

ON THE APPLICABILITY OF THE WANG TRANSFORM FOR PRICING FINANCIAL RISKS

BY

ANTOON PELSSER

ABSTRACT

In an arbitrage-free economy, it is well-known that financial risks can be priced using equivalent martingale measures. We establish in this paper that, for general stochastic processes, the Wang Transform does not lead to a price which is consistent with the arbitrage-free price. Based on these results we must conclude that the Wang Transform cannot be a universal framework for pricing financial and insurance risks.

1. INTRODUCTION

In two recent papers, Wang (2000) and Wang (2002), Shaun S. Wang postulates a general framework for pricing financial and insurance risks. This framework is based on a distortion operator which acts on the probability distribution of the risk one seeks to price. This distortion operator is now known in the actuarial literature as the “Wang Transform”.

In an arbitrage-free economy, it is well-known that financial risks can be priced using equivalent martingale measures. In this paper we investigate whether the Wang Transform is consistent with the arbitrage-free pricing approach for pricing financial risks. Wang shows that the Wang Transform is consistent with arbitrage-free pricing in the setting of Geometric Brownian Motion with constant coefficients. However, this result does not hold in general. We establish in this paper that, for general stochastic processes, the Wang Transform does not lead to a price which is consistent with the arbitrage-free price. Based on these results we must conclude that the Wang Transform cannot be a universal framework for pricing financial and insurance risks.

A related result has also been reported in Ruhm (2003). Using a different probabilistic setting, his paper proves the result that distribution-based pricing formulas for European-style payoffs cannot consistently produce arbitrage-free prices. In this paper we consider the setting of continuous-time diffusion processes to derive our result.

JEL classification code: G13.

The remainder of this paper is organised as follows. In Section 2 we give a very brief introduction to stochastic processes, stochastic calculus and the related partial differential equations. In Section 3 we provide a brief introduction to arbitrage-free pricing. Armed with this knowledge we then proceed in Section 4 with our main result: the analysis of the Wang Transform. First we analyse which change of probability measure is actually implied by the Wang Transform. Then we analyse under which conditions the "Wang change of measure" is consistent with the change of measure required for arbitrage-free pricing. A simple example where the Wang Transform is not consistent with arbitrage-free pricing is provided in Section 5.

2. STOCHASTIC PROCESSES

In this section we provide a very brief introduction in stochastic processes and stochastic calculus. We will only review those results needed for the proof of our claim in Section 4. All theorems in this section are provided without proofs. For a more formal introduction to this subject we refer to Karatzas & Shreve (1991).

We begin with the premise that the evolution of traded assets' prices in an economy are described by stochastic differential equations. Suppose we have a traded asset X_t in the economy whose price follows the stochastic differential equation

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \quad (1)$$

where W_t denotes Brownian Motion. The dW_t -term can be interpreted as a "noise term" with the following properties: $\mathbb{E}[dW_t] = 0$ and $\mathbb{E}[(dW_t)^2] = dt$. Please note that a realisation of a path W_t of a Brownian Motion is a continuous function in time that is *nowhere* differentiable. As a consequence, the normal rules of integration and differentiation cannot be applied to Brownian Motion.

As X_t is the price of a traded asset, a financial derivative (or derivative) is an asset in the economy whose value at some time point $T > t$ is given by a function $f(X_T)$. Hence, the value $f(X_T)$ of the derivative at time T is derived from the value of the underlying asset X_T . The expected value at time t of the payoff $f(X_T)$ is given by the function $u(t, x) = \mathbb{E}[f(X_T) | X_t = x]$. The function $u(t, x)$ satisfies a partial differential equation:

Kolmogorov's Backward Equation. *Let the stochastic differential equation for the process X_t be given by (1). Define $u(t, x) = \mathbb{E}[f(X_T) | X_t = x]$. Then $u(t, x)$ is a solution to the partial differential equation*

$$\frac{\partial}{\partial t} u(t, x) + \mu(t, x) \frac{\partial}{\partial x} u(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2}{\partial x^2} u(t, x) = 0 \quad (2)$$

for $t < T$ and boundary condition $u(T, x) = f(x)$.

A special case of a derivative is the indicator variable $\mathbf{1}_{(X_T \leq y)}$. By taking an expectation of this indicator function we find the distribution function $F(t, x; T, y)$ of the process X_t :

$$F(t, x; T, y) = \Pr[X_T \leq y \mid X_t = x] = \mathbb{E}[\mathbf{1}_{(X_T \leq y)} \mid X_t = x].$$

Hence, the distribution function also satisfies Kolmogorov’s Backward Equation (2) with boundary condition $F(T, x; T, y) = \mathbf{1}_{(x \leq y)}$.

Let the stochastic process Y_t be defined as a function $Y_t = g(t, X_t)$ of the process X_t for all t . Then the stochastic differential equation for Y_t is given by:

Ito’s Lemma. *Let the stochastic differential equation for the process X_t be given by (1). Let the function $g(t, x)$ be twice continuously differentiable. Define $Y_t = g(t, X_t)$, then the stochastic differential equation for Y_t is given by*

$$dY_t = \left[\frac{\partial}{\partial t} g(t, X_t) + \mu(t, X_t) \frac{\partial}{\partial x} g(t, X_t) + \frac{1}{2} \sigma^2(t, X_t) \frac{\partial^2}{\partial x^2} g(t, X_t) \right] dt + \sigma(t, X_t) \frac{\partial}{\partial x} g(t, X_t) dW_t. \tag{3}$$

Another concept we need to introduce is a *martingale*. A martingale is a stochastic process that stays on average at its current level. If we would track the gains and losses from a fair game over time, then we expect on average not to gain or loose money systematically. This is an example of a martingale. In mathematical terms we define:

Martingale. A stochastic process is a martingale if and only if

$$\mathbb{E}[X_T \mid X_t = x] = x \quad \forall T \geq t.$$

A stochastic differential equation with a dt -term equal to zero describes a martingale process¹.

Because the Wang Transform is basically a method for changing the probability distribution of a stochastic process, we want to conclude this section with Girsanov’s Theorem. This is a fundamental result in stochastic analysis. The Girsanov Theorem states that if we change the probability distribution of a stochastic process, we only change the dt -coefficient of the stochastic differential equation. Hence, our mathematical analysis of the Wang Transform will be built on the implied Girsanov transformation of the stochastic process.

¹ This statement is not entirely correct as we are ignoring the difference between “local martingales” and “true martingales”, which is a mathematical technicality. In the remainder of this paper we are implicitly assuming that all martingales we encounter are “true martingales”.

Girsanov's Theorem. For any stochastic process K_t , such that $\int_0^t K_s^2 ds < \infty$ with probability one, consider the stochastic process R_t given by

$$R_t = \exp \left\{ \int_0^t K_s dW_s - \frac{1}{2} \int_0^t K_s^2 ds \right\}, \quad (4)$$

where W_t is Brownian Motion under the probability measure \mathbb{Q} . Define the probability measure \mathbb{Q}^* as $d\mathbb{Q}^* = R_t d\mathbb{Q}$. Under the probability measure \mathbb{Q}^* the process

$$W_t^* = W_t - \int_0^t K_s ds \quad (5)$$

is also a Brownian Motion.

The stochastic process R_t is called the Radon-Nikodym derivative and is often denoted by $R_t = \frac{d\mathbb{Q}^*}{d\mathbb{Q}}$. With the help of Ito's Lemma it is straightforward to show that the Radon-Nikodym derivative R_t is a martingale under the probability measure \mathbb{Q} with stochastic differential equation $dR_t = K_t R_t dW_t$ with a dt -term is equal to 0. The stochastic process K_t is often called the Girsanov kernel or Girsanov exponent.

To analyse the impact of a change of probability measure on a stochastic process X_t we proceed as follows. A change in probability measure implies a Radon-Nikodym derivative R_t . If we apply Ito's Lemma to the stochastic process R_t we obtain its stochastic differential equation and we can infer the Girsanov kernel K_t . The stochastic differential equation for X_t under the new probability measure \mathbb{Q}^* can now be obtained by substituting $dW_t = dW_t^* + K_t dt$ (which is equation (5) rewritten in differential form). Let us consider an example: suppose the stochastic differential equation for X_t is given by (1) under the probability measure \mathbb{Q} . Under the new probability measure \mathbb{Q}^* , the stochastic differential equation for X_t is equal to

$$\begin{aligned} dX_t &= \mu(t, X_t) dt + \sigma(t, X_t) (dW_t^* + K_t dt) \\ &= (\mu(t, X_t) + \sigma(t, X_t) K_t) dt + \sigma(t, X_t) dW_t^* \end{aligned} \quad (6)$$

and we see indeed that the change in probability measure only affects the dt -term of the process X_t .

We can now use the Kolmogorov Backward Equation (2) to find the partial differential equation that describes the probability distribution of X_t under the new probability measure \mathbb{Q}^* . This is used in Section 4 to find which change of probability measure is actually implied by the Wang Transform.

3. ARBITRAGE-FREE PRICING

Again, in this section we provide a very brief introduction into arbitrage-free pricing. All theorems in this section are provided without proofs. For a more formal introduction in this subject we refer to Björk (1998).

To calculate the prices of financial derivatives, we can use a very elegant mathematical theory which is based on the elements of stochastic analysis outlined in the previous section.

Suppose we have a traded asset X_t in the economy whose price follows the stochastic differential equation (1). Suppose we also have an asset N_t , which is an asset with a strictly positive price. We can choose the asset N_t as a *numeraire*, i.e. as a unit of measurement. In the remainder of the paper, we will express the prices of all traded assets in terms of the numeraire. So, X_t is the price of the traded asset expressed in units of N_t . The euro-value is therefore equal to $X_t N_t$.

The first property we want to impose on our economy is a no-arbitrage property. An *arbitrage opportunity* is a trading strategy which costs no money at time t and has a non-negative value at time $T > t$ and a positive probability of a strictly positive value. Such an arbitrage opportunity would be a "money-machine" which is clearly something we would like to avoid in modeling the economy. An economy that admits no arbitrage opportunities is called *arbitrage-free*.

The second property we want to impose on an economy is the completeness property. In the previous section we defined a financial derivative as a traded asset which has a value of $f(X_T)$ at time T . Given the price processes of the two traded assets X_t and N_t , we can ask the question whether it is possible to find a trading strategy using only X_t and N_t that replicates the value of the derivative $f(X_T)$ at time T . The economy is called *complete* if for any derivative such a replicating trading strategy exists.

If a replicating trading strategy can be found for any derivative $f(X_T)$ then, in order to avoid arbitrage opportunities, the price at time $t < T$ for the derivative is given by the value of the replicating trading strategy at time t . This is known as *arbitrage-free pricing*.

The question whether an economy is arbitrage-free and complete can be determined with the help of the following theorem:

Unique Martingale Measure. *An economy is arbitrage-free and complete if and only if there exists a unique probability measure such that the prices of all traded assets divided by a numeraire are martingales.*

Let us investigate the economy with X_t and N_t we have defined above. To establish that our economy is arbitrage-free and complete we must find a unique probability measure such that the process X_t is a martingale. Suppose the process X_t follows (1). If we apply a Girsanov transformation to the process X_t with Girsanov kernel K_t , then the process follows (6) under the new measure \mathbb{Q}^* .

There is only one possible choice for the Girsanov kernel that nullifies the dt -coefficient and turns the process X_t into a martingale:

$$K_t = -\frac{\mu(t, X_t)}{\sigma(t, X_t)} \quad (7)$$

and this expression is known as the *market price of risk*. This establishes that our economy is arbitrage-free and complete if and only if $\sigma(t, X_t) \neq 0 \forall t$.

Now that we have established that our economy is arbitrage-free and complete, we also know that the prices divided by N_t for all traded assets will be martingales under the probability measure \mathbb{Q}^* . In particular, the price of a financial derivative divided by the numeraire $f(X_T)/N_T$ is also martingale under \mathbb{Q}^* . Let us denote this price process by f_t/N_t for $t < T$. From the martingale definition we obtain that the price f_t/N_t at time $t < T$ is given by

$$\frac{f_t}{N_t} = \mathbb{E}^* \left[\frac{f(X_T)}{N_T} \mid X_t \right] \quad (8)$$

where $\mathbb{E}^*[\cdot \mid X_t]$ denotes an expectation under the measure \mathbb{Q}^* conditional on observing the price of the asset X_t . This expectation can be evaluated using the probability distribution $F^*(t, x; T, y)$ for the process X_t under the measure \mathbb{Q}^* .

4. WANG TRANSFORM

In two recent papers, Wang (2000) and Wang (2002), an alternative framework for pricing financial risks is postulated. This framework is based on a distortion operator which acts directly on the probability distribution of the asset one seeks to price.

For the economy we have defined in the previous section, the distortion operator is defined as follows. Given the probability distribution $F(t, x; T, y)$ for the traded asset X_t , the Wang Transform constructs a new distribution $F^W(t, x; T, y)$ via the distortion operator

$$F^W(t, x; T, y) = \Phi(\Phi^{-1}(F(t, x; T, y)) - \lambda(t, T)) \quad (9)$$

where $\lambda(t, T)$ is a deterministic function. The function $\Phi(\cdot)$ denotes the standard normal distribution function.

The function $\lambda(t, T)$ is determined by adjusting the distribution $F^W(t, x; T, y)$ in such a way that the martingale condition $\mathbb{E}^*[X_T \mid X_t = x] = x$ is fitted for all T . Hence, $\lambda(t, T)$ is obtained by solving (numerically) the equation:

$$\begin{aligned} \mathbb{E}^W[X_T \mid X_t = x] &= \int_{-\infty}^{\infty} X_T dF^W(t, x; T, X_T) = \\ & \int_{-\infty}^{\infty} X_T d\Phi(\Phi^{-1}(F(t, x; T, X_T)) - \lambda(t, T)) = \mathbb{E}^*[X_T \mid X_t = x] = x \quad \forall T > t \end{aligned} \quad (10)$$

As $F^W(\cdot)$ is a monotonic function in $\lambda(t, T)$ this equation will always have a unique solution for all $T > t$ with initial condition $\lambda(t, t) = 0$.

Before we proceed, let us briefly introduce the line of reasoning we will use. First we analyse which change of probability measure (i.e. which Girsanov

Transformation) is actually implied by the Wang Transform. Then we analyse under which conditions the “Wang change of measure” is consistent with the change of measure required for arbitrage-free pricing.

To investigate which Girsanov Transformation is implied by the Wang Transform we have to deduce which stochastic differential equation gives rise to the distribution $F^W(\cdot)$. We know that a Girsanov Transformation only affects the dt -coefficient of the stochastic process. Hence, using (6), the Girsanov Transformation implied by the Wang Transform will lead to the following stochastic differential equation for the process X_t :

$$dX_t = (\mu(t, X_t) + \sigma(t, X_t)K_t^W)dt + \sigma(t, X_t)dW_t^W \tag{11}$$

where W_t^W denotes Brownian Motion under the “Wang probability measure” \mathbb{Q}^W and K_t^W denotes the Girsanov kernel associated with the Wang Transform.

We also know that the distribution function $F^W(t, x; T, y)$ satisfies Kolmogorov’s Backward Equation (2):

$$\begin{aligned} \frac{\partial}{\partial t} F^W(t, x; T, y) + (\mu(t, x) + \sigma(t, x)K_t^W) \frac{\partial}{\partial x} F^W(t, x; T, y) \\ + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2}{\partial x^2} F^W(t, x; T, y) = 0. \end{aligned} \tag{12}$$

Hence, we obtain the following expression for $\mu(t, x) + \sigma(t, x) K_t^W$:

$$(\mu(t, x) + \sigma(t, x) K_t^W) = \frac{\frac{\partial}{\partial t} F^W(t, x; T, y) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2}{\partial x^2} F^W(t, x; T, y)}{-\frac{\partial}{\partial x} F^W(t, x; T, y)}. \tag{13}$$

Given the expression (9) for the Wang Transform, we obtain the following expressions for the partial derivatives (note that we will suppress the arguments of $F(t, x; T, y)$ and $\lambda(t, T)$ hereafter to lighten the notation):

$$\begin{aligned} \frac{\partial}{\partial t} F^W(t, x; T, y) &= \frac{\phi(\Phi^{-1}(F) - \lambda)}{\phi(\Phi^{-1}(F))} \left(\frac{\partial}{\partial t} F - \phi(\Phi^{-1}(F)) \frac{\partial}{\partial t} \lambda \right) \\ \frac{\partial}{\partial x} F^W(t, x; T, y) &= \frac{\phi(\Phi^{-1}(F) - \lambda)}{\phi(\Phi^{-1}(F))} \frac{\partial}{\partial x} F \\ \frac{\partial^2}{\partial x^2} F^W(t, x; T, y) &= \frac{\phi(\Phi^{-1}(F) - \lambda)}{\phi(\Phi^{-1}(F))} \left(\frac{\lambda}{\phi(\Phi^{-1}(F))} \left(\frac{\partial}{\partial x} F \right)^2 + \frac{\partial^2}{\partial x^2} F \right). \end{aligned}$$

Substituting these expressions into (13) and simplifying yields:

$$\sigma(t, x) K_t^W = \left[\left(\frac{\phi(\Phi^{-1}(F))}{\frac{\partial}{\partial x} F} \right) \frac{\partial}{\partial t} \lambda - \frac{1}{2} \sigma^2(t, x) \left(\frac{\frac{\partial}{\partial x} F}{\phi(\Phi^{-1}(F))} \right) \lambda \right]. \quad (14)$$

If we want to establish under which conditions the Wang Transform is consistent with arbitrage-free pricing, we can substitute the market price of risk (7) for K_t^W and we obtain $\sigma(t, x) K_t^W = -\mu(t, x)$ and we can rewrite (14) as a differential equation in $\lambda(t, T)$:

$$\frac{\partial}{\partial t} \lambda(t, T) = - \left[\mu(t, x) \frac{\frac{\partial}{\partial x} F}{\phi(\Phi^{-1}(F))} \right] + \frac{1}{2} \left[\sigma(t, x) \frac{\frac{\partial}{\partial x} F}{\phi(\Phi^{-1}(F))} \right]^2 \lambda(t, T). \quad (15)$$

This differential equation can be solved by $\lambda(t, T)$ if and only if the coefficients in the equation are functions of time only. We summarise this result in the following corollary:

Validity of Wang Transform. *The Wang Transform is consistent with arbitrage-free pricing if and only if the following conditions on $\mu(t, x)$ and $\sigma(t, x)$ are satisfied*

$$\frac{\partial}{\partial x} \left[\mu(t, x) \frac{\frac{\partial}{\partial x} F(t, x; T, y)}{\phi(\Phi^{-1}(F(t, x; T, y)))} \right] = 0 \quad (16)$$

and

$$\frac{\partial}{\partial x} \left[\sigma(t, x) \frac{\frac{\partial}{\partial x} F(t, x; T, y)}{\phi(\Phi^{-1}(F(t, x; T, y)))} \right] = 0. \quad (17)$$

Note that these conditions are very restrictive. For example, a necessary condition for the coefficients in (15) to be functions of time only, is that the ratio $\mu(t, x) / \sigma(t, x)$ is a function of time only.

We must therefore conclude that the Wang Transform is, in general, not consistent with arbitrage-free pricing and can therefore not constitute a universal framework for pricing financial and insurance risks.

5. COUNTER-EXAMPLE FOR WANG TRANSFORM

Suppose we have an economy where the stock price process S_t follows the stochastic differential equation:

$$dS_t = (\mu + a(\mu t - \ln S_t)) S_t dt + \sigma S_t dW_t \quad (18)$$

where μ , a and σ are constants. Such a specification is consistent with a model for stock prices where the return on stock prices is mean reverting towards the level μ . The speed of mean reversion of the returns is governed by the parameter a . Note that for $a = 0$ the model is equal to the standard Black-Scholes model. Furthermore, let the bond price B_t be given by

$$dB_t = rB_t dt \tag{19}$$

where the rate of interest r is a constant. The bond has a strictly positive price and can be used as a numeraire. Hence, if we set $X_t = S_t / B_t$, then we have

$$dX_t = (\mu - r + a((\mu - r)t - \ln X_t)) X_t dt + \sigma X_t dW_t. \tag{20}$$

Note that this specification for the X_t process does not satisfy the necessary condition for validity of the Wang Transform.

The distribution function $F(t, x; T, y)$ is a solution of Kolmogorov's Backward Equation (2). If we consider the logarithm of the process X_t defined here, the distribution function of $\ln X_t$ can be expressed in closed form as

$$F(t, x; T, y) = \Phi \left(\frac{y - m(t, x; T)}{\frac{\sigma^2}{2a} (1 - e^{-2a(T-t)})} \right) \tag{21}$$

$$m(t, x; T) = e^{-a(T-t)} x + \int_t^T e^{-a(T-s)} \left(\mu - r - \frac{1}{2} \sigma^2 + a(\mu - r) s \right) ds$$

and we see that $\ln X_T$ has a normal distribution with mean $m(t, x; T)$ and variance equal to $\frac{\sigma^2}{2a} (1 - e^{-2a(T-t)})$. Note that for $T \rightarrow \infty$ the variance is bounded by $\sigma^2 / 2a$.

The economy we have defined here is arbitrage-free and complete if and only if there is a unique probability measure \mathbb{Q}^* such that the process X_t is a martingale. It is easy to see that the process X_t is a martingale only for the probability measure \mathbb{Q}^* defined by the Girsanov kernel

$$K_t = - \frac{\mu - r + a((\mu - r)t - \ln X_t)}{\sigma}. \tag{22}$$

Under the martingale measure \mathbb{Q}^* the price process X_t follows the stochastic differential equation

$$dX_t = \sigma X_t dW_t^*. \tag{23}$$

Under the measure \mathbb{Q}^* the process $\ln X_t$ therefore has a distribution function given by

$$F^*(t, x; T, y) = \Phi \left(\frac{y - x}{\sigma^2(T - t)} \right). \quad (24)$$

In particular we see that the variance of $\ln X_T$ under the arbitrage-free pricing measure \mathbb{Q}^* is given by $\sigma^2(T - t)$ which is not bounded for $T \rightarrow \infty$.

If we apply the Wang Transform we have to solve $\lambda(t, T)$ from equation (10). For the distribution function(21) we find the following solution for $\lambda(t, T)$:

$$\lambda(t, T) = \frac{x - m(t, x; T)}{\frac{\sigma^2}{2a} (1 - e^{-2a(T-t)})} \quad (25)$$

and we obtain the following distribution function for the process X_t under the Wang Transform

$$F^W(t, x; T, y) = \Phi \left(\frac{y - x}{\frac{\sigma^2}{2a} (1 - e^{-2a(T-t)})} \right). \quad (26)$$

Note that under the Wang Transform, the variance of the process X_t is different than the correct variance under the arbitrage-free pricing measure which may lead to considerable mispricing of financial risks based on the Wang Transform. This concludes our counter-example.

6. ACKNOWLEDGEMENTS

The author would like to thank seminar participants at the University of Bonn, the ICMS Workshop on the Interface between Quantitative Finance and Insurance, David Ruhm and an anonymous referee for comments and helpful suggestions. All remaining errors are the responsibility of the author.

REFERENCES

- BJÖRK, T. (1998) *Arbitrage Theory in Continuous Time*, Oxford University Press, Oxford.
- KARATZAS, I. and S.E. SHREVE (1991) *Brownian Motion and Stochastic Calculus*, Second Edition, Springer Verlag, New York.
- RUHM, D. (2003) "Distribution-Based Pricing Formulas Are Not Arbitrage-Free", *Casualty Actuarial Society Proceedings*, XC, 97-129.
- WANG, S. (2000) "A Class of Distortion Operators for Pricing Financial and Insurance Risks", *Journal of Risk and Insurance*, 67(1), 15-36.

WANG, S. (2002) "A Universal Framework for Pricing Financial and Insurance Risk", *ASTIN Bulletin*, 32(2), 213-234.

ANTOON PELSSER

University of Amsterdam and NETSPAR

Dept. of Quantitative Economics

Roetersstraat 11

1018 WB Amsterdam

The Netherlands

Tel.: (31)20-525 4210

Fax: (31)20-525 4349

E-Mail: a.a.j.pelsser@uva.nl

Home-page: <http://www.ase.uva.nl/pp/apelsser>