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Andreas Würth

Hans Schumacher

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Abstract

This paper generalizes the notion of risk aversion for functions which are not necessarily differentiable nor strictly concave. Using an approach based on superdifferentials, we define the notion of a risk aversion measure, from which the classical absolute as well as relative risk aversion follows as a Radon-Nikodym derivative if it exists. Using this notion, we are able to compare risk aversions for nonsmooth utility functions, and to extend a classical result of Pratt to the case of nonsmooth utility functions. We prove how relative risk aversion is connected to a super-power property of the function. Furthermore, we show how boundedness of the relative risk aversion translates to the one of the conjugate function. We propose also a weaker ordering of the risk aversion, denoted by essential bounds for the risk aversion, which requires only that bounds of the (absolute or relative) risk aversion have to hold up to a certain tolerance.

Keywords: Risk aversion, Utility maximization

1. Introduction

In the economic literature, the notion of risk aversion plays a quite large role in characterizing investor preferences. Risk aversion is often defined as the Arrow-Pratt coefficient of absolute or relative risk aversion. However, the classical definition of this coefficient assumes twice differentiability, which is not always satisfied in examples. One typical example would be a piecewise linear utility function. Our aim is therefore to give a definition which coincides with the classical one in the case of twice differentiability, but is also applicable in all other cases.

Nonsmooth utility functions have been applied in some articles, for example in Bouchard et al. (2004). However, to our knowledge, a definition of the absolute or relative risk aversion for such cases has not been formulated before.

*A.M. Würth, CentER, Department of Econometrics and Operations Research, Tilburg University, Tilburg, the Netherlands, Tel.: +41-76-3814579, Email: a.m.wurth@uvt.nl.

†J.M. Schumacher, CentER, Department of Econometrics and Operations Research, Tilburg University, Tilburg, the Netherlands, Tel.: +31-13-4662050, Email: j.m.schumacher@uvt.nl. Research supported in part by Netspar.

Why is the question of risk aversion for nonsmooth utility functions interesting? In Segal and Spivak (1997), the authors argue that a nondecreasing concave utility function is differentiable almost everywhere, and non-differentiability is therefore of no importance in expected utility theory. However, in the same paper, it is noted that first-order risk aversion arises in some non-expected utility theories, and they prove that locally approximating utility functions must in this case be nondifferentiable. Furthermore, a smooth utility function can be reconstructed from its risk aversion. This cannot be done for piecewise linear concave but not linear utility functions without considering the nullset where the function is not differentiable. The risk aversion of such a function is almost everywhere 0, and a reconstruction would give a linear utility function, which is not what we want. Moreover, the fact that a piecewise linear utility function has the same risk aversion as a linear one is not sensible even in the case of expected utility theory, and even if the random variables would have a continuous distribution. It follows that the risk aversion at the nullset where the function is nondifferentiable has also to be considered.

For nonsmooth functions, there are several ways of defining derivatives. The general method uses distribution theory. Another generalization can be made specifically for concave functions, for which there exists the theory of the generalized first differential (subdifferential for convex functions, see for example Rockafellar (1970)), as well as some literature about a generalized second derivative such as in Rockafellar (1985). The idea of defining the second differential for convex functions as a Lebesgue-Stieltjes measure, which we will partially follow, is also not new. In principle, one could try to use one of those definitions for generalizing the classical formula for risk aversion. However, this would still require an appropriate definition of a quotient of those generalized derivatives, an issue which is not trivial.

In this paper, we follow a slightly different approach. Instead of defining a quotient of the generalized first and second differential, we define the risk aversion directly as a Lebesgue-Stieltjes measure. This measure is generated from the logarithm of the marginal utility (defined again by subdifferential calculus). The absolute as well as relative risk aversion is then defined as a Radon-Nikodym derivative provided it exists. If the utility function is indeed strictly increasing and twice differentiable, this Radon-Nikodym derivative coincides with the classical definition of the (absolute or relative) risk aversion. Therefore, our definition is indeed an extension of the classical one.

Our definition allows us to compare the risk aversion measures of different utility functions. This gives us the opportunity to compare risk aversions of different investors. Those comparisons always hold for absolute as well as for relative risk aversion. Using this comparison, we prove an extension of a classical result of Pratt (1964), which connects the fact that u_1 is more risk averse than u_2 to the existence of a concave function T such that $u_1(x) = T(u_2(x))$, for the case of nonsmooth utility functions. We show also that our definition is consistent with another definition about what more risk averse means, and which does not assume utility functions.

For the absolute risk aversion, we can typically take the whole real line as domain. However, the relative risk aversion has to been taken relative to a specific wealth. Typically, it is calculated relative to the current wealth, by the formula that relative risk aversion is current wealth times absolute risk aversion. As a consequence, the relative risk aversion is always 0 at wealth 0. It can therefore not be expected that there would be a good definition for utility functions which are defined on the positive as well as on the negative domain. Typically, the relative risk aversion is taken for functions which are defined only for positive wealth. In our paper, we will focus also on this case, but take alternatively also the case where it is negative. We will treat both cases in a unifying way when this is possible, and separately if not.

We also introduce a second, weaker notion of ordering between risk aversions, which we call essential bounds for the risk aversion. We will give an example of a piecewise linear utility function which has essentially constant relative risk aversion. Furthermore, we will show that strict bounds are always essential bounds for the risk aversion, and therefore the definition of essential bounds is indeed a relaxation of the former definition.

Constant relative risk aversion is connected to power utilities. In the same way, constant absolute risk aversion is connected to exponential utility functions. We will generalize this feature to functions whose associated risk aversion is bounded from above or below, where we do not have anymore the power (or exponential) property, but at least an inequality which gives in some sense a super-power, respectively super-exponential property. We will formulate this issue firstly for strict bounds for the risk aversion, as well as later also for essential bounds.

Finally, we will show that the upper bound of the relative risk aversion of such a function translates into a lower bound of the concave conjugate function, and vice versa, an issue which is well-known for power utility functions. We will do this for strict as well as for essential bounds.

The outline of the paper is as follows: In section 2, we treat smooth utility functions, where we review inequality relationships between absolute risk aversion and exponential functions (as well as relative risk aversion and power functions). In section 3, we will give our generalized definition of the risk aversion measure, and prove the connection to power utility as well as the translation to bounds of the conjugate function. In section 4, we will do the same for essential bounds for the risk aversion. Section 5 concludes.

2. Definition and relationships for smooth utility functions

The classical definition of risk aversion from the literature is the following:

Definition 2.1. [Absolute and relative risk aversion] Let $u(x)$ be strictly increasing and concave for all $x \in D$, where D is the domain, and at least twice differentiable. Then the *absolute risk aversion* of $u(x)$ for $D \subset \mathbb{R}$ is defined through

$$ara(x) := -\frac{u''(x)}{u'(x)} \tag{2.1}$$

and the *relative risk aversion* of $u(x)$ for $D \subset]0, \infty[$ through

$$rra(x) := -\frac{xu''(x)}{u'(x)} \tag{2.2}$$

We will call the functions $ara(x)$ and $rra(x)$ risk aversion densities, for reasons which will become clear later. It is a well-known fact that constant absolute risk aversion density is connected to exponential utility functions, whereas constant relative risk aversion density connects to power utility functions. The following proposition says that for bounded risk aversion density, one has a super-exponential respectively a super-power property. Furthermore, for the relative risk aversion density, one has a translation of the risk aversion density to the one of its dual function.

Proposition 2.2. *Let the utility function be strictly increasing, concave and twice differentiable. Furthermore, assume that it is defined and strictly larger than* $-\infty$ *on the whole* $\mathbb{R}$ *for absolute risk aversion, and on* $]0,\infty[$ *for the relative risk aversion case. Then*

1. *The absolute risk aversion density is bounded from below by a constant* γ *if and only if the function* $e^{\gamma x}u'(x)$ *is nonincreasing, that is*

$$\frac{u'(y)}{u'(x)} \leq e^{-\gamma(y-x)} \tag{2.3}$$

for all $x < y$, $x, y \in \mathbb{R}$. *Analogously, it is bounded from above by a constant* γ *if and only if* $e^{\gamma x}u'(x)$ *is nondecreasing.*

2. *The relative risk aversion density is bounded from below by a constant* γ *if and only if the function* $x^\gamma u'(x)$ *is nonincreasing, that is*

$$\frac{u'(y)}{u'(x)} \leq \left(\frac{y}{x}\right)^{-\gamma} \tag{2.4}$$

for all $x < y$, $x, y \in]0,\infty[$. *Analogously, it is bounded from above if and only if* $x^\gamma u'(x)$ *is nondecreasing.*

3. *If the utility function satisfies the Inada conditions, that is* $u :]0,\infty[\to \mathbb{R}$ *is strictly increasing and concave, with* $u'(0^+) = \infty$ *and* $u'(x) \downarrow 0$ *as* $x \to \infty$, *then the dual function* $u^*(y) := \inf_x(xy - u(x))$ *is strictly increasing and twice differentiable too and for the relative risk aversion density, one has*

$$rra_{u^*}(y) = \frac{1}{rra_u((u^*)'(y))}$$

Remark 2.3. A relationship for the conjugate function does not hold for the absolute risk aversion density.

Proof

1. Because $u'(x)e^{\gamma x}$ is differentiable, this function is nonincreasing if and only if

$$u''(x)e^{\gamma x} + \gamma u'(x)e^{\gamma x} = e^{\gamma x}\left(u''(x) + \gamma u'(x)\right) \leq 0$$

which is equivalent to the condition that the term in the brackets on the right-hand side is smaller than or equal to 0. This is equivalent to

$$\gamma \leq -\frac{u''(x)}{u'(x)} = ara(x)$$

by the fact that $u'(x) > 0$.

2. Again by the differentiability, the condition that $x^\gamma u'(x)$ is nonincreasing is equivalent to

$$u''(x)x^\gamma + u'(x)\gamma x^{\gamma-1} = x^{\gamma-1}\left(xu''(x) + \gamma u'(x)\right) \leq 0$$

which is equivalent to the condition that the term in the brackets on the right-hand side is smaller than or equal to 0, by the fact that $x > 0$. This is equivalent to

$$\gamma \leq \frac{-xu''(x)}{u'(x)} = rra(x)$$

3. By the fact that u is smooth, the infimum of $xy - u(x)$ is at the point $y = u'(x)$, and it follows that

$$u^*(y) = y(u')^{-1}(y) - u((u')^{-1}(y))$$

where the inverse exists by the fact that u satisfies the Inada conditions. Furthermore, one has $(u')^{-1}(y) = (u^*)'(y)$ by standard differential calculus, from which it follows that u^* is strictly increasing and twice differentiable. Applying formula (2.2) to $u^*(y)$ one obtains

$$rra_{u^*}(y) = -\frac{y(u^*)''(y)}{(u^*)'(y)} = -\frac{u'(x)}{xu''(x)} = \frac{1}{rra_u(x)} = \frac{1}{rra_u((u^*)'(y))}$$

by standard differential calculus and the fact that $(u')^{-1}(y) = x$.

□

The following Corollary shows how the proposition can be used to make a connection from bounded risk aversion density to a super-exponential or super-power property of the utility function.

Corollary 2.4.

1. *Assume that relationship (2.3) holds with $\gamma > 0$, and that $u(\infty) = 0$. Then, for all $x < y$, $x, y \in \mathbb{R}$, we have*

$$u(y) \geq u(x)e^{-\gamma(y-x)}$$

2. *Assume that relationship (2.4) holds, and assume that $\gamma < 1$ (an analogous statement holds also for $\gamma > 1$, which we will state later in a more general context). Assume furthermore that $u(0) = 0$. Then, for all $x < y$, $x, y \in]0, \infty[$, we have*

$$u(y) \leq u(x)\left(\frac{y}{x}\right)^{1-\gamma}$$

Proof

1. Let $x < y < \infty$, then

$$\begin{aligned}-u(x) &= -u(y) + \int_x^y u'(\xi)d\xi \geq -u(y) + u'(y)\int_x^y e^{-\gamma(\xi-y)}d\xi \\ &= -u(y) + \frac{u'(y)}{-\gamma}\left(1 - e^{-\gamma(x-y)}\right)\end{aligned}$$

by equation (2.3) and the fact that $\xi < y$ in the integration area. Furthermore,

$$-u(y) = \int_y^\infty u'(\xi)d\xi \leq u'(y)\int_y^\infty e^{-\gamma(\xi-y)}d\xi = \frac{u'(y)}{\gamma}$$

because here, $\xi > y$, and again (2.3). It follows that

$$\begin{aligned}-u(x) &\geq -u(y) + \frac{u'(y)}{\gamma}\left(e^{-\gamma(x-y)} - 1\right) \geq -u(y)\left(1 + e^{-\gamma(x-y)} - 1\right)\\ &= -u(y)e^{-\gamma(x-y)}\end{aligned}$$

by the fact that $e^{-\gamma(x-y)} > 1$ for $y > x$. Putting the exponential function to the left-hand side, the result follows.

2. Let $0 < x < y$, then

$$\begin{aligned}u(y) &= u(x) + \int_x^y u'(\xi)d\xi \leq u(x) + u'(x)\int_x^y \left(\frac{\xi}{x}\right)^{-\gamma} d\xi\\ &= u(x) + \frac{u'(x)x}{1-\gamma}\left(\left(\frac{y}{x}\right)^{1-\gamma} - 1\right)\end{aligned}$$

because $\xi > x$ in the integration area and (2.4). Furthermore, again by (2.4),

$$u(x) = \int_0^x u'(\xi)d\xi \geq u'(x)\int_0^x \left(\frac{\xi}{x}\right)^{-\gamma} d\xi = \frac{u'(x)x}{1-\gamma}$$

because here, $\xi < x$. It follows that

$$\begin{aligned}u(y) &\leq u(x) + \frac{u'(x)x}{1-\gamma}\left(\left(\frac{y}{x}\right)^{1-\gamma} - 1\right) \leq u(x)\left(1 + \left(\frac{y}{x}\right)^{1-\gamma} - 1\right)\\ &= u(x)\left(\frac{y}{x}\right)^{1-\gamma}\end{aligned}$$

because $(\frac{y}{x})^{1-\gamma} > 1$.

□

Remark 2.5. A normalization to $u(\infty) = 0$ is possible if the utility function is bounded from above, which is for example the case if the absolute risk aversion density is bounded from below by a positive constant. Indeed, $u'(x)e^{\gamma x}$ nonincreasing implies that $u'(x)e^{\gamma x} \leq c$ with the constant $c = u'(0) > 0$ for $x > 0$, which implies that $u(x) \leq u(0) + \frac{c}{\gamma}(1 - e^{-\gamma x})$.

Similarly, a normalization to $u(0) = 0$ is possible in the case where the domain is $]0, \infty[$ if the utility function is bounded from below. This is for example the case if the relative risk aversion density is bounded from above and $\gamma < 1$. Because $u'(x)x^\gamma$ is nondecreasing, we have $u'(x)x^\gamma \leq u'(1) =: c$, and therefore $u(1) - u(\epsilon) \leq \frac{c}{1-\gamma}(1 - \epsilon^{1-\gamma})$ As $\epsilon \downarrow 0$, the left-hand side must remain bounded.

3. A generalized definition of risk aversion

3.1. Assumptions

We have to specify firstly the domain. For absolute risk aversion, the domain may be the whole $\mathbb{R}$. On the other hand, the typical case where relative risk aversion makes sense is when the utility function is concave and defined on the positive domain. This is the typical case that we will treat with most emphasis. Alternatively, we will also treat the case where the utility function is only defined on the negative domain. We will see that mathematically, this is the same as if we talk about risk loving instead of risk averse investors, and a utility function which is convex and defined on the positive domain.

Assumption 3.1. If we talk about absolute risk aversion, the wealth can be any value in $\mathbb{R}$, that is $D = \mathbb{R}$. On the other hand, for relative risk aversion, the wealth is either positive or negative, that is the domain is either $D =]0, \infty[$ or $D =]-\infty, 0[$.

Assumption 3.2. The utility function $u : D \to \mathbb{R} \cup \{-\infty\}$ is nondecreasing, concave and upper semicontinuous on D.

Assumption 3.3. The utility function u is proper, that is there exists a point $x \in D$ with $u(x) > -\infty$.

Remark 3.4. If $D =]-\infty, 0[$ and $u(x)$ satisfies Assumption 3.2, then $\tilde{u}(x) := -u(-x)$ is defined on the positive domain, and is nondecreasing, convex and lower semicontinuous.

3.2. Risk aversion measure

The aim of this section is to provide a definition of the risk aversion which applies for all utility functions satisfying Assumptions 3.1 and 3.2. For this, we will introduce a measure which we denote risk aversion measure, of which the absolute as well as relative risk aversion will turn out to be a Radon-Nikodym derivative if they exist.

Definition 3.5. [Superdifferential] Let $u : D \subset \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a concave function. Then the *superdifferential* of u at a point $x \in D$ is the set

$$\{s \in \mathbb{R} \cup \{\infty\} \mid u(y) - u(x) \leq s(y - x) \;\forall\; y \in D\}$$

By the fact that u is concave, we have that the superdifferential $\delta u(x)$ always exists and is nonempty in the region where $u(x)$ is finite, and by Assumption 3.2, it is nonincreasing in the sense that for $x < y$ we have $z_x \geq z_y$ for every $z_x \in \delta u(x)$, $z_y \in \delta u(y)$. We may therefore uniquely define

$$u'_r(x) := \inf_y \{y \in \delta u(x)\} \tag{3.5}$$

It is easy to show that $u'_r(x) \in \delta u(x)$ and the function $u'_r(x)$ is right-continuous and monotonically decreasing and therefore of finite variation, and we may define the (positive) Lebesgue-Stieltjes measure $-du'_r(x)$ without ambiguity if $-u(x) < \infty$. It is clear that if $x = \sup\{y \mid -u(y) = \infty\}$, we must have $\sup_y\{y \in \delta u(x)\} = \infty$. For consistency and for preserving the monotonicity of u'_r, we define therefore $u'_r = \infty$ for all x with $-u(x) = \infty$. If u would be twice continuously differentiable and strictly increasing, we would have, by rearranging the terms in equation (2.2), that $-d \ln u'$ is absolutely continuous with respect to $\operatorname{sgn}(x) d \ln |x|$, and there exists a unique Radon-Nikodym derivative $\gamma(x)$. The function $\operatorname{sgn}(x)$ is 1 if $x > 0$ (positive wealth) and -1 if $x < 0$ (negative wealth). This can also be expressed in the following way: The absolute risk aversion density on an interval I is $\gamma(x)$ if

$$\int_B -d\ln u'_r = \int_B \gamma(x)dx \tag{3.6}$$

for every Borel set $B \subset I$, or in differential notation

$$-d\ln u'_r = \gamma(x)dx$$

and the relative risk aversion density is $\gamma(x)$ on an interval $I \subset D$ if

$$\int_B -d\ln u'_r = \int_B \gamma(x)\,\mathrm{sgn}(x)d\ln|x| \tag{3.7}$$

for every Borel set $B \subset I$, or in differential notation

$$-d\ln u'_r = \mathrm{sgn}(x)\gamma(x)d\ln|x|$$

This definition is almost surely equivalent to 2.1. It becomes now clear why we denote $\gamma(x)$ as risk aversion density.

The measure $-d\ln u'_r$ is defined in principle for all utility functions satisfying Assumptions 3.1 and 3.2, but not necessarily on the whole domain D. Indeed, if $u(x)$ is constant after some point, or if $-u(x) = \infty$, the measure $-d\ln u'_r$ is not defined. We define therefore for the utility function u the domains

$$\begin{array}{lcl} D^u_{\text{eff}} & := & \mathrm{int}(D \setminus (D^u_{\text{inf}} \cup D^u_{\text{sup}})) \\ D^u_{\text{ra}} & := & D \setminus \mathrm{int}(D^u_{\text{sup}}) \end{array} \tag{3.8}$$

where the function int(.) means the interior of the set, and

$$\begin{array}{lcl} D^u_{\text{inf}} & := & \{x \in D \mid u(x) = -\infty\} \\ D^u_{\text{sup}} & := & \{x \in D \mid u(x) = \sup_{z\in D} u(z)\} \setminus D^u_{\text{inf}} \end{array} \tag{3.9}$$

Remark 3.6. The effective domain for a utility function u as defined above is

$$D^u_{\text{eff}} = \{x \in D \mid \exists\epsilon > 0 : 0 < u'_r(y) < \infty \ \forall y \in B_\epsilon(x)\}$$

It follows that the measure $-d\ln u'_r$ is sigma-finite on D_{eff}. Actually the measure is finite on all compact intervals of D^u_{eff}.

We turn now to the general definition of the risk aversion measure.

Definition 3.7. [Risk aversion measure] Let $u(x)$ be a utility function satisfying Assumptions 3.1 and 3.2. Then the *risk aversion measure* of u is defined as the following measure ρ on the Borel sets of D^u_{ra}: $\rho = -d\ln(u'_r)$ on D^u_{eff}, and for all $x \in D^u_{\text{ra}} \setminus D^u_{\text{eff}}$, $\rho(\{x\}) = \infty$.

Remark 3.8. It follows that ρ is sigma-finite on D_{eff}, whereas on $D_{\text{ra}} \setminus D_{\text{eff}}$, it is obviously not sigma-finite.

If this measure is absolutely continuous with respect to $d\ln|x|$, then u has a relative risk aversion density $\gamma(x)$ on an interval $I \subset D$ if the measure $-d\ln u'_r$ satisfies equation (3.7) on every Borel subset $B \subset I$. The same holds for the absolute risk aversion density if the measure is absolutely continuous with respect to dx.

Remark 3.9. If $u'(x)$ exists, the measure $-d\ln u'(x)$ is identical to $-d\ln u'_r(x)$. But replacing $-d\ln u'(x)$ by $-d\ln u'_r(x)$, we may apply the notion of risk aversion measures for all utility functions satisfying Assumptions 3.1 and 3.2.

In the classical utility theory, there is a one-to-one relationship between risk aversions and equivalence classes of utility functions, where two utility functions are equivalent if they can be mapped to each other by a positive affine transformation. This has been pointed out for instance in Pratt (1964). Similarly, using our definition of the risk aversion measure, one has a one-to-one relationship between a suitable class of measures and the set of equivalence classes of utility functions. This issue will be treated in the sequel.

Definition 3.10. A *risk aversion set* $D_{\rm ra}$ associated to a given domain D is a subset of the domain D as stated in Assumption 3.1, which is of the form $D_{\rm ra} = D\cap]-\infty, b]$.

Definition 3.11. A measure μ on the Borel sets of a risk aversion set $D_{\rm ra} = D\cap]-\infty, b]$ is said to be a *ρ-finite* measure if there exists a constant $a \le b$ such that $\mu(\{x\}) = \infty$ if $x \in D_{\rm ra}\setminus]a, b[$ and for each compact subset $K \subset]a, b[$ we have $\mu(K) < \infty$.

The classical definition of equivalence classes of utility functions is repeated here:

Definition 3.12. Two utility functions u and v satisfying Assumptions 3.1 and 3.2 are *equivalent* if there exist constants $c \in \mathbb{R}$ and $d > 0$ such that $u = c + dv$.

Proposition 3.13. *For each equivalence class of utility functions, there exists a unique risk aversion set $D_{\rm ra}$, and a unique ρ-finite measure μ on $D_{\rm ra}$, such that $D_{\rm ra} = D^u_{\rm ra}$ according to equation (3.8) and μ is the risk aversion measure of any utility function u of this class. On the other hand, for each risk aversion set $D_{\rm ra}$ and each ρ-finite measure μ on it, there exists a utility function u, unique up to equivalence, such that $D_{\rm ra} = D^u_{\rm ra}$ and μ is the risk aversion measure of u.*

Proof For a specific utility function u, we have that $D^u_{\rm ra}$ from equation (3.8) is a risk aversion set, because $D^u_{\sup}$ must by the concavity be of the form $[b, \infty[$, with $b \le \infty$. It follows from what has been done before that there exists a measure μ which is ∞ at each point of $D^u_{\rm ra} \setminus D_{\rm eff}$, where $D_{\rm eff} =]a, b[$ is an open interval, and with $\mu = -d\ln u'_r$ on $D_{\rm eff}$. Let $K \subset]a, b[$ be compact. Then there exist constants $a < \gamma < \delta < b$ with $K \subset]\gamma, \delta]$, and $\mu(]\gamma, \delta]) = \ln u'_r(\gamma) - \ln u'_r(\delta) < \infty$, that is μ is a ρ-finite measure. If v is of the same equivalence class as u, then $D^u_{\rm ra} = D^v_{\rm ra}$, $D^u_{\rm eff} = D^v_{\rm eff}$ and $v = c + du$ with $d > 0$, $v'_r = du'_r$, and $\ln v'_r = \ln d + \ln u'_r$. It follows that $d\ln v'_r = d\ln u'_r$ on $D^u_{\rm eff} = D^v_{\rm eff}$, and therefore the risk aversion measures ρ_u and ρ_v corresponding to u and v are the same.

Let now a risk aversion set $D_{\rm ra}$ and a ρ-finite measure μ be given. Let us firstly assume that $D_{\rm eff} \neq \emptyset$, where $D_{\rm eff} := \{x \in D \mid \mu(\{x\}) < \infty\}$. Then, for an $x_0 \in]a, b[= D_{\rm eff}$, we define a function

$$F(x) := \begin{matrix} \int_{]x_0,x]} d\mu(\xi) & \text{if} \quad x > x_0 \\ -\int_{]x,x_0]} d\mu(\xi) & \text{if} \quad x < x_0 \end{matrix}$$

and $F(x_0) = 0$, for all $x \in]a, b[$. It follows that $F(x)$ is right continuous, finite for all $x \in]a, b[$ and nondecreasing. Now we define $g(x) := e^{-F(x)}$, then it follows that $g(x)$

is right continuous, strictly positive, nonincreasing and finite on $]a,b[$, and therefore also integrable on compact sets in $]a,b[$. Define

$$u(x) := \begin{array}{lll} \lim_{\xi\uparrow b} u(\xi) & \text{if} & x \geq b \\ \int_{x_0}^{x} g(\xi)d\xi & \text{if} & a < x < b \\ \lim_{\xi\downarrow a} u(\xi) & \text{if} & x = a \\ -\infty & \text{if} & x < a \end{array}$$

Then u is nondecreasing, concave and upper semicontinuous, and because $u(x) = -\infty$ for $x < a$ and $u(x)$ is constant for $x \geq b$, $D^u_{\text{eff}} \subset]a,b[$. It is clear that g is in the superdifferential of u, and because it is nonincreasing and right-continuous, $u'_r = g$ on $]a,b[$. It follows that $0 < u'_r < \infty$ on $]a,b[$, and $D^u_{\text{eff}} =]a,b[= D_{\text{eff}}$, and $D_{\text{sup}} = [b,\infty[$, from which it follows that $D_{\text{ra}} = D^u_{\text{ra}}$ according to equation (3.8). Furthermore, on $]a,b[$, $\ln u'_r = -F$, and therefore, for a half-open interval $]x,y] \subset]a,b[$, $x > a$, one has $-d\ln u'_r(]x,y]) = F(y) - F(x) = \mu(]x,y])$. By the right continuity of F and the properties of measures, this must also hold as $x \downarrow a$.

It remains to show that for any other v satisfying Assumptions 3.1 and 3.2, for which $\rho_v = \mu$, it follows that $v = c + du$ with $c \in \mathbb{R}$ and $d > 0$. We have that $D_{\text{ra}} = D^u_{\text{ra}} = D^v_{\text{ra}}$, and from the fact that the risk aversion measures of u and v are the same, it follows that the domains $D_{\text{eff}} =]a,b[$ must coincide. Let now $a < x < y < b$. Then

$$\ln v'_r(x) - \ln v'_r(y) = -d\ln v'_r(]x,y]) = -d\ln u'_r(]x,y]) = \ln u'_r(x) - \ln u'_r(y)$$

and therefore (because $u'_r > 0$, $v'_r > 0$)

$$v'_r(y) = \frac{v'_r(x)}{u'_r(x)} u'_r(y)$$

Fixing an $x_0 \in]a,b[$, and applying the fundamental theorem of differential calculus Berberian (1999) we have

$$v(x) = v(x_0) + \int_{x_0}^{x} v'_r(\xi)d\xi = v(x_0) + \frac{v'_r(x_0)}{u'_r(x_0)} \int_{x_0}^{x} u'_r(\xi)d\xi = v(x_0) + \frac{v'_r(x_0)}{u'_r(x_0)}(u(x) - u(x_0))$$

so that v is equivalent to u on D_{eff}. For $x \geq b$, it follows by the concavity and monotonicity as well as the fact that $D \backslash D_{\text{ra}} = \text{int}(D_{\text{sup}})$ that $v(x)$ must be constant as well for $x \geq b$, and $v(x) = \lim_{\xi\uparrow b} v(\xi) = c + d\lim_{\xi\uparrow b} u(\xi) = c + du(x)$. For $x = a$, one has the limiting argument ($\xi \downarrow a$) by the fact that the functions are nondecreasing and upper semicontinuous. For $x < a$, both utility functions are $-\infty$ because $D^u_{\text{eff}} = D^v_{\text{eff}}$, and the result still holds.

If $D_{\text{eff}} = \emptyset$, one has the following cases:

1. $D_{\text{ra}} = \emptyset$, then any utility function u must be constant.

2. $D_{\text{ra}} = D$, then $u(x) = -\infty$.

3. $D_{\text{ra}} =]-\infty, b]$, then $u(x) = -\infty$ for $x < b$, and constant for $x \geq b$

In all cases, those properties require uniqueness up to positive affine transformations. □

3.3. Comparison of risk aversions

If one has the classical absolute or relative risk aversion density, one has a partial ordering according to which u_1 is more risk averse than u_2 if their absolute or relative risk aversion densities γ_1 and γ_2 satisfy $\gamma_1 \geq \gamma_2$. With the notion of the risk aversion measure, one can extend this definition to a partial ordering of the risk aversions for all utility functions satisfying Assumptions 3.1 and 3.2, and which coincides with the classical ordering in the case of absolute continuity.

Definition 3.14. [Comparison of risk aversions] Let $I \subset D$ be an interval, and $u_1(x), u_2(x)$ two utility functions with ρ_1 and ρ_2 their associated risk aversion measures. Then we say that $\rho_1 \leq \rho_2$ on I if $D^{u_1}_{\mathrm{ra}} \cap I = D^{u_2}_{\mathrm{ra}} \cap I =: D_{\mathrm{ra}} \cap I$ and for all Borel sets $B \subset D_{\mathrm{ra}} \cap I$ we have that

$$\rho_1(B) \leq \rho_2(B)$$

Remark 3.15. If γ_1 and γ_2 are the (relative or absolute) risk aversion densities corresponding to ρ_1 and ρ_2, it follows that $\gamma_1 \leq \gamma_2$ on I if $\rho_1 \leq \rho_2$ on I, provided the densities exist.

Remark 3.16. It follows that if $\gamma_1 \leq \gamma_2$, then ρ_1 is absolutely continuous with respect to ρ_2. Furthermore, if ρ_{u_2} is absolutely continuous with respect to $\mathrm{sgn}(x)d\ln|x|$, it follows that $\gamma_1(x) \leq \gamma_2(x)$, almost surely with respect to the measure $\mathrm{sgn}(x)d\ln|x|$, on I.

One may think about a general definition of the notion "more risk averse" without use of utility functions. Such a definition could be given in the following way:

Definition 3.17. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\succeq$ a partial ordering on its random variables. Then a random variable X on it is *unacceptable* if for all random variables Y we have $Y \succeq X$ and there exists a random variable $\hat{Y}$ with $\hat{Y} \succ X$.

Definition 3.18. [Comparison of risk aversions without utility functions] Let there be two investors, I_1 and I_2, say, and let there be a probability space $(\Omega, \mathcal{F}, P)$. Let there be preference relationships $\succeq_1$ and $\succeq_2$ on the set of random variables on $(\Omega, \mathcal{F}, P)$, corresponding to the preferences of I_1 and I_2. Then I_1 is *more risk averse* than I_2 if for all constants $w \in D$ and for all random variables X on $(\Omega, \mathcal{F}, P)$ which map to D one has

$$w \succeq_2 X \Rightarrow w \succeq_1 X \tag{3.10}$$

as well as, for all X on $(\Omega, \mathcal{F}, P)$ which are not unacceptable for investor I_1 one has

$$w \preceq_1 X \Rightarrow w \preceq_2 X \tag{3.11}$$

In the case in which preferences are given by expected utilities, this implication may also be written in the following way:

$$\{x \mid v(x) \geq Ev(X)\} \subset \{x \mid u(x) \geq Eu(X)\} \tag{3.12}$$

for all random variables X on $(\Omega, \mathcal{F}, P)$, and

$$\{x \mid u(x) \leq Eu(X)\} \subset \{x \mid v(x) \leq Ev(X)\} \tag{3.13}$$

for all random variables X with $Eu(X) > -\infty$, where investor 1 and investor 2 have utility functions $u(\cdot)$ and $v(\cdot)$ respectively. In this case we shall also simply say that u is more risk averse than v.

Roughly speaking, this definition says that a more risk averse investor prefers always a certain outcome to the risk if the less risk averse investor does.

We will show now that Definition 3.14 is equivalent to Definition 3.18 in the case where preferences are expressed in terms of expected utilities, no matter whether or not the utility functions are smooth. Actually, we prove a theorem given firstly by Pratt (1964), but now without any assumption about differentiability or strict monotonicity. This gives another argument that our definition of the risk aversion measure is sensible.

Theorem 3.19. *Let u and v be two utility functions that satisfy assumptions 3.1 and 3.2. Then the following statements are equivalent:*

1. $D^u_{\mathrm{ra}} = D^v_{\mathrm{ra}}$ *and on this domain,* $\rho_u \geq \rho_v$*, where* ρ_u *and* ρ_v *are the risk aversion measures corresponding to u and v.*

2. *There exists a nondecreasing concave function* $T : \mathbb{R} \cup \{-\infty\} \mapsto \mathbb{R} \cup \{-\infty\}$ *with* $u(x) = T(v(x))$ *for all* $x \in D$*, which is strictly increasing on* $v(D)$ *unless* $T(y) = -\infty$*, that is for all* $y_1 < y_2 \in v(D)$ *we have either* $T(y_1) < T(y_2)$ *or* $T(y_2) = -\infty$*.*

Assume furthermore that we have a probability space $(\Omega, \mathcal{F}, P)$ *which admits a continuously distributed random variable. Then statement 2 is equivalent to*

3. *u is more risk averse than v in the sense of Definition 3.18.*

Remark 3.20. Statement 3 is similar to the ones about risk premium and certainty equivalent of Pratt. However, our statement connects to a situation which does not use utility functions and can therefore be applied to more general situations.

The theorem uses some technical lemmas. The first is about chain rule of superdifferentials. A lemma for subdifferentials for nondecreasing convex functions has been stated in Hiriart-Urruty and Lemarechal (1996). We will refer to this lemma.

Lemma 3.21. *Let* $v : U \subset \mathbb{R} \to V \subset \mathbb{R} \cup \{-\infty\}$ *and* $T : V \to W \subset \mathbb{R} \cup \{-\infty\}$ *be two upper semicontinuous, concave and nondecreasing functions, and* $u(x) := T(v(x))$*. Then, for any* $x \in \mathbb{R}$*,*

$$\{\delta u(x)\} = \{\Delta_1 \Delta_2 : \Delta_1 \in \delta T(v(x)), \Delta_2 \in \delta v(x)\}$$

Proof Set $f(x) := -T(-x)$ and $g(x) := -v(x)$. Then f is a nondecreasing convex and g a convex function, and Theorem 4.3.1 in Hiriart-Urruty and Lemarechal (1996) about the chain rule for subdifferentials can be applied. From this, the statement of Lemma 3.21 follows. □

Lemma 3.22. *If u is more risk averse than v, then* $D^v_{\inf} \subset D^u_{\inf}$ *and* $D^v_{\sup} = D^u_{\sup}$*.*

Proof If $D^v_{\sup} \not\subset D^u_{\sup}$, then there exist x_1 and x_2 in D such that $u(x_2) > u(x_1)$ while $v(x_2) = v(x_1)$. We have $x_1 \in \{x \mid v(x) \geq v(x_2)\}$ while $x_1 \notin \{x \mid u(x) \geq u(x_2)\}$, so that (3.12) is violated when X is the degenerate random variable that always takes the value x_2.

If $D^u_{\sup} \not\subset D^v_{\sup}$, then there exist x_1 and x_2 in D such that $u(x_2) = u(x_1) > -\infty$ while $v(x_2) > v(x_1)$. It follows that $x_2 \in \{x \mid u(x) \leq u(x_1)\}$ but $x_2 \notin \{x \mid v(x) \leq v(x_1)\}$, so that (3.13) is violated with $X = x_1$.

If $D^v_{\inf} \not\subset D^u_{\inf}$, then there exists $x \in D$ such that $u(x) > -\infty$ whereas $v(x) = -\infty$. It follows that $\{y \mid v(y) \geq v(x)\} = D$ but, for $x_1 < x$, we have $u(x_1) < u(x)$. This is so because otherwise, $x \in \mathrm{int}(D^u_{\sup})$ but also $x \in D^v_{\inf}$, which cannot be by the former considerations. The inclusion (3.12) is therefore violated for the degenerate random variable $X = x$, because $x_1 \notin \{y \mid u(y) \geq Eu(X)\}$. □

Lemma 3.23. *If the risk aversion measure of u is larger than or equal to the one of v, then $D^u_{\mathrm{eff}} \subset D^v_{\mathrm{eff}}$.*

Proof It is clear from the definition that $D^u_{\mathrm{ra}} = D^v_{\mathrm{ra}}$. Let x be in D^u_{eff}. It follows that $x \in D^v_{\mathrm{ra}}$, and that $\{x\}$ is a compact subset of D^u_{eff}. Therefore the risk aversion measure of u, ρ_u, satisfies $\rho_u\{x\} < \infty$. If $x \in (D^v_{\mathrm{eff}})^c$, then $\rho_v(\{x\}) = \infty$, which is a contradiction. □

Proof [Proof of Theorem 3.19] We will firstly prove the equivalence of statements 1 and 2, because they do not use any probability space. After this, we will prove the equivalence of the statements 2 and 3.

Let statement 2 be correct, that is there is a concave function T with $u(x) = T(v(x))$. Then, by Lemma 3.21, we must have that $u'_r(x) = T'_r(v(x))v'_r(x)$, from which it follows for any interval $]x, y] \subset D^u_{\mathrm{eff}}$ that

$$-d \ln u'_r(]x, y]) = -d \ln v'_r(]x, y]) + \ln\left(\frac{T'_r(v(x))}{T'_r(v(y))}\right)$$

If $y \in D^u_{\sup}$, then $y \in D^v_{\sup}$ because T is strictly increasing on $v(D)$. It follows that $D^u_{\mathrm{ra}} = D^v_{\mathrm{ra}}$. Otherwise, if $y \in D^u_{\mathrm{eff}}$, we have $0 < u'_r(y) = T'_r(v(y))v'_r(y) < \infty$, and therefore the argument of the logarithm is strictly larger than 0 and therefore a valid expression. If $y \in (D^u_{\mathrm{eff}})^c$, then statement 1 follows because $\rho_u(]x, y]) = \infty$. We can therefore restrict to values of $y \in D^u_{\mathrm{eff}}$.

Because T'_r is nonincreasing and $v(x)$ is nondecreasing, it follows from $y > x$ that $-d \ln u'_r(]x, y]) \geq -d \ln v'_r(]x, y])$ on $D^u_{\mathrm{eff}} \cap D^v_{\mathrm{eff}}$. This holds for all intervals in this set. If B is a general Borel set in $D^u_{\mathrm{eff}} \cap D^v_{\mathrm{eff}}$, the result follows from Lemma 3.27, which we will prove in the next section. But $D^v_{\mathrm{eff}} \supset D^u_{\mathrm{eff}}$ because T is monotonic and concave, and we must have that $T(-\infty) = -\infty$ and therefore $D^v_{\inf} \subset D^u_{\inf}$. If B is not in D^u_{eff}, then $\rho_u(B) = \infty$, and therefore the result still must be true. The inequality holds therefore for all cases, and the implication is proved.

Conversely, let statement 1 be correct. Define the measure ϕ by

$$\phi(B) := \rho_u(B) - \rho_v(B)$$

for all Borel sets B on D_{ra}. Obviously, ϕ is a positive measure. Let x_0 be in D_{eff} with respect to u and v. If such a point would not exist, then, because $D^u_{\mathrm{eff}} \subset D^v_{\mathrm{eff}}$ by Lemma 3.23, then $D^u_{\mathrm{eff}} = \emptyset$. In this case, we have either $D_{\mathrm{ra}} = D$ and $u(x) = -\infty$, $T(y) = -\infty$ satisfy the requirements, or $D_{\mathrm{ra}} =]-\infty, b] \cap D$, and $u(x) = -\infty$ for $x < b$, $u(x) = \mathit{const}$ for $x \geq b$ is the utility function corresponding to statement 1, and $T(y) = -\infty$ for $y < v(b)$, $T(y) = \mathit{const}$ for $y \geq v(b)$ the corresponding concave function. We can therefore for the rest of the proof assume that $D^u_{\mathrm{eff}} \cap D^v_{\mathrm{eff}}$ is not empty, and define the function F by $F(x_0) := \ln u_r'(x_0) - \ln v_r'(x_0)$, and

$$F(x) := \begin{array}{ll} F(x_0) + \phi(]x, x_0]) & \text{if} \quad x < x_0 \\ F(x_0) - \phi(]x_0, x]) & \text{if} \quad x > x_0 \end{array}$$

from which it follows that F is nonincreasing and that $F(x) - F(y) = \phi(]x, y])$ for all $x < y$, and

$$u_r'(x) = e^{F(x)} v_r'(x)$$

On D_{eff}^v, $v : D_{\text{eff}} \to v(D_{\text{eff}})$ is invertible, with $D_{\text{eff}} =]a, b[$ and $v(D_{\text{eff}}) =]v(a), v(b)[$ is an open interval, and $u_r' = 0$ as well as $v_r' = 0$ for $x \geq b$, and $u_r'(x) = \infty$ as well as $v_r'(x) = \infty$ for $x < a$, and for $x \downarrow a$ both functions converge to their value at a. We can therefore define the function g by

$$g(y) = \begin{array}{lll} \infty & \text{if} & y < v(a) \\ \lim_{x \downarrow v(a)} g(x) & \text{if} & y = v(a) \\ e^{F(v^{-1}(y))} & \text{if} & v(a) < y < v(b) \\ 0 & \text{if} & y \geq b \end{array}$$

from which it follows that g is nonincreasing, nonnegative and strictly positive on $]v(a), v(b)[$ and

$$u_r'(x) = g(v(x)) v_r'(x)$$

Take now x_0 again a point at D_{eff}, with respect to both functions u and v. If we define $G(x) := u(x_0) + \int_{v(x_0)}^{x} g(\xi) d\xi$, then G is a nondecreasing concave function, strictly increasing on $[v(a), v(b)] \supset v(D) \setminus \{G = -\infty\}$, which can be chosen to be upper semicontinuous, and we obtain

$$u(x) = u(x_0) + \int_{x_0}^{x} u_r'(\xi) d\xi = u(x_0) + \int_{x_0}^{x} g(v(\xi)) v_r'(\xi) d\xi = G(v(x))$$

by the differential calculus Berberian (1999).

Let now statement 2 be satisfied, then there exists a nondecreasing, concave and upper semicontinuous function T, strictly increasing on $v(D) \setminus \{T = -\infty\}$, with $u(x) = T(v(x))$ for all $x \in D$. Let X be an arbitrary random variable on $(\Omega, \mathcal{F}, P)$, and let x be in the set $\{y \mid v(y) \geq Ev(X)\}$. Because T is nondecreasing and concave, we have

$$u(x) = T(v(x)) \geq T(Ev(X)) \geq E[T(v(X))] = Eu(X)$$

and therefore $x \in \{y \mid u(y) \geq Eu(X)\}$, so that inequality (3.12) is satisfied. Let X be a random variable with $Eu(X) > -\infty$, and x be in the set $\{y \mid u(y) \leq Eu(X)\}$. Then

$$T(v(x)) = u(x) \leq Eu(X) = ET(v(X)) \leq T(Ev(X))$$

It follows that $T(Ev(X)) > -\infty$, and by the fact that T is strictly increasing on $v(D) \setminus \{T = -\infty\}$, we have $v(x) \leq Ev(X)$, and inequality (3.13) is satisfied.

Let now statement 3 be satisfied. Define

$$\begin{array}{lll} T(y) & := & \tilde{T}(y_+) \\ \tilde{T}(y) & := & \inf\{u(z_-) \mid v(z) \geq y\} \end{array} \tag{3.14}$$

The definition is understood in the sense that $\tilde{T}(y) = \sup_{z\in D} u(z)$ when $y > \sup_{z\in D} v(z)$. Furthermore, if $D =]0,\infty[$, u and v have to be interpreted as the extension to the whole real line, with values $-\infty$ for x negative. To show that $u(x) = T(v(x))$ for all $x \in D$, we need to show that

$$u(x) = \lim_{\epsilon\downarrow 0} \inf\{u(z_-) \mid v(z) \geq v(x) + \epsilon\} \tag{3.15}$$

for all $x \in D$. First of all, consider the situation in which v is constant, say $v(x) = v_0$. In that case it follows from Lemma 3.22 that $D^u_{\text{sup}} = D$ so that u is constant as well, say $u(x) = u_0$ for all $x \in D$. Since $\inf_{z\in D} u(z) = \sup_{z\in D} u(z) = u_0$, it follows from the definition (3.14) that $T(y) = u_0$ for all y so that the relation $u(x) = T(v(x))$ is satisfied for all x.

In the rest of the proof, we assume that v is not constant. First consider the case in which $x \in D^v_{\text{sup}}$. The set D^v_{sup} is nonempty if and only if the function v has a saturation point. In this case, write $D^v_{\text{sup}} = [b,\infty[$, respectively $[b,0[$ if $D =]-\infty,0[$. For all $x \in D^v_{\text{sup}}$, we have $\{z \mid v(z) \geq v(x)+\epsilon\} = \emptyset$ for all $\epsilon > 0$. Because $D^v_{\text{sup}} = D^u_{\text{sup}}$ by Lemma 3.22, we must have $u(x) = \sup_{z\in D} u(z) = u(b)$ for $x \in D^v_{\text{sup}}$, and by definition $T(v(x)) = \sup_x u(x) = u(b)$ for $x \in D^v_{\text{sup}}$.

Next assume that $x \in D^v_{\text{ra}} \setminus \text{int}(D^v_{\text{inf}})$. Because v is not constant, this set is nonempty and is an interval closed in D. If we add the limit points of D which may be $\pm\infty$, we have without loss of generality that this interval is $[a,b]$. On this interval, the function $v : [a,b] \to [v(a),v(b)]$ is continuous and strictly increasing, and has therefore an inverse function $v^{-1} : [v(a),v(b)] \to [a,b]$ which is again continuous and strictly increasing. If $x \in D^v_{\text{ra}} \setminus \text{int}(D^v_{\text{inf}}) \setminus \{b\}$, we have for ϵ small enough still that $v(x)+\epsilon < v(b)$ and therefore $a < v^{-1}(v(x)+\epsilon) < b$. It follows that $\tilde{T}(v(x)+\epsilon) = u(v^{-1}(v(x)+\epsilon)_-) \geq u(v^{-1}(v(x))) = u(x)$ by the monotonicity of u and the strict monotonicity of v^{-1}. By the right-continuity of u which is the same as upper semicontinuity for nondecreasing functions, the left-hand side converges to $u(x)$ as $\epsilon \downarrow 0$, so that the relation (3.15) holds.

Finally, consider the situation in which $x \in D^v_{\text{inf}}$. This means that $v(x) = -\infty$ for $x \in D^v_{\text{inf}}$. Since $D^u_{\text{inf}} \supset D^v_{\text{inf}}$, we also have $u(x) = -\infty$ for $x \in D^v_{\text{inf}}$. The definition (3.14) implies that $T(-\infty) = -\infty$, so that also in this case the relation $u(x) = T(v(x))$ is satisfied.

It is clear that T is nondecreasing, because if $y_1 < y_2$, the set from definition (3.14) for y_1 is a superset of the one for y_2, and in general for two sets $A \subset B$ one has $\inf A \geq \inf B$. If a function is nondecreasing, upper semicontinuity is equivalent to right-continuity. But right-continuity follows directly from definition (3.14).

Next we show that T is strictly increasing on $v(D) \cap \{T = -\infty\}$. Let $y_1 < y_2$ in $v(D)$. If such two values do not exist, then nothing is to prove. If $y_1 = -\infty$, we have $T(y_1) = -\infty$, and the statement is trivially satisfied. If $y_1 > -\infty$, then $y_1 \in [v(a),v(b)]$, where v is strictly increasing and continuous on $[a,b]$. In this case, $\tilde{T}(y_i) = u(v^{-1}(y_i)_-)$. But in this range, also v^{-1} is strictly increasing and continuous, and $v^{-1}(y_1) < v^{-1}(y_2)$. Let $\tilde{T}(y_1) = \tilde{T}(y_2) > -\infty$. Then $u(v^{-1}(y_1)) = u(v^{-1}(y_2))$, and $v^{-1}(y_1) \in D^u_{\text{sup}}$ but not in D^v_{sup}, a contradiction to Lemma 3.22. It follows that $\tilde{T}(y_1) < \tilde{T}(y_2)$ for $y_1 < y_2$ unless $\tilde{T}(y_i) = -\infty$. If we choose $\epsilon > 0$ so small that $y_1 + \epsilon < y_2$, then

$$T(y_1) \leq \tilde{T}(y_1 + \epsilon) < \tilde{T}(y_2) \leq T(y_2)$$

It remains to show that T is concave. If not, then there exist $y_1, y_2 \in \mathbb{R}$, $t \in]0,1[$ with

$$tT(y_1) + (1-t)T(y_2) > T(ty_1 + (1-t)y_2) \tag{3.16}$$

Without loss of generality, we may assume that $y_1 < y_2$. If u is constant, it follows that T is constant, and we have a direct contradiction. If not, we have from the arguments above and a little extension that there exists an interval $[a, b]$ such that $T(y) = u(v^{-1}(y))$ for $y \in [v(a), v(b)]$, $T(y) = u(b)$ for $y > v(b)$ and $T(y) = -\infty$ for $y < v(a)$.

Now assume that $y_1 < v(a)$. It follows that $T(y_1) = -\infty$, which is a contradiction to (3.16). Assume now that both y_1 and y_2 are in $[v(a), v(b)]$. Then inequality (3.16) says

$$tu(v^{-1}(y_1)) + (1-t)u(v^{-1}(y_2)) > u(v^{-1}(ty_1 + (1-t)y_2))$$

If $X := v^{-1}(y_1)$ with probability t, $v^{-1}(y_2)$ with probability $1-t$ (such a random variable is possible because the probability space admits a continuous distribution), this equation means

$$Eu(X) > u(v^{-1}(Ev(X)))$$

where by the assumption that $y_1, y_2 \in [v(a), v(b)]$, also $Ev(X) \in [v(a), v(b)]$. Define $x^* := v^{-1}(Ev(X))$. It is obvious that $v(x^*) \geq Ev(X)$. But on the other hand, $Eu(X) > u(x^*)$. This is a contradiction to statement 3.

If $v(b) < y_1 < y_2$, we have again that $T(y_1) = T(y_2)$, and equation (3.16) gives a direct contradiction. It remains to show the case where $v(a) \leq y_1 \leq v(b) < y_2$. But then we have

$$tT(y_1)+(1-t)T(v(b)) = tT(y_1)+(1-t)T(y_2) > T(ty_1+(1-t)y_2) \geq T(ty_1+(1-t)v(b))$$

that means inequality (3.16) is also satisfied if we replace y_2 by $v(b)$. But this case we have already treated before. □

3.4. Connection to power utility functions

In this section, we will show that the connection to power utility functions, as established in Proposition 2.2 for smooth utility functions, continues to hold in the general case, with our generalized definition of risk aversion. A similar connection holds also for the absolute risk aversion. We omit here the proof and restrict to the case of relative risk aversion.

Firstly, we reconsider from Definition 3.14 what bounded risk aversion means.

Definition 3.24. Let u be a utility function with risk aversion measure ρ, and $\gamma \geq 0$ a constant. Then the absolute risk aversion is *bounded from above by* γ on an interval I if ρ satisfies

$$\rho \leq \gamma dx \quad \text{on} \quad I$$

with dx the Lebesgue measure. It has a relative risk aversion *bounded from above by γ* on I if ρ satisfies

$$\rho \leq \operatorname{sgn}(x)\gamma d\ln(|x|) \quad \text{on} \quad I$$

Boundedness from below is defined analogously with the reverse inequalities.

Remark 3.25. γdx and $\operatorname{sgn}(x)\gamma d\ln(|x|)$ are risk aversion measures on the Borel sets of $\mathbb{R}$, respectively on $]0,\infty[$ or $]-\infty,0[$, and γ is the corresponding Radon-Nikodym derivative. Definition 3.24 is therefore consistent with Definition 3.14. Furthermore, from Definition 3.24 it follows that I cannot contain elements of D^u_{sup}, because the utility function v corresponding to the risk aversion measure on the right-hand side of the inequality is an exponential or power utility function, and therefore $D^v_{\text{sup}} = \emptyset$. But according to Definition 3.14, the risk aversion domains must coincide on I, that is $D^u_{\text{ra}} \cap I = D^v_{\text{ra}} \cap I$.

Proposition 3.26. *The relative risk aversion of a utility function u satisfying Assumptions 3.1 and 3.2 is uniformly bounded from below (above) by a constant $\gamma > 0$ in a region $R :=]a,b[\subset D$ if and only if $R \subset D^u_{\text{ra}}$ and for every $a < x \leq y < b$ we have the inequality*

$$u'_r(y) \leq \begin{array}{ll} u'_r(x)\left(\frac{y}{x}\right)^{-\gamma} & \text{if} \quad D =]0,\infty[\\ u'_r(x)\left(\frac{y}{x}\right)^{\gamma} & \text{if} \quad D =]-\infty,0[\end{array} \tag{3.17}$$

or inequality in the other direction if bounded from above.

Proof By Definition 3.24, it is clear that $R \subset D^u_{\text{ra}}$. Let firstly x be in D^u_{eff}, and $x < y < b$, then $y \in D^u_{\text{eff}}$. By assumption (Definition 3.24), and because $]x,y] \subset D_{\text{eff}}$, we have

$$\rho(]x,y]) = -d\ln u'_r(]x,y]) \geq \operatorname{sgn}(\xi)\gamma d\ln|\xi|(]x,y])$$

on $]x,y]$. If $D =]0,\infty[$, then by integration

$$\ln u'_r(y) - \ln u'_r(x) \leq -\gamma\left(\ln y - \ln x\right)$$

and by the rules of the logarithm and the monotonicity of the exponential function

$$\frac{u'_r(y)}{u'_r(x)} \leq \left(\frac{y}{x}\right)^{-\gamma}$$

which is equation (3.17) for positive wealth. For $D =]-\infty,0[$ the arguments are similar.

If $x \in (D^u_{\text{eff}})^c$, then $u'_r = \infty$ from which the equation follows, or $\xi \in D^u_{\text{eff}}$ for all $\xi > x$. In the latter case, also inequality (3.17) holds for all $y > \xi > x$ by the arguments before, and by the right-continuity of u'_r the inequality then also holds for x.

Let now equation (3.17) hold and $R \subset D^u_{\text{ra}}$. Consider the interval $]x,y] \subset R$. We consider firstly the case where $x,y \in D^u_{\text{eff}}$. On D^u_{eff} the measures $-d\ln u'_r$ and $\operatorname{sgn}(x)d\ln|x|$ are sigma-finite. By taking the logarithm which is monotonic we have

$$\ln u'_r(y) - \ln u'_r(x) \leq \operatorname{sgn}(x)\gamma\left(\ln y - \ln x\right)$$

on every half-open interval in $D_{\text{eff}} \cap]a,b[$. The result follows by the following Lemma 3.27 for sigma-finite measures.

If $x \in (D^u_{\text{eff}})^c$, then $\rho(\{x\}) = \infty$ but $\operatorname{sgn}(y)\gamma d \ln|y|(\{x\}) = 0$, for every $\gamma > 0$, and therefore the relative risk aversion must be bounded from below by any constant $\gamma > 0$.

□

Lemma 3.27. *Let μ_1 and μ_2 two sigma-finite measures on the Borel set with $\mu_1(I) \le \mu_2(I)$ for every half-open interval I. Then $\mu_1(B) \le \mu_2(B)$ for every Borel set B.*

Proof Let μ_1 and μ_2 be two sigma-finite measures on the Borel set with $\mu_1(I) \le \mu_2(I)$ for all half-open intervals I. Sigma-finiteness of the measures μ_j on the Borel sets of a set D_{eff} means that there are disjoint subsets A_i with $\cup_{i\ge 1} A_i = D_{\text{eff}}$ and $\mu_j(A_i) < \infty$ for all i, for both $j = 1$ and $j = 2$. If there would be a Borel set B on which $\mu_1(B) > \mu_2(B)$, there would at least be one set A_i with $\mu_1(A_i \cap B) > \mu_2(A_i \cap B)$, and on A_i the measures are finite. We may therefore assume that both measures are finite. We define then a signed measure $\lambda := \mu_2 - \mu_1$. This measure is obviously countably additive, positive on all half-open intervals, and negative on B. Because of the additivity of λ, we have that $\lambda \ge 0$ for all finite unions of half-open intervals, which form an algebra. Because this algebra is a subset of the Borel sets, λ is also countably additive on this algebra. By Carathéodory's extension theorem, see Williams (1991), one can therefore extend λ (defined on this algebra, where it is positive) to a positive measure $\tilde{\lambda}$, defined on the whole Borel set. It follows that $\tilde{\lambda}(B) \ge 0$. The two measures λ and $\tilde{\lambda}$ coincide on a π-system generating the Borel sets, and by the uniqueness lemma, see Williams (1991), which holds also for signed measures, it follows that $\lambda = \tilde{\lambda}$ on the Borel set, therefore $\lambda(B) \ge 0$, a contradiction.

□

In the case of constant relative risk aversion, inequality (3.17) becomes an equality, and by integrating both sides firstly with respect to x, then with respect to y, we obtain (we treat here the case $D =]0, \infty[$)

$$u(y)\left(x^{1-\gamma} - y_0^{1-\gamma}\right) - u(y_0)x^{1-\gamma} = u(x)\left(y^{1-\gamma} - y_0^{1-\gamma}\right) - u(y_0)y^{1-\gamma}$$

where y_0 is the lower integration bound. Treating y_0 and x as two different constants in the interval $[a, b]$ where the relative risk aversion is γ, it follows for suitable constants A and B that

$$u(y) = A + By^{1-\gamma}$$

that means constant relative risk aversion means precisely the power property, a well-known result. When the relative risk aversion is only bounded but not constant, inequality (3.17) gives a super-power property of the utility function.

3.5. Relative risk aversion of the conjugate function

The connection between the relative risk aversion of a utility function and its dual stated in Proposition 2.2 for smooth utility functions is not so easy any more for the general case. The reason is that u'_r may not be invertible any more, and as a consequence, if we have $rra_{u^*}(y) = \frac{1}{rra_u(x)}$, it is not clear which x one has to take for a specific y. However, it is still possible to make a statement about the bounds of the relative risk aversion. This is the aim of this section.

Definition 3.28. Let $u(x)$ be a utility function satisfying Assumptions 3.1, 3.2 and 3.3. We extend the domain in the following way:

$$u_{ext}(x) := \begin{array}{lll} -\infty & \text{on} &]-\infty,0[\quad \text{if} \quad D=]0,\infty[\\ u(-x) & \text{on} &]0,\infty[\quad \text{if} \quad D=]-\infty,0[\\ u(x) & \text{on} & D \end{array} \tag{3.18}$$

and at 0 such that the function becomes upper semicontinuous. Then its *dual function* is defined as the concave conjugate

$$u^*(y) := \inf_{x\in\mathbb{R}} (xy - u(x)) \tag{3.19}$$

Lemma 3.29. *Let $u(x)$ satisfy Assumptions 3.1, 3.2 and 3.3, with domain D. Then its dual function $u^*(y)$ satisfies them too on D. Furthermore, the extension of u^* following (3.18) is $(u^*)_{ext} = (u_{ext})^*$.*

Proof From Rockafellar (1970), it follows that $u^*(y)$ defined on $\mathbb{R}$ is a concave, proper and upper semicontinuous function. (This theorem is proved in Rockafellar (1970) for convex instead of concave functions. Note that closedness is equivalent to upper semicontinuity for concave functions). It remains to show that $u^*(y)$ is nondecreasing on D. Let $y_1 < y_2 \in D$. Then for all $x > 0$, it follows

$$xy_1 - u_{ext}(x) \le xy_2 - u_{ext}(x)$$

and the same for the infimum over all positive x. The result follows by recognizing that for $y \in D$,

$$u^*(y) = \inf_{x\in\mathbb{R}} (xy - u_{ext}(x)) = \inf_{x>0} (xy - u_{ext}(x))$$

The final statement is clear for $y \in D$. For $y < 0$ and $D =]0,\infty[$, expression (3.19) can be made arbitrarily small when $x \to \infty$. For $y > 0$ and $D =]-\infty,0[$, we have

$$\begin{aligned} (u_{ext})^*(y) &= \inf_{x\in\mathbb{R}} (xy - u_{ext}(x)) = \inf_{x\in\mathbb{R}} ((-x)(-y) - u_{ext}(-x)) \\ &= \inf_{-x\in\mathbb{R}} ((-x)(-y) - u_{ext}(-x)) = u^*(-y) \end{aligned}$$

which is by definition $(u^*)_{ext}$. □

Proposition 3.30. *Let u be a utility function satisfying Assumptions 3.1, 3.2 and 3.3. Then its dual function $u^*(y)$ has a relative risk aversion bounded from above (below) by $\frac{1}{\gamma}$ on $]a,b[$ if $u(x)$ has a relative risk aversion bounded from below (above) by γ on $](u^*)'_r(b),(u^*)'_r(a)[$ ($[(u^*)'_r(b),(u^*)'_r(a)]$) if $D=]0,\infty[$, respectively on $]-(u^*)'_r(a),-(u^*)'_r(b)[$ ($[-(u^*)'_r(a),-(u^*)'_r(b)]$) if $D=]-\infty,0[$, provided this set is nonempty.*

Proof Let $a < x < y < b$ and assume firstly that $x, y \in D^{u^*}_{\text{eff}}$. We have that

$$\int_{]x,y]} -d\ln(u^*)'_r = -\ln(u^*)'_r(y) + \ln(u^*)'_r(x)$$

and

$$\int_{](u^*)'_r(y),(u^*)'_r(x)]} d\ln|\xi| = \ln(u^*)'_r(x) - \ln(u^*)'_r(y)$$

for $D =]0, \infty[$, respectively for $D =]-\infty, 0[$

$$\int_{]-(u^*)'_r(x), -(u^*)'_r(y)]} -d\ln|\xi| = \ln(u^*)'_r(x) - \ln(u^*)'_r(y)$$

and therefore the integrals are the same. Because on D the measure $d\ln|x|$ is absolutely continuous with respect to the Lebesgue measure, we may exclude the point at the right end of the integration interval without changing the value of the integral. Let firstly $D =]0, \infty[$. Then, by assumption, if $](u^*)'_r(y), (u^*)'_r(x)[\subset D^u_{\text{eff}}$

$$-\int_{]x,y]} d\ln(u^*)'_r = \int_{](u^*)'_r(y),(u^*)'_r(x)[} d\ln x \le \frac{1}{\gamma} \int_{](u^*)'_r(y),(u^*)'_r(x)[} -d\ln u'_r$$

and the right-hand side is then equal to

$$-\frac{1}{\gamma}\left(\ln u'_r((u^*)'_r(x)_-) - \ln u'_r((u^*)'_r(y))\right) \le \frac{1}{\gamma}(\ln y - \ln x) = \frac{1}{\gamma}\int_{]x,y]} d\ln|\xi|$$

which is the required result. The first inequality follows from the fact that

$$(u^*)'_r(x) \in \delta u^*(x) \Rightarrow x \in \delta u((u^*)'_r(x))$$

by the general duality rules of superdifferentials, and $u'_r((u^*)'_r(x)_-)$ is the supremum of those superdifferentials, and therefore larger. This holds by the general rule that $u'_r(z_-) = \sup\{\delta u(z)\}$. On the other hand, by the same argument, we have that $y \in \delta u((u^*)'_r(y))$ and therefore larger than or equal than the infimum of the superdifferential, which is $u'_r((u^*)'_r(y))$. If $](u^*)'_r(y), (u^*)'_r(x)[\not\subset D^u_{\text{eff}}$, then $u'_r((u^*)'_r(y)) = \infty$, and by the same rules as before, $y = \infty$, a contradiction to the assumption that $y \in D$. The other case, where $D =]-\infty, 0[$, goes essentially the same way, where one has to apply u_{ext} and u^*_{ext}, and the fact that by the symmetry from Definition 3.28, $y \in \delta u(z) \Rightarrow -y \in \delta u(-z)$.

We will show now that $x, y \in D^{u^*}_{\text{eff}}$ is always true if u has a relative risk aversion bounded from below, and the other assumptions of Proposition 3.30 are satisfied. We have in general $D_{\text{eff}} =]d_{min}, d_{max}[$. Let us firstly consider the case when $D =]-\infty, 0[$. If $d_{max} < b \le 0$, we have that $0 \in \delta u^*(d_{max})$ and thus $\pm d_{max} \in \delta u(0)$. It follows that $u(x) \le u(0) - d_{max}x$ and because u'_r is monotonically decreasing we have $u'_r \ge -d_{max}$. For $\epsilon > 0$, it follows that

$$\int_{]\epsilon,0[} -d\ln u'_r \le \frac{1}{-d_{max}} \int_{]\epsilon,0[} -du'_r = \frac{1}{-d_{max}}\left(u'_r(-\epsilon) - u'_r(0_-)\right)$$

which must be bounded if $\epsilon > 0$ is small enough due to the fact that u is proper. On the other hand, $d\ln x(]-\epsilon, 0[) = \infty$ for every $\epsilon > 0$. It follows that the relative risk aversion of u cannot be bounded from below.

Let on the other side $0 > d_{min} > a$. It follows that $(u^*)'_r(a) = \infty$ and therefore $a \in \delta u(\infty)$ or by the symmetry $-a \in \delta u(-\infty)$. It follows that $u'_r(x) \le -a$ for all $x \in D$. By the fact that u'_r is nonincreasing, it follows that $u'_r(x)$ converges as $x \to -\infty$. For all $\epsilon > 0$, there exists an $x \in]-\infty, -(u^*)'_r(b)[$ with

$$\int_{x-1}^{x} -d\ln u'_r \le \frac{1}{u'_r(c)} \int_{x-1}^{x} -du'_r \le \frac{u'_r(x-1) - u'_r(x)}{u'_r(c)} < \frac{\epsilon}{u'_r(c)}$$

for $x < c < -(u^*)'_r(b)$ and c such that $u'_r(c) > 0$, which must be possible because otherwise D^u_{ra} is empty, and therefore the relative risk aversion cannot be bounded anywhere. On the other hand, the measure $-d\ln|\xi|(]x-1,x])$ as a function of x is bounded from below for $x < c$. Again, it follows that the relative risk aversion of u cannot be bounded from below.

The case $D =]0,\infty[$ follows similar arguments, but for completeness we will show it too. Let us firstly assume that $a < d_{min}$. Then we have $(u^*)'_r(a) = \infty$ and therefore $a \in \delta u(\infty)$. It follows that $](u^*)'_r(b), (u^*)'_r(a_-)[$ is an interval of the form $]c,\infty[$ with $c < \infty$, because otherwise the interval would be empty. By the fact that u'_r is nonincreasing, we must have $u'_r(x) \geq a$ for all $x > 0$, and by the monotonicity and the boundedness $u'_r(x)$ converges as $x \to \infty$. This means that for all $\epsilon > 0$ there must be an $x > c$ with $|u'_r(x) - u'_r(x+1)| < \epsilon$. It follows

$$-\int_x^{x+1} d\ln u'_r \leq \frac{1}{-a}\int_x^{x+1} du'_r < \frac{\epsilon}{a}$$

On the other hand, $d\ln|\xi|(]x,x+1]) \geq d\ln|\xi|(]c,c+1]) > 0$, and therefore bounded from below. It follows that the relative risk aversion of u cannot be bounded from below.

For the last case, let us assume that $d_{max} < b$. Then $(u^*)'_r(b) = 0$ and $b \in \delta u(0)$. The interval $](u^*)'_r(b), (u^*)'_r(a_-)[$ is of the form $]0,c[$, with $c > 0$ because otherwise it is empty. It follows by the rules of superdifferentials that $u(x) - u(0) \leq bx$ and by the monotonicity $u'_r(x) \leq b\ \forall x$. That means that $u'_r(x)$ is monotonically decreasing and bounded from above, and therefore

$$-\int_{\epsilon_1}^{\epsilon_2} d\ln u'_r \leq -\frac{1}{u'_r(c_1)}\int_{\epsilon_1}^{\epsilon_2} du'_r \leq \frac{u'_r(\epsilon_1) - u'_r(\epsilon_2)}{u'_r(c_1)}$$

where $0 < c_1 < c$ is a constant with $u'_r(c_1) > 0$ which exists if u is not constant, which cannot be because then $D^u_{\mathrm{ra}} = \emptyset$. The right-hand side of the inequality tends to 0 for all $0 < \epsilon_1 < \epsilon_2 < c_1$ as $c_1 \to 0$, but the measure $d\ln|x|(]\epsilon_1,\epsilon_2[)$ remains bounded from below for a suitable sequence of ϵ_1, ϵ_2. Again, it follows that the relative risk aversion of u cannot be bounded from below.

We have therefore that the boundedness from above holds on any half-open interval, and because $]a,b[\subset D^u_{\mathrm{eff}}$, the measures $d\ln x$ and $d\ln(u^*)'_r$ are sigma-finite on $]a,b[$. The result follows now by Lemma 3.27.

□

4. Essential bounds for the risk aversion

4.1. Definition

Definitions 3.14 and 3.24 would imply that there cannot be jumps in u'_r if the risk aversion is bounded from above. It is clear that the (absolute or relative) risk aversion is not bounded from above at the jumps. On the other hand, Definitions 3.7, 3.14 and 3.24 are too strict. It would still mean that the risk aversion cannot be bounded by a constant either from above or from below for all piecewise linear utility functions. Furthermore, for having "essentially" a super-power property in the sense that a slight modification of inequality (3.17) holds, it is only necessary that the relative risk aversion is bounded from below (or bounded from above) up to a certain tolerance.

A first idea for a weaker ordering of risk aversions would be if we would say that a utility function u is essentially not more risk averse than v on $]a,b[\subset D$ if $]a,b[\cap D^u_{\text{ra}} =]a,b[\cap D^v_{\text{ra}}$ and there exists a constant $C < \infty$ such that

$$\rho_u(B') \leq \rho_v(B') + C$$

for all $B' \subset \mathcal{B}]a,b[$.

For obtaining again an if and only if statement analogous to Proposition 3.26 (super-power resp. super-exponential property), as well as for being able to have essentially bounded risk aversion also for piecewise linear functions, we define it again slightly more generally.

Definition 4.1. [Weak comparison of risk aversions] A utility function $u(x)$ is *essentially not more risk averse* than a utility function $v(x)$ on an interval $]a,b[\subset D$ if $]a,b[\cap D^u_{\text{ra}} =]a,b[\cap D^v_{\text{ra}}$ and there exists a constant $0 < C < \infty$ such that

$$\rho_u([x,y]) \leq \rho_v([x,y]) + C \tag{4.20}$$

for all intervals $[x,y] \subset]a,b[$. Analogously to the open interval $]a,b[$, weak comparison for risk aversions is also defined on closed or half-open intervals.

Remark 4.2. From Definition 4.1, it follows that inequality 4.20 holds for all (open, closed, half-open) intervals, by the fact that one can create those intervals as a countable union of closed intervals, and by the monotone convergence of measures, see Williams (1991).

Remark 4.3. In order to show that u is essentially not more risk averse than v, one can also show inequality (4.20) for half-open intervals $]x,y]$ and, in the case of the closed interval $[a,b]$, additionally for the sets $\{a\}$ and $\{b\}$. The reason is that a closed interval can be written as a countable intersection of half-open intervals. The result follows then by the monotone convergence of measures, see Williams (1991), and the fact that each closed interval in D^u_{eff} with finite endpoints is compact.

Indeed, given an open interval $]a,b[$ and $x,y \in D^u_{\text{eff}}$ with $]x,y] \subset]a,b[$, then $[x,y] \subset D^u_{\text{eff}}$. Assume firstly that $[x,y] \subset D^v_{\text{eff}}$, then there exists an N such that $]x-\frac{1}{n},y] \subset]a,b[$ for all $n \geq N$ and furthermore a subset of D^u_{eff} as well as D^v_{eff}, and the monotone convergence of measures yields that the constant C for closed sets is bounded by the one for half-open sets. If $[x,y]$ would not be a subset of D^v_{eff}, by the fact that $\rho_u([x,y]) < \infty$, inequality (4.20) would trivially be satisfied.

If x or y would not be in D^u_{eff}, then either $\rho_u(]x,y]) = \infty$ or $\rho_u(\{x\}) = \infty$, by the ρ-finiteness of the measure ρ_u (Definition 3.11). It follows that in the first case, $\rho_v(]x,y]) = \infty$ and the inequality (4.20) still holds. In the second case, if x would be in D^v_{eff}, there must be a $\xi < x$ with $[\xi,x] \subset D^v_{\text{eff}}$, and by the ρ-finiteness of the measure ρ_v, it follows that $\rho_v(]\xi,x]) < \infty$, a contradiction to $\rho_u(]\xi,x]) = \infty$ and the fact that (4.20) holds for half-open intervals. It follows that $\rho_v(\{x\}) = \infty$, and therefore inequality (4.20) still holds.

Given a closed interval $[a,b]$, the constant C in (4.20) for intervals $[x,y]$ must be smaller than or equal to the sum of the one for intervals $]a,b[$ and the ones for the sets $\{a\}$ and $\{b\}$.

Analogous to Definition 3.24, we may also state what is meant with essential bounds for the risk aversion.

Definition 4.4. [Essential bounds for risk aversion] A utility function $u(x)$ satisfying Assumptions 3.1 and 3.2 *has essentially a relative risk aversion bounded from above by* γ, $0 < \gamma < \infty$, on an interval $]a, b[\subset D$, if $]a, b[\subset D^u_{\text{ra}}$ and there exists a constant $0 < C < \infty$ such that

$$\int_{]x,y]} d\rho_u \leq \int_{]x,y]} \text{sgn}(x)\gamma d\ln|x| + C \tag{4.21}$$

holds for all intervals $]x, y] \subset]a, b[$. It has a relative risk aversion *essentially bounded from below by* γ if $]a, b[\subset D^u_{\text{ra}}$ and there exists a constant $0 < C < \infty$ such that

$$\int_{]x,y]} d\rho_u \geq \int_{]x,y]} \text{sgn}(x)\gamma d\ln|x| - C \tag{4.22}$$

for all intervals $]x, y] \subset]a, b[$. It has essentially a relative risk aversion of γ if both (4.21) and (4.22) are valid. Analogously, essential bounds for the absolute risk aversion are defined, with the measure $\text{sgn}(x)\gamma d\ln|x|$ replaced by the measure dx.

Remark 4.5. We have defined the essential bounds here by use of half-open intervals, because this will be more convenient for the proofs later. This is possible, by the Remarks 4.2 and 4.3. If we look at the interval $[a, b]$ instead of the open one, we would have to add, as in Remark 4.3, the requirement that $\rho_u(\{a\}) < \infty$ in the case of bounded from above.

Example 4.6. For $n \geq 0$ define

$$d\rho_u = -d\ln u'_r(x) = \begin{matrix} \ln 2 & \text{if} & x = \pm\frac{3}{4}2^{-n} \\ 0 & & \text{otherwise} \end{matrix}$$

where the plus and minus sign depends on the domain. Then we have that $\int_B(d\rho_u - \text{sgn}(x)\gamma d\ln|x|)$ is unbounded from above as well as from below with respect to all $B \in \mathcal{B}(D)$. But looking only at intervals, one can see that the function $u(x)$ has essentially a relative risk aversion of 1 in the sense of Definition 4.4. Looking only at intervals, the positive and negative parts of the 'measure' $-d\ln u'_r - \text{sgn}(x)\gamma d\ln|x|$ cancel out to a uniformly bounded number, even if this (signed) 'measure' maps some Borel sets to $+\infty$ and some others to $-\infty$ and does therefore not define a true signed measure.

Remark 4.7. A sufficient condition for u having a risk aversion essentially bounded from above is that there exists a Borel-measurable set $C \subset]a, b[$ such that

$$\int_{B'} d\rho_u \leq \int_{B'} \text{sgn}(x)\gamma d\ln|x| \tag{4.23}$$

(relative risk aversion case), respectively

$$\int_{B'} d\rho_u \leq \int_{B'} \gamma dx \tag{4.24}$$

(absolute risk aversion case) for each Borel set $B' \subset (]a, b[\setminus C)$, where

$$\int_C d\rho_u < \infty \tag{4.25}$$

Example 4.8. If u'_r has a finite amount of jumps in $a < x_1 < ... < x_n < b$ it satisfies assumption (4.25) if $u'_r(x_n) > 0$ and $u'_r(x_1^-) < \infty$. By the transformation of variable formula for finite variation processes equation (4.25) then gives for $C = \{x_1, ..., x_n\}$

$$\int_C d\rho_u = \int_C -d\ln u'_r(x) = \sum_i \ln\left(\frac{u'_r(x_i-)}{u'_r(x_i)}\right) < \infty$$

Example 4.9. The function $u(x) := -|x|^{\frac{1}{-x}}$ is concave in a region around 0, for both cases positive and negative wealth. But the relative risk aversion according to to Definition 4.4 is not essentially bounded from above on any interval $]-\epsilon, 0[$ nor $]0, \epsilon[$.

Remark 4.10. A sufficient condition for u having a risk aversion essentially bounded from below is that there exists a Borel set $C \subset]a, b[$ such that

$$\int_{B'} d\rho_u \geq \operatorname{sgn}(x)\gamma \int_{B'} d\ln|x| \tag{4.26}$$

(relative risk aversion case), respectively

$$\int_{B'} d\rho_u \geq \gamma \int_{B'} dx \tag{4.27}$$

(absolute risk aversion case) for each Borel set $B' \subset (]a, b[\backslash C)$, where

$$\int_C \operatorname{sgn}(x) d\ln|x| < \infty \tag{4.28}$$

for the relative risk aversion case, and $\mu(C) < \infty$ for the absolute risk aversion case, where μ is the Lebesgue measure.

In particular, together with Remark 4.7, the risk aversion is always essentially bounded if it is bounded.

Example 4.11. Condition (4.28) is satisfied for a finite union of closed intervals in D_{eff}: $C = [x_1, y_1] \cup ... \cup [x_n, y_n]$ with $a < x_1 < y_1 < x_2 < y_2 < ... < y_n < b$, and $D_{\text{eff}} =]a, b[$.

Example 4.12. The functions $u(x) = -x \ln x$ for $D =]0, \infty[$, and $u(x) := \frac{-x}{\ln|x|}$ for $D =]-\infty, 0[$ are concave if $|x|$ is sufficiently small. The relative risk aversion is not essentially bounded from below by a constant $\gamma > 0$ on any interval $]0, \epsilon[$ for positive wealth nor $]-\epsilon, 0[$. The functions are asymptotically linear.

4.2. Connection to power utility

We now reformulate Proposition 3.26 for the case of essentially bounded relative risk aversion. For the essentially bounded absolute risk aversion, a similar statement holds. We will omit here the proof of that case, and focus on relative risk aversion.

Proposition 4.13. *Let u be a utility function satisfying Assumptions 3.1 and 3.2. Then the relative risk aversion of u is essentially bounded from above (below) by a constant $0 < \gamma < \infty$ for $x \in]a, b[\subset D$, if and only if $]a, b[\subset D^u_{\text{eff}}$ and there exists a constant $K > 0$ such that for all $0 < x < y < b$, we have the inequality*

$$u'_r(y) \geq \begin{array}{ll} K u'_r(x) \left(\frac{y}{x}\right)^{-\gamma} & \text{if} \quad D =]0, \infty[\\ K u'_r(x) \left(\frac{y}{x}\right)^{\gamma} & \text{if} \quad D =]-\infty, 0[\end{array} \tag{4.29}$$

respectively inequality in the other direction if bounded from below.

Proof Let $u(x)$ satisfy equation (4.21) of Definition 4.4 on $]a, b[$. Let $a < x < y < b$. Because $]x, y] \subset]a, b[$ is an interval, we have

$$\int_{]x,y]} d\rho_u \leq \gamma \operatorname{sgn}(x) \int_{]x,y]} d\ln|x| + C$$

where C is the constant from equation (4.21). It follows by the rules of the logarithm that if $x, y \in D^u_{\text{eff}}$

$$-\ln u_r'(y) \leq -\ln u_r'(x) \pm \gamma \ln\left(\frac{y}{x}\right) + C$$

where the plus is for the case $D =]0, \infty[$ and the minus for the other case. By the monotonicity of the exponential function, equation (4.29) follows with the constant $K = \exp(-C)$. If there would be an $x \in]a, b[$ with $x \in (D^u_{\text{eff}})^c$, then there exists a half-open interval $]\xi_1, \xi_2] \subset]a, b[$ with $\rho_u(\{\xi\}) = \infty$ for all $\xi \in]\xi_1, \xi_2]$, by the fact that $]a, b[$ is open. Because $\operatorname{sgn}(x) d\ln|x|(\{]\xi_1, \xi_2]\})$ is finite, this contradicts definition (4.21) of the essential upper bound. It follows that $]a, b[\subset D^u_{\text{eff}}$.

On the other hand, let there be a constant $K > 0$ such that for all $a < x < y < b$, equation (4.29) is satisfied. Because $]a, b[\subset D^u_{\text{eff}}$, we can always take the logarithm. Then, by doing this, we have

$$\ln u_r'(y) - \ln u_r'(x) \geq \mp\gamma(\ln y - \ln x) + \ln K$$

It follows that

$$-\ln K \geq \int_{]x,y]} -d\ln u_r' - \gamma \int_{]x,y]} \operatorname{sgn}(x) d\ln|x| = \int_{]x,y]} d\rho_u - \gamma \int_{]x,y]} \operatorname{sgn}(x) d\ln|x|$$

Therefore, equation (4.21) is satisfied, with the constant $C = -\ln K$. We have only proved it for intervals $]x, y]$ with $a < x$, but the bound for intervals of the form $]a, y]$ is the same, by the monotone convergence of measures Williams (1991).

The statement for the case with boundedness from below is proved in the same way.

□

As for smooth utility functions (Corollary 2.4), one can see how Proposition 4.13 connects essential bounds of the relative risk aversion to an essential super-power property.

Corollary 4.14. *Let u be a utility function satisfying Assumptions 3.1 and 3.2 with relative risk aversion essentially bounded from above by a constant γ on $]a, b[$.*

1. *If $D =]0, \infty[$, $\gamma < 1$ and $a = 0$, and assume that $u(0^+) = 0$, then for all $0 < x < y < b$ the following inequality holds:*

$$u(y) \geq K u(x) \left(\frac{y}{x}\right)^{1-\gamma} \tag{4.30}$$

2. *If $D =]0, \infty[$, $\gamma > 1$ and $b = \infty$, and assume that $u(\infty^-) = 0$, then for all $a < x < y < \infty$ the following inequality holds:*

$$-u(y) \geq -u(x) K \left(\frac{y}{x}\right)^{1-\gamma} \tag{4.31}$$

3. *If $D =]-\infty, 0[$ and $b = 0$, and assume that $u(0^-) = 0$, then for all $a < x < y < 0$ the following inequality holds:*

$$-u(y) \geq -K u(x) \left(\frac{y}{x}\right)^{\gamma+1} \tag{4.32}$$

Proof Because $u'_r(x)$ is nonincreasing and $]a,b[\in D^u_{\text{eff}}$, it is continuous with exception of at most countably many points, which are a Lebesgue nullset. It follows that almost surely (with respect to the Lebesgue measure), the function u is differentiable and $u' = u'_r$. One can therefore apply the fundamental theorem of calculus even if u'_r is not continuous, that is

$$\int_{x_0}^{x} u'_r(\xi)d\xi = u(x) - u(x_0) \tag{4.33}$$

See for example Berberian (1999), Theorem 5.10.1.

Applying this fundamental theorem, the proofs are almost the same as the ones for smooth utility functions (Corollary 2.4).

□

Remark 4.15. For $\gamma < 1$, we have already discussed in section 2 under which conditions a normalization of u to $u(0) = 0$ is possible. For $\gamma > 1$, a normalization is possible if u is bounded from above, which is the case of the relative risk aversion being essentially bounded from below. Indeed, by equation (4.29), and with an $x_0 \in D^u_{\text{eff}}$, it follows that $u'_r(y) \leq Cy^{-\gamma}$ for a constant $C > 0$, and by the fundamental theorem, see Berberian (1999), this means $u(y) \leq u(x_0) + \frac{C}{\gamma-1}\left(x_0^{1-\gamma} - y^{1-\gamma}\right)$ for $y > x_0$.

For the case when $D =]-\infty, 0[$, a normalization to $u(0) = 0$ is always possible, which follows directly by the concavity of u.

If the utility function is invertible, one can from Corollary 4.14 also establish some relationships for the inverse function. This is the issue of the next proposition.

Proposition 4.16. *Let $]a,b[\subset D^u_{\text{eff}}$, then the function $u :]a,b[\to]u(a),u(b)[$ is invertible. Furthermore, let the assumptions of Corollary 4.14 be satisfied, in particular the inequalities (4.30) to (4.32) under the corresponding assumptions. Then the inverse function $u^{-1} :]u(a),u(b)[\to]a,b[$ satisfies for all $\xi < \eta \in]u(a),u(b)[$ the following inequalities:*

1. *Case 1:*
$$\frac{u^{-1}(\eta)}{u^{-1}(\xi)} \leq \frac{1}{K'}\left(\frac{\eta}{\xi}\right)^{\frac{1}{1-\gamma}} \tag{4.34}$$
2. *Case 2:*
$$\frac{u^{-1}(\eta)}{u^{-1}(\xi)} \geq \frac{1}{K'}\left(\frac{\eta}{\xi}\right)^{\frac{1}{1-\gamma}} \tag{4.35}$$
3. *Case 3:*
$$\frac{u^{-1}(\eta)}{u^{-1}(\xi)} \leq \frac{1}{K'}\left(\frac{\eta}{\xi}\right)^{\frac{1}{\gamma+1}} \tag{4.36}$$

Proof Because $]a,b[\subset D^u_{\text{eff}}$, the function u is continuous on $]a,b[$, and therefore surjective. Furthermore, $u'_r > 0$, from which it follows that u must be injective. The inequalities (4.34) to (4.36) then follow from equations (4.30) to (4.32) by setting $x = u^{-1}(\xi)$ and $y = u^{-1}(\eta)$, and by the assumptions about the normalization of u from this Corollary, $\xi, \eta > 0$ in case 1 and $\xi, \eta < 0$ in case 2 and 3.

□

Remark 4.17. For the case 3 in Proposition 4.16, but with risk loving investors and positive capital, we have from $\tilde{u}(x) = -u(-x)$ that $\tilde{u}^{-1}(\xi) = -u^{-1}(-\xi)$ is the inverse.

Proposition 4.18. *The relative risk aversion of a utility function u satisfying Assumptions 3.1, 3.2 and 3.3 is essentially bounded from below by a constant* γ *on a set* $]a,b[$ *if its conjugate function* $u^*(y)$ *has a relative risk aversion essentially bounded from above by the constant* $\frac{1}{\gamma}$ *on the set* $[u_r'(b), u_r'(a)]$ *for* $D =]0,\infty[$*, respectively* $[-u_r'(a), -u_r'(b)]$ *for* $D =]-\infty, 0[$*. Furthermore, the statement holds also if we exchange the words 'above' and 'below', then even with the intervals* $]u_r'(b), u_r'(a)[$ *and* $]-u_r'(a), -u_r'(b)[$*, respectively, provided these sets are nonempty.*

Proof Let $a < x < y < b$ be given and let firstly $x, y \in D^u_{\text{eff}}$. We define the interval

$$\operatorname{sgn}(x)]u_r'(y), u_r'(x)] := \begin{array}{ll}]u_r'(y), u_r'(x)] & \text{if} \quad D =]0,\infty[\\ ]-u_r'(x), -u_r'(y)] & \text{if} \quad D =]-\infty, 0[\end{array}$$

and analogous for closed or open intervals. We have

$$\int_{]x,y]} -d\ln u_r' = \ln u_r'(x) - \ln u_r'(y) = \int_{\operatorname{sgn}(x)]u_r'(y),u_r'(x)]} \operatorname{sgn}(x) d\ln|x|$$

and on the right-hand side we can take the closed interval instead, because the difference is only a Lebesgue-Nullset.

By the fact that u^* has a relative risk aversion essentially bounded from above by $\frac{1}{\gamma} > 0$, we have in particular that the interval considered is a subset of $D^{u^*}_{\text{eff}}$, and therefore $\rho_{u^*} = -d\ln(u^*)_r'$, and

$$\int_{\operatorname{sgn}(x)[u_r'(y),u_r'(x)]} \operatorname{sgn}(x) d\ln|x| + \gamma K \geq \gamma \left(\int_{\operatorname{sgn}(x)[u_r'(y),u_r'(x)]} -d\ln(u^*)_r' \right)$$

where K is the supremum in equation (4.21), and therefore independent of the choice of x and y. A similar argument as in the proof of Proposition 3.30 gives that

$$\begin{array}{rcl} \gamma K + \int_{]x,y]} -d\ln u_r' & \geq & \gamma \int_{\operatorname{sgn}(x)[u_r'(y),u_r'(x)]} -d\ln(u^*)_r' \geq \gamma \operatorname{sgn}(x)(\ln|y| - \ln|x|) \\ & = & \gamma \int_{]x,y]} \operatorname{sgn}(\xi) d\ln|\xi| \end{array}$$

from which equation (4.22) follows, because this holds uniformly for all intervals $]x,y]$.

For intervals which contain elements in $(D^u_{\text{eff}})^c$, equation (4.22) holds trivially, because by definition the measure ρ_u is infinite on this set, whereas the measure $d\ln|x|$ is finite. □

Remark 4.19. It may be that the closed intervals $[u_r'(b), u_r'(a)]$, respectively $[-u_r'(a), -u_r'(b)]$ are not subsets of D. This is the case if $u_r'(a) = \infty$ or $u_r'(b) = 0$. In this case, it is enough to assume that the relative risk aversion is essentially bounded from above on the intersection of this interval and D. Indeed, the requirement of being essentially bounded from above is only needed if $x, y \in D^u_{\text{eff}}$. If $u_r'(x) < u_r'(a)$ and $u_r'(y) > u_r'(b)$, the restriction of the assumption to D (that is to the open interval) is enough. But if $u_r'(y) = 0$ or $u_r'(x) = \infty$, x or y are not in D^u_{eff}, and therefore equation (4.22) holds trivially.

5. Conclusion

In this paper, we have generalized the notion of risk aversion to nonsmooth utility functions. For this, we have introduced the concept of risk aversion measures, from which the classical absolute as well as relative risk aversion, denoted here as risk aversion density, is calculated as Radon-Nikodym derivative provided it exists. However, the advantage of the risk aversion measure is that it can be defined for all, also nonsmooth, utility functions.

It turns out that the one-to-one relationship between equivalence classes of utility functions and risk aversion densities, a well-known result in the smooth case, can be extended to the nonsmooth case considering a suitable class of measures. The equivalence class of utility functions is defined in the classical way through positive monotone transformations, but without any assumption about differentiability or strict monotonicity.

Using this notion of risk aversion measures, we define an ordering for risk aversions of different investors, that is of different, nonsmooth, utility functions. For the case where the utility functions are smooth, this ordering coincides with the classical one for densities. Furthermore, we prove an extension of a classical result of Pratt for nonsmooth utility functions. The connection between u_1 is more risk averse than u_2 and the existence of a concave function T with $u_1(x) = T(u_2(x))$ does still hold for nonsmooth utility functions, if we express "more risk averse" in terms of risk aversion measures. Furthermore, we have shown that the notion of more risk averse in terms of risk aversion measures is also consistent with a reasonable alternative definition which does not necessarily make use of utility functions.

Typically, relative risk aversion makes sense for utility functions which are defined on the positive domain, that is for positive wealth. We did not try to extend this concept to utility functions which are defined for positive as well as for negative wealth. However, we treated, alternatively to the typical case, also the one where the wealth is always negative. We could give a unifying definition for both cases. However, for some specific proofs, we had to treat both cases separately. For absolute risk aversion, one can take the whole real line as domain, and the restriction mentioned above does not apply.

We have proposed a weaker ordering which we called essential bounds for the risk aversion, and which requires only that the risk aversion is bounded up to a certain tolerance. We have formalized this informal statement, and have shown that a strict bound is always an essential bound. We have given examples where there is no essential bound, as well as where even a piecewise linear function has essentially constant relative risk aversion.

Like the constant relative risk aversion property is equivalent to a power property of the utility function, we have shown that a bound of the relative risk aversion is equivalent to a super-power property of the function. The same holds for absolute risk aversion and exponential property. We have also shown that this equivalence continues to hold for nonsmooth utility functions using our definition, and holds not only for strict but also for essential bounds of the risk aversion, when the super-power property is relaxed appropriately.

Finally, it has been shown how the relative risk aversion translates into the one of the concave conjugate function, for strict bounds as well as also for essential bounds.

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