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Abstract

This paper discusses the valuation of wage-indexed pension fund liabilities. Valuation by replication with market instruments is typically not possible as there are no wage-indexed assets. This paper discusses several methods to find a value in such incomplete markets and advocates utility-based valuation. This approach implies a simple adjustment on the discount factor that can be used to calculate the value of wage indexed liabilities.

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1 Introduction

The valuation of liabilities of pension funds is in a state of flux. Traditional actuarial rules prescribe discounting of the expected cash flows at a fixed interest rate (typically 4%), although it is unclear whether this should be applied to the nominal or indexed pension claims. Recently, due to changes in the supervisory and accounting rules, pension funds are required to value their liabilities at market prices. Typically, this is done by discounting the liabilities with a market interest rate for loans with the same maturity and risk characteristics. In practice, this approach poses some challenges, as pension fund liabilities are typically not marketed assets. The long maturity of the claims and, perhaps most importantly, the indexation to price or wage growth make it impossible to find a portfolio of perfectly matching market instruments. Therefore, finding the right discount rate is problematic.

In the literature on financial asset pricing (see e.g. Duffie, 1996), the situation where a payoff cannot be replicated perfectly using market instruments is referred to as a situation of an incomplete market. In such a setting, part of the payoff risk on the derivative is uncorrelated to the price changes in the replicating assets. Reasons why replication may be imperfect are the lack of traded instruments, infrequent (non-continuous) trading and transaction costs. For the case of pension liabilities, the major source of market incompleteness is the absence of financial market instruments that exactly replicate the price or wage index that the pensions are linked to. Although there may be products that correlate strongly with the price level, such as index linked bonds, this is definitely not the case for wage-linked payoffs. Hence, for the valuation of wage linked pension liabilities one cannot use a replicating argument directly.

The academic literature offers a few methods for the pricing of assets in incomplete markets. A first approach is to find a super-replicating portfolio.[1] This is a portfolio whose payoffs are always (in any state of the world) at least as big as the payoff of the derivative. The value of the derivative is then bounded by the value of the super-replicating portfolio. Although for some products, such as options, this may be a useful approach, for pension fund liabilities this approach is not very helpful as the unhedgeable wage growth is in principle unbounded and no super-replicating portfolio can be found. A second approach, developed by Cochrane and Saá-Requejo (2000)

[1]For applications of this idea see e.g. Cvitanic, Pham and Touzi (1998, 1999).

weakens the no-arbitrage argument and finds arbitrage strategies that have bounded Sharpe ratio's. Cochrane and Saá-Requejo claim that even in incomplete markets, this argument may lead to very tight bounds on option values. We shall investigate this method later in the paper for the case of wage risk.

The final approach that we consider is utility-based valuation. The investor in the pension liability assumes unhedgeable risk, which will affect the probability distribution of his consumption and final wealth level, and hence his utility. The certainty equivalent wealth of the expected utility then is the value an investor is prepared to pay for the claim. Conversely, we may ask the question how much wealth the pension fund should invest in the market to give their members the same utility as a fully guaranteed wage indexed pension. This wealth then is the "shadow" market value of the (partially) unhegdeable claim.

In this paper, we shall further develop the second and third approach for the specific case of pension liabilities and (partly) unhedgeable wage risk. We shall demonstrate that finding a value for the pension liability amounts to finding a value for the price of the residual wage risk. We show that this valuation method can be implemented in a simple way by adjusting the discount rate.

2 Valuation principles

The workhorse of asset pricing is the pricing kernel or stochastic discount factor. This is a random variable that can be used to price any asset in the economy. In shorthand notation, the price of an asset is given as[2]

$$P_0 = \mathsf{E}\left[\sum_{t=1}^{T} M_t X_t\right] \tag{1}$$

where X_t are the (random) cash flows generated by the asset at time t and M_t is the pricing kernel. It can be demonstrated that such a pricing kernel exists in any economy where the law of one price holds. If we also assume absence of arbitrage, the pricing kernel is always positive, i.e. a positive payoff will always have a positive price. The pricing kernel is not necessarily unique; there may be many M's that give the correct prices. Only in the case where markets are complete the pricing kernel is unique.

[2]Unless otherwise indicated, all expectations are as of time zero ($t = 0$).

The pricing kernel can be interpreted as a stochastic discount factor, which gives values to payoffs in each possible state of the world. To simplify matters, suppose there is only one payoff, X_T, at time T and S possible values for X_T. The price of this instrument then is given by

$$P_0 = \mathsf{E}\left[M_T X_T\right] = \sum_{s=1}^{S} p(s) M_T(s) X_T(s) \tag{2}$$

where $p(s)$ is the probability of state s. We can interpret $M_T(s)$ as a weight attached to the payoff $X_T(s)$, and the price as the expectation of this weighted payoff.

If there is a risk free asset in the economy, the expectation of the pricing kernel is simply the usual risk-free discount factor. For the risk free asset, the payoff is constant: $X_T^f(s) = 1$ for all s. The price obviously should be $P = \exp(-rT)$ where r is the (continuously compounded) risk-free interest rate. Hence, we must have that

$$\mathsf{E}[M_T X_T^f] = \mathsf{E}\left[M_T\right] = \exp(-rT) \tag{3}$$

Of course, for finding the value of a risk-free asset, the pricing kernel approach is not really interesting. The interesting applications concern the pricing of assets with random payoffs and derivatives of these assets.

An important example of such a setting is the Black and Scholes (1973) asset pricing model. In this model, the prices of the basic asset (a stock) follows a geometric Brownian motion with expected return μ and volatility σ:

$$dS/S = \mu dt + \sigma dZ \tag{4}$$

In addition to this stock, there exists a risk free asset with constant return r. Cochrane (2001) shows that the pricing kernel in this model is

$$M_t = \exp(-rt) \exp\left(-\tfrac{1}{2}\lambda^2 - \lambda Z_t\right) \tag{5}$$

where Z_t is the Brownian motion driving the stock price, and λ is the Sharpe ratio of the stock

$$\lambda = \frac{\mu - r}{\sigma} \tag{6}$$

The interpretation for this pricing kernel is simple. The first part is the risk-free discount factor, and the second part is random variable which puts a high weight on payoffs that occur at low values of the stock (i.e. low values of Z_t). This is particularly relevant for the pricing of derivatives. Let's consider a derivative whose payoff is a function of the stock price at time T (any European option, for example) and denote the payoff as $X_T = f(S_T)$. The price of this derivative is then given by

$$P_0 = \mathsf{E}\left[M_T X_T\right] = \mathsf{E}\left[M_T f(S_t)\right] = \mathsf{E}\left[\exp(-rt)\exp\left(-\tfrac{1}{2}\lambda^2 - \lambda Z_t\right) f\left(S_t\right)\right] \tag{7}$$

Payoffs in states with high stock prices will receive a low weight (as long as $\lambda > 0$) because of the $-\lambda Z_t$ term in the SDF. Vice versa, payoffs in low stock price states will have a high weight. Put options will therefore be relatively expensive, compared to their (undiscounted) expected payoff, and call options relatively cheap.[3] The economic logic for this is quite simple: put options provide an insurance against low stock prices, and the market is prepared to pay a price for that insurance.

Finally, we note that the dynamics of the SDF are given by

$$dM_t/M_t = -rdt - \lambda dZ_t \tag{8}$$

This equation only depends on the risk-free interest rate, which captures the time value of money, and the Sharpe ratio of the stock, which captures the risk-return tradeoff for stock market exposure. This property holds more generally: with multiple sources of risk in the economy, the stochastic discount factor can typically be written as a function of the risk free rate, the Sharpe ratio's of marketed assets and the shocks to the asset prices. Typically, the dynamics of the SDF are given by

$$dM_t/M_t = -rdt - \lambda' dZ_t \tag{9}$$

where λ and dZ_t are now vectors of Sharpe ratio's and stochastic shocks, respectively.

2.1 Asset pricing in incomplete markets: Good deal bounds

In the Black-Scholes example, there is only one source of uncertainty, the stock price, and the market is (dynamically) complete, since the payoff of any stock price derivative can be attained by a trading strategy in the stock and the risk-free asset. This is of course a very stylized setting and in practice, the payoff of claims such as pension liabilities depend on a multitude of risk factors. Moreover, some of these risk factors cannot be replicated by trading in financial market instruments. How to deal with pricing and valuation in such a situation?

[3]For simple put and call options, working out the expectations in equation (7) leads to the well-known Black-Scholes option pricing formula.

One guideline is given by Cochrane and Saá-Requejo (2000).[4] They study the pricing of a claim which can not be replicated perfectly by trading in financial market assets. Suppose the uncertainty in the payoffs is generated by the stock price from equation (4) and an additional state variable, whose dynamics can be described by

$$dV_t/V_t = \mu_V + \sigma_V dZ_t + \sigma_{Vw} dZ_t^w \tag{10}$$

where Z_t^w is a Brownian motion that is independent of Z_t. The coefficients σ_V and σ_{Vw} denote the exposure of the state variables to the stock price shocks and the independent Brownian motion.

Cochrane and Saá-Requejo (2000) show that the pricing kernel in this economy has dynamics

$$dM_t/M_t = -rdt - \lambda dZ_t - \lambda_w dZ_t^w \tag{11}$$

where, as before, λ is the Sharpe ratio of the stock and λ_w is an unknown coefficient. The no-arbitrage framework doesn't say anything about the value of λ_w, so that we're left with an infinity of possible SDF's and, by implication, infinitely many possible values for contingent claims that depend on the value of the state variable V. The insight of Cochrane and Saá-Requejo (2000) is that λ_w can be interpreted as the Sharpe ratio of a (hypothetical) asset that has a non-zero exposure to dZ_t^w and no exposure to dZ_t. Now Cochrane and Saá-Requejo (2000) suggest to put a bound on the absolute value of this Sharpe ratio, such that too 'good deals', i.e. assets or positions with too high (absolute) Sharpe ratios, are ruled out. The bounds on the Sharpe ratio then imply an upper and lower bound for the value of any contingent claim. Of course, the magnitude of the bound on the Sharpe ratio is arbitrary, but CS show that even fairly generous values for the bound may lead to very tight ranges for the implied prices of derivatives such as options.

The question now is how useful this framework is for the valuation of pension fund liabilities. The main source of unhedgeable risk seems to be real wage growth. As this applies to the full value of the liabilities, the implied bounds can actually be pretty wide. To give an example, suppose that wages are not correlated with stock prices, hence $\sigma_V = 0$, and the Sharpe ratio of wage-linked assets is bounded by $|\lambda_w| < A$. Then the discount rate to be applied to the expected payoff of a wage linked bond is bounded between $(r - A\sigma_w, r + A\sigma_w)$, where σ_w is the volatility of wage growth. Estimates in

[4] Related ideas are presented by Roorda, Schumacher and Engwerda (2004).

section 4 show that the real interest rate (in excess of the real wage growth) is around 1.5%, and the long-run annualized variance of the real wage growth is 4.5%. We choose $A = 0.2$, which corresponds roughly to the equity market Sharpe ratio.[5] With these numbers, the discount rate is in the range $(0.6\%, 2.4\%)$. Assuming a liability of 100 million euro with a duration of 20 years, this implies a range of present values between 62.23 and 88.72 million. This is a fairly wide bound and may be too wide to get good judgements of the solvency of the fund. For example, the requirement that the fund has a solvency ratio of 105% will translate into a required asset value of 65 million (lower bound) or 93 million (upper bound), roughly a fifty percent difference.

2.2 Utility-based pricing

A third approach to pricing in incomplete markets is based on the utility a representative agent can derive from investing in the untraded claim. This approach has been used extensively in the literature, for example by Svensson and Werner (1993), Henderson (2002, 2005) and Pelsser (2005). These papers study a situation where a new derivative is added to the existing portfolio of an investment bank or a new product is added to the portfolio of an insurer, and part of the risk on the new derivative or product cannot be hedged using market instruments. Our analysis builds on these papers and is also closely related to work on asset allocation in incomplete markets, see for example the work by Brennan and Xia (2002), Munk and Sørensen (2005) and Van Hemert, De Jong and Driessen (2005). We now first discuss the general setup of the utility maximization problem and show how we use it to find the value for a non-traded claim. In the next section we present a detailed model that is explicitly targeted at valuing long-dated pension claims.

The setting we study is a stylized representation of an individual saving for retirement. The main simplification is that we consider a finite horizon, terminal wealth problem and ignore intermediate consumption and labour income.[6] Thus, the economic

[5]Here we use an estimate of the equity premium $\mu_S - r$ equal to 4% as in Fama and French (2002), and a stock price volatility σ_S of 20%. Together, this implies a Sharpe ratio $\lambda_S = 0.04/0.20 = 0.2$.

[6]Brennan and Xia (2002) show that ignoring interim consumption does not affect the results of the optimal investment policy much if the horizon T is chosen appropriately. Ignoring labour income matters more, as this provides an automatic hedge against wage risk. However, analytical solutions for the optimal investment strategy and the utility level do not exist for such a setting. Papers in this literature therefore rely on numerical results instead, see e.g. Munk and Sørensen (2005), and Koijen,

agent maximizes expected utility of terminal wealth over all the possible investment strategies. First consider the complete market problem, where any value of terminal wealth can be attained by an investment strategy in financial market instruments. In that case, the agent solves the problem

$$\max_{W_T} \mathsf{E}\left[U(W_T)\right] \quad \text{s.t.} \quad \mathsf{E}[W_T M_T] = W_0 \tag{12}$$

where $U(W)$ is the utility function, W_0 the initial wealth and the latter expression is the budget constraint. Denote the solution to this problem by W_T^*. From the mathematical finance literature (see e.g. Pliska, 1997) it is well known that the optimal investment strategy is given by

$$W_T^* = U'^{-1}\left(\ell M_T\right) \tag{13}$$

where ℓ is the Lagrange multiplier of the budget constraint.

For the incomplete markets, we study the special case where terminal wealth can be decomposed as the product of a perfectly hedgeable wealth component W_T and an independent (residual) component Y_T. As a normalization, we assume $\mathsf{E}[Y_T] = 1$. The agent then solves the problem over all feasible values for W_T, subject to a budget constraint

$$\max_{W_T} \mathsf{E}\left[U(W_T Y_T)\right] \quad \text{s.t.} \quad \mathsf{E}[W_T M_T] = W_0 \pi_0 \tag{14}$$

In equation (14), π_0 denotes the additional initial wealth needed to give the agent the same utility as in the complete markets case. Denoting the optimal wealth strategy of the incomplete markets problem by $\hat{W}_T$, this leads to the condition

$$\mathsf{E}\left[U(\hat{W}_T Y_T)\right] = \mathsf{E}\left[U(W_T^*)\right] \tag{15}$$

We now make one more (restrictive) assumption, and that is that the utility function allows the separation of W_T and Y_T as follows

$$U(W_T Y_T) = U(W_T) U(Y_T) \tag{16}$$

For example, the constant relative risk aversion utility function (CRRA) has this property. With this type of utility function, we can show a few things. First, the optimal wealth strategy in the incomplete markets case is a scaled version of the optimal wealth strategy in the complete markets case

$$\hat{W}_T = c W_T^* \tag{17}$$

Nijman and Werker (2005).

The linearity of the budget constraints immediately implies that $c = \pi_0$. The condition that the expected utilities are equal can now be written as

$$\mathsf{E}\left[U(W_T^*)\right] = \mathsf{E}\left[U(\hat{W}_T Y_T)\right] = U(\pi_0)\mathsf{E}\left[U(W_T^*)\right]\mathsf{E}\left[U(Y_T)\right] \tag{18}$$

from which follows

$$\pi_0 = U^{-1}\left(\frac{1}{\mathsf{E}[U(Y_T)]}\right) \tag{19}$$

It is easy to show that $\pi_0 > 1$ for any utility function that exhibits risk aversion, i.e. $\mathsf{E}[U(Y_T)] < U(\mathsf{E}[Y_T])$. Hence, the investor should invest more than in the complete markets case to obtain the same expected utility.

3 Model

We now present a specific model for a long-horizon investor who faces wage growth risk. We build on the model and notation of Brennan and Xia (2002). Their model consists of equations for four state variables: the stock price S, the instantaneous real interest rate r, the instantaneous expected inflation π^e, and the price level Π. We leave out the equations for the inflation and price level, as we're in a real economy with index linked bonds. We extend the model with an equation for the real wage, w. The equations driving these variables are

$$dS/S = (r + \lambda_S \sigma_S)dt + \sigma_S dZ_S \tag{20a}$$

$$dr = \kappa(\bar{r} - r)dt + \sigma_r dZ_r \tag{20b}$$

$$dw/w = g dt + \theta' dZ + \theta_w dZ_w \tag{20c}$$

where the vector $Z = (Z_S, Z_r)$ and the covariance matrix of dZ is ρ. In the last equation, g denotes the expected real wage growth. The term $\theta' dZ$ denotes the exposure of real wage growth to the shocks in stock returns, interest rates and inflation. The final term $\theta_w dZ_w$ denotes the idiosyncratic wage risk, which is by assumption uncorrelated with all the other risk factors. The assumption in equation (20c) is that the real wage is serially uncorrelated. The model can easily be extended to serially correlated real wage growth, at the expense of more heavy notation.

We assume that in the economy there is a full menu of index linked bonds, so that real interest rate risk can be hedged perfectly. Price indexed asset values are determined from the real pricing kernel

$$dM_{1t}/M_{1t} = -rdt + \phi' dZ \tag{21}$$

where $\phi = -\rho^{-1}\lambda$ and λ is the vector of real market prices of risk or Sharpe ratio's. Brennan and Xia (2002) show that the dynamics of the real value of an index linked bond (ILB) are given by

$$dP_{ILB}/P_{ILB} = [r - B\sigma_r\lambda_r]dt - B\sigma_r dZ_r \tag{22}$$

with and B is a function of the time to maturity of the bond, that follows from the Vasicek (1977) model.[7]

3.1 Optimal wealth and portfolio strategy

We assume that the objective of the consumer is to maximize the utility of wage-deflated terminal wealth. This is different from the Brennan and Xia (2002) model, where the agent maximizes the utility of price deflated wealth. This is our way of introducing wage risk into the model. The optimization problem is

$$\max \mathsf{E}\left[U(W_T/w_T)\right], \text{ s.t. } \mathsf{E}\left[W_T M_{1T}\right] = W_0 \tag{23}$$

where W_T denotes real wealth, w_T the real wage level index, and M_{1T} is the real pricing kernel. The real wage level is decomposed as $w_t = w_{1t}w_{2t}$ with[8]

$$dw_1/w_1 = (g - \theta_w^2)dt + \theta' dZ \tag{24a}$$

$$dw_2/w_2 = \theta_w^2 dt + \theta_w dZ_w \tag{24b}$$

According to Brennan and Xia (2002), the optimal real wealth strategy W_T^* is independent of w_{2T} because this is just orthogonal background risk that cannot be hedged and therefore does not affect the optimal wealth strategy.[9] Solving for the optimal real

[7]The exact expression is $B(\tau) = (1 - \exp(-\kappa\tau))/\kappa$, where τ is the time to maturity of the bond; λ_r is the element of λ that concerns the real interest rate risk.

[8]To be consistent with the general structure in the previous section, we shall assume $\mathsf{E}\left[\frac{1}{w_{2T}}\right] = 1$, hence the term $\theta_w^2 dt$ in the dynamics of w_2.

[9]The key for this to hold is the CRRA utility function and the multiplicative nature of the unhedgeable risk. The unhedgeable risk therefore does not influence the marginal utility of the hedgeable wealth component (separability). This will be different for the models of Henderson (2002, 2005) and Pelsser (2005), where the utility function is CARA and the risk is additive and therefore does impact the marginal utility of hedgeable wealth.

wealth process gives

$$W_T^*/w_{1T} = \ell M_{1T}^{-1/\gamma} \tag{25}$$

where ℓ is a Lagrange multiplier that is independent of the process for w_{2T}. The indirect utility function therefore can be written as

$$J = \mathsf{E}\left[\left(\frac{W_T^*}{w_{1T}}\right)^{1-\gamma}\right] \mathsf{E}\left[\left(w_{2T}^{-1}\right)^{1-\gamma}\right] /(1-\gamma) \tag{26}$$

Using that $-\ln w_{2T} \sim N(-\frac{1}{2}\theta_w^2,\ \theta_w^2 T)$ we find

$$J = W_0^{1-\gamma} F_1 \exp\left\{-\tfrac{1}{2}(1-\gamma)\theta_w^2 T + \tfrac{1}{2}(1-\gamma)^2\theta_w^2 T\right\}/(1-\gamma) \tag{27}$$

where F_1 is independent of the parameters for w_{2T}. The certainty equivalent wealth of the optimal portfolio strategy is

$$W_{CE} = W_0 F_1^{\frac{1}{1-\gamma}} \exp\left(-\tfrac{1}{2}\gamma\theta_w^2 T\right) \equiv W_0 F_1^{\frac{1}{1-\gamma}} \exp(-\delta T) \tag{28}$$

Hence, the certainty equivalent value of the optimal wealth is equal to the value of the hedgeable part times a discount factor with implied discount rate $\delta = \frac{1}{2}\gamma\theta_w^2$. This discount rate will be positive (i.e. unhedgeable wealth is worth less than fully hedgeable wealth) for any risk averse investor ($\gamma > 0$). The result can also be interpreted in a different way: to achieve the same certainty equivalent wealth as in the complete markets case, one needs to invest a fraction $\pi_0 = \exp(\delta T) > 1$ more in the incomplete markets case.

Adapting the results of Brennan and Xia (2002) and De Jong (2003), we can also show that the optimal portfolio weights are given as

$$x^* = \frac{1}{\gamma}\left(\sigma\rho\sigma'\right)^{-1}\sigma\lambda + \left(1-\frac{1}{\gamma}\right)\left(\sigma\rho\sigma'\right)^{-1}\sigma\rho\left[\theta - \sigma_r B(T) e_r\right] \tag{29}$$

where σ is the matrix of exposure coefficients of the assets included in the portfolio problem, typically stocks and price indexed bonds. The first component of this expression is the usual mean-variance optimal speculative portfolio, and the second component contains the hedges against wage risk and interest rate risk. If a wage indexed bonds would exist, the hedge portfolio would solely consist of a zero-coupon wage indexed bond with maturity T, and the hedge against wage and interest risk would be perfect. Without such a wage linked bond, however, the hedge portfolio is the combination of stocks and bonds that shows the highest possible correlation with the (hypothetical) wage linked bond.

3.2 Wage Linked Bonds

The preceding results allow us to define a full pricing kernel for general price and wage-linked claims as

$$dM_t/M_t = -rdt + \phi' dZ + \phi_w dZ_w \tag{30}$$

with $\phi_w = \frac{1}{2}\gamma\theta_w$. The proof is provided in appendix A. The real value of a wage linked bond (WLB) obeys

$$dP_{WLB}/P_{WLB} = [r - B\sigma_r\lambda_r + \theta'\lambda + \theta_w\lambda_w]dt - B\sigma_r dZ_r + \theta' dZ + \theta_w dZ_w \tag{31}$$

where λ is the vector of real risk premiums, and $\lambda_w = -\phi_w = -\frac{1}{2}\gamma\theta_w$ is the (shadow) market price of real wage risk. The risk premium due to the idiosyncratic wage risk therefore is $\theta_w\lambda_w = -\frac{1}{2}\gamma\theta_w^2 < 0$. This negative risk premium implies that wage linked bonds are more expensive that other bonds with the same expected payoff, and reflects the price a risk averse consumer is willing to pay to perfectly hedge wage risk.

We can express this higher price in terms of a lower discount rate used to discount wage indexed claims. The real value of the WLB scaled with the real wage level w follows

$$\frac{d(P_{WLB}/w)}{P_{WLB}/w} = [r - g - B\sigma_r\hat{\lambda}_r + \theta'\lambda + \theta_w\lambda_w]dt - B\sigma_r dZ_r \tag{32}$$

where $\hat{\lambda} = \lambda - \rho\theta$ is the vector of real market prices of risk corrected for correlation with wage growth. This expression can be rewritten as

$$\frac{d(P_{WLB}/w)}{P_{WLB}/w} = [r^w - B\sigma_r\hat{\lambda}_r]dt - B\sigma_r dZ_r \tag{33}$$

with

$$r^w = r - g + \theta'\lambda + \theta_w\lambda_w \tag{34}$$

The variable r^w can be interpreted as the wage-corrected instantaneous real interest rate, i.e. a discount rate for wage indexed claims. For comparison, the price dynamics of a real (price index linked) bond are

$$\frac{dP}{P} = [r - B\sigma_r\lambda_r]dt - B\sigma_r dZ_r \tag{35}$$

Hence, the difference in the discount rate for a WLB and a real bond is

$$\frac{d(P_{WLB}/w)}{P_{WLB}/w} - \frac{dP}{P} = [r^w - r - B\sigma_r(\hat{\lambda}_r - \lambda_r)]dt \tag{36}$$

$$= [-g + \theta'\lambda + \theta_w\lambda_w - B\sigma_r(\hat{\lambda}_r - \lambda_r)]dt \tag{37}$$

Compared to the interest rate on a real (price index linked) bond, the interest rate on a WLB is different in four respects

(i) The discount rate is lower by the expected real wage growth g; this is similar to the classic Gordon stock valuation model model, where the discount rate equals the cost of capital minus the expected dividend growth.

(ii) The discount rate is different because of the correlations between wage growth and the other risk factors, $\theta'\lambda$, which depends on the risk premium on the other risk factors. This term reflects the risk premium earned on the 'best' (but imperfect) hedging portfolio for the wage linked bond.[10]

(iii) The discount rate is lower because of an additional (negative) risk premium $\theta_w\lambda_w = -\frac{1}{2}\gamma\theta_w^2$, which prices the idiosyncratic wage risk.

(iv) Finally, the term premium is different by a factor $B\sigma_r(\lambda_r - \hat{\lambda}_r) = B\sigma_r(\theta'\rho e_r)$, which depends on the maturity of the bond, and reflects the difference in the market prices of risk for price and wage deflated claims.

4 Calibration

In this section, we calibrate the model with wage growth to data from the Netherlands. The sample period is 1950-2002 (annual data), obtained from Statistics Netherlands. Real wage growth is constructed as nominal wage growth minus realized inflation. The data are graphed in Figure 1. It is clear that real wage growth fluctuates quite a bit and also exhibits some persistence, i.e. years with high wage growth are more likely to be followed by another year of high wage growth. The mean and standard deviation of annual wage growth are 1.5% and 2.65%, respectively. The sum of the coefficients in an AR(3) model (selected by the Akaike information criterion) is 0.55, indicating some persistence in the real wage growth process. As this persistence is not explicitly modelled in the model of this paper, we correct the standard deviation of wage growth to correspond to the annualized standard deviation of the 30-year wage growth. For this, we use the formulas of the variance ratio in Campbell, Lo and MacKinlay (1997,

[10]This part of the risk premium can also be written as $\theta'\lambda = \beta'(\mu - r)$, where $\beta = \sigma(\sigma'\sigma)^{-1}\theta$ are the regression coefficients of wage growth on the asset returns and $\mu - r = \sigma\lambda$ is the vector of risk premiums earned on the assets with an exposure σ to the risk factors dZ.

p.49). Corrected for serial correlation, the annualized standard deviation of real wage growth is 4.5%.

The exposure coefficients θ_S and θ_r are estimated using the residual covariance matrix of a VAR(1) model for excess stock returns, the ex-post real interest rate (constructed as the difference between the nominal short rate and inflation) and the real wage growth. The estimates are reported in Table 1. These estimates imply a positive but small exposure of wage growth to stock returns and interest rates. Using an equity Sharpe ratio of 0.2, the stock market exposure translates into a risk premium of $\theta_S \lambda_S = 0.0030 * 0.20 = 0.0006$, or only 6 basis points, which is almost negligible.[11] The real interest rate exposure is also small and leads to a negligible addition to the risk premium.

To give a numerical example, assume a risk aversion $\gamma = 5$ and an idiosyncratic wage risk $\theta_w = 4.5\%$, based on the estimate of real wage volatility corrected for serial correlation. We assume a real interest rate of 3% and an expected real wage growth of 1.5%. The discount rate for wage-linked bonds then is

$$\begin{aligned} r^w &= r - g + \theta' \lambda - \tfrac{1}{2} \gamma \theta_w^2 \\ &= 3.0\% - 1.5\% + 0\% - 0.5 * 5 * (0.045)^2 \\ &= 1.5\% - 0.5\% = 1.0\% \end{aligned} \tag{38}$$

The additional term caused by the market incompleteness is -0.5%, which seems small, but with such low interest rates it has a significant impact on the valuation. The shadow market value of the 100 million liability from the earlier example will be 81.95 million. Ignoring the extra term of the market incompleteness (i.e. assuming a discount rate of 1.5%), the value is 74.25, which makes about a 10% difference.

[11] This result is much smaller than the estimate in Bohn (2005), who reports a long-horizon correlation of 0.41 between GDP growth (which is closely related to wage growth) over a 30-year horizon and capital returns. This correlation and the standard deviations Bohn reports translate to an exposure coefficient $\theta_S = 0.04$, implying a risk premium of 0.8%. In their valuation of wage indexed pension liabilities, Westerhout et al. (2004) use a risk premium due to correlation between wages and the stock market of 1.5%. This number seems too high. They also ignore the negative risk premium due to market incompleteness.

5 Conclusion and implications for pension funds

Pension funds with wage indexed liabilities face two issues in valuating the liabilities. The first is that the expected wage growth has to be taken into account. This can be achieved fairly simply by correcting the real discount rate for the expected real wage growth rate. The second is issue is more complex and arises because real wage risk cannot be hedged perfectly. In other words, the pension fund cannot replicate a wage exactly with its asset mix. Although the asset mix may be chosen strategically to include assets that show correlation with wage growth, a non-hedgeable residual wage risk remains. Then, to offer the pension fund members the same expected utility as a perfect wage indexed pension, the fund should invest initially more money (and hence pay out more on average) to compensate for this risk. In this paper, we show that this additional value can be calculated fairly easily by lowering the discount rate of the future claims by 0.5%. In our stylized pension fund example, taking the unhedgable nature of wage risk into account implies around 10% higher value for wage linked liabilities compared to discounting the expected payoff with market interest rates.

A Pricing kernel with real wage risk

Consider the valuation of a payoff perfectly linked to the wage level. This instrument will raise the certainty equivalent wealth of the investor in the incomplete market to the certainty equivalent wealth of the complete markets case. The relative increase in wealth is $\exp(\delta T)$. Denoting the payoff of this instrument by P, we can split this in a hedgeable part P_1 and an unhedgeable part P_2. The value will be determined by

$$P = \mathsf{E}[M_{1T} P_{1T}] \mathsf{E}[M_{2T} P_{2T}] \tag{39}$$

where $M = M_1 M_2$ is the full pricing kernel. The latter part of the expression should equal the increase in certainty equivalent wealth for the investor, hence

$$\mathsf{E}[M_{2T} P_{2T}] = \exp(\delta T) \tag{40}$$

Given that $dP_2/P_2 = \theta_w dZ^w$ and assuming $dM_2/M_2 = \phi_w dZ^w$ we find that

$$\theta_w \phi_w = \delta = \tfrac{1}{2} \gamma \theta_w^2 \tag{41}$$

and hence $\phi_w = \frac{1}{2} \gamma \theta_w$.

An alternative way to find the pricing kernel parameter follows an argument by Sangvinatsos and Wachter (2005). They argue that the right parameter ϕ_w is the one that makes the demand for Wage Linked Bonds by the investor equal to zero. Brennan and Xia (2002) derive that the ratio of the certainty equivalent wealth for the investor in a market with a WLB over a market without a WLB is

$$\frac{W_{CE}^{WLB}}{W_{CE}} = \exp\left\{ \left(\hat{\phi}_w - \gamma \theta_w \right)^2 \right\} \tag{42}$$

where $\hat{\phi}_w = \phi_w + \theta_w$ is the parameter of dZ_w in the pricing kernel for wage-deflated claims. For $\hat{\phi}_w = \gamma \theta_w$ the utility will be the same in both cases and hence the demand for WLB's will be zero. Sangvinatsos and Wachter (2005) argue that this value for defines the 'shadow' price of the unhedgeable risk. The unhedgeable real wage risk premium in that case is $\lambda_w = -\phi_w = (1 - \gamma)\theta_w^2$. The difference with the result based on the equivalent utility argument is explained by the fact that under the Sangvinatsos-Wachter argument, the investor optimally chooses a position in WLB's, whereas in the first argument, the original position in assets is augmented with WLB's such as to compensate for the utility cost of the unhedged risk.

B Human capital

The approach of Sangvinatsos and Wachter (2005) is also useful when we want to include human capital into the model. Suppose the individual has an exogenous stream of labor income (net of consumption) w_t for $0 < t < T$. Human capital is defined as the present value of the stream of labor income

$$H = \mathsf{E}\left[\int_0^T M_t w_t dt\right] \tag{43}$$

where M_t is the real pricing kernel, given by

$$dM_t/M_t = -rdt + \phi' dZ \tag{44}$$

where $dZ = (dZ_S, dZ_r, dZ_w)'$. Given the real wage process (20c), human capital then is

$$H_t = w_t \int_t^T \exp\left\{-(r_s - g + \theta'\lambda + \theta_w\lambda_w)\right\} ds \tag{45}$$

where r_s is the yield on a real bond with time to maturity s. The dynamics of human capital are

$$dH/H = dw/w - B_H\sigma_r dZ_r = gdt + \theta' dZ + \theta_w\lambda_w - B_H\sigma_r dZ_r \tag{46}$$

where B_H is the real interest rate duration of human capital.

In a complete financial market, the optimal real wealth strategy of the investor is determined from[12]

$$d\ln G = [..]dt - \frac{1}{\gamma}\phi' dZ + \left(1 - \frac{1}{\gamma}\right)\left[\theta' dZ - B(T)\sigma_r dZ_r\right] \tag{47}$$

The actual real wealth dynamics for portfolio weights x are given by

$$d\ln W = [..]dt + w_F x'\sigma + (1 - w_F)\left[\theta' dZ - B_H\sigma_r dZ_r\right] \tag{48}$$

with $w_F \equiv \frac{F}{F+H}$ the fraction of financial assets in total wealth. Equating these two expressions gives the optimal portfolio

$$x^* = (\sigma\rho\sigma')^{-1}\sigma\rho\left[-\frac{1}{\gamma w_F}\phi + \left(1 - \frac{1}{\gamma w_F}\right)\theta + be_r\right] \tag{49}$$

[12]We simplify notation by including in ϕ, θ and dZ the element referring to the idiosyncratic real wage risk.

where b is a complicated expression involving the real interest rate duration of human capital. With the asset menu containing a wage linked bond with zero duration as the last asset, the last element of this equation simplifies to

$$x_w^* = -\frac{1}{\gamma w_F}\phi_w + \left(1 - \frac{1}{\gamma w_F}\right)\theta_w \tag{50}$$

The principle of Sangvinatsos and Wachter (2005) is to find a value ϕ_w such that the demand for wage linked bonds is zero, hence $x_w^* = 0$. The solution to this equality is

$$\hat{\phi}_w = \phi_w + \theta_w = w_F\gamma\theta_w \tag{51}$$

This value is smaller than in the no human capital case ($w_F = 1$).

Suppose the pricing in the economy is determined by the average investor with human capital to total wealth ratio h^* and wage linked bonds are in zero net supply (and risk aversion γ is the same for all investors). Then investors with human capital to total wealth ratio below h^* (typically these are the older investors) will have a positive demand for wage linked bonds, whereas young investors with high human capital will sell (short) wage linked bonds.

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Table 1: Selected parameter estimates

This table shows selected estimates of the parameters in the model, obtained from annual data for the Netherlands, 1950-2002. The subindex S refers to the excess stock return, r to the ex-post real interest rate (nominal short rate minus inflation) and w to real wage growth.

σ_S	0.2051	θ_S	0.0035
σ_r	0.0237	θ_r	0.0030
ρ_{Sr}	-0.19	θ_w	0.0248

Figure 1: Real wage growth in the Netherlands

This figure shows annual real wage growth rates in the Netherlands, 1950–2002. Source: Statistics Netherlands.