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Andreas Würth

Hans Schumacher

CVaR Pricing and Hedging in Unit-Linked Insurance Products

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Andreas Würth[*] Hans Schumacher[†]

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Abstract

This paper describes a way how to find the minimal seller's price for a unit-linked insurance product in order to make the claim acceptable, under the assumption that information about the insurance process is only available at the time of maturity. For the general case, the price calculated here provides still an upper bound. Acceptability is defined through the CVaR criterion. Furthermore, the paper shows how to find the corresponding hedging strategy. We show how CVaR pricing is connected to earlier results of Föllmer/Leukert about minimization of Expected Shortfall, and apply and extend those results for the case of discrete insurance probabilities. We arrive at an algorithm which is straightforward and does not involve any optimization problem. For an example of a unit-linked survival insurance, we provide analytical formulas for the corresponding hedge, as well as an explicit numerical solutions for the CVaR price.

Keywords: Risk measure pricing, unit-linked insurance products, incomplete markets, CVaR, utility maximization

1 Introduction

Unit linked insurance products become more and more popular in insurance industry, because they combine the classical coverage against risks such as death,

[*]A.M. Würth, CentER, Department of Econometrics and Operations Research, Tilburg University, Tilburg, the Netherlands, Tel.: +41-76-3814579, Email: a.m.wurth@uvt.nl.

[†]J.M. Schumacher, CentER, Department of Econometrics and Operations Research, Tilburg University, Tilburg, the Netherlands, Tel.: +31-13-4662050, Email: j.m.schumacher@uvt.nl. Research supported in part by Netspar.

longevity and disability in life insurance with the possible chance of large capital earnings that traditionally banks offer.

The pricing of such insurance products needs a combination of classical actuarial principles as well as principles from financial mathematics. Such combinations have been treated in Møller (2002), where the focus was mainly on pricing using a standard deviation principle, but also some hints for a general utility function were given.

In general, financial valuation principles are based on a replication of a claim, whereas in insurance, a risk-loading is charged, because the claims cannot be hedged. For unit-linked insurance products, one can assume that a full hedge is not possible, because there is a nonhedgeable component. The insurance company will therefore still ask for a risk loading. However, because of the financial component of those products, at least a partial hedge should be possible, which gives the ability to reduce the minimal necessary risk loading. The general aim of this paper is to find a minimal price as well as the corresponding hedging strategy which make a unit-linked insurance claim acceptable for the insurer, where acceptability is defined in terms of coherent risk measures, as in Artzner et al. (1999). We will take CVaR as risk measure. From a practical point of view, this pricing method has the advantage that it is more closely related to the cost of capital method than other pricing rules are, a method which is typically applied in insurance industry.

The topic of risk measure pricing and minimization has been treated recently in several papers. One of the first papers which presented this idea was Carr et al. (2001). In the sequel, general principles of risk measure pricing have been developed in Klöppel & Schweizer (2007), Xu (2006), Cheridito & Kupper (2006) as well as Jobert & Rogers (2008), see also Cherny (2008). The pricing principles are similar to the one in this paper. An abstract framework for pricing on the basis of coherent risk measures has been firmly established in the cited papers. However, to calculate prices in specific situations one typically still needs to solve an optimization problem. In this paper, we address the optimization problem for the case of unit-linked insurance products, and for CVaR as specific risk measure.

A concrete result for the problem of minimization of Worst Conditional Expectation has been developed in Sekine (2004). In this paper, the author obtained a formula for the solution of the minimization problem. However, the problem stated there is not the same as the one in this paper. We will talk about this issue again later in the paper. In Ilhan et al. (2008), the authors solved, apart from theoretical considerations, the problem of numerical risk measure pricing in the example of expected shortfall in the sense of Föllmer

& Leukert (2000). For their solution, a Hamilton-Jacobi-Bellman method has been applied, which leads to a nonlinear partial differential equation with two variables. The authors pointed out there the computational challenge for doing this.

Risk-minimizing strategies for unit-linked insurance products have already been treated in Møller (2001), where risk-minimizing is understood in the sense of local risk minimization, see for example Schweizer (1991). Other papers of the same author consider the variance as definition of risk. Minimization of value at risk in unit-linked insurance products as well as corresponding pricing principles have been looked at in Melnikov & Skornyakova (2004), where the authors have mainly focused on the case with only one insured person, and in this way obtained analytic formulas. For the case of many insured persons, they obtain bounds which are derived by considering the financial and insurance risk separately.

As already stated, in our paper, we consider CVaR as risk measure. We are interested in a risk minimizing strategy when the financial and insurance risk are considered in an integrated manner. Assuming, as in Melnikov & Skornyakova (2004), that all information about the insurance process is arriving only at the end of the time period, we obtain the minimal price making the claim acceptable as well as the corresponding hedge. It turns out that with this simplification, the problem becomes easy to calculate, and for the specific model we use, we obtain analytic formulas. And even in situations where this assumption is unrealistic, the method presented here still leads to an upper bound for the CVaR price, including the corresponding hedge.

Actually, the results presented in this paper can be applied not only for unit-linked insurance products, but also for other situations of CVaR pricing in incomplete markets. The key issue is only the pricing of a payoff which depends on a complete financial market, as well as on another source of uncertainty, independent of the financial market, which cannot be replicated. One may think about the option of a company to buy a specific commodity at a specific time in a specific currency, where the decision whether or not to buy depends on the foreign exchange rate, but also on other circumstances which are independent of the financial market.

Our approach is based on a result of Rockafeller & Uryasev (2002), with which we can connect the problem of CVaR pricing to the earlier results of Föllmer & Leukert (1999) and Föllmer & Leukert (2000). These papers connect the problem of minimization of expected shortfall to the Neyman-Pearson theory, see Witting (1985), and for some specific cases they develop analytic formulas. We further develop those results for the specific case of unit-linked insurance products. In particular, we show in general how those papers can be connected for obtaining a CVaR price for unit-linked insurance products, as well as in a specific example. Furthermore, we extend a theorem presented in Föllmer & Leukert (2000) for the case where the insurance probabilities are discrete. Finally, we apply the results obtained to a specific unit-linked insurance model, and show explicitly the formulas and the numerical results.

The structure of the paper is as follows: In section 2, we formulate the general model, as well as the CVaR pricing principle. In section 3, we present an algorithm for calculating the CVaR price under the additional assumption of continuous distribution. We give an example in which we approximate the discrete insurance probabilities by a normal approximation. In section 4, we prove an extension of the theorem mentioned, in order to be able to apply the result for discrete probabilities. With this result, we are able to obtain analytical formulas for some specific models, or to solve the problem numerically. In section 5, we present again an algorithm for the calculation of the CVaR price, without assumption of continuous distributions. We present a specific example of a unit linked survival insurance, where we obtain analytical formulas. We give an explicit numerical example for the CVaR price, as well as an analytical formula for the corresponding hedge. Section 6 concludes.

2 Problem specification and general statements

2.1 Insurance model and problem specification

We are dealing with a probability space $(\Omega, \mathcal{F}, P)$. On this probability space, a vector-valued stochastic process Z_t is defined representing the insurance state process. The states of the financial market are represented by another vector-valued process X_t on $(\Omega, \mathcal{F}, P)$. The filtration $\mathcal{F}_t$ is given by the natural filtration generated by X_t and Z_t. It is assumed that Z_t and X_t are Markovian, X_t is continuous, and that this market is complete, in the sense that every contingent claim $F(X_T)$ can be replicated by a suitable trading strategy

$$F(X_T) = S_0 + \int_0^T \pi_t dS_t$$

where S_t is the vector-valued process representing the available financial assets. It is assumed that this process is a vector-valued function of time and the financial state variables, that is

$$S_t = \tilde{S}(t, X_t)$$

with $\tilde{S}$ a measurable function, such that S_t is a vector-valued continuous semi-martingale. A sufficient condition for this would be, by the Itô formula, that

X_t is a continuous semimartingale and $\tilde{S}$ is twice differentiable. It is assumed that only the assets S_t can be used for trading. Throughout the whole paper, it is assumed that there exists an equivalent measure Q such that S_t is a local martingale. It is assumed furthermore that X_t and Z_t are independent under P.

The option payoff due to a unit-linked insurance product at the terminal time T is given by a nonnegative product-measurable function $g(X_T, Z_T)$, depending on the financial market as well as on the insurance process.

Remark 2.1. By extending the state space, it is always possible that the payoff depends on some states at $t < T$. Therefore, to restrict to payoffs depending only on states at time T is not really a restriction.

Remark 2.2. A genuine restriction is that payoffs can only take place at time T, even if they may depend on earlier times. This has to be assumed because we aim to calculate the CVaR at the terminal time T. It is in general not clear how to define the CVaR if there are different payoff times. In some specific examples, it may be sensible to divide the payments at all times by a numéraire (a reasonable choice may be a zero bond with expiry at time T), and take the CVaR at the fixed time T. This situation is also covered by our model.

We repeat at this point the CVaR risk measure from Rockafeller & Uryasev (2002):

Definition 2.3. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then $CVaR_\beta(X)$ at a certain level β is the mean of the distribution function

$$\psi_X(\xi) := \begin{array}{ll} 0 & \text{if} \quad \xi < \xi_\beta \\ (\phi_X(\xi) - \beta)/(1-\beta) & \text{if} \quad \xi \geq \xi_\beta \end{array}$$

where ϕ_X is the cumulative distribution function of X, and ξ_β the Value-at-Risk at level β, that is

$$\xi_\beta = \min\{\xi \mid \phi_X(\xi) \geq \beta\}$$

The credit-constrained CVaR pricing rule, which we will denote in the sequel for simplicity CVaR price, is now the following:

Definition 2.4. Let the CVaR level β be given, as well as a self-financing portfolio B_t. Then the CVaR price is the minimal capital V_0 such that there exists a self-financing predictable strategy π with respect to the filtration $\mathcal{F}_t$ such that

$$CVaR_\beta\left[(g(X_T, Z_T) - Y_T^\pi)\right] \leq 0 \tag{2.1}$$

and such that $Y_t^\pi \geq B_t$ for all t, where the wealth process Y_t^π is defined as

$$Y_t^\pi = V_0 + \int_0^t \pi_u dS_u$$

The problem is now to find the CVaR price due to definition 2.4, as well as the corresponding strategy π.

Remark 2.5. In the context of the general theory of coherent risk measures, see Artzner et al. (1999), equation (2.1) means that the risk is acceptable for the insurance company.

Remark 2.6. At this stage, it is not clear that such a CVaR price exists. This is the issue of the Proposition 2.15.

Remark 2.7. In contrast to Klöppel & Schweizer (2007) or Xu (2006), we consider only strategies with wealth processes uniformly (that is independent of the specific process) bounded from below by a self-financing portfolio. Typically, one may think about a zero-bond. Economically, this makes sense, because no insurance company has an unlimited credit line. A lower bound of $-cB_t$, c a constant and B_t a zero-bond, means then the limit until which the insurance company can use credit.

Mathematically, one needs this uniform bound, because Assumption 5.4 from Klöppel & Schweizer (2007) (or Assumption 2.3 from Xu (2006)) is not satisfied by the CVaR in (for instance) the Black-Scholes model. Assumption 5.4 says, in our terminology, that $\inf_{g \in \mathcal{C}} \rho(-g) > -\infty$, where ρ is the coherent risk measure and $\mathcal{C}$ is the set of superreplicable claims at 0 wealth. Essentially the same problem leads also to CVaR prices which are not necessarily market consistent, an issue which will be discussed in the subsequent section.

As a consequence of the limited credit condition, the CVaR price is not translation invariant, in contrast to the risk measure prices in Klöppel & Schweizer (2007).

Remark 2.8. Mathematically, a uniform lower bound B_t, where B_t is a general self-financing strategy, leads to the same problem as the assumption that the wealth process Y_t^π remains nonnegative. Indeed, adding capital B_0 to the initial available capital, and B_T to the terminal payoff, the restriction of a nonnegative wealth process Y_t of this modified problem is the same as the restriction that $Y_t \geq B_t$ of the original problem. In the sequel, we will therefore always assume that $B_t = 0$, that is that the wealth cannot be negative.

Remark 2.9. Conversely, for a fixed initial capital V_0, one can also ask the question what is the minimal possible CVaR and the corresponding hedging strategy. This question is sensible if the market is competitive and an insurance company is not able to price independently its products.

Remark 2.10. The pricing method described here differs completely from the one by using an equivalent martingale measure. Actually, there does typically not exist any absolutely continuous probability measure such that the CVaR price of any insurance claim is given as an expectation under this measure. To see this, consider a simple economy which consists only of two insurance states which have both 50% probability, and only one risk-free financial asset with return 0. For an insurance option which pays 1 in the first state and 0 in the second, one can see that the CVaR price must be 1. As a consequence, the only probability measure under which the expected payoff is equal to the CVaR price is $Q = (1, 0)$. On the other hand, when considering an option which pays 0 in the first state and 1 in the second, the only possible probability measure is $Q = (0, 1)$.

Because the price of risk is typically obtained from the pricing measure, it follows that no market price of risk for the insurance variable Z_T can be defined. If we define $Z_T := 1_{\text{state } 1}$, the Sharpe ratio, from the point of view of the insurer, would be 1 for the claim Z_T as well as for $1 - Z_T$.

2.2 Market consistent CVaR

The pricing rule from Definition 2.4 does not necessarily produce a market consistent price. Indeed, consider a Black-Scholes model, and a replicable claim $g(X_T) = 1_{X_T<c}$ with c a very small constant. Take the constant a such that $Q(X_T < c) = -aQ(X_T > c)$, with Q the martingale measure. Then the replicating hedge of the claim $-a1_{X_T>c}$ gives a CVaR strictly smaller than 0, that is $CVaR(1_{X_T<c} + a1_{X_T>c}) < 0$, as can be checked by calculation. Obviously, by definition, $-a1_{X_T>c}$ has the same price as $1_{X_T<c}$. It follows that there is a price $\pi < Q(X_T < c)$ at which the claim is still acceptable.

This is a problem of the CVaR risk measure, which does not occur in Klöppel & Schweizer (2007), because market consistency of the prices has been proved there under the assumption that the infimum attainable risk over all admissible strategies starting with 0 wealth is bounded (cf. again Assumption 5.4 of Klöppel & Schweizer (2007)). This assumption is not satisfied by the CVaR risk measure.

To guarantee market consistency, one has to change the risk measure. One possible way to do this is by adding the measure Q^* to the set of test measures in the dual representation of CVaR. This measure is defined as the one which gives the original probabilities for the insurance process and the risk-neutral ones for the financial process, in such a way that X_t and Z_t are independent under Q^*. This gives a coherent risk measure, and the new pricing rule is market consistent and is nothing else than what practitioners applying the CVaR criterion for pricing would probably do: When the price obtained by Definition 2.4 would have a negative risk loading, they would set the risk loading to 0, otherwise, they would take the price as obtained by the CVaR criterion. We refer to this new risk measure as market consistent CVaR. Formally, we have the following

Proposition 2.11. *The risk measure price with respect to the market consistent CVaR, denoted by V_M, is*

$$V_M(g(X_T, Z_T)) = \max\left(E^{Q^*}[g(X_T, Z_T)], V_0(g(X_T, Z_T))\right)$$

where V_0 is the CVaR price.

Proof. It is clear that $V_M \geq \max(E^{Q^*}[g], V_0)$. Let $E^{Q^*}[g(X_T, Z_T)] \leq V_0(g(X_T, Z_T))$. Then there exists a strategy π such that $CVaR(g(X_T, Z_T) - V_T^\pi) \leq 0$. But by assumption $E^{Q^*}[g(X_T, Z_T) - V_T^\pi] = E^{Q^*}[g(X_T, Z_T)] - V_0(g(X_T, Z_T)) \leq 0$. It follows that the risk is also acceptable with respect to the new measure, and therefore $V_M \leq V_0$.

Let now $E^{Q^*}[g(X_T, Z_T)] > V_0(g(X_T, Z_T))$. Then there exists a strategy which makes the risk CVaR-acceptable at a price $V_0 < E^{Q^*}[g]$. By the monotonicity of coherent risk measures, the risk is also CVaR acceptable at the price $E^{Q^*}[g]$. It follows that $E^{Q^*}[g]$ is a price which makes the risk acceptable with respect to the new risk measure, and therefore $V_M \leq E^{Q^*}[g]$. □

In practice, the situations when this new measure leads to different prices than the original one depends on the CVaR level β, the credit limit as well as on the payoff g. In many situations, and in particular in all of our examples, there is no difference between the price with respect to this new measure and the CVaR price.

2.3 Connection to minimization of Expected Shortfall

It follows from Rockafeller & Uryasev (2002) that the conditional value at risk of a random variable X at a level β is given by

$$CVar(X) = \min_a \left(a + \frac{1}{1-\beta} E\left[(X-a)^+\right]\right) \tag{2.2}$$

Using this, the problem of CVaR pricing from Definition 2.4 can be reformulated to the following: Find the minimal initial capital V_0 such that there exist a allowed strategy π and a parameter a with

$$f(a, \pi; V_0) \leq -a(1-\beta) \tag{2.3}$$

where $f(a, \pi; V_0)$ is given by

$$f(a, \pi; V_0) := E\left[\left((g(X_T, Z_T) - a)^+ - Y_T^\pi\right)^+\right] \tag{2.4}$$

Remark 2.12. From (2.2), or from (2.3) and (2.4), it is clear that a must be nonpositive. It follows that $g(X_T, Z_T) - a \geq 0$ is always satisfied if the insurance claim is nonnegative.

Remark 2.13. We have that $((g(X_T, Z_T) - a)^+ - Y_T^\pi)^+ = (g(X_T, Z_T) - a - Y_T^\pi)^+$, because of the nonnegativity of Y_T^π. The reason why we write the expectation as in (2.4) is because it helps to guarantee nonnegativity of Y_T^π.

Remark 2.14. By Rockafeller & Uryasev (2002), the minimum in a is always attained. It follows that the minimum in V_0 under condition (2.1) is attained if the minimum $V_0(a)$ under condition (2.3) is attained for all fixed a.

For fixed a, we can define $V_0(a)$ as the minimal initial capital such that there exists a strategy π which satisfies (2.3).

Proposition 2.15. *The following is true:*

1. *For each a, the minimum $V_0(a)$ is attained.*
2. *The function $V_0(a)$ is convex in a.*
3. *If a^* minimizes $V_0(a)$ and π^* is the strategy which minimizes $V_0(a^*)$ for the given a^*, then π^* is the strategy which makes the claim at initial capital $V_0(a^*)$ acceptable in the sense of criterion (2.1).*

Proof. For the first statement, we follow the arguments of Föllmer & Leukert (1999), which are based on the Neyman-Pearson lemma. For each a and given strategy π, we can define an $\mathcal{F}_T$-measurable random variable $\phi \in [0, 1]$ by

$$\left((g(X_T, Z_T) - a)^+ - Y_T^\pi\right)^+ = (1 - \phi)(g(X_T, Z_T) - a)^+$$

The variable ϕ has been called the success ratio in Föllmer & Leukert (1999). We are interested in the success ratio ϕ which minimizes the price of a superhedge, that is

$$\sup_{Q \in \mathcal{Q}} E^Q[\phi(g(X_T, Z_T) - a)^+]$$

where $\mathcal{Q}$ is the set of all equivalent martingale measures, under the condition that equation (2.3) is satisfied, that is

$$\hat{E}[\phi] \geq \frac{1 + a(1 - \beta)}{E[(g(X_T, Z_T) - a)^+]}$$

where $\hat{E}$ denotes the expectation under the measure $\hat{P}$, defined by

$$\frac{d\hat{P}}{dP} = \frac{(g(X_T, Z_T) - a)^+}{E[(g(X_T, Z_T) - a)^+]}$$

Existence of an optimal ϕ follows now by the same argument as in Föllmer & Leukert (1999) which is based on the Neyman-Pearson lemma. The corresponding optimal strategy is then given as the superhedge of the claim $\phi(g(X_T, Z_T) - a)^+$.

For the second statement, let V_1 be the minimal required capital for a_1 and V_2 the same for a_2, where a_1 and a_2 are arbitrary real numbers, and let π_1 and π_2 be the corresponding strategies. Then (2.3) is satisfied for (a_1, π_1, V_1) as well as for (a_2, π_2, V_2), and for an arbitrary $t \in [0, 1]$ we have

$$tf(a_1, \pi_1; V_1) + (1-t)f(a_2, \pi_2; V_2) \leq -(ta_1 + (1-t)a_2)(1-\beta)$$

By the convexity of the function $x \to x^+$, it follows that the left-hand side is larger than or equal to $f(ta_1 + (1-t)a_2, t\pi_1 + (1-t)\pi_2; tV_1 + (1-t)V_2)$, so that

$$f(ta_1 + (1-t)a_2, t\pi_1 + (1-t)\pi_2; tV_1 + (1-t)V_2) \leq -(ta_1 + (1-t)a_2)(1-\beta)$$

It follows that at capital $tV_1 + (1-t)V_2$, there exists a strategy such that this equation is satisfied, and the minimal capital must therefore be smaller or equal. The third statement is obvious from (2.2). □

It follows that we can firstly minimize the required capital $V_0(a)$ for a fixed a, and subsequently minimize this expression with respect to a. But the first one is a problem of the type discussed in Föllmer & Leukert (2000), namely the minimization of the capital required provided an expected shortfall constraint. We can therefore apply the considerations made in this paper.

It will sometimes be easier to consider the related problem, namely to minimize $f(a, \pi; V_0)$ with respect to π at a given initial capital V_0. It is clear that

$$f_{min}(a, V_0) := \min_{\pi} f(a, \pi; V_0) \tag{2.5}$$

is a nonincreasing function in V_0 for given a. If we have $f_{min}(a, V_0)$ for all a and V_0, we can take the minimal V_0 such that

$$f_{min}(a, V_0) \leq -a(1-\beta)$$

An advantageous situation occurs if f_{min} is continuous in V_0. In this case, we can replace the inequality sign by equality.

If we aim to minimize the CVaR at a given initial capital V_0, we can again apply equation (2.5). By Rockafeller & Uryasev (2002), the function

$$a + \frac{1}{1-\beta} f_{min}(a, V_0)$$

is convex in a, and we can again do the minimization over all values of a.

It becomes clear that the essential problem is (2.5), which is a problem of minimizing expected shortfall in the sense of Föllmer & Leukert (2000), and from which everything else follows. In the sequel, we will therefore focus on this problem. As in Föllmer & Leukert (1999) and Föllmer & Leukert (2000), we reformulate the problem of minimizing expected shortfall as a problem of maximizing a state-dependent utility function. We write

$$E[(g(X_T, Z_T) - Y_T^\pi)^+] = E[g(X_T, Z_T)] - E[g(X_T, Z_T) \wedge Y_T^\pi]$$

We can therefore, instead of solving the minimization problem (2.5), maximize

$$E[(g(X_T, Z_T) - a)^+ \wedge Y_T^\pi] \tag{2.6}$$

under the condition that

$$B_0 E^Q[Y_T^\pi] \le V_0$$

for all equivalent martingale measures Q, where B_0 is the value of the zero bond with expiry time T, which is taken here as numéraire. In the sequel, we will take expression (2.6) as objective function.

2.4 Insurance information at the end of the period

As stated already in the introduction, one general assumption of this paper is that nonfinancial information is only available at the terminal time T. This assumption has not been used until now, but will be used for the rest of the paper.

The idea when insurance information arrives only at the end of the period is that, by a similar idea as in Föllmer & Leukert (1999), we can integrate out the insurance random variable. For any strategy we follow up to time T^-, the objective function at time T^- is given by

$$E[(g(X_T, Z_T) - a)^+ \wedge Y_T^\pi | X_{T^-}, Y_{T^-}^\pi]$$

and by the Markov property and the predictability of X_t and Y_t^π, this expression is equal to $\alpha(X_T, Y_T^\pi)$, where

$$\alpha(x, y) := \int \left((g(x, z) - a)^+ \wedge y\right) dP(z) \tag{2.7}$$

where $dP(z)$ is the distribution function of Z. By assumption, Z is independent of $\mathcal{F}_{T^-}$, and therefore in particular independent of the strategy. The function α is concave in the second argument, and the optimization problem is now

$$\max_{\pi} E[\alpha(X_T, Y_T^{\pi})] \tag{2.8}$$

subject to

$$B_0 E^Q[Y_T^{\pi}] \leq V_0$$

Because this problem is independent of $\sigma(Z)$, we are in a complete model, and the martingale measure Q is unique. Because the number of insured persons has a maximum, we can, in life insurance, mostly assume that

$$u(x) := \sup_z g(x, z) < \infty$$

and that

$$V_{sup} := B_0 E^Q[u(X_T)] < \infty$$

In this case, a superhedge is possible at a finite initial capital, and we can apply to a large extent the theory developed in Föllmer & Leukert (1999) or Föllmer & Leukert (2000). We will show how this is done in the subsequent section.

Remark 2.16. Even if we have reduced the problem to one of a complete market, the results of Sekine (2004) are not applicable here. Firstly, the risk measures are not the same. In Sekine (2004), the author minimizes the expression, with our terminology,

$$\sup_{A \in \mathcal{F}: P(A) \geq 1-\beta} E\left[\left(g - V_0 - \int_0^T \pi dS\right)^+ \mid A\right]$$

which is not equal to CVaR and is in fact not a coherent risk measure due to the $(.)^+$ function in the expectation.

Moreover, if we would integrate out the random variable Z_T in the framework of Sekine (2004), this must be done for all test probability measures $\tilde{P}$ which give the dual representation of the CVaR risk measure. To be specific, $\frac{d\tilde{P}}{dP} = \frac{1_A}{P(A)}$ with A the sets from above. The expected payoff is then (with x the state variable, y the terminal wealth)

$$\tilde{g}(x, y) = -\int g(x, z) d\tilde{P}(z) + y$$

This is a payoff in a complete market, but it depends on the test probability measure. As a consequence, it does not seem obvious how to translate the idea of Sekine (2004) to our paper.

3 Continuous probability distributions

3.1 Assumptions

In this section, we study the case with continuous insurance probabilities, which leads in some way to a simplified problem. The general problem is then studied in the subsequent sections.

Throughout this section, we have the following assumptions:

Assumption 3.1. The insurance variable Z_T has a continuous distribution, and the law of the financial variable X_T is absolutely continuous with respect to the Lebesgue measure.

Assumption 3.2. For almost every x, the payoff $g(x, Z_T)$ has a continuous distribution. Almost surely is meant with respect to the law of X_T.

Assumption 3.3. For almost all x, the values of $g(x, Z_T)$ lie in $[0, H(x)]$, where H is a given function such that $H(X_T)$ is integrable.

Assumption 3.4. The Radon-Nikodym density of the financial market admits small values, that is for all $\epsilon > 0$ the set $\{\frac{dQ}{dP} < \epsilon\}$ has strictly positive probability.

Assumption 3.5. There exists a measurable function $q(x)$ such that the Radon-Nikodym density of the financial market is equal to $q(X_T)$. Furthermore, $q(X_T)$ has zero measure for all level sets, that is $P[q(X_T) = c] = 0$ for all constants $c \geq 0$.

Remark 3.6. All those assumptions are satisfied in the Black-Scholes market.

For being able to explore the continuity fully, we need some technical lemmas.

Lemma 3.7. *Let X and Z be two independent random variables, and $g(x, z)$ a measurable function. If for almost every x, $g(x, Z)$ has a continuous distribution, then the distribution of $g(X, Z)$, that is $F(y) := P[g(X, Z) \leq y]$, is continuous too.*

Proof. Define $f(x; y) := P[g(x, Z) \leq y]$ then $f(X, y)$ is a version of the conditional probability $P[g(X, Z) \leq y \mid \sigma(X)]$. Let $y_n \to y$ be a sequence, then $f(X; y_n) \to f(X, y)$ almost surely by the continuity of the distribution of $g(x, Z)$. But $|f(X; y_n)| \leq 1$ because it is a probability. By the dominated convergence theorem, it follows that

$$P[g(X, Z) \leq y_n] = E[f(X; y_n)] \to E[f(X; y)] = P[g(X, Z) \leq y]$$

□

Lemma 3.8. *Let $f(x,y)$ be a nonincreasing function in y on $\mathbb{R}_+$, $d\nu(x)$ a measure which is absolutely continuous with respect to the Lebesgue measure, and $\int f(x,0^+)d\nu(x) < \infty$. Let $g(x)$ be a function, with zero $d\nu$-measure on all level sets, that is $\nu(\{x \mid g(x) = c\}) = 0$, for all $c \in \mathbb{R}$. Then the function*

$$\gamma \mapsto \int f(x, \gamma g(x))d\nu(x)$$

is continuous.

Proof. Because f is nonincreasing in y, there are at most countably many points of discontinuity. Let $\gamma_n \to \gamma$. Then, for $d\nu(x)$-almost all x, the function $f(x, \gamma_n g(x)) \to f(x, \gamma g(x))$, because convergence can only fail at countably many values of $g(x)$, and by the assumption that it has zero $d\nu$-measure on level sets, the points x at which convergence fails build also a $d\nu(x)$-nullset. By the fact that $f(x, 0^+)$ is integrable, one can apply the dominated convergence theorem which yields the result. □

3.2 Calculation of CVaR price

Theorem 3.9. *Let the insurance and financial process Z and X as well as the insurance claim $g(X_T, Z_T)$ satisfy the model assumptions stated in 2.1 and 2.4. Let furthermore Assumptions 3.1 to 3.5 be valid. Then the CVaR price as formulated in Definition 2.4 is given by*

$$V = E^Q[I_a(X_T, \gamma q(X_T))] \tag{3.1}$$

where I_a is the inverse function given by

$$I_a(x,y) := \inf\{z \geq 0 \mid P[g(x, Z_T) > z + a] < y\} \tag{3.2}$$

and a is given by

$$a = -\frac{E\left[(g(X_T, Z_T) - I_0(X_T, \gamma q(X_T)))^+\right]}{1 - \beta - P[q(X_T) > \frac{1}{\gamma}]} \tag{3.3}$$

and γ is determined by the equation

$$\gamma = 1 - \beta + \epsilon_\gamma \tag{3.4}$$

with

$$\epsilon_\gamma = \gamma E[q(X_T) 1_{q(X_T) > \frac{1}{\gamma}}] - P[q(X_T) > \frac{1}{\gamma}]$$

or, otherwise formulated

$$\gamma = \frac{1 - \beta - P[q(X_T) > \frac{1}{\gamma}]}{Q[q(X_T) < \frac{1}{\gamma}]} \tag{3.5}$$

Remark 3.10. From (3.4), it follows that γ depends only on the financial process X, on T and on the value β. In particular it depends neither on the insurance process Z, nor on the payoff $g(x,z)$.

Remark 3.11. For many reasonable parameter values, $\epsilon \approx 0$ and equation (3.4) gives even an explicit equation, which does not need recursion to solve it. The reason is that the CVaR level β is typically near to 1, so that γ is small. It follows that $q(X_T) > \frac{1}{\gamma}$ happens only in a few events, with respect to probability Q as well as to probability P.

Remark 3.12. From (3.5), the parameter γ does not depend on the constraint (3.3). It follows that minimizing the initial wealth such that the corresponding $CVaR$ is smaller than or equal to any constant would lead to the same parameter γ. This means also that minimization of $CVaR$ with any given initial wealth V_0 would lead to the same value of γ.

The proof is based on a theorem stated in Föllmer & Leukert (2000), which we present here in an abbreviated version. It gives an optimal solution for the problem

$$u(z) = \sup_Z \{E[U(Z,.)] | 0 \le Z \le H \quad \text{and} \quad E^*[Z] \le z\} \tag{3.6}$$

where $U(Z,.)$ is a state-dependent utility function which is nondecreasing and concave in the second variable, as well as strictly concave and differentiable on $]0, H(\omega)[$, and E^* is the expectation under the risk-neutral measure, which is unique in the complete market. We use here the same notations as in Föllmer & Leukert (2000).

Theorem 3.13 (Theorem 7.1 of Föllmer & Leukert (2000))**.** *For each $z \le E^*[H]$ there is a unique solution $\tilde{Z}$ such that $u(z) = E[U(\tilde{Z},.)]$. It takes the form*

$$\tilde{Z}(\omega) = I(y(z)\rho^*(\omega), \omega) \wedge H(\omega)$$

where $y(z)$ is the solution of

$$E^*[I(y(z)\rho^*(\omega), \omega) \wedge H(\omega)] = z$$

The function ρ^* plays the role of the density of the risk neutral measure with respect to the original one, and the function I is the inverse of U', that is

$$I(y,\omega) = \inf\{z \in [0, H(\omega)] | U'(z,\omega) < y\}$$

Remark 3.14. The theorem holds also if the concavity is not strict, as will be the case in our application. Only the uniqueness may fail, but this does not affect the statements of Theorem 3.9. We will omit here the proof for the relaxed assumption, because this situation is anyway covered in the more extended Theorem 4.2, which does not even need differentiability of U.

For proving Theorem 3.9, we will prove firstly a further lemma:

Lemma 3.15. *Let the function* α *be given by*

$$\alpha(y) := E[Y \wedge y]$$

with Y *a random variable with continuous distribution. Then*

$$\frac{\partial}{\partial y}\alpha(y) = P[Y > y] \tag{3.7}$$

In particular, α *is differentiable and concave, and strictly concave as long as the density of* Y *exists and is strictly positive.*

Proof. Let $\Delta y > 0$, the other case leads to a similar argument. Then

$$\begin{aligned} \tfrac{1}{\Delta y}\left(\alpha(y+\Delta y) - \alpha(y)\right) \quad = \quad & \tfrac{1}{\Delta y}\textstyle\int_{Y\in[y,y+\Delta y]}(Y-y)\,dP(Z_T) \\ & + \textstyle\int_{Y>y+\Delta y} dP(Z_T) \end{aligned}$$

The second expression on the right-hand side is equal to $P[Y > y + \Delta y]$, the first one can be estimated by

$$\frac{1}{\Delta y} \mid \int_{Y\in[y,y+\Delta y]} (Y-y)\,dP(Z_T) \mid \leq \int_{Y\in[y,y+\Delta y]} dP(Z_T)$$

As $\Delta y \to 0$, the latter converges to 0 by the continuity of the distribution of Y, the first to $P[Y > y]$.

If the density of Y exists and is strictly positive, then $P[Y > y]$ is strictly decreasing, and $\alpha(y)$ strictly concave. □

Proof of Theorem 3.9. Let V_0 be the CVaR price. Then, by the considerations in section 2.3, there exist an a and a strategy π^* such that $f_{min} \leq -a(1-\beta)$, where as in (2.5),

$$f_{min} = E[\left(g(X_T, Z_T) - a - Y_T^{\pi^*}\right)^+]$$

By (2.7) and (2.8), this can be written as

$$f_{min} = E[g(X_T, Z_T) - a] - \max_{\pi} E[\alpha(X_T, Y_T^{\pi})]$$

By Lemma 3.15, α is differentiable and the derivative given by $\frac{\partial\alpha}{\partial y}(x, y) = P[g(x, Z_T) > y + a]$, and concave in the second variable. Furthermore, the second term involves an optimization problem in a complete market setting. Theorem 3.13 states then that, for a fixed a, there exists a constant γ such that

$$f_{min}(a, V_0) = E[g(X_T, Z_T) - a] - E[\alpha\left(X_T, I_a(X_T, \gamma q(X_T))\right)]$$

By the convexity and the finiteness of f_{min} in a, it is clear that it is continuous in a. We will now show that it is also continuous in V_0. Firstly, f_{min} is continuous in γ, because $h(x, y) := \alpha(x, I_a(x, y))$ is nonincreasing in y and Lemma 3.8. Furthermore, $V_0(\gamma) := E^Q[I_a(X_T, \gamma q(X_T))]$ is by Lemma 3.8 continuous too, and furthermore strictly decreasing, for $\gamma > 0$. Indeed, $I_a(x, y)$ is strictly decreasing in y for $y < 1$ and $I_a(x, y) = 0$ for $y \geq 1$. Because of Assumption 3.4, the set of all X_T with $q(X_T) < \frac{1}{\gamma}$ has positive Q-probability, and therefore $V_0(\gamma)$ is strictly decreasing by the monotonicity of the expectation. The inverse $\gamma(V_0)$ is therefore continuous, and therefore also $f_{min}(a, V_0)$ in V_0.

It follows that

$$E[(g(X_T, Z_T) - a - I_a(X_T, \gamma q(X_T)))^+] = -a(1 - \beta) \tag{3.8}$$

holds, because if this would be a strict inequality, there would be by the continuity a $V < V_0$ such that $f_{min}(a, V) \leq -a(1 - \beta)$ would still be satisfied, that is there would exist an allowed strategy at initial wealth V, and therefore V_0 could not be the CVaR price.

From equation (3.2), one can check that

$$I_a(x, y) = \begin{array}{ll} I_0(x, y) - a & \text{if} \quad y \leq 1 \\ 0 & \text{if} \quad y > 1 \end{array}$$

From this, it follows that equation (3.8) is equivalent to

$$-a(1 - \beta) = E\left[(g(X_T, Z_T) - I_0(X_T, \gamma q(X_T)))^+\right] - aP[q(X_T) > \frac{1}{\gamma}]$$

This equation can be solved and yields (3.3).

By Assumptions 3.1 and 3.2 and Lemma 3.7, the random variable $R := g(X_T, Z_T) - I_a(X_T, \gamma q(X_T))$ has a continuous distribution. This means that the Value-at-Risk a^* at level β of R is given by

$$P[R > a^*] = P[R \geq a^*] = 1 - \beta$$

We will show that $f_{min}(a^*, V_0) = -a^*(1 - \beta)$. We know that for a, the equality is satisfied, and by Rockafellar & Uryasev (2000), a^* gives a minimum and therefore $f_{min}(a^*, V_0) \leq -a^*(1 - \beta)$. On the other hand, let the inequality be strict. By the continuity of f_{min} in V, there must exist a $V < V_0$ such that the inequality is still satisfied for V. But this means again that there is an allowed strategy at the price $V < V_0$, and V_0 could not be the CVaR price. We can therefore take $a = a^*$. By Assumptions 3.1 and 3.2, we have

$$P[g(x, Z_T) > a + I_a(x, y)] = y \wedge 1$$

It follows that

$$1 - \beta = P[R > a] = P[g(X_T, Z_T) - I_a(X_T, \gamma q(X_T)) > a] = E[\gamma q(X_T) \wedge 1]$$

Equations (3.4) and (3.5) now follow immediately. □

3.3 Algorithm

Theorem 3.9 suggests now the following algorithm for calculating the CVaR price:

1. Calculate γ from (3.5)
2. Determine $I_0(x, y)$ by (3.2)
3. Determine a by equation (3.3)
4. Determine the CVaR price by equation (3.1)

Remark 3.16. Even if we had originally an optimization problem, this algorithm is a straightforward calculation and does not require any optimization algorithm anymore.

3.4 Application to a unit-linked insurance model

We will apply now Theorem 3.9 to the case of a unit-linked survival insurance, where the stock price S_T is paid at time T if the person is still alive, and nothing if the insured has died before this time. The process S_t is assumed to follow a geometric Brownian motion, that is

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.9}$$

and the amount of survivors a binomial distribution. It is assumed that the insurance outcome is independent of the Brownian motion process. For simplicity, we assume the risk-free interest rate to be 0. If n is the amount of insured persons, and p the probability of surviving, then the amount of survivors N_T is

$$N_T \sim BIN(n, p) \tag{3.10}$$

The total insurance payoff is

$$g(S_T, N_T) = S_T N_T \tag{3.11}$$

It is furthermore assumed that the information about the survivors is firstly revealed at time T. For large values of n, it seems reasonable to approximate the amount of survivors by a truncated normal distribution, that is

$$N_T \sim \mathcal{N}_{trunc}(np, np(1-p)) \tag{3.12}$$

The density of the truncated normal distribution is defined as

$$f_{trunc}(z) := \begin{array}{ll} 0 & \text{if} \quad z < 0 \\ cf(z) & \text{if} \quad 0 \leq z \leq n \\ 0 & \text{if} \quad z > n \end{array} \tag{3.13}$$

where $f(z)$ is the density of the normal distribution with mean np and variance $np(1-p)$ and c is a normalization constant. With equation (3.7), one has

$$\frac{\partial}{\partial y}\alpha(x,y) = P[N_T x > y+a] = 1-c\left(\Phi\left(\frac{\frac{y+a}{x}-np}{\sqrt{np(1-p)}}\right)-\Phi\left(\frac{-np}{\sqrt{np(1-p)}}\right)\right)\wedge 1$$

with $c^{-1} = \Phi\left(\sqrt{\frac{n(1-p)}{p}}\right) - \Phi\left(-\sqrt{\frac{np}{1-p}}\right)$.

As can be checked for the Black-Scholes model (3.9), the Radon-Nikodym density is given by $q(S_T) = S_0^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T - \frac{\mu}{2}T} S_T^{-\frac{\mu}{\sigma^2}}$.

Let us apply now Theorem 3.9 to this unit-linked insurance model. By (3.4), γ is given by

$$\gamma = 1 - \beta + \epsilon \tag{3.14}$$

where

$$\epsilon = \gamma\Phi\left(\frac{\ln\left(\frac{c_0}{S_0}\right)+\frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\ln\left(\frac{c_0}{S_0}\right)-(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

with

$$c_0 := \gamma^{\frac{\sigma^2}{\mu}} S_0 e^{\frac{1}{2}(\mu-\sigma^2)T}$$

By (3.2), we have

$$I_a(x,y) = \begin{array}{ll} nx - a & \text{if} \quad y = 0 \\ npx - a + \sqrt{np(1-p)}x\Phi^{-1}\left(\frac{1-y}{c} + \Phi\left(-\sqrt{\frac{np}{1-p}}\right)\right) & \text{if} \quad 0 < y < 1 \\ 0 & \text{if} \quad y \geq 1 \end{array} \tag{3.15}$$

By (3.3), we must calculate by numerical integration

$$a = -\frac{E[f(S_T)]}{1-\beta-\Phi\left(\frac{\ln\left(\frac{c_0}{S_0}\right)-(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)} \tag{3.16}$$

Table 3.1: Minimal capital for different parameters, $\beta = 95\%$

n	p	γ	a	V_0	*Load*
1000	0.5	0.05	−6.62	532.61	6.5%
1000	0.1	0.05	−3.97	119.56	19.6%
50	0.5	0.05	−1.48	32.29	29.2%
50	0.1	0.05	−0.89	9.38	87.6%

where $f(x)$ is determined by integration of (3.3)

$$\begin{aligned} f(x) &= \sqrt{np(1-p)}x \times \\ & \left[\frac{c}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}\left(\Phi^{-1}\left(\frac{1-\gamma q(x)}{c}+\Phi_0\right)\right)^2} - e^{-\frac{n(1-p)}{2p}} \right) - \gamma q(x)\Phi^{-1}\left(\frac{1-\gamma q(x)}{c} + \Phi_0\right) \right] \end{aligned}$$

if $\gamma q(x) < 1$ and npx otherwise, where $\Phi_0 = \Phi(-\sqrt{\frac{np}{1-p}})$. Finally, we can now perform again a numerical integration for solving (3.1).

3.5 Numerical results

We solved the problem numerically using the parameters $\mu = 0.07$, $\sigma = 0.2$, $T = 1$, $S_0 = 1$, $\beta = 0.95$. The resulting parameter γ from formula (3.14) is $\gamma = 0.05$ with $\epsilon = 0$, and one can check that indeed $\epsilon = 5.0 \times 10^{-20} \approx 0$ for this choice of parameters. The results for the CVaR price are in the following Table 3.1. The last column, *Load*, is the risk loading as percentage of the pure premium. The optimal hedge is then the complete delta hedge of the payoff $I_a(S_T, \gamma S_0^{\frac{\mu}{\sigma^2}} e^{\frac{T}{2}(-\mu+(\frac{\mu}{\sigma^2})^2)} S_T^{-\frac{\mu}{\sigma^2}})$, with a and the corresponding γ given in Table 3.1.

4 Optimal solution for nonsmooth state dependent utility function

In typical situations of life insurance, we have a discontinuous distribution function for the amount of survivors. Furthermore, in particular Assumption 3.2 is too restrictive. Applying (2.7) to discrete distributions leads to a piecewise linear function $\alpha(x, y)$, which is therefore not differentiable nor strictly concave. We can therefore not apply theorem 3.13.

In this section, we prove essentially an extension of Theorem 3.13 to the case of discrete probabilities. However we use for this a formulation which is more similar to Proposition 5.14 in Föllmer & Leukert (1999), because this is more related to our application. The result shows then how to find the optima. The extension with respect to Föllmer & Leukert (1999) is that the state dependent utility function is not required to be strictly concave nor differentiable.

Firstly, we state here a condition which will often be used throughout this section. We denote this condition FHFC (full hedge with finite capital), because, economically speaking, this condition means that one can make a full hedge using only a finite amount of capital. In life insurance, this condition is mostly satisfied, because the assumption that everyone survives gives the worst case, or that everyone dies for a death insurance.

Definition 4.1. A function $\alpha(x, y)$ is said to satisfy the FHFC condition with respect to the probability measure μ on the Borel set $\mathcal{B}(A)$, $A \subset \mathbb{R}^n$, if there exists a measurable and μ-integrable function $h(x) > 0$ such that $\sup_y \alpha(x, y) = \alpha(x, h(x))$.

Theorem 4.2. *Let $A \subset \mathbb{R}^n$ an interval, and let ν and μ be two finite equivalent measures on $\mathcal{B}(A)$. Let $\alpha : D := A \times [0, \infty) \to \mathbb{R}$ be a function which is concave, nondecreasing in the second argument, and satisfies the FHFC condition with respect to μ, and let $\alpha(x, h(x))$ be ν-integrable. Define $\alpha(x, y) := -\infty$ for all $y < 0$: then the concavity holds for all real numbers.*

Let $v : A \to [0, \infty)$ be a function in C, where C is the set of all Borel-measurable functions

$$f : A \to \mathbb{R}$$

with the property that

$$||f|| := \sup_{x \in A} \left| \frac{f(x)}{h(x) + 1} \right| < \infty \tag{4.1}$$

with $h(x)$ the function from the FHFC condition.

Then the following statements are equivalent:

1. *There exists a function $\beta : D \to \mathbb{R}$ such that for each $(x, y) \in D$ $\beta(x, y)$ is a point in the superdifferential of α with respect to the second argument, and such that*

   $$\beta(x, v(x)) = \gamma \frac{d\mu}{d\nu}(x) \tag{4.2}$$

   for a constant $\gamma > 0$.

2. *The function $v(x)$ optimizes*

   $$\int_A \alpha(x, f(x)) d\nu(x)$$

over all functions $f \in C$ with

$$\int_A f(x)d\mu(x) \leq \int_A v(x)d\mu(x) \tag{4.3}$$

Remark 4.3. Economically speaking, this is a functional analytical version of the statement that in the optimum, the marginal price is proportional to the marginal utility, where μ is the pricing functional of f, and $\nu(\alpha(.,f))$ is its utility.

Proof. We first prove the easy direction from (1) to (2). Let $f(x)$ be any nonnegative function in C. We define the function

$$f_\lambda(x) := (1-\lambda)v(x) + \lambda f(x)$$

It is clear that for all $1 \geq \lambda \geq 0$, $f_\lambda \in C$ and nonnegative. By the concavity of α in the second argument, we have, for any choice $\beta(x,y)$ for the superdifferential, that

$$\alpha(x, f_\lambda(x)) - \alpha(x, f_0(x)) \leq \beta(x, f_0(x))(f(x) - v(x))\lambda$$

and therefore, because α satisfies FHFC and $\alpha(x, h(x))$ is integrable, we have that $\alpha(x, f_\lambda(x)) \leq \alpha(x, h(x))$ is integrable for all $\lambda \geq 0$ and

$$\int_A \alpha(x, f(x))d\nu(x) - \int_A \alpha(x, v(x))d\nu(x) \leq \int_A \beta(x, v(x))(f(x) - v(x))d\nu(x) \tag{4.4}$$

Now let the superdifferential satisfy property (4.2). Then it follows for the right-hand side of equation (4.4) that

$$\int_A \beta(x, v(x))(f(x) - v(x))d\nu(x) = \gamma\left(\int_A f(x)d\mu(x) - \int_A v(x)d\mu(x)\right) \leq 0$$

where the last inequality follows if f satisfies property (4.3), and the integrability is again guaranteed by FHFC with respect to μ. It follows from (4.4) that

$$\int_A \alpha(x, f(x))d\nu(x) \leq \int_A \alpha(x, v(x))d\nu(x)$$

and therefore $v(x)$ is optimal for all nonnegative functions $f \in C$. For $f(x) < 0$ on a set with ν-positive measure, $\alpha(x, f(x)) = -\infty$ on a set with ν-positive measure, and the integral is $-\infty$, which cannot be optimal.

Now let us turn to the other direction. Here we need functional analytical arguments from infinite-dimensional convex analysis. We have that C, with the norm from (4.1) is a convex normed vector space. Furthermore, the function F on C defined by

$$F(f) := \int_A \alpha(x, f(x))d\nu(x)$$

is a concave function, which follows easily by the concavity of α. The function $f \to \int_A f(x)d\mu(x)$ is a continuous linear functional on C. Furthermore, if $v(x)$ is not identically 0, the Slater condition is satisfied, and there exists a point f such that F is continuous in f, for example $f(x) = 1 + h(x)$. For being able to apply the Kuhn-Tucker theorem, it remains to show, by Theorem 9.6.1 of Attouch et al. (2005), that $F(f)$ is closed. We will show that the set $\{F(f) < \tilde{\alpha}\}$ is open for all $\tilde{\alpha} \in \mathbb{R}$. Indeed, let firstly f be nonnegative. Then, by the definition of C, for $g \in C$ with $||g - f|| \leq t$,

$$g(x) \leq f(x) + t(1 + h(x))$$

Furthermore, as $t \downarrow 0$, $\alpha(x, f(x) + t(1 + h(x))) \downarrow \alpha(x, f(x))$ almost surely, by the fact that α is nondecreasing and right-continuous in the second variable for $y \geq 0$. By FHFC, $\alpha(x, f(x)+t(1+h(x))) \leq \alpha(x, h(x))$ which is ν-integrable. By the dominated convergence theorem, we must have that $F(f+t(1+h)) \downarrow F(f)$. For each $\epsilon > 0$, we can therefore find a $\delta > 0$ such that $F(g) \leq F(f+t(1+h)) < F(f) + \epsilon$ for all $||g - f|| \leq t < \delta$. Because $F(f) < \tilde{\alpha}$, we find an $\epsilon > 0$ such that $F(f) + \epsilon < \tilde{\alpha}$, and therefore $F(g) < \tilde{\alpha}$ for all g with $||f - g|| < \delta$. If f is not nonnegative, there exists a set $A' \subset A$, $\nu(A') > 0$, such that $f(x) \leq -\epsilon < 0$ on A'. Because $h(x)$ is finite, we may furthermore find a further subset with nonzero $d\nu$-measure, again denoted by A', such that $h(x) \leq K < \infty$ for all $x \in A'$. Let now $||f - g|| < \frac{\epsilon}{2K+2}$. Then, on the set A',

$$g(x) \leq f(x) + \frac{\epsilon}{2K + 2}(1 + h(x)) \leq -\epsilon + \frac{\epsilon}{2} < 0$$

It follows that $\alpha(x, g(x)) = -\infty$ on a set with positive $d\nu$-measure, and therefore $F(g) = -\infty < \tilde{\alpha}$ for all $\tilde{\alpha}$. As a consequence, $\{F(f) < \tilde{\alpha}\}$ is open, from which it follows that F is closed.

By the Kuhn-Tucker theorem in infinite dimensions (Theorem 9.6.1 of Attouch et al. (2005)), there must exist a continuous linear functional $\phi \in \delta F(v)$ in the superdifferential and a constant $\gamma > 0$ such that

$$\phi = \gamma\mu$$

By the fact that μ is absolutely continuous with respect to ν, it follows that

$$\phi(f) = \int_A \gamma \frac{d\mu}{d\nu}(x) f(x) d\nu(x)$$

which means that ϕ is even in $L^1(A, \nu)$.

Now we define, for a function $f \in C$, the new function

$$g(t) := F(v + tf)$$

Then $\phi(f)$ must be in the superdifferential of $\delta g(0)$, because we have

$$g(t) - g(0) = F(v + tf) - F(v) \leq \phi(tf) = t\phi(f)$$

by the fact that ϕ is in the superdifferential of F. Let now f be such that g is continuous in 0 (that is $v + tf \geq 0$ for $|t|$ small enough). Then

$$g(t) - g(0) = \int_A [\alpha(x, v(x) + tf(x)) - \alpha(x, v(x))] \, d\nu(x) \leq t\phi(f) \ \forall t \in B_\epsilon(0)$$

must be satisfied. But for $t \downarrow 0$, we have

$$\lim_{t \downarrow 0} \frac{g(t) - g(0)}{t} = \lim_{t \downarrow 0} \int_A \frac{1}{t} [\alpha(x, v(x) + tf(x)) - \alpha(x, v(x))] \, d\nu(x)$$

The integrand converges $d\nu$-a.s. to

$$\beta_-(x, v(x)) f^+(x) - \beta_+(x, v(x)) f^-(x)$$

where $\beta_-(x, y)$ is the right limit of the difference quotient of $\alpha(x, y)$ in y, and $\beta_+(x, y)$ the left limit, and f^+ and f^- are the nonnegative, respectively non-positive, parts of f. Similarly, for $t \uparrow 0$, the integrand converges to

$$\beta_+(x, v(x)) f^+(x) - \beta_-(x, v(x)) f^-(x)$$

By the fact that $g(t)$ is concave and that there exists an $\epsilon > 0$ with $g(-\epsilon) > -\infty$ the dominated convergence theorem yields

$$\begin{aligned} \phi(f) \ \in \ & [\textstyle\int_A (\beta_-(x, v(x)) f^+(x) d\nu(x) - \beta_+(x, v(x)) f^-(x) d\nu(x)), \\ & \textstyle\int_A (\beta_+(x, v(x)) f^+(x) d\nu(x) - \beta_-(x, v(x)) f^-(x) d\nu(x))] \end{aligned} \tag{4.5}$$

Let now $\hat{\phi}(x) = \gamma \frac{d\mu}{d\nu}(x)$ be the function with $\phi(f) = \int_A \hat{\phi}(x) f(x) d\nu(x)$. Assume that on a set $A' \subset A$ with $v(x) > 0$ on A' and $\nu(A') > 0$, we have $\hat{\phi}(x) > \beta_+(x, v(x))$. Then we find a subset of A', denoted again by A', on which $v(x) \geq \epsilon > 0$, with $\nu(A') > 0$. The function $1_{A'}$ is obviously in C and nonnegative, and for this function, $g_{1_{A'}}(t)$ is continuous at 0. It follows from (4.5) that

$$\phi(1_{A'}) \leq \int_{A'} \beta_+(x, v(x)) d\nu(x)$$

But by definition,

$$\begin{aligned} \phi(1_{A'}) \ = \ & \textstyle\int_{A'} \hat{\phi}(x) d\nu(x) = \int_{A'} \beta_+(x, v(x)) d\nu(x) \\ & + \textstyle\int_{A'} \left(\hat{\phi}(x) - \beta_+(x, v(x)) \right) d\nu(x) > \int_{A'} \beta_+(x, v(x)) d\nu(x) \geq \phi(1_{A'}) \end{aligned}$$

a contradiction. In the same way, the assumption $\hat{\phi}(x) < \beta_-(x, v(x))$ leads to a contradiction on $\{v(x) > 0\}$.

On $\{v(x) = 0\}$, if $f(x) \geq 0$ but $g(t)$ is not necessarily continuous at 0, we can still take the right limit $g(t) \downarrow 0$, and the monotone convergence theorem yields

$$\phi(f) \geq \int_A \beta_-(x, v(x)) f(x) d\nu(x)$$

Furthermore, we have $\beta_+(x, 0) = \infty$ by the definition of α. We must have that $\beta_-(x, 0) < \infty$, because otherwise, defining $f(x) = 1_{v(x)=0}(x)$, we would have $\phi(f) = \infty$, a contradiction to the continuity of ϕ. Let now $\hat{\phi}(x) < \beta_-(x, v(x))$ on a subset A' of $\{v(x) = 0\}$, with $\nu(A') > 0$. Again,

$$\begin{aligned} \int_{A'} \hat{\phi}(x) d\nu(x) &= \int_{A'} \beta_-(x, v(x)) d\nu(x) - \int_{A'} \left(\beta_-(x, v(x)) - \hat{\phi}(x)\right) d\nu(x) \\ &< \int_{A'} \beta_-(x, v(x)) d\nu(x) \leq \phi(1_{A'}) \end{aligned}$$

which is a contradiction. It follows that always

$$\hat{\phi}(x) = \gamma \frac{d\mu}{d\nu}(x) \in [\beta_-(x, v(x)), \beta_+(x, v(x))]$$

For each $x \in A$, defining $y = v(x)$, we may therefore find a point $\beta(x, y)$ in this interval such that (4.2) holds. But by definition, this interval coincides precisely with the superdifferential of $\alpha(x, y)$ at point y. For a y for which no x exists with $v(x) = y$, we may choose an arbitrary point $\beta(x, y)$ in the superdifferential of α. It follows that the $\beta(x, y)$ defined in this way satisfies property (4.2), and the theorem is now completely proved. □

Remark 4.4. For a continuous and strictly concave function and with $\mu = \nu$, this theorem is essentially proposition 5.14 in Föllmer & Leukert (1999).

Remark 4.5. If $\alpha(x, y)$ is strictly concave, Theorem 4.2 gives an algebraic equation from which we can find the optimal function $v(x)$.

Indeed the optimal function exists, which can be proved in the same way as in Föllmer & Leukert (1999). Theorem 4.2 is therefore just an answer to the question how an optimum may be found. Similar to Föllmer & Leukert (1999), we state also the existence theorem.

Theorem 4.6. *Let μ and ν be two finite equivalent measures, and let $\alpha(x, y)$ be as in Theorem 4.2. Let $V_0 < \mu(h)$ be larger than 0. Then there exists a measurable function $v(x) \in C$, C defined as in Theorem 4.2, such that*

$$\int_A \alpha(x, v(x)) d\nu(x) = \sup_f \int_A \alpha(x, f(x)) d\nu(x)$$

where the supremum is taken over all measurable functions f with

$$\mu(f) \leq V_0$$

Proof. Because $\alpha(x, y) = -\infty$ for $y < 0$, we can restrict to nonnegative functions f. Furthermore, we can restrict to functions in C, because $h(x) \in C$, and any nonnegative measurable function f satisfies

$$\int_A \alpha(x, f(x))d\nu(x) = \int_A \alpha(x, f(x) \wedge h(x))d\nu(x)$$

so that f can be even chosen bounded by $h(x)$. We can furthermore choose f such that $\mu(f) = V_0$, because if $\mu(f) < V_0 < \mu(h)$, $f_t(x) := f(x)+t(h(x)-f(x))$ is for $t \in [0, 1]$ still a nonnegative function bounded by h, and by the fact that $\alpha(x, y)$ is nondecreasing in y,

$$\int_A f_t(x)d\nu(x)$$

is a nondecreasing function in t, and

$$g(t) := \int_A f_t(x)d\mu(x)$$

is a continuous (linear) function with $g(0) = \mu(f)$ and $g(1) = \mu(h)$. By standard real analysis, there exists a $0 < t < 1$ with $\mu(f_t) = V_0$. If we define now the set

$$C' := \{0 \leq f(x) \leq h(x) : \mu(f) = V_0\}$$

then C' is a convex set which is weakly compact in $\mathcal{L}^1$, and we are precisely in the same situation as in Föllmer & Leukert (1999). Existence follows now by the same arguments. □

Corollary 4.7. *Let $(\Omega, \mathcal{F}_T, \mathcal{F}_t, P)$ be a filtered probability space, and X_t a continuous semimartingale with values in a convex set $A \subset \mathbb{R}^n$. Let there exist a unique equivalent local martingale measure Q, and assume that $\frac{dQ}{dP}$ is $\sigma(X_T)$-measurable. Let $\alpha(x, y)$ satisfy the properties of Theorem 4.2, with μ and ν the laws of X_T under Q and P, respectively. Then the hedge which optimizes $E[\alpha(X_T, V_T)]$ at initial capital V_0 is given by the hedge of the claim $v(X_T)$, with v from the Theorems 4.2 and 4.6.*

Proof. The proof follows similar arguments as given in Föllmer & Leukert (1999). Let π be any admissible strategy, and let its value process be

$$V_t = V_0 + \int_0^t \pi_s dX_s$$

At time T, we define the X_T-measurable random variable $f(X_T) := E[V_T|X_T]$. By the concavity of α in the second argument we have that

$$E[\alpha(X_T, V_T)] \leq E[\alpha(X_T, E[V_T|X_T])] = E[\alpha(X_T, f(X_T))]$$

But by the fact that $\frac{dQ}{dP}$ is $\sigma(X_T)$-measurable, we have

$$E^Q[f(X_T)] = E[\frac{dQ}{dP}E[V_T|X_T]] = E[E[\frac{dQ}{dP}V_T|X_T]] = E^Q[V_T] = V_0$$

If v is optimal in the sense of Theorem 4.2 or 4.6, it follows that

$$\begin{aligned} E[\alpha(X_T, V_T)] &\leq E[\alpha(X_T, f(X_T))] = \int_A \alpha(x, f(x))d\nu(x) \\ &\leq \int_A \alpha(x, v(x))d\nu(x) = E[\alpha(X_T, v(X_T))] \end{aligned}$$

and therefore the replication of the claim $v(X_T)$ is optimal. □

5 CVaR pricing with discrete insurance probabilities

5.1 Problematics and assumptions

The idea of this section is to develop an algorithm for obtaining the CVaR price analogous to section 3, but without the assumptions 3.1 and 3.2. The first idea would be to apply Theorem 4.2 in the same way as Theorem 3.13 has been applied for proving Theorem 3.9. However, there is one further problem. The derivation of equations (3.4) and (3.5) relies on the assumption that $P[g(x, Z_T) - I_a(x, y) = a] = 0$, which is typically not satisfied when the distribution of $g(x, Z_T)$ is not continuous. Actually, there may be cases where this probability is even quite large as will be shown in the example later in this section.

However, the formulas (3.4) and (3.5) hold also in many cases where $P[g(x, Z_T) - I_a(x, y) = a] > 0$, so that the continuity of the distribution helps to prove the theorem, but is not the key assumption. We will derive here another argument, which relies on the classical first-order condition of a minimization problem, that is $V_0'(\gamma) = 0$ at the minimum, where the minimal price V_0 is calculated as a function of γ, and satisfying the constraint (3.8).

To develop general conditions under which Theorem 3.9 extends to the case of possibly discontinuous distributions is a technically delicate issue, and would exceed the scope of this paper. In particular, one would have to analyze carefully the boundary regions of sets $\{x : \gamma q(x) \in I\}$, where I is an interval in $[0, 1]$, and q again such that $q(X_T)$ is equal to the Radon-Nikodym density. We will prove a simplified version of this theorem under some additional assumptions, which cover at least the Black-Scholes model with a finite insurance state space.

Assumption 5.1. The insurance state space is finite, that is the set of all insurance outcomes is $\{Z_1, ..., Z_n\}$. We assume that all those have a positive probability.

Assumption 5.2. For the claim $g(x, Z)$, there is a uniform ordering, that is one can order $Z_1, Z_2, ..., Z_n$ in a way that

$$g(x, Z_1) \leq g(x, Z_2) \leq ... \leq g(x, Z_n)$$

uniformly for all x.

Assumption 5.3. The state space variable X_T has a density function which is continuous, under the original measure P as well as under the martingale measure Q.

Assumption 5.4. There exists a differentiable function $q(x)$ such that the Radon-Nikodym density of the financial market $\frac{dQ}{dP} = q(X_T)$.

Assumption 5.5. The state space variable X_T is one-dimensional, and the density $q(x)$ is a strictly decreasing function, and surjective on $]0, \infty[$.

Assumption 5.6. The Radon-Nikodym density of the financial market admits small values, that is for all $\epsilon > 0$ the set $\frac{dQ}{dP} < \epsilon$ has strictly positive probability.

Remark 5.7. The assumption that the financial state variable is one-dimensional seems not to be essential. However, it facilitates considerably the proof. Without this assumption, one has precisely the technical problems mentioned above.

Remark 5.8. In the Black-Scholes model, $q(x) = cx^{-\frac{\mu}{\sigma^2}}$ is, with a suitable constant c, equal to the Radon-Nikodym density, and therefore assumption 5.5 is satisfied.

Lemma 5.9. *Let Assumptions 5.1 to 5.2 be satisfied. Then the inverse function $I(x, y)$, defined in (3.2), is given by*

$$I(x, y) = \sum_{k=1}^{n} g(x, Z_k) 1_{y \in]P_{k+1}, P_k]} - a 1_{y<1} \tag{5.1}$$

where P_k are given by

$$P_k = P[\{Z_k, Z_{k+1}, .., Z_n\}] \tag{5.2}$$

Proof. Let $y \in]P_{k+1}, P_k]$. Then, for $\xi < g(x, Z_k) - a$, it follows by Assumption 5.2 that $\xi < g(x, Z_j) - a$ for all $j \geq k$, and therefore $P[g(x, Z) - a > \xi] \geq P_k$. On the other hand, if $\xi > g(x, Z_k) - a$, then $\xi < g(x, Z_j) - a$ is only possible for $j \geq k + 1$, and therefore $P[g(x, Z) - a > \xi] < y$ for all $y > P_{k+1}$. The infimum of all those ξ is consequently $\xi = g(x, Z_k) - a$. Equation (5.1) follows. □

Lemma 5.10. *Let the insurance and financial process Z and X as well as the insurance claim $g(X_T, Z_T)$ satisfy the model assumptions stated in 2.1 and 2.4. Let Assumptions 5.1 to 5.5 be satisfied. Then, for fixed a, $\gamma \mapsto V_0(\gamma) = E^Q[I_a(X_T, \gamma q(X_T))]$ is a nonincreasing differentiable function. Furthermore, $\gamma \mapsto E[(g(X_T, Z_T) - I_0(X_T, \gamma q(X_T)))^+]$, is differentiable in γ. The differentials are given by*

$$\frac{d}{d\gamma} E^Q[I_a(X_T, \gamma q(X_T))] = \sum_{k=2}^{n} f(c_k) q(c_k) \left(g(c_k, Z_{k-1}) - g(c_k, Z_k)\right) \frac{dc_k}{d\gamma} - (g(c_1, Z_1) - a) f(c_1) q(c_1) \frac{dc_1}{d\gamma} \tag{5.3}$$

$$\frac{d}{d\gamma} E[(g(X_T, Z_T) - I_0(X_T, \gamma q(X_T)))^+] = \sum_{k=2}^{n} P_k \left(g(c_k, Z_k) - g(c_k, Z_{k-1})\right) f(c_k) \frac{dc_k}{d\gamma} + g(c_1, Z_1) f(c_1) \frac{dc_1}{d\gamma} \tag{5.4}$$

where

$$c_k := q^{-1}\left(\frac{P_k}{\gamma}\right) \tag{5.5}$$

Proof. By Lemma 5.9, we know the formula of the integrand. Furthermore, by Assumption 5.5, we know that $\gamma q(X_T) \in]P_{k+1}, P_k]$ if and only if $X_T \in [c_k, c_{k+1}[$ with c_k given by (5.5). By Assumption 5.3, it follows that the integrals

$$E^Q[g(X_T, Z_k) 1_{X_T \in [c_k, c_{k+1}[}] = \int_{c_k}^{c_{k+1}} f(x) q(x) g(x, Z_k) dx$$

are differentiable functions, and by the fundamental theorem of differential calculus and the chain rule one has

$$\frac{d}{d\gamma} \int_{c_k}^{c_{k+1}} f(x) q(x) g(x, Z_k) dx = f(c_{k+1}) q(c_{k+1}) g(c_{k+1}, Z_k) \frac{dc_{k+1}}{d\gamma} - f(c_k) q(c_k) g(c_k, Z_k) \frac{dc_k}{d\gamma} \tag{5.6}$$

Summation and reordering of the terms yields the result (5.3), considering that q is differentiable and invertible and therefore $\frac{dc_k}{d\gamma} = \frac{1}{q'\left(q^{-1}\left(\frac{P_k}{\gamma}\right)\right)} \left(-\frac{P_k}{\gamma^2}\right)$. That the function is nonincreasing follows from the fact that all $f(c_k) \geq 0$, and by

Assumption 5.5 $\frac{dc_k}{d\gamma} > 0$, and because $q(X_T)$ is equal to the density function and therefore $q(c_k) > 0$, because of the equivalence of the two measures. By Assumption 5.2, all terms in the sum of (5.3) are nonpositive. The proof of equation (5.4) follows essentially the same arguments. □

5.2 Calculation of the CVaR price for general insurance state models

Theorem 5.11. *Let the insurance and financial process Z and X as well as the insurance claim $g(X_T, Z_T)$ satisfy the model assumptions stated in 2.1 and 2.4. Let furthermore Assumptions 5.3, 5.4 and 5.6 be valid. Moreover, let* $\sup_{Z_T} g(x, Z_T)$ *be finite for each x. Then the CVaR price V_0 is again given by (3.1), together with (3.2) and a is given by (3.3). The parameter γ is determined by a minimization of V_0 with respect to γ. If all assumptions (5.1) to (5.6) are valid, also formulas (3.4) or (3.5) are valid.*

Remark 5.12. As for the case of continuous distributions, we have often that $P[q(X_T) > \frac{1}{\gamma}] \approx 0$ and $Q[q(X_T) \leq \frac{1}{\gamma}] \approx 1$, so that formula (3.4) holds with $\epsilon \approx 0$. In this case, γ can be obtained explicitly, otherwise a zero-finding algorithm is needed.

Remark 5.13. Again, equations (3.4) and (3.5) depend neither on the insurance process Z_t, nor on the payoff $g(X_T, Z_T)$.

We prove here firstly a further lemma:

Lemma 5.14. *The function $E[(g(X_T, Z_T) - a - I_a(X_T, \gamma q(X_T)))^+]$ can be written as*

(5.7)
$$E[(g(X_T, Z_T) - a - I_a(X_T, \gamma q(X_T)))^+] = E[(g(X_T, Z_T) - I_0(X_T, \gamma q(X_T)))^+] - aP[q(X_T) \geq \tfrac{1}{\gamma}]$$

In particular, the function is continuous in a for fixed γ.

Proof. We have still $I_a(x, y) = (I_0(x, y) - a)1_{y \leq 1}$ from the proof of Theorem 3.9. Taking the expectation and conditioning gives

$$E[(g(X_T, Z_T) - a - I_0(X_T, \gamma q(X_T)) + a)^+ 1_{\gamma q(X_T) \leq 1}] + E[(g(X_T, Z_T) - a)1_{\gamma q(X_T) > 1}]$$

Recognizing that $I_0(x, y) = 0$ if $y > 1$, the result follows. □

Proof of Theorem 5.11. Let V_0 be the CVaR price. Then, by the arguments of section 2.3, there exist an a and a strategy π^* such that

$$f_{min}(a, V_0) = f(a, \pi^*, V_0) = E[g(X_T, Z_T) - a] - E[\alpha(X_T, Y_T^{\pi^*})] \leq -a(1 - \beta)$$

where $\alpha(x, y)$ is nondecreasing and concave. Furthermore the FHFC condition is satisfied. From Theorem 4.2, together with Corollary 4.7, it follows that there exists a $\gamma > 0$ with

$$f_{min}(a, V_0) = E[g(X_T, Z_T) - a] - E[\alpha(X_T, I_a(X_T, \gamma q(X_T)))]$$

and $V_0(\gamma) = E^Q[I_a(X_T, \gamma q(X_T))]$.

We will now show equality (3.3). The inequality

$$E[(g(X_T, Z_T) - a - I_a(X_T, \gamma q(X_T)))^+] \leq -a(1-\beta) \tag{5.8}$$

follows from the acceptability condition. If this would be a strict inequality, then $a < 0$, because on the left-hand side, the expression cannot be negative. By increasing a, we have that

$$V_0(\gamma) = E^Q[I_0(X_T, \gamma q(X_T))] - aQ(X_T \leq \frac{1}{\gamma})$$

is strictly decreasing in a, because of Assumption 5.6. On the other hand, by Lemma 5.14, the left-hand side of (5.8) is continuous in a, and therefore the inequality would still be satisfied for a strictly larger a. But this means that V_0 cannot be the CVaR price. The inequality must therefore be an equality. Again by Lemma 5.14, it is clear that we can solve (5.8) for a. If we insert (3.3) into (3.1), we obtain for each γ a price $V_0(\gamma)$ which satisfies constraint (2.3). and is therefore acceptable. The CVaR price is therefore the one which minimizes $V_0(\gamma)$.

Let now all assumptions 5.1 to 5.6 be satisfied. By Lemmas 5.10 and 5.14, it follows that a is differentiable in γ with

$$\frac{da}{d\gamma} = -\frac{\frac{d}{d\gamma}E[(g(X_T, Z_T) - I_0(X_T, \gamma q(X_T)))^+]}{1-\beta - P[q(X_T) > \frac{1}{\gamma}]} + a\frac{\frac{d}{d\gamma}P[q(X_T) > \frac{1}{\gamma}]}{1-\beta - P[q(X_T) > \frac{1}{\gamma}]}$$

and

$$V_0'(\gamma) = \frac{d}{d\gamma}E^Q[I_a(X_T, \gamma q(X_T))] - Q[q(X_T) \leq \frac{1}{\gamma}]\frac{da}{d\gamma}$$

Inserting (5.3) and (5.4) and recognizing that $q(c_k) = \frac{P_k}{\gamma}$ leads to the expression

$$\frac{dV_0}{d\gamma} = (\text{nonpositive terms})\left(\frac{1}{\gamma} - \frac{Q[q(X_T) \leq \frac{1}{\gamma}]}{1-\beta - P[q(X_T) > \frac{1}{\gamma}]}\right)$$

and γ given by equations (3.4) and (3.5) must be a minimum for $V_0(\gamma)$. □

5.3 Application to the unit-linked insurance model

We take again the same model for the financial market as in section 3 as well as the same unit-linked survival insurance, but now with a discrete distribution of the amount of the survivors N_T which is not specified at this stage. It follows that all assumptions 5.1 to 5.6 are satisfied. We can therefore apply Theorem 5.11 and follow the same algorithm as presented in section 3.3.

We have again $q(x) = X_0^{\frac{\mu}{\sigma^2}} e^{\frac{1}{2}(-\mu+(\frac{\mu}{\sigma})^2)T} x^{-\frac{\mu}{\sigma^2}}$, and therefore, and because of the assumption of a geometric Brownian motion, we have by (3.5)

$$\gamma = \frac{1 - \beta - \Phi\left(\frac{\ln\left(\frac{c_0}{X_0}\right) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)}{\Phi\left(\frac{\ln\left(\frac{X_0}{c_0}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)} \tag{5.9}$$

where Φ is the cumulative normal distribution function, and the payoff to be hedged is

$$v(X_T) := \sum_{k=0}^{n-1}(kX_T - a)1_{c_k<X_T<c_{k+1}} + (nX_T - a)1_{X_T>c_n} \tag{5.10}$$

with

$$c_k := q^{-1}\left(\frac{P[N_T \geq k]}{\gamma}\right) = \left(\frac{\gamma}{\sum_{j=k}^{n} p_j}\right)^{\frac{\sigma^2}{\mu}} X_0 e^{\frac{T}{2}(\mu-\sigma^2)} \tag{5.11}$$

The parameter a is given by

$$a = \frac{a_0}{1 - \beta - \Phi\left(\frac{\ln\left(\frac{c_0}{X_0}\right) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)} \tag{5.12}$$

where a_0 is given by

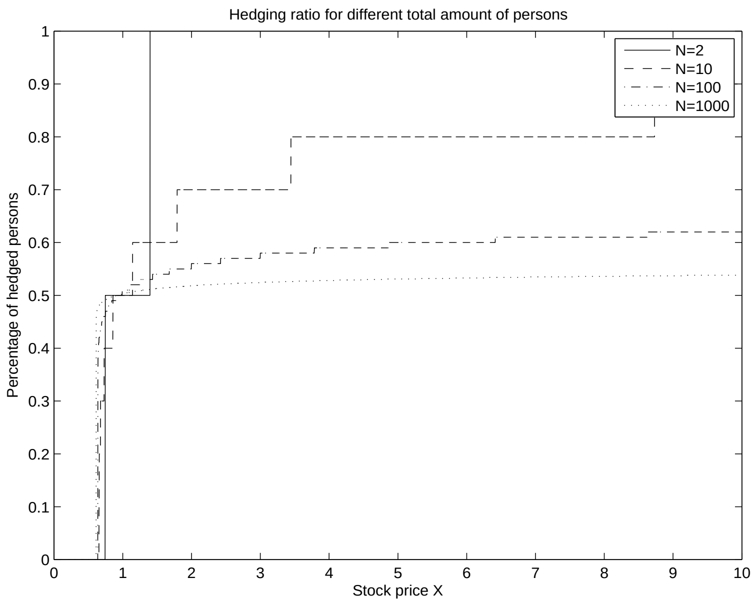
$$a_0 = X_0 e^{\mu T} \sum_{k=1}^{n}\left(\sum_{l=k}^{n} p_l\right)\Phi\left(\frac{\ln\left(\frac{c_k}{X_0}\right) - (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \tag{5.13}$$

Finally, we obtain the CVaR price by

$$V_0(a) = \sum_{k=1}^{n} X_0 \Phi\left(\frac{\ln\left(\frac{X_0}{c_k}\right) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) - a\Phi\left(\frac{\ln\left(\frac{X_0}{c_0}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) \tag{5.14}$$

In this context, we used the notation $c_{n+1} := \infty$.

Figure 5.1: Optimal payoff for different amount of insured persons, $a = 0$



Remark 5.15. Even if under the original probability measure the processes N_t and X_t are independent, the optimal strategy is not simply the delta hedge corresponding to the payoff npX_T; rather a higher survival rate is hedged for larger values of X_T. This means also that under the worst-case martingale measure as defined in Föllmer & Leukert (1999) the two events are not independent any more.

Remark 5.16. As for the case with continuous distributions, with the approximation $\epsilon \approx 0$, and inserting (5.9) into (5.11) and this into (5.12), we obtain an analytical formula for the CVaR price inserting those equations into (5.14).

As an illustration, Figure 5.1 shows the hedge ratio of the optimal payoff to be hedged, that is $\frac{v(x)}{nx}$, as a function of the terminal stock price, for different amounts of insured persons. $v(x)$ is the function given in (5.10), where for simplicity we took $a = 0$. From this, one can see that the optimal payoff is a sum of knock-in options. We took for the insurance probabilities a binomial distribution with n persons and $p = 0.5$.

For only few persons (in practice, one can think about an special insurance for few persons with very large payoffs), zero payoff is hedged if the stock price is below a limit c_1, whereas a full hedge is implemented if the stock price is large enough. The reason is that, if the stock price is large, the risk that there are more survivors than expected plays a much larger role than for small

Table 5.1: Minimal capital for different parameters, $\beta = 0.95$

n	p	γ	a	V_0	*Load*
1000	0.5	0.05	-6.62	532.60	6.5%
1000	0.1	0.05	-4.19	120.0	20.0%
50	0.5	0.05	-1.45	32.24	29.0%
50	0.1	0.05	-1.07	9.76	95.2%

stock prices. If the amount of insured persons increases, the optimal hedge ratio converges more and more to the one which is usually done by actuaries in practice, namely the hedge of the expected amount of survivors.

5.3.1 Numerical results

Even if CVaR is a translation invariant risk measure, it is not the same as to calculate only the minimal CVaR at capital 0 and take this as the minimal capital required. The reason is that for larger initial capitals, more trading strategies are allowed.

We repeat the numerical example from section 3, with the same parameters as there. With those parameters, we obtain

$$\Phi\left(\frac{\ln\left(\frac{c_0}{X_0}\right) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) = 1.2 \times 10^{-18} \approx 0$$

and

$$\Phi\left(\frac{\ln\left(\frac{X_0}{c_0}\right) - \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right) = 1 - 2.6 \times 10^{-17} \approx 1$$

which justifies the approximation $\epsilon \approx 0$. The results for the discrete probability case are in Table 5.1. Comparing the results with the ones of section 3, one can see that the differences to the normal approximation are rather small, even for the case of only 50 persons.

For looking at the sensitivity of the risk loading with respect to the CVaR level β, we have done the calculations also for $\beta = 0.99$. This gives a risk loading for 50 persons of 37% when $p = 0.5$ and 127% for $p = 0.1$.

6 Conclusion

In this paper, we have shown how the CVaR price can be obtained in order to make an insurance payoff acceptable for the seller, under the assumption that nonfinancial information is only obtained at the time of maturity. Using this assumption, the problem simplifies to a complete market problem, and the CVaR pricing problem including the corresponding hedging strategy become almost explicit formulas.

The approach we have chosen for this problem is to apply the relationship between expected shortfall and CVaR given in Rockafeller & Uryasev (2002), and to connect the CVaR pricing problem to the one of minimization of expected shortfall treated in Föllmer & Leukert (2000). For the case where the insurance claim has a continuous distribution function, the state dependent utility function satisfies the requirements given there, and one can apply the results of Föllmer and Leukert directly. We have shown that for this situation, and under some additional technical assumptions which are satisfied in the Black-Scholes model, we have approximately an analytical formula for the CVaR price. Without approximation, we have at least a straight forward algorithm which does not require any optimization procedure.

However, in life insurance, the probabilities are typically discrete, and even if they were continuous, the insurance payoff may have some points with nonzero probability. As a consequence, one cannot apply directly the results from before. Therefore, a considerable part of the paper was devoted to extending the result of Föllmer and Leukert to the case where the state dependent utility function is not necessarily differentiable nor strictly concave, in order to be able to apply the result also for discrete probabilities. The result has been held quite general, and applies to any state dependent utility maximization problem in complete markets.

With this additional result, we could formulate, under some additional assumptions, an algorithm similar to the one for the case of continuous distribution functions. However, to find necessary conditions in order to have the algorithm which does not require any numerical optimization is a technically delicate problem, which we left open for further research. Instead, we proved a result under some more restrictive assumptions which are satisfied by the Black-Scholes model. Using again an approximation, we have again an almost analytical formula for the CVaR price.

We applied the theoretical results to the case of a unit-linked survival insurance, where we applied a truncated normal approximation for being able to apply the result for continuous distribution, as well as a binomial model where we applied the result for discrete distributions. We obtained analytical formulas for the hedge, as well as for the CVaR price, using again the approximation which is for reasonable parameters very accurate. Otherwise, if this approximation is not good enough, one has to solve numerically an equation in one variable, which is also a feasible problem.

In practice, the assumption that insurance information arrives only at the end of the period may be unrealistic. However, the price obtained by this assumption can be thought as an upper bound for the actual CVaR price, because for the hedge, it is always possible to ignore additional information. The question arrives how good this upper bound is. The extension from insurance information at the end to a general information structure will be covered in future work. Similar to the considerations of Föllmer and Leukert, the idea is to introduce more and more information steps, and to prove the convergence when information arrives continuously. However, our first numerical examples show that there is only little gain when considering more than only information at the end. On the other hand, there is no analytical solution anymore when considering more information steps, and the numerics becomes more difficult.

We have here only considered insurance payoffs which occur at a specific terminal time. However, in practice, one has often payment processes, such as pension rents. In such a situation, one has to reconsider what is meant with the requirement that the wealth process is bounded from below, because the insurance payoffs may reduce the wealth at a time before the terminal one. This has also consequences with respect to the assumption that information about the insurance process arrives only at the end, because if state dependent payments have to been done before the terminal time, one knows obviously more about the insurance process. The study of such payment processes, as well as a reasonable redefinition of the corresponding CVaR price, would be another issue of further research.

A natural next step in further research would also be to consider the problem of satisfying the CVaR criterion and maximizing the profits in the sense of Basak & Shapiro (2001). This is a typical problem a financial institution has to deal with.

The application has essentially been done with a specific model for the financial and the insurance process. The general considerations, however, could be applied also in other situations. It would be interesting to apply them in particular to a situation where there exists no analytical solution for the model, and where the solution has to be found numerically, such as financial models with nonconstant coefficients or with stochastic volatility.

Finally, one could think about more general risk measures than the CVaR. In Föllmer & Schied (2004), it has been shown that the CVaR is a building block for all law-invariant risk measures. Even if it is not obvious how to combine the theories in order to calculate a risk measure price for a general law invariant risk measure, it would be interesting to see whether an extension in this direction is possible.

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