Closed-form Approximations to Optimal Investment Policies in Markets with Frictions

Thijs Kamma[1,2] Antoon Pelsser[1,2]

[1]Maastricht University
Dept. of Quantitative Economics
Maastricht, The Netherlands

[2]NETSPAR
Tilburg University
Tilburg, The Netherlands

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Introduction

- For investment problems in markets with frictions
  - Unable to employ standard machinery (martingale methods)
  - Instead: solve auxiliary problem including so-called shadow price
  - Difficulty consists in finding expressions for shadow price
  - So far, solutions only available for standard preferences (e.g. CRRA) in simple market models (e.g. Black-Scholes)
- Auxiliary problem: suggests "structure" of optimal solution
  - Namely: relaxes frictions
  - Analytical expression hence available
- Our paper develops a method to approximate – analytically – the optimal trading policies

Financial Market Model

- Model is based on Davis and Norman (1990) and Cvitanić and Karatzas (1996)
- Financial market model: fix probability space $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ for horizon $T > 0$, and let $\{W_t\}_{t\in[0,T]}$ be a scalar-valued standard Brownian motion. Define risk-free and risky asset:

$$\frac{\mathrm{d}B_t}{B_t} = r_t \mathrm{d}t, \quad \text{and} \quad \frac{\mathrm{d}S_t}{S_t} = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \tag{1}$$

  - Interest rate: r_t; expected return and volatility: μ_t and σ_t
  - Bank account and stock processes
  - Prop. transaction costs: $\lambda_1 \in [0, 1)$ and $\lambda_2 > 0$ for, resp. selling and buying. Bid and ask price: $S_t^B = (1 - \lambda_1) S_t$ and $S_t^A = (1 + \lambda_2) S_t$

Problem Specification

- Let $f(x) = (1-\lambda_1)x\mathbb{1}_{\{x>0\}} + (1+\lambda_2)x\mathbb{1}_{\{x\leq 0\}}$. The investor maximises utility from $X_T + f(Y_T)$, in relation to a predetermined stochastic benchmark Π_T by opting for $\{p_t, q_t\}_{t\in[0,T]}$:

$$
\begin{aligned}
\sup_{\{p_t,q_t\}_{t\in[0,T]}\in\mathcal{A}_{X_0,Y_0}} \quad & \mathbb{E}\left[U\left(X_T + f(Y_T), \Pi_T\right)\right] \\
\text{s.t.} \quad & \mathrm{d}X_t = S_t^B \mathrm{d}q_t - S_t^A \mathrm{d}p_t + r_t X_t \mathrm{d}t, \\
& \mathrm{d}Y_t = S_t\left[\mathrm{d}p_t - \mathrm{d}q_t\right] + \left[p_t - q_t\right]\mathrm{d}S_t
\end{aligned}
\tag{2}
$$

  - Cash-holdings and stock-holdings: X_t and Y_t
  - Cumulative sales and purchases of stock: p_t and q_t
  - Admissibility set, $\mathcal{A}_{X_0,Y_0}$: all $\{p_t, q_t\}_{t\in[0,T]}$ s.t. $X_t + f(Y_t) \geq 0$

- Generally **not** possible to solve (2) for $\{p_t\}_{t\in[0,T]}$ and $\{q_t\}_{t\in[0,T]}$

Convex Duality

- Resort to application of convex duality to problem at hand, cf. Klein and Rogers (2007) and Rogers (2003, 2013)

- Introduce two semi-martingale processes $\left\{Z_t^0\right\}_{t\in[0,T]}$ and $\left\{Z_t^1\right\}_{t\in[0,T]}$ such that $\mathrm{d}Z_t^0 = Z_t^0\left[\alpha_t \mathrm{d}t + \beta_t \mathrm{d}W_t\right]$ and $\mathrm{d}Z_t^1 = Z_t^1\left[\gamma_t \mathrm{d}t + \delta_t \mathrm{d}W_t\right]$. Then, the dual problem lives for $\widehat{X}_0 = X_0 + (1-\lambda_1)\, Y_0$ by:

$$\inf_{Z_T^0 \in \mathcal{Z}(Z_0)} \mathbb{E}\left[V\left(Z_T^0, \Pi_T\right)\right] + \widehat{X}_0 Z_0^0, \tag{3}$$

  - Duality set, $\mathcal{Z}(Z_0)$, contains all Z_T^0 such that:
  - $B_t Z_t^0 = Z_0^0 \frac{\mathrm{d}\mathbb{Q}_\lambda}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \mathbb{E}\left[Z_T^0 \mid \mathcal{F}_t\right]$
  - and $S_t^B \le \frac{Z_t^1 S_t}{Z_t^0} \le S_t^A$ (which must bind at $t = T$) hold for $\mathbb{Q}_\lambda \sim \mathbb{P}$
  - and Z_t^1 that satisfies $\mathbb{E}\left[Z_T^1 S_T \mid \mathcal{F}_t\right] = Z_t^1 S_t \;\forall\; t \in [0, T]$

Shadow Price Process

- Based on the foregoing dual problem specification:
  - We may formulate a frictionless auxiliary market and a corresponding unconstrained investment problem
  - This problem involves so-called **shadow price process**, which serves as "fictitious" risky asset
  - Fact that finite-horizon investor is unconstrained in alternative environment facilitates recovery of closed-form expressions
- Shadow price process lives by $\tilde{S}_t = \frac{Z_t^1}{Z_t^0} S_t$
  - Must attain values in set $\mathcal{S}$, containing all $\left\{\tilde{S}_t\right\}_{t\in[0,T]}$, such that
  - $S_t^B \leq \tilde{S}_t \leq S_t^A \; \forall \; t \in [0, T]$ and $\tilde{S}_t Z_t^0$ is a $\mathbb{P}$-martingale
- Each feasible $\tilde{S}_t$ spawns **upper bound** on optimal value function

Auxiliary Problem

- Unconstrained investment problem in auxiliary market that excludes transaction costs is for any $\left\{\tilde{S}_t\right\}_{t\in[0,T]} \in \mathcal{S}$ given by:

$$\begin{aligned} \sup_{\{\pi_t\}_{t\in[0,T]}\in\widehat{\mathcal{A}}_{X_0,Y_0}} \quad & \mathbb{E}\left[U\left(H_T, \Pi_T\right)\right] \\ \text{s.t.} \quad & \mathrm{d}H_t = H_t\Big[\left(1-\pi_t\right) {B_t}^{-1}\mathrm{d}B_t + \pi_t \tilde{S}_t^{-1}\mathrm{d}\tilde{S}_t\Big], \end{aligned} \tag{4}$$

  - Problem (4) allows for recovery of controls in closed-form:
  - Analytics available for $\{H_t^{\mathrm{opt}}\}_{t\in[0,T]}$ and $\{\pi_t^{\mathrm{opt}}\}_{t\in[0,T]}$
  - However, shadow price, $\{\tilde{S}_t\}_{t\in[0,T]}$, is not yet analytically specified
  - Identification of primal-controls via π_t^{opt} and $\tilde{S}_t^{\mathrm{opt}}$

- **Difficult** to find closed-form $\{\tilde{S}_t\}_{t\in[0,T]}$ that minimises utility in (4) $\rightarrow$ closed-form expressions for $p_t^{\mathrm{opt}}, q_t^{\mathrm{opt}}$ hence not available

Rationale Approximate Method

- Auxiliary problem demonstrates few crucial points:
  - Controls for optimal transactions are available in closed-form, though **up to** shadow price process
  - Thus: structure of controls must certainly abide by the one that is spelled out by solutions to auxiliary problem
  - Tractable **approximation** to shadow price process would respect this "structure", and clarify underlying dynamics
- Suppose: we choose analytical approximation to shadow price
  - Continuity of approximation causes infeasible controls in original market
  - **Discretise** ensuing analytical expressions
  - Comparison of resulting **bounds** on optimal value function, enables us to quantify accuracy
- Threefold procedure: (i) approximate shadow price, (ii) discretise artificial controls, (iii) compare primal-dual bounds

Technicalities Approximate Method

- Let $\widehat{\mathcal{S}} \subseteq \mathcal{S}$, containing approximate shadow prices. Define $\{\tau_i\}_{i=1}^{M}$ as M state-dependent discrete time-points, and $\theta_{\tau_i} = \pi^*_{\tau_i} \tilde{S}^{-1}_{\tau_i} H^*_{\tau_i}$, where asterisk indicates it establishes an approximation. Approximate method works in step-wise manner as follows:

1. Approximate shadow price process as:
   $\{\tilde{S}_t\}_{t\in[0,T]} = \arg\inf_{\{\tilde{S}_t\}_{t\in[0,T]}\in\widehat{\mathcal{S}}} \sup_{\{\pi_t\}_{t\in[0,T]}\in\widehat{\mathcal{A}}_{X_0,Y_0}} \mathbb{E}\left[U\left(H_T, \Pi_T\right)\right]$
2. Step 1 implies artificial-sub-optimal controls that we project into the primal-admissibility region as follows:
   $p^*_t = \left(\theta_{\tau_i} - \theta_{\tau_{i-1}}\right)^+ + p^*_{\tau_{i-1}}, q^*_t = q^*_{\tau_{i-1}} - \left(\theta_{\tau_i} - \theta_{\tau_{i-1}}\right)^-$
3. We compare the discrepancy between the lower and upper bounds inherent in, resp, steps 1 and 2, by means of "compensating variation":
   $J_P\left(\widehat{X}_0 + \mathcal{CV}\right) = \inf_{\{\tilde{S}_t\}_{t\in[0,T]}\in\widehat{\mathcal{S}}} J_D\left(\widehat{X}_0, \{\tilde{S}_t\}_{t\in[0,T]}\right)$

Remarks Approximate Method

- Closed-form nature of approximate controls:
  - Consists in controls embedded in recursive relationships for p_t^*, q_t^*
  - These are available in **closed-form**: no numerical effort required

- Admissibility requirements can be troublesome:
  - If investor is forced at $t = 0$ to take a **leveraged** position in stock, the requirement $X_t + f(Y_t) > 0$ may not be satisfied for all $\omega \in \Omega$
  - However, if this is not the case, it is possible to construct trading dates such that $X_t + f(Y_t) > 0$ holds at all times
  - Also, – constituting an addition to our framework – one could adjust the shadow price such that $X_t + f(Y_t) > 0$ indeed holds

- Quantification duality gap via compensating variation:
  - Annualised counterpart: annual management fee, due to which one would obtain truly optimal terminal liquidated wealth

Numerical Examples

- To evaluate the method's accuracy, we use:
  - A one-dimensional standard Black-Scholes model
  - A CRRA utility function : $-x\frac{U_x''(x,y)}{U_x'(x,y)} = \gamma$ for all $x, y \in \mathbb{R}_+$ and $\gamma > 1$
  - A dual-CRRA utility qualification : $-x\frac{U_x''(x,y)}{U_x'(x,y)} = \gamma_d \mathbb{1}_{\{\frac{x}{y} \leq 1\}} + \gamma_u \mathbb{1}_{\{\frac{x}{y} > 1\}}$ for all $x, y \in \mathbb{R}_+$ and $\gamma_d, \gamma_u > 1$, cf. Kamma and Pelsser (2019)

- Concerning confined dual-set for shadow prices:
  - We let the local drift and diffusion terms in the semi-martingale process, $\{\tilde{S}_t\}_{t\in[0,T]}$, abide by deterministic **constants**
  - Confinement mainly based on analytical tractability

- Regarding the approximate trading dates:
  - Trading dates: adjust holdings at the start of each year, and at the start of each second year; also, twice and five times over the trading interval; and buy-and-hold

CRRA Utility

γ	$T = 10$ 1.5	2	5	10	$T = 40$ 1.5	2	5	10
TC 1%								
B&H	1.900	2.585	3.392	3.185	3.575	7.784	11.178	9.764
2-year	3.042	3.484	2.916	2.276	3.491	4.330	3.878	3.007
1-year	3.777	4.503	3.604	2.671	4.101	5.230	4.503	3.355
TC 5%								
B&H	2.215	2.750	3.170	2.977	3.889	7.946	11.123	9.712
2-year	10.561	11.654	8.700	6.300	12.928	16.522	13.757	9.956
1-year	15.660	17.367	12.838	9.144	17.335	22.621	18.780	13.457
TC 10%								
B&H	2.708	2.950	2.931	2.776	4.759	8.474	11.185	9.734
2 times	10.026	9.916	7.047	5.368	7.504	10.175	10.627	8.833
5 times	22.168	21.779	14.777	10.653	13.552	16.331	13.767	10.372

Table 1. Annual welfare losses for CRRA agents.

Dual-CRRA Utility

	$T = 10$				$T = 40$			
γ_d, γ_u	$10, 2$	$6, 4$	$4, 6$	$2, 10$	$10, 2$	$6, 4$	$4, 6$	$2, 10$
TC 1%								
B&H	2.998	3.019	3.118	10.492	10.349	11.813	12.722	12.075
2-year	2.092	2.423	2.670	5.660	3.600	4.584	5.565	6.984
1-year	2.487	3.025	3.470	6.641	3.952	5.126	6.304	7.905
TC 5%								
B&H	2.806	2.828	2.875	8.254	10.268	11.703	12.611	11.907
2-year	6.131	7.692	9.186	15.186	10.503	13.525	16.359	19.149
1-year	8.975	11.439	13.830	22.470	13.999	18.100	21.891	25.377
TC 10%								
B&H	2.629	2.636	2.660	5.900	10.255	11.687	12.638	12.073
2 times	5.222	6.280	7.376	11.321	9.351	10.976	12.340	12.938
5 times	10.507	13.253	16.114	23.675	10.883	13.601	16.165	18.748

Table 2. Annual welfare losses for dual-CRRA agents.

Conclusion

- Proposal of approximating procedure
  - Recovery of approximate optimal controls in closed-form
  - Accompanied by strong guarantee as to accuracy
- Method is based on a threefold routine:
  - First: confine space of shadow price processes
  - Second: discretise the ensuing artificial-optimal investment policies
  - Third: assess quality of approximation via duality bounds
- In the numerical examples that we considered:
  - Approximation succeeds in rendering **near-optimal** strategies
  - Buy-and-hold performs quite well under all circumstances
  - Frequency of trades, risk-aversion etc. all have sensible effects on decision rules

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