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Abstract

In this paper we show how pay-as-you-go pension schemes impact on the individual's optimal investment portfolio. Introducing a pay-as-you-go pension scheme implies that human wealth of young generations is transferred to retired generations. As a consequence, individuals will in general invest less conservatively. These portfolio effects gradually disappear at the end of life.

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1 Introduction

An individual saving for his pension has to decide how to invest his pension savings. A pay-as-you-go (PAYG) pension scheme interferes with this portfolio decision by reducing net wage income when young and providing pension income when retired. In fact, introducing a pay-as-you-go pension scheme implies that human wealth of young generations is partly transferred to retired generations. As noted by Merton (1983) and Persson (2002), a PAYG-pension scheme may actually be considered as a quasi-asset relevant for the optimal portfolio allocation. The main question we address in this paper is how a PAYG pension scheme impacts on the optimal financial investment over lifetime. To this end we will analyze how human wealth from (net) wages and pension wealth from PAYG-pension benefits impact on the optimal allocation of financial wealth over the individual's lifecycle.

The literature on the optimal funding of pensions has mainly focussed on the question whether pensions should be financed on a funded or a pay-as-you-go basis[1]. However, more recently optimal portfolio theory from finance literature has shed a new light on the issue of PAYG versus funding. Merton (1983) already argued that a mix of PAYG and funding might be optimal for reasons of diversification. Persson (2002) noted that a PAYG-scheme can be considered as an asset in the overall investment portfolio. By formalizing this argument in a two period overlapping generations model, Matsen and Thøgersen (2004) demonstrated that even a low-yielding PAYG-scheme can benefit individuals if it contributes to hedge other risks to their lifetime resources.

Our paper is in two aspects an extension of the analysis of Bodie et al. (1992). The first extension is that pension wealth from PAYG pension schemes is explicitly taken into account when determining the optimal allocation of financial wealth. This matters especially after retirement when an individual's human wealth is fully depleted. Apart from financial wealth, total wealth then consists of pension wealth from future PAYG pension benefits. The second extension is the inclusion of inflation as a risk factor. Uncertainty about future inflation implies uncertainty about future real wealth, which is relevant for consumption and investment decisions. Apart from inflation we also include productivity growth as a risk factor in our model. We take the existence of a PAYG pension scheme as given and do not analyse the welfare effects of a PAYG pension scheme[2].

The paper is organized as follows. The next section presents our basic theoretical model framework. Applying the dynamic programming approach (Campbell and Viceira, 2002), we derive the individual's optimal consumption and portfolio allocation over the lifecycle. In section 3, we extend the model with stochastic wage income and analyse how human wealth from stochastic wages matters for the optimal allocation of financial wealth in complete markets. In section 4, we extend the model with a PAYG pension scheme and analyse how

[1] See Matsen and Thøgersen (2004) for an overview of the literature.

[2] There is a rich literature about the welfare effects of PAYG schemes, see for instance Bohn (2001) and Sánchez-Marcos et al. (2006).

this impacts on the optimal allocation of financial wealth. We consider both a defined contribution and a defined benefit PAYG-pension scheme and show how pension wealth from PAYG-schemes matters for the optimal allocation of financial wealth. In section 5, we calibrate the model to quantify the effects predicted by the theoretical model. The results of the paper are summarized in section 6, while appendices A-G provide the detailed formal derivations as well as a table of notation.

2 Model

In this section we explore the lifetime consumption and portfolio choice model of Fischer (1975) to derive optimal portfolio investment over the life cycle. Individuals live from time 0 to D (being deterministic) and derive utility from consumption. Initially, we assume that individuals only receive income as a (stochastic) return on tradable assets. In sections 3 and 4, we extend the model with stochastic wages and pay-as-you-go pension benefits, respectively.

2.1 Price dynamics and asset returns

Initially, we assume that the individual's investment portfolio can contain three assets: price indexed bonds, equity and nominal bonds. The portfolio can be adjusted instantaneously and costlessly and there are no nonnegativity constraints on the asset holdings.

Suppose the behavior of the price level P follows a geometric Brownian motion:

$$\frac{dP}{P} = \pi dt + s_1 dz_1 \tag{1}$$

with π being the *expected* price increase per unit of time[3]. Here, dz_1 is a Wiener process with s_1^2 being the variance of the change in the price level per unit of time. The real return on price indexed bonds (Q_1) equals:

$$\frac{d\left(Q_1/P\right)}{Q_1/P} = r_1 dt \tag{2}$$

Suppose that real returns on equity also follow a geometric Brownian motion. The expected real return on equity (Q_2) equals r_2. Stochastic changes in the real returns on equity are assumed to be related to productivity growth. On its turn, productivity growth is supposed to be driven by the Wiener process dz_2. This gives:

$$\frac{d\left(Q_2/P\right)}{Q_2/P} = r_2 dt + s_2 dz_2 \tag{3}$$

[3] Some models in the literature apply stochastic expected inflation, see for instance Brennan and Xia (2002). However, in our model expected inflation is deterministic for reasons of tractability.

where s_2^2 denotes the variance of real equity returns per unit of time.

The nominal return on nominal bonds (Q_3) equals R_3 implying that the *real* return on nominal bonds can be expressed as (Fischer (1974)):

$$\frac{d\left(Q_3/P\right)}{Q_3/P} = r_3 dt + s_3 dz_3 \tag{4}$$

where

$$\begin{aligned} r_3 &= R_3 - \pi + s_1^2 \\ dz_3 &= dz_1 \\ s_3 &= -s_1 \end{aligned}$$

2.2 Budget constraints and individual's choice problem

Let α_1, α_2 and α_3 be the proportions of the portfolio held in price indexed bonds, equity and nominal bonds, respectively. This implies that the stock budget constraint is

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{5}$$

The flow budget constraint, describing the change in real wealth (W), can be defined as:

$$dW = \sum_{i=1}^{3} \alpha_i r_i W dt - C dt + \sum_{i=2}^{3} \alpha_i s_i dz_i W \tag{6}$$

with C being consumption per unit of time. Note that the last term on the right-hand side of (6) captures the stochastic change in real wealth due to holdings of equity and nominal bonds.

Substituting for α_1 from (5) into (6) gives

$$dW = \left[\left[r_1 + \alpha_2 \left(r_2 - r_1\right) + \alpha_3 \left(r_3 - r_1\right)\right] W - C\right] dt + \left[\alpha_2 s_2 dz_2 + \alpha_3 s_3 dz_1\right] W \tag{7}$$

Merton (1969) demonstrates that the optimal portfolio and consumption problem of individuals living from time 0 to D, is to find

$$\max_{C,\alpha_i} E_0 \int_0^D U\left[C\left(t\right), t\right] dt \tag{8}$$

subject to (7) and $W(0) = W_0 > 0$, with $C\left(t\right) \geq 0$ and $U\left(\cdot\right)$ strictly concave in C.

Let $J\left[W\left(t\right), t\right]$ denote the derived utility function of this problem:

$$J\left[W\left(t\right), t\right] \equiv \max_{C,\alpha_i} E_t \int_{\varphi=t}^{\varphi=D} U\left[C\left(\varphi\right), \varphi\right] d\varphi \tag{9}$$

subject to the transversality condition

$$J\left[W\left(D\right),D\right]=0 \tag{10}$$

The Bellman-Dreyfus fundamental equation of optimality yields:

$$0=\max_{C,\alpha_i}\left\{U\left[C\left(t\right),t\right]+\frac{1}{dt}E_t\left[dJ\left(W,t\right)\right]\right\} \tag{11}$$

The Bellman principle implies that at the optimum, the investor has traded off the value of present and future consumption perfectly.

Applying Itô's lemma to $J\left[W\left(t\right),t\right]$ gives:

$$dJ\left[W\left(t\right),t\right]=J_W dW+J_t dt+\frac{1}{2}J_{WW}dW^2 \tag{12}$$

where subscripts denote partial derivatives. Substituting (12) into the Bellman equation (11) and computing the first-order conditions by taking derivatives with respect to C,α_1 and α_2, gives three expressions for optimal consumption and portfolio choice[4]

$$U_C\left(C,t\right) = J_W \tag{13}$$

$$\alpha_2 = -\frac{J_W}{J_{WW}W}\left[\frac{\left(r_2-r_1\right)}{\left(1-\rho_{12}^2\right)s_2^2}-\frac{\left(r_3-r_1\right)\rho_{12}}{\left(1-\rho_{12}^2\right)s_2 s_3}\right] \tag{14}$$

$$\alpha_3 = -\frac{J_W}{J_{WW}W}\left[\frac{\left(r_3-r_1\right)}{\left(1-\rho_{12}^2\right)s_3^2}-\frac{\left(r_2-r_1\right)\rho_{12}}{\left(1-\rho_{12}^2\right)s_2 s_3}\right] \tag{15}$$

subject to the transversality condition $J\left[W\left(D\right),D\right]=0$ and with ρ_{12} being the instantaneous correlation coefficient between Wiener processes dz_1 and dz_2 ($|\rho_{12}|<1$).

Equation (13) determines the optimal consumption policy and is known as the envelope condition, stating that at the optimum an extra unit of consumption is as valuable as an extra unit of wealth to finance future consumption.

Equations (14) and (15) determine the optimal portfolio allocation to equity and nominal bonds, respectively. For both equations, the coefficient of the first term in brackets ($-\frac{J_W}{J_{WW}W}$) is the inverse of the degree of relative risk aversion of the individual. Suppose for the moment that $\rho_{12}=0$. Then the demand for both equity and nominal bonds depends in an intuitively plausible way on their expected excess real return over price indexed bonds (r_3-r_1 respectively r_2-r_1) and their return variances (s_2^2 and s_3^2). Moreover, for $\rho_{12}<0$ and $s_3<0$, equity and nominal bonds are substitutes in that the demand for each is negatively related to the expected real return of the other.

As equations (13)-(15) depend on the indirect utility function J, they do not offer the complete solution to the model. In order to find a complete solution, we will now consider a specific utility function with constant relative risk aversion γ (CRRA):

[4] See appendix A for the formal derivation.

$$U\left[C\left(t\right),t\right]=\exp\left[-\delta t\right]\frac{\left[C\left(t\right)\right]^{1-\gamma}}{1-\gamma} \tag{16}$$

where δ is the rate of time preference.

As a trial solution for the indirect utility function we take:

$$J\left[b\left(t\right),W\left(t\right),t\right]=\frac{b\left(t\right)}{1-\gamma}\exp\left(-\delta t\right)\left[W\left(t\right)\right]^{1-\gamma} \tag{17}$$

By substituting the trial solution (17) into the Bellman-Dreyfus equation of optimality (11), we find that $b\left(t\right)$ must satisfy the following ordinary differential equation:

$$\frac{db}{dt}=\left(1-2\gamma\right)b^{-\frac{1-\gamma}{\gamma}}+bX \tag{18}$$

where X is a constant[5].

A solution to (18) satisfying the transversality condition ($J\left[b\left(D\right),W\left(D\right),D\right]=0$) is:

$$b\left(t\right)=\left(\frac{\left(2\gamma-1\right)\left[1-\exp\left(\frac{t-D}{\gamma}X\right)\right]}{X}\right)^{\gamma} \tag{19}$$

Optimal consumption over time can now be expressed as a function of real wealth:

$$C\left(t\right)=\left[b\left(t\right)\right]^{-\frac{1}{\gamma}}W\left(t\right) \tag{20}$$

while the optimal portfolio shares are now equal to:

$$\alpha_1 = 1-\frac{1}{\gamma}\left[\frac{\left(r_2-r_1\right)}{\left(1-\rho_{12}^2\right)s_2^2}+\frac{\left(r_3-r_1\right)}{\left(1-\rho_{12}^2\right)s_3^2}-\frac{\left(r_2+r_3-2r_1\right)\rho_{12}}{\left(1-\rho_{12}^2\right)s_2s_3}\right] \tag{21}$$

$$\alpha_2 = \frac{1}{\gamma}\left[\frac{\left(r_2-r_1\right)}{\left(1-\rho_{12}^2\right)s_2^2}-\frac{\left(r_3-r_1\right)\rho_{12}}{\left(1-\rho_{12}^2\right)s_2s_3}\right] \tag{22}$$

$$\alpha_3 = \frac{1}{\gamma}\left[\frac{\left(r_3-r_1\right)}{\left(1-\rho_{12}^2\right)s_3^2}-\frac{\left(r_2-r_1\right)\rho_{12}}{\left(1-\rho_{12}^2\right)s_2s_3}\right] \tag{23}$$

Here, α_1 is determined as the residual investment following from (22) and (23). The interpretation of the optimal portfolio shares in case of CRRA-utility is the same as the general case ((14)-(15)). However, it is important to note that the optimal portfolio shares are now independent of the level of W. This implies that the optimal allocation of *total* real wealth is constant over lifetime. We will explore this result in sections 3 and 4, when we extend the analysis with human wealth from wages and pension wealth from pension benefits.

[5] See appendix B for the derivation.

3 Wage income

So far we have assumed that (gross) returns from assets are the only source of income. As an extension, we now also introduce wage income from inelastically supplied labour[6]. Moreover, suppose the individual retires at the statutory retirement age $T(< D)$. The present value of future wages can be considered as human wealth (Bodie et al. (1992)). This implies that total wealth can be defined as the sum of financial wealth F and human wealth from wages H^w:

$$W(t) = F(t) + H^w(t) \tag{24}$$

If we manage both to value this human wealth as well as to find a replicating portfolio in terms of financial assets, we can determine the optimal portfolio allocation of (financial) wealth by using equations (21)-(23).

We suppose that real wages are subject to both inflationary and productivity risks. Hence, we suppose that real wages w follow the geometric Brownian motion

$$\frac{dw}{w} = \omega dt + s_{w1} dz_1 + s_{w2} dz_2 \tag{25}$$

with s_{w1}^2 being the variance of the change in real wages per unit of time related to price changes and s_{w2}^2 being the variance of the change in real wages per unit of time related to productivity changes. In order to value the human wealth from this wage income, we now apply a similar approach as taken by Bodie et al. (1992).

Proposition 1 *The real value of human wealth from wages (H^w) equals the present value of future wages discounted at a risk-adjusted discount rate μ_w:*

$$H^w[w(t), t] = \frac{w(t)}{\mu_w}\left[1 - \exp\left[-\mu_w\left(\widehat{t} - t\right)\right]\right] \tag{26}$$

with

$$\widehat{t} = \max[t, T]$$

$$\mu_w = r_1 + \lambda_{w2}(r_2 - r_1) + \lambda_{w1}(r_3 - r_1) - \omega$$

$$\lambda_{w1} = \frac{s_{w1}}{s_3}$$

$$\lambda_{w2} = \frac{s_{w2}}{s_2}$$

Proof. *See appendix C.* ■

[6] The option to substitute labour for leisure (Bodie et al. (1992)) is not taken into account here.

In (26), the discount factor μ_w is equal to the real return on price indexed bonds plus the risk-adjusted equity and nominal bond premiums respectively, minus the real wage drift. Guiding this risk adjustment are the factors λ_{w1} and λ_{w2}, where λ_{w1} reflects the relative sensitivity of real wages for inflationary shocks and λ_{w2} the relative sensitivity of real wages for productivity shocks. At birth human wealth is at its maximum, while it is gradually depleted thereafter. Note that $H^w = 0$ once the individual is retired (i.e. for $t > T$, $\widehat{t}$ equals t).

Given the expression we derived to value human wealth, we now need a technique to find equivalents of this value in terms of financial assets. An applicable technique is contingent claim analysis (Merton, 1990).

Proposition 2 *The value of human wealth from wages H^w is economically equivalent to a portfolio with $\lambda_{w2}H^w$ invested in equity, $\lambda_{w1}H^w$ invested in nominal bonds and $(1-\lambda_{w1}-\lambda_{w2})\,H^w$ invested in price indexed bonds.*
Proof. *See appendix D.* ■

Armed with propositions 1 and 2 and applying equation (24), we can now express the explicit euro investment in equity $(Z_{Q_2}(t))$ as:

$$\begin{aligned} Z_{Q_2}(t) &= \alpha_2 W(t) - \lambda_{w2} H^w(t) \\ &= \alpha_2 F(t) + (\alpha_2 - \lambda_{w2})\, H^w(t) \end{aligned} \tag{27}$$

where the explicit investment in equity equals the desired gross investment minus the implicit investment via human wealth from wages. Equivalently, the explicit euro investment in nominal bonds $(Z_{Q_3}(t))$ can be expressed as:

$$\begin{aligned} Z_{Q_3}(t) &= \alpha_3 W(t) - \lambda_{w1} H^w(t) \\ &= \alpha_3 F(t) + (\alpha_3 - \lambda_{w1})\, H^w(t) \end{aligned} \tag{28}$$

Finally, the explicit euro investment in price indexed bonds $(Z_{Q_1}(t))$ can be expressed as:

$$Z_{Q_1}(t) = \alpha_1 F(t) + (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2}))\, H^w(t) \tag{29}$$

Hence, the explicit investment in price indexed bonds equals the desired gross investment minus the implicit investment via human wealth from wages.

This implies that the optimal portfolio shares of *financial* wealth $F(t)$ invested in equity, nominal and price indexed bonds respectively (denoted by $\widehat{\alpha}_2$, $\widehat{\alpha}_3$ and $\widehat{\alpha}_1$ respectively) can be expressed as

$$\begin{aligned} \widehat{\alpha}_2(t) &\equiv \frac{Z_{Q_2}(t)}{F(t)} \\ &= \alpha_2 + (\alpha_2 - \lambda_{w2})\,\frac{H^w(t)}{F(t)} \end{aligned} \tag{30}$$

$$\widehat{\alpha}_3(t) = \frac{Z_{Q_3}(t)}{F(t)} = \alpha_3 + (\alpha_3 - \lambda_{w1}) \frac{H^w(t)}{F(t)} \quad (31)$$

and

$$\widehat{\alpha}_1(t) \equiv \frac{Z_{Q_1}(t)}{F(t)} = \alpha_1 + (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2})) \frac{H^w(t)}{F(t)} \quad (32)$$

where the second terms on the right-hand side of expressions (30)-(32) allow for an analysis of the incremental effects of human wealth on the optimal portfolio allocation. Note that the optimal portfolio allocation of financial wealth is now time-dependent (i.e. upon the value of H^w and F). Hence, in response to changes in either or H^w or F, a continuous reallocation of *financial* wealth may be required under constant relative risk aversion in order to keep the allocation of *total* wealth constant over the lifecycle. Early in the lifecycle total wealth consists predominantly of human wealth. Hence, it might be optimal for the individual to borrow early in the lifecycle. We will calibrate the model in section 5 in order to quantify the incremental portfolio effects as predicted by equations (30)-(32). However, we will first derive the portfolio effects of a pay-as-you-go pension scheme.

4 Pay-as-you-go pension scheme

In this section we will consider how a PAYG pension scheme impacts on the optimal portfolio derived before. Under a PAYG pension scheme, income from the young is taxed in order to finance an old-age pension benefit for retirees. The present value of future pension benefits can be considered as a form of positive pension wealth for the individual. Equivalently, the present value of future tax payments can be considered as a form of negative pension wealth for the individual. In fact, introducing a pay-as-you-go pension scheme implies that human wealth of young generations is partly transferred to retired generations.

We start by considering a defined contribution pension scheme in section 4.1, before proceeding to a defined benefit pension scheme (section 4.2).

4.1 Defined contribution pension scheme

Suppose the government introduces a defined contribution (DC) PAYG pension scheme providing retirees with a real pension benefit $\theta(t)$. The scheme is financed by a fixed proportional wage tax τ. The flow budget constraint of this DC PAYG pension scheme is

$$n^o(t)\,\theta(t) = n^y(t)\,\tau w(t) \tag{33}$$

Suppose that the ratio of the number of young persons $n^y(t)$ over the number of old persons $n^o(t)$ (the so-called support ratio k)[7] remains constant. Using (33), we can now express the real pension benefit as

$$\theta(t) = \tau k w(t) \tag{34}$$

implying that the DC pension benefit depends on current wages.

Total wealth now equals the sum of financial wealth (F^{DC}), human wealth and *net* pension wealth, the latter being equal to pension wealth from future pension benefits (H^θ) minus the government's tax claim on human wealth (τH^w):

$$W(t) = F^{DC}(t) + H^w(t) + H^\theta(t) - \tau H^w(t) \tag{35}$$

In order to analyze the impact of the DC pension scheme on total wealth and the optimal allocation of financial wealth, we need to value pension wealth from pension benefits.

Proposition 3 *The value of pension wealth from DC pension benefits as defined in (34) $\left(H^\theta\right)$ equals the present value of future pension benefits discounted at the risk-adjusted discount rate μ_w*

$$H^\theta[\theta(t), t] = \tau k \frac{w(t)}{\mu_w}\left[\exp\left[-\mu_w\left(\widehat{t} - t\right)\right] - \exp\left[-\mu_w(D - t)\right]\right] \tag{36}$$

with:

$$\widehat{t} = \max[t, T]$$

$$\mu_w = r_1 + \lambda_{w2}(r_2 - r_1) + \lambda_{w1}(r_3 - r_1) - \omega$$

$$\lambda_{w1} = \frac{s_{w1}}{s_3}$$

$$\lambda_{w2} = \frac{s_{w2}}{s_2}$$

Proof. *See appendix E.* ■

Note that the development of pension wealth differs from human wealth over lifetime. After being born, the value of human wealth gradually diminishes until pension age T when it is fully depleted. In the same period, the value of pension wealth gradually *increases* until it reaches its maximum at retirement age T. After being retired, the value of pension wealth gradually diminishes until $t = D$ when it is fully depleted.

[7] Note that the support ratio is the inverse of the so called old-age dependency ratio, being the ratio of number of old over the number of young persons.

Armed with equation (35) and proposition 3, we can distinguish two potential effects of the DC PAYG pension scheme on total wealth. First of all, as wage income is taxed individuals may save less over life causing $F^{DC}(t) < F(t)$. Such crowding out of financial wealth reduces total wealth. Secondly, total wealth changes with the value of net pension wealth $(H^{\theta}(t) - \tau H^{w}(t))$. Using propositions 1 and 3, we can show that net pension wealth at birth is zero if[8]:

$$(1 + k^*) \exp[-\mu_w T] - k^* \exp[-\mu_w D] = 1 \tag{37}$$

By applying a second order Maclaurin expansion on equation (37), we can derive the following approximation for the equilibrium support ratio k^*:

$$k^* \approx \frac{T}{D - T} \phi(\mu_w) \tag{38}$$

with

$$\phi(\mu_w) = \frac{1 - \frac{1}{2}\mu_w T}{1 - \frac{1}{2}\mu_w (T + D)}$$

Net pension wealth at birth will be positive for $k > k^*$ and negative for $k < k^*$. Let us now take a closer look at equation (38) by considering the case of a dynamically efficient economy (implying $\mu_w > 0$) with no population growth (implying a constant support ratio k). In this case the support ratio k will be equal to $\overline{k} = \frac{T}{D-T}$ as every dying generation is exactly replaced by a new generation. Moreover, we can show that $\phi(\mu_w) > 1$ for small values of μ_w[9], implying $k = \overline{k} < k^*$. Hence, net pension wealth at birth will always be negative in this case. Put differently, net pension wealth will only be positive when population growth causes $k > k^* > \overline{k}$. In fact, equation (38) reflects the Aaron-condition (Aaron, 1966) stating that a PAYG pension scheme in a dynamically efficient economy is only a Pareto improvement if population growth exceeds the (positive) difference between the return on financial wealth and real wage growth (i.e. μ_w).

The replicating portfolio of pension wealth can again be determined with contingent claim analysis:

Proposition 4 *The value of pension wealth from pensions H^{θ} is economically equivalent to a portfolio with $\lambda_{w2} H^{\theta}$ invested in equity, $\lambda_{w1} H^{\theta}$ invested in nominal bonds and $(1 - \lambda_{w1} - \lambda_{w2}) H^{\theta}$ invested in price indexed bonds.*
Proof. *See appendix E.* ■

Let α_i^{DC} denote the optimal fraction of financial wealth invested in asset i under a DC PAYG pension scheme. Armed with propositions 3 and 4 and equation (35), we can now express the incremental *changes* in the optimal financial portfolio due to the introduction of the DC PAYG pension scheme as:

[8] See appendix E.

[9] Mathematically, $\phi(\mu_w)$ has a vertical asymptote at $\mu_w = \frac{2}{T+D}$ and a horizontal asymptote at $\phi = 1$. Hence, $\phi(\mu_w) > 1$ for $\mu_w < \frac{2}{T+D}$.

$$\alpha_1^{DC}(t) - \widehat{\alpha}_1(t) = (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2}))\left[\Theta_1(t) + \Theta_2(t)\right] \quad (39)$$

$$\alpha_2^{DC}(t) - \widehat{\alpha}_2(t) = (\alpha_2 - \lambda_{w2})\left[\Theta_1(t) + \Theta_2(t)\right] \quad (40)$$

$$\alpha_3^{DC}(t) - \widehat{\alpha}_3(t) = (\alpha_3 - \lambda_{w1})\left[\Theta_1(t) + \Theta_2(t)\right] \quad (41)$$

for

$$\Theta_1(t) = \frac{H^{\theta}(t) - \tau H^{w}(t)}{F^{DC}(t)} \quad (42)$$

$$\Theta_2(t) = \frac{H^{w}(t)}{F^{DC}(t)} - \frac{H^{w}(t)}{F(t)} \quad (43)$$

Equations (39)-(41) show that we can break up the incremental portfolio changes in two terms. In the first term, $\Theta_1(t)$ reflects the direct effect of net DC PAYG pension wealth on the optimal composition of financial wealth. In the second term, $\Theta_2(t)$ reflects the indirect effect arising in case of crowding out of financial wealth (i.e. $F^{DC}(t) < F(t)$). In that case, the incremental portfolio effects of human wealth as derived in the previous section are amplified. We will calibrate the model in the next section in order to quantify these incremental portfolio effects.

4.2 Defined benefit pension scheme

In this subsection, we will consider a defined benefit (DB) pension scheme with a fixed real pension benefit $\overline{\theta}$ but a variable tax rate ($\tau(t)$). The flow budget constraint of this DB PAYG pension scheme is

$$n^{o}(t)\,\overline{\theta} = n^{y}(t)\,\tau(t)\,w(t) \quad (44)$$

With a constant support ratio k, the tax rate can be expressed as

$$\tau(t) = \frac{\overline{\theta}}{kw(t)} \quad (45)$$

Hence, for a given real pension benefit, keeping a balanced budget implies that the tax rate under a DB pension scheme becomes a function of the wage level.

Apart from creating an asset in the form of pension wealth from pension benefits, the DB pension scheme also creates a stochastic liability for the young individual in the form of a future tax payments, which can be considered as a government's tax claim on the individual. Unlike a DC pension scheme, the tax claim is no longer a fixed proportion of human wealth under a DB pension scheme with stochastic tax rates (see equation (45)).

Total real wealth can now be expressed as the sum of financial wealth (F^{DB}), human wealth and net pension wealth, being gross pension wealth from DB pension benefits ($H^{\overline{\theta}}$) minus the government's tax claim on human wealth (H^{ψ}):

$$W(t) = F^{DB}(t) + H^{w}(t) + H^{\overline{\theta}}(t) - H^{\psi}(t) \tag{46}$$

In order to analyze the impact of the DB-pension scheme on total wealth and the optimal portfolio allocation, we need to value both DB pension wealth as well as the government's tax claim on human wealth.

The value of pension wealth from the fixed real DB pension benefits is simply the present value of these benefits discounted at the riskfree real interest rate:

$$H^{\overline{\theta}}(t) = \frac{\overline{\theta}}{r_1}\left[\exp\left[-r_1\left(\widehat{t}-t\right)\right] - \exp\left[-r_1(D-t)\right]\right] \tag{47}$$

with

$$\widehat{t} = \max[t, T]$$

Note that the value of DB pension wealth is deterministic. This also implies that the replicating portfolio of $H^{\overline{\theta}}(t)$ consists solely out of price indexed bonds. Put differently, the value of DB pension benefits $H^{\overline{\theta}}(t)$ is economically equivalent to a portfolio with $H^{\overline{\theta}}(t)$ invested in price indexed bonds[10].

Define the individual's instantaneous payable taxes under a DB pension scheme as:

$$\psi(t) = \tau(t)\, w(t) \tag{48}$$

Using (45), we can rewrite this as:

$$\psi(t) = \psi = \frac{\overline{\theta}}{k} \tag{49}$$

Proposition 5 *The value of the government's tax claim under a DB pension scheme (H^{ψ}) equals the present value of future payable taxes ψ discounted at the discount rate r_1*

$$H^{\psi}[\psi(t), t] = \frac{\overline{\theta}}{kr_1}\left[1 - \exp\left[-r_1\left(\widehat{t}-t\right)\right]\right] \tag{50}$$

with

$$\widehat{t} = \max[t, T]$$

Proof. *See appendix F.* ■

The value of the tax claim is perfectly hedged for changes in the wage level, given that the tax rate is perfectly inversely related to wages (see equation (45)). Put differently, wage shocks are compensated for by inverse tax rate shocks keeping instantaneous payable taxes constant.

[10] See appendix F for the formal proof.

Armed with equations (46), (47) and (50), we can distinguish two potential effects of introducing a DB PAYG pension scheme on total wealth. First of all, as wage income is taxed individuals may save less over life causing $F^{DB}(t) < F(t)$. Such crowding out of financial wealth reduces total wealth. Secondly, total wealth changes with the value of net pension wealth $(H^{\overline{\theta}}(t) - H^{\psi}(t))$. Using propositions 1 and 5, we can show that net pension wealth at birth is zero if[11]:

$$(1+k^*)\exp[-r_1 T] - k^* \exp[-r_1 D] = 1 \tag{51}$$

This implies that net pension wealth at birth will be positive for $k > k^*$ and negative for $k < k^*$. However, analogously to the DC-case, we can show that net DB PAYG pension wealth will only be positive if population growth causes $k > k^* > \overline{k}$. Thus, equation (51) again reflects the Aaron-condition (Aaron, 1966).

The replicating portfolio of the government's tax claim can again be determined with contingent claim analysis:

Proposition 6 *The the present value of payable taxes under a DB pension scheme ($-H^{\psi}$) is economically equivalent to a portfolio with $-H^{\psi}$ invested in price indexed bonds.*
Proof. *See appendix F.* ■

Armed with propositions 5 and 6 and equation (46), we can now express the incremental *changes* in the optimal financial portfolio due to the introduction of the DB PAYG pension scheme as:

$$\alpha_1^{DB}(t) - \widehat{\alpha}_1(t) = (\alpha_1 - 1)\overline{\Theta}_1(t) + (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2}))\overline{\Theta}_2(t) \tag{52}$$

$$\alpha_2^{DB}(t) - \widehat{\alpha}_2(t) = \alpha_2\overline{\Theta}_1(t) + (\alpha_2 - \lambda_{w2})\overline{\Theta}_2(t) \tag{53}$$

$$\alpha_3^{DB}(t) - \widehat{\alpha}_3(t) = \alpha_3\overline{\Theta}_1(t) + (\alpha_3 - \lambda_{w1})\overline{\Theta}_2(t) \tag{54}$$

with

$$\overline{\Theta}_1(t) = \frac{H^{\overline{\theta}}(t) - H^{\psi}(t)}{F^{DB}(t)}$$

$$\overline{\Theta}_2(t) = \frac{H^{w}(t)}{F^{DB}(t)} - \frac{H^{w}(t)}{F(t)}$$

where α_i^{DB} denotes the optimal fraction of financial wealth invested in asset i under a DB PAYG pension scheme[12]. Equations (52)-(54) show that we can break up the incremental portfolio changes in two terms. In the first term, $\overline{\Theta}_1(t)$ reflects the direct effect of net DB PAYG pension wealth on the optimal composition of financial wealth. In the second term, $\overline{\Theta}_2(t)$ reflects an indirect

[11] See appendix F.
[12] See appendix F for the formal derivation.

effect arising in case of crowding out of financial wealth (i.e. $F^{DB}(t) < F(t)$). In that case, the incremental portfolio effects of human wealth as derived in the previous section are amplified. We will calibrate the model in the next section in order to quantify these incremental portfolio effects.

5 Calibration

In this section we will calibrate the model to get an idea of the quantitative magnitude of the effects as predicted by the model. Based on the revealing study by Dimson, Marsh and Staunton (2002) (DMS), providing long run statistical properties of different countries' asset market performance, we set the expected real return on price indexed bonds (r_1) equal to 1.5% and on equity (r_2) equal to 5.8% with a standard deviation (s_2) of 21%. Also based on DMS, the expected *nominal* return (R_3) on nominal bonds is set equal to 4.2%, while expected inflation (π) is set at 2%, with a standard deviation (s_1) of 5%. This yields an expected *real* return on nominal bonds (r_3) of 2.4%. The puzzling tendency of real equity returns to covary negatively with inflation has been extensively documented in the literature[13]. Marshall (1992) estimates the value for the correlation coefficient between real equity returns and inflation (ρ_{12}) at -0.29. De Jong (2005) estimates annual real wage growth in the Netherlands in the period 1950-2002 at 1.4% with a total standard deviation of 4.5%. Based on these estimates, we set expected real wage growth (ω) equal to 1.4%, s_{w1} equal to 2.3% and s_{w2} equal to 3.2% (yielding a total standard deviation of 4.5%). The values for the rate of relative risk aversion (γ) and time preference (σ) are derived from the literature review undertaken by Teulings and De Vries (2006) (TDV). Following TDV we set the retirement age (T) at 40 and the time of death (D) at 55.

Table 1: Parameter calibration

Asset	*Expected real return*	*Stand. dev.*
Price-indexed bonds	0.015	0
Equity	0.058	0.21
Nominal bonds	0.024	0.05
Inflation	0.02	0.05
Correlationcoefficient (ρ_{12})	-0.29	
Real wages	0.014	0.045
Preferences		
Risk aversion (γ)	5	
Time preference (δ)	0.02	
Life cycle		
Retirement age (T)	40	
Time of death (D)	55	

[13]See Marshall (1992) for an overview of the literature. The negative covariance suggests that supply shocks dominate demand shocks.

The optimal financial portfolio in the absence of human and PAYG pension wealth is summarized in table 2. Approximately a quarter of financial wealth is invested in price-indexed bonds and equity respectively, and the remaining half in nominal bonds. As demonstrated before, the incremental change in the optimal financial portfolio in case of human and PAYG pension wealth depends *inter alia* on the sensitivity of real wages to economic shocks. The relative sensitivity of real wages to inflationary shocks (λ_{w1}) equals 0.46, implying that $\alpha_3 > \lambda_{w1}$. The relative sensitivity of real wages to productivity shocks (λ_{w2}) equals 0.19, implying that $\alpha_2 > \lambda_{w2}$. As a consequence, $1 - \lambda_{w1} - \lambda_{w2}$ equals 0.35, implying that $\alpha_1 < 1 - \lambda_{w1} - \lambda_{w2}$. The value of μ_w, the discount factor used to calculate *inter alia* the value of human wealth, equals 0.013. This implies the economy under consideration is dynamically efficient.

Table 2: optimal portfolio and implied values

Optimal portfolio	*Fraction*
Price-indexed bonds (α_1)	0.28
Equity (α_2)	0.26
Nominal bonds (α_3)	0.47
Implied values	
λ_{w1}	0.46
λ_{w2}	0.19
$1 - \lambda_{w1} - \lambda_{w2}$	0.35
μ_w	0.013

We ran the model 10000 times to simulate the impact of human and pay-as-you-go pension wealth (both DC and DB) on optimal portfolio and consumption behaviour. We set initial financial wealth equal to 1 and initial wages equal to 20. Below, we will discuss the *average* outcomes of our simulations in terms of wealth dynamics, optimal consumption and the incremental change in the optimal financial portfolio.

Figure 1 shows the wealth dynamics when there is only human wealth (section 3). Human wealth is gradually depleted until the retirement age, while financial wealth follows the typical hump-shaped pattern as predicted by standard life cycle theory (Ando and Modigliani, 1963). As a result, total wealth increases until midlife and is depleted thereafter. Consumption develops smoothly over life. However, at the end of life consumption steeply increases as the remaining financial wealth (partly being precautionary savings for unexpected shocks) is depleted.

Figure 2 shows the incremental effects on the optimal financial portfolio due to the inclusion of human wealth in overall wealth (derived in equations (30)-(32)). The fraction of financial wealth invested in equity and (too a lesser degree) nominal bonds is increased, while the fraction invested in price indexed bonds is decreased. Put differently, our simulations predict that an individual will invest less conservatively due to human wealth. These portfolio effects peak at birth, when the ratio of human wealth over financial wealth is at its maximum, and gradually decline thereafter.

Figure 1: Wealth dynamics with human wealth

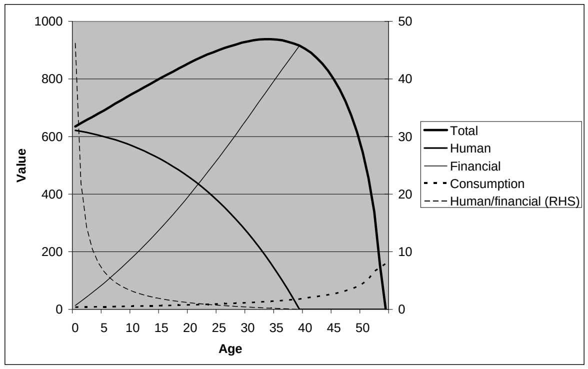
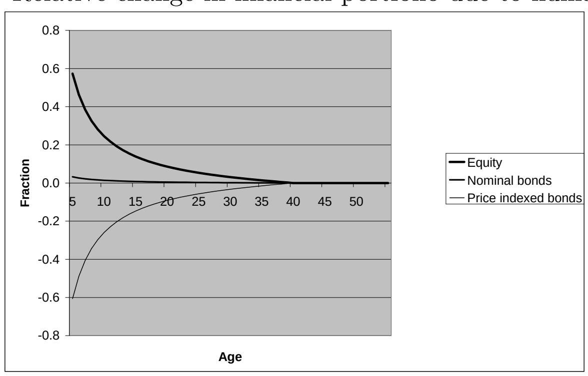


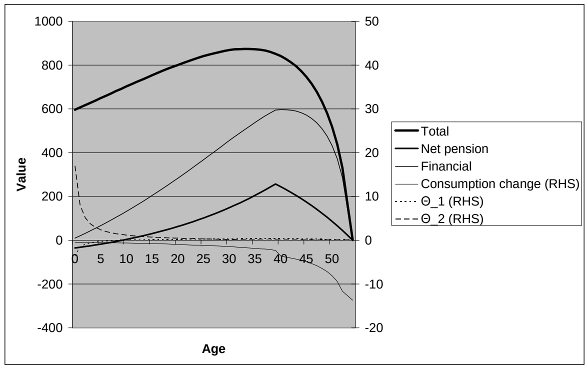
Figure 2: Relative change in financial portfolio due to human wealth



Now we will consider the portfolio effects of a DC PAYG pension scheme as modelled in section 4.1 with a tax rate of 20% (i.e. $\tau = 0.2$). Assuming no population growth, the support ratio k equals $\overline{k} = T/(D-T)$=2.7. Using equation (37), we can calculate that k^{*}=3.8 implying $k = \overline{k} < k^{*}$: net pension wealth at birth is negative in this case. Figure 3 shows the wealth dynamics over life. As predicted net pension wealth at birth is indeed negative. However, it gradually increases in value until the retirement age and gradually falls thereafter. Financial wealth again follows the typical hump-shaped pattern as predicted by life-cycle theory.

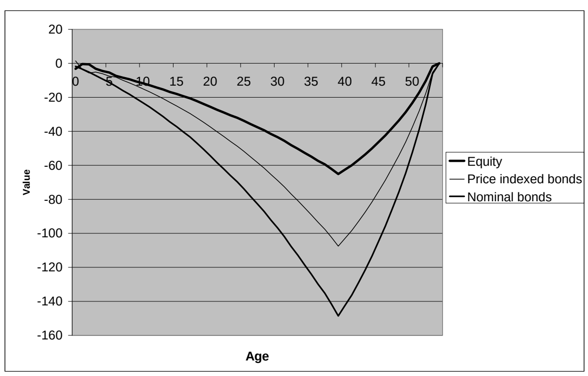
However, comparing figures 1 and 3 shows that financial savings over lifetime fall as a consequence of the DC PAYG pension scheme. Apart from reducing lifetime consumption (figure 3), this financial crowding out also implies a reduction in the absolute investments in all three assets (figure 4). In fact, these reductions mirror the implicit portfolio inherent to the DC PAYG pension scheme. It can be shown that there is complete neutrality if net DC PAYG pension wealth

Figure 3: Wealth dynamics with DC PAYG pension wealth



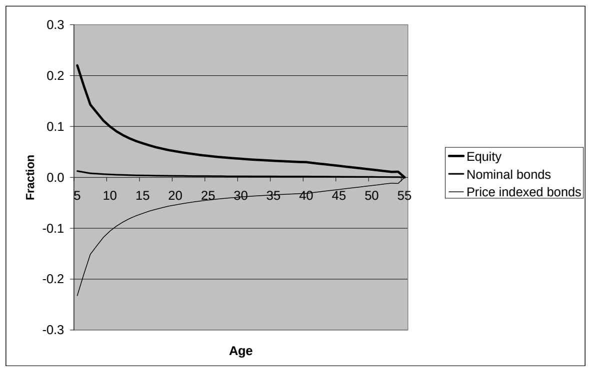
is zero at birth (i.e. if $k = k^*$). In that case, the implicit portfolio arising due to the DC PAYG pension scheme is exactly mirrored by the absolute change in the financial portfolio.

Figure 4: Absolute changes in financial portfolio due to DC PAYG pension scheme



Using the analysis of section 4.1, we will now consider the relative changes in the financial portfolio. The term guiding the incremental portfolio effects in equations (39)-(41) $(\Theta_1(t) + \Theta_2(t))$ remains positive over life. Put differently, the initial direct effect of negative net DC PAYG pension wealth $(\Theta_1(t))$ is dominated by the positive indirect crowding out effect $(\Theta_2(t))$. Consequently, the DC PAYG pension scheme amplifies the incremental portfolio changes related to human wealth. Hence, the fraction of financial wealth invested in equity and (too a lesser degree) nominal bonds is increased, while the fraction invested in price indexed bonds is decreased (see figure 5). Summarizing, our simulations predict that an individual will invest his financial wealth less conservatively due to the DC PAYG pension scheme. These portfolio effects peak at birth and gradually disappear thereafter.

Figure 5: Relative changes in financial portfolio due to DC PAYG pension scheme



Instead of a DC PAYG pension scheme, we will now consider the portfolio effects of introducing a DB PAYG pension scheme as modelled in section 4.2. More specifically, we suppose the DB PAYG pension scheme provides retirees with a real pension benefit of 12 (i.e. a replacement rate of 60% in terms of initial real wage). Again assuming no population growth, the support ratio k equals $\overline{k} = T/(D-T)$=2.7. Using equation (51), we can calculate that k^{*}=4.1 implying $k = \overline{k} < k^{*}$. Hence, net DB pension wealth at birth will be negative in this case. Figure 6 shows the wealth dynamics in case of a DB PAYG pension scheme. Net pension wealth at birth is indeed negative as predicted above. However, it gradually increases in value until the retirement age and gradually falls thereafter. Financial wealth follow the hump-shaped pattern as predicted by life-cycle theory.

However, comparing figures 1 and 6 shows that financial savings over lifetime are reduced as a consequence of the DB PAYG pension scheme. Hence, just as with the DC PAYG pension scheme there is substantial crowding out of financial wealth. Apart from causing a reduction of lifetime consumption (figure 6), this crowding out of financial wealth also imply a change in absolute investments in all three assets (see figure 7). These changes mirror the implicit portfolio inherent to the DB PAYG pension scheme. It can be shown that there is complete neutrality if net DB PAYG pension wealth is zero at birth (i.e. if $k = k^{*}$). In that case, the implicit portfolio arising due to the DB PAYG pension scheme is exactly mirrored by the absolute change in the financial portfolio.

Using the analysis of section 4.2, we will now consider the relative changes in the financial portfolio. Figure 8 shows the changes optimal financial portfolio due to the introduction of the DB pension scheme as derived in equations (52)-(54). Early in life the fraction of financial wealth invested in price-indexed bonds and equity is increased, while the fraction invested in nominal bonds is

Figure 6: Wealth dynamics with DB pension wealth

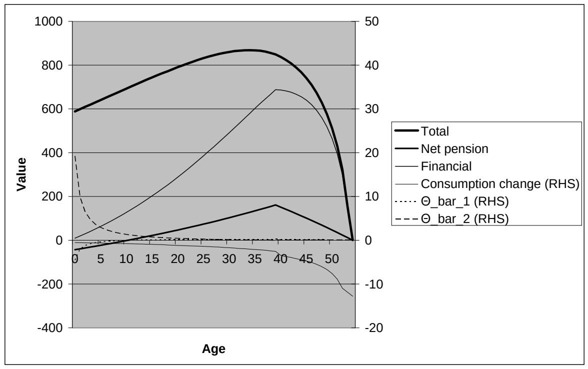
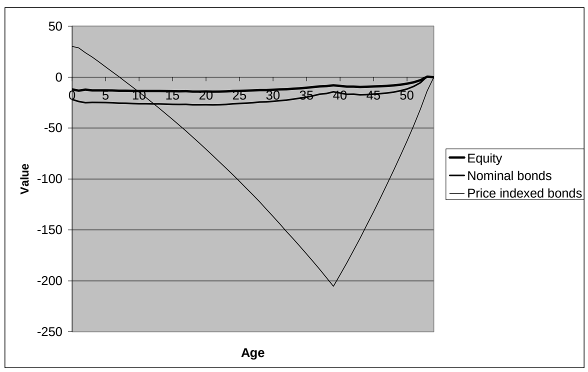
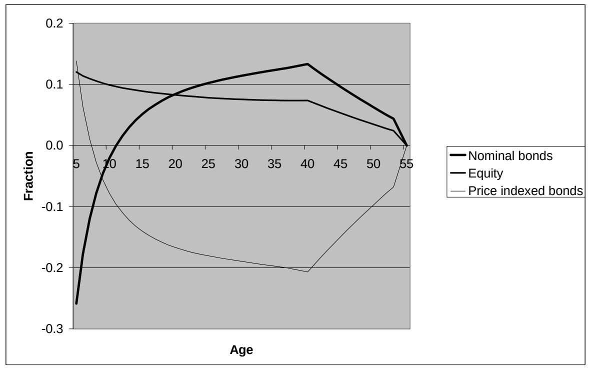


Figure 7: Absolute changes in financial portfolio due to DB PAYG pension scheme



decreased. However, for bonds these changes are reversed after a certain age as net pension wealth becomes positive and indirect crowding out effects gain importance. Consequently, ultimately the individual will invest less conservatively due the the introduction of the DB PAYG pension scheme.

Figure 8: Relative changes in financial portfolio due to DB PAYG pension scheme



6 Summary and conclusions

In this paper, we have analyzed the impact of a PAYG pension schemes on the optimal allocation of financial wealth. Our model contains some simplifications of reality. First of all, demographic risks due to changes in birth rates, longevity or migration are not taken into account. Moreover, the hold-up risk inherent to pay-as-you-go schemes is not included and important PAYG pension parameters as the tax rate (DC), the replacement rate (DB) and the retirement age are exogenously determined which might not be optimal for the individual. Besides, we take the existence of a PAYG pension scheme as given and do not justify it in welfare terms. Finally, we abstain from credit constraints in modelling the optimal investment behaviour of an individual. Although certainly being relevant, including these factors in the model would take us away from the main question we attempt to address in this paper.

We have shown that taking into account the replicating portfolios inherent to human wealth and pension wealth impacts on the individual's optimal investment portfolio. Introducing a pay-as-you-go pension scheme implies that human wealth of young generations is partly transferred to retired generations. As a consequence, individuals will in general invest less conservatively. These portfolio effects gradually disappear at the end of life. As pointed out by Bodie et al. (1992), ignoring human wealth constitutes an "omitted variable" problem. The same holds true for pension wealth from PAYG pension schemes. This is not only relevant for individuals saving and investing for their pension, but also for pension funds investing on behalf of the individual.

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A Optimal portfolio allocation

Inserting equation (12) in the Bellman equation (11):

$$0 = \max_{C,\alpha_i} \left\{ U\left[C\left(t\right), t\right] + \frac{1}{dt} E_t \left[J_W dW + J_t dt + \frac{1}{2} J_{WW} dW^2 \right] \right\}$$

In order to develop $E_t\left[\cdot\right]$, we can use the following multiplication rules for multivariate Itô-processes (Magill and Quinzy, 2002):

$$\begin{aligned} dz_i dt &= 0 \\ dz_i^2 &= dt \\ dz_1 dz_2 &= \rho_{12} dt \end{aligned}$$

with ρ_{12} being the instantaneous correlation coefficient between Wiener processes dz_1 and dz_2.

Applying these multiplication rules gives:

$$\begin{aligned} E_t\left[dW\right] &= \left[\left[r_1 + \alpha_2\left(r_2 - r_1\right) + \alpha_3\left(r_3 - r_1\right)\right] W - C\right] dt \\ E_t\left[dW^2\right] &= \left[\alpha_2^2 s_2^2 + \alpha_3^2 s_3^2 + 2\alpha_2\alpha_3 s_2 s_3 \rho_{12}\right] W^2 dt \end{aligned}$$

Inserting gives the Bellman-Dreyfus fundamental equation of optimality:

$$0 = \max_{C,\alpha_i} \left\{ \begin{array}{c} U\left(C,t\right) + J_W\left[\left[r_1 + \alpha_2\left(r_2 - r_1\right) + \alpha_3\left(r_3 - r_1\right)\right] W - C\right] + J_t \\ + \frac{1}{2} J_{WW}\left[\alpha_2^2 s_2^2 + \alpha_3^2 s_3^2 + 2\alpha_2\alpha_3 s_2 s_3 \rho_{12}\right] W^2 \end{array} \right\}$$

First-order conditions:

C:

$$0 = U_C - J_W$$

α_2:

$$0 = \frac{\partial}{\partial \alpha_2} \left[\begin{array}{c} J_W\left[\left[r_1 + \alpha_2\left(r_2 - r_1\right) + \alpha_3\left(r_3 - r_1\right)\right] W - C\right] + J_t \\ + \frac{1}{2} J_{WW}\left[\alpha_2^2 s_2^2 + \alpha_3^2 s_3^2 + 2\alpha_2\alpha_3 s_2 s_3 \rho_{12}\right] W^2 \end{array} \right]$$

Implicit solution:

$$\alpha_2 = -\frac{J_W}{J_{WW} W} \frac{r_2 - r_1}{s_2^2} - \frac{\alpha_3 \rho_{12} s_2 s_3}{s_2^2}$$

Same for α_3:

$$0 = \frac{\partial}{\partial \alpha_3} \left[\begin{array}{c} J_W\left[\left[r_1 + \alpha_2\left(r_2 - r_1\right) + \alpha_3\left(r_3 - r_1\right)\right] W - C\right] + J_t \\ + \frac{1}{2} J_{WW}\left[\alpha_2^2 s_2^2 + \alpha_3^2 s_3^2 + 2\alpha_2\alpha_3 s_2 s_3 \rho_{12}\right] W^2 \end{array} \right]$$

Implicit solution:

$$\alpha_3 = -\frac{J_W}{J_{WW}W}\frac{r_3 - r_1}{s_3^2} - \frac{\alpha_2 \rho_{12} s_2 s_3}{s_3^2}$$

Combining both implicit solutions gives explicit solutions for α_2 and α_3:

$$\begin{aligned}
\alpha_2 &= -\frac{J_W}{J_{WW}W}\left(\frac{(r_2 - r_1)}{(1-\rho_{12}^2)\, s_2^2} - \frac{(r_3 - r_1)\,\rho_{12}}{(1-\rho_{12}^2)\, s_2 s_3}\right) \\
\alpha_3 &= -\frac{J_W}{J_{WW}W}\left(\frac{(r_3 - r_1)}{(1-\rho_{12}^2)\, s_3^2} - \frac{(r_2 - r_1)\,\rho_{12}}{(1-\rho_{12}^2)\, s_2 s_3}\right)
\end{aligned}$$

Finally, using the stock budget constraint (5), we can derive:

$$\begin{aligned}
\alpha_1 &= 1 - \left(-\frac{J_W}{J_{WW}W}\left(\frac{(r_2 - r_1)}{(1-\rho_{12}^2)\, s_2^2} - \frac{(r_3 - r_1)\,\rho_{12}}{(1-\rho_{12}^2)\, s_2 s_3}\right)\right) \\
&\quad - \left(-\frac{J_W}{J_{WW}W}\left(\frac{(r_3 - r_1)}{(1-\rho_{12}^2)\, s_3^2} - \frac{(r_2 - r_1)\,\rho_{12}}{(1-\rho_{12}^2)\, s_2 s_3}\right)\right) \\
&= 1 + \frac{J_W}{J_{WW}W}\left(\frac{(r_2 - r_1)}{(1-\rho_{12}^2)\, s_2^2} - \frac{(r_3 - r_1)\,\rho_{12}}{(1-\rho_{12}^2)\, s_2 s_3}\right) + \frac{J_W}{J_{WW}W}\left(\frac{(r_3 - r_1)}{(1-\rho_{12}^2)\, s_3^2} - \frac{(r_2 - r_1)\,\rho_{12}}{(1-\rho_{12}^2)\, s_2 s_3}\right)
\end{aligned}$$

B Optimal consumption

We start with the trial solution:

$$J\left[W\left(t\right),t\right]=\frac{b\left(t\right)}{1-\gamma}\exp\left(-\delta t\right)\left[W\left(t\right)\right]^{1-\gamma}$$

implying:

$$\begin{aligned}J_W &= b\left(t\right)\exp\left(-\delta t\right)\left[W\left(t\right)\right]^{-\gamma}\\ J_W &= \left(1-\gamma\right)\frac{J}{W}\end{aligned}$$

and:

$$\begin{aligned}J_{WW} &= \frac{\left(1-\gamma\right)^2 J}{W^2}-\frac{\left(1-\gamma\right)J}{W^2}\\ J_{WW} &= -\frac{J}{W^2}\gamma\left(1-\gamma\right)\end{aligned}$$

and:

$$J_t=\left[\frac{b_t}{b}-\delta+\left(1-\gamma\right)\frac{W_t}{W}\right]J$$

Inserting J_t in the Bellman-Dreyfus fundamental equation of optimality (11) gives:

$$\begin{aligned}0 &= \exp\left[-\delta t\right]\frac{\left[C^*\right]^{1-\gamma}}{1-\gamma}+J_W\left[\left[r_1+\alpha_2^*\left(r_2-r_1\right)+\alpha_3^*\left(r_3-r_1\right)\right]W-C^*\right]+\left[\frac{b_t}{b}-\delta+\left(1-\gamma\right)\frac{W_t}{W}\right]J\\ &\quad +\frac{1}{2}J_{WW}\left[\left(\alpha_2^*\right)^2 s_2^2+\left(\alpha_3^*\right)^2 s_3^2+2\left(\alpha_2^*\right)\left(\alpha_3^*\right)s_2 s_3\rho_{12}\right]W^2\end{aligned}$$

Recall:

$$E_t\left[W_t\right]=\left[r_1+\alpha_2\left(r_2-r_1\right)+\alpha_3\left(r_3-r_1\right)\right]W-C^*$$

Inserting gives:

$$\begin{aligned}0 &= \exp\left[-\delta t\right]\frac{\left[C^*\right]^{1-\gamma}}{1-\gamma}+\left[-J_W-\left(1-\gamma\right)\frac{J}{W}\right]C^*+J_W W\left[r_1+\alpha_2^*\left(r_2-r_1\right)+\alpha_3^*\left(r_3-r_1\right)\right]\\ &\quad +\left[\frac{b_t}{b}-\delta+\left(1-\gamma\right)\left[r_1+\alpha_2\left(r_2-r_1\right)+\alpha_3\left(r_3-r_1\right)\right]\right]J\\ &\quad +\frac{1}{2}J_{WW}\left[\left(\alpha_2^*\right)^2 s_2^2+\left(\alpha_3^*\right)^2 s_3^2+2\left(\alpha_2^*\right)\left(\alpha_3^*\right)s_2 s_3\rho_{12}\right]W^2\end{aligned}$$

Recall the first order conditions:

$$
\begin{aligned}
0 &= U_C - J_W \\
\alpha_2 &= -\frac{J_W}{J_{WW} W}\left(\frac{(r_2 - r_1)}{(1-\rho_{12}^2) s_2^2} - \frac{(r_3 - r_1)\rho_{12}}{(1-\rho_{12}^2) s_2 s_3}\right) \\
\alpha_3 &= -\frac{J_W}{J_{WW} W}\left(\frac{(r_3 - r_1)}{(1-\rho_{12}^2) s_1^2} - \frac{(r_2 - r_1)\rho_{12}}{(1-\rho_{12}^2) s_2 s_3}\right)
\end{aligned}
$$

implying:

$$
\begin{aligned}
C^*(t) &= [J_W \exp[\delta t]]^{-\frac{1}{\gamma}} \\
\alpha_2^* &= -\frac{J_W A_2}{J_{WW} W} \\
\alpha_3^* &= -\frac{J_W A_3}{J_{WW} W}
\end{aligned}
$$

for

$$
\begin{aligned}
A_2 &= \frac{(r_2 - r_1)}{(1-\rho_{12}^2) s_2^2} - \frac{(r_3 - r_1)\rho_{12}}{(1-\rho_{12}^2) s_2 s_3} \\
A_3 &= \frac{(r_3 - r_1)}{(1-\rho_{12}^2) s_3^2} - \frac{(r_2 - r_1)\rho_{12}}{(1-\rho_{12}^2) s_2 s_3}
\end{aligned}
$$

Inserting $C^*, \alpha_2^*, \alpha_3^*$ gives:

$$
\begin{aligned}
0 = {} & \left[\frac{\gamma}{1-\gamma} J_W - (1-\gamma)\frac{J}{W}\right][J_W \exp[\delta t]]^{-\frac{1}{\gamma}} + J_W W r_1 + \left[\frac{b_t}{b} - \delta\right] J \\
& -(1-\gamma)\frac{J_W J}{J_{WW} W}[r_1 + (r_2 - r_1) A_2 + (r_3 - r_1) A_3] \\
& + \frac{(J_W)^2}{J_{WW}}\left[\frac{1}{2} s_2^2 A_2^2 + \frac{1}{2} s_3^2 A_3^2 + A_2 A_3 s_2 s_3 \rho_{12} - (r_2 - r_1) A_2 - (r_3 - r_1) A_3\right]
\end{aligned}
$$

Inserting J_W and J_{WW} gives:

$$
\begin{aligned}
0 = {} & -(1-2\gamma)\frac{1}{W}\left[\left((1-\gamma)\frac{J}{W}\right)\exp[\delta t]\right]^{-\frac{1}{\gamma}} + \frac{b_t}{b} - \delta \\
& -\frac{(1-\gamma)}{\gamma}\left[\frac{1}{2} s_2^2 A_2^2 + \frac{1}{2} s_3^2 A_3^2 + A_2 A_3 s_2 s_3 \rho_{12} - (r_2 - r_1) A_2 - (r_3 - r_1) A_3\right] \\
& + (1-\gamma) r_1 + \frac{1-\gamma}{\gamma}[r_1 + (r_2 - r_1) A_2 + (r_3 - r_1) A_3]
\end{aligned}
$$

Finally, inserting J yields:

$$0 = -(1-2\gamma)[b]^{-\frac{1}{\gamma}} + \frac{b_t}{b} - \delta - \frac{(1-\gamma)}{\gamma}Y + \frac{1-\gamma^2}{\gamma}r_1$$

with:

$$Y = \frac{1}{2}s_2^2A_2^2 + \frac{1}{2}s_3^2A_3^2 + A_2A_3s_2s_3\rho_{12} - 2(r_2 - r_1)A_2 - 2(r_3 - r_1)A_3$$

Implying:

$$\frac{db}{dt} = (1-2\gamma)b^{-\frac{1-\gamma}{\gamma}} + bX$$

with:

$$X = \delta + \frac{1-\gamma}{\gamma}Y - \frac{1-\gamma^2}{\gamma}r_1$$

and

$$C^*(t) = [b(t)]^{-\frac{1}{\gamma}}W$$

$$\alpha_2^* = \frac{1}{\gamma}A_2$$

$$\alpha_3^* = \frac{1}{\gamma}A_3$$

Implying:

$$\begin{aligned}\alpha_1^* &= 1 - \alpha_2 - \alpha_3 \\ &= 1 - \frac{1}{\gamma}(A_2 + A_3)\end{aligned}$$

A solution for the ordinary differential equation $\frac{db}{dt}$ is:

$$b(t) = \left(\frac{(2\gamma-1)\left[1-\exp\left(\frac{t-D}{\gamma}X\right)\right]}{X}\right)^{\gamma}$$

As $b(D) = 0$, this solution also satisfies the transversality condition $J[b(D), W(D), D] = 0$.

C Valuing wealth H^x from income x

We will determine the value of human wealth applying a similar approach as Bodie et al. (1992). However, we will do this by presenting a general method to determine the wealth of stochastic income x (based on Merton (1990), section 12.2) as this allows us to apply this method to other stochastic incomes as well.

Suppose x follows a geometric Brownian motion with drift χ and diffusion terms $s_{x1}dz_1$ and $s_{x2}dz_2$:

$$\frac{dx}{x} = \chi dt + s_{x1}dz_1 + s_{x2}dz_2$$

with dz_1 and dz_2 being the Wiener processes as defined before.
Define:

$$\lambda_{x1} = \frac{s_{x1}}{s_3}$$
$$\lambda_{x2} = \frac{s_{x2}}{s_2}$$

allowing us to express dx/x as

$$\frac{dx}{x} = \chi dt + \lambda_{x1}s_3dz_1 + \lambda_{x2}s_2dz_2$$

Recall:

$$\frac{d\left(Q_2/P\right)}{Q_2/P} = r_2dt + s_2dz_2$$

$$\frac{d\left(Q_3/P\right)}{Q_3/P} = r_3dt + s_3dz_1$$

implying:

$$s_2dz_2 = \frac{d\left(Q_2/P\right)}{\left(Q_2/P\right)} - r_2dt$$
$$s_3dz_1 = \frac{d\left(Q_3/P\right)}{\left(Q_3/P\right)} - r_3dt$$

Inserting gives:

$$\begin{aligned}\frac{dx}{x} &= \chi dt + \lambda_{x2}\left[\frac{d\left(Q_2/P\right)}{\left(Q_2/P\right)} - r_2dt\right] + \lambda_{x1}\left[\frac{d\left(Q_3/P\right)}{\left(Q_3/P\right)} - r_3dt\right] \\ &= \left(\chi - \lambda_{x1}r_3 - \lambda_{x2}r_2\right)dt + \lambda_{x1}\frac{d\left(Q_3/P\right)}{\left(Q_3/P\right)} + \lambda_{x2}\frac{d\left(Q_2/P\right)}{\left(Q_2/P\right)}\end{aligned}$$

Hence, we have found an expression for $\frac{dx}{x}$ as a function of $\frac{d(Q_3/P)}{(Q_3/P)}$ and $\frac{d(Q_2/P)}{(Q_2/P)}$. The next step is to find an expression for $x(t)$ as a function of (Q_3/P) and (Q_2/P):

Proposition 7 *$x(t)$ can be expressed as a function of the value of equity and nominal bonds:*

$$x(t) = x(0) \exp[\kappa_x t] \left[\frac{(Q_3/P)(t)}{(Q_3/P)(0)}\right]^{\lambda_{x1}} \left[\frac{(Q_2/P)(t)}{(Q_2/P)(0)}\right]^{\lambda_{x2}}$$

with

$$\kappa_x = \chi - \left[\lambda_{x1} r_3 + \lambda_{x2} r_2 - \frac{1}{2}\lambda_{x1}(1-\lambda_{x1}) s_3^2 - \frac{1}{2}\lambda_{x2}(1-\lambda_{x2}) s_2^2 + \lambda_{x1}\lambda_{x2}(r_3 r_2 + s_3 s_2 \rho_{12})\right]$$

Proof. We will proof that $x(t)$ is compatible with ■

$$\frac{dx}{x} = \chi dt + \lambda_{x1} s_3 dz_1 + \lambda_{x2} s_2 dz_2$$

as derived above.

Applying Itô's lemma to $x(t)$ gives:

$$\frac{dx}{x} = \lambda_{x1}\frac{d(Q_3/P)}{(Q_3/P)} + \lambda_{x2}\frac{d(Q_2/P)}{(Q_2/P)} + \kappa_x dt - \frac{1}{2}\lambda_{x1}(1-\lambda_{x1})\left[\frac{d(Q_3/P)}{(Q_3/P)}\right]^2 - \frac{1}{2}\lambda_{x2}(1-\lambda_{x2})\left[\frac{d(Q_2/P)}{(Q_2/P)}\right]^2 + \lambda_{x1}\lambda_{x2}\frac{d(Q_3/P)}{(Q_3/P)}\frac{d(Q_2/P)}{(Q_2/P)}$$

We can derive:

$$\left[\frac{d(Q_3/P)}{(Q_3/P)}\right]^2 = s_3^2 dt$$

and:

$$\left[\frac{d(Q_2/P)}{(Q_2/P)}\right]^2 = s_2^2 dt$$

and:

$$\frac{d(Q_3/P)}{(Q_3/P)}\frac{d(Q_2/P)}{(Q_2/P)} = (r_3 r_2 + s_3 s_2 \rho_{12}) dt$$

Inserting gives:

$$\frac{dx}{x} = \left[\lambda_{x1} r_3 + \lambda_{x2} r_2 - \frac{1}{2}\lambda_{x1}(1-\lambda_{x1}) s_3^2 - \frac{1}{2}\lambda_{x2}(1-\lambda_{x2}) s_2^2 + \lambda_{x1}\lambda_{x2}(r_3 r_2 + s_3 s_2 \rho_{12}) + \kappa_x\right] dt$$
$$+ \lambda_{x1} s_3 dz_1 + \lambda_{x2} s_2 dz_2$$
QED

Given that we have expressed $x(t)$ as a function of $(Q_2/P)(t)$, the next step is to express the wealth of $x(t)$ (i.e. $H^x(t)$) as a function G of $(Q_3/P)(t), (Q_2/P)(t)$ and t:

$$H^x(t) = G[(Q_3/P)(t), (Q_2/P)(t), t]$$

Suppose the investor receives an instantaneous payout x from H^x:

$$dH^x = (r_{H^x} H^x - x)\, dt + H^x \left(s_{H_1^x} dz_{H_1^x} + s_{H_2^x} dz_{H_2^x}\right)$$

Applying Itô's lemma on $G[(Q_3/P)(t), (Q_2/P)(t), t]$ gives:

$$\begin{aligned} dG &= G_{(Q_3/P)} d(Q_3/P) + G_{(Q_2/P)} d(Q_2/P) + G_t dt \\ &+\frac{1}{2}\left[G_{(Q_3/P)(Q_3/P)} d(Q_3/P) + G_{(Q_2/P)(Q_2/P)} d(Q_2/P) + 2G_{(Q_3/P)(Q_2/P)} d(Q_3/P)\, d(Q_2/P)\right] \end{aligned}$$

recall:

$$d(Q_3/P)^2 = s_3^2 (Q_3/P)^2 dt$$

$$d(Q_2/P)^2 = s_2^2 (Q_2/P)^2 dt$$

$$d(Q_3/P)\, d(Q_2/P) = (r_3 r_2 + s_3 s_2 \rho_{12})(Q_3/P)(Q_2/P)\, dt$$

Combining:

$$dG = \left[\begin{array}{c} G_{(Q_3/P)} r_3 (Q_3/P) + G_{(Q_2/P)} r_2 (Q_2/P) + G_t + \frac{1}{2} G_{(Q_3/P)(Q_3/P)} s_3^2 (Q_3/P)^2 \\ +\frac{1}{2} G_{(Q_2/P)(Q_2/P)} s_2^2 (Q_2/P)^2 + G_{(Q_3/P)(Q_2/P)} (r_3 r_2 + s_3 s_2 \rho_{12})(Q_3/P)(Q_2/P) \end{array}\right] dt$$
$$+ G_{(Q_3/P)} s_3 (Q_3/P)\, dz_1 + G_{(Q_2/P)} s_2 (Q_2/P)\, dz_2$$

Recall we can express:

$$dH^x = [r_{H^x} H^x - x]\, dt + s_{H_1^x} H^x dz_{H_1^x} + s_{H_2^x} H^x dz_{H_2^x}$$

By definition:

$$dH^x(t) = dG[(Q_3/P)(t), (Q_2/P)(t), t]$$

implying the equalities:

$$
\begin{aligned}
r_{H^x} H^x &\equiv G_{(Q_3/P)} r_3 (Q_3/P) + G_{(Q_2/P)} r_2 (Q_2/P) + G_t + \frac{1}{2} G_{(Q_3/P)(Q_3/P)} s_3^2 (Q_3/P)^2 \\
&+ \frac{1}{2} G_{(Q_2/P)(Q_2/P)} s_2^2 (Q_2/P)^2 + G_{(Q_3/P)(Q_2/P)} (r_3 r_2 + s_3 s_2 \rho_{12}) (Q_3/P)(Q_2/P) + x \\
s_{H_1^x} H^x &\equiv G_{(Q_3/P)} s_3 (Q_3/P) \\
dz_{H_1^x} &\equiv dz_1 \\
s_{H_2^x} H^x &\equiv G_{(Q_2/P)} s_2 (Q_2/P) \\
dz_{H_2^x} &\equiv dz_2
\end{aligned}
$$

Now, suppose we construct a zero net investment portfolio with β_1 the number of euro's invested in Q_1 (indexed bonds), β_2 the number of euro's invested in Q_2 (equity), β_3 the number of euro's invested in Q_3 (riskless asset) respectively) and β_4 the number of euro's invested in H^x. Zero net investment implies:

$$
\begin{aligned}
\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0 \Leftrightarrow \\
\beta_1 = -\beta_2 - \beta_3 - \beta_4
\end{aligned}
$$

Now, the real return of this portfolio can be expressed as:

$$
\begin{aligned}
&\beta_1 \frac{d(Q_1/P)}{Q_1/P} + \beta_2 \frac{d(Q_2/P)}{Q_2/P} + \beta_3 \frac{d(Q_3/P)}{Q_3/P} + \beta_4 \frac{dH^x + x dt}{H^x} \\
&= [\beta_2 (r_2 - r_1) + \beta_3 (r_3 - r_1) + \beta_4 (r_{H^x} - r_1)] dt + \left(\beta_4 s_{H_1^x} + \beta_3 s_3\right) dz_1 \\
&+ \left(\beta_4 s_{H_2^x} + \beta_2 s_2\right) dz_2
\end{aligned}
$$

where we used the identities:

$$
\begin{aligned}
dz_{H_1^x} &\equiv dz_1 \\
dz_{H_2^x} &\equiv dz_2
\end{aligned}
$$

Zero inflation risk implies

$$\beta_3 = -\beta_4 \frac{s_{H_1^x}}{s_3}$$

while zero productivity risk implies

$$\beta_2 = -\beta_4 \frac{s_{H_2^x}}{s_2}$$

No arbitrage implies:

$$\beta_2 (r_2 - r_1) + \beta_3 (r_3 - r_1) + \beta_4 (r_{H^x} - r_1) = 0$$

Note that we have now three equalities and three unknowns (β_2,β_3, β_4). Combining gives:

$$\frac{-s_3 (r_2 - r_1) s_{H_2^x} H^x - s_2 (r_3 - r_1) s_{H_1^x} H^x}{s_3 s_2} = r_1 H^x - r_{H^x} H^x$$

Implying:

$$\begin{aligned}
&\frac{s_3 (r_2 - r_1) G_{(Q_2/P)} s_2 (Q_2/P) + s_2 (r_3 - r_1) G_{(Q_3/P)} s_3 (Q_3/P)}{s_3 s_2} \\
&= G_{(Q_3/P)} r_3 (Q_3/P) + G_{(Q_2/P)} r_2 (Q_2/P) + G_t \\
&+ \frac{1}{2} G_{(Q_3/P)(Q_3/P)} s_3^2 (Q_3/P)^2 + \frac{1}{2} G_{(Q_2/P)(Q_2/P)} s_2^2 (Q_2/P)^2 \\
&+ G_{(Q_3/P)(Q_2/P)} (r_3 r_2 + s_3 s_2 \rho_{12}) (Q_3/P) (Q_2/P) + x - r_1 G \Longleftrightarrow
\end{aligned}$$

$$\begin{aligned}
0 &= G_{(Q_3/P)} r_1 (Q_3/P) + G_{(Q_2/P)} r_1 (Q_2/P) + G_t \\
&+ \frac{1}{2} G_{(Q_3/P)(Q_3/P)} s_3^2 (Q_3/P)^2 + \frac{1}{2} G_{(Q_2/P)(Q_2/P)} s_2^2 (Q_2/P)^2 \\
&+ G_{(Q_3/P)(Q_2/P)} (r_3 r_2 + s_3 s_2 \rho_{12}) (Q_3/P) (Q_2/P) + x - r_1 G
\end{aligned}$$

This is the fundamental partial differential equation (fundamental PDE) G has to satisfy. However, given that $H^x(t)$ is by definition equal to $G[(Q_3/P)(t), (Q_2/P)(t), t]$, $H^x(t)$ also has to satisfy the fundamental PDE.

Proposition 8

$$H^x[x(t), t] = \frac{x(t)}{\mu_x} [1 - \exp[-\mu_x (D - t)]]$$

with

$$\mu_x = r_1 + \lambda_{x2} (r_2 - r_1) + \lambda_{x1} (r_3 - r_1) - \chi$$

Proof. Recall: ■

$$x(t) = x(0) \exp[\kappa_x t] \left[\frac{(Q_3/P)(t)}{(Q_3/P)(0)}\right]^{\lambda_{x1}} \left[\frac{(Q_2/P)(t)}{(Q_2/P)(0)}\right]^{\lambda_{x2}}$$

with

$$\kappa_x = \chi - \left[\lambda_{x1} r_3 + \lambda_{x2} r_2 - \frac{1}{2} \lambda_{x1} (1 - \lambda_{x1}) s_3^2 - \frac{1}{2} \lambda_{x2} (1 - \lambda_{x2}) s_2^2 + \lambda_{x1} \lambda_{x2} (r_3 r_2 + s_3 s_2 \rho_{12})\right]$$

implying:

$$H^x[(Q_2/P)(t), (Q_3/P)(t), t] = \frac{x(0)}{\mu_x} \exp[\kappa_x t] [1 - \exp[-\mu_x (D - t)]] \left[\frac{(Q_3/P)(t)}{(Q_3/P)(0)}\right]^{\lambda_{x1}} \left[\frac{(Q_2/P)(t)}{(Q_2/P)(0)}\right]^{\lambda_{x2}}$$

H^x should satisfy the fundamental PDE:

$$\begin{aligned}0 \quad = \quad & H^x_{(Q_3/P)} r_1 (Q_3/P) + H^x_{(Q_2/P)} r_1 (Q_2/P) + H^x_t + \frac{1}{2} H^x_{(Q_3/P)(Q_3/P)} s_3^2 (Q_3/P)^2 \\ & + \frac{1}{2} H^x_{(Q_2/P)(Q_2/P)} s_2^2 (Q_2/P)^2 + H^x_{(Q_3/P)(Q_2/P)} (r_3 r_2 + s_3 s_2 \rho_{12}) (Q_3/P)(Q_2/P) + x - r_1 H^x\end{aligned}$$

We can derive:

$$\begin{aligned}H^x_{(Q_3/P)} &= \frac{\lambda_{x1}}{(Q_3/P)} H^x \\ H^x_{(Q_2/P)} &= \frac{\lambda_{x2}}{(Q_2/P)} H^x \\ H^x_{(Q_2/P)(Q_2/P)} &= -\frac{\lambda_{x2}}{(Q_2/P)^2} H^x + \frac{\lambda_{x2}^2}{(Q_2/P)^2} H^x \\ &= \frac{H^x}{(Q_2/P)^2} [-\lambda_{x2} (1 - \lambda_{x2})] \\ H^x_{(Q_3/P)(Q_3/P)} &= -\frac{\lambda_{x1}}{(Q_3/P)^2} H^x + \frac{\lambda_{x1}^2}{(Q_3/P)^2} H^x \\ &= \frac{H^x}{(Q_3/P)^2} [-\lambda_{x1} (1 - \lambda_{x1})] \\ H^x_{(Q_2/P)(Q_3/P)} &= \frac{\lambda_{x2}}{(Q_2/P)} \frac{\lambda_{x1}}{(Q_3/P)} H^x \\ H^x_t &= \frac{x(0)}{\mu_x} \left[\frac{(Q_2/P)(t)}{(Q_2/P)(0)}\right]^{\lambda_{x2}} \left[\frac{(Q_3/P)(t)}{(Q_3/P)(0)}\right]^{\lambda_{x1}} \\ &\times \{\kappa_x \exp[\kappa_x t] [1 - \exp[-\mu_x (D - t)] + \exp[\kappa_x t] [-\exp[-\mu_x (D - t)]] \mu_x]\} \\ &= \frac{x(0)}{\mu_x} \exp[\kappa_x t] [1 - \exp[-\mu_x (D - t)]] \left[\frac{(Q_2/P)(t)}{(Q_2/P)(0)}\right]^{\lambda_{x2}} \left[\frac{(Q_3/P)(t)}{(Q_3/P)(0)}\right]^{\lambda_{x1}} \\ &\times \left\{\kappa_x - \mu_x \frac{\exp[-\mu_x (D - t)]}{1 - \exp[-\mu_x (D - t)]}\right\} \\ &= H^x \left\{\kappa_x - \mu_x \frac{\exp[-\mu_x (D - t)]}{1 - \exp[-\mu_x (D - t)]}\right\}\end{aligned}$$

Inserting in the fundamental PDE gives:

$$
\begin{aligned}
0 &= H^x_{(Q_3/P)} r_1 (Q_3/P) + H^x_{(Q_2/P)} r_1 (Q_2/P) + H^x_t + \frac{1}{2} H^x_{(Q_3/P)(Q_3/P)} s_3^2 (Q_3/P)^2 \\
&+ \frac{1}{2} H^x_{(Q_2/P)(Q_2/P)} s_2^2 (Q_2/P)^2 + H^x_{(Q_3/P)(Q_2/P)} (r_3 r_2 + s_3 s_2 \rho_{12}) (Q_3/P)(Q_2/P) + x - r_1 H^x \Longleftrightarrow \\
x(t) &= H^x \left\{ \begin{array}{c} \frac{\mu_x \exp[-\mu_x (D-t)]}{1-\exp[-\mu_x (D-t)]} + (1 - \lambda_{x2} - \lambda_{x1}) r_1 + \frac{1}{2} \lambda_{x2} (1 - \lambda_{x2}) s_2^2 + \frac{1}{2} \lambda_{x1} (1 - \lambda_{x1}) s_3^2 \\ -\lambda_{x1} \lambda_{x2} (r_3 r_2 + s_3 s_2 \rho_{12}) - \kappa_x \end{array} \right\} \Leftrightarrow \\
x(t) &= \frac{x(t)}{\mu_x} \left\{ (1 - \lambda_{x2} - \lambda_{x1}) r_1 + \frac{1}{2} \lambda_{x2} (1 - \lambda_{x2}) s_2^2 + \frac{1}{2} \lambda_{x1} (1 - \lambda_{x1}) s_3^2 - \lambda_{x1} \lambda_{x2} (r_3 r_2 + s_3 s_2 \rho_{12}) - \kappa_x \right\} \Leftrightarrow \\
\mu_x &= (1 - \lambda_{x2} - \lambda_{x1}) r_1 + \frac{1}{2} \lambda_{x2} (1 - \lambda_{x2}) s_2^2 + \frac{1}{2} \lambda_{x1} (1 - \lambda_{x1}) s_3^2 - \lambda_{x1} \lambda_{x2} (r_3 r_2 + s_3 s_2 \rho_{12}) - \kappa_x \\
&= r_1 + \lambda_{x2} (r_2 - r_1) + \lambda_{x1} (r_3 - r_1) - \chi \\
& QED
\end{aligned}
$$

Replacing x by w, χ by ω, κ_x by κ_w and λ_x by $\lambda_w \left(= \frac{s_w}{s_2}\right)$ in propositions 7 and 8 gives:

$$
w(t) = w(0) \exp[\kappa_w t] \left[\frac{(Q_2/P)(t)}{(Q_2/P)(0)} \right]^{\lambda_{w2}} \left[\frac{(Q_3/P)(t)}{(Q_3/P)(0)} \right]^{\lambda_{w3}}
$$

with

$$
\kappa_w = \omega - \left[\lambda_{w1} r_3 + \lambda_{w2} r_2 - \frac{1}{2} \lambda_{w1} (1 - \lambda_{w1}) s_3^2 - \frac{1}{2} \lambda_{w2} (1 - \lambda_{w2}) s_2^2 + \lambda_{w1} \lambda_{w2} (r_3 r_2 + s_3 s_2 \rho_{12}) \right]
$$

and

$$
H^w [w(t), t] = \frac{w(t)}{\mu_w} \left[1 - \exp\left[-\mu_w \left(\widehat{t} - t \right) \right] \right]
$$

with

$$
\widehat{t} = \max[t, T]
$$

$$
\mu_w = r_1 + \lambda_{w2} (r_2 - r_1) + \lambda_{w1} (r_3 - r_1) - \omega
$$

$$
\lambda_{w1} = \frac{s_{w1}}{s_3}
$$

$$
\lambda_{w2} = \frac{s_{w2}}{s_2}
$$

D Contingent claim analysis

Now that we have derived an expression to value wealth from $x(t)$, the next step is to find a replication portfolio for H^x. We will do this by applying the contingent claim analysis as outlined in section 13.2 of Merton (1990).

Suppose we follow an investment strategy where we invest $G_{(Q_2/P)}(Q_2/P)$ euro's of H^x in equity, $G_{(Q_3/P)}(Q_3/P)$ euro's of H^x in nominal bonds and $H^x - G_{(Q_2/P)}(Q_2/P) - G_{(Q_3/P)}(Q_3/P)$ euro's of H^x in price indexed bonds. Moreover, suppose the investor receives an instantaneous payout $D(t)$ from this portfolio. We will prove that this investment and payout strategy is consistent with our definition of $G[(Q_2/P)(t), (Q_3/P)(t), t]$.

The dynamics of this portfolio are:

$$dH^x = G_{(Q_2/P)} d(Q_2/P) + G_{(Q_3/P)} d(Q_3/P) + \left(H^x - G_{(Q_2/P)}(Q_2/P) - G_{(Q_3/P)}(Q_3/P)\right) r_1 dt - D(t)\, dt$$

Recall that Itô's lemma implies:

$$\begin{aligned} dG &= G_{(Q_3/P)} d(Q_3/P) + G_{(Q_2/P)} d(Q_2/P) + G_t dt \\ &\quad + \frac{1}{2}\left[\begin{array}{c} G_{(Q_3/P)(Q_3/P)} s_3^2 (Q_3/P)^2 + G_{(Q_2/P)(Q_2/P)} s_2^2 (Q_2/P)^2 \\ +2G_{(Q_3/P)(Q_2/P)} (r_3 r_2 + s_3 s_2 \rho_{12}) (Q_3/P)(Q_2/P) \end{array}\right] dt \end{aligned}$$

while from the fundamental PDE we know:

$$\begin{aligned} & G_t + \frac{1}{2}\left[\begin{array}{c} G_{(Q_3/P)(Q_3/P)} s_3^2 (Q_3/P)^2 + G_{(Q_2/P)(Q_2/P)} s_2^2 (Q_2/P)^2 \\ +2G_{(Q_3/P)(Q_2/P)} (r_3 r_2 + s_3 s_2 \rho_{12}) (Q_3/P)(Q_2/P) \end{array}\right] \\ = & -G_{(Q_3/P)} r_1 (Q_3/P) - G_{(Q_2/P)} r_1 (Q_2/P) + r_1 G - x \end{aligned}$$

Combining gives:

$$dG = G_{(Q_2/P)} d(Q_2/P) + G_{(Q_3/P)} d(Q_3/P) + \left[-G_{(Q_2/P)} r_1 (Q_2/P) - G_{(Q_3/P)} r_1 (Q_3/P) + r_1 G - x\right] dt$$

This implies that:

$$\begin{aligned} dH^x - dG &= G_{(Q_2/P)} d(Q_2/P) + G_{(Q_3/P)} d(Q_3/P) + \left(H^x - G_{(Q_2/P)}(Q_2/P) - G_{(Q_3/P)}(Q_3/P)\right) r_1 dt \\ &\quad - D(t)\, dt - G_{(Q_2/P)} d(Q_2/P)\, dt + G_{(Q_3/P)} d(Q_3/P)\, dt \\ &\quad + \left[-G_{(Q_2/P)} r_1 (Q_2/P) - G_{(Q_3/P)} r_1 (Q_3/P) + r_1 G - x\right] dt \end{aligned}$$

Now, suppose that $D(t) = x(t)$ implying that our portfolio fully replicates the real returns of wealth H^x. In that case:

$$dH^x - dG = (H^x - G)\, r_1 dt$$

Implying:

$$H^x(t) - G[(Q_2/P)(t), (Q_3/P)(t), t] = [H^x(0) - G(0)] \exp[r_1 t]$$

For $H^x(0) = G(0)$:

$$H^x(t) = G[(Q_2/P)(t), (Q_3/P)(t), t]$$

Hence, following the dynamic investment strategy were we invest $G_{(Q_2/P)}(Q_2/P)$ euro's of H^x in equity, $G_{(Q_3/P)}(Q_3/P)$ euro's of H^x in nominal bonds and $H^x - G_{(Q_2/P)}(Q_2/P) - G_{(Q_3/P)}(Q_3/P)$ euro's of H^x in price indexed bonds, is consistent with our definition of $G[(Q_2/P)(t), (Q_3/P)(t), t]$ .We know:

$$\begin{aligned} \frac{G_{(Q_2/P)}(Q_2/P)}{H^x} &= \frac{H^x_{(Q_2/P)}(Q_2/P)}{H^x} \\ &= \lambda_{x2} \end{aligned}$$

and:

$$\begin{aligned} \frac{G_{(Q_3/P)}(Q_3/P)}{H^x} &= \frac{H^x_{(Q_3/P)}(Q_3/P)}{H^x} \\ &= \lambda_{x1} \end{aligned}$$

and:

$$\begin{aligned} 1 - \frac{G_{(Q_2/P)}(Q_2/P)}{H^x} - \frac{G_{(Q_3/P)}(Q_3/P)}{H^x} &= 1 - \frac{H^x_{(Q_2/P)}(Q_2/P)}{H^x} - \frac{G_{(Q_3/P)}(Q_3/P)}{H^x} \\ &= 1 - \lambda_{x2} - \lambda_{x1} \end{aligned}$$

Implication: the value of wealth from income x (H^x) is economically equivalent to a portfolio with $\lambda_{x2}H^x$ invested in equity, $\lambda_{x1}H^x$ invested in nominal bonds and $(1 - \lambda_{x1} - \lambda_{x2}) H^x$ invested in price indexed bonds.

Replacing x by w and χ by ω as defined in (25), λ_{x1} by $\lambda_{w1}\left(= \frac{s_{w1}}{s_3}\right)$ and λ_{x2} by $\lambda_{w2}\left(= \frac{s_{w2}}{s_2}\right)$, implies that the value of human wealth from wages w (H^w) is economically equivalent to a portfolio with $\lambda_{w2}H^w$ invested in equity, $\lambda_{w1}H^w$ invested in nominal bonds and $(1 - \lambda_{w1} - \lambda_{w2}) H^w$ invested in price indexed bonds (proposition 2).

E Valuing pension wealth from a DC pension scheme

Now, we apply the analysis of appendices C and D to value pension wealth from a defined contribution pension scheme. We start by determining an expression for $\theta(t)$.

Applying Itô's lemma to (34) (and dropping time index t):

$$\begin{aligned} \frac{d\theta}{\theta} &= \frac{1}{\theta}\left\{\theta_t dt + \theta_w dw + \frac{1}{2}\theta_{ww} dw^2\right\} \\ &= \frac{dw}{w} \end{aligned}$$

Recall:

$$\lambda_{w2} = \frac{s_{w2}}{s_2}$$

and

$$\lambda_{w1} = \frac{s_{w1}}{s_3}$$

Implying:

$$\frac{dw}{w} = \omega dt + \lambda_{w2} s_2 dz_2 + \lambda_{w1} s_3 dz_1$$

Inserting in the expression found for $\frac{d\theta}{\theta}$ gives:

$$\begin{aligned} \frac{d\theta}{\theta} &= \frac{dw}{w} + \frac{dk}{k} + \frac{dk}{k}\frac{dw}{w} \\ &= \omega dt + \lambda_{w2} s_2 dz_2 + \lambda_{w1} s_3 dz_1 \end{aligned}$$

Hence, the evolution of the DC pension benefit can be expressed as a geometric Brownian motion with drift ω and diffusion $\lambda_{w2} s_2 dz_2 + \lambda_{w3} s_1 dz_1$.

Replacing x by θ, χ by ω, κ_x by κ_θ, and λ_x by λ_w in propositions 7 and 8 of appendix C gives:

$$\theta(t) = \theta(0)\exp[\kappa_\theta t]\left[\frac{(Q_2/P)(t)}{(Q_2/P)(0)}\right]^{\lambda_w}$$

with

$$\kappa_\theta = \omega - \left[\lambda_{w1} r_3 + \lambda_{w2} r_2 - \frac{1}{2}\lambda_{w1}(1-\lambda_{w1}) s_3^2 - \frac{1}{2}\lambda_{w2}(1-\lambda_{w2}) s_2^2 + \lambda_{w1}\lambda_{w2}(r_3 r_2 + s_3 s_2 \rho_{12})\right]$$

and

$$H^{\theta}\left[\theta(t),t\right]=\tau k\frac{w\left(t\right)}{\mu_{w}}\left[1-\exp\left[-\mu_{\theta}\left(D-t\right)\right]\right] \text{ for t>T}$$

with

$$\mu_{w}=r_{1}+\lambda_{w2}\left(r_{2}-r_{1}\right)+\lambda_{w1}\left(r_{3}-r_{1}\right)-\omega$$

The next step is to find a general expression for H^{θ} for all t.
Define:

$$H_{B}^{\theta}\left[\theta(t),t\right]=H_{C}^{\theta}\left[\theta(t),t\right]-H_{A}^{\theta}\left[\theta(t),t\right]$$

where:

$$\begin{aligned}H_{C}^{\theta}\left[\theta(t),t\right] &= \tau k\frac{w\left(t\right)}{\mu_{w}}\left[1-\exp\left[-\mu_{w}\left(D-t\right)\right]\right]\\ H_{A}^{\theta}\left[\theta(t),t\right] &= \tau k\frac{w\left(t\right)}{\mu_{w}}\left[1-\exp\left[-\mu_{w}\left(\widehat{t}-t\right)\right]\right]\end{aligned}$$

with

$$\widehat{t}=\max\left[t,T\right]$$

Combining:

$$\begin{aligned}H^{\theta}\left[\theta(t),t\right] &= H_{B}^{\theta}\left[\theta(t),t\right]\\ &= \tau k\frac{w\left(t\right)}{\mu_{w}}\left[\exp\left[-\mu_{w}\left(\widehat{t}-t\right)\right]-\exp\left[-\mu_{w}\left(D-t\right)\right]\right]\end{aligned}$$

Net pension wealth can now be expressed as:

$$H^{\theta}\left(t\right)-\tau H^{w}\left(t\right)=\tau\frac{w\left(t\right)}{\mu_{w}}\left\{(1+k)\exp\left[-\mu_{w}\left(\widehat{t}-t\right)\right]-k\exp\left[-\mu_{w}\left(D-t\right)\right]-1\right\}$$

Implying that net pension wealth at birth equals:

$$H^{\theta}\left(t\right)-\tau H^{w}\left(t\right)=\tau\frac{w_{0}}{\mu_{w}}\left\{(1+k)\exp\left[-\mu_{w}T\right]-k\exp\left[-\mu_{w}D\right]-1\right\}$$

Net pension wealth at birth is zero if:

$$(1+k^{*})\exp\left[-\mu_{w}T\right]-k^{*}\exp\left[-\mu_{w}D\right]=1$$

Developing a second order Maclaurin series gives:

$$(1+k^*)\left[1-\mu_w T+\frac{1}{2}\mu_w^2 T^2\right]-k^*\left[1-\mu_w D+\frac{1}{2}\mu_w^2 D^2\right]\approx 1$$

implying

$$k^*\approx\frac{T}{D-T}\phi\left(\mu_w\right)$$

for

$$\phi\left(\mu_w\right)=\frac{1-\frac{1}{2}\mu_w T}{1-\frac{1}{2}\mu_w\left(T+D\right)}$$

As a final step, we determine the replication portfolio of H^θ by applying the results of the contingent claim analysis of appendix D by replacing θ for x and λ_w for λ_x. Then the value of pension wealth from pensions H^θ is economically equivalent to a portfolio with $\lambda_{w2}H^\theta$ invested in equity and $\lambda_{w1}H^\theta$ invested in nominal bonds and $(1-\lambda_{w1}-\lambda_{w2})\,H^\theta$ price indexed bonds.

We can now express the explicit euro investment in equity ($Z_{Q_2}(t)$) as:

$$\begin{aligned}
Z_{Q_2}(t) &= \alpha_2 W(t)-\lambda_{w2}H^w(t)-\lambda_{w2}\left[H^\theta(t)-\tau H^w(t)\right]\\
&= \alpha_2 F^{DC}(t)+(\alpha_2-\lambda_{w2})\,H^w(t)+(\alpha_2-\lambda_{w2})\left[H^\theta(t)-\tau H^w(t)\right]
\end{aligned}$$

where the explicit investment in equity equals the desired gross investment minus the implicit investment via human wealth and pension wealth. Equivalently, the explicit euro investment in nominal bonds ($Z_{Q_3}(t)$) can be expressed as:

$$\begin{aligned}
Z_{Q_3}(t) &= \alpha_3 W(t)-\lambda_{w1}H^w(t)-\lambda_{w1}\left[H^\theta(t)-\tau H^w(t)\right]\\
&= \alpha_3 F^{DC}(t)+(\alpha_3-\lambda_{w1})\,H^w(t)+(\alpha_3-\lambda_{w1})\left[H^\theta(t)-\tau H^w(t)\right]
\end{aligned}$$

Finally, the explicit euro investment in price indexed bonds ($Z_{Q_1}(t)$) can be expressed as:

$$Z_{Q_1}(t)=\alpha_1 F^{DC}(t)+(\alpha_1-(1-\lambda_{w1}-\lambda_{w2}))\,H^w(t)+(\alpha_1-(1-\lambda_{w1}-\lambda_{w2}))\left[H^\theta(t)-\tau H^w(t)\right]$$

Hence, the explicit investment in price indexed bonds equals the desired gross investment minus the implicit investment via human and pension wealth.

This implies that the optimal portfolio shares of financial wealth $F^{DC}(t)$ in equity, nominal and price-indexed bonds (denoted by $\widehat{\alpha}_2^{DC}$, $\widehat{\alpha}_3^{DC}$ and $\widehat{\alpha}_1^{DC}$ respectively) can be expressed as

$$\begin{aligned}
\widehat{\alpha}_2^{DC}(t) &\equiv \frac{Z_{Q_2}(t)}{F^{DC}(t)}\\
&= \alpha_2+(\alpha_2-\lambda_{w2})\frac{H^w(t)}{F^{DC}(t)}+(\alpha_2-\lambda_{w2})\frac{H^\theta(t)-\tau H^w(t)}{F^{DC}(t)}
\end{aligned}$$

$$\begin{aligned}
\widehat{\alpha}_3^{DC}(t) &\equiv \frac{Z_{Q_3}(t)}{F^{DC}(t)} \\
&= \alpha_3 + (\alpha_3 - \lambda_{w1}) \frac{H^w(t)}{F^{DC}(t)} + (\alpha_3 - \lambda_{w1}) \frac{H^\theta(t) - \tau H^w(t)}{F^{DC}(t)}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\alpha}_1^{DC}(t) &\equiv \frac{Z_{Q_1}(t)}{F^{DC}(t)} \\
&= \alpha_1 + (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2})) \frac{H^w(t)}{F^{DC}(t)} + (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2})) \frac{H^\theta(t) - \tau H^w(t)}{F^{DC}(t)}
\end{aligned}$$

Finally, we can express:

$$\begin{aligned}
\alpha_1^{DC}(t) - \widehat{\alpha}_1(t) &= (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2})) \left[\frac{H^\theta(t) - \tau H^w(t)}{F^{DC}(t)} + \left(\frac{H^w(t)}{F^{DC}(t)} - \frac{H^w(t)}{F(t)} \right) \right] \\
\alpha_2^{DC}(t) - \widehat{\alpha}_2(t) &= (\alpha_2 - \lambda_{w2}) \left[\frac{H^\theta(t) - \tau H^w(t)}{F^{DC}(t)} + \left(\frac{H^w(t)}{F^{DC}(t)} - \frac{H^w(t)}{F(t)} \right) \right] \\
\alpha_3^{DC}(t) - \widehat{\alpha}_3(t) &= (\alpha_3 - \lambda_{w1}) \left[\frac{H^\theta(t) - \tau H^w(t)}{F^{DC}(t)} + \left(\frac{H^w(t)}{F^{DC}(t)} - \frac{H^w(t)}{F(t)} \right) \right]
\end{aligned}$$

F Valuing pension wealth and the government's tax claim under a DB pension scheme

F.1 Pension wealth

Replacing x by $\overline{\theta}$, and χ, κ_x and λ_x by 0 in propositions 7 and 8 of appendix C gives:

$$\theta(t) = \overline{\theta}$$

$$H^{\overline{\theta}}(t) = \frac{\overline{\theta}}{r_1}\left[1 - \exp\left[-r_1\left(D - t\right)\right]\right] \text{ for t>T}$$

giving

$$H^{\overline{\theta}}(t) = \frac{\overline{\theta}}{r_1}\left[1 - \exp\left[-r_1\left(D - t\right)\right]\right] \text{ for t>T}$$

The next step is to find a general expression for $H^{\overline{\theta}}$ for all t.
Define:

$$H_B^{\overline{\theta}}(t) = H_C^{\overline{\theta}}\left[\overline{\theta}(t), t\right] - H_A^{\overline{\theta}}\left[\overline{\theta}(t), t\right]$$

where:

$$\begin{aligned} H_C^{\overline{\theta}}(t) &= \frac{\overline{\theta}}{r_1}\left[1 - \exp\left[-r_1\left(D - t\right)\right]\right] \\ H_A^{\overline{\theta}}(t) &= \frac{\overline{\theta}}{r_1}\left[1 - \exp\left[-r_1\left(\widehat{t} - t\right)\right]\right] \end{aligned}$$

with

$$\widehat{t} = \max\left[t, T\right]$$

Combining:

$$\begin{aligned} H^{\overline{\theta}}(t) &= H_B^{\overline{\theta}}(t) \\ &= \frac{\overline{\theta}(t)}{r_1}\left[\exp\left[-r_1\left(\widehat{t} - t\right)\right] - \exp\left[-r_1\left(D - t\right)\right]\right] \end{aligned}$$

As a final step, we determine the replication portfolio of $H^{\overline{\theta}}$ by applying the results of the contingent claim analysis of appendix D by replacing $\overline{\theta}$ for x and 0 for λ_x. Then the value of pension wealth from pensions $H^{\overline{\theta}}$ is economically equivalent to a portfolio with $H^{\overline{\theta}}$ invested in price indexed bonds.

F.2 Government's tax claim on the individual

Recall equation (49)

$$\psi(t) = \frac{\overline{\theta}}{k}$$

Replacing x by ψ, χ by 0, κ_x by κ_θ, and λ_x by 0 in propositions 7 and 8 of appendix C gives:

$$H^{\psi}(t) = \frac{\overline{\theta}}{r_1 k}\left[1 - \exp\left[-r_1\left(\widehat{t} - t\right)\right]\right]$$

Now we can express the value of net pension wealth as:

$$\begin{aligned} H^{\overline{\theta}}(t) - H^{\psi}(t) &= \frac{\overline{\theta}}{r_1}\left[\exp\left[-r_1\left(\widehat{t} - t\right)\right] - \exp\left[-r_1\left(D - t\right)\right]\right] - \frac{\overline{\theta}}{kr_1}\left[1 - \exp\left[-r_1\left(\widehat{t} - t\right)\right]\right] \\ &= \frac{\overline{\theta}}{r_1}\left\{\frac{1+k}{k}\exp\left[-r_1\left(\widehat{t} - t\right)\right] - \exp\left[-r_1\left(D - t\right)\right] - \frac{1}{k}\right\} \end{aligned}$$

This implies that net pension wealth at birth equals:

$$H^{\overline{\theta}}(0) - H^{\psi}(0) = \frac{\overline{\theta}}{r_1}\left\{\frac{1+k}{k}\exp\left[-r_1 T\right] - \exp\left[-r_1 D\right] - \frac{1}{k}\right\}$$

implying that net pension wealth at birth is zero if:

$$\frac{1+k^*}{k^*}\exp\left[-r_1 T\right] - \exp\left[-r_1 D\right] = \frac{1}{k^*}$$

which equals

$$\left(1+k^*\right)\exp\left[-r_1 T\right] - k^*\exp\left[-r_1 D\right] = 1$$

As a final step, we determine the replication portfolio of H^{ψ}(being the present value of payable taxes) by applying the results of the contingent claim analysis of appendix D by replacing ψ for x. The present value of payable taxes under a DB pension scheme ($-H^{\psi}$) is economically equivalent to a portfolio with $-H^{\psi}$ invested in price indexed bonds.

Recalling that the value of human wealth from wages H^w is economically equivalent to a portfolio with $\lambda_{w2}H^w$ invested in equity, $\lambda_{w1}H^w$ invested in nominal bonds and $\left(1 - \lambda_{w1} - \lambda_{w2}\right)H^w$ invested in price indexed bonds bonds (proposition 2), we can now express the explicit euro investment in equity ($Z_{Q_2}(t)$) as:

$$\begin{aligned} Z_{Q_2}(t) &= \alpha_2 W(t) - \lambda_{w2}H^w(t) \\ &= \alpha_2\left[F^{DB}(t) + H^{\overline{\theta}}(t) - H^{\psi}(t)\right] + \left(\alpha_2 - \lambda_{w2}\right)H^w \end{aligned}$$

and the explicit euro investment in nominal bonds as:

$$
\begin{aligned}
Z_{Q_3}(t) &= \alpha_3 W(t) - \lambda_{w1} H^w(t) \\
&= \alpha_3 \left[F^{DB}(t) + H^{\overline{\theta}}(t) - H^{\psi}(t)\right] + (\alpha_3 - \lambda_{w1}) H^w
\end{aligned}
$$

Equivalently, the explicit euro investment in price indexed bonds ($Z_{Q_1}(t)$) can be expressed as:

$$
\begin{aligned}
Z_{Q_1}(t) &= \alpha_1 W(t) - (1 - \lambda_{w1} - \lambda_{w2}) H^w(t) + H^{\psi}(t) - H^{\overline{\theta}}(t) \\
&= \alpha_1 F^{DB}(t) + (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2})) H^w(t) + (\alpha_1 - 1)\left[H^{\overline{\theta}}(t) - H^{\psi}(t)\right]
\end{aligned}
$$

Hence, the explicit investment in price indexed bonds equals the desired gross investment minus the implicit investment via human wealth, the present value of payable taxes and pension wealth.

This implies that the optimal portfolio shares of financial wealth $F^{DB}(t)$ in price indexed bonds and equity (denoted by α_1^{DB} and α_2^{DB} respectively) can be expressed as

$$
\begin{aligned}
\alpha_1^{DB}(t) &\equiv \frac{Z_{Q_1}(t)}{F^{DB}(t)} \\
&= \alpha_1 + (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2})) \frac{H^w(t)}{F^{DB}(t)} + (\alpha_1 - 1) \frac{H^{\overline{\theta}}(t) - H^{\psi}(t)}{F^{DB}(t)}
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_2^{DB}(t) &\equiv \frac{Z_{Q_2}(t)}{F^{DB}(t)} \\
&= \alpha_2 + (\alpha_2 - \lambda_{w2}) \frac{H^w}{F^{DB}(t)} + \alpha_2 \frac{H^{\overline{\theta}}(t) - H^{\psi}(t)}{F^{DB}(t)}
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_3^{DB}(t) &\equiv \frac{Z_{Q_3}(t)}{F^{DB}(t)} \\
&= \alpha_3 + (\alpha_3 - \lambda_{w1}) \frac{H^w(t)}{F^{DB}(t)} + \alpha_3 \frac{H^{\overline{\theta}}(t) - H^{\psi}(t)}{F^{DB}(t)}
\end{aligned}
$$

Finally, we can express:

$$\begin{aligned}
\alpha_1^{DB}(t) - \widehat{\alpha}_1(t) &= (\alpha_1 - (1 - \lambda_{w1} - \lambda_{w2}))\left[\frac{H^w(t)}{F^{DB}(t)} - \frac{H^w(t)}{F(t)}\right] + (\alpha_1 - 1)\,\frac{H^{\overline{\theta}}(t) - H^{\psi}(t)}{F^{DB}(t)} \\
\alpha_2^{DB}(t) - \widehat{\alpha}_2(t) &= (\alpha_2 - \lambda_{w2})\left[\frac{H^w(t)}{F^{DB}(t)} - \frac{H^w(t)}{F(t)}\right] + \alpha_2\frac{H^{\overline{\theta}}(t) - H^{\psi}(t)}{F^{DB}(t)} \\
\alpha_3^{DB}(t) - \widehat{\alpha}_3(t) &= (\alpha_3 - \lambda_{w1})\left[\frac{H^w(t)}{F^{DB}(t)} - \frac{H^w(t)}{F(t)}\right] + \alpha_3\frac{H^{\overline{\theta}}(t) - H^{\psi}(t)}{F^{DB}(t)}
\end{aligned}$$

G Table of notation

G.1 Subscripts

1.	price indexed bond
2.	equity
3.	nominal bond

G.2 Alphabetical

k	support ratio
n^y	number of young individuals
n^o	number of old individuals
r_i	(expected) real return on asset i
s_1^2	variance of prices
s_2^2	variance of real return on equity
s_3^2	variance of real return on nominal bonds
s_{w1}^2	variance of real wages due to inflation shocks
s_{w2}^2	variance of real wages due to productivity shocks
t	time
w	real wages
x	income
dz_1	Wiener process generating prices (equal to dz_3)
dz_2	Wiener process generating productivity growth (and indirectly equity returns and wages)

A_2	constant introduced in section 2
A_3	constant introduced in section 2
C	consumption
D	(deterministic) life expectancy
E	expectation operator
F	real value of financial wealth
G	function introduced in appendix C equal to H^x
H^x	real value of wealth from income x
H^w	real value of human wealth from wages
H^θ	real value of pension wealth from DC pension benefits
$H^{\overline{\theta}}$	real value of pension wealth from DB pension benefits
H^ψ	real value of tax claim on human wealth under DB pension scheme
J	indirect utility function
K	constant used in appendix B
P	prices
Q_i	nominal value asset i
R_i	(expected) nominal return asset i
T	retirement age
U	utility function
W	real value of total wealth
X	constant introduced in section 2
Y	constant introduced in section 2
Z	actual number of euros invested in asset i
$\widetilde{Z}$	desired number of euros invested in asset i

α_i	optimal fraction of total wealth invested in asset i
$\widehat{\alpha}_i$	optimal fraction of financial wealth invested in asset i when total wealth includes human wealth
α_i^{DB}	idem, including (net) DB pension wealth
α_i^{DC}	idem, including (net) DC pension wealth
β_i	number of euros invested in asset i in contingent claim analysis
δ	rate of time preference
γ	risk-aversion parameter
κ_x	constant introduced in appendix C (also with x replaced by by w, θ and $\overline{\theta}$)
θ	DC pension benefit
$\overline{\theta}$	DB pension benefit
λ_w	ratio of standard deviation of wages over standard deviation of equity return
μ_x	(risk-adjusted) discount rate x (also with x replaced by by w, θ and $\overline{\theta}$)
π	expected price increase
ρ_{12}	instantaneous correlation coefficient between Wiener processes dz_1 and dz_2
τ	proportional wage tax
φ	time index
χ	expected increase in x
ω	expected wage increase
ψ	payable taxes under DB pension scheme
ϕ	function introduced in section 4.1
Θ	function introduced in section 4.1
$\overline{\Theta}_i$	function introduced in section 4.2 for $i = 1, 2$