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Abstract: This paper reviews the investment policy of collective pension plans. We focus on funds with a collective Defined Contribution character. We suggest two reasons to invest in equities: the lack of a well-developed market in index linked bonds, and deliberate exposure to equity returns to collect the equity risk premium. Furthermore, this paper assesses the value of limited or conditional indexation options found in many plans.

Keywords: Pension funds, Portfolio choice, Valuation of liabilities.

JEL codes: G11, G23

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"Even if only nominal bonds are available, conservative long-term investors should hold large positions in long-term bonds if they believe that inflation risk is low, as we have estimated it to be in the USA in the period 1983-99. In this sense, the message of this chapter might be summarized as 'Bonds, James, Bonds'. Inflation risk is however a serious caveat." John Campbell and Luis Viceira, *Strategic Asset Allocation*, p.87.

1 Introduction

Several countries have organized their pension system by means of fully funded pension plans. The nature of the plans can be quite different. Some are Defined Benefit, i.e. the participants are promised a pension that is linked to the wage earned before retirement. But increasingly, pension plans are of a Defined Contribution nature, either with individual accounts or with a collective investment pool, see e.g. Cairns (2003). A collective investment pool is a feature of occupational pension plans in the UK and industry pension plans in the Netherlands. The contributions are typically a fixed fraction of wage income. In such plans, the pension is a form of deferred wage and the sponsor has no obligation to support the fund. The residual claimant are the pension plans members.

In a DB scheme there is no direct link between the benefits (pensions) and the returns earned on the investment portfolio. Exley (2001) and others have argued that this feature implies that pension funds should invest in a portfolio that exactly matches the liability. Essentially, this portfolio would consist of 100% long term index linked bonds. For collective DC plans, this is however not true and from a financial investments perspective, the fund may well find it optimal to invest in equities. Nevertheless, the quote from Campbell and Viceira (2002) also suggests that long-term investors, such as pension funds, should take substantial positions in bonds.

In practice, we observe quite a variety of pension fund portfolio compositions, but typically funds hold between 40% and 60% of their wealth in equities and real estate, and sometimes even more, especially in the US and the UK. The remainder is typically invested in medium-term nominal bonds, and only a small fraction of pension fund assets is in index linked bonds. The risk-return profile of this investment portfolio is therefore quite different from the risk and return profile of the pension plan's liabilities. If there is a positive equity premium, the expected return on the actual portfolio is higher than the expected return on index linked bonds. But the risk is also larger and especially the inflation hedge of the portfolio is rather weak. This has lead to a lot of criticism on pension fund investment managers, arguing they take too many risks at the expense of the pension fund participants who see their benefits endangered by the risk of low returns. The melt-down of the stock market in the early 2000's has made this point very clear, with many pension plans technically underfunded.[1] Several pension funds put structures into place to mitigate the risks, such as limited indexation clauses conditional on the level of the funds assets.

The aim of this paper is to introduce a framework for thinking about the investment and risk management policies of pension funds. I analyze these questions using models from the recent literature on long-term investments, which is very nicely presented in the book by Campbell and Viceira (2002). In particular I will use the continuous time model by Brennan and Xia (2002), who study the optimal investment portfolio of an individual long-term investor in an setting with inflation risk.[2] In the paper I apply this framework to two problems. First, I review the optimal investment policy and assess the case for investing in equities and other risky assets such as real estate. Second, I show how to value liabilities under limited or conditional inflation indexation rules. Finally, I tie these two together and show how the investment policy has an impact on the value of future pension benefits.

The structure of the paper is as follows. First, I give a formal theoretical structure for the optimal investment portfolio of a long-term investor, and calibrate the model to obtain some numerical results. Given the insights of this model, I discuss the arguments for equity investments in more detail. I then turn to the valuation of pension fund liabilities with (conditional) indexation. Finally, I discuss some avenues for further research.

[1] Estimates in the beginning of 2003 of the pension fund supervisor in the Netherlands show that one quarter of pension funds has assets that are smaller then the present value of liabilities.

[2] This paper also relates to the work of Munk, Sørensen and Vinter (2004), who study the optimal portfolio of a long-term investor in a model that is slightly different from Brennan and Xia's, but has the same key risk factors.

2 Long term investments: a review of the theory

In this section I review the most important results of the recent literature on long-term investments. The exact model structure presented here is based on Brennan and Xia (2002), but other models give similar results. I picked the Brennan-Xia model because of the explicit solutions for optimal portfolio's for investors with a long but *finite* investment horizon. The Brennan-Xia model considers an investor with investment horizon T, which one could think of as the retirement date. The objective of the investor is to maximize the utility of end-of-period real wealth. The investor is assumed to have a CRRA utility function with relative risk aversion parameter γ. Formally, the problem of the investor is

$$\max \mathsf{E}\left[U\left(W_T/\Pi_T\right)\right], \qquad U(w) = \frac{w^{1-\gamma}}{1-\gamma} \tag{1}$$

where W_T denotes end-of-period nominal wealth, and Π_T the price level. The budget constraint of the investor is given by the initial wealth W_0 and the nominal wealth dynamics equation

$$dW/W = [x'\mu + (1-\iota'x)R_f] + x'\sigma dZ \tag{2}$$

In this expression, x is the vector of portfolio weights on the risky assets, with expected return μ, and $1-\iota'x$ is the weight on the nominally riskless asset, with return R_f. The matrix σ denotes the exposure of the asset returns to the risk factors dZ, which will be specified shortly. Notice that the wealth dynamics can also be written as

$$dW/W = [R_f + x'\sigma\lambda] + x'\sigma dZ \tag{3}$$

where λ is the vector of market prices of risk and $\mu - R_f = \sigma\lambda$ is the vector of asset risk premia. The risk factor dynamics in the Brennan-Xia model can be summarized in the following state variables: the stock price S, the instantaneous real interest rate r, the instantaneous expected inflation π and the price level Π. Stock prices follow a geometric Brownian motion, like in the Black and Scholes (1973) model. The real interest rate and expected inflation follow Ornstein-Uhlenbeck processes, and are therefore normally distributed around a fixed long run mean. Finally, the actual inflation equals the expected inflation plus a random shock. Formally, the equations driving the state variables are

$$dS/S \;=\; \mu_S dt + \sigma_S dZ_S \tag{4a}$$

$$dr = \kappa(\bar{r} - r)dt + \sigma_r dZ_r \tag{4b}$$

$$d\pi = \alpha(\bar{\pi} - \pi)dt + \sigma_\pi dZ_\pi \tag{4c}$$

$$d\Pi/\Pi = \pi dt + \sigma_\Pi dZ_\Pi \tag{4d}$$

It is sometimes useful to orthogonalize the equation for unexpected inflation

$$d\Pi/\Pi = \pi dt + \xi_S dZ_S + \xi_r dZ_r + \xi_\pi dZ_\pi + \xi_u dZ_u = \xi' dZ \tag{5}$$

where dZ_u is the part of dZ_Π orthogonal to (dZ_S, dZ_r, dZ_π).[3]

The investment vehicles in this model are stocks, nominal bonds and index linked bonds. The price dynamics of the stock are given by the first equation of this system. The bond price dynamics follow from the Vasicek (1977) model. The price dynamics of a nominal zero-coupon bond is given by (see Brennan and Xia, 2000, p. 1207)

$$dP/P = [R_f - B(\tau)\sigma_r\lambda_r - C(\tau)\sigma_\pi\lambda_\pi]dt - B(\tau)\sigma_r dZ_r - C(\tau)\sigma_\pi dZ_\pi \tag{6}$$

and the *nominal* price dynamics for an Index Linked Bond (ILB) are

$$dILB/ILB = [r + \pi - B(\tau)\sigma_r\lambda_r]dt - B(\tau)\sigma_r dZ_r + \sigma_\Pi dZ_\Pi \tag{7}$$

where τ is the time-to-maturity of the bond and

$$B(\tau) = \frac{1 - e^{-\kappa\tau}}{\kappa}, \quad C(\tau) = \frac{1 - e^{-a\tau}}{a}$$

These equations show that a nominal bond provides a hedge against real interest rate and expected inflation risk, but not against unexpected inflation risk. The ILB price moves one-for-one with realized inflation, and depends on the level of the real interest rate. By construction, the risk premia in this model are time-invariant.[4] The risk premium on stocks is given by $\mu_S - R_f = \lambda_S\sigma_S$, and the risk premium on long term bonds follows from the drift of equation (6),

$$\mu_B - R_f = -B(\tau)\sigma_r\lambda_r - C(\tau)\sigma_\pi\lambda_\pi \tag{8}$$

where τ denotes the maturity of the bond. The risk premium on an index linked bond is derived from equation (7) and substituting using $R_f = r + \pi + \lambda_\Pi\sigma_\Pi$:

$$\mu_{ILB} - R_f = -B(\tau)\sigma_r\lambda_r - \lambda_\Pi\sigma_\Pi \tag{9}$$

[3] In this notation, σ denotes the exposure to all the risk factors, including the unexpected inflation risk, and ρ denotes the correlation matrix of $dZ = (dZ_S, dZ_r, dZ_\pi, dZ_\Pi)'$.

[4] See Munk, Sørensen and Vinter (2004) for a model with time-varying risk premia. A model of that type can incorporate "mean reversion" in stock returns.

Notice that ILBs don't have a risk premium for expected inflation risk, but they do have a risk premium for unexpected inflation.

Brennan and Xia (2002) and the Appendix show that the optimal portfolio composition for the long-term investor is

$$x^{opt} = \frac{1}{\gamma}(\sigma\rho\sigma')^{-1}\sigma\lambda + \left(1 - \frac{1}{\gamma}\right)(\sigma\rho\sigma')^{-1}\sigma\rho(b+\xi)' \tag{10}$$

where $b = (0, -B(T)\sigma_r, 0, 0)'$ and $\xi = (\xi_S, \xi_r, \xi_\pi, \xi_u)'$. The intuition for this portfolio is as follows. The portfolio consists of a speculative part and a hedge part. The speculative part is equal to the optimal portfolio of Merton's (1969) investment problem, and gives the usual mean-variance tradeoff between risk and risk premium (in terms of nominal returns). The hedge part gives the minimum variance (least squares) hedge against the long term risks for the investor: $b'dZ$ is the long term real interest rate risk, and $\xi'dZ$ is the unexpected inflation risk exposure. We can express the effectiveness of this hedge in a coefficient of determination measure

$$R^2_{hedge} = \frac{(b+\xi)'\rho\sigma'(\sigma\rho\sigma')^{-1}\sigma\rho(b+\xi)}{(b+\xi)'\rho(b+\xi)} \tag{11}$$

The long term risk of the investor is exactly the risk exposure of an ILB with maturity equal to the investment horizon T. The proof is simple: we can write the nominal price dynamics of an ILB as

$$dILB/ILB = [..]dt + (b+\xi)'dZ \tag{12}$$

This insight also implies that if there is an ILB, it is an ideal hedge instrument. Suppose that there are n assets and let the ILB be the last element in the vector of assets. Then one can write its row in the return exposure matrix σ as $(b+\xi)'$. From this it follows that

$$b + \xi = \sigma' \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

and the optimal portfolio simplifies to

$$\begin{pmatrix} x \\ x_{ILB} \end{pmatrix} = \frac{1}{\gamma}(\sigma\rho\sigma')^{-1}\sigma\lambda + \left(1 - \frac{1}{\gamma}\right)\begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \tag{13}$$

Hence, the hedge part of the portfolio is completely dominated by an ILB with maturity T, which is the perfect hedge instrument ($R^2_{hedge} = 1$).

Which fraction of wealth the investor puts into each of these two portfolios is determined by the inverse of the risk aversion parameter, $1/\gamma$. A log-utility investor ($\gamma = 1$)

will invest everything in the speculative portfolio. A more risk averse investor will invest only a fraction of his wealth in the speculative portfolio and the remainder in the hedge portfolio. An extremely risk averse investor (γ going to infinity) will invest everything in the hedge portfolio.

Young workers typically have little financial wealth. Most of their wealth is human capital, i.e. the present value of their future labor income. Assuming that labor income is exogenous, the optimal long-term investment framework can easily adapted to a situation with human capital. The Appendix shows that the optimal investment portfolio becomes

$$x^{opt} = \frac{1}{1-h}(\sigma\rho\sigma')^{-1}\left[\frac{1}{\gamma}\sigma\lambda + \left(1-\frac{1}{\gamma}\right)\sigma\rho(b_T+\xi) - h\sigma\rho\sigma_H'\right] \quad (14)$$

where $h = H/W$ is the ratio of human capital to total wealth and σ_H is the exposure of human capital to the risk factors. The term $\frac{1}{1-h}$ is a leverage effect, and the last term in this expression is the hedge that human capital provides against the risk factors.

If wages are perfectly indexed against inflation, and have no exposure to stock returns, human capital is like an indexed bond with duration τ. In that case, $\sigma_H' = b_\tau + \xi$ and the optimal portfolio becomes after some re-writing

$$x^{opt} = (\sigma\rho\sigma')^{-1}\left[\frac{1}{\gamma_h}\sigma\lambda + \left(1-\frac{1}{\gamma_h}\right)\sigma\rho(b_S+\xi)\right] \quad (15)$$

where $\gamma_h = \gamma(1-h)$ and

$$b_S = \frac{\left(1-\frac{1}{\gamma}\right)b_T - hb_\tau}{\left(1-\frac{1}{\gamma}\right) - h} \quad (16)$$

is the desired real interest rate exposure. Notice that this optimal portfolio is different from the one without human capital because the investor is willing to take more risk ($\gamma_h \leq \gamma$) and the real interest rate exposure is smaller ($b_S \leq b_T$) because of the automatic hedge the human capital provides against interest rate risk and inflation.

3 Calibration

In this section I investigate the model presented in the previous section in more detail. I calibrate the model by imposing parameter values and calculate the associated optimal portfolios. For this exercise I use the model parameter values reported in Brennan and Xia (2002), with two modifications. First, Brennan and Xia report a high and a low value for the mean reversion parameter of the real interest rate. With the high value, the interest rate hedge component is very small. I select the more realistic low mean reversion parameter. The second modification is in the market prices of risk. Brennan and Xia report values that imply fairly high risk premia on stocks and bonds (5.5% per annum for stocks and 3% per annum for 10-year bonds). In the current market circumstances, these values seem unrealistically high. I therefore pick lower values for the market prices of risk. The parameter values for the calibration are summarized in Table 1.

Brennan and Xia don't give an estimate of the market price of risk for unexpected inflation. I take a conservative approach (i.e. biased against holding ILBs in the portfolio) by assuming that $\lambda_{\Pi} = 0$. I also assume that unexpected inflation is uncorrelated with stock returns, expected inflation and the real interest rate, so that stocks and nominal bonds provide no hedge against unexpected inflation.

I now consider two financial market settings, one with nominal bonds and one with index linked bonds. I also consider a case with borrowing constraints and human capital.

3.1 Nominal bonds

For the first situation, I assume that the investor can invest in cash, one nominal bond and stocks. In the model, a linear combination of two nominal bonds can hedge perfectly against expected inflation risk. However, this combination requires a short position in one of the bonds. This makes the strategy infeasible for a pension fund that is typically restricted to long positions. Therefore, I don't consider the case with two nominal bonds. Instead, I consider the optimal choice of maturity for the single bond that the fund can buy, with the restriction that the position in this bond does not exceed 100% of the invested wealth.

Table 2 shows the composition of the optimal portfolio for an investment horizon of 20 years and a bond maturity of 5 years. The speculative portfolio is highly leveraged, but has stocks and bonds in almost 50-50 proportions. The hedge portfolio is tilted towards bonds and cash, with only very small position in stocks because of the positive correlation between stock and bond returns. Recall here that stocks don't provide a direct hedge against unexpected inflation and the correlation between expected inflation and stock returns in the data is extremely weak. Stocks therefore don't provide an inflation hedge in this calibration. The effectiveness of the hedge, i.e. the squared correlation between the long term risk and the hedge portfolio return, is small, it takes the value 0.337.

Of course, other maturities for the bond can be chosen. Figure 1 shows the optimal position in the nominal bond and the hedge effectiveness as a function of the bond maturity. The figure shows that for longer maturities the optimal position in bonds is smaller and that the hedge effectiveness decreases. This is an immediate result of the high expected inflation risk of long term nominal bonds. This can be seen by writing the hedge effectiveness for the simplified case where $\rho_{Sr} = \rho_{S\pi} = 0$ and also $\rho_{r\pi} = 0$, which is approximately true in the data. In that case, the optimal position in the bond is

$$x_B^{hedge} = \frac{B(\tau)B(T)\sigma_r^2}{B(\tau)^2\sigma_r^2 + C(\tau)^2\sigma_\pi^2 + \sigma_\Pi^2} \tag{17}$$

The hedge effectiveness is

$$R_{hedge}^2 = \frac{B(\tau)^2B(T)^2\sigma_r^4}{(B(\tau)^2\sigma_r^2 + C(\tau)^2\sigma_\pi^2 + \sigma_\Pi^2)(B(T)^2\sigma_r^2 + \sigma_\Pi^2)} \tag{18}$$

Given the parameter values from Table 1, $C(\tau)$ converges much slower to its maximum value than $B(\tau)$ because of the slow mean reversion in expected inflation. The R_{hedge}^2 is therefore decreasing in τ and the model suggests to invest in short-term nominal bonds. However, to get the right exposure to the long-term real interest rate risk a highly leveraged position is needed. Only for bond maturities of approximately 4 years or higher, no leverage is needed in the hedge portfolio. The hedge effectiveness is therefore also bounded to around $R_{hedge}^2 = 0.35$.

3.2 Index linked bonds

The second situation I consider is cash, an index linked bond and stock. Table 3 shows the relevant data for this situation. In the table, it is assumed that the ILB has a maturity equal to the investment horizon, 20 years. Obviously, this ILB provides a perfect hedge to the long term risk and therefore completely dominates the hedge portfolio. The ILB also enters the speculative portfolio because of its positive risk premium, caused by the real interest rate risk premium. The speculative investment in ILBs is smaller than the speculative investment in nominal bonds, mainly because the risk premium on nominal bonds is higher due to the unexpected inflation risk that these carry.

In practice, ILBs with a very long maturity may be unavailable and the fund has to invest in shorter maturity ILBs. Figure 2 shows the optimal fraction of ILBs and the hedge effectiveness for shorter maturities. It turns out that even with reasonable short maturities, around 5 years, the hedge effectiveness is already close to 1. This is a result of the relatively quick mean reversion of the real interest rate. The portfolio has substantial leverage, however.

Let's now turn to the overall optimal portfolio composition. The overall optimal portfolio depends of course on the mix between the speculative and the hedge portfolio. Theoretically, this depends on the investor's risk aversion. A pension fund invests on behalf of its participants and if it invests all their wealth, it will inherit the participants' risk aversion. I proxy this by the $\gamma = 2$ (for a reasonably aggressive investor) and $\gamma = 5$ (for a reasonably conservative investor) scenarios in the tables. The optimal portfolio for $\gamma = 2$ shows a substantial investment in stocks, around 60%, and a substantial leverage. Notice that this holds even with our relatively conservative values for equity and bond risk premia (3.16% and around 1%, respectively). The optimal portfolio also has a substantial leverage, around 50% borrowed cash. On the other hand, if the pension fund invests only a fraction of the investor's wealth, its investment target is more likely to be a replication of defined benefits, with the portfolio tilted towards the hedge portfolio. We proxy this by the $\gamma = 10$ scenario in the tables. This more conservative assumption generates portfolio's with around 15% in stocks (more when the bond is nominal), between 80% and 100% in bonds, and a small amount of leverage.

3.3 Borrowing constraints

In several occasions the optimal portfolio, which is a combination of the speculative and the hedge portfolio, contains a short position in cash. This is due to the substantial leverage in the speculative part of the portfolio. Many pension funds are restricted to take only long positions, either by regulators or by their own investment charters. I therefore consider also an optimal portfolio composition with a borrowing constraint, i.e. the restriction $x_{cash} \geq 0$. The Appendix shows how to derive optimal the portfolio under borrowing constraints. As it turns out, imposing a borrowing constraint is sufficient to rule out all short positions in the calibration examples. Imposing the borrowing constraint makes the optimal equity investments around 10% smaller. The effect on bond investments is bigger, and bonds and cash are somehow substitutes: the net position in bonds with borrowing constraints is about the same as the sum of the bond and cash positions in the no-borrowing-constraints case.

3.4 Human capital

Table 4 shows optimal portfolios for several levels of human capital, ranging from $h = 0.2$ to $h = 0.8$. The general conclusion of the table is that the investor uses human capital to leverage up the stock weight in the portfolio. For $h = 0.5$, the portfolio consists almost entirely of stocks. The investor also leverages up the bond weight, but to a lesser extent than the stock weight. This is the result of the automatic hedge that human capital provides against real interest rate changes. The investor also increases his borrowing at the risk free rate. Without a risk free asset, the investor still leverages up the stock weight, at the expense of bond investments. So, the predominant effect of human capital is to leverage up the equity share in the portfolio, to give the right exposure to the equity premium. Given the desired equity exposure, the investor chooses the best portfolio of bonds and cash. This results holds both for the situation with nominal bonds and for the case with index linked bonds.

4 The case for equity investments

Let's now return to the original question and review the arguments for equity investments. My arguments here assume that a pension fund is like a long-term investor with a relatively constant investment horizon. For a pension fund in a steady state with a regular inflow of new, young participants and an outflow of old participants this may be a good first approximation. For such a pension fund, an average maturity of claims of 20 years seems reasonable. I therefore use the results developed in the calibration part for an individual investor to get some insights for the investments of pension funds.

The long-term optimal investment portfolio equation (10) gives two reasons to hold equity (or any other asset for that matter): (i) as an element of the speculative part of the portfolio. Here the key determinants of the amount of equity are the risk (variance) and the risk premium or Sharpe ratio; (ii) as an element of the hedge portfolio. Here the key determinants of the amount of equity are the correlation with long-term real interest rate risk and unexpected inflation risk. The stronger the correlation, the better the hedge and the higher the portfolio weight.

Following these insights, there are two main arguments for large equity and real estate investments for pension funds. The first argument is that the market for index linked bonds is severely underdeveloped, effectively preventing pension funds from investing large fractions of their wealth in ILBs. Instead, they invest in a second best portfolio of more liquid assets that replicates the risk profile of ILBs as closely as possible. Potentially, equity and real estate are an important part of this portfolio. A second reason for a substantial investment in equity is that pension funds deliberately take more risks than a pure Defined Benefit scheme would impose. They invest more in equities to reap the equity premium. This gain is balanced against the larger risk, but leads to a higher fraction of equity investments. I now go into more detail on each of these two points.

4.1 Replication of Index Linked Bonds

One could argue that a Defined Benefit pension plan is like a long term riskless investment. So, it provides the hedge part of the optimal portfolio. The optimal investment for the pension fund is therefore 100% in index linked bonds. In practice, ILBs may be unavailable to the investor. Then the second best strategy of the fund is then to match the long term risks as closely as possible by choosing assets that have the highest correlation with the long term risk exposures, i.e. real interest rate risk and inflation. This is exactly the second part of expression (10), the least squares hedge portfolio. The composition of this portfolio depends on the exposures of the asset returns to the risk factors σ and the correlations between the risk factors ρ, and the correlation of the assets with long term real interest rate risk (b) and unexpected inflation risk (ξ). From the calibrations of the model a few results can be established:

- Nominal bonds provide a good hedge against interest rate risk, but no hedge against unexpected inflation risk.
- A roll over of short term bonds provides a good hedge against unexpected inflation risk, but almost no hedge against interest rate risk
- For stocks (and other risky assets) everything depends on their correlation with interest rates and inflation. Empirically, these correlations are weak and some-

times even negative.[5] Moreover, stock returns have a fairly high variance. All this suggest that the role of stocks in the hedge portfolio will at best be very limited.

- A more useful risky asset may be real estate. Theebe (2002) shows that in the long run the value of real estate is strongly correlated with the price level. However, like stocks real estate returns have a high variance and that will limit the holdings in the hedge portfolio.

In the absence of ILBs the composition of the hedge portfolio has to strike a balance between the inflation hedge of short term bonds and the interest rate hedge of long term bonds, with a very limited role for stocks and other risky assets. The best hedge portfolio will be a medium term nominal bond portfolio, with duration a function of the relative magnitude of interest rate risk to inflation risk.

4.2 Exposure to the equity risk premium

The previous discussion assumed that pension plans basically provides the hedge part of the optimal investment portfolio. For many individuals, the accumulated pension rights are a substantial part of their total wealth, larger than private savings and investments, and second only to the value of housing. Especially in countries with a relatively generous pension system this will be the case. For example, in the Netherlands the pension benefit of a retiree with 40 years of labor history is 70% of the final wage. If the individual is not extremely risk averse, he might be interested in investing a part of this wealth in the speculative portfolio, to reap the benefits of the risk premia on risky assets.

In a recent study, Blake, Cairns and Dowd (2003) investigated this argument for a retiree. They compare the utility of investing in traditional nominally risk free annuities and equity linked annuities (ELAs). Such ELAs are more risky that traditional annuities but also provide a higher expected payoff. Blake et al. show that the utility gains of investing in ELAs can be substantial. This argument also implies that pension plan participants may be interested in giving up some part of a DB pension to invest in risky assets. The implication of this is that the optimal pension structure would not be Defined Benefit, but more like a mixture of DB and (collective) DC. Of course,

[5] These estimates are for annual data. For longer horizons, the correlations may be stronger. This aspect has to be studied in more detail.

one could question whether collective pension plans should provide the speculative part of the portfolio or whether this is best left to the individuals. An important reason for individuals to participate in collective pension plans is that these give an automatic annuitization of cash flows. When indexed annuities are unavailable in the marketplace, pension funds have a useful role in effectively providing these. Lopes (2002) emphasizes the benefits of index linked annuities for retired individuals.

5 The value of conditional indexation

An important practical complication of the investment policy for pension funds is the regulatory environment. Typically, regulators require the pension fund's assets to be worth at least as much as the present value of the pension liabilities. Such solvency requirements in principle can be added to the optimization problem. Basak and Shapiro (2001) have investigated optimal investment strategies under Value-at-Risk constraints. A similar approach could be taken for pension funds if regulators apply related tools such as Solvency-at-Risk constraints. Technically, an analysis of optimal long-term investments under such constraints is very difficult and beyond the scope of this paper. However, the regulatory environment is clearly important and affects the payout policies of pension funds.[6] In this section I take a more ad-hoc approach and assume an indexation policy that depends on the solvency of the fund, in order to get some insights about the impact of regulatory constraints on the value of pensions.

In principle, most DB pensions are indexed to either price or wage inflation. In practice, however, indexation is often limited. In the UK, indexation is typically restricted to the range of 0% to 5% per year. In the Netherlands, indexation is granted only if the asset value of the fund is sufficient to cover all future obligations. Hence, indexation is conditional and has option-like features.

Pricing contingent claims in the Brennan-Xia model is relatively straightforward, because the pricing kernel takes a very convenient form. The (real) pricing kernel[7] in

[6] This assumes no role for the plan sponsor; all risk in the pension fund is shared among the participants. This is obviously a simplification, but I argue that it is a good first approximation for industry-wide or government operated pension plans. Blake (1998) provides an analysis of the role of the sponsor in company pension plans.

[7] In the actuarial literature, the pricing kernel is usually called the deflator.

the Brennan-Xia model is given by

$$M_T^* = \exp\left\{\int_0^T (-r(s) - \tfrac{1}{2}\phi'\rho\phi)ds + \int_0^T \phi' dZ\right\} \tag{19}$$

where ϕ is related to the (real) market prices of risk by $\lambda^* = -\rho\phi$. The price level is given by equation (5) and equals

$$\Pi_T = \exp\left\{\int_0^T (\pi(s) - \tfrac{1}{2}\xi'\rho\xi)ds + \int_0^T \xi' dZ\right\} \tag{20}$$

The nominal pricing kernel is given by

$$M_T = M_T^*/\Pi_T = \exp\left\{\int_0^T (-r(s) - \pi(s) - \tfrac{1}{2}\phi'\rho\phi + \tfrac{1}{2}\xi'\rho\xi)ds + \int_0^T (\phi - \xi)' dZ\right\} \tag{21}$$

where we normalized $M_0 = \Pi_0 = 1$. The nominal pricing kernel can also be written as

$$M_T = \exp\left\{\int_0^T (-R_f(s) - \tfrac{1}{2}(\phi - \xi)'\rho(\phi - \xi))ds + \int_0^T (\phi - \xi)' dZ\right\} \tag{22}$$

with $R_f = r + \pi - \xi'\lambda$ the nominal risk free rate and $\lambda = -\rho(\phi - \xi)$ the vector of nominal market prices of risk. Any payoff at time T can be valued using these pricing kernels. For example, a nominal payoff X_T has time 0 value

$$X_0 = \mathsf{E}[M_T X_T] \tag{23}$$

Complicated, path dependent claims can be valued by Monte Carlo simulation methods. These methods simulate paths for the pricing kernel (deflator) and the option payoff, and average the product of these over many Monte Carlo replications.[8]

Now consider a Defined Benefit pension with full indexation. In a very stylized setting, the payoff of this pension can be written as $L \cdot \Pi_T$, where L is the present value of all the rights built up so far. The value of this claim is

$$L_0 = L\mathsf{E}[M_T \Pi_T] = L\mathsf{E}[M_T^*] \tag{24}$$

which is equal to the price of an index linked bond with face value L paying off at time T. This is of course not surprising as this ILB is the perfect hedge instrument for this particular claim.

Things become a little more interesting when indexation is limited. Again, as a very stylized example consider a pension where indexation is limited to a maximum

[8] See e.g. Duffie (1996), Chapter 11, or Hull (2003), Chapter 18, for details on Monte Carlo valuation of contingent claims.

of 5% per year (continuously compounded) on average. The payoff of this pension is $L\min\{\Pi_T, \exp(0.05T)\}$, which can be written as

$$X_T = L\Pi_T - \max\{\Pi_T - \exp(0.05T), 0)\} \tag{25}$$

The conditional indexation therefore works like a call option on inflation, written to the fund by the pension plan participant. In practice, the check for indexation is every year. This will lead to a path dependent option, with payoff

$$X_T = L\prod_{t=1}^{T}\min\{\Pi_t/\Pi_{t-1}, 1.05\} \tag{26}$$

Numerical valuation of this option is straightforward by Monte Carlo simulation. Figure 4 shows the value of the implicit call written by the participant to its pension fund, as a function of the maturity. The figure shows that the limited indexation option is quite valuable, up to 10% of the pension value for a maturity of 20 years.

Many pension schemes also have a downside protection, in the sense that with negative inflation, the nominal pensions are not reduced. Effectively, this works as a zero downside limit on the indexation. Combined with an upward limit, the indexation has a "collar" structure with inflation as the underlying value. Figure 3 graphs this indexation rule. Mathematically, the structure can be expressed as

$$X_T = L\prod_{t=1}^{T}\max\{1.0, \min\{\Pi_t/\Pi_{t-1}, 1.05\}\} \tag{27}$$

Again, the value of this indexation has to be determined numerically. Figure 5 shows the resulting value (the net value of the put option and the written call option). Somewhat surprisingly, the value of the "collar" type pension is about the same as the value of a fully indexed pension (without a nominal floor). The reason is that, with our parameters, the probability of negative inflation is quite high and therefore the put option that provides the nominal floor is expensive.

In the Netherlands, the decision to grant indexation depends on the funding ratio, defined as the ratio of assets to liabilities

$$FR = A/L$$

If the funding ratio falls below threshold level (e.g. 100%) indexation is limited or skipped altogether.[9] We can formalize this indexation structure as a call option on

[9]The pension contracts in the Netherlands are not well specified, but this rule conforms to the observed policy and is effectively imposed by the pensions regulators.

inflation triggered by the funding ratio:

$$X_T = L \prod_{t=1}^{T} \max\{\Pi_t/\Pi_{t-1} I(FR_t > 1.0), 1.0\}$$

As the funding ratio depends on the investment policy, we now get a complicated interaction between investment policy and the value of the pension deal. Figure 6 shows the present value of a pension for four example portfolios, picked from the optimal portfolios in Table 2 (stocks and nominal bonds) and Table 3 (stocks and ILBs) for two values of risk aversion, $\gamma = 2$ and $\gamma = 5$. The figure shows that with only nominal bonds in the portfolio, the value of the pension is much lower than when ILBs are available. Moreover, under the more aggressive investment strategy (the $\gamma = 2$ case), the pension is worth less than under the more conservative strategies ($\gamma = 5$). The intuition for this result is straightforward: conditional indexation works as a written call option, which becomes more valuable when the volatility of the investment returns is higher. The conditional indexation makes the pension less valuable, especially when the volatility of the investment returns is high. This would be an argument in favor of a more conservative investment portfolio than the analysis for the utility maximizing long-term investor would suggest.

A possible counterargument is that the equity premium leads to high expected funding surpluses. The expected funding surplus is often the focus of the literature on asset-liability for pension funds, see e.g. Boender (1997). Figure 7 shows the expected surplus of the fund, defined as the expected value of assets minus liabilities, for the four example portfolio strategies. Indeed, the expected surplus is higher with a more aggressive investment policy. An interesting result is that the expected surplus of the portfolios with ILBs is about the same as the expected surplus for the portfolio with nominal bonds. Hence, the superiority of the stock-plus-ILB portfolio in terms of the present value of the pension doesn't come at a cost in terms of expected surplus. However, I find the focus on high expected surpluses somewhat beside the point and potentially misleading. The high values of the surplus coincide with good returns on the stocks, and hence a low value of the deflator. Therefore, the present value of the surplus is not necessarily bigger when volatility is higher. Actually, the option valuation presented just before shows that the present value of the pension is lower with a high volatility.

6 Conclusion and avenues for further research

In this paper I considered the optimal investment policy of pension funds. Using a continuous time long-term investments framework, I showed that the optimal portfolio consists of two parts, a speculative part and a hedge part that covers the long term interest rate and price risks. The speculative part is a stock-bond portfolio with roughly a 55-45 mix for stocks and medium term nominal bonds, or a 65-35 mix for stocks and index linked bonds, plus a substantial leverage if borrowing at the risk free rat is allowed. The hedge part depends on whether index linked bonds are available. With nominal bonds only, the calibrations of the model suggest to invest in medium term, around 5 year, nominal bonds. The hedge effectiveness of this portfolio is low because of the substantial unhedgeable inflation risk that nominal bonds carry. For the hedge portfolio, an index linked bond with the same maturity as the pension liability is the ideal asset as it provides the pension fund with a perfect hedge instrument. With human capital, the investor leverages up the speculative portfolio and shortens the duration of the hedge portfolio.

The overall optimal portfolio depends on the risk aversion assumed for the fund. A very conservative fund that aims to replicate Defined Benefit guarantees should invest almost exclusively in long-term index linked bonds. Without ILBs, medium term nominal bonds are the best alternative, but the hedge effectiveness of this policy is very limited with large unhedged inflation and interest rate exposures. A fund with less risk averse or younger members should invest more in stocks and, if allowed, take on some leverage. Effectively then the fund partially runs a speculative investment portfolio on behalf of its participants. The drawback of a more aggressive investment strategy is that the implicit limited indexation options in the plans become more valuable, decreasing the present value of the future pension payouts to the participants. For an optimal design of a pension fund investment strategy, both features have to be traded off.

In future research, we plan to explore a number of issues. First, the model so far is based on price indexation, whereas in practice many pension plans are based on wage indexation. Also, the asset allocation model ignores the value of human capital of the participants. Adding a wage equation and the value and correlation of human capital with other risk factors will add realism to the model. A second important extension will be to look at the intergenerational transfers of wealth and risk within a pension fund. A pension plan invests on behalf of many generations, who may have different risk preferences, wealth and investment horizons. The conditional indexation options are written by future generations to the present generations, and valuing these options is important to judge how attractive participation in a pension plan is for an individual. Cui, de Jong, and Ponds (2005) provide a first analysis of these questions. A third extension is to explicitly add regulatory constraints to the model, and determine the optimal investment policy that satisfies these constraints.

Appendix

In this appendix we show how to derive the optimal portfolio of stocks, nominal bonds and index linked bonds, with and without cash positions. The basis is the dynamics of optimal real wealth, derived by Brennan and Xia (2002)

$$d\ln G_t = [..]dt + \left[-\frac{1}{\gamma}\phi' - \left(1-\frac{1}{\gamma}\right)\sigma_{tT}\right] dZ_t \equiv c' dZ \tag{28}$$

where $\sigma_{tT} = (0, \sigma_r B(T-t), 0, 0)'$. This dynamics has to be equated with the actual (feasible) wealth dynamics at $t = 0$, the time of planning the wealth.

The menu of assets consists of risky assets, with portfolio weights x and nominal price dynamics

$$dP/P = [..]dt + \sigma dZ \tag{29}$$

Therefore, the real wealth dynamics are given by

$$dX^*/X^* = [..]dt + (x'\sigma - \xi')dZ \tag{30}$$

where $\xi = (\xi_S, \xi_r, \xi_\pi, \xi_u)'$ are the loadings of the price level on the factors. The optimal portfolio is given by minimizing the norm of the difference between optimal and feasible wealth dynamics

$$\min_x ||(x'\sigma - \xi' - c')dZ|| \tag{31}$$

with $||a'dZ|| = a'\rho a$. If there is a nominally risk free asset, x is unconstrained and the first order condition is

$$\sigma\rho(\sigma' x - \xi - c) = 0 \tag{32}$$

and hence the optimal portfolio rule follows by substituting c as

$$\begin{aligned} x^{opt} &= (\sigma\rho\sigma')^{-1}\left[-\frac{1}{\gamma}\sigma\rho\phi - \left(1-\frac{1}{\gamma}\right)\sigma\rho\sigma_{0T}'\right] + (\sigma\rho\sigma')^{-1}\sigma\rho\xi \\ &= (\sigma\rho\sigma')^{-1}\left[\frac{1}{\gamma}\sigma\lambda + \left(1-\frac{1}{\gamma}\right)\sigma\rho(b_T+\xi)\right] \end{aligned} \tag{33}$$

where $b_T = (0, -B(T)\sigma_r, 0, 0)'$ and $\lambda = -\rho(\phi - \xi)$.

With human capital, the actual real wealth dynamics become

$$dW/W = (1-h)dX^*/X^* + hdH/H = [..]dt + (1-h)(x'\sigma - \xi')dZ + h(\sigma_H - \xi')dZ \quad (34)$$

where $h = H/W$ is the ratio of human capital to total wealth and σ_H is the exposure of human capital to the risk factors. The optimal portfolio then becomes

$$x^{opt} = \frac{1}{1-h}(\sigma\rho\sigma')^{-1}\left[\frac{1}{\gamma}\sigma\lambda + \left(1 - \frac{1}{\gamma}\right)\sigma\rho(b_T + \xi) - h\sigma\rho\sigma_H'\right] \quad (35)$$

Without a risk free asset, we have to impose the constraint $\iota' x = 1$. The first order conditions for optimality are

$$\sigma\rho(\sigma' x - \xi - c) - \ell\iota = 0 \quad (36)$$

where ℓ is the Lagrange multiplier for the constraint $\iota' x = 1$. Solving for ℓ gives

$$\ell = \frac{1 - \iota' x_{opt}}{\iota'(\sigma\rho\sigma')^{-1}\iota} \quad (37)$$

The optimal portfolio with constraints then is

$$x^* = x^{opt} + (1 - \iota' x^{opt})x^{min}, \quad x^{min} = \frac{(\sigma\rho\sigma')^{-1}\iota}{\iota'(\sigma\rho\sigma')^{-1}\iota} \quad (38)$$

where x^{min} is the minimum variance portfolio of the risky assets.

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Table 1: **Parameters for the Brennan and Xia model**

The table shows the parameter values from Brennan and Xia (2002), and alternative values, where applicable, as used in this paper.

	Brennan-Xia	alternative
α	0.027	
κ	0.630	0.105
σ_S	0.158	
σ_r	0.026	0.013
σ_π	0.014	
σ_Π	0.013	
ρ_{Sr}	-0.129	
$\rho_{S\pi}$	-0.024	
$\rho_{r\pi}$	-0.061	
λ_S	0.343	0.200
λ_r	-0.209	-0.100
λ_π	-0.105	-0.050
λ_Π	NA	0

Table 2: **Optimal portfolio of stocks and nominal bonds**

The table shows the optimal portfolio composition in the Brennan and Xia (2002) model for an investor with CRRA risk aversion coefficient γ and investment horizon of 20 years. The menu of assets is stocks, nominal bonds (5 year or 20 year maturity) and cash. The rows x^{opt} show the unrestricted optimal portfolio, the rows x^{optc} show the optimal portfolio with borrowing constraints ($x_{cash} \geq 0$) for different values of the relative risk aversion coefficient γ.

	stock	5yr	cash	stock	20yr	cash
risk premium	3.16	0.83		3.16	2.17	
st. dev.	15.80	8.03		15.80	23.61	
correlation		0.101			0.081	
x^{spec}	1.21	1.05	-1.26	1.23	0.32	-0.55
x^{hedge}	0.05	0.78	0.17	0.07	0.18	0.75
R^2_{hedge}			0.337			0.170
x^{opt}						
$\gamma = 1$	1.21	1.05	-1.26	1.23	0.32	-0.55
$\gamma = 2$	0.63	0.91	-0.54	0.65	0.25	0.10
$\gamma = 5$	0.28	0.83	-0.11	0.30	0.21	0.49
$\gamma = 10$	0.17	0.80	0.03	0.18	0.20	0.62
x^{optc}						
$\gamma = 1$	0.99	0.01	0	0.84	0.16	0
$\gamma = 2$	0.53	0.47	0	0.65	0.25	0.10
$\gamma = 5$	0.26	0.74	0	0.30	0.21	0.49
$\gamma = 10$	0.17	0.80	0.03	0.18	0.20	0.62

Table 3: **Optimal portfolio of stocks and 20 year index linked bond**

The table shows the optimal portfolio composition in the Brennan and Xia (2002) model for an investor with CRRA risk aversion coefficient γ and investment horizon of 20 years. The menu of assets is stocks, an index linked bond (20 years maturity) and cash. The rows x^{opt} show the unrestricted optimal portfolio, the rows x^{optc} show the optimal portfolio with borrowing constraints ($x_{cash} \geq 0$) for different values of the relative risk aversion coefficient γ.

	stock	ILB	cash
risk premium	3.16	1.09	
st. dev.	15.80	10.94	
correlation		0.128	
x^{spec}	1.21	0.68	-0.89
x^{hedge}	0.00	1.00	0.00
R^2_{hedge}			1.000
x^{opt}			
$\gamma = 1$	1.21	0.68	-0.89
$\gamma = 2$	0.60	0.84	-0.44
$\gamma = 5$	0.24	0.94	-0.18
$\gamma = 10$	0.12	0.97	-0.09
x^{optc}			
$\gamma = 1$	0.94	0.06	0
$\gamma = 2$	0.47	0.53	0
$\gamma = 5$	0.19	0.81	0
$\gamma = 10$	0.09	0.91	0

Table 4: **Optimal portfolio of stocks and bonds with human capital**

The table shows the optimal portfolio composition in the Brennan and Xia (2002) model for an investor with CRRA risk aversion coefficient γ an investment horizon of 20 years and human capital ratio h. The menu of assets is stocks, bonds (nominal 5 year or index linked 20 year maturity) and, in the first panel, cash. The rows x^{opt} show the unrestricted optimal portfolio, the rows x^{optc} show the optimal portfolio with borrowing constraints ($x_{cash} \geq 0$) for different values of the relative risk aversion coefficient γ.

	stock	bond	cash		stock	ILB	cash
x^{opt}							
				$h = 0.0$			
$\gamma = 2$	0.63	0.91	-0.57		0.60	0.84	-0.44
$\gamma = 5$	0.28	0.83	-0.11		0.24	0.94	-0.18
				$h = 0.5$			
$\gamma = 2$	1.22	1.25	-1.48		1.21	0.94	-1.15
$\gamma = 5$	0.53	1.09	-0.61		0.48	1.13	-0.61
				$h = 0.8$			
$\gamma = 2$	3.01	2.27	-4.28		3.01	1.23	-3.25
$\gamma = 5$	1.26	1.86	-2.12		1.21	1.71	-1.91
x^{optc}							
				$h = 0.0$			
$\gamma = 2$	0.53	0.47	0		0.47	0.53	0
$\gamma = 5$	0.26	0.74	0		0.19	0.81	0
				$h = 0.5$			
$\gamma = 2$	0.96	0.04	0		0.86	0.14	0
$\gamma = 5$	0.42	0.58	0		0.30	0.70	0
				$h = 0.8$			
$\gamma = 2$	2.24	-1.24	0		2.04	-1.04	0
$\gamma = 5$	0.88	0.12	0		0.63	0.37	0

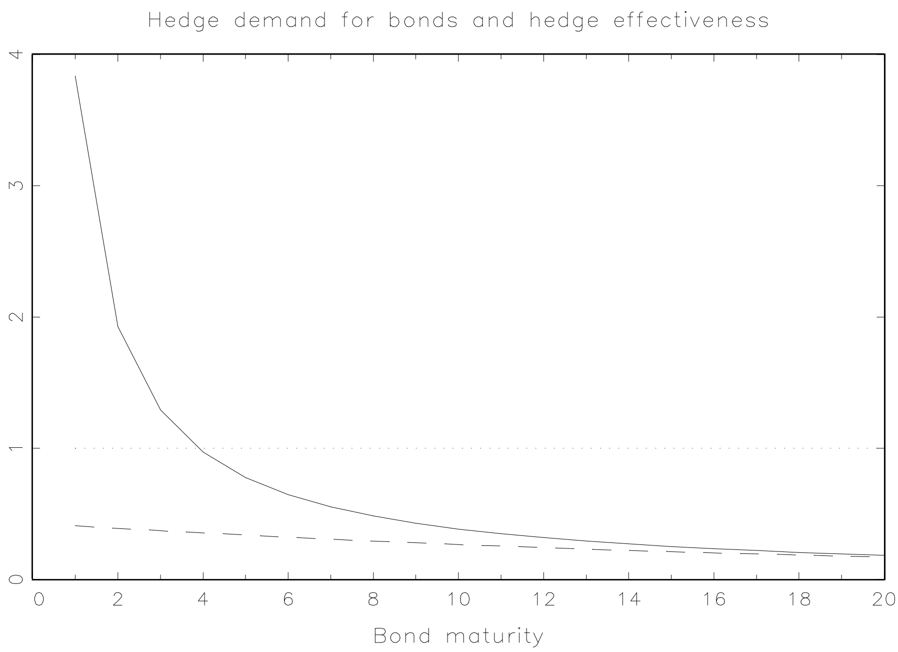

Figure 1: **Optimal bond weight in hedge portfolio**
The solid line graphs the optimal weight of a nominal bond in the hedge portfolio, as a function of the bond maturity. The dashed line graphs the hedge effectiveness (R^2_{hedge}).

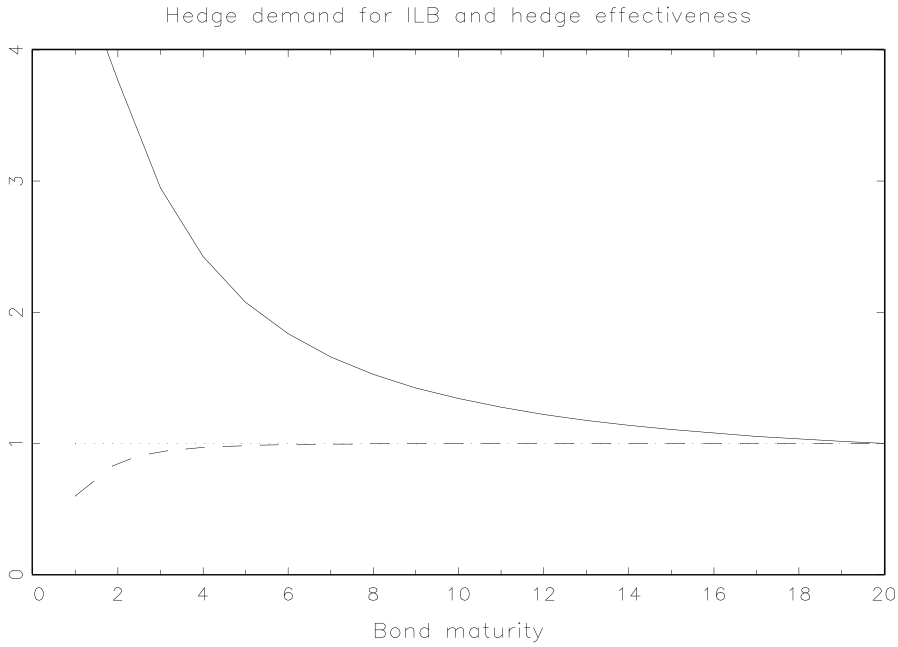

Figure 2: **Optimal index linked bond weight in hedge portfolio**
The solid line graphs the optimal weight of an index linked bond in the hedge portfolio, as a function of the bond maturity. The dashed line graphs the hedge effectiveness (R^2_{hedge}).

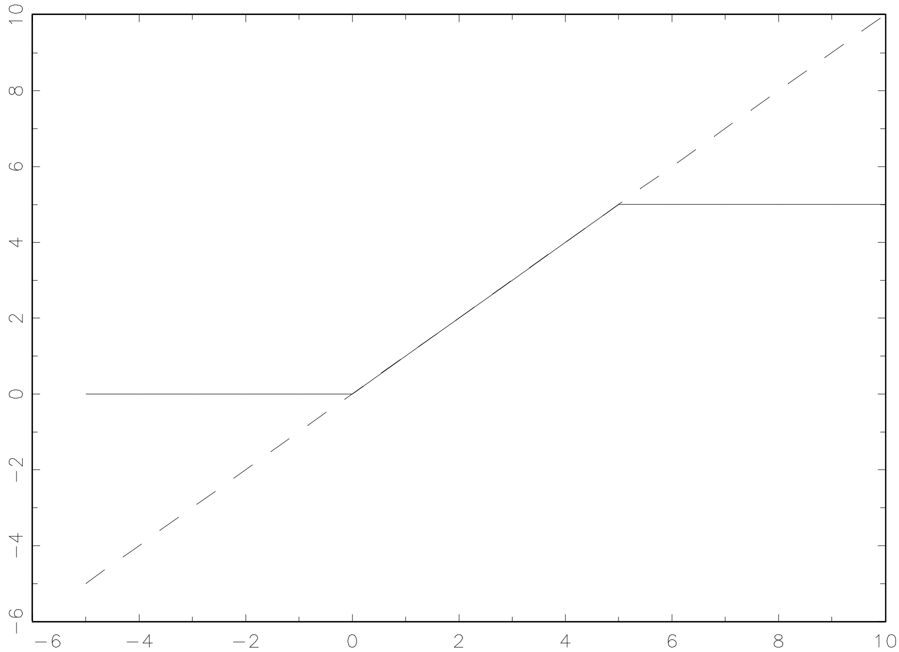

Figure 3: **Collar type indexation rule, limiting indexation between 0% and 5%**

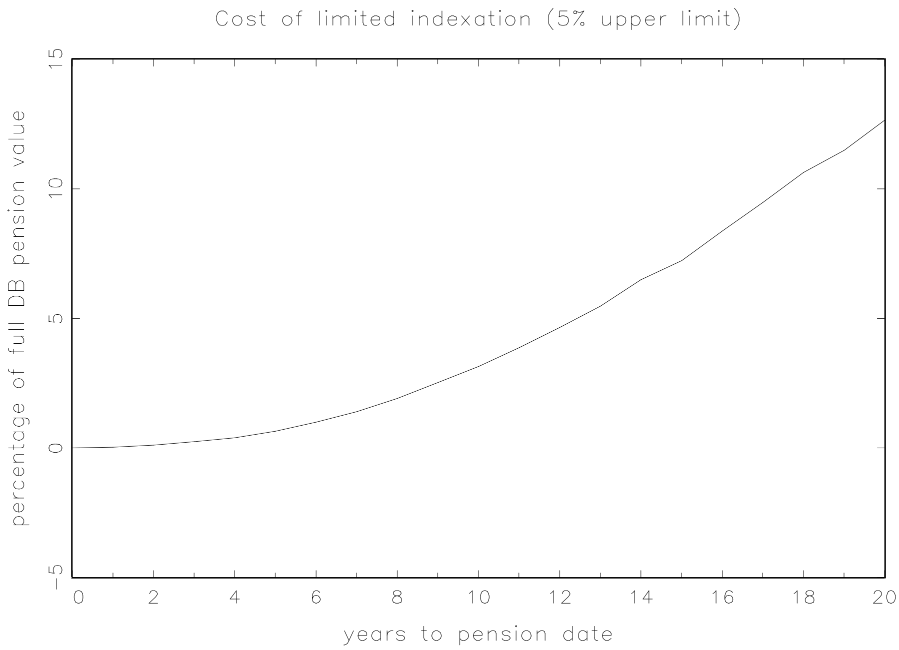

Figure 4: **Value of limiting indexation to 5%**
This figure graphs the value of the limited indexation option as a function of the years to the pension date. The indexation is capped at 5% every year.

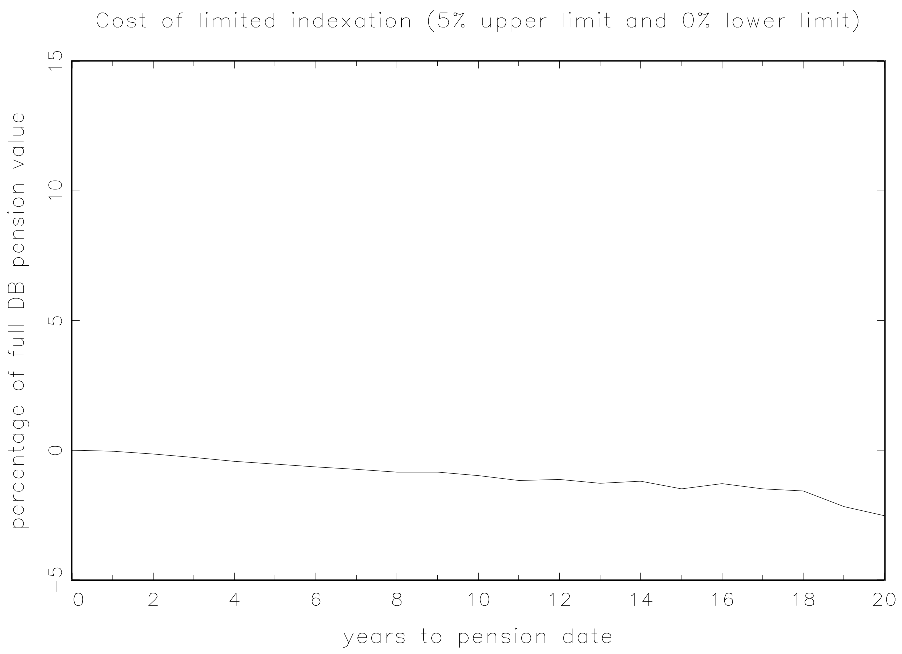

Figure 5: **Value of limiting indexation between 0% and 5%**
This figure graphs the value of the limited indexation option as a function of the years to the pension date. The indexation is capped at 5% every year, and has a floor of 0%.

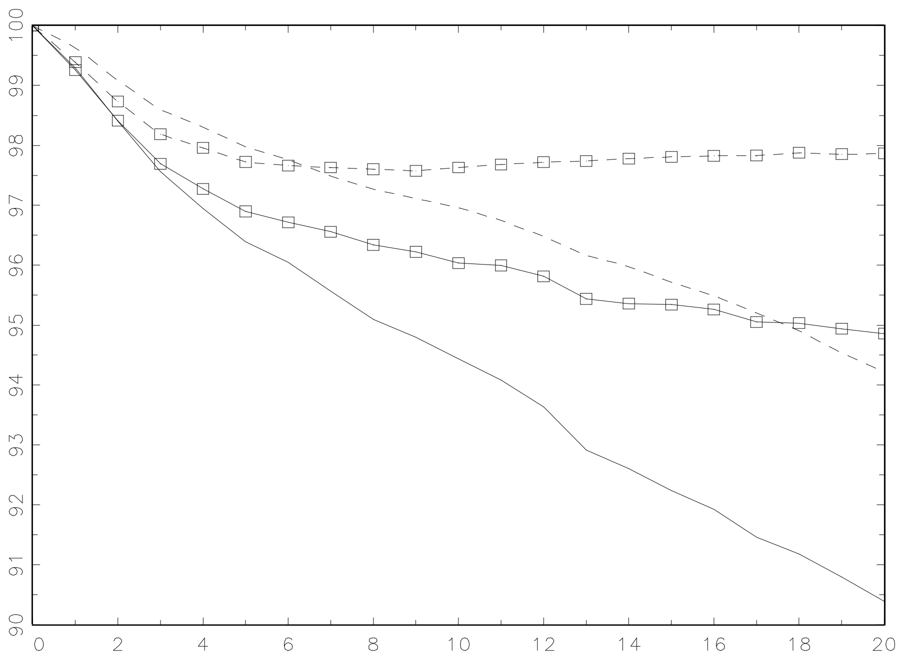

Figure 6: **Present value of pension with conditional indexation**
The value is expressed as a percentage of a fully indexed pension. Straight lines $\gamma = 2$ policy; dashed lines $\gamma = 5$ policy; lines with squares are with ILBs in the portfolio.

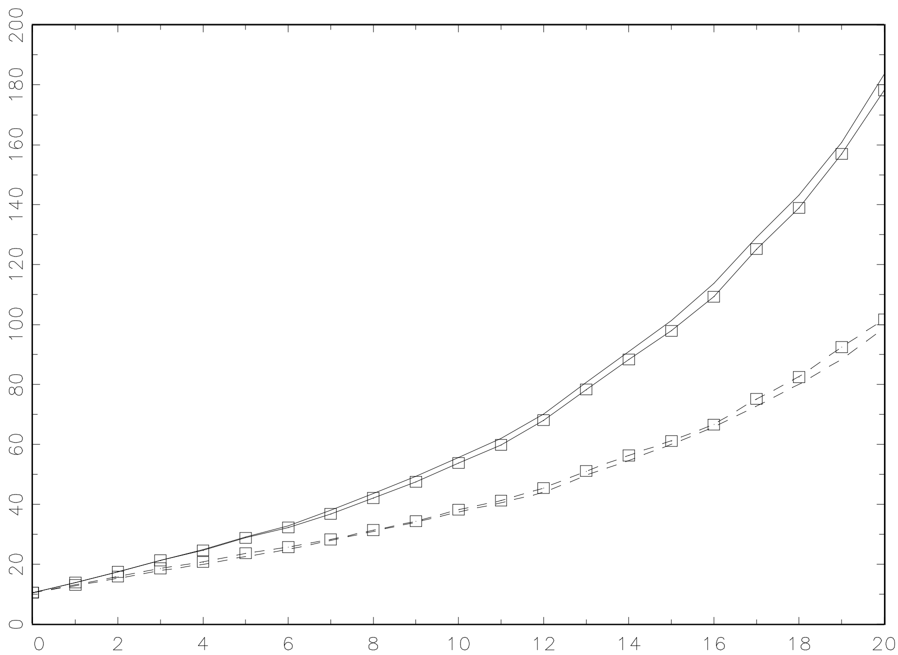

Figure 7: **Expected surplus with conditional indexation**
The expected surplus is expressed as a percentage of a fully indexed pension. Straight lines $\gamma = 2$ policy; dashed lines $\gamma = 5$ policy; lines with squares are with ILBs in the portfolio.