Secular Stagnation? Growth, Asset Returns and Welfare in the Next Decades*

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September 4, 2020

Preliminary and Incomplete

Abstract

Ongoing demographic change will lead to a relative scarcity of raw labor to the effect that output growth will be decreasing in the next decades, a secular stagnation. As physical capital will be relatively abundant, this decrease of output will be accompanied by reductions of asset returns. We quantify these effects for the US economy by developing an overlapping generations model with risky and risk-free assets.

In an earlier version of this paper we obtained the following results: Without adjustments of human capital, risky returns decrease until 2035 by about 0.7 percentage point, and the risk-free rate by about one percentage point, leading to substantial welfare losses for asset rich households. Per capita output is reduced by 6%. Endogenous human capital adjustments strongly mitigate these effects. We conclude that human capital policies will be crucial in the context of labor shortages.

JEL classification: E17, C68, G12

Keywords: secular stagnation; demographic change; overlapping generations; natural rate; equity premium; growth; welfare; human capital

*We thank Martin Barbie, Axel Börsch-Supan, Helge Braun, Wouter den Haan, Daniel Harenberg, Philip Jung, Tom Krebs, Wolfgang Kuhle, and Edgar Vogel for helpful discussions and seminar/conference participants at various places for valuable comments. We also gratefully acknowledge financial support by the German National Research Foundation (SPP 1578) and by the Research Center SAFE, funded by the State of Hessen initiative for research LOEWE. The usual disclaimer applies. In particular, the views expressed in this paper do not represent the official positions of the OECD or the governments of the OECD member countries.

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1 Introduction

In his famous keynote addresses at the IMF in 2013 and at the NABE Policy conference in 2014, Summers (2013, 2014) argued in terms of three propositions as follows: in consequence of "secular stagnation"—a long-run or trend reduction of growth[1]—(i) it will be "increasingly difficult" to achieve "adequate growth" in the US and other industrialized economies; (ii) that these developments will likely be accompanied by "a substantial decline in the equilibrium or natural rate of interest"; and (iii) that "addressing these challenges requires different policy approaches than are represented by the current conventional wisdom".[2]

The present paper analyzes the role of demographic change for a secular stagnation along all three propositions. First, we ask if—and if so how strongly and for how long—demographic change will effect growth of the US economy in the next decades. We do so by developing a structural model of an economy with a production sector and a household sector in order to capture how the relative demands and supplies of production factors change in consequence of the demographic evolution over the next decades. In order to realistically capture these demographic developments, we develop an overlapping generations model featuring a very detailed description of the aging process[3] and its role for household decisions.

Second, we investigate whether the potential reduction of growth is accompanied by a decline of real rates of return. To do so we explicitly distinguish between equilibrium risky asset returns and the returns of a (one period ahead) risk-free bond. While standard predictions suggest that demographic change leads to a shortage of labor and an abundance of physical capital to the effect that marginal productivities and hence asset returns decline, these effects may be very different across asset classes. Older households may have a higher preference for relatively risk-free investments which would increase relative demand for bonds. Therefore, bond returns may decrease by more than risky asset returns. Also, our distinction between different assets enables us to be explicit about the notion of the "natural rate" which provides important guidance for central banks for their interest rate policies. According to Wicksell (1898), the natural rate of interest is the interest rate which is compatible with a stable price level. In monetary theory the workhorse model of central bank behavior is the Tayor rule (Taylor 1993) which models the nominal interest rate as the

[1] The term secular is used in contrast to cyclical or short-term, and suggests a change of fundamental dynamics. The term was originally coined by Hansen (1938).

[2] See Summers (2014, p. 66). As Summers (2014) further points out, one reason for secular stagnation might be the demographic development over the next decades leading to an increasing shortage of labor thereby depressing output.

[3] We use the terms "demographic change" and "aging" interchangeably.

natural rate plus some deviation terms. Hence, an appropriate model of the development of the natural rate is required for fine tuning monetary policy instruments. We deliver such a forecasting instrument by modeling the supply side of assets through a production sector on one hand and government debt on the other. Equity is held in the form of risky productive capital and bonds constitute safe government debt. The demand side for differential assets stems from the household sector where intra-generational heterogeneity—induced by stochastic income processes—and intergenerational heterogeneity (by age) gives rise to trade across households in both assets.

Third, in terms of policy implications, we emphasize that human capital policies may play an important role in the next decades. While we do not model these policies explicitly, we consider two polar human capital scenarios. In our first scenario we restrict human capital adjustments. Then, accumulated losses in per capita output until year 2035—resulting from the demography induced reduction of growth rates—stand at roughly 6 percent relative to a constant growth environment. Risky asset returns decrease by 0.7 and bond returns by 1 percentage points until 2035. These relatively strong effects would lead to quite strong welfare losses for middle aged and old households who hold substantial physical and financial wealth. In our second scenario households react to increasing life-expectancy and increasing wages as well as falling real asset returns by increasing their human capital investments. Then per capita output decreases by only 2 percent (relative to a long-run path with constant growth rates). Risky asset returns decrease by roughly 0.16 and risk-free returns by 0.24 percentage points. These quantitative effects are rather mild. In light of real world frictions on markets for human capital (which we do not model)[4] we argue that our two scenarios bracket the evolution of future asset returns, growth and welfare. Furthermore, we show

These results are driven by demographic developments which we take as exogenous. According to our demographic projections, the working age population ratio[5] will decrease by roughly 10 percentage points. The ensuing shortage of the quantity of raw labor and the accompanying relative abundance of physical capital leads to decreasing output, increasing gross wages and falling asset returns. These relative price movements—in combination with the increasing life-time horizon resulting from increasing life expectancies—will lead households to increase their human capital investments because relative human capital returns increase. As a result of human capital adjustments, the reductions of output and of capital productivity as well as equilibrium asset returns will be mitigated. Furthermore, our model predicts that average portfolio shares of investments in risky assets are decreasing over the

[4]Such elements are not included in the model for technical reasons. As further discussed below, we achieve computational tractability by using a specific framework giving rise to closed form solutions of households' policy functions. These analytical results would seize to exist in a setup with frictions.

[5]I.e., the fraction of the population in working age (age 20-64) relative to the total adult population.

life-cycle. Because demographic change shifts population shares so that there are relatively more elderly households—the old-age dependency ratio[6] increases by more than 15 percentage points—the demand for bonds relative to equities increases. This reduces bond returns more strongly, hence the equity premium increases.

We contribute to a growing literature on secular stagnation. A collection of overview articles on the topic is given in Teulings and Baldwin (2014). Eggertsson and Mehrotra (2014) develop a New Keynesian model with overlapping generations in which a deleveraging shock leads to an oversupply of savings triggering low interest rates. In their framework, the same effects would be caused by a drop of the population growth rate. Eggertsson, Mehrotra, and Summers (2016) employ a two-country model to study how capital flows transmit shocks in a low interest rate environment. Relative to this literature, we emphasize demographics as the key source of the joint phenomenon of low growth and low interest rates. For simplicity, we look at a closed economy model.[7] Similar to our contribution, Carvalho, Ferrero, and Nechio (2016) investigate whether demographic developments drive down real interest rates. They focus on a deterministic environment with one asset and accordingly cannot study differential asset returns as we do.

Our work also relates to a relatively large literature that employs variants of Auerbach and Kotlikoff (1987) overlapping generations models to quantitatively evaluate the consequences of demographic change for growth and welfare, cf., e.g., Börsch-Supan, Ludwig, and Winter (2006), Attanasio, Kitao, and Violante (2007), Krueger and Ludwig (2007), Ludwig, Schelkle, and Vogel (2012) and the literature cited therein. Relative to this literature, the main novel aspect in our work is to explicitly model differential asset returns. We thereby relate to a literature on aging and the equity premium (Bakshi and Chen 1994; Brooks 2004; Börsch-Supan, Ludwig, and Sommer 2003; Geanakoplos, Magill, and Quinzii 2004; Kuhle 2008) which has not reached a consensus on the quantitative effects of demographic change on differential asset returns. While Brooks (2004) reports substantial increases in the equity premium, the approximate calculations in Börsch-Supan, Ludwig, and Sommer (2003) rather suggest a small increase. Geanakoplos, Magill, and Quinzii (2004) conclude that "the equity premium is smaller when the population of savers is older" which the authors interpret as a contradiction to the findings of Bakshi and Chen (1994) and Brooks (2004). These papers all employ relatively stylized overlapping generations models with few generations. Such a periodicity severely restricts households to re-balance their portfolios. We avoid such

[6]I.e., the population in retirement age (age 65 and older) as a fraction of the working age population.

[7]This is a fair approximation for the US economy. E.g., in a purely deterministic multi-country overlapping generations model with international capital flows driven by heterogenous demographic developments, Krueger and Ludwig (2007) show that world interest rates faced by the US alter relatively little between closed and open economy variants of their model.

restrictions by employing a large scale overlapping generations model that runs at an annual frequency. To significantly reduce computational costs we adopt the risky human capital framework developed in Krebs (2003) and Krebs and Wilson (2004). This setup gives rise to closed form solutions of households' policy functions for consumption and total saving, conditional on the law of motion of the aggregate state of the economy and the solution for optimal portfolio shares.[8]

This paper proceeds as follows. Section 2 develops the large scale quantitative overlapping generations model. Section 3 describes our approach to numerically solve this model as well as the model's calibration.

2 Quantitative Model

We extend the classical Diamond (1965) economy to a multi-period setup as in Auerbach and Kotlikoff (1987) with idiosyncratic risk. On the household side, labor income is a choice variable which we implement by adopting the human capital framework developed in Krebs (2003) and Krebs and Wilson (2004) in an overlapping generations setup. In each period, a household of a given age chooses to invest a fraction of her overall wealth in human capital, respectively financial capital. As for the fraction of wealth invested in financial assets, the household solves a standard portfolio allocation problem by choosing how much to invest into risky physical capital and risk-free bonds. Adopting the framework developed in Angeletos (2007) risky returns on physical capital arise form risky idiosyncratic return rates. Consequently, there are three assets in the economy: risky human capital, risky physical capital and risk-free bonds. Our setup is such that, once portfolio allocation decisions are made, household consumption and savings policies are linear age-dependent functions of wealth, cf. Merton (1969) and Samuelson (1969). Therefore, the household problem is easy to solve. Linear policy functions paired with the lack of aggregate uncertainty lead to aggregate dynamics being unaffected by idiosyncrasies. This feature of the model is particularly useful because it enables us to solve a large-scale OLG model with idiosyncratic risk and rather complex dynamics without incurring large computational costs.

2.1 Demographics

Time is discrete and runs from $t = 0, \dots, \infty$. In the economy there are overlapping generations. Each generation is populated by a continuum of uniformly distributed households.

[8] Human capital is modeled as an asset and in suitable transformation of our model reduces to a standard portfolio choice model. Once portfolio shares are computed, policy functions of savings and consumption are linear in wealth, as in the seminal work by Merton (1969) and Samuelson (1969).

The dual index j, i denotes the i'th household within the population of age j. Let $N_{t,j}$ denote the population mass of households of age j in time period t. Households born at t live until turning $J_t + 1$. J_t changes over time, capturing changes in life expectancy. The process governing J_t is assumed to be non-stochastic. For a discussion how we deal with sub-period changes see section A.1 in the appendix. We assume $J_{t+1} - J_t \geq -1$. Households enter the model at the age of 20 ($j = 0$). Given the population dynamics, the economy is populated with a changing number of overlapping generations. Population mass is given recursively as

$$N_{t,j} = \begin{cases} N_{t-1,j-1} & \text{for } j \in \{q : 0 \leq q \leq J_{t-q}\}^9 \\ newborns_t & \text{for } j = 0, \\ 0 & \text{for all other } j \end{cases} \tag{1}$$

where $newborns_t$ denotes the mass of the age 0 generation newly entering the economy. The underlying population dynamics is the exogenous driving force of the model.

2.2 Preferences

We assume Epstein-Zin-Weil recursive preferences (Epstein and Zin 1989; Epstein and Zin 1991; Weil 1989). Let θ be a measure of risk-aversion and ξ denote the elasticity of intertemporal substitution. Epstein-Zin preferences then write as

$$u_{t,j,i} = \left[c_{t,j,i}^{\frac{1-\theta}{\gamma}} + \beta \cdot \left(\mathbb{E}[u_{t+1,j+1,i}^{1-\theta}] \right)^{\frac{1}{\gamma}} \right]^{\frac{\gamma}{1-\theta}}$$

where $\gamma \equiv (1-\theta)/(1-1/\xi)$. $0 < \beta < 1$ is the standard discount factor. For $\theta = 1/\xi$ we have $\gamma = 1$ and are back at standard CRRA preferences. β is the raw time discount factor and $c_{t,j,i}$ is consumption at time t, age j of household i. $\mathbb{E}$ is the expectations operator and expectations are taken with respect to idiosyncratic shocks to human capital and technology. The felicity function in the last period an individual is still alive is given by $u_{t+J_t+1,J_t,i} = c_{t+J_t+1,J_t,i}$. Also observe that $u_{t,j,i} > 0$ for $c_{t,j,i} > 0$. This implies that life-time utility is increasing in J_t.[10]

[9] $j \in \{q : 0 \leq q \leq J_{t-q}\}$ is equivalent to $j \leq \bar{J}_{t-1}$, where $\bar{J}_{t-1} = \max\{q : 0 \leq q \leq J_{t-q}\}$, i.e. there are no holes in the age set. This follows from assumption $J_{t+1} - J_t \geq -1$.

[10] For a rigorous discussion on Epstein-Zin-Weil utility and the value of life see, e.g., Córdoba and Ripoll (2016) and Bommier, Harenberg, and Legrand (2016).

2.3 Budgets

Households consume $c_{t,j,i}$, hold risky physical capital $k_{t,j,i}$ with capital income $\pi_{t,j,i}(k_{t,j,i})$ (to be specified below), bonds $b_{t,j,i}$ which accrue risk-free interest at the rate r_t^f and human capital $h_{t,j,i}$ earning a gross rate of return of r_t^H. Additionally, our model features a pay as you go social security system. Social security benefits p_t, which are the same across all households and age groups, are lump-sum and are financed by contributions to labor income τ_t. At each age a household works for fraction $\omega_{t,j} \in [0,1]$ of her time and is retired with fraction $1-\omega_{t,j}$. By letting $\omega_{t,j} = 0$ for j considerably before retirement and letting $\omega_{t,j}$ gradually rise to 1 around the official retirement age we can accommodate the fact that actual age at retirement varies across individuals. Furthermore, by letting $\omega_{t,j}$ increase with t, we can allow for a rising share of elderly employed due to an increase in the official retirement age and improving health condition. With $\ell_{t,j,i} = \omega_j \cdot h_{t,j,i}$ we denote the laborable human capital, i.e. the human capital supplied to the labor market by the household. The budget constraint of household i of age j in period t is given by

$$\begin{aligned} b_{t+1,j+1,i} + k_{t+1,j+1,i} = {} & b_{t,j,i}(1+r_t^f) + \pi_{t,j,i}(k_{t,j,i}) + \\ & h_{t,j,i}(1-\tau_t)\omega_j r_t^H + (1-\omega_j)p_t - c_{t,j,i} - e_{t,j,i} \end{aligned} \tag{2}$$

When entering the economy at age $j = 0$, any household i is endowed with an initial level of human capital, $h_{t,0,i} = h_0$, physical capital $k_{t,0,i} = 0$ and bond holdings $b_{t,0,i} = 0$ for all t (there are no bequest flows to households).

2.4 Technology

Every household i owns a firm that produces a final good according to the Cobb-Douglas production function

$$y_{t,j,i} = \psi \zeta_{t,j,i} k_{t,j,i}^{\alpha} (\Upsilon_t \ell_{ji}^D)^{1-\alpha},$$

The household supplies her capital holdings $k_{t,j,i}$ to her firm and hires labor $\ell_{t,j,i}^D$, measured in human capital units, at the market rate r_t^H. $\ell_{t,j,i}^D$ differs from $\ell_{t,j,i}$, the laborable human capital supplied by household i. Human capital is both, supplied to and demanded from the labor market by households and their firms. Households do not employ their human capital directly in their firms. Υ_t is a human capital augmenting productivity parameter which grows at the exogenous constant rate g to capture the observed trend growth of GDP and ψ is a scaling parameter. $\zeta_{t,j,i}$ is an idiosyncratic firm specific physical capital shock:

$$\zeta_{t,j,i} \sim \mathcal{D}(1, \sigma_\zeta^2),$$

where $\mathcal{D}$ is some distribution with mean one, further specified in Section 3.

In order to work with stationary variables we de-trend the production function by dividing it by Υ_t, yielding

$$\tilde{y}_{t,j,i} = \psi \zeta_{t,j,i} \tilde{k}_{t,j,i}^{\alpha} (\ell_{ji}^{D})^{1-\alpha},$$

where $\tilde{y}_{t,j,i} = \frac{y_{t,j,i}}{\Upsilon_t}$ and $\tilde{k}_{t,j,i} = \frac{k_{t,j,i}}{\Upsilon_t}$.

Whereas $\tilde{k}_{t,j,i}$ is determined before the firm specific TFP shock realizes, $\ell_{t,j,i}$ is determined after its realization. Hence, the firm chooses $\ell_{t,j,i}^{D}$ to maximize (de-trended) capital income (net of labor costs and depreciation of physical capital at rate δ^K)

$$\tilde{\pi}_{t,j,i}(\tilde{k}_{t,j,i}) = \max_{\ell} \tilde{y}_{t,j,i} - \tilde{r}_t^H \ell_{t,j,i}^{D} + (1-\delta^K)\tilde{k}_{t,j,i},$$

given the amount of physical capital, where $\tilde{r}_t^H = \frac{r_t^H}{\Upsilon_t}$. This results in labor demand

$$\ell_{t,j,i}^{D} = \left((1-\alpha)\psi \frac{\zeta_{t,j,i}}{\tilde{r}_t^H}\right)^{\frac{1}{\alpha}} \tilde{k}_{t,j,i} \tag{3}$$

output

$$\tilde{y}_{t,j,i} = (1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\psi \zeta_{t,j,i} (\tilde{r}_t^H)^{\alpha-1}\right)^{\frac{1}{\alpha}} \tilde{k}_{t,j,i} \tag{4}$$

and capital income

$$\tilde{\pi}_{t,j,i} = \left[1 + \alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\psi \zeta_{t,j,i} (\tilde{r}_t^H)^{\alpha-1}\right)^{\frac{1}{\alpha}} - \delta^k\right] \tilde{k}_{t,j,i},$$

which is proportional to $\tilde{k}_{t,j,i}$. Hence we can write capital income as

$$\tilde{\pi}_{t,j,i}(\tilde{k}_{t,j,i}) = (1 + r_{t,j,i}^K)\tilde{k}_{t,j,i} \tag{5}$$

with an idiosyncratic rate of return on physical capital

$$r_{t,j,i}^K = \alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\psi \zeta_{t,j,i} (\tilde{r}_t^H)^{\alpha-1}\right)^{\frac{1}{\alpha}} - \delta^K. \tag{6}$$

Dividing household budget (2) by Υ_t and using equation (5) in place of capital income we get

$$\tilde{b}_{t+1,j+1,i} + \tilde{k}_{t+1,j+1,i} = \frac{1}{1+g}\Big(\tilde{b}_{t,j,i}(1+r_t^f) + \tilde{k}_{t,j,i}(1+r_{t,j,i}^K) + \\ (1-\tau_t)\omega_j \tilde{r}_t^H h_{t,j,i} + (1-\omega_j)\tilde{p}_t - \tilde{c}_{t,j,i} - \tilde{e}_{t,j,i}\Big), \tag{7}$$

where $\tilde{b}_{t,j,i} = \frac{b_{t,j,i}}{\Upsilon_t}, \tilde{c}_{t,j,i} = \frac{c_{t,j,i}}{\Upsilon_t}$ and $\tilde{p}_t^H = \frac{p_t^H}{\Upsilon_t}$.

2.5 Human Capital Accumulation

We assume that human capital depreciates at the individual level by the age-specific deterministic rate δ_j^h. The age-profile of $\{\delta_j^h\}_{j=1}^{j_r}$ enables us to calibrate the model such that it mimics decreasing returns to human capital accumulation as assumed elsewhere in the literature (e.g., Huggett, Ventura, and Yaron (2011)). Furthermore, we let human capital depreciation increase with the aggregate level of laborable human capital, L_t. This reflects the fact that the more aggregate knowledge is in working society, the more an household's knowledge depreciates in relative terms because it has to keep up with the knowledge stock. Introducing this aggregate laborable human capital dependent depreciation is necessary as the pension payments introduce an externality of L_t into the household's budget constraint. The larger L_t, the higher the average income and thus the higher p_t (see section 2.8). We assume the following functional form

$$\delta_j^h = -\chi_0 + \exp(\chi_1 \cdot j) + \chi_2 L_t, \quad \chi_0 > 0, \chi_1 \geq 0, \chi_2 > 0,$$

which is monotonically increasing in j so that $1 - \chi_0 \leq \delta_j^h \leq \delta_{j+1}^h$ for all j. χ_1 is the rate at which the household's human capital depreciation accelerates when getting older and χ_2 determines the sensitivity to the aggregate level of laborable human capital.

After the return to human capital is paid the household is hit by an additive idiosyncratic shock to its human capital holdings:

$$\eta_{t,j,i} \sim \mathcal{D}(0, \sigma_\eta^2)$$

where $\mathcal{D}$ is some distribution with mean zero, further specified in Section 3.

Collecting these elements, the human capital accumulation equation in period t, age j, of household i is given by

$$h_{t+1,j+1,i} = h_{t,j,i} \cdot (1 - \delta_j^h + \eta_{t,j,i}) + \tilde{e}_{t,j,i}^h, \qquad h_{t,j,i} \geq 0 \;\; \forall\, t, j, i, \tag{8}$$

where $\tilde{e}_{t,j,i}^h \equiv e_{t,j,i}^h / \Upsilon_t$.[11] Note that all variables in (8) are trend-stationary.[12]

2.6 Pension Stock

Denote by

$$\begin{aligned} ps_{t,j} =& \frac{(1-\omega_{t,j})p_t}{1+r_t^f} + \frac{(1-\omega_{t+1,j+1})p_{t+1}}{(1+r_t^f)(1+r_{t+1}^f)} + \dots + \frac{(1-\omega_{t+J_{t-j}+1-j, J_{t-j}+1})p_{t+J_{t-j}+1-j}}{\prod_{s=0}^{J_{t-j}+1-j}(1+r_{t+s}^f)} \\ =& \sum_{s=0}^{J_{t-j}+1-j} (1-\omega_{j+s})p_{t+s} \prod_{q=0}^{s} \left(1 + r_{t+q}^f\right)^{-1} \end{aligned} \tag{9}$$

the discounted value of social security benefits for households of age j in period t (before interest accrues). Thereby the sequence of future bond rates, which are deterministic and thus known, feature as discount rate. Hereafter $ps_{t,j}$ is called pension stock.

Recursively the pension stock is given by

$$ps_{t+1,j+1} = ps_{t,j}(1 + r_t^f) - (1 - \omega_{t,j})p_t.$$

De-trended pension stock thus evolves according to

$$\widetilde{ps}_{t+1,j+1} = \frac{1}{1+g} \left(\widetilde{ps}_{t,j}(1 + r_t^f) - (1 - \omega_{t,j})\tilde{p}_t \right). \tag{10}$$

with pension stock after death being zero, i.e.

$$\widetilde{ps}_{t+J_{t-j}+2, J_{t-j}+2} = 0. \tag{11}$$

[11] We assume that costs for human capital investment, grow with the same rate as Υ_t.

[12] As the return to human capital r_t^H already exhibits a trend growth along with Υ_t, human capital must be trend stationary in order to assure that gross human capital earnings, $h_{t,j,i} \cdot r_t^H$, grow at the same rate as Υ_t over time.

2.7 Transformations and Recursive Household Problem

Transformations. Add $\frac{1}{1+g}$ times equation (8) and equation (10) to the de-trended dynamic budget constraint (7) yielding

$$\tilde{b}_{t+1,j+1,i} + \tilde{k}_{t+1,j+1,i} + \frac{h_{t+1,j+1,i}}{1+g} + \widetilde{ps}_{t+1,j+1,i} =$$

$$\frac{1}{1+g}\Big(\tilde{b}_{t,j,i}(1+r_t^f) + \tilde{k}_{t,j,i}(1+r_{t,j,i}^K) + h_{t,j,i}\left(1+(1-\tau_t)\omega_j\tilde{r}_t^H - \delta_j^H + \eta_{t,j,i}\right) + \widetilde{ps}_{t,j}(1+r_t^f) - \tilde{c}_{t,j,i}\Big). \tag{12}$$

Now, we can define total household wealth as the sum of holdings of physical capital, bond holdings, (growth rate adjusted) human capital and pension stock, $\tilde{w}_{t,j,i} = \tilde{b}_{t,j,i} + \tilde{k}_{t,j,i} + \frac{h_{t,j,i}}{1+g} + \widetilde{ps}_{t,j,i}$. Let $\hat{\alpha}_{t,j,i}^B = \frac{b_{t,j,i}}{w_{t,j,i}}$, $\hat{\alpha}_{t,j,i}^K = \frac{k_{t,j,i}}{w_{t,j,i}}$, $\hat{\alpha}_{t,j,i}^H = \frac{h_{t,j,i}/(1+g)}{w_{t,j,i}}$, $\hat{\alpha}_{t,j,i}^{PS} = \frac{ps_{t,j,i}}{w_{t,j,i}}$ denote the household i, period t, age j holdings of bonds, physical capital, human capital and pension stock relative to total wealth, respectively. Further let $\hat{\alpha}_{t,j,i}^f = \hat{\alpha}_{t,j,i}^B + \hat{\alpha}_{t,j,i}^{PS}$ denote the fraction of risk-free holdings in total wealth. Then $\hat{\alpha}_{t,j,i}^f = 1 - \hat{\alpha}_{t,j,i}^K - \hat{\alpha}_{t,j,i}^H$. Additionally define by

$$1 + \hat{r}_{t,j,i}^H \equiv (1+g)(1+(1-\tau_t)\omega_j\tilde{r}_t^H - \delta_j^H + \eta_{t,j,i})$$

a growth rate adjusted gross return on human capital. Now we can rewrite (12) in terms of total wealth accumulation as

$$\tilde{w}_{t+1,j+1,i} = \tilde{w}_{t,j,i} \cdot \frac{1}{1+g}\left(1 + r_t^f + \hat{\alpha}_{t,j,i}^K \cdot (r_{t,j,i}^K - r_t^f) + \hat{\alpha}_{t,j,i}^H \cdot (\hat{r}_{t,j,i}^H - r_t^f)\right) - \frac{\tilde{c}_{t,j,i}}{1+g} \tag{13}$$

Finally, define the gross portfolio return as

$$\hat{r}_{t,j,i} \equiv r_t^f + \hat{\alpha}_{t,j,i}^K \cdot (r_{t,j,i}^K - r_t^f) + \hat{\alpha}_{t,j,i}^H \cdot \left(\hat{r}_{t,j,i}^H - r_t^f\right) \tag{14}$$

and let cash-on-hand as wealth cum interest be $\tilde{x}_{t,j,i} \equiv \tilde{w}_{t,j,i}(1+\hat{r}_{t,j,i})$. With this definition rewrite (13) as

$$\tilde{x}_{t+1,j+1,i} = (\tilde{x}_{t,j,i} - \tilde{c}_{t,j,i}) \cdot \frac{1}{1+g} \cdot (1 + \hat{r}_{t+1,j+1,i}). \tag{15}$$

Going from (13) to (15) is the second key transformation. As a consequence of this transformation all additive terms have vanished. This, in combination with homothetic preferences, will give rise to the closed form solutions stated in the subsequent proposition. Also notice that period t age j choice variables are $\tilde{c}_{t,j,i}, \tilde{x}_{t+1,j+1,i}$ as well as the portfolio shares $\hat{\alpha}_{t+1,j+1,i}^K, \hat{\alpha}_{t+1,j+1,i}^H$ which together determine the period $t+1$, age $j+1$ holdings of physical capital, human capital and risk-free holdings. The social security benefits p_t are either exogenously given or are indirectly determined by the social contribution rate τ_t. Given the sequence of p_t and r^f_t, pension stock holdings and shares in total wealth are pinned down. This allows us to determine bond holdings by $\hat{\alpha}^B_{t+1,j+1,i} = \hat{\alpha}^f_{t+1,j+1,i} - \hat{\alpha}^{PS}_{t+1,j+1,i}$.

Recursive Household Problem. We now define the household problem recursively. It is convenient to express next period's values with symbol $'$, irrespective of whether they are only time-dependent or both, age- and time-dependent. The household problem is solved contingent on age j, the endogenous idiosyncratic state of (de-trended) cash-on-hand, $\tilde{x}$ and prices next period $\phi'_t = \{r^{f\prime}_t, \tilde{r}^{H\prime}_t, p'_t, \tau'_t\}$.

$$
\begin{aligned}
v(t,j,\tilde{x}) = & \max_{\tilde{c},\tilde{x}',\widehat{\alpha}^{k\prime},\widehat{\alpha}^{h\prime}} \left\{ \tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot \left(\mathbb{E}[v(t+1,j+1,\tilde{x}')^{1-\theta}]\right)^{\frac{1}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
\text{s.t. } & \tilde{x}' = \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot \left(1 + \widehat{r}(\phi'_t, \eta', \zeta', j+1)\right) \\
& \widehat{r}(\phi'_t, \eta', \zeta', j+1) = r^{f\prime}_t + \hat{\alpha}^{K\prime} \cdot \left(r^K(\tilde{r}^{H\prime}_t, \zeta') - r^{f\prime}_t\right) + \hat{\alpha}^{H\prime} \cdot \left(\hat{r}^H(\tilde{r}^{H\prime}_t, \tau'_t, \eta', j+1) - r^{f\prime}_t\right) \\
& t \mapsto \phi'_t \\
& \eta' \sim \mathcal{D}(0, \sigma^2_\eta) \\
& \zeta' \sim \mathcal{D}(1, \sigma^2_\zeta),
\end{aligned}
$$

where $\widehat{\beta} \equiv \beta \cdot (1+g)^{\frac{1-\theta}{\gamma}}$.

Proposition 1. *Conditional on the portfolio shares $\hat{\alpha}$, policy functions for consumption are linear in cash-on-hand*

$$\tilde{c}(t,j,\tilde{x}) = m(t,j)\tilde{x}. \tag{17}$$

Furthermore the shares $\hat{\alpha}^K(t,j,\tilde{x}) = \hat{\alpha}^K(t,j)$ and $\hat{\alpha}^H(t,j,\tilde{x}) = \hat{\alpha}^H(t,j)$ solve

$$\mathbb{E}[(1+\widehat{r}')^{-\theta} \cdot (r^{K\prime} - r^{f\prime})] = 0 \tag{18a}$$

$$\mathbb{E}[(1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{H\prime} - r^{f\prime})] = 0 \tag{18b}$$

and the marginal propensity to consume out of cash-on-hand is given by

$$m = \frac{(\beta^\gamma \cdot \mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}} \cdot m'}{1 + (\beta^\gamma \cdot \mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}} \cdot m'}. \tag{19}$$

Proof. See Section A.2 in the Appendix. □

Note that both, marginal propensity to consume m as well as portfolio shares $\hat{\alpha}^K$ and $\hat{\alpha}^H$ are independent of the household specific state realizations in η, ζ and $\tilde{x}$.

2.8 Bond Supply and Social Security System

We assume a zero-net-supply of bonds,

$$B_t = 0, \tag{20}$$

where B_t are the aggregate bonds supplied in period t.

The social security system is exclusively financed by labor income contributions. Benefits are linked to labor income through $\tilde{p}_t = \rho_t \cdot \frac{(1-\tau_t)\cdot \tilde{r}_t^H \cdot L_t}{\sum_j N_{t,j}\cdot \omega_{t,j}}$, with ρ_t being the replacement rate and L_t the aggregate labor employed in the economy as measured in human capital units. That is, the pension payments are equal to the replacement rate times the average net labor income. Closed financing of the pension system implies

$$\left(\sum_j N_{t,j} \cdot \omega_{t,j}\right) \tau_t \tilde{r}_t^H L_t = \left(\sum_j N_{t,j} \cdot (1-\omega_{t,j})\right) \cdot \rho_t \cdot (1-\tau_t) \cdot \tilde{r}_t^H \cdot L_t. \tag{21}$$

Ater simplification it implies that in equilibrium the labor contribution rate is

$$\tau_t = \frac{\sum_j N_{t,j}(1-\omega_{t,j})\rho_t}{\sum_j N_{t,j}\omega_{t,j} + \sum_j N_{t,j}(1-\omega_{t,j})\rho_t} \tag{22}$$

or equivalently the replacement rate is

$$\rho_t = \frac{\tau_t}{1-\tau_t} \cdot \frac{\sum_j N_{t,j}\omega_{t,j}}{\sum_j N_{t,j}(1-\omega_{t,j})}. \tag{23}$$

Note, that given the other, the labor contribution rate (τ_t) or the replacement rate (ρ_t) are just determined by the demographics, independent of any prices or aggregates. It is the population measures of workers and retirees which is driving τ_t and ρ_t.

2.9 Equilibrium

Aggregate quantities are stated in capital letters. Before we give a definition of the equilibrium we determine aggregate labor demand by aggregating equation (3) across ages j and firms i:

$$\begin{aligned} L_t^D =& \sum_j N_{t,j} \cdot \int [(1-\alpha)\psi]^{\frac{1}{\alpha}} \left(\frac{\zeta_{t,j,i}}{\tilde{r}_t^H}\right)^{\frac{1}{\alpha}} \tilde{k}_{t,j,i}\ di \\ =& [(1-\alpha)\psi]^{\frac{1}{\alpha}}\, \mathbb{E}[\zeta^{\frac{1}{\alpha}}](\tilde{r}_t^H)^{-\frac{1}{\alpha}} \widetilde{K}_t, \end{aligned} \tag{24}$$

by the law of large numbers as $\zeta_{t,j,i}$ and $\tilde{k}_{t,j,i}$ are independently distributed. Aggregating equation (4) gives aggregate output as

$$\widetilde{Y}_t = (1-\alpha)^{\frac{1-\alpha}{\alpha}} \psi^{\frac{1}{\alpha}} \mathbb{E}[\zeta^{\frac{1}{\alpha}}](\tilde{r}_t^H)^{\frac{\alpha-1}{\alpha}} \widetilde{K}_t. \tag{25}$$

Note that like in the case with a representative firm with Cobb-Douglas technology, the labor income share (here the share of employed human capital) is $1-\alpha$ and the physical capital income share (including depreciated capital) is α.

Similarly, the aggregate pension stock is determined by aggregating the individual pension stocks across ages j:

$$\widetilde{PS}_t = \sum_j N_{t,j} \widetilde{ps}_{t,j}, \tag{26}$$

where $\widetilde{ps}_t$ is determined by equation (9).

Age dependent accumulation of cash-on-hand determines the aggregate law of motion. Average cash-on-hand holdings of age j generation are given by aggregating the individual cash-on-hand accumulation equation (15) across households, i.e.

$$\begin{aligned} \widetilde{X}_{t+1,j+1} =& \frac{1}{1+g} \cdot \int \tilde{x}_{t,j,i} \cdot (1-m_{t,j}) \cdot (1+\hat{r}_{t+1,j+1,i})\ di \\ =& \frac{1}{1+g} \cdot \widetilde{X}_{t,j} \cdot (1-m_{t,j}) \cdot (1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t+1,j+1,i}\right]), \end{aligned} \tag{27}$$

which follows from $\hat{\alpha}^f, \hat{\alpha}^K$ and m not being functions of cash on hand and $(\eta_{t,j,i}, \zeta_{t,j,i})$ being independently distributed from $\tilde{x}_{t,j,i}$.

The equilibrium in the economy is defined recursively and presented in de-trended form, cf. Section 2.7. It requires market clearing in all periods while household optimality and aggregation conditions have to hold. In the following, $'$ indicates next period's variables.

Definition 1. *A recursive competitive equilibrium is a value function $v(t,j,\tilde{x})$ and policy functions, $\widehat{\alpha}^{K\prime}(t,j)$, $\widehat{\alpha}^{H\prime}(t,j)$, $m(t,j)$ for the household, functions for aggregate labor demand $L^D(\tilde{r}^H, \widetilde{K})$, bond supply $B'(t)$, physical capital holdings $K'(t)$, aggregate pension stock $\widetilde{PS}(t)$, aggregate cash-on-hand holdings of generation j, $\widetilde{X}_j'(\tilde{X}_j, \phi')$, pricing functions*

$\phi'(t) = \{r^{f\prime}(t), \tilde{r}^{H\prime}(t), p'(t), \tau'(t)\}$ *and the demographic distribution,* $N(t,j)$*, such that for all* t

1. $v(\cdot)$ *satisfies the household's recursive problem, and* $\widehat{\alpha}^{K\prime}(\cdot)$*,* $\widehat{\alpha}^{H\prime}(\cdot)$*,* $m(\cdot)$ *are the associated policy functions following from the conditions in Proposition 1, given* $\phi'(t)$
2. *aggregate labor demand* $L^D(\tilde{r}^H, \widetilde{K})$ *is given by* (24)*,*
3. *aggregate pension stock is given by* (26) *and thus a function of future benefits and risk-free rates,* $\widetilde{PS}(\{\rho(s), \tau(s), \tilde{r}^H(s), r^f(s)\}_t^{t+J_t})$*,*
4. *the aggregate bond supply is given by* (20)*, determining next period's bond supply as a function of the risk free rate and current bond supply,* $\tilde{B}'(r^f, \tilde{B})$
5. *Pension payments are financed through labor income contributions as given in equation* (21)*.*
6. *Risk free assets and labor markets clear:*

$$\widetilde{B}'(r^f(t), Y(t), B(t)) + \widetilde{PS}'(\{\rho(s), \tau(s), \tilde{r}^H(s), r^f(s)\}_t) = \\ \frac{1}{1+g} \cdot \sum_j N(t,j) \cdot \widetilde{X}_j(\widetilde{X}_{j-1}(t-1), \phi(t)) \cdot (1 - m(t,j)) \cdot (1 - \widehat{\alpha}^{K\prime}(t,j) - \widehat{\alpha}^{H\prime}(t,j))^{13} \tag{28a}$$

$$L^{D\prime}(\tilde{r}^{H\prime}(t), \widetilde{K}'(t)) = \\ L^{S\prime}(t) \equiv \sum_j N(t,j) \cdot \widetilde{X}_j(\widetilde{X}_{j-1}(t-1), \phi(t)) \cdot (1 - m(t,j)) \cdot \hat{\alpha}^{H\prime}(t,j) \cdot \omega_{t+1,j+1},^{13} \tag{28b}$$

7. *aggregate law of motion* $\widetilde{X}_j'(\widetilde{X}_j, \phi')$ *is given by equation* (27)*, where*

$$\widetilde{X}_0(t) = h_0 \cdot (1 + (1-\tau(t))\omega_{t,0}\tilde{r}^H(t) - \delta_0^H) + \widetilde{ps}_{t,0} \cdot (1 + r_t^f). \tag{29}$$

8. *and either* $\rho(t)$ *or* $\tau(t)$ *is given.*

[14]

Lemma 1 (Walras' Law)**.** *The definition of a recursive equilibrium above ensures that the aggregate resource constraint holds:*

$$\widetilde{Y}(t) = \widetilde{C}(t) - N(t,0) \cdot \widetilde{B}(t+1,0) + \widetilde{I}^K(t) + I^H(t), \tag{30}$$

[13] Aggregation results are based on the fact that $m(j,\cdot), \hat{\alpha}^{f\prime}(\cdot)$ and $\hat{\alpha}^{K\prime}(\cdot)$ functions are independent of individual cash-on-hand levels.

[14] Additionally, the pension stocks across generations in the initial period, $\{\widetilde{ps}_{0,j}\}_{j=0}^{J_0}$, and in the final period, $\{\widetilde{ps}_{T,j}\}_{j=T}^{J_T}$ are given. See appendix B.1.1, and there especially equation (**??**) (combined with equation (10)).

where $\tilde{I}^K(t)$ and $I^H(t)$ are aggregate (de-trended) investment into physical capital and human capital.

Proof. See Section A.3 in the Appendix. □

Definition 2. *A stationary recursive competitive equilibrium is a special case of the equilibrium described above. It is characterized by time-constant prices $\phi(t)$. This requires a time-constant demographic distribution, $N(t)$.*

3 Solution Method and Calibration

Solution Method. As shown in proposition 1 the households' portfolio shares and marginal propensities to consume are independent of individual variables (shock realizations and cash-on-hand). Instead they only depend on time t and age j. By the law of large number the idiosyncratic shocks cancel each other out on the cohort level. This allows us to to treat the model at the aggregate level as a deterministic OLG model with one representative agent per cohort, where as sole state variable we have time and cohort specific (de-trended) cash-on-hand, $\widetilde{X}_{t,j}$. We solve for an initial stationary equilibrium in the year 1960 and a final one in the year 2500. Starting with the initial stationary equilibrium in 1960 we solve the model for the transitional path until 2500. For more details on the computational approach refer to appendix B.

Calibration. Calibration of the model is in part by reference to other studies and in part by informal matching of moments procedures. The period length is one year. Table 1 summarizes structural model parameters where target values refer to the referenced study or in the other cases to the average of years $1960-2010$. The additional parameters governing stochastic and demographic processes are described in the text.

The elasticity of inter-temporal substitution, ξ, equals 0.5. It lies in the range considered in the asset pricing literature (cf. the discussion in Bansal and Yaron (2004, pp. 1492-93)). The value of households' raw time discount factor, β is calibrated to yield a capital-output ratio of 2.65, as measured in NIPA data, cf., e.g., Ludwig, Schelkle, and Vogel (2012).

Due to the homotheticity of preferences, the initial level of human capital h_0 is irrelevant and we normalize human capital by setting $h_0 = 1$.

The value of the capital share parameter, $\alpha = 0.36$, is based on an estimation of the aggregate production function for the US, cf. Krueger and Ludwig (2007), and lies in the usual range considered in the literature. We set the average risk-free rate to 2.5%, the approximate average US bank lending rates in the 1960s according to Worldbank data.

Table 1: First and Second Stage Parameters

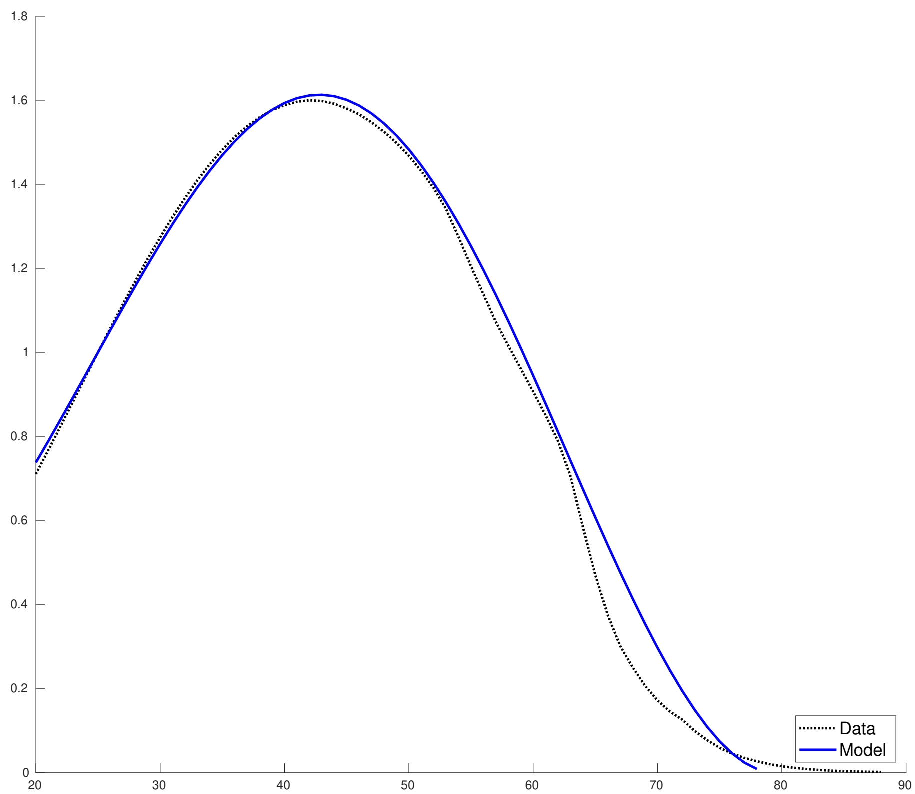
Parameter	Value	Target	Target Source, Comment
Preferences			
Elasticity inter-temp. substit.: ξ	0.5	1^{st} stage	Bansal and Yaron (2004)
Time discount factor: β	0.911	$K/Y = 2.86$	NIPA
Relative risk aversion: θ	7.17	$\mathbb{E}[r]^s - r^f = 0.0398$	PST
Budgets			
Endowment: $\{h_0, k_0\}$	$\{1.0, 0.0\}$	1^{st} stage	normalization
Technology			
Capital share: α	0.36	1^{st} stage	wage share (NIPA)
Technological progress: g	0.018	1^{st} stage	TFP growth (NIPA)
Depreciation rate K: δ_0^K	0.0786	$\mathbb{E}[r^s] = 0.0223$	PST
TFP Scaling factor: ψ	1	–	
TFP shock std.: σ_ζ	0.0841	$std(r^k) = 0.0841$ (std. of stock returns)	PST
Human Capital			
Depreciation rate h intercept: χ_0	0.333	$\frac{r^H \cdot h_{40}}{r^H \cdot h_{25}} = 1.49$	PSID
Depreciation rate h slope: χ_1	0.00354	$\frac{r^H \cdot h_{60}}{r^H \cdot h_{25}} = 1.29$	PSID
Hum cap. shock std.: σ_η	0.756	$std(\Delta \log \widetilde{inc}) = 0.15$	Krebs (2003)

Source: Baseline model: The target year is 2010. *Notes*: PST $\hat{=}$ Piazzesi, Schneider, and Tuzel (2007). We target the average of the post-Second World War risk-free rates of PST and Shiller (2015).

Piazzesi, Schneider, and Tuzel (2007) report average risky stock and housing returns of 6.94% and 2.52%, respectively. Following Glover, Heathcote, Krueger, and Rios-Rull (2014), we set the risk premium as the difference between the return of an equally weighted portfolio of stocks and housing minus the risk-free rate, yielding roughly 2.2%. The standard deviation of the equally weighted portfolio is 8.41%. The risk premium and the standard deviation act as targets in the calibration of the depreciation rate of physical capital, δ_0^K, and the standard deviation of the TFP shock, σ_ζ, respectively, where we choose the TFP shocks to be log-normally distributed, $\zeta_{t,j,i} \sim \mathcal{N}(0, \sigma_\zeta^2)$. The TFP scaling factor, ψ, is set such that the average return on human capital (net of depreciation) is 12%.

We calibrate the human capital depreciation rate, δ^h, by setting the corresponding parameters, χ_0 and χ_1, such that the model matches observed labor income profiles based on PSID data, as estimated in Ludwig, Schelkle, and Vogel (2012). Figure 1 shows the profiles in the data and of the calibrated steady state in the model. We choose human capital shocks to be normally distributed, $\eta_{t,j,i} \sim \mathcal{N}(0, \sigma_\eta^2)$, and pin down σ_η such that we match observed labor income risk by requiring labor income growth to have an average standard deviation of 0.15, following Krebs (2003). Appendix B.3.2 gives details on how to compute the variance of income growth. Given the distributional assumptions on $\zeta_{t,j,i}$ and $\eta_{t,j,i}$ we get approximate closed form solutions of the portfolio choice problem as shown in appendix B.2.

Figure 1: Labor Income Profile in 2010



Notes: The data profile is based on PSID data averaged from 1960-2010.

Time-specific cohort life-expectancies are computed from mortality rates taken from the Human Mortality Database (2008). Newborn data is taken from the United Nations' population projections (United Nations 2007) and normalized by the US population size in 1960. OECD data is used to set time-age-specific participation rates, $\omega_{t,j}$.

The social security system is switched off for now. It will be introduced in a later version of this paper.

4 Results

4.1 Cross-Sectional Profiles in 2010

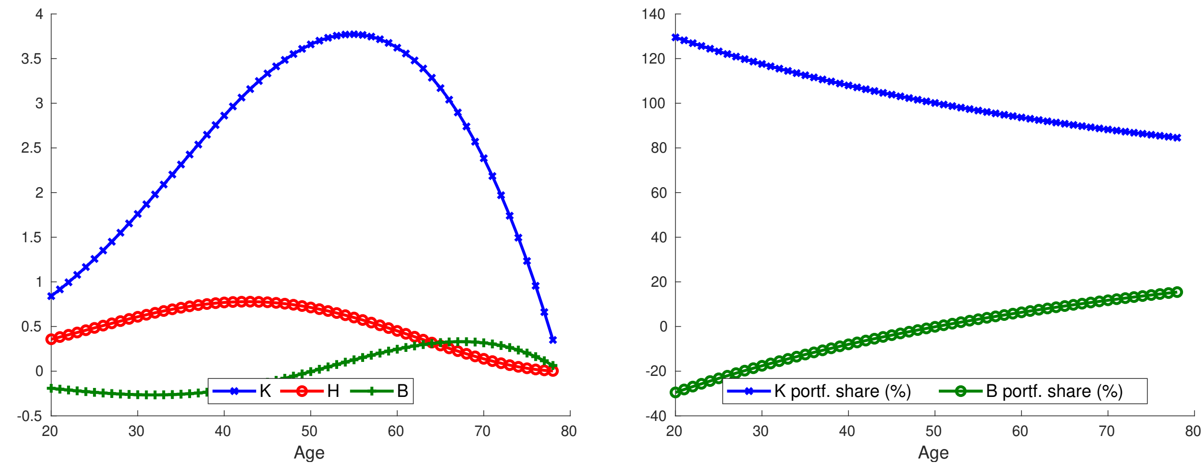
Figure 2 shows cross-sectional age profiles of the model economy in year 2010. The left panel depicts the portfolio allocation of households by age. Households enter their economically relevant lifetime with zero financial assets but positive human capital. Human capital follows a hump-shaped pattern over the working life which results in a corresponding pattern in the age-earnings profile (as shown earlier in figure 1). This is a target in the calibration.

The share of financial asset holdings in physical capital, $\alpha^K_{t,j}$, is shown in the right panel. This share is defined as $\alpha^K_{t,j} = \frac{K_{t,j}}{B_{t,j}+K_{t,j}} = \frac{\hat{\alpha}^K_{t,j} W_{t,j}}{B_{t,j}+K_{t,j}}$. The portfolio share in total wealth (including human capital), $\hat{\alpha}^K_{t,j}$, is roughly constant over the life-cycle. This is a well-known feature of portfolio choice models such as ours, cf., e.g., Campbell and Viceira (2002, ch. 6). Horizon effects in our model arising from the finite horizon of the life-cycle model are small, also see Barberis (2000). In consequence, the dynamics of $\alpha^K_{t,j}$ over the life-cycle are mainly driven by the dynamics of the share of financial wealth in total wealth, $\frac{B_{t,j}+\tilde{K}_{t,j}}{\tilde{W}_{t,j}}$. As households accumulate more and more financial wealth over their life-cycle and because human wealth is decumulated after about age 40, the share of financial wealth in total wealth is increasing over the life-cycle.

4.2 Macroeconomic Aggregates and Asset Returns

We now turn to results on key macroeconomic aggregates and differential asset returns and how they are affected by the demographic transition. Figure 3 summarizes the key summary statistics on the demographic transition for the US economy, which we take as the exogenous driving force. Following the model setup we define the working(-age) population as $\sum_j N_{t,j} \cdot \omega_{t,j}$ and the old-age population as $\sum_j N_{t,j} \cdot (1 - \omega_{t,j})$. The working age to population ratio— the ratio of the working population to the total adult population of age 20 and above—is projected to decrease by about 5 percentage points until 2050. The

Figure 2: Cross-Sectional Profiles in 2010



Assets

Financial Portfolio Shares

Notes: Baseline model in year 2010: Selected average cross-sectional age profiles.

mirror image of this development is the evolution of the old age dependency ratio—the ratio of the old-age population to the working-age population—which is projected to increase by almost 10 percentage points over the same period.

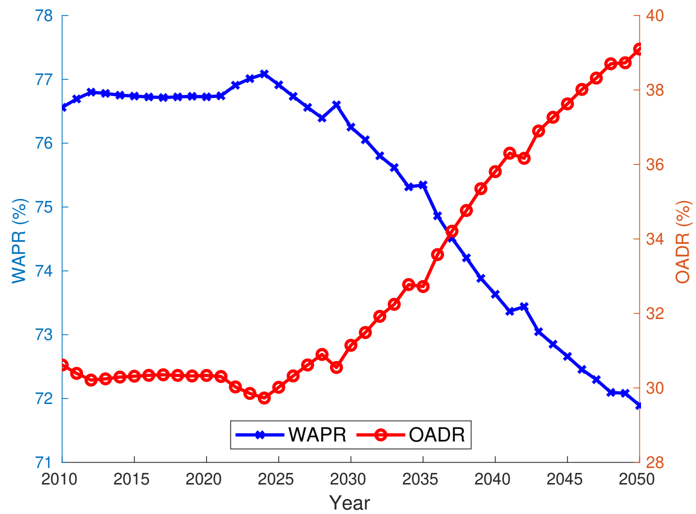
When we analyze the macroeconomic consequences of these developments by using our model with endogenous human capital formation, we distinguish between two variants of our model. As a first variant we consider a specification where we hold constant the human capital shares, $\hat{\alpha}^H_{t,j}$, of year 1960 in all periods. We thereby approximate a model without human capital adjustments.[15] Our second variant is the full-blown model with fully flexible human capital adjustments. The comparison across these two model variants enables us to illustrate the mitigating effects of endogenous human capital adjustments for the dynamics of aggregate measures and asset returns along the demographic transition. As both models are extreme variants—on the one hand, the restricted model likely too severely restricts economic adjustments, on the other hand, the fully endogenous model overstates such adjustments because it does not feature any frictions on the market for human capital—our approach enables us to bracket the likely evolution of these variables over the next decades.

4.2.1 Macroeconomic Aggregates

We first turn in Figure 4 to showing the evolution of the ratio of physical capital to output, K/Y, and of human capital to output, H/Y. K/Y is shown in blue (left scales) in the two panels of the figure whereas H/Y is depicted on the right scales as red lines. Conventional

[15] To preserve the closed form solutions in our model, we have to fix $\hat{\alpha}^H_{t,j}$ and cannot directly hold constant $h_{t,j}$. We therefore speak of an approximation to a model with constant human capital.

Figure 3: Working Age-to-Population Ratio



Source: Own calculations based on United Nations (2007) and Human Mortality Database (2008).
Notes: The working age-to-population ratio equals the working-age population ($\sum_j N_{t,j} \cdot \omega_{t,j}$) over the total population of the model economy. The old-age-dependency-ratio equals the old-age population ($\sum_j N_{t,j} \cdot (1 - \omega_{t,j})$) over the working-age population.

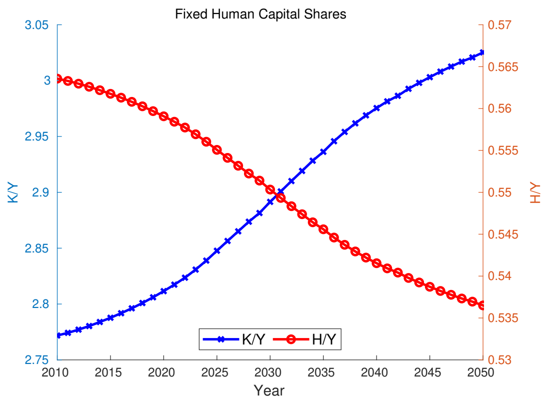
analyses suggest that aging induces a relative shortage of labor and a relative abundance of physical capital in the economy. This leads to an increase in the physical capital-output ratio and a decrease in the human capital-output ratio. Our model with fixed human capital shares, shown in Panel (a) of the figure, is in line with these conventional analyses. It predicts an increase of the capital output ratio from 2.77 to 3.02 until 2050, a change of roughly 9%. Likewise the human capital to output ratio decreases substantially. With fully endogenous human capital adjustment, shown in Panel (b), we observe very mild changes in these variables over time. The endogenous adjustment of human capital dampens the reduction of the human capital to output ratio because human capital shares per worker go up. This also stabilizes output which in turn dampens the increase of the physical capital to output ratio.

4.2.2 Asset Returns

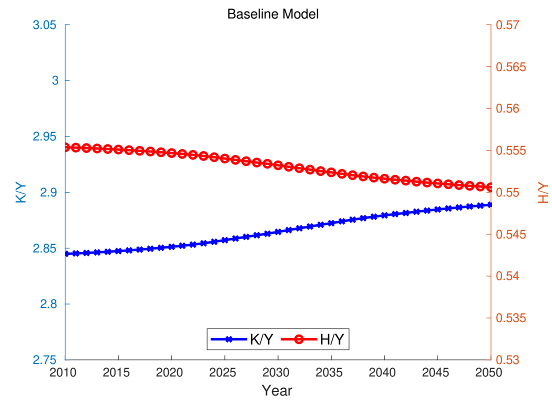
We now turn to the key quantitative question of this paper, namely how demographic change affects the risky and risk-free rate. Figure 5 accordingly plots the time paths for returns to physical capital (blue lines, left scale), returns to bonds (green lines, left scale) and returns to human capital (red lines, right scale). While human capital returns increase over time, both returns to physical capital and bonds decline over time. Again effects are much larger in

Figure 4: Macroeconomic Aggregates: Physical and Human Capital

(a) Fixed Human Capital Shares



(b) Flexible Human Capital Adjustments

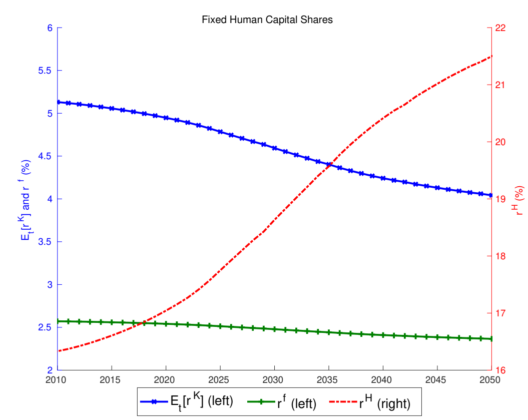


Notes: Model with fixed human capital shares, at respective values in 1960, in Panel (a) and flexible human capital adjustments in Panel (b).

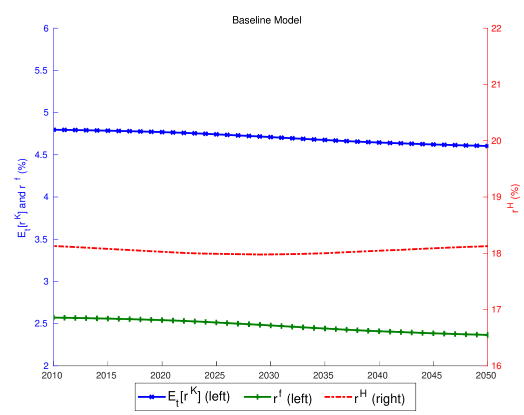
the model with constant human capital shares. These predictions suggest that low financial market returns can be expected for (many) decades to come, irrespective of the risk nature of the asset.

Figure 5: Return to Equity and Risk-Free Rate

(a) Fixed Human Capital Shares



(b) Flexible Human Capital Adjustments



Notes: Model with fixed human capital shares, at respective values in 1960, in Panel (a) and flexible human capital adjustments in Panel (b).

A Theoretical Appendix

A.1 Sub-period life expectancy

As sub-period life expectancy we understand fractional final years. E.g. a life expectancy of 77.4 has a 0.4 sub-period in the final period. In order to capture that policy rules of an individual born at period t are constructed as a weighted average of the policy rules of an individual with maximum age $77 - 20 = 57$ and another with maximum age $78 - 20 = 58$. The weight ψ_t in this example is 0.4. Hence $m_{j,t} = \psi_{t-j} \cdot m^1_{t,j} + (1 - \psi_{t-j})m^2_{t,j}$, $\hat{\alpha}^{K\prime}_{j,t} = \psi_{t-j} \cdot \hat{\alpha}^{K\prime,1}_{t,j} + (1 - \psi_{t-j})\hat{\alpha}^{K\prime,2}_{t,j}$ and $\hat{\alpha}^{H\prime}_{j,t} = \psi_{t-j} \cdot \hat{\alpha}^{H\prime,1}_{t,j} + (1 - \psi_{t-j})\hat{\alpha}^{H\prime,2}_{t,j}$, where $m^1, \hat{\alpha}^{K\prime,1}, \hat{\alpha}^{H\prime,1}$ are the policy functions of an individual born at t with maximum age J_t and $m^2, \hat{\alpha}^{K\prime,2}, \hat{\alpha}^{H\prime,2}$ the policy functions of an individual born at t with maximum age $J_t + 1$.

A.2 Solution of the Household Problem

Proof of Proposition 1. We guess that $v = m^l \cdot \tilde{x}$ where l is some parameter to be determined below and m is the deterministic marginal propensity to consume out of $\tilde{x}$ and show below that this is indeed true. From the guess it follows that

$$
\begin{aligned}
v &= \max_{\tilde{c}, \tilde{x}', \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + \widehat{\beta} \cdot (\mathbb{E}[(m^{\prime l} \cdot \tilde{x}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}} \text{ s.t. } \tilde{x}' = \frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}) \cdot (1 + \widehat{r}') \\
&= \max_{\tilde{c}, \widehat{\alpha}^{s\prime}, \widehat{\alpha}^{h\prime}} \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot m^{\prime l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}}
\end{aligned}
$$

Next, we compute the first-order conditions (FOCs) with respect to $\tilde{c}, \widehat{\alpha}^{k\prime}, \widehat{\alpha}^{h\prime}$:

- FOC with respect to consumption:

$$
\begin{aligned}
0 =& \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot m^{\prime l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta} - 1} \\
& \cdot \{\frac{1-\theta}{\gamma} \cdot \tilde{c}^{\frac{1-\theta-\gamma}{\gamma}} - \frac{1-\theta}{\gamma \cdot (1+g)} \cdot (\frac{1}{1+g} \cdot (\tilde{x} - \tilde{c}))^{\frac{1-\theta-\gamma}{\gamma}} \cdot \widehat{\beta} \cdot m^{\prime l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}}\} \\
\tilde{c} =& (\tilde{x} - \tilde{c}) \cdot (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m^{\prime l \cdot \frac{1-\theta}{1-\theta-\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}
\end{aligned}
$$

  Defining $n := \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m^{\prime l \cdot \frac{1-\theta}{1-\theta-\gamma}} \cdot (\mathbb{E}[(1 + \widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$, $o := (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}$, and $m := \frac{o \cdot n}{1 + o \cdot n}$, where m is a deterministic function of age and time only, we get

$$\tilde{c} = m \cdot \tilde{x}.$$

- FOC with respect to risky portfolio share:

$$
\begin{aligned}
0 =& \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot m'^{l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\
& \cdot (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot \frac{1}{\gamma} \cdot m'^{l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}-1} \\
& \cdot \mathbb{E}[(1-\theta) \cdot (1+\widehat{r}')^{-\theta} \cdot (r^{k\prime} - r^{f\prime})] \\
0 =& \mathbb{E}[(1+\widehat{r}')^{-\theta} \cdot (r^{k\prime} - r^{f\prime})],
\end{aligned}
$$

- FOC with respect to human capital portfolio share:

$$
\begin{aligned}
0 =& \frac{\gamma}{1-\theta} \cdot \{\tilde{c}^{\frac{1-\theta}{\gamma}} + (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot m'^{l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}}\}^{\frac{\gamma}{1-\theta}-1} \\
& \cdot (\frac{1}{1+g} \cdot (\tilde{x}-\tilde{c}))^{\frac{1-\theta}{\gamma}} \cdot \widehat{\beta} \cdot \frac{1}{\gamma} \cdot m'^{l \cdot \frac{1-\theta}{\gamma}} \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{\gamma}-1} \\
& \cdot \mathbb{E}[(1-\theta) \cdot (1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{h\prime} - r^{f\prime})] \\
0 =& \mathbb{E}[(1+\widehat{r}')^{-\theta} \cdot (\widehat{r}^{h\prime} - r^{f\prime})],
\end{aligned}
$$

What is left is to show that indeed $v = m^l \cdot \tilde{x}$. Using $\tilde{c} = m \cdot \tilde{x}$, $n = \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m'^{l \cdot \frac{1-\theta}{1-\theta-\gamma}} \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$, $m = \frac{o \cdot n}{1+o \cdot n}$, and $o = (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}}$ in u we get:

$$
\begin{aligned}
v &= \left\{ (m \cdot \tilde{x})^{\frac{1-\theta}{\gamma}} + \left(\frac{1}{1+g} \cdot (\tilde{x} - m \cdot \tilde{x}) \right)^{\frac{1-\theta}{\gamma}} \cdot n^{\frac{1-\theta-\gamma}{\gamma}} \right\}^{\frac{\gamma}{1-\theta}} \\
&= \tilde{x} \cdot \left\{ \frac{(o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}}{(1+o \cdot n)^{\frac{1-\theta-\gamma}{\gamma}}} \right\}^{\frac{\gamma}{1-\theta}} = \tilde{x} \cdot m^{\frac{1-\theta-\gamma}{1-\theta}}
\end{aligned}
$$

Hence, $v = m^l \cdot \tilde{x}$ where $l = \frac{1-\theta-\gamma}{1-\theta}$.

Defining $\wp := \mathbb{E}[(m'^{\frac{1-\theta-\gamma}{1-\theta}} \cdot (1+\widehat{r}'))^{1-\theta}]$, Noting that $n = \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}$ the marginal propensity to consume equals:

$$
\begin{aligned}
m &= \frac{o \cdot n}{1+o \cdot n} = \frac{(\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}}{1 + (\frac{1}{1+g})^{\frac{1-\theta}{1-\theta-\gamma}} \cdot \widehat{\beta}^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}} \\
&= \frac{\beta^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}}{1 + \beta^{\frac{\gamma}{1-\theta-\gamma}} \cdot m' \cdot (\mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}}} = \frac{(\beta^{\gamma} \cdot \mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}} \cdot m'}{1 + (\beta^{\gamma} \cdot \mathbb{E}[(1+\widehat{r}')^{1-\theta}])^{\frac{1}{1-\theta-\gamma}} \cdot m'}
\end{aligned}
$$

□

A.3 Derivation of the Aggregate Resource Constraint

Proof of Lemma 1. Using the recursive pension equation (10) in the dynamic budget constraint (12) we get

$$\tilde{b}_{t+1,j+1,i} + \tilde{k}_{t+1,j+1,i} + \frac{h_{t+1,j+1,i}}{1+g} =$$
$$\frac{1}{1+g}\left(\tilde{b}_{t,j,i}(1+r_t^f) + \tilde{k}_{t,j,i}(1+r_{t,j,i}^K) + h_{t,j,i}\left(1+(1-\tau_t)\omega_{t,j}\tilde{r}_t^H - \delta_j^H + \eta_{t,j,i}\right) + (1-\omega_{t,j})\tilde{p}_t - \tilde{c}_{t,j,i}\right).$$

Now, in order to derive the aggregate resource constraint, we take the population weighted sum of above individual budget constraints (pre-multiplied by $(1+g)$). Note that it is understood that we sum over all individuals of each age bin characterized by the idiosyncratic mean-zero shock η and idiosyncratic mean-one shock ζ without making this explicit.[16] We then get

$$\begin{aligned}
(1+g)\cdot \sum_j N_{t,j} \cdot \widetilde{B}_{t+1,j+1} + (1+g)\cdot \sum_j N_{t,j} \cdot \widetilde{K}_{t+1,j+1} + \sum_j N_{t,j} \cdot H_{t+1,j+1} \\
= \sum_j N_{t,j} \cdot \widetilde{B}_j(1+r_t^f) + \sum_j N_{t,j} \cdot \widetilde{K}_j(1+r_t^K) \\
+ \sum_j N_{t,j} \cdot H_{t,j} \cdot \left(1+(1-\tau_t)\omega_{t,j}\tilde{r}_t^h - \delta_j^h + \int \eta_{t,j,i}\, di\right) \\
+ \sum_j N_{t,j} \cdot (1-\omega_{t,j})\tilde{p}_j - \sum_j N_{t,j} \cdot \widetilde{C}_j,
\end{aligned}$$

where $\widetilde{B}_{t,j}, \widetilde{K}_{t,j}, H_{t,j}, \widetilde{C}_{t,j}$ are the generation j averages of $\tilde{b}_{t,j,i}, \tilde{k}_{t,j,i}, h_{t,j,i}, \widetilde{c}_{t,j,i}$, $r_t^K = \int r_{t,j,i}^K di$ and $\int \eta_{t,j,i}\, di = 0$ due to the law of large numbers. Noting that $L_t = \sum_j N_{t,j} \cdot H_{t,j} \cdot \omega_{t,j}$ we

[16] This is possible as the idiosyncratic shocks $\eta_{t,j,i}$ and $\zeta_{t,j,i}$ are independent of $\tilde{k}_{t,j,i}$ and $h_{t,j,i}$.

can rewrite above equation in terms of aggregates as

$$
\begin{aligned}
(1+g)\cdot\widetilde{B}_{t+1} + (1+g)\cdot\widetilde{K}_{t+1} + H_{t+1} - N_{t,0}\cdot(1+g)\cdot\left(\widetilde{B}_{t+1,0} + \widetilde{K}_{t+1,0} + \frac{H_{t+1,0}}{1+g}\right)& \\
= \ \widetilde{B}_t(1+r_t^f) + \widetilde{K}_t(1+r_t^k) + \sum_j N_{t,j}\cdot H_{t,j}(1-\delta_j^h) + L_t(1-\tau_t)\tilde{r}_t^h& \\
+ \sum_j N_{t,j}\cdot(1-\omega_{t,j})\tilde{p}_j - \widetilde{C}_t& \\
\Leftrightarrow \ \left[(1+g)\cdot\widetilde{B}_{t+1} - \widetilde{B}_t(1+r_t^f)\right] + \left[(1+g)\cdot\left(\widetilde{K}_{t+1} - N_{t,0}\cdot\widetilde{K}_{t+1,0}\right) - (1-\delta^k)\widetilde{K}_t\right]& \\
+ \left[H_{t+1} - N_{t,0}\cdot H_{t+1,0} - \sum_j N_{t,j}\cdot H_{t,j}(1-\delta_j^h)\right] - (1+g)\cdot N_{t,0}\cdot\widetilde{B}_{t+1,0} + \widetilde{C}_t& \\
= \ \left[\widetilde{K}_t(r_t^k+\delta^k) + L_t\tilde{r}_t^h\right] + \left[\sum_j N_{t,j}\cdot(1-\omega_{t,j})\tilde{p}_t - L_t\tilde{r}_t^h\tau_t\right]&
\end{aligned}
\tag{31}
$$

The definition of the capital income function and equation (5) imply that $\tilde{k}_{t,j,i}(r_{t,j,i}^K + \delta^k) + \ell_{t,j,i}\tilde{r}_t^H = \tilde{y}_{t,j,i}$ and thus in the aggregate $\widetilde{K}_t(r_t^K + \delta^k) + L_t\tilde{r}_t^H = \tilde{Y}_t$. Note, that $\sum_j N_{t,j}\cdot\widetilde{B}_{t+1,j+1} = \widetilde{B}_{t+1} - N_{t,0}\cdot\widetilde{B}_{t+1,0}$, $\sum_j N_{t,j}\cdot\widetilde{K}_{t+1,j+1} = \widetilde{K}_{t+1} - N_{t,0}\cdot\widetilde{K}_{t+1,0}$ and $\sum_j N_{t,j}\cdot H_{t+1,j+1} = H_{t+1} - N_{t,0}\cdot H_{t+1,0}$ as $\tilde{w}_0 > 0$. Using this together with the bond supply equation (20), pension system financing (21) and definitions for physical capital investment, $\widetilde{I}_t^K = (1+g)\cdot\left(\widetilde{K}_{t+1} - N_{t,0}\cdot\widetilde{K}_{t+1,0}\right) - (1-\delta^k)\widetilde{K}_t$, and human capital investment, $I_t^H = H_{t+1} - N_{t,0}\cdot H_{t+1,0} - \sum_j N_{t,j}\cdot H_{t,j}(1-\delta_j^h)$ in equation (31) yields the aggregate resource constraint

$$
\widetilde{I}_t^K + I_t^H - (1+g)\cdot N_{t,0}\cdot\widetilde{B}_{t+1,0} + \widetilde{C}_t = \widetilde{Y}_t.
$$

□

B Computational Appendix

B.1 Solving for the equilibrium

Let $\Phi = \{\phi_t\}_{t=1}^T$ with $\phi_t = \{\tilde{r}_t^h, r_t^f\}$, $Q = \{Q_t\}_{t=1}^T$ with $Q_t = \{\widetilde{B}_t, \widetilde{K}_t, L_t\}$. Collect equations (20) and (24) and rearrange them so that the risk-free rate and the return on human capital are on the left-hand side. This gives us a two-dimensional equation in prices that we call institutional side, $\Phi = IS(Q) = \{IS_t(Q)\}_t$, where

$$\phi_t = IS_t(Q) = \begin{cases} r_t^f = \frac{(1+g)\widetilde{B}_{t+1}}{\widetilde{B}_t} - 1 \\ \tilde{r}_t^h = (1-\alpha)\psi\left(\frac{\widetilde{K}_t}{L_t}\right)^\alpha \mathbb{E}\left[\zeta^{\frac{1}{\alpha}}\right]^\alpha \end{cases} \tag{32}$$

The other side is the household side $Q = HS(P) = \{HS_t(P)\}_t$, where we collect equations (26), (28) and additionally the equation for aggregate physical capital holdings, yielding

$$Q_t = HS_t(\Phi) = \begin{cases} \widetilde{B}_{t+1} = \frac{1}{1+g}\sum_j N_{t,j}\cdot\tilde{X}_{t,j}\cdot(1-m_{t,j})\cdot(1-\hat{\alpha}_{t,j}^{K\prime}-\hat{\alpha}_{t,j}^{H\prime}) - \sum_j N_{t+1,j}\widetilde{ps}_{t+1,j+1} \\ \widetilde{K}_{t+1} = \frac{1}{1+g}\sum_j N_{t,j}\cdot\tilde{X}_{t,j}\cdot(1-m_{t,j})\cdot\hat{\alpha}_{t,j}^{K\prime} \\ L_{t+1} = \sum_j N_{t,j}\cdot\tilde{X}_{t,j}\cdot(1-m_{t,j})\cdot\hat{\alpha}_{jt}^{H\prime}\cdot\omega_{t+1,j+1}, \end{cases} \tag{33}$$

where $\{m_{t,j}, \hat{\alpha}_{t,j}^K, \hat{\alpha}_{t,j}^H\}_{t,j}$ are given by the policy functions (17) and (18) and $\widetilde{ps}_{t,j}$ is recursively given by equations (10) and (11).

$\{\tilde{X}_{t,j}\}_{t,j}$ is determined by the aggregate law of motion (27) with starting values given by equation (29). In order to determine the starting values $\{\tilde{X}_{t,0}\}_t$, the initial pension stock $\widetilde{ps}_{t,0}$ needs to be determined, which is done using equation (9), which in its detrended form and with pension payments substituted, i.e. $\tilde{p}_t = \rho_t \cdot \frac{(1-\tau_t)\cdot\tilde{r}_t^H\cdot L_t}{\sum_j N_{t,j}\cdot\omega_{t,j}}$, is

$$\begin{aligned} \widetilde{ps}_{t,0} &= \sum_{s=0}^{J_{t-j}+1}(1-\omega_s)\tilde{p}_{t+s}\prod_{q=0}^{s}\left(1+r_{t+q}^f\right)^{-1} \\ &= \sum_{s=0}^{J_{t-j}+1}(1-\omega_s)\rho_{t+s}\cdot\frac{(1-\tau_{t+s})\cdot\tilde{r}_{t+s}^H\cdot L_{t+s}}{\sum_j N_{t,j}\cdot\omega_{t,j}}\prod_{q=0}^{s}\left(1+r_{t+q}^f\right)^{-1}. \end{aligned} \tag{34}$$

But $\{L_t\}_t$ is itself a function of $\widetilde{ps}_{t,0}$ as it depends on $\{\tilde{X}_{t,j}\}_{t,j}$, which itself depends on the initial pension stock. Hence we have to solve jointly for $\{L_t\}_t$ and $\{\widetilde{ps}_{t,0}\}_t$. We do this by a fixed point iteration iterating over equation (34) for the initial pension stock and the cash-on-hand accumulation equation (27), computing $\{L_t\}_t$ using equation (28b). Note that the household problem, i.e. determining $\{m_{t,j}\}_{t,j}$ and the portfolio shares, is independent of $\{\widetilde{ps}_{t,0}\}_t$. Thus, in the above fixed point iteration we do not have to recompute the household problem in every iteration. We only iterate over equations (34) and (27), given the values for $\{m_{t,j}\}_{t,j}$ and the portfolio shares. Instead of using equation (34) we can equivalently determine $\widetilde{ps}_{t,0}$ by using equation (10) recursively backwards together with the final condition (11).

The equilibrium is then given by the fixed point $\Phi^\star$ in

$$\Phi^\star = IS(HS(\Phi^\star)).$$

We determine $\Phi^\star$ by using the Gauss-Seidel algorithm with one-parameter dampening where each iteration n is given by

$$\Phi^{n+1} = \Phi^n - w(\Phi^n - IS(HS(\Phi^n))),$$

with w being the dampening factor.

Let steady state values be denoted by a bar. Furthermore, initial (final) steady state values are denoted by a superscript 1 (2). See below for a detailed description of the steady states. The economy is assumed to start form the initial steady state, i.e. $N_{0,j} = \bar{N}_j^1, \widetilde{X}_{0,j} = \bar{\widetilde{X}}_j^1$ and $\widetilde{B}_0 = \bar{\widetilde{B}}^1$. Additionally we assume prices and laborable human capital after period T to be according to the final steady state, i.e. $\phi_t = \bar{\phi}^2$ for $t \in \{T+1, \dots, T+J_T+1\}$.

B.1.1 Initial Pension Stock

The pension payments depend on the aggregate labor supply. But aggregate labor supply also depends on the pension payments, on the initial pension stock. Human capital holdings depend on the initial wealth, which depends on the initial pension stock. Hence, we have to jointly solve for the aggregate labor supply and the initial pension stock. As shown here, we get a closed form solution for this.

Iterating over the average cash-on-hand accumulation equation (15) we get cash-on-hand as a function of initial wealth:

$$\widetilde{W}_{t,j} = \frac{\prod_{s=1}^{j}(1 - m_{t-s,j-s}) \cdot (1 + \mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t-s,j-s,i}\right])}{(1+g)^j} \widetilde{W}_{t-j,0},$$

where

$$\widetilde{W}_{t,0} = \frac{h_0}{1+g} + \widetilde{ps}_{t,0}.$$

is the average initial wealth level. Plugging this into the aggregate labor supply as given in equation (28b) we get

$$L_t = \sum_{j=0}^{J+1} N_{t,j} \cdot \frac{\prod_{s=1}^{j}(1 - m_{t-s,j-s}) \cdot (1 + \mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t-s,j-s,i}\right])}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^H \cdot \omega_{t,j} \cdot \left(\frac{h_0}{1+g} + \widetilde{ps}_{t-j,0}\right), \quad (35)$$

where we define $\prod_{s=1}^{j}(1-m_{t-s,j-s})\cdot(1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t-s,j-s,i}\right])\equiv 1$. This gives us aggregate labor supply L_t as a function of the initial pension stock $\widetilde{ps}_{t-(J+1)}$ to $\widetilde{ps}_t$. Note that L_t is linear in the initial pension stocks.

Equation (9) give us the initial pension stock as

$$\widetilde{ps}_{t,0}=\sum_{j=0}^{J_t+1}(1+g)^j\cdot(1-\omega_{t+j,j})\tilde{p}_{t+j}\cdot\prod_{s=0}^{j}\left(1+r_{t+s}^f\right)^{-1}.$$

Pension market clearing implies that pension benefits are given by

$$\tilde{p}_t=\frac{\tau_t\cdot\tilde{r}_t^H\cdot L_t}{\sum_j N_{t,j}(1-\omega_{t,j})},$$

yielding

$$\widetilde{ps}_{t,0}=\sum_{j=0}^{J_t+1}(1+g)^j\cdot(1-\omega_{t+j,j})\cdot\frac{\tau_{t+j}\tilde{r}_{t+j}^H}{\sum_s N_{t+j,s}(1-\omega_{t+j,s})}\cdot L_{t+j}\cdot\prod_{s=0}^{j}\left(1+r_{t+s}^f\right)^{-1} \tag{36}$$

Combining equations (35) and (36) we get the equilibrium quantities of L_t and $\widetilde{ps}_{t,0}$. Defining

$$\mathcal{A}_{t,j}=N_{t,j}\cdot\frac{\prod_{s=1}^{j}(1-m_{t-s,j-s})\cdot(1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{t-s,j-s,i}\right])}{(1+g)^{j-1}}\cdot\hat{\alpha}_j^H\cdot\omega_{t,j}$$

$$\mathcal{B}_{t,j}=(1+g)^j\cdot(1-\omega_{t+j,j})\cdot\frac{\tau_{t+j}\tilde{r}_{t+j}^H}{\sum_s N_{t+j,s}(1-\omega_{t+j,s})}\cdot\prod_{s=0}^{j}\left(1+r_{t+s}^f\right)^{-1}$$

the condition for labor supply is

$$L_t=\sum_{j=0}^{J+1}\mathcal{A}_{t,j}\cdot\frac{h_0}{1+g}+\sum_{j=0}^{J+1}\mathcal{A}_{t,j}\sum_{q=0}^{J_{t-j}}\mathcal{B}_{t-j,q}\cdot L_{t-j+q}. \tag{37}$$

Above equation gives us labor supply as a difference equation, where in general current labor supply depends both on past and future labor supply. If, once retired, households do not work anymore, current labor supply only depends on future labor supply.

Steady State: In steady state equations (35) and (36) simplify to

$$L = N \cdot \sum_{j=0}^{J+1} \frac{\prod_{s=1}^{j}(1-m_{j-s}) \cdot (1+\mathbb{E}_{\eta,\zeta}[\hat{r}_{j-s,i}])}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^H \cdot \omega_j \cdot \left(\frac{h_0}{1+g} + \widetilde{ps}_0\right) \tag{38}$$

$$\widetilde{ps}_0 = \frac{\tau \cdot \tilde{r}^H}{1+r^f} \cdot \frac{1}{N} \cdot \frac{\sum_{j=0}^{J+1}\left(\frac{1+g}{1+r^f}\right)^j (1-\omega_j)}{\sum_{j=0}^{J+1}(1-\omega_j)} \cdot L, \tag{39}$$

resulting in the equivalent of equation (37) giving us aggregate labor supply as

$$L = \frac{N \cdot \left(\sum_{j=0}^{J+1} \frac{\prod_{s=1}^{j}(1-m_{j-s})\cdot\left(1+\mathbb{E}_{\eta,\zeta}[\hat{r}_{j-s,i}]\right)}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^H \cdot \omega_j\right)}{1 - \left(\sum_{j=0}^{J+1} \frac{\prod_{s=1}^{j}(1-m_{j-s})\cdot\left(1+\mathbb{E}_{\eta,\zeta}[\hat{r}_{j-s,i}]\right)}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^H \cdot \omega_j\right) \cdot \frac{\tau \cdot \tilde{r}^H}{1+r^f} \cdot \frac{\sum_{j=0}^{J+1}\left(\frac{1+g}{1+r^f}\right)^j (1-\omega_j)}{\sum_{j=0}^{J+1}(1-\omega_j)}} \cdot \frac{h_0}{1+g}. \tag{40}$$

If the denominator is not positive, no steady state equilibrium exists. For a discussion see REFER TO MY PAPER.

With subperiod life-expectancy: As discussed in the appendix A.1 we model subperiod life-expectancy by having two types of households in the model with different life-expectancies. In the implementation of the model we have to take account of that, resulting in slightly different equations than above. Here only the equations for the steady state are reported. Let $\ell \in 1, 2$ denote the type of household.

The equivalent of equations (38) and (39) are

$$L = \sum_{\ell=1}^{2} \left\{ N^\ell \cdot \sum_{j=0}^{J^\ell+1} \frac{\prod_{s=1}^{j}(1-m_{j-s}^\ell) \cdot \left(1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{j-s,i}^\ell\right]\right)}{(1+g)^j} \cdot (1+g) \cdot \hat{\alpha}_j^{H,\ell} \cdot \omega_j \cdot \left(\frac{h_0}{1+g} + \widetilde{ps}_0^\ell\right) \right\}$$

$$\widetilde{ps}_0^\ell = \frac{\tau \cdot \tilde{r}^H}{1+r^f} \cdot \frac{\sum_{j=0}^{J^\ell+1}\left(\frac{1+g}{1+r^f}\right)^j (1-\omega_j)}{N^1 \sum_j^{J^1+1}(1-\omega_j) + N^2 \sum_j^{J^2+1}(1-\omega_j)} \cdot L,$$

where the second equation results from pension market clearing implying

$$\tilde{p}_t = \frac{\tau_t \cdot \tilde{r}_t^H \cdot L_t}{N^1 \sum_j^{J^1+1}(1-\omega_j) + N^2 \sum_j^{J^2+1}(1-\omega_j)}.$$

The equivalent of (40) is thus

$$L = \frac{\sum_{\ell=1}^{2}\left(N^{\ell}\cdot\sum_{j=0}^{J^{\ell}+1}\frac{\prod_{s=1}^{j}(1-m_{j-s}^{\ell})\cdot\left(1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{j-s,i}^{\ell}\right]\right)}{(1+g)^{j}}\cdot(1+g)\cdot\hat{\alpha}_{j}^{H,\ell}\cdot\omega_{j}\right)}{1-\sum_{\ell=1}^{2}\left\{\left(N^{\ell}\cdot\sum_{j=0}^{J^{\ell}+1}\frac{\prod_{s=1}^{j}(1-m_{j-s}^{\ell})\cdot\left(1+\mathbb{E}_{\eta,\zeta}\left[\hat{r}_{j-s,i}^{\ell}\right]\right)}{(1+g)^{j}}\cdot(1+g)\cdot\hat{\alpha}_{j}^{H,\ell}\cdot\omega_{j}\right)\cdot\frac{\tau\cdot\tilde{r}^{H}}{1+r^{f}}\cdot\frac{\sum_{j=0}^{J^{\ell}+1}\left(\frac{1+g}{1+r^{f}}\right)^{j}(1-\omega_{j})}{N^{1}\sum_{j}^{J^{1}+1}(1-\omega_{j})+N^{2}\sum_{j}^{J^{2}+1}(1-\omega_{j})}\right\}}\cdot\frac{h_{0}}{1+g}.$$

B.1.2 Steady State

A steady state is characterized by $N_{t,j} = \bar{N}_j$ $\phi_t = \bar{\phi} = \{\bar{\tilde{r}}^h, \bar{r}^f\}$, $Q_t = \bar{Q} = \{\bar{\tilde{B}}, \bar{\tilde{K}}, \bar{H}\}$, $\widetilde{X}_{t,j} = \bar{X}_j$, $m_{jt} = \bar{m}_j$, $\hat{\alpha}_{t,j}^K = \bar{\hat{\alpha}}_j^K$ and $\hat{\alpha}_{t,j}^H = \bar{\hat{\alpha}}_j^H$ for all t.[17] Steady state analogues of equations (32) and (33) are

$$\bar{\phi} = IS(\bar{Q}) = \begin{cases} \bar{r}^f = g \\ \bar{\tilde{r}}^h = (1-\alpha)\psi\left(\frac{\bar{\tilde{K}}}{\bar{L}}\right)^{\alpha}\mathbb{E}\left[\zeta^{\frac{1}{\alpha}}\right]^{\alpha} \end{cases} \tag{41}$$

and

$$\bar{Q} = HS(\bar{\phi}) = \begin{cases} \bar{\tilde{B}} = \frac{1}{1+g}\sum_j \bar{N}_j\cdot\bar{\tilde{X}}_j\cdot(1-\bar{m}_j)\cdot(1-\bar{\hat{\alpha}}_j^{K\prime}-\bar{\hat{\alpha}}_j^{H\prime}) - \sum_j N_j\widetilde{ps}_{j+1} \\ \bar{\tilde{K}} = \frac{1}{1+g}\sum_j \bar{N}_j\cdot\bar{\tilde{X}}_j\cdot(1-\bar{m}_j)\cdot\bar{\hat{\alpha}}_j^{K\prime} \\ \bar{L} = \sum_j \bar{N}_j\cdot\bar{\tilde{X}}_j\cdot(1-\bar{m}_j)\cdot\bar{\hat{\alpha}}_j^{H\prime}\cdot\bar{\omega}_{j+1}. \end{cases}$$

The steady state equilibrium is then given by

$$\bar{\phi}^{\star} = IS(HS(\bar{\phi}^{\star})),$$

which is also solved by using the Gauss-Seidel algorithm.

The Initial steady state corresponds to constant fertility rates, $\bar{f}_j^1 = f_{0,j}$, life expectancy $\bar{J}^1 = J_0$, working time shares $\bar{\omega}_j^1 = \omega_{0,j}$ as of the initial period. The final steady state corresponds to these quantities as of the final period, i.e. $\bar{f}_j^2 = f_{T,j}$, $\bar{J}^2 = J_T$ and $\bar{\omega}_j^2 = \omega_{T,j}$.

B.2 Approximate Solution of the Portfolio Choice Problem

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[17]Furthermore the distribution of cash-on-hand holdings across households, $\{\tilde{x}_{t,j,i}\}$, is constant across time. But this distributions does not have any effect on aggregate prices and quantities.

B.3 Calibration

We calibrate the steady state equilibrium. In steady state the risk-free rate is pinned down by the growth rate and the moment condition on the capital-output ratio pins down the capital-labor ratio and thus the return on human capital. This can be seen by referring to equation (41) and by combining equations (24) and (25) yielding

$$\frac{\widetilde{K}_t}{L_t} = \left(\psi \mathbb{E}\left[\zeta^{\frac{1}{\alpha}}\right]^{\alpha} \frac{\widetilde{K}_t}{\widetilde{Y}_t}\right)^{\frac{1}{1-\alpha}}.$$

Hence, the moment conditions pin down the equilibrium prices as given in equation (41).

Given these prices we search for the parameter values such that the moment conditions are met. This entails computing the household model and aggregating resulting quantities. But we do not need to solve for the equilibrium, as we already set prices to equilibrium prices.

Two parameters, ψ and σ_{ζ}^2, can even be set without having to compute the household model. To see that we plug the market clearing condition for the return on human capital from equation (41) into the return on physical capital as given in equation (6) yielding

$$r_{t,j,i}^K = \alpha\psi\zeta_{t,j,i}^{\frac{1}{\alpha}} \mathbb{E}\left[\zeta_{t,j,i}^{\frac{1}{\alpha}}\right]^{\alpha-1} \left(\frac{\widetilde{K}}{\widetilde{L}}\right)^{\alpha-1} - \delta^K$$

Hence

$$\begin{aligned}
\mathbb{E}\left[r_{t,j,i}^K\right] &= \alpha\psi \mathbb{E}\left[\zeta_{t,j,i}^{\frac{1}{\alpha}}\right]^{\alpha} \left(\frac{\widetilde{K}}{\widetilde{L}}\right)^{\alpha-1} - \delta^K \\
&= \alpha\frac{\widetilde{Y}}{\widetilde{K}} - \delta^K.
\end{aligned}$$

and

$$\begin{aligned}
Var\left(r_{t,j,i}^K\right) &= \left(\alpha\psi \mathbb{E}\left[\zeta_{t,j,i}^{\frac{1}{\alpha}}\right]^{\alpha-1} \left(\frac{\widetilde{K}}{\widetilde{L}}\right)^{\alpha-1}\right)^2 \cdot Var\left(\zeta_{t,j,i}^{\frac{1}{\alpha}}\right) \\
&= \left(\alpha \mathbb{E}\left[\zeta_{t,j,i}^{\frac{1}{\alpha}}\right]^{-1} \frac{\widetilde{Y}}{\widetilde{K}}\right)^2 \cdot Var\left(\zeta_{t,j,i}^{\frac{1}{\alpha}}\right).
\end{aligned}$$

Thus, the moment condition on the capital-output ratio plus the ones on the expectation and variance of the physical capital return pin down ψ and σ_{ζ}^2.

B.3.1 Setting the Demographic Model Parameters

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B.3.2 Determining Average Variance of Income Growth

Using the definition of cash-on-hand, its transitional equation (15), $\frac{h_{t,j,i}}{1+g} = \hat{\alpha}^H_{t,j}$ and $c_{t,j,i} = m_{t,j} x_{t,j,i}$ we have

$$h_{t+1,j+1,i} = \frac{1}{1+g} \frac{\hat{\alpha}^H_{t+1,j+1}}{\hat{\alpha}^H_{t,j}} (1 - m_{t,j})(1 + \hat{r}_{t,j,i}(\eta_{t,j,i})) h_{t,j,i},$$

where $\hat{r}_{t,j,i}(\eta_{t,j,i})$ is the portfolio return as given in equation (14), explicitly denoted as a function of the human capital shock $\eta_{t,j,i}$. Let $\widetilde{inc}_{t,j,i} = \omega_j \tilde{r}^H_t h_{t,j,i}$ denote the (de-trended) gross labor income. Hence, the growth rate of labor income, as approximated by the difference in logs is

$$\begin{aligned} \Delta \log \widetilde{inc}_{t+1,j+1,i} &= \Delta \log(\omega_{j+1} \tilde{r}^H_{t+1}) + \Delta \log h_{t+1,j+1,i} \\ &= \Delta \log(\omega_{j+1} \tilde{r}^H_{t+1}) - \log(1+g) + \log \frac{\hat{\alpha}^H_{t+1,j+1}}{\hat{\alpha}^H_{t,j}} + \log(1 - m_{t,j}) + \log(1 + \hat{r}_{t,j,i}(\eta_{t,j,i}, \zeta_{t,j,i})) \\ &\approx D_{t,j} + E_{t,j} (\zeta_{t,j,i})^{\frac{1}{\alpha}} + F_{t,j} \eta_{t,j,i}, \end{aligned}$$

where $D_t = \Delta \log(\omega_{j+1} \tilde{r}^H_{t+1}) - \log(1+g) + \log \frac{\hat{\alpha}^H_{t+1,j+1}}{\hat{\alpha}^H_{t,j}} + \log(1 - m_{t,j}) + (1 - \hat{\alpha}^K_{t,j} - \hat{\alpha}^H_{t,j}) r^f_t - \hat{\alpha}^K_{t,j} \delta^K + (1+g)(1 + (1 - \tau_t)\omega_j \tilde{r}^H_t - \delta^H_j) - 1$, $E_{t,j} = \hat{\alpha}^K_{t,j} \alpha (1-\alpha)^{\frac{1-\alpha}{\alpha}} \left(\psi (\tilde{r}^H_t)^{\alpha - 1} \right)^{\frac{1}{\alpha}}$ and $F_{t,j} = (1+g) \hat{\alpha}^H_{t,j}$. The last line results from the approximation $\log(1+x) \approx x$ and the equations for portfolio, physical capital and human capital returns. Hence, variance of income growth of an individual aged j in period t is

$$Var(\Delta \log \widetilde{inc}_{t+1,j+1,i}) \approx E^2_{t,j} \sigma^2_{\zeta^{\frac{1}{\alpha}}} + F^2_{t,j} \sigma^2_\eta + 2 E_{t,j} F_{t,j} \sigma_{\zeta\eta}.$$

The average variance at time t is thus

$$\begin{aligned} \overline{Var(\Delta \log \widetilde{inc}_{t+1,j+1,i})} &= \frac{1}{N} \sum_j N_{t,j} Var \left(\Delta \log \widetilde{inc}_{t+1,j+1,i} \right) \\ &\approx \frac{1}{N} \sum_j N_{t,j} \left(E^2_{t,j} \sigma^2_{\zeta^{\frac{1}{\alpha}}} + F^2_{t,j} \sigma^2_\eta + 2 E_{t,j} F_{t,j} \sigma_{\zeta\eta} \right) \end{aligned}$$

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