ratio, F^0_t, to the underfunding threshold, τ, is relevant for determining the underfunding probabilities and recovery fractions, not their absolute value. Thus, for given moments of the log funding ratio return s^0_{t+s},

[15] We ignore in this Taylor series expansion the fact that v_t depends on r^{Δ}_{t+k}-r^f_{t+k} as well (as shown below). This is justified by the fact that since the elements of v_t sum up to unity, the impact of this simplification is negligible. Later we will see that the dependency of v_t on the funding spread, which itself will depend on r^{Δ}_{t+k}-r^f_{t+k} and the chosen asset allocation, has no consequences for the resulting optimal asset allocation.

[16] In the appendix, we provide a similar derivation for the case (10) in which the sponsoring company is liable to fund pension liabilities.

the parameter constellation $F_t^0 = 1$ and $\tau = 0.8$ will give the same values for π_t^s and λ_t^s as the constellation $F_t^0 = 1.25$ and $\tau = 1.0$.

2.3 Asset Allocation

In the previous subsection, we derived the funding spreads (18) which are dependent on the underfunding probabilities (23) and recovery fractions in the case of underfunding (24). Because the latter two variables depend on the future funding ratio of assets to RF liabilities, the funding spreads are a function of the current asset allocation. Hence, the current and future values of FRA pension liabilities, (17) and (19), depend on the chosen asset allocation of the pension plan.[17] A pension plan simply needs to evaluate the underfunding probabilities and recovery fractions at the implemented asset allocation in order to obtain both the appropriate term structure of funding spreads and the FRA liability value of interest.

This, however, assumes that the asset allocation is chosen independently of the liability valuation process which, of course, is not likely to be optimal in general. It is true that the pension industry uses asset-liability management tools to manage the corporate sponsor's funding risk. Typically, an objective function in the future funding ratio of plan assets to plan liabilities is maximized by means of an optimal liability-driven asset allocation. If the funding ratio is defined with respect to RF liabilities, this problem can be solved after the liability valuation exercise because RF liabilities are independent on the chosen asset allocation. However, if the funding ratio is defined in terms of FRA liabilities, the problem needs to be solved jointly with the liability valuation exercise, because FRA liabilities depend on the chosen asset allocation. We argued in the introduction that all stakeholders of a pension plan have good reasons to be interested in the FRA liability value (not least to provide a comparison with the RF liability value). For this reason, we propose an objective function in the future funding ratio of assets to FRA liabilities. We compare this objective function with the more conventional one in the future funding ratio of assets to RF liabilities. In the empirical part of the paper, we will analyze the asset allocation consequences and the implied term structures of funding spreads for the two approaches for a stylized pension plan.

In line with Hoevenaars et al. (2005) and Van Binsbergen and Brandt (2007), we assume that the pension plan maximizes the conditional time-t expectation of a utility function in the terminal funding ratio at horizon k. This is a natural extension of an objective function in the expected utility of final wealth – considered, for example, by Campbell and Viceira (2005) –

[17] This result will always hold when the liability depends on future funding ratios. The case of conditionally indexed pensions discussed in footnote 7 is another example of an asset-allocation-dependent liability value.

to the case of an institutional investor who cares about liabilities. Assuming power utility with a constant coefficient of relative risk aversion, γ, and lognormally distributed funding ratio returns, the optimization problem is equivalent to the following mean-variance optimization problem in log funding ratio returns (see Hoevenaars et al., 2005)

$$w_t^0 = \arg\max_{w_t} \left\{ E_t\left[s_{t+k}^0\right] + \frac{1}{2}(1-\gamma)V_t\left[s_{t+k}^0\right] \right\} \tag{25}$$

$$w_t^\Delta = \arg\max_{w_t} \left\{ E_t\left[s_{t+k}^\Delta\right] + \frac{1}{2}(1-\gamma)V_t\left[s_{t+k}^\Delta\right] \right\}. \tag{26}$$

Problem (25) refers to a pension plan which values the pension liability using a term structure of RF discount rates, while problem (26) holds for a pension plan which uses a term structure of FRA discount rates for liability valuation purposes. Problem (25) and its solution is due to Hoevenaars et al. (2005), while problem (26) and its solution is our contribution to the literature.[18] Although similar at first sight, it should by now be clear that solving problem (26) is much more demanding than solving problem (25) because of the presence of endogenous funding spreads which depend on the chosen asset allocation. We will use (25) as a benchmark for comparing the results obtained from (26) in the empirical part of the paper.

The conditional moments of the cumulative log funding ratio returns at horizon k (which are also relevant for expressions (22) – (24) for k = s) are

$$\begin{aligned} E_t\left[s_{t+k}^0\right] = {} & E_t\left[r_{t+k}^f\right] + w_t' E_t\left[r_{t+k}^e\right] + 0.5 w_t' \mathrm{diag}\left(V_t\left[r_{t+k}^e\right]\right) - 0.5 w_t' V_t\left[r_{t+k}^e\right] w_t \\ & - u_t' E_t\left[r_{t+k}^0\right] - 0.5 u_t' \mathrm{diag}\left(V_t\left[r_{t+k}^0 - r_{t+k}^f 1_v\right]\right) + 0.5 u_t' V_t\left[r_{t+k}^0 - r_{t+k}^f 1_v\right] u_t \end{aligned} \tag{27}$$

$$\begin{aligned} V_t\left[s_{t+k}^0\right] = {} & V_t\left[r_{t+k}^f\right] + w_t' V_t\left[r_{t+k}^e\right] w_t + 2 w_t' \mathrm{cov}_t\left[r_{t+k}^e, r_{t+k}^f\right] + u_t' V_t\left[r_{t+k}^0\right] u_t \\ & - 2\,\mathrm{cov}_t\left[r_{t+k}^f, r_{t+k}^0\right] u_t - 2 w_t' \mathrm{cov}_t\left[r_{t+k}^e, r_{t+k}^0\right] u_t \end{aligned} \tag{28}$$

$$\begin{aligned} E_t\left[s_{t+k}^\Delta\right] = {} & E_t\left[r_{t+k}^f\right] + w_t' E_t\left[r_{t+k}^e\right] + 0.5 w_t' \mathrm{diag}\left(V_t\left[r_{t+k}^e\right]\right) - 0.5 w_t' V_t\left[r_{t+k}^e\right] w_t \\ & - v_t' E_t\left[r_{t+k}^\Delta\right] - 0.5 v_t' \mathrm{diag}\left(V_t\left[r_{t+k}^\Delta - r_{t+k}^f 1_v\right]\right) + 0.5 v_t' V_t\left[r_{t+k}^\Delta - r_{t+k}^f 1_v\right] v_t \end{aligned} \tag{29}$$

$$\begin{aligned} V_t\left[s_{t+k}^\Delta\right] = {} & V_t\left[r_{t+k}^f\right] + w_t' V_t\left[r_{t+k}^e\right] w_t + 2 w_t' \mathrm{cov}_t\left[r_{t+k}^e, r_{t+k}^f\right] + v_t' V_t\left[r_{t+k}^\Delta\right] v_t \\ & - 2\,\mathrm{cov}_t\left[r_{t+k}^f, r_{t+k}^\Delta\right] v_t - 2 w_t' \mathrm{cov}_t\left[r_{t+k}^e, r_{t+k}^\Delta\right] v_t\,. \end{aligned} \tag{30}$$

[18] Since the conditional underfunding probability, π^s_t, and the recovery fraction, λ^s_t, in case of underfunding are closely related to the risk measures Value at Risk (VaR) and Expected Shortfall (ES), we can justify Campbell and Viceira's (2005) assumption of a buy-and-hold investment strategy for our approach: both VaR and ES are usually computed under the assumption that the chosen asset allocation will be maintained over the forecast period, as emphasized by Cuoco et al. (2005).

Since the funding spread only involves conditional expectations and conditional probabilities (which are non-stochastic), we can simplify (29) and (30) by recognizing the identities $V_t\left[r^{\Delta}_{t+k}\right] = V_t\left[r^{0}_{t+k}\right]$, $\text{cov}_t\left[r^{f}_{t+k}, r^{\Delta}_{t+k}\right] = \text{cov}_t\left[r^{f}_{t+k}, r^{0}_{t+k}\right]$, and $\text{cov}_t\left[r^{e}_{t+k}, r^{\Delta}_{t+k}\right] = \text{cov}_t\left[r^{e}_{t+k}, r^{0}_{t+k}\right]$. Apart from the difference in the liability portfolio weights u_t and v_t,[19] the only other difference between (27) and (28), on the one hand, and (29) and (30), on the other hand, is that between $E_t\left[r^{0}_{t+k}\right]$ in (27) and $E_t\left[r^{\Delta}_{t+k}\right]$ in (29). We note that the vector $E'_t\left[r^{\Delta}_{t+k}\right] = \left(E_t\left[r^{\Delta,1}_{t+k}\right], \ldots, E_t\left[r^{\Delta,T}_{t+k}\right]\right)$ has individual elements $E_t\left[r^{\Delta,s}_{t+k}\right] = E_t\left[r^{0,s}_{t+k}\right] + E_t\left[\ln D^{s}_{t+k}\right] - E_t\left[\ln D^{s}_{t}\right]$, the latter two components of which are non-stochastic, thus $E_t\left[\ln D^{s}_{t}\right] = \ln D^{s}_{t}$ and

$$\begin{aligned} E_t\left[\ln D^{s}_{t+k}\right] &= E_t\left[\ln\left(1+\theta^{s}_{t+k}\right) + \ln\left(1-\pi^{s}_{t+k} + \pi^{s}_{t+k}\lambda^{s}_{t+k}\right)\right] \\ &\approx \ln\left(1+E_t\left[\theta^{s}_{t+k}\right]\right) + \ln\left(1 - E_t\left[\pi^{s}_{t+k}\right] + E_t\left[\pi^{s}_{t+k}\right]E_t\left[\lambda^{s}_{t+k}\right]\right) \\ &= \ln\left(1+\theta^{k+s}_{t}\right) + \ln\left(1-\pi^{k+s}_{t} + \pi^{k+s}_{t}\lambda^{k+s}_{t}\right) = \ln D^{k+s}_{t} \end{aligned} \tag{31}$$

where the second line follows from Jensen's inequality[20] and the third line follows from the law of iterated expectations. So the main distinction between a pension plan treating future pension payments as risk free and a pension plan taking into account the fact that future pension payments are subject to funding risk lies in the term spread, Δterm

$$\begin{aligned} \Delta\text{term} &\equiv v'_t E_t\left[r^{0}_{t+k}\right] - v'_t E_t\left[r^{\Delta}_{t+k}\right] = -\sum_{s=1}^{T} v^{s}_{t}\left(\ln D^{k+s}_{t} - \ln D^{s}_{t}\right) \\ &= \sum_{s=1}^{T} v^{s}_{t}\left((k+s)\ln\left(1+\Delta^{k+s}_{t}\right) - s\ln\left(1+\Delta^{s}_{t}\right)\right) \end{aligned} \tag{32}$$

of the funding spread curve which will be maximized as part of the optimization exercise (26). This is an important result. The pension plan that uses FRA liability valuation methods acts to increase the steepness of the funding-spread curve. Nearby pension payments are treated as relatively more safe, while distant pension payments are treated as relatively less safe, in comparison with a pension plan using RF valuation: in other words, distant uncertainty is pe-

[19] From experimentally using u_t in (26) instead of v_t we find that this difference has a negligible impact on asset allocation. See also footnote 15.

[20] We follow Greene (1990, chapter 3.9.3) who approximates the expectation of a nonlinear function g( ) of a random variable X with the nonlinear function of the expectation of the random variable, thus E[g(X)] ≈ g(E[X]). From Jensen's inequality we know that we overstate E[g(X)] if g( ) is a concave function and understate E[g(X)] if g( ) is a convex function. Because the funding-risk premium, θ^{s}_{t+k}, is a convex function of the funding-risk adjustment factor, Θ^{s}_{t+k}, which itself is a nonlinear function of π^{s}_{t} and λ^{s}_{t}, and the log function is concave, we conjecture that the approximation errors cancel each other out in our case

nalized more than nearby uncertainty. The most likely consequence of this is a lower allocation to risky assets such as stocks in comparison with a pension plan using RF valuation. We will return to this conjecture about the optimal behavior of a pension plan with FRA liabilities in the empirical section below.

The optimal asset allocation w_t^0 can be readily derived in closed form as

$$w_t^0 = \frac{1}{\gamma} V_t^{-1}\left[r_{t+k}^e\right]\left(E_t\left[r_{t+k}^e\right] + 0.5\text{diag}\left(V_t\left[r_{t+k}^e\right]\right)\right) + \left(1 - \frac{1}{\gamma}\right) V_t^{-1}\left[r_{t+k}^e\right] \text{cov}_t\left[r_{t+k}^e, r_{t+k}^0 - r_{t+k}^f 1_v\right] u_t \,. \tag{33}$$

This is the optimal portfolio obtained by Hoevenaars et al. (2005) and is a weighted average of two components, one related to speculative asset demand and the other to liability-hedging demand. An infinitely risk-averse investor only pays attention to the latter component, a result that has been known since Sharpe and Tint (1990).

By contrast, the optimal asset allocation w_t^Δ has to be obtained by numerical methods since both the liability portfolio weights, v_t, and the term spread, Δterm, depend on w_t^Δ. Recall that the outcome of this optimization exercise not only determines the asset allocation but simultaneously the term structure of funding spreads (18) and the FRA liability value (17). The traditional sequence of liability valuation followed by the determination of the asset allocation can and, in our view, should be replaced by a single optimization exercise determining simultaneously the optimal asset allocation and the appropriate set of FRA discount rates and hence liability value, both of which are economically consistent with each other.

3. Return Dynamics

We now specify an econometric model to determine the conditional moments of future asset and liability log returns. The data source is described and estimation results are presented.

3.1 The Model

We follow Campbell et al. (2003), Campbell and Viceira (2005) and Hoevenaars et al. (2005) and model the return dynamics by means of a vector autoregression with one lag, VAR(1).[21] From equations (22) – (24) and (27) – (30), it is clear that we need k-period ahead forecasts of log asset returns and up to $(k+s)$-period ahead forecasts of the log returns on a default-

[21] Adopting a similar approach, Van Binsbergen and Brandt (2005) present an asset-liability model which builds on a VAR(1) model as a data generating process for asset and liability log returns.

free zero-coupon bond with maturity s, for $s = 1, \ldots, T$. Since T will be small in our application (we use the CRSP Fama-Bliss zero-coupon data where the maximum maturity equals five years), we will not specify an arbitrage-free term structure model[22] that generates yields with long maturities. At their simplest, the liabilities of a pension plan could consist of a single discounted pension payment at maturity s. Adding payments with additional maturities to the liability structure of the pension plan will make little difference to the resulting asset allocation unless asset and liability log return correlations vary substantially with maturity. For this reason we can omit a term structure model from the current analysis.

Like Campbell and Viceira (2005), we assume that the risk premium on quarterly stock returns is driven by the dividend-price ratio, q_t. But unlike these authors, we do not include the spread between the yield on a 5-year zero-coupon bond and the nominal short-term interest rate. This is because we are already using log returns on zero-coupon bonds with maturities up to T years in our regression for the liability return vector, r^0_{t+1}. We do, however, use the log return on T-bills for r^f_{t+1}, and stock and bond excess log returns for r^e_{t+1}. The next subsection discusses the data in more detail.

Collecting all required data for every quarter $t = 1, \ldots, Q-1$ in the vector $z'_{t+1} = \left[r^f_{t+1} \quad r'^e_{t+1} \quad q_{t+1} \quad r'^0_{t+1} \right]$, we can write a homoskedastic VAR(1) model as

$$z_{t+1} = \Phi_0 + \Phi_1 z_t + \varepsilon_{t+1} = A x_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \sim N(0, \Sigma) \tag{34}$$

where $A = (\Phi_0 \quad \Phi_1)$ and $x_t = (1 \quad z'_t)'$. Stability (and thus stationarity) of the VAR(1) system requires that the maximum eigenvalue of Φ_1 is smaller than unity. An OLS estimator of the matrix A can be obtained from $\hat{A} = ZX'(XX')^{-1}$ with $Z = (z_2, \ldots, z_Q)$ and $X = (x_1, \ldots, x_{Q-1})$.[23] For the stabilizing transformation of $\hat{\alpha} = \mathrm{vec}(\hat{A})$, we have $\sqrt{Q-1}(\hat{\alpha} - \alpha) \xrightarrow{d} N\left(0, (XX')^{-1} \otimes \Sigma\right)$. Σ can be estimated by the moment estimator $\frac{1}{Q-1}\hat{\varepsilon}\hat{\varepsilon}'$ where $\hat{\varepsilon} = (\hat{\varepsilon}_2, \ldots, \hat{\varepsilon}_Q)$ comprises the estimated residuals of the VAR(1) model.

Campbell and Viceira (2004) derive the conditional first and second moments of cumulative k-period log returns (and state variables) implied by the VAR(1) model (34) which are

$$E_t\left[z_{t+1} + \ldots + z_{t+k}\right] = \left[\sum_{i=0}^{k-1} (k-1)\Phi_1^i\right]\Phi_0 + \left[\sum_{i=1}^{k} \Phi_1^i\right] z_t \tag{35}$$

[22] This would exclude arbitrage possibilities across different maturities. A possible candidate for a term structure model could come from the family of essentially affine term structure models proposed by Duffee (2002), which generate time-varying risk premia along the lines of the VAR(1) model considered here.

[23] The OLS estimator is equal to the GLS estimator because the regressors are the same in all equations.

$$V_t\left[z_{t+1}+\ldots+z_{t+k}\right]=\sum_{j=1}^{k}\left(\left(\sum_{i=0}^{j-1}\Phi_1^i\right)\Sigma\left(\sum_{i=0}^{j-1}\Phi_1^i\right)'\right). \tag{36}$$

Note that these conditional moments are investment horizon (k) dependent. In order to generate representative asset allocations for our sample, we will evaluate (35) at the average value of the state variables in Section 4. Using (35) and (36) we can generate all required moments.

3.2 The Data

We consider a simple pension plan which is independent of its sponsoring company and faces pension payments in 1, 2, 3, 4 and 5 years' time; thus, $T = 5$ years in our model.[24] The choice of $T = 5$ is dictated by the availability of zero-coupon bond price data which we obtained from the CRSP Fama and Bliss files for the period 1952:Q2 –2005:Q4. We use the same time period for our other quarterly data.

We assume the pension plan only considers investing in the principal asset classes of cash, bonds and stocks. This is in line with Campbell and Viceira (2005).[25] We use the same quarterly data on interest rates, bond and stock returns as Goyal and Welch (2007). Thus, we compute continuously compounded quarterly returns of the S&P500 index including dividends. The dividend-price ratio is based on 12-month moving sums of dividends paid on the S&P500 index.[26] Bond returns refer to long-term government bonds. Stock and bond excess returns are based on the nominal T-bill return. We use the nominal T-bill rate instead of the real T-bill rate because the pension plan promises pension payments in nominal terms.

[24] Note also, that we can still generate term structures of funding spreads from (15) which go beyond T = 5.

[25] Hoevernaars et. al. (2005) show that additional asset classes such as credit, commodities, real estate and hedge funds have positive demands from institutional investor using the portfolio rule (33). However, we disregard these findings and instead focus on the asset allocation implications of the endogenous funding spread.

[26] The data are available from Amit Goyal's webpage http://goizueta.emory.edu/faculty/AmitGoyal/. Goyal and Welch (2005) present a critical view on the explanatory power of the dividend-price ratio in predictive regressions for the equity premium and encourage the profession to come up with new variables that have more robust predictive power. Until such variables become available, the dividend-price ratio (or its near neighbor the earnings-price ratio) remains the most accepted summary measure of current investment opportunities. We need to keep in mind that the principal purpose of our paper is to show the asset allocation consequences of an endogenous funding spread. This is achieved by comparing the optimal asset allocation from our model with the optimal asset allocation from an identical model but with a zero spread. The underlying data-generating process is of only secondary importance for such a comparison.

Table 1 shows annualized descriptive statistics for the implied log excess returns which are used in our regression. According to these statistics, our extended sample has a somewhat higher expected bond excess return than the samples considered by Campbell et al. (2003) and Campbell and Viceira (2005), but also higher volatility, although the implied Sharpe ratio turns out to be similar to that used in the earlier studies.

Table 1 also shows yields on zero-coupon bonds with maturities from 1 to 5 years. Mean yields increase from 5.56% for the shortest maturity to 6.14% for the longest maturity. Short-term yields are slightly more volatile than long-term yields. Based on these yields, we compute prices $P_t^s = \left(1 + Y_t^s\right)^{-s}$ for every period t and maturity s and constant-maturity returns $R_{t+k}^{0,s} = P_{t+k}^s / P_t^s$. Table 1 shows the log returns which are part of the VAR(1) model. The mean of constant-maturity zero-coupon returns is approximately zero at all maturities. The volatility increases from 1.75% for the shortest maturity to 6.24% for the longest maturity. Due to the constant-maturity assumption, the value of the pension plan liability will not change in expectation over time, but will, given these bond return volatilities, be subject to considerable interest rate risk.

3.3 Estimation Results

Table 2 shows the OLS estimation results for the parameter matrix A in (34). Table 3 presents the standard deviations and correlations implied by the moment estimator of Σ in (32). The maximum eigenvalue of the matrix Φ_1 is 0.9805. Thus, the system is both stable and stationary. However, some variables like the log dividend-price ratio, and the 3- and 4-year zero-coupon bond log returns are highly persistent with AR(1) parameters close to unity.[27]

Looking first at the overall goodness of fit of the VAR(1) model, the last column in Table 2 displays R^2 and p-values of an F-test of the joint significance of the slope parameters for all equations. While each equation is statistically significant, the R^2 values are usually around 10%, with the exception of the log T-bill return (94%) and the log dividend-price ratio (96%). Of note is the performance as a state variable of the 4-year zero-coupon bond log return, which has a particularly strong positive impact on future log bond excess returns. Table 3 shows that the 4-year zero-coupon bond log return and the contemporaneous bond excess log return are positively correlated as well. Thus, the 4-year zero-coupon bond log return has similar predictive properties for bond excess log returns as the log yield spread in Campbell et al. (2003) and Campbell and Viceira (2005).

[27] Note that in contrast with a stationary single equation AR(1) model, a stationary VAR(1) model can have AR(1) coefficients larger than unity in absolute value.

We also confirm previous findings of these authors regarding the impact of the log dividend-price ratio on stock excess log returns. High dividend-price ratios are negatively correlated with contemporaneous stock excess returns, but significantly predict positive future stock excess returns. As a result, the dividend-price ratio causes mean reversion in stock excess returns.

While the overall predictive success of the VAR(1) model is modest in a statistical sense, Campbell and Thompson (2007) show that even a small amount of predictability as measured by a small R^2 can nevertheless have important consequences for the resulting asset allocations in an economic sense. We will use the estimated model to generate the conditional moments involved in equations (22) – (24) and (27) – (30) from the (35) and (36).

4. Optimization Results

Based on the estimated conditional moments of future asset and liability returns, we can calculate the optimal asset allocation, w_t^0, for a pension fund, which values liabilities using the term structure of RF yields from (33). The optimal asset allocation, w_t^Δ, for a pension plan which values liabilities using a term structure of FRA yields that include funding spreads of the form (18) is computed from solving (26) using numerical methods.[28]

Before comparing the two resulting asset allocations, we investigate the implied term structures of funding spreads for the two pension plans.

4.1 The Term Structure of Funding Spreads

As we saw above, our proposed method of optimizing an objective function in the log return on the funding ratio of assets to FRA liabilities automatically generates a term structure of funding spreads (18) evaluated at the optimal asset allocation, w_t^Δ. We can also manually calculate the funding spreads at the asset allocation outcome, w_t^0, from optimizing an objective function in the terminal log return on the funding ratio of assets to RF liabilities. By comparing the difference between the resulting term structures of funding spreads, Δ_t^s, and its components, π_t^s and λ_t^s, we can assess the funding risks implied by the two optimal portfolios. We will try this for different combinations of the parameters involved in the two objective functions: k, the investment horizon, ϕ, the relative consumption growth, γ, the coefficient of

[28] We use the optimization module "optmum" in Gauss 8.0 with default settings and (33) as starting values. We confirm that the optimal asset allocation is not dependent on the starting values. Convergence is achieved in under a minute of computation time on a standard office notebook of 2004 vintage for all parameter constellations.

relative risk aversion, and F_t^0, the initial funding ratio (which we assume to be the same in both cases). Recall that only the ratio of the initial funding ratio F_t^0 to the funding threshold τ is relevant for determining the underfunding probabilities and recovery fractions, not the absolute value of the two parameters. For this reason, we decided to set $\tau = 1$ and vary F_t^0.

Table 4 shows funding spreads, funding-risk premia, underfunding probabilities and recovery fractions for a fixed maturity of one year (four quarters) for different values of k, ϕ, γ and F_t^0. What is clear is that – for the parameter constellations under considerations – the resulting funding spreads resulting from maximizing objective function (26) based on FRA liabilities are smaller than the funding spreads derived by maximizing an objective function (25) in RF liabilities. From an examination of the decomposition of the spreads into underfunding probabilities and recovery fractions, it turns out that the size of both components confirms the benefits from using our proposed optimizing method. The pension plan which determines the optimal asset allocation using (26) will be less prone to underfunding and usually achieves higher recovery fractions than the plan adopting (25).[29]

The first panel of Table 4 shows the impact of a variation in the initial funding ratio on the funding spread for a given investment horizon of $k = 5$ years, relative consumption growth $\phi = 1.04$, and relative risk aversion $\gamma = 5$. F_t^0 assumes values ranging between 0.75, which corresponds to a situation of serious underfunding relative to the threshold $\tau = 1$, and 1.0, which corresponds to a situation of full funding. It is not surprising that the funding spread decreases with increasing initial funding from $\Delta_t^4 = 0.208$ when $F_t^0 = 0.75$ to $\Delta_t^4 = 0.012$ when $F_t^0 = 1.0$. The appropriate funding spread for a pension plan with 100% initial funding reflects an underfunding probability of $\pi_t^4 = 0.1855$ and a recovery fraction of $\lambda_t^4 = 0.9446$. To assess the magnitude of these funding spreads, we compute the spreads between the yields on long-term corporate bonds with maturities of 20 years and above and the yield on a Treasury bond with a constant maturity of 10 years.[30] Depending on the credit rating of the issuing companies, we obtain the following average spreads from annual data between 1980 and 2006: 0.0105 (AAA), 0.0137 (AA), 0.0167 (A) and 0.0214 (BAA). Thus, the one-year spread for a 100%-funded pension plan approximately corresponds to the long-term average yield spread on AAA-rated corporate bonds. A 90%-funded pension plan would need a funding spread that is more than twice the average yield spread on BAA-rated bonds to appropriately reflect the risk underlying the promised pension payment at the one-year horizon.

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[29] The differences in this example are, however, economically small.

[30] The corporate bond yields are derived from Moody's long-term corporate bond yield index available from Moody's web-site. Yields on Treasury bonds with a constant maturity of 10 years are available from the Federal Reserve statistical release (http://federalreserve.gov/releases/h15/data.htm).

The second panel of Table 4 shows the effect of varying the relative risk aversion parameter on the funding spread for a given investment horizon of $k = 5$ years, relative consumption growth $\phi = 1.04$, and initial funding ratio $F_t^0 = 1.0$. As might be anticipated, higher relative risk aversion parameters imply lower funding spreads, mostly because underfunding probabilities are reduced as a result of the adoption of a more conservative asset allocation. A pension plan with relative risk aversion $\gamma = 2$ should use a one-year funding spread of 0.041, which decreases to 0.012 for $\gamma = 5$ and to 0.005 for $\gamma = 10$.

The third panel of Table 4 shows the impact of a variation in relative consumption growth, ϕ, in the state of overfunding to the state of underfunding for given investment horizon of $k = 5$ years, relative risk aversion $\gamma = 5$ and initial funding ratio $F_t^0 = 1.0$. Recall that ϕ is the unknown parameter of the stochastic discount factor. Hence, a variation of ϕ directly affects the funding-risk premium, θ_t^s. Increasing the relative consumption growth from $\phi = 1.02$ – which is consistent with 4% consumption growth in the state of overfunding and 2% consumption growth in the state of underfunding – to $\phi = 1.06$ – which is consistent with 8% consumption growth in the state of overfunding and 2% consumption growth in the state of underfunding – increases the funding-risk premium from $\theta_t^s = 0.001$ to $\theta_t^s = 0.003$. The increase in the funding-risk premium leads to an increase in the funding spread by a corresponding amount. While the recovery fraction, λ_t^4, remains unchanged, the underfunding probability decreases from $\pi_t^4 = 0.185$ when $\theta_t^s = 0.001$ to $\pi_t^4 = 0.183$ when $\theta_t^s = 0.003$. This is a consequence of a modest change in the optimal asset allocation caused by the variation of the funding-risk premium.

Finally, the fourth panel of Table 4 contains results for a variation in the investment horizon, k, for given relative consumption growth $\phi = 1.04$, relative risk aversion $\gamma = 5$ and initial funding ratio $F_t^0 = 1.0$. We see that a pension plan with a longer investment horizon should discount one-year liabilities using larger funding spreads of 0.019 for k = 10 years, compared with 0.016 for k = 7.5 years and 0.012 for k = 5 years. This is because one-year underfunding probabilities increase and one-year recovery fractions decrease with an increasing investment horizon. The explanation for this is that the long-term investor attempts to exploit mean reversion in stock returns with a large holding in stocks, which favorably impacts the long-term Sharpe ratio for stocks, but has little effect at the one-year horizon. Thus, at the one-year horizon, the asset allocation of a long-term investor is actually quite risky and this is reflected in the higher funding spread.

Figures 1 and 2 present the term structure of funding spreads (18) for maturities of up to 10 years for a pension plan with an investment horizon corresponding to the liability horizon, $k = 5$, and relative risk aversion of $\gamma = 5$. In Figure 1, we consider an initial funding ratio of

$F_t^0 = 1.0$, while in Figure 2, the initial funding ratio is reduced to $F_t^0 = 0.75$. The upper graph in each figure shows the funding spread and funding-risk premium curves, while the lower graph displays the corresponding underfunding probabilities and recovery fractions. We present the spreads automatically generated from computing w_t^Δ and the spreads manually calculated from w_t^0. Although both funding spreads are close to each other, the spread generated from the portfolio w_t^0 always exceeds the spread related to w_t^Δ. Note that the figures show annualized spreads obtained by dividing the quarter s in equation (18) by four.[31] Funding-risk premia are annualized as well.

Figure 1 shows that funding spreads decline rapidly towards zero with increasing maturity. This is because the liabilities in our example do not grow systematically over time and are only affected by interest rate risk, while the assets have positive expected returns. Thus, the probability of underfunding decreases with maturity. This can be seen clearly from the lower graph in Figure 1. Underfunding probabilities decrease from about 0.3 at the one-quarter horizon to about zero at a horizon of 40 quarters. Underfunding probabilities for w_t^0 are always slightly larger than those for w_t^Δ. Recovery fractions first decrease with maturities below about 2.5 years and then increase afterwards. They always stay in a fairly narrow range of 0.935 to 0.965. The initial decrease is due to the volatility of the funding ratio return, while the subsequent increase is due to the effect of positive expected asset returns on raising the expected funding ratio and hence λ_t^s. Recovery fractions for w_t^0 are always smaller than those for w_t^Δ by about 0.5 percentage points.

The funding spreads for the seriously underfunded pension plan in Figure 2 start at a much higher level than those for the fully funded pension plan in Figure 1, but decline even faster to zero with increasing maturity. This is because of underfunding probabilities, which are very close to unity for the early quarters (recall $F_t^0 = 0.75$ and $\tau = 1$), but subsequently decline rapidly to about zero at a horizon of 40 quarters as the expected funding ratios increase. The term structure of recovery fractions is upward sloping in the range 0.77 to 0.94 which can again be explained by the increase in expected funding ratios. The maximum recovery fraction in Figure 2 is only slightly higher than the minimum recovery fraction in Figure 1 reflecting the 30% difference in initial funding.

The funding-risk premium, θ_t^s, declines with maturity and increasing recovery fractions in Figure 1 (for $F_t^0 = 1.0$) from about 0.006 at the one-quarter horizon to zero at the four-year horizon. For all maturities, the risk premia are too small to have a sizable impact on funding-

[31] A quarterly spread for a maturity of 20 quarters becomes an annual spread for a maturity of 5 years. Thus, the quarterly spread for, e.g., a horizon of one quarter corresponding to an annual spread of 1.8 depicted in Figure 2 is –1+(1+1.8)^0.25 = 0.29.

risk spreads. From Figure 2 (for $F_t^0 = 0.75$), we see that the funding-risk premium first increases from zero to about 0.003 at the two-year horizon and then decreases again to zero at the seven-year horizon. This pattern can be explained by our comparative statics results for the funding-risk premium in Section 2.2. Recall that the funding-risk premium decreases with increasing maturity and an increasing recovery fraction. The funding-risk premium increases with a decreasing underfunding probability whenever $\phi^{-\gamma} < \lambda_t^s \left(\pi_t^s\right)^2 \left(1-\pi_t^s\right)^{-2}$. For maturities below two years, the latter effect dominates the effects of an increasing maturity and an increasing recovery fraction on the risk premium. For maturities above two years, the relative magnitude of the two effects reverses. Moreover, for maturities less than or equal to three years, we find that $\phi^{-\gamma} < \lambda_t^s \left(\pi_t^s\right)^2 \left(1-\pi_t^s\right)^{-2}$, while the inequality is reversed for maturities above three years. Thus, for $s \geq 13$ quarters, the funding-risk premium unambiguously decreases with increasing maturity and recovery fraction and decreasing underfunding probability, and rapidly declines to zero.

The existence of a term structure of funding spreads as shown in Figures 1 and 2 brings into question the current practice in the U.S. of using maturity-invariant discount rates for liability valuation purposes (often the yield on AA-rated long-term corporate bonds). A constant discount rate only can be justified in the unlikely case where the slope of the term structure of interest rates exactly offsets the slope of the term structure of funding spreads.

4.2 Optimal Asset Allocation

We compute the optimal asset allocations w_t^0 and w_t^Δ for a wide range of initial funding ratios between $F_t^0 = 0.025$ and $F_t^0 = 1.625$ for fixed $\tau = 1$ and $\phi = 1.04$ and a baseline parameter specification with investment horizon $k = 5$ and relative risk aversion $\gamma = 5$. In addition, we provide two comparative statics results for an increase in the relative risk aversion, to $\gamma = 10$, and in the investment horizon, to $k = 10$, leaving all other parameters unchanged. Figures 3 to 5 display optimized outcomes for the three parameter constellations. Specifically, the upper graphs in the three figures contain plots of the optimal allocation to stocks and the one-year funding spread over the range of initial funding ratios, while the lower graphs contain plots of the one-year underfunding probability and the one-year recovery fraction over the range of initial funding ratios. We present results for both w_t^0 and w_t^Δ.

The optimal allocation to stocks in portfolio w_t^0 is independent of the initial funding ratio as is clear from (33). In the upper graph in Figure 3, this allocation is represented by the horizontal line at a level of 73% stocks. By contrast, the optimal allocation to stocks in portfolio w_t^Δ depends on the initial funding ratio, since this affects both the underfunding probability (23) and the recovery fraction (24). From Figures 3 to 5, we see that the optimal allocation to stocks in portfolio w_t^{Δ} is a highly non-linear function of the initial funding ratio. We first describe the function and then provide an explanation for it.

The w_t^{Δ} function has four segments which correspond to four states of the initial funding ratio. Consider first Figure 3 which depicts the baseline parameter specification of $k = 5$ and $\gamma = 5$. State 1, which we define as a state of "critical underfunding", holds for initial funding ratios below 40% ($F_t^0 \leq 0.4$). In this segment, the allocation to stocks is both relatively low (around 25%) and invariant to the size of the initial funding ratio below the critical threshold. State 2, which we define as a state of "severe underfunding", holds for initial funding ratios between 40% and 75% ($0.40 < F_t^0 \leq 0.75$). The moment the initial funding ratio rises above 40% initial funding, the allocation to stocks jumps to a level of about 90%, which is above the optimal allocation to stocks in portfolio w_t^0 (around 73%). In this segment, the optimal allocation to stocks in portfolio w_t^{Δ} is a concave function of the initial funding ratio, always lying *above* the corresponding allocation in w_t^0, reaching a peak of 97% at an initial funding ratio of 50%, before falling back to 73% when the initial funding ratio reaches 75%. State 3, which we define as a state of "moderate underfunding", holds for initial funding ratios between 75% and 100% ($0.75 < F_t^0 \leq 1$). In this segment, w_t^{Δ} is a convex function of the initial funding ratio and always lies *below* w_t^0. The allocation to stocks reaches a local minimum in this state of approximately 67% at an initial funding ratio of 90%. State 4 is a state of "overfunding" ($F_t^0 > 1$). The optimal allocation to stocks in portfolio w_t^{Δ} is an increasing function of the initial funding ratio and rapidly approaches the optimal allocation to stocks in portfolio w_t^0. For initial funding ratios larger than 120%, the two allocations coincide exactly.

What determines this functional form for the optimal allocation to stocks in portfolio w_t^{Δ}? The lower graph in Figure 3 helps explain the optimal asset allocation in state 1 (critical underfunding). We see that the one-year underfunding probability equals unity below an initial funding threshold of 65%. Below 40%, all underfunding probabilities equal unity for horizons up to and including the maturity of the pension liability. In this state, the initial funding ratio is so low that the asset allocation cannot influence underfunding probabilities for nearby horizons. Thus, the asset allocation remains invariant to changes in the initial funding ratio. The funding spreads, reflecting the unit underfunding probabilities, are extremely high for short maturities. Figure 2 shows that the one-quarter funding spread for an initial funding ratio of 75% almost equals 2: for funding ratios below 40%, the spread is considerably higher. As a consequence, the term spread Δterm in (32) becomes significantly negative which results in the expected terminal funding ratio decreasing substantially. This, in turn, leads to a significant reduction in expected utility for an investor with a mean-variance objective function of the type (26). For given risk aversion, the pension plan responds to the lower expected utility by choosing an asset allocation with a relatively low equity exposure.

The lower graph in Figure 3 also helps explain the asset allocation in state 4 (overfunding). Since funding spreads rapidly approach zero for initial funding ratios in excess of 100%, the stock allocation in portfolio w_t^{Δ} converges to that in portfolio w_t^0. This is because the objective functions (25) and (26) only differ by the presence of funding spreads in the latter function.

The asset allocations associated with states 1 and 4 could be readily predicted. More interesting, because they are less predictable, are the asset allocations associated with states 2 and 3. In state 2 (severe underfunding), the initial funding ratio is sufficiently large that a change in the asset allocation has an impact on underfunding probabilities for longer horizons. In this state, the pension plan bets on the equity premium in order to increase the expected funding ratio at the investment horizon. This results in a higher equity weighting for w_t^{Δ} compared with w_t^0. If a pension plan finds itself in state 3 (moderate underfunding), it wants to reduce the risk of making the deficit even worse. It therefore takes a more cautious approach than it would in state 2 and chooses a lower equity weighting than it would in that state, one which is even lower than the equity weighting it would have selected had it adopted objective function (25).

Either increasing the coefficient of relative risk aversion from $\gamma = 5$ in our baseline specification to $\gamma = 10$ or increasing the investment horizon from $k = 5$ in our baseline specification to $k = 10$ does not change the general picture as Figures 4 and 5 show. The four states identified in Figure 3 are clearly discernible in these figures too, although the corresponding initial funding ratio boundaries differ slightly. The resulting stock allocations in Figure 4 are much lower than those in Figure 2, which is a direct consequence of the increased relative risk aversion. By contrast, the pension plan with a longer investment horizon in Figure 5 chooses a larger equity exposure than the pension plan in our baseline specification since mean reversion in stock returns renders stocks more attractive for the long-term investor. This result is in line with earlier findings by Campbell et al. (2003), Campbell and Viceira (2005), and Hoevenaars et al. (2005).

While the optimal asset allocations clearly differ between the three cases depicted in Figures 3 to 5, funding spreads, underfunding probabilities and recovery fractions are very similar. In all three cases, the one-year funding spreads decrease from a level in excess of 3.5 at $F_t^0 = 0.2$ to below 0.1 at an initial funding ratio of around 80%. A closer look at the data underlying these graphs reveals that the funding spreads for w_t^{Δ} slightly exceed those for w_t^0 in state 1. Since one-year underfunding probabilities for w_t^0 and w_t^{Δ} in this state both equal zero, the difference in funding spreads is explained by one-year recovery fractions which are slightly higher for w_t^0 than for w_t^Δ. However, in states 2 to 4, a pension plan using the proposed asset allocation w_t^Δ will always exhibit a smaller funding spread than a pension plan using w_t^0.

Tables 5 and 6 shed a more quantitative light on the asset allocation "wave" in states 2 to 4 than can be gleaned from Figures 3 to 5 alone. Table 5 shows the optimal allocations to cash, bonds, and stocks in portfolios w_t^0 and w_t^Δ, as well as the term spread Δterm defined in equation (32). We argued in the section above discussing equation (32) that Δterm, which describes the term spread of the funding spread curve across maturities (of pension payments), is likely to be the principal cause of any differences in the two asset allocations w_t^0 and w_t^Δ. We anticipated that Δterm for w_t^Δ will always exceed Δterm for w_t^0,[32] which implies a steeper funding spread curve for the former portfolio than the latter: short-term pension payments become relatively more safe and long-term pension payments relatively less safe. Table 5 confirms this conjecture although the differences are economically small.

Table 6 presents the first two moments of the terminal funding ratios for RF liabilities, F_{t+k}^0, and FRA liabilities, F_{t+k}^Δ, respectively. Portfolio w_t^0 involves an optimal return-risk trade-off between the moments $E_t\left[F_{t+k}^0\right]$ and $V_t^{0.5}\left[F_{t+k}^0\right]$, while portfolio w_t^Δ optimally balances the moments $E_t\left[F_{t+k}^\Delta\right]$ and $V_t^{0.5}\left[F_{t+k}^\Delta\right]$. For the sake of comparison, we also evaluate $E_t\left[F_{t+k}^0\right]$ and $V_t^{0.5}\left[F_{t+k}^0\right]$ for portfolio w_t^Δ and $E_t\left[F_{t+k}^\Delta\right]$ and $V_t^{0.5}\left[F_{t+k}^\Delta\right]$ for portfolio w_t^0.

In Tables 5 and 6, we present results from the optimization exercise for fixed $\tau = 1$ and $\phi = 1.04$ and for a range of initial funding ratios, $F_t^0 \in (0.75, 0.90, 1.00, 1.10)$, falling within states 2 to 4 for our baseline parameter specification, $k = 5$ and $\gamma = 5$, together with comparative statics results for $\gamma = 10$ and $k = 10$. We can see from Table 5 that a 100% funded pension plan, which adopts the optimal portfolio for FRA liabilities, w_t^Δ, would allocate 69.1% of its assets to stocks in our baseline parameter specification. A pension plan selecting portfolio w_t^0, which is optimal for RF liabilities, would allocate 73.4% to stocks. From Table 4, we know that this stock exposure translates into a one-year funding spread of 0.012 for w_t^Δ and 0.014 for w_t^0 because the underfunding probability is smaller and the recovery fraction is larger for w_t^Δ. The 4.3 percentage point difference between the two stock allocations is distributed to cash (2.6) and bonds (1.7) in portfolio w_t^Δ.

Increasing the coefficient of relative risk aversion to $\gamma = 10$ decreases the optimal stock allocations to 40.6% in portfolio w_t^Δ and to 41.7% in portfolio w_t^0. On the other hand, in-

[32] We can manually evaluate (32) at the optimal portfolio w_t^0. Obviously, (32) is zero for the objective function (25).

creasing the investment horizon to twice the maturity of the liabilities, $k = 10$, increases stock allocations to 89.1% (w_t^{Δ}) and 93.0% (w_t^{0}), respectively.

Table 6 reveals that, as expected, higher allocations to stocks increase both the expected terminal funding ratio and its standard deviation. The higher risk of portfolio w_t^0 for our baseline parameter specification with 100% initial funding is reflected in higher expected terminal funding ratios with $E_t\left[F_{t+k}^0\right] = 1.612$ and $E_t\left[F_{t+k}^{\Delta}\right] = 1.603$ evaluated at w_t^0 compared with $E_t\left[F_{t+k}^0\right] = 1.594$ and $E_t\left[F_{t+k}^{\Delta}\right] = 1.586$ evaluated at w_t^{Δ}.[33] But this comes at the cost of higher terminal funding ratio volatilities which are $V_t^{0.5}\left[F_{t+k}^0\right] = 0.316$ and $V_t^{0.5}\left[F_{t+k}^{\Delta}\right] = 0.314$ evaluated at w_t^0, compared with $V_t^{0.5}\left[F_{t+k}^0\right] = 0.294$ and $V_t^{0.5}\left[F_{t+k}^{\Delta}\right] = 0.293$ evaluated at w_t^{Δ}.

5. Conclusion

In this paper, we critically review the current practice of pension liability valuation using discount rates which do not reflect the risk inherent in the pension promise. We propose a new approach to the valuation of pension obligations which depends on the term structure of funding-risk-adjusted discount rates that, in turn, depend on the asset allocation of the pension fund. When the asset allocation is based on an objective function in assets and funding-risk-adjusted liabilities, discount rates and portfolio weights become completely interdependent and can be determined in a single optimization step. We demonstrate this for a pension plan which optimizes the expected utility in a terminal funding ratio (of assets to funding-risk-adjusted liabilities) at a finite horizon. Since our approach does not require the pension plan to make *any* assumptions about the chosen discount rate for liability valuation purposes, we effectively remove an important degree of freedom from the plan sponsor's set of choices on how to measure liabilities and, hence, reduce the likelihood of the liabilities being reported in a strategic way to satisfy goals that might conflict with the proper objective of managing the plan in the best interests of the plan's beneficiaries. On the contrary, we believe that our proposed valuation method has advantages for all stakeholders of the pension plan, including the sponsoring company and its shareholders, and this is because it increases transparency with respect to the plan's expected future funding position.

On the basis of applying our model to U.S. data, we can conclude that a pension plan with an initial funding ratio that is not critically low and which maximizes our proposed objective

[33] When liabilities are valued using the FRA approach, we always observe lower expected terminal funding ratios compared to RF valuation. This is a direct consequence of higher discount rates.

function will always exhibit lower underfunding probabilities, higher recovery fractions, and lower funding spreads than a pension plan which maximizes a conventional objective function in the log return of assets over risk-free liabilities. The funding spreads also vary with maturity, which brings into question the current practice of using constant discount rates. The optimal asset allocation varies with the initial funding ratio of the pension plan. A plan which follows our methodology will usually allocate a smaller weighting of plan assets to equities unless the initial funding ratio is severe (a state of underfunding which lies between critical and moderate). A severely underfunded pension plan tries to benefit from the equity premium, while a moderately underfunded pension plan takes a more cautious approach and tries to avoid a further reduction of the funding ratio by choosing a lower equity exposure.

Finally, we note that our methodology is very easy to implement and converges very quickly. We would therefore expect pension plan stakeholders and regulators, pension accounting standard setting bodies and the financial markets to show interest in the funding-risk-adjusted measure of the pension liabilities of private companies in due course.

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Appendix

This appendix derives exact forms for the underfunding probability (8) and the recovery fraction in the case of underfunding (9) under the assumption that log returns are normally distributed. In contrast with the exact forms (23) and (24) provided in the text, we assume here that the regulatory authority obliges the sponsoring company to close any funding gap. In this case, the funding threshold is modeled according to (10).

Assume that the net worth of the plan sponsor in period $t+k$ can be written as $N_{t+k} = N_t R^N_{t+k}$. Define the sum of assets of the pension plan and the net worth of the sponsoring company as $V_t = A_t + N_t$ (total assets) and the funding ratio in terms of total assets as $G^0_t = (A_t + N_t)/L^0_t$. Let $a_t = A_t/(A_t + N_t)$ denote the fraction of the plan assets in total assets, V_t. Then the cumulative k-period return on total assets follows as

$$R^V_{t+k} = \frac{V_{t+k}}{V_t} = \frac{A_{t+k} + N_{t+k}}{A_t + N_t} = \frac{A_t R^A_{t+k} + N_t R^N_{t+k}}{A_t + N_t} = a_t R^A_{t+k} + (1-a_t)R^N_{t+k} = R^N_{t+k} + a_t\left(R^A_{t+k} - R^N_{t+k}\right).$$

Using the Campbell and Viceira (2002) approximation of the log portfolio return we obtain

$$r^V_{t+k} = r^N_{t+k} + a_t\left(r^A_{t+k} - r^N_{t+k}\right) + 0.5 a_t (1-a_t) V_t\left[r^A_{t+k} - r^N_{t+k}\right].$$

Assuming that the difference between the log return on total assets and the log return of the RF liabilities (6), $x^0_{t+s} = r^V_{t+s} - r^{L,0}_{t+s}$, is normally distributed and denoting the log initial funding ratio as $g^0_t = \ln G^0_t$, we find

$$\ln\left(\frac{A_{t+s} + N_{t+s}}{L^0_{t+s}}\right) = \ln\left(G^0_t \frac{R^V_{t+s}}{R^{L,0}_{t+s}}\right) = g^0_t + x^0_{t+s} \sim N\left(g^0_t + E_t\left[x^0_{t+s}\right], V_t\left[x^0_{t+s}\right]\right).$$

From here we obtain the expressions of interest similar to equations (23) and (24) in the text

$$\pi^s_t = \Pr_t\left(\frac{A_{t+s} + N_{t+s}}{L^0_{t+s}} < 1\right) = \Pr_t\left(\ln\left(\frac{A_{t+s} + N_{t+s}}{L^0_{t+s}}\right) < 0\right) = \Pr_t\left(g^0_t + x^0_{t+s} < 0\right) = \Phi\left(-\frac{\left(g^0_t + E_t\left[x^0_{t+s}\right]\right)}{V^{0.5}_t\left[x^0_{t+s}\right]}\right)$$

$$\lambda^s_t = E_t\left[\frac{A_{t+s} + N_{t+s}}{L^0_{t+s}} \mid \frac{A_{t+s} + N_{t+s}}{L^0_{t+s}} < 1\right]$$

$$= \frac{1}{\pi^s_t}\exp\left(g^0_t + E_t\left[x^0_{t+s}\right] + 0.5 V_t\left[x^0_{t+s}\right]\right)\cdot\Phi\left(\frac{-\left(g^0_t + E_t\left[x^0_{t+s}\right]\right) - V_t\left[x^0_{t+s}\right]}{V^{0.5}_t\left[x^0_{t+s}\right]}\right).$$

Tables

Table 1: Descriptive statistics

	Mean	St.dev.
Nominal T-bill return	5.21	1.41
Bond excess return	1.53	9.78
Stock excess return	7.05	15.74
Dividend-price ratio	-3.47	0.40
1-yr zero-coupon bond yield	5.56	2.92
2-yr zero-coupon bond yield	5.76	2.88
3-yr zero-coupon bond yield	5.93	2.81
4-yr zero-coupon bond yield	6.06	2.78
5-yr zero-coupon bond yield	6.14	2.74
1-yr zero-coupon bond return	-0.03	1.75
2-yr zero-coupon bond return	-0.04	3.15
3-yr zero-coupon bond return	-0.04	4.29
4-yr zero-coupon bond return	-0.03	5.32
5-yr zero-coupon bond return	-0.01	6.24

Note: With the exception of the dividend-price ratio, all statistics are annualized percentages. Mean log returns are adjusted by half of their variance to reflect mean gross returns. Quarterly data from 1952:Q2 –2005:Q4.

Table 2: VAR(1) estimation results

	Const.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	R^2
(1) Nominal T-bill return	0.00	0.97	0.01	0.00	0.00	-0.13	0.03	0.05	-0.05	-0.01	0.94
	0.65	*54.40*	*1.33*	*2.23*	*0.38*	*-2.25*	*0.59*	*1.16*	*-1.77*	*-0.30*	*0.00*
(2) Bond excess return	-0.01	0.02	-0.02	-0.06	-0.01	-1.58	0.20	-1.89	2.79	-0.84	0.09
	-0.42	*0.04*	*-0.12*	*-1.39*	*-0.57*	*-0.97*	*0.13*	*-1.48*	*3.31*	*-1.18*	*0.02*
(3) Stock excess return	0.17	-2.06	-0.16	0.07	0.04	-2.57	4.11	-4.93	2.43	0.56	0.11
	3.30	*-2.60*	*-0.51*	*0.99*	*2.79*	*-1.00*	*1.76*	*-2.44*	*1.81*	*0.49*	*0.00*
(4) Dividend-price ratio	-0.13	1.34	0.21	-0.06	0.97	2.77	-4.22	4.46	-2.24	-0.49	0.96
	-2.36	*1.65*	*0.64*	*-0.84*	*68.49*	*1.05*	*-1.76*	*2.15*	*-1.63*	*-0.42*	*0.00*
(5) 1-yr zero-coupon bond return	0.00	0.16	-0.03	-0.01	0.00	-0.37	-0.11	-0.17	0.43	-0.07	0.13
	-0.59	*1.81*	*-0.96*	*-1.91*	*-0.30*	*-1.31*	*-0.41*	*-0.78*	*2.90*	*-0.55*	*0.00*
(6) 2-yr zero-coupon bond return	-0.01	0.22	-0.05	-0.03	0.00	-0.21	-0.46	-0.32	0.72	-0.08	0.10
	-0.54	*1.41*	*-0.78*	*-1.92*	*-0.33*	*-0.41*	*-0.97*	*-0.79*	*2.66*	*-0.37*	*0.01*
(7) 3-yr zero-coupon bond return	-0.01	0.24	-0.03	-0.04	0.00	-0.47	0.03	-1.01	1.18	-0.24	0.11
	-0.52	*1.11*	*-0.31*	*-1.92*	*-0.36*	*-0.66*	*0.05*	*-1.83*	*3.24*	*-0.76*	*0.01*
(8) 4-yr zero-coupon bond return	-0.01	0.25	-0.03	-0.05	0.00	-1.10	0.34	-0.73	1.01	-0.28	0.09
	-0.50	*0.91*	*-0.28*	*-2.02*	*-0.38*	*-1.24*	*0.42*	*-1.05*	*2.19*	*-0.72*	*0.03*
(9) 5-yr zero-coupon bond return	-0.01	0.26	-0.01	-0.05	0.00	-0.94	0.09	-0.88	1.68	-0.71	0.09
	-0.52	*0.81*	*-0.06*	*-1.94*	*-0.42*	*-0.91*	*0.10*	*-1.08*	*3.13*	*-1.55*	*0.02*

Note: Column labeled 'const.' contains the OLS estimate of the intercept vector Φ_0, while columns (1) to (9) contain the OLS estimate of the matrix Φ_1, which contains the slope parameters for the vector of lagged dependent variables. The numbers in italics in the first 10 columns are t-values. The maximum eigenvalue of the estimated Φ_1 matrix is 0.9805. The final column contains R^2 and (in italics) p-values from an F-test of joint significance for each equation. All variables are in logs. Q = 214 quarterly observations.

Table 3: Residual correlation matrix

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(1) Nominal T-bill return	0.17	-0.50	-0.17	0.18	-0.78	-0.71	-0.64	-0.61	-0.57
(2) Bond excess return		4.67	0.19	-0.20	0.76	0.85	0.88	0.91	0.93
(3) Stock excess return			7.41	-0.98	0.16	0.14	0.12	0.13	0.11
(4) Dividend-price ratio				7.59	-0.17	-0.16	-0.14	-0.14	-0.13
(5) 1-yr zero-coupon bond return					0.82	0.96	0.93	0.89	0.87
(6) 2-yr zero-coupon bond return						1.49	0.98	0.96	0.95
(7) 3-yr zero-coupon bond return							2.03	0.99	0.98
(8) 4-yr zero-coupon bond return								2.54	0.99
(9) 5-yr zero-coupon bond return									2.97

Note: Correlation matrix implied by the estimated residual variance-covariance matrix Σ of the VAR(1) model. Main diagonal elements contain standard deviations of quarterly residuals in %.

Table 4: One-year funding spreads, funding-risk premia, underfunding probabilities and recovery fractions

				Δ_t^4		ϕ_t^4		π_t^4		λ_t^4	
k	ϕ	γ	F_t^0	w_t^0	w_t^Δ	w_t^0	w_t^Δ	w_t^0	w_t^Δ	w_t^0	w_t^Δ
5	1.04	5	0.75	0.208	0.208	0.002	0.002	0.956	0.956	0.822	0.822
			0.90	0.053	0.050	0.004	0.004	0.530	0.545	0.914	0.920
			1.00	0.014	0.012	0.002	0.002	0.194	0.184	0.941	0.945
5	1.04	2	1.00	0.047	0.041	0.003	0.002	0.302	0.291	0.859	0.871
		5		0.014	0.012	0.002	0.002	0.194	0.184	0.941	0.945
		10		0.005	0.005	0.001	0.001	0.111	0.107	0.969	0.970
5	1.02	5	1.00	0.013	0.011	0.001	0.001	0.194	0.185	0.941	0.945
	1.04			0.014	0.012	0.002	0.002	0.194	0.184	0.941	0.945
	1.06			0.015	0.013	0.003	0.003	0.194	0.183	0.941	0.945
5	1.04	5	1.00	0.014	0.012	0.002	0.002	0.194	0.184	0.941	0.945
7.5				0.017	0.016	0.002	0.002	0.212	0.205	0.932	0.936
10				0.020	0.019	0.003	0.003	0.226	0.220	0.924	0.928

Note: k is the investment horizon in years, ϕ relative consumption growth, γ the coefficient of relative risk aversion and F_t^0 the initial funding ratio (assuming $\tau = 1$). Δ_t^4 denotes the one-year (four-quarter) funding spread, θ_t^4 the one-year funding-risk premium, π_t^4 the one-year underfunding probability and λ_t^4 the one-year recovery fraction. These are valued at w_t^0 and w_t^Δ, the optimal portfolios for RF and FRA liabilities, respectively.

Table 5: Optimal asset allocation

			cash		bonds		stocks		Δ term	
k	γ	F_t^0	w_t^0	w_t^Δ	w_t^0	w_t^Δ	w_t^0	w_t^Δ	w_t^0	w_t^Δ
5	5	0.75	-0.024	-0.045	0.290	0.307	0.734	0.738	-0.084	-0.084
		0.90	-0.024	0.013	0.290	0.317	0.734	0.669	-0.020	-0.019
		1.00	-0.024	0.002	0.290	0.307	0.734	0.691	-0.005	-0.005
		1.10	-0.024	-0.013	0.290	0.297	0.734	0.717	-0.001	-0.001
5	10	0.75	0.235	0.203	0.348	0.352	0.417	0.445	-0.090	-0.089
		0.90	0.235	0.247	0.348	0.359	0.417	0.394	-0.014	-0.014
		1.00	0.235	0.241	0.348	0.353	0.417	0.406	-0.001	-0.001
		1.10	0.235	0.235	0.348	0.348	0.417	0.416	0.000	0.000
10	5	0.75	-0.145	-0.139	0.215	0.240	0.930	0.899	-0.090	-0.090
		0.90	-0.145	-0.120	0.215	0.245	0.930	0.875	-0.026	-0.025
		1.00	-0.145	-0.126	0.215	0.235	0.930	0.891	-0.009	-0.008
		1.10	-0.145	-0.134	0.215	0.226	0.930	0.908	-0.003	-0.003

Note: k is the investment horizon in years, γ the coefficient of relative risk aversion and F_t^0 the initial funding ratio (assuming $\tau = 1$). Cash, bonds and stocks refer to the allocations to these asset classes. Δ term is a measure of the term spread of the funding spread curve described in the text. These are valued at w_t^0 and w_t^Δ, the optimal portfolios for RF and FRA liabilities, respectively. The parameter of relative consumption growth is set to $\phi = 1.04$.

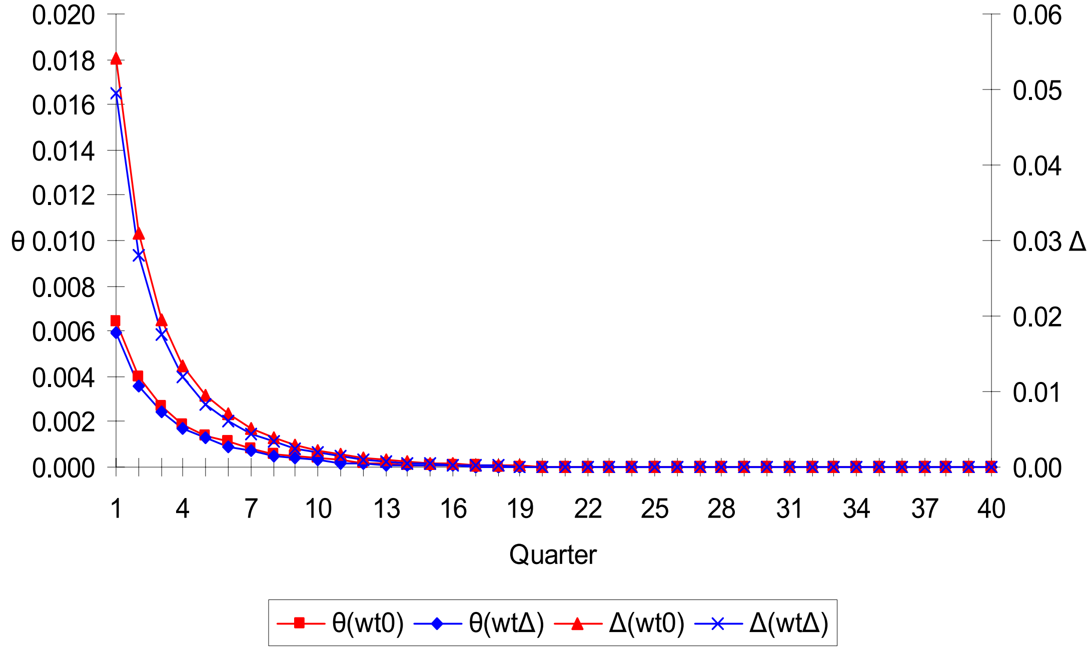
Table 6: Moments of the terminal funding ratio

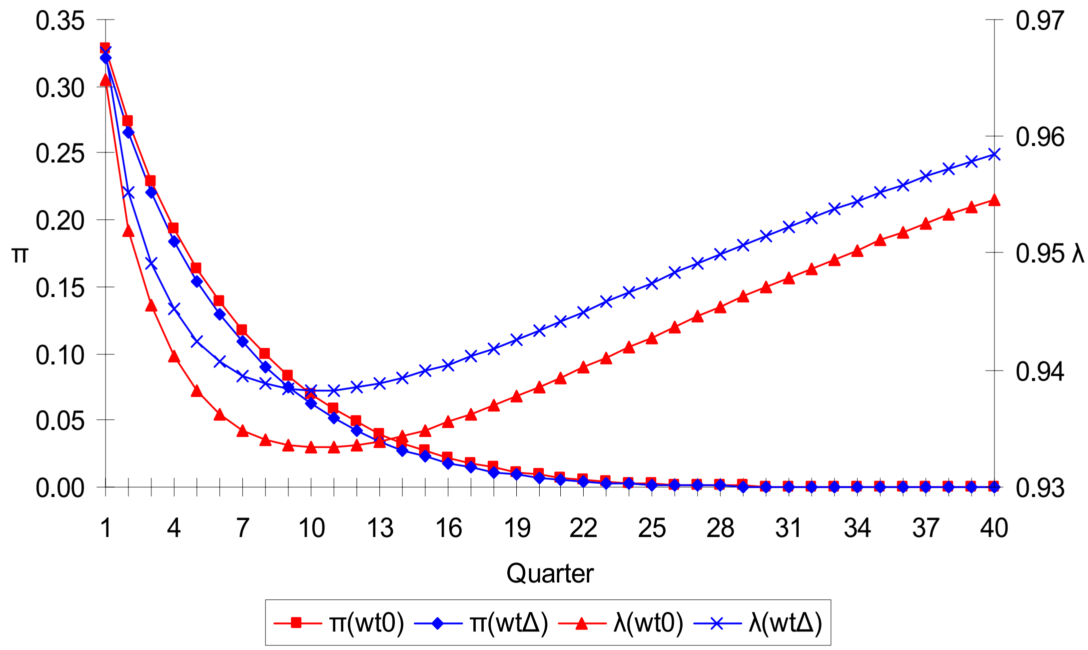
			$E_t[F^0_{t+k}]$		$V^{0.5}_t[F^0_{t+k}]$		$E_t[F^\Delta_{t+k}]$		$V^{0.5}_t[F^\Delta_{t+k}]$	
k	γ	F^0_t	w^0_t	w^Δ_t	w^0_t	w^Δ_t	w^0_t	w^Δ_t	w^0_t	w^Δ_t
5	5	0.75	1.209	1.210	0.237	0.238	1.109	1.110	0.217	0.218
		0.90	1.451	1.427	0.284	0.255	1.420	1.399	0.278	0.250
		1.00	1.612	1.594	0.316	0.294	1.603	1.586	0.314	0.293
		1.10	1.773	1.766	0.348	0.338	1.771	1.764	0.347	0.338
5	10	0.75	1.117	1.124	0.132	0.139	1.019	1.026	0.121	0.127
		0.90	1.340	1.331	0.159	0.150	1.320	1.312	0.156	0.148
		1.00	1.489	1.484	0.176	0.172	1.487	1.482	0.176	0.172
		1.10	1.638	1.638	0.194	0.194	1.638	1.637	0.194	0.194
10	5	0.75	1.997	1.967	0.549	0.518	1.815	1.789	0.498	0.470
		0.90	2.396	2.340	0.658	0.600	2.327	2.277	0.639	0.584
		1.00	2.663	2.617	0.731	0.684	2.634	2.592	0.723	0.677
		1.10	2.929	2.904	0.805	0.779	2.920	2.896	0.802	0.776

Note: k is the investment horizon in years, γ the coefficient of relative risk aversion and F^0_t the initial funding ratio (assuming $\tau = 1$). $E_t[F^0_{t+k}]$ and $V^{0.5}_t[F^0_{t+k}]$ are the expectation and standard deviation of the terminal funding ratio in assets over RF liabilities. $E_t[F^\Delta_{t+k}]$ and $V^{0.5}_t[F^\Delta_{t+k}]$ are the expectation and standard deviation of the terminal funding ratio in assets over FRA liabilities. These are valued at w^0_t and w^Δ_t, the optimal portfolios for RF and FRA liabilities, respectively. The parameter of relative consumption growth is set to $\phi = 1.04$.

Figures

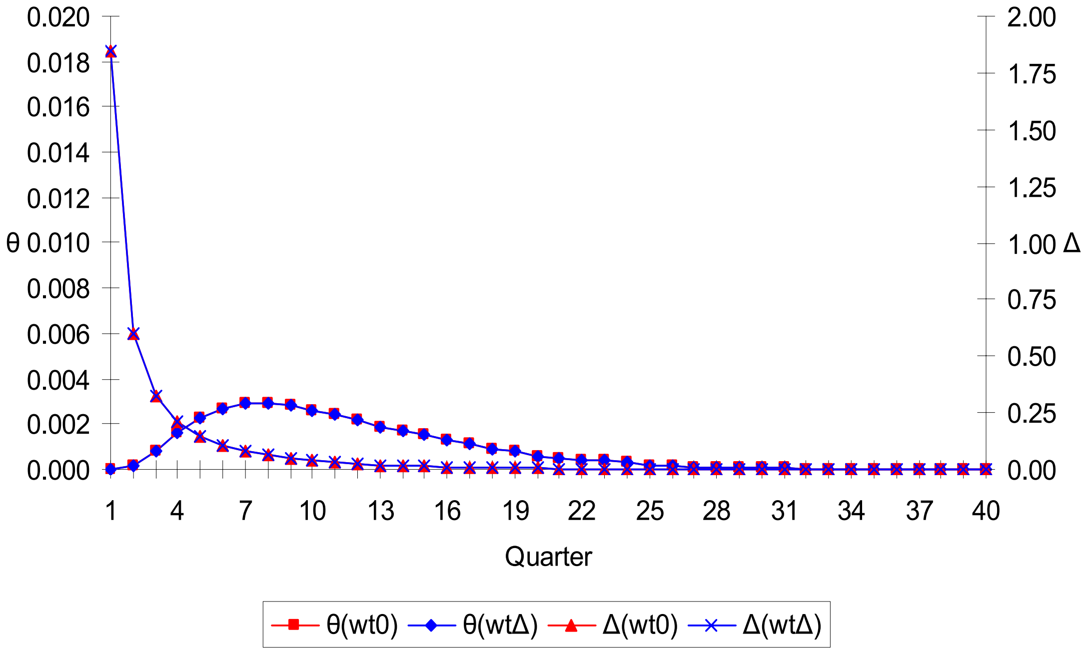
Figure 1: Term structure of funding spreads, funding-risk premia, underfunding probabilities, and recovery fractions for $k = 5$ years, $\phi = 1.04$, $\gamma = 5$, $\tau = 1$, $F_t^0 = 1.0$

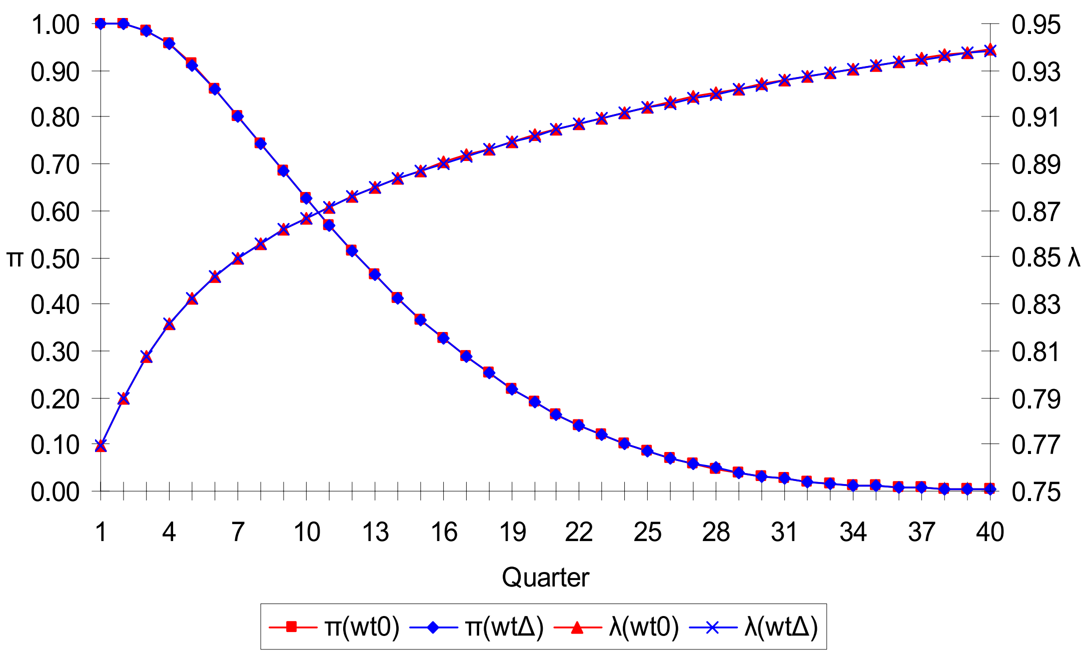




Note: The graphs show annualized funding spreads, annualized funding-risk premia, underfunding probabilities, and recovery fractions over 40 quarters. These are valued at w_t^0 and w_t^{Δ}, the optimal portfolios for RF and FRA liabilities, respectively.

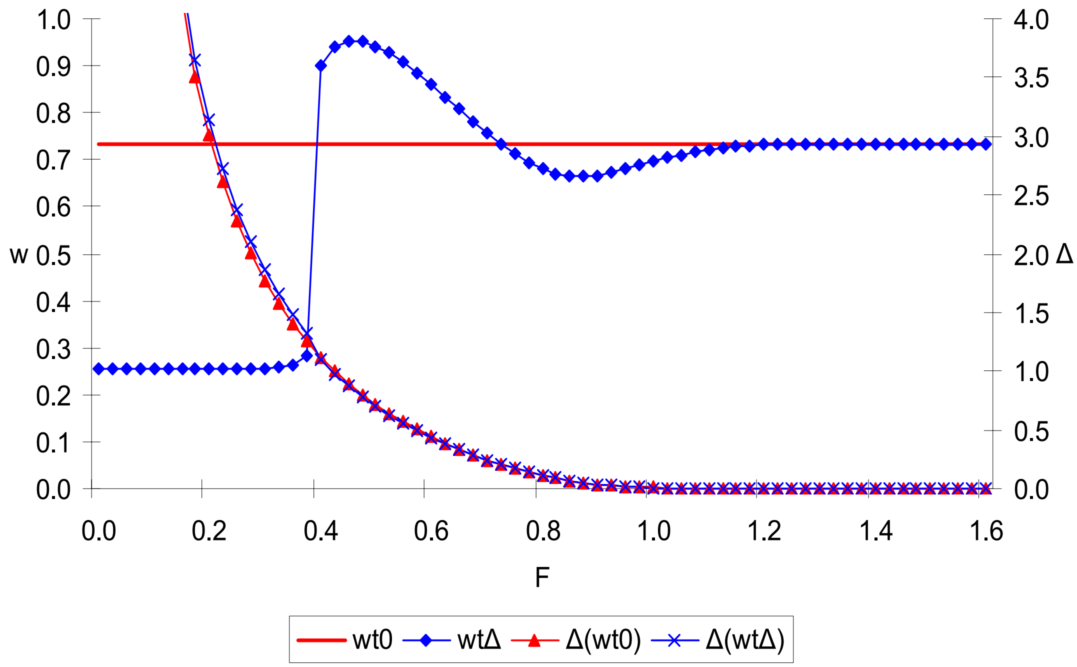
Figure 2: Term structure of funding spreads, funding-risk premia, underfunding probabilities, and recovery fractions for $k = 5$ years, $\phi = 1.04$, $\gamma = 5$, $\tau = 1$, $F_t^0 = 0.75$

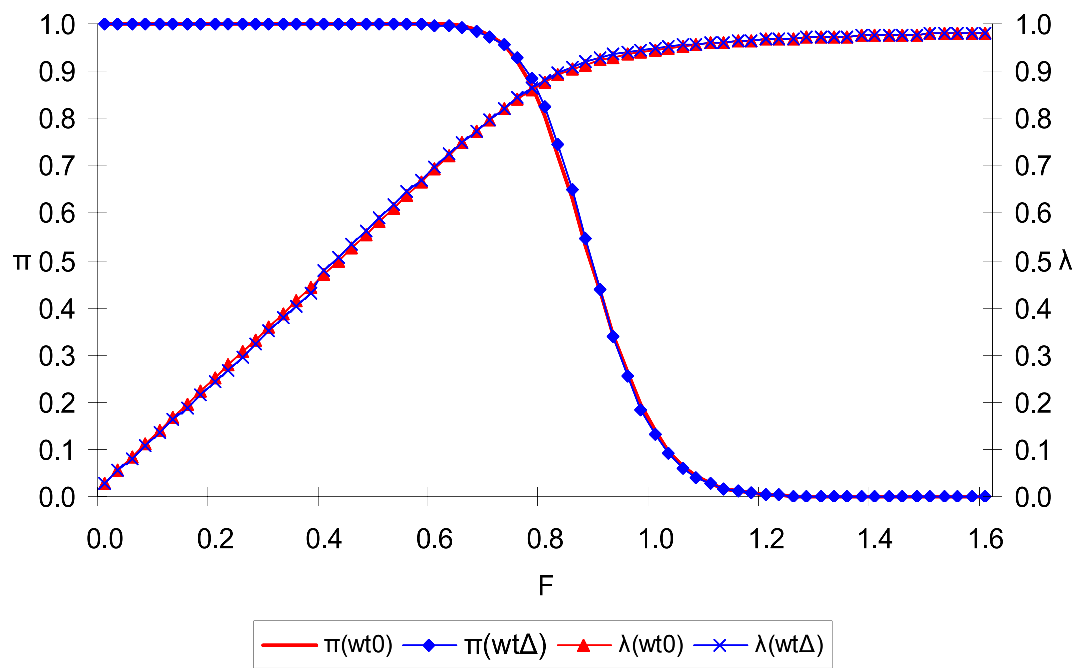




Note: As per Figure 1.

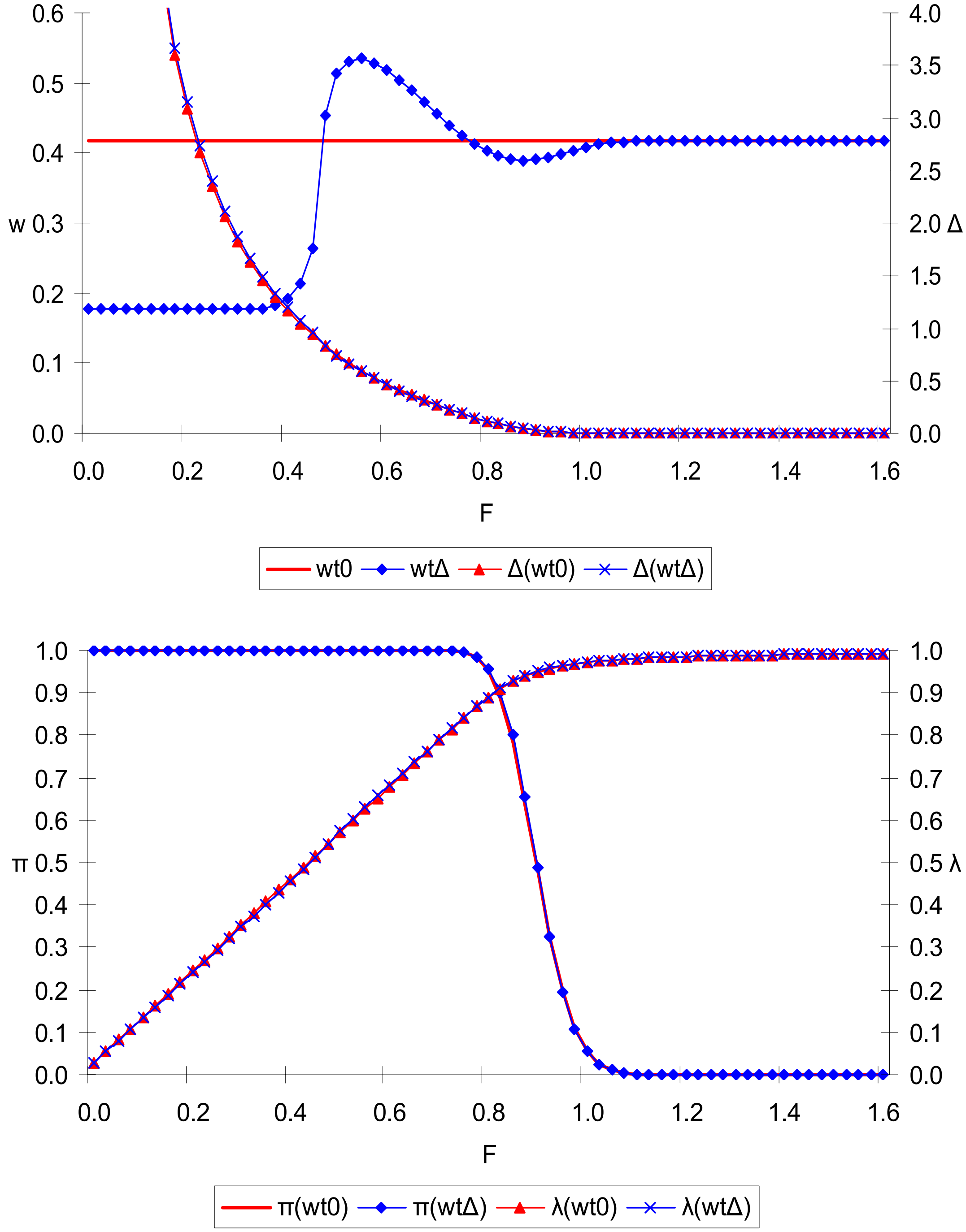
Figure 3: Stock allocations, funding spreads, underfunding probabilities and recovery fractions as a function of the initial funding ratio for $k = 5$ years, $\phi = 1.04$, $\gamma = 5$, $\tau = 1$





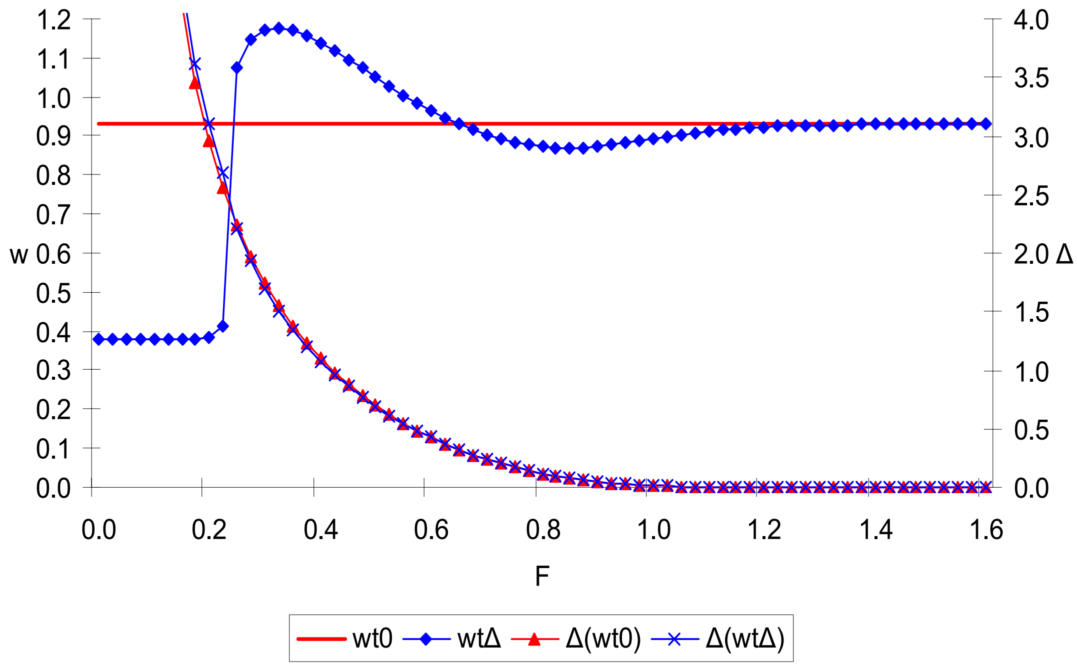
Note: The graphs show the optimal stock allocations and annualized funding spreads (up to a ceiling of 4.0), underfunding probabilities and recovery fractions at 1-year maturity. w_t^0 and w_t^Δ are the optimal portfolios for RF and FRA liabilities, respectively.

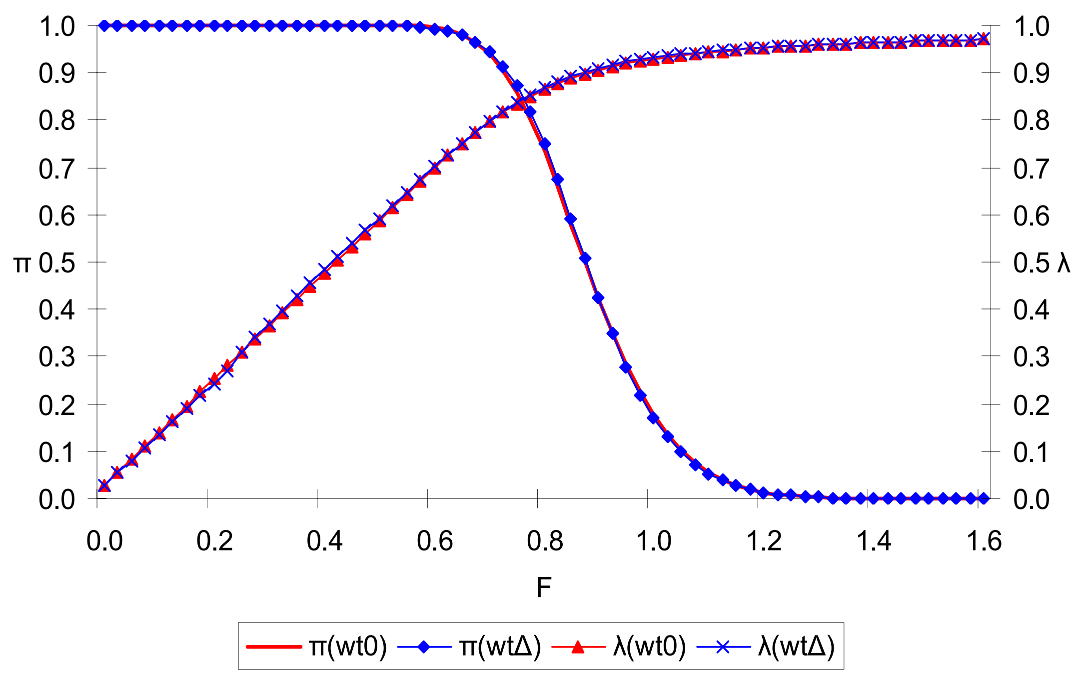
Figure 4: Stock allocations, funding spreads, underfunding probabilities and recovery fractions as a function of the initial funding ratio for $k = 5$ years, $\phi = 1.04$, $\gamma = 10$, $\tau = 1$



Note: As per Figure 3.

Figure 5: Stock allocations, funding spreads, underfunding probabilities and recovery fractions as a function of the initial funding ratio for $k = 10$ years, $\phi = 1.04$, $\gamma = 5$, $\tau = 1$





Note: As per Figure 3.