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Safety First Approach in a Life Cycle Model

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Abstract

Since it was proposed by Roy (1952), the idea of safety first has seen an increasing popularity, especially in case of retirement. This thesis solves a life cycle model of consumption and portfolio choice in the presence of a safety first constraint on the intermediate consumption. Because of the absence of a closed-form solution for the optimization problem, we impose an assumption that the investors are myopic and we use an approximation based on the log-linearization method in Campbell and Viceira (1999) and Viceira (2001). The model predicts the incorporation of the safety first idea into the individual's decision making has an impact on both the optimal asset allocation and consumption. The safety first concern depresses the consumption motive and raises the conservativeness of the portfolio choice, which partially provides an explanation for the " limited stock holding".

Keywords: Life cycle; Safety first; Optimal consumption; Optimal portfolio choice.

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1. Introduction

Starting from the late 1960's with papers by Merton (1969) and Samuelson (1969), there have been an increasing number of papers on individual optimal saving and investment over the life cycle. Earlier studies predict that the optimal fraction to equity should either be a constant (Merton, 1969), or decline with age (Bodie et al, 1992), which coincides with the well known strategy suggested by financial advisors that investors should place (100-age) % of their wealth in a well-diversified equity portfolio (Malkiel, 1996). However, the predictions of these models are still at odds with the empirical evidence on stock holdings, namely the limited stock market participation (Bertaut and Haliassos, 1995, Blume and Zeldes, 1993), a hump-shaped life cycle pattern (Ameriks and Zeldes, 2001, Faig and Shum, 2002, Heaton and Lucas, 2000, and Poterba and Samwick, 2001), and a significant heterogeneity of stockholdings (Curcuru, Heaton, Lucas and Moore, 2006). Therefore, many theorists have been inspired to develop optimization models that capture important features of reality, such as stochastic labor income, borrowing constraints and transaction costs, or preferences other than the Expected Utility family. But the literature does not reach a consensus and the gap has not yet been filled completely, which still calls for new researches to bridge the theoretical models and the real world.

The idea of safety first was first proposed by Roy (1952). According to his idea, individuals consider outcomes below a certain value as a "disaster". He argues that when making decisions about uncertain prospects, individuals are intended to minimize the probability of reaching disaster. Based on Roy's argument, Levy and Levy (2009) examine the idea of "safety first" both theoretically and experimentally. They conclude the important role of safety first in agents' decision-making.

Moreover, safety first is empirically meaningful, especially in case of retirement. As suggested by Bodie (2008), a minimum income guarantee should be the default option in retirement saving and investment. Guarantees create certainty, which has a profound impact on the practice since less financial education is needed and the information cost is lower. Besides, an on-going discussion on redesigning Dutch pension schemes advocates that the pension plan should switch from Defined Contribution to Defined Benefit as people age – a real application of safety first idea into the life cycle.

This thesis aims to contribute to the literature by combining the safety first approach with the life cycle model. It will focus on answering the two core questions: what is the optimal portfolio allocation and what is the optimal consumption path over the life span, for the standard investors and for the safety first investors, respectively. By a comparison of the individuals' decisions, we are able to reach the conclusion on whether the incorporation of the safety first approach into the individual's decision making has an impact on the optimal asset allocation and consumption over the lifespan, which might further provide an explanation for the limited stock participation.

The literature (for example, Davis and Willen, 2006) state that the dynamic optimization under a borrowing constraint has no closed-form solution. An optimum can only be obtained when the numerical methods are used. Moreover, the introduction of an extra nonlinear safety first constraint adds more complications to solving the problem, so that even the numerical methods will be much more difficult to execute. Given the consideration of the computational feasibility and that the focus of this thesis is to investigate the role of safety first in agent's decision making, we simplify the dynamic optimization problem into a sequence of two period optimizations, by assuming that investors are myopic, having a planning period of one year. This assumption is generated in the light of Bernartzi and Thaler (1995), who propose the idea of myopic investor and point out that individuals have a short evaluation horizon. Of course, it also brings a limitation to the model: the solution is not an optimum throughout the life cycle, but only a static and periodical optimization.

In our model, at each decision period, individuals are supposed to make their decision based on the consideration on the current period and one period ahead. Such a decision process is rolled over until the agents' life end. We further impose an extra constraint for the safety first investors: to control the probability of reaching disaster, that is, the consumption-wage ratio falling below the exogenously determined reference point, to be smaller than a critical level. Due to the nonlinearity of the safety first constraint, no exact analytical solution is available, while a numerical method is time consuming and inefficient. Therefore, we follow Campbell and Viceira (1999) and Viceira (2001), and use a log-linearization approximation method to work out an analytical solution.

According to our model, safety first does influence both the optimal consumption and optimal portfolio choice. When having concerns on the tail risk of relative consumption to income, agents become more risk averse and allocate a less proportion of their financial wealth to the risky asset. Thus, on average, safety first investors have less equity participation. Note that the optimal portfolio choice stays constant over time, because the investment opportunity set is not time varying.

In addition, agents are reluctant to consume at the current period so as to accumulate enough wealth for ensuring a certain level of consumption in the next period. Given our assumption on the decision horizon, which is one year ahead, agent starts making adjustment for their retirement at age 64. However, the safety first investors demonstrate a completely different consumption behavior form the standard investors. Instead of investing, safety first investors would rather consume more and have a smaller exposure to financial market, while standard investors consume less and save more financial wealth to invest, which can hedge the loss of income. The opposite decision is actually driven by the safety first concern of the investors: the portfolio composition is fixed over time, thus as less financial wealth is invested in the financial market, the absolute amount that is allocated into the risky asset is less, which results in smaller risk exposure.

Nevertheless, the model has its limitations. First of all, neither wage risk nor longevity risk is included in our analysis, for the sake of simplicity. Secondly, we assume a simple setting of the financial market, which does not incorporate the mean reversion of the equity premium or the stochastic interest rate. Last but not least, our result is not robust to the value of some parameters, which leaves room for future research.

The thesis will be presented in the following structure. Section 2 gives a literature review on the life cycle model and the safety first approach. We present our model and solve the optimal path analytically in Section 3. Section 4 visualizes the optimal consumption and portfolio choice when the model is calibrated. Finally, we conclude in Section 5.

2. Literature Review

In this section, we will briefly give a literature review of life cycle models and applications of the safety first approach.

2.1 Life cycle

Merton's continuous finance theory (1969, 1971, 1973, 1975) establishes a general framework which discusses the optimal consumption and portfolio choice over the lifetime. Given certain model assumptions[1], it is claimed that the demand for the risky assets is constant over time. One of the crucial, but unrealistic assumptions is the absence of labor income. Bodie et al. (1992) further incorporates the flexibility of labor supply into the individual's decision making. They conclude that the fraction of an individual's financial wealth optimally invested in equity should "normally" decline with age, which coincides with the well known strategy suggested by financial advisor that investors should place (100-age) % of their wealth in a well-diversified equity portfolio (Malkiel, 1996).

However, these predictions are inconsistent with the empirical evidence. First, it is well documented that a large fraction of the US population holds little or no stocks (Bertaut and Haliassos 1995, Blume and Zeldes, 1993). Secondly, several studies report that typically the risky asset holding is low at young ages and then either increasing or hump-shaped over the life cycle (see, for example Ameriks and Zeldes, 2001, Faig and Shum, 2002, Heaton and Lucas, 2000, and Poterba and Samwick, 2001). Furthermore, there is a significant heterogeneity of stockholdings among the stock market participants (Curcuru, Heaton, Lucas, and Moore, 2006).

In response, great efforts have been made to reveal the portfolio problem with a focus on 1) incomplete markets in which background risks can affect portfolio rules, and 2) alternative specifications of preference.

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[1] Asset returns are independently and identically distributed, agents have hyperbolic or constant absolute risk aversion utility function, no labor income and markets are frictionless and complete.

2.1.1 Incomplete market

The incompleteness of a market can stem from several sources, namely stochastic labor income, a limitation of individuals' market trading activity, for example, short sale constraints, transaction cost and illiquidity, etc.

Wage risk

In a life cycle asset allocation context, labor income plays a crucial role since it might result in wealth accumulation, and the level and risk of the labor income varies over the lifespan, which to a large extent provides the rationale for age-dependent portfolio choice.

Apart from Bodie et al. (1992), other papers which consider the labor income risk include Campbell et al. (2001), Cocco et al. (2005), Davis and Willen (2000), Haliassos and Michaelides (2003), Viceira (2001), etc. These studies show that labor income can act as a substitute for risk-free asset holdings, which makes the individuals in the pursuit of more aggressive investment in the risky assets so as to acquire a sufficient amount of risk. Only counterfactually high correlations between shocks to labor income and stock returns, or the possibility of disastrous labor income shocks (see, for example, Cocco et al., 2005) can explain the limited stock market participation.

However, the labor income specification in these models may be unnecessarily restrictive (Benzoni et al., 2007), because the instantaneous correlation between stock returns and the changes to labor income at the aggregated level is insignificant, resulting in a low long-time correlation as well. On the contrary, Benzoni et al. (2007) specify aggregate labor income to be cointegrated with dividends and predict a hump-shaped equity holding over the life cycle. Because of the long run cointegration of labor income and stock market performance, the human capital has pronounced stock-like features and commands a higher discount rate for young agents, whereas it acquires bond-like properties and thus is discounted at a lower rate for older agents. Their recent work (Benzoni and Chyruk, 2009) further discusses how long-run labor income risk helps to explain the limited stock market participation puzzle. Yet, not all the literature reaches the same conclusions that the long-run correlation of shocks to labor income and stock returns is positive and high. For instance, the model in Lustig and Van Nieuwerburgh

(2006) indicates that the innovations in human wealth and financial asset returns are negatively correlated.

Transaction Cost

A large literature has suggested that small entry costs can be consistent with the observed low stock market participation rates. For example, Vissing-Jorgensen (2002) documents the evidence of structural state dependence in the stock market participation decision, supporting the importance of fixed transactions costs. Gomes and Michaelidis (2005) introduce a fixed entry cost for agents that want to invest in risky assets for the first time and find that, to some extent, this constrains young households from investing in the stock market. More prudent investors are the ones who participate in the stock market as they accumulate more wealth and, therefore, have strong incentives to pay the fixed entry cost. However, their model still counterfactually predicts that young agents that have already paid the participation cost invest most of their portfolio in equities, which leaves the limited participation puzzle unexplained.

Borrowing Constraints and Housing

Another way to explain the limited or non participation in the stock market is the borrowing and short sale constraints. As reported by, for instance, Jappelli (1990) and Duca and Rosenthal (1993), there is a declining incidence of binding borrowing constraints with age. Thus, young agents can hardly borrow against the collateral of their human capital due to the adverse selection and moral hazard problem. Moreover, when their borrowing rates are higher than the expected equity return, their demand for equity holding is discouraged (Davis and Willen, 2006).

Cocco et al. (2005) examine the life cycle portfolio implications of endogenous borrowing limits and document that investors with a bounded income process hold negative wealth when young and do not invest in equities. Davis and Willen (2006) conclude that realistic borrowing costs dramatically reduce equity holdings. On the contrary, Bovenberg et al. (2007) discuss the case with an exogenous borrowing constraint and find the restricted access of younger workers to capital markets harms their welfare because it constrains not only risk taking but also intertemporal consumption smoothing for young agents.

Other researches also examine the crowding-out effect from housing when the investor is borrowing constrained. Flavin and Yamashita (2002) suggest that young agents typically have large holdings of real estate relative to their net worth, which creates a highly leveraged position for them. This negatively affects the agents' risk tolerance and forces them to use their net worth to either pay down their mortgage or buy bonds instead of buying stocks. Cocco (2005) shows that house price risk crowds out stockholdings, and this crowding out effect is larger for lower financial net-worth. Yao and Zhang (2005) incorporate a rental market for housing services and examine the optimal dynamic portfolio decision for both owners and renters. They find that when being indifferent between renting and owning, the homeowner holds a lower equity proportion in his financial net worth as a result of the substitution effect between housing and stocks. Yao and Zhang (2008) introduce costly refinancing of existing mortgage and argue that the investor with a high housing value-net worth ratio has a large portion of his wealth tied to illiquid home equity. Hence, they reduce the fraction of net worth invested in stocks when borrowing against stocks or human capital is not possible. However, their predicted net worth equity proportion and liquid wealth equity proportion are still higher than the observed counterparts.

2.1.2 Alternative Preference Specification

So far, the majority of researches in this field use the standard Expected Utility, for example, the Constant Relative Risk Aversion (CRRA) utility. One disadvantage of this representation is that it links risk preferences with time preferences. Specifically, the coefficient of relative risk aversion (RA) is the reciprocal of the elasticity of intertemporal substitution (EIS). What this implies is that if an individual is averse to variation of consumption across different states at a particular point of time then he will also be averse to consumption variation over time. However, there is no fundamental economic reason why this must be so (Mehra and Prescott, 2003). Therefore, an alternative venue to explain the "stockholding puzzle" is to explore various specifications of individuals' preferences, which are not in the standard Expected Utility family.

Gomes and Michaelides (2005) take advantage of the separation of risk aversion coefficient and elasticity of intertemporal substitution (Epstein and Zin, 1991) to build up a heterogeneous agent model, which predicts that households with low RA and low EIS smooth idiosyncratic earnings shocks with a small buffer stock of assets and most of them never invest in equities. Another way of changing the classical preference structure is to introduce habit formation. Gomes and Michaelides (2003) consider the internal ratio-habit formation preference in a life cycle model. They conclude that households increase wealth accumulation early in life because the presence of the habit term, which leads to a stronger incentive to smooth consumption. Thus, the habit formation model decreases its ability to match the observed empirical regularities. On the contrary, Polkovnichenko (2007) explores the implication of additive and internal habit formation preferences and shows that young investors should hold more conservative portfolios than middle-aged investors because they have not yet accumulated enough wealth to sustain consumption sufficiently above habit.

In addition to the standard utility functions, which are smooth and twice differentiable, the prospect theory developed by Tversky and Kahneman (1992) suggests a kinked utility function, with the kink at a reference point. Unlike the preceding two specifications, the kinked utility function has not been adequately investigated in a life cycle framework. However, it has been well incorporated and applied in the asset pricing and optimal portfolio choice literature. Bernatzi and Thaler (1995) argue that people are reluctant to invest in stocks because of a combination of loss aversion and a short planning period, thus requiring a higher risk premium on equity. Barberis, Huang, and Santos (2001) explore a setting in which investors derive direct utility from not only consumption but also from fluctuations in their financial wealth. Berkelaar, Kouwenberg, and Post (2004) also document the substantial effect of loss aversion on the optimal investment strategy. However, their findings that the initial portfolio weight of stocks of a loss-averse investor typically increases with the investment horizon are still at odds with the empirically limited participants of young agents.

2.2 Application of Safety First

The idea of safety first was first proposed by Roy (1952). According to his idea, individuals consider outcomes below a certain value as a "disaster". He argues that when making decisions about uncertain prospects, the individuals' first consideration is to minimize the probability of reaching disaster.

Since then, the idea of safety first started to appear in various strands of the literature. Telser (1955) gives the first extension and application of the safety first idea and investigates the economic implication on hedging. Pyle and Turnovsky (1970) study the relationship between safety first criteria and standard mean-variance optimization. They find, in the absence of a riskless asset, that a correspondence can be established between the safety first criterion and expected utility maximization when that maximization results in concave indifference curves in the mean-standard deviation space. In addition, Levy and Sarnat (1972) also examine the comparison between expected utility and the safety first principle. Arzac and Bawa (1977) develop optimal portfolio choice using the safety first rules. Their conclusion is that the CAPM seems to be robust to safety first investors under the assumption of a normal return distribution. All in all, the early works on safety first approaches highlight its generalization potential and relevance to the traditional decision making criteria under uncertainty, for example, the stochastic dominance criteria (Bawa, 1978).

Later, several studies extend the idea of safety first into a multi-period setting. Goetzmann and Broadie (1992) combine elements of the single period safety first idea with multiperiod insurance strategies like constant proportion portfolio insurance (CPPI) and time invariant portfolio protection (TIPP) and demonstrate how a dynamic insurance program can be implemented within a mean-variance framework. Milevsky (1999) extend the classical results of Samuelson (1969) and Merton (1971), which are derived under conventional utility assumptions, to an individual optimizing a safety first objective function. He explores the effect of the investment horizon on the asset allocation choice of a safety first investor. Stutzer (2003) develops a new investment decision framework related to safety first rules: the investor maximizes the probability that the growth rate of the invested wealth will exceed some pre-determined target growth rate.

Given the burst of financial crises, the safety first idea again attracts a lot of attention. Based on Roy’s argument, Levy and Levy (2009) examine “safety first” both theoretically and experimentally. They propose an expected utility-safety first (EU-SF) model where decisions are made based on a weighted average of the safety first criterion and standard expected utility maximization.

3. Model

In this section, we are going to present different life cycle models to make a comparison. We start with the theoretical benchmark model in subsection 3.1, which is consistent with the dynamic optimization in a continuous time setting. In subsection 3.2, we illustrate a simple two period model with a safety first constraint on the intermediate consumption.

3.1 Theoretical Benchmark Model

Following Merton (1969) and Bodie et al. (1992), one of the most straightforward ways of modeling life cycle optimization problem is to use a continuous time setting, for the reason that a closed form solution on the basis of a dynamic optimization is available, if there is no additional constraint other than the budget constraint. (For example, when the borrowing constraint is imposed, only numerical solution is achievable.) Based on Huang, Milevsky, and Wang (2008), we will first examine the continuous model as our theoretical benchmark model,

In this section, we make the following assumptions: individuals are utility maximizers. The utility of the individual takes the form of a standard power utility, which has a constant relative risk aversion coefficient of γ. The individual's objective is to maximize the discounted lifetime expected utility. Let β stand for the subjective time discount factor. C_t is the consumption in period t. Agents start working at time t, retire at time R, and live up to time T. Then the objective function is given by:

$$E_t\left[\int_t^T e^{-\beta s}U(C_t)ds\right], \text{ where } U(C_t)=\frac{C_t^{1-\gamma}}{1-\gamma}.$$

Individuals make decision on consumption and investment. In the financial market, there are only two assets that individuals can choose from, a riskless and a risky one. We assume they can be traded continuously without any transaction cost or tax. For the riskless asset, with price B_t, the return is a constant r_f. The initial price of the riskless asset is normalized to 1, i.e., $B_0 = 1$. At each period, the payoff is generated by the product of the risk free rate and the initial price, which is expressed in the following equation:

$$dB_t = r_f B_t dt .$$

The price S_t of a risky asset follows a geometric Brownian Motion, with a drift μ and volatility σ. The only uncertainty is generated by dZ, which is a standard Wiener process. In the simulation, we start with an initial price of 100 for the risky, i.e., $S_0 = 100$.

$$dS_t = \mu S_t dt + \sigma S_t dZ .$$

Furthermore, we assume that there exists a riskless non-capital gain Y_t from current time t to retirement period R, which can be regarded as labor income, with a constant growth rate of g (for the sake of simplicity, we assume g is larger than the risk free rate r_f), satisfying

$$dY_t = gY_t dt, \text{ for } t < R .$$

Define W as an individual's financial wealth and let ω be the fraction of savings that is invested in the risky asset. Then the dynamics of the financial wealth reads:

$$dW_t = [(1-\omega_t)r_f + \omega_t\mu]W_t dt + (Y_t - C_t)dt + \sigma\omega_t W_t dZ , \tag{1}$$

for $t < R$, and

$$dW_t = [(1-\omega_t)r_f + \omega_t\mu]W_t dt - C_t dt + \sigma\omega_t W_t dZ , \tag{2}$$

for $R \le t \le T$.

The first part on the right hand side of (1) is the expected return on the investment, the second part is the change in wealth from saving, and the last term reflects the volatility of the risky asset. The equation for the retirement is almost the same, except that in retirement, no labor income will be realized ($Y_t = 0$). They are the budget constraints of the maximization problem for pre-retirement and post-retirement period, respectively.

Since the wealth dynamics of the post-retirement period in equation (2) is just a special case of the pre-retirement dynamic in (1), where $Y_t = 0$, I will focus on the optimal solution under the budget constraint (1).

To apply the stochastic dynamic programming, the value function is defined as:

$$J(W,Y,t) \quad \equiv \quad \max E_t\left[\int_t^T e^{-\beta s}U(C_t)ds\right],$$

which is subject to the constraint (1) and (2), when the individuals are in the employment and the retirement period, respectively.

The control variables are the consumption C_t and the portfolio allocation parameter ω_t, while the state variables are the current value of total wealth W_t and the wage Y_t. Then we derive the Hamilton-Jacobi-Bellman (HJB) differential equation to solve the dynamic optimization (see Appendix A for the detailed derivation).

Applying Ito's lemma, the optimality conditions of $J(W,Y,t)$ is then formulated in terms of the HJB equation (3). Time subscripts are dropped for simplicity, and the remaining subscripts denote partial derivatives.

$$0 = \max\left\{e^{-\beta t}U(C) + J_W\left[[(1-\omega)r_f + \omega\mu]W - C + Y\right] + J_t + J_Y gY + 0.5\omega^2\sigma^2W^2J_{WW}\right\} \quad (3)$$

Based on the HJB equation, we can further compute the optimum for the control variable C_t and ω_t .We take the first order condition (FOC) on the control variables of the HJB equation (3), which gives

$$\begin{aligned}(C_t^*)^{-\gamma} &= e^{\beta t}J_W \\ \omega^*W &= -\frac{J_W}{J_{WW}}\frac{\mu - r_f}{\sigma^2}.\end{aligned} \quad (4)$$

Thus, substituting the optimal value for the control variables in (4) into the HJB condition (3) yields:

$$0 = \frac{\gamma}{1-\gamma}e^{-\beta t}(e^{\beta t}J_W)^{\frac{\gamma-1}{\gamma}} + J_W(rW+Y) + J_t + J_Y gY - 0.5\frac{J_W^2}{J_{WW}}\frac{(\mu - r_f)^2}{\sigma^2} \quad (5)$$

Observe the presence of the J function in (5), the form of the J function should be obtained in order to work out the optimal condition. Motivated by the closed-form solution for the constant wage used in Merton (1971), we assume that the J function takes the following form, which is common in the literature (for example, Huang, Milevsky, and Wang, 2008).

$$J(W,Y,t)=\frac{h(t)[W+k(t)Y]^{1-\gamma}}{1-\gamma}. \tag{6}$$

After guessing the form of the J function, we substitute (6) into (5) and derive the exact expression for the J function. For the employment period,

$$J(W,Y,t)=e^{-\beta t}\left(\frac{e^{A(T-t)}-1}{A}\right)^{\gamma}\frac{[W_t+\frac{1}{g-r_f}(1-e^{(g-r\ )(R-t)})Y_t]^{1-\gamma}}{1-\gamma}\text{, when } t<R,$$

and for post-retirement period, it is

$$J(W,Y,t)=e^{-\beta t}\left(\frac{e^{A(T-t)}-1}{A}\right)^{\gamma}\frac{W_t^{1-\gamma}}{1-\gamma}\text{, when } R\leq t<T.$$

where

$$A=-\frac{\beta}{\gamma}+\frac{1-\gamma}{\gamma}\left(r_f+0.5\frac{(\mu-r_f)^2}{\gamma\sigma^2}\right).$$

Given the formula for the derived utility function $J(W,Y,t)$, the corresponding optimal consumptions and allocation to the risky asset are determined. According to (4), we have

$$\begin{aligned} C_t^* &= \frac{A}{e^{A(T-t)}-1}\left[W_t+\frac{1}{g-r_f}(1-e^{(g-r_f)(R-t)})Y_t\right], \\ \omega_t^* W_t &= \left[W_t+\frac{1}{g-r_f}(1-e^{(g-r_f)(R-t)})Y_t\right]\frac{(\mu-r_f)}{\gamma\sigma^2}, \end{aligned} \tag{7}$$

for $t<R$, and

$$\begin{aligned} C_t^* &= \frac{A}{e^{A(T-t)}-1}W_t, \\ \omega_t^* W_t &= W_t\frac{(\mu-r_f)}{\gamma\sigma^2}, \end{aligned} \tag{8}$$

for $R\leq t<T$.

One caveat worthy our attention is that the borrowing constraint is not imposed in this model, which may result in an unrealistically high consumption at the beginning of the agent's career. As a consequence, the optimal decision for a young worker will be borrowing money to sustain the desired level of consumption, which may cause a negative financial wealth. In fact, a stream of literature tries to illustrate this issue by introducing a borrowing constraint into the life cycle model (see the literature review in section 2.1.1). However, a more realistic constraint comes at the expense of the computational simplicity: a constraint on borrowing adds the complications of the model, which leads to no closed-form solution. As a consequence, an optimum is only available when a numerical method is conducted, which is more time consuming (Davis and Willen, 2006). Given that the focus of this thesis is to investigate the role of safety first in agent's decision making, we will only discuss the intuition for this benchmark model, and no simulation or the graphical comparison will be provided.

We first discuss the optimal choice for the retirement period. It is notable that, for the post-retirement case, when there is no (wage) income, the optimal portfolio choice is just the same as that in Merton's problem (1971), with a constant proportion of the financial wealth invested in the risky asset. Since there is no income after retirement, we cannot derive any expression for the consumption-income ratio. Instead, only the consumption-to-financial wealth ratio is available, which is an increasing function of time. The reason is that, during retirement, when there is no wage income, agents' consumption is getting more and more heavily dependent on their financial wealth. Since the financial wealth will decrease, the increasing proportion provides evidence for the individual's incentive of consumption smoothing.

On the other hand, the optimal choice is more complicated for the employment period. We rearrange the expression in (7) to derive the optimal consumption-income ratio and optimal portfolio composition in the form of the wealth-income ratio:

$$\begin{aligned} \frac{C_t^*}{Y_t} &= \frac{A}{e^{A(T-t)}-1}\left[\frac{W_t}{Y_t}+\frac{1}{g-r_f}(1-e^{(g-r_f)(R-t)})\right] \\ \omega_t^* &= \frac{(\mu-r_f)}{\gamma\sigma^2}+\frac{1}{g-r_f}(1-e^{(g-r_f)(R-t)})\frac{(\mu-r_f)}{\gamma\sigma^2}\frac{Y_t}{W_t} \end{aligned} \tag{7'}$$

According to (7'), the portfolio choice before retirement is time dependent. More precisely, the allocation of the risky asset is a linear function of the inversed wealth-income ratio. The intercept is indeed the same as the optimal portfolio choice in the retirement period, when labor income diminishes to zero. Agents in the employment period are intended to invest an additional amount to the risky asset, which depends on the inversed wealth-income ratio. The slope is positive and increasing as time approaches to retirement (the growth rate of wage is assumed to be larger than the risk free rate). The intuition is simple. When people get closer to the end of employment, they become more sensitive to the proportion between their incomes relative to their financial wealth in making the investment decision. All in all, the optimal allocation of financial wealth to the risky asset contains a constant portfolio, which is regardless of the income, and a hedge portfolio against the changes in the human capital, with respect to the financial wealth.

As to the optimal consumption, the first expression in (7) indicates that it is a time varying proportion of the total wealth, which consists of the financial wealth and the human capital of the agents. The proportion also increases over time, once more implying an increasing dependence of consumption on the total wealth when the agent ages. Rewriting the expression of optimal consumption path in (7) gives the first expression in (7'). The pre-retirement consumption-income ratio is a linear function of wealth-income ratio. As the slope is positive (since A is negative, both the nominator and denominator are negative) and time dependent, the effect of financial wealth on the optimal choice of consumption reinforces over time. It is due to the fact that before retiring, the human capital of an individual has been realized gradually while the financial wealth has been accumulated, thus, the effect of the financial wealth becomes more dominating.

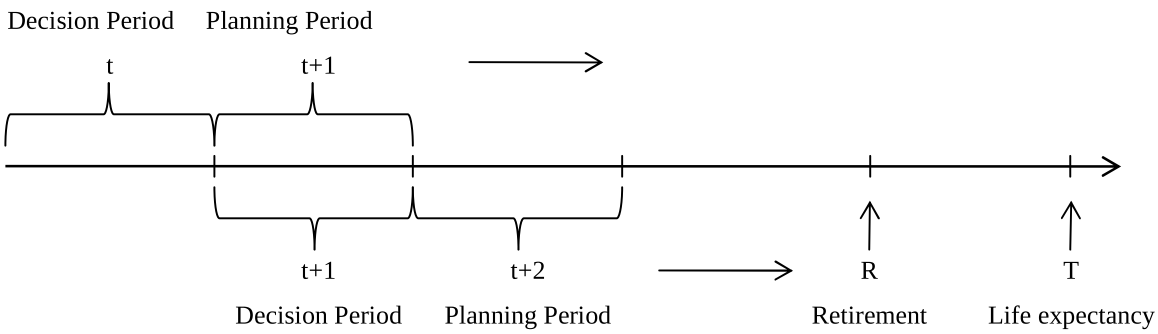
3.2 Two Period Model

As discussed in the previous section that the dynamic optimization under a borrowing constraint has no closed-form solution, there is no doubt that after holding an extra and nonlinear safety first constraint, even the numerical method will be much more difficult to execute. Based on the consideration of the computational feasibility, we simplify the dynamic optimization problem by putting forward the following assumption.

We assume each investor is myopic, having a planning period (one year) and a perfect foresight for his planning period. This implies that when making a decision, he only considers the current and the next period. The actual consumption or investment of the next period may be different from his plan or expectation. Figure 1 illustrates the rolling planning procedure of the myopic investors. Starting from period t, when the agent enters the employment period, he makes the decision on the consumption and portfolio allocation for the decision year (current year), based on the consideration of the decision year (current year) and the planning year (next year). When he moves to the next year, the same decision process is repeated, his decision period is then $t+1$, and his planning period is rolled over to period $t+2$. We assume that the decision making process will continue up to his death at period T, and even during retirement, he retains this habit for decision making.

Figure 1 Rolling Two Period Model

This figure demonstrates the decision making process of the agents, which is rolled over repeatedly throughout the life time.



Through such an assumption, we are able to simplify the optimization problem over the life cycle into a sequence of two period optimization problems. Indeed, this assumption is generated in the light of Bernartzi and Thaler (1995), who propose the idea of a myopic investor and point out that individuals have a short evaluation horizon. Of course, this assumption brings a limitation to the model: the solution is not an optimum throughout the life cycle, but only a static and periodical optimization.

The assumptions on the individuals' working life are the same as that in section 3.1, except that we assume that individuals are entitled to receive a constant annual endowment (or pension income) during their retirement, which is equivalent to a specified replacement rate η times the wage income of the last working period,

$$Y_t = \eta Y_{R-1}, \; R \le t < T.$$

Following the same assumption of the utility function, the individual's objective in the two period model is to maximize the sum of the utility on the consumption in the first period, and the expected value function of available wealth in the second period. Here M_{t+1} stands for all the wealth that the agent has in the second period.

$$\frac{C_t^{1-\gamma}}{1-\gamma} + \beta E_t \frac{M_{t+1}^{1-\gamma}}{1-\gamma}. \tag{9}$$

During the first period, agents receive labor income Y_t, which is riskless and has a growth rate of G until retirement. After substracting the consumption for that period, C_t, agents allocate their saving $Y_t - C_t$ to the financial assets. A borrowing constraint is imposed to reflect more realistic problems for the individuals, especially the younger. In the second period, they are capable to use all the wealth M_{t+1}, which consists of a realization of labor income Y_{t+1}, and the accumulated gain on the financial market. Let the return on the portfolio that they have chosen in the first period yield a gross return of R_{t+1}^p. Thus, the agents are subject to the intertemporal budget constraint:

$$(C_{t+1} =) M_{t+1} = (Y_t - C_t) R_{t+1}^p + Y_{t+1}, \tag{10}$$

$$\text{with} \quad Y_{t+1} = (1+G) Y_t. \tag{11}$$

The assumption of the financial market is the same as section 3.1. Denote R_f as the one period gross return on the riskless asset, which is equal to e^{r_f} when it is continuously compounded, and let R_{t+1}^e be the gross return of the risky asset. Using Ito's lemma, the return of the risky asset R_{t+1}^e is log-normally distributed. In addition, the mean and the variance of its log return, defined as $r_{t+1}^e = \log(R_{t+1}^e)$, is $\mu - 0.5\sigma^2$ and σ^2.

Assume ω is the proportion of savings invested in the risky asset at time t. The one period return on the portfolio is given by:

$$R_{t+1}^p = R_f + \omega (R_{t+1}^e - R_f). \tag{12}$$

Last but not least, safety first investors are subject to a safety-first constraint. From the definition of safety first, people consider the outcome of an uncertain prospect under a certain value as a disaster, and they want to minimize the probability of reaching disaster. Therefore, there is a large variety of the format of the safety first constraint. In our case, we impose the safety first constraint on the intermediate consumption, which takes the form

$$\Pr_t\left(\frac{C_{t+1}}{Y_{t+1}} < c\right) \le \alpha . \tag{13}$$

Here c refers to an exogenously determined reference point, and α is the investor's critical level on the probability of reaching disaster. A disaster is defined as the situation when the consumption-wage ratio in the planning period falls below the reference point. Agents' safety first preference is reflected in their intention to control the probability of reaching disaster to be lower than the critical level α. In order to draw the most common conclusion, we further impose α to be smaller than 0.5, which means that agents only care about the left tail risk.

In fact, the safety first constraint is formulated similarly to a value-at-risk constraint. Inspired from the application of value-at-risk constraint in the asset pricing field (for instance, Basak and Shapiro, 2001; Krokhmal, Palmquist, and Uryasev, 2001), we choose this specification to keep consistent with the literature and to use the techniques that have already been developed.

3.2.1 Log-Linear Approximation Method

There are no exact closed-form solutions for this two period optimization problem due to the presence of non-linear terms. Solving for the optimal policies requires some numerical methods, but it will be extremely time consuming when the two-period model is rolled over. Therefore, we choose to follow Campbell and Viceira (1999) and Viceira (2001), and use a log-linear approximation method to solve the problem.

Similar to Campbell and Viceira (1999), the problem can be log-linearized because the assumptions on preferences, labor income, and the investment opportunity set ensure that the consumption and savings to invest are strictly positive. The log-linear approximation to the intertemporal budget constraint is given by (see Appendix B for the detailed derivation)

$$k + g + \rho_{t+1}(c_{t+1} - y_{t+1}) - \rho_t(c_t - y_t) - r_{t+1}^P \approx 0 \tag{14}$$

Here, lowercase letters denote the variables in logs and $g = \log(1+G)$. Furthermore, k , ρ_{t+1} and ρ_t are the log-linearization constants (see Appendix B forthe details)

$$\rho_t = \frac{\exp[E(c_t - y_t)]}{\exp[E(c_t - y_t)] - 1},\ \rho_{t+1} = \frac{\exp[E(c_{t+1} - y_{t+1})]}{\exp[E(c_{t+1} - y_{t+1})] - 1},$$

$$k = -\rho_{t+1}\log\rho_{t+1} + (\rho_{t+1} - 1)\log(\rho_{t+1} - 1) + (1 - \rho_t)\log(1 - \rho_t) + \rho_t\log(-\rho_t).$$

An approximate expression for the expected log return of the portfolio is also derived according to Campbell and Viceira (1999)

$$E_t r_{t+1}^P \approx r_f + \omega(\mu - r_f) + 0.5\omega(1-\omega)\sigma^2 \tag{15}$$

Combining (14) and (15), we get the log budget constraint in a linear form of asset returns, consumption, and labor income. In addition to the intertemporal budget constraint, we have a safety-first constraint on the planned consumption-wage ratio as formulated in (13). Since the return on the portfolio R_{t+1}^P follows a lognormal distribution and the labor income is nonstochastic, the distribution of the planned consumption-wage ratio C_{t+1}/Y_{t+1} is lognormal. We substitute the expression of the planned consumption-wage ratio in (13) based on the intertemporal budget constraint (10), use the quantile as an alternative expression of the probability, and then take the logarithm on both sides. After several steps of rearrangement (see Appendix B for the derivation), the resulting safety-first constraint is

$$E_t r_{t+1}^P + Z_\alpha\sqrt{Var_t(r_{t+1}^P)} \le \log(c-1) + g - \log[1 - \exp(c_t - y_t)] \tag{16}$$

where Z_α is the α-th quantile of a standard normal distribution.

Observe that the log-term $\log[1 - \exp(c_t - y_t)$ appears again in the expression, we once again apply the log-linearization method to approximate, which will be discussed in section 3.2.3. Before that, we will first discuss the case of a standard investor, who has no safety first constraint.

3.2.2 A Standard Investor's Problem

In this section, we are going to derive the optimal path without including a safety-first constraint, as a benchmark for the two-period model. In case of no safety constraint, the Euler equation of the optimization problem in (9) takes the standard form

$$1 = E_t(\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{t+1}^i),$$

where $i = e, f, P$, standing for returns on risky asset, risk free asset, and the overall portfolio.

Taking the logarithm on both sides, we get the exact formula

$$0 = \log\beta - \gamma E_t(c_{t+1} - c_t) + E_t r_{t+1}^i + \frac{1}{2}Var_t[r_{t+1}^i - \gamma(c_{t+1} - c_t)]. \qquad (17)$$

Substracting the log Euler equation for $i = f$ form that for $i = e$, the following equation holds

$$E_t r_{t+1}^e - r_f + \frac{1}{2}Var_t(r_{t+1}^e) = \gamma Cov_t(c_{t+1} - c_t, r_{t+1}^e). \qquad (18)$$

Moreover, substituting c_{t+1} with the log budget constraint (14) gives

$$E_t r_{t+1}^e - r_f + \frac{1}{2}Var_t(r_{t+1}^e) = \gamma\omega Var_t(r_{t+1}^e),$$

and the optimal portfolio rule

$$\omega^* = \frac{\mu - r_f + \frac{1}{2}\sigma^2}{\gamma\sigma^2} = \frac{\lambda}{\gamma\sigma}, \qquad (19)$$

where $\lambda = \frac{\mu - r_f + \frac{1}{2}\sigma^2}{\sigma}$ stands for the Sharpe ratio of the risky asset.

Since the assumption on the financial market implies that the investment opportunity set is constant, the optimal portfolio allocates a fixed fraction to the risky assets, which is comparable to the well-known Merton's portfolio. When agents get wealthier, the total wealth that they invest in the risky asset will increase accordingly.

The optimal consumption policy is given by the log Euler equation (17) for $i = p$. After substitute $E_t c_{t+1}$ using the log budget constraint (14), we have

$$c_t^* = \frac{1}{\rho_{t+1} - \rho_t} \times \left((1 - \frac{\rho_{t+1}}{\gamma}) E_t r_{t+1}^P - \frac{\rho_{t+1}}{2\gamma} (1 - \frac{\gamma}{\rho_{t+1}})^2 Var_t (r_{t+1}^P) - \frac{\rho_{t+1}}{\gamma} \log \beta - k + (\rho_{t+1} - 1) g \right) + y_t . \quad (20)$$

The optimal log consumption in a two-period model is a linear function of wage income. The slope of optimal consumption with respect to wage income is exactly equal to one, which implies that the consumption-wage ratio is a constant. Note that such a result might be the consequence of the assumptions on the model setup, for example, the investment opportunity set remains constant over periods and there is no stochastic wage.

However, in the last working period, individuals are aware that there will be less income in the next period because their income will be the product of the replacement ratio and their income in the last employment period, i.e., the growth rate of income g becomes negative. As a result, the consumption-to-wage ratio would be lower according to (20). Such a change in the consumption behavior leads to an increased proportion of income that is invested in the risky assets, since the optimal weight remains constant. It is driven by the investors' hedging demand: in the last working period, people want to hedge as much as possible the reduction in income through investment. Nevertheless, the short reaction is purely driven by our assumption that individuals are only capable to forecast one year before. We expect that the solution for the dynamic optimization over the life cycle will predict a much earlier start of an adjustment and preparation for the retirement.

When agents arrive at their retirement, they receive a constant endowment annually. The expression (20) predicts that the consumption-wage ratio increases from the level of age 64 to a certain point, which must be lower than that in the working period because there is no growth in their income. When agents move to the retirement, the hedging demand diminishes because income is stabilized again. Since the portfolio composition is still the same as before, the total investment in the risky assets reduces because the income has been decreased.

3.2.3 A Safety First Investor's Problem

When the constraint in (16) is binding[2], we have

$$E_t r_{t+1}^P + Z_\alpha \sqrt{Var_t(r_{t+1}^P)} = \log(c-1) + g - \log[1-\exp(c_t - y_t)]. \tag{21}$$

Taking into account that the non-linearity in constraint (13) may lead to no analytical solution of the optimal path, we again apply Campbell and Viceira (1999)'s approximation technique, and get

$$E_t r_{t+1}^P + Z_\alpha \sqrt{Var_t(r_{t+1}^P)} - m - \log(c-1) - g + \rho_t(c_t - y_t) = 0. \tag{22}$$

where $m = (1-\rho_t)\log(1-\rho_t) + \rho_t \log(-\rho_t)$.

Using the same approximation as (15), the expression for the mean and the variance of the log portfolio return r_{t+1}^p are obtained, which can be used to derive the formula of c_t

$$E_t r_{t+1}^p \approx r_f + \omega(\mu - r_f) + 0.5\omega(1-\omega)\sigma^2,\ Var_t(r_{t+1}^p) \approx \omega^2 Var_t(r_{t+1}^e) = \omega^2\sigma^2.$$

In order to get the expression of c_{t+1}, we combine (22) with the log-linearized budget constraint (14), and find

$$k + \rho_{t+1} E_t(c_{t+1} - y_{t+1}) + Z_\alpha \sqrt{Var_t(r_{t+1}^P)} - m - \log(c-1) = 0. \tag{23}$$

After obtaining the functions of c_t and c_{t+1} in terms of ω, we plug them into the optimization problem in (9), and then the resulting first order condition with respect to ω is

$$C_t^{1-\gamma}[(\mu - r_f) + 0.5\sigma^2 - \omega\sigma^2 + Z_\alpha\sigma] + \beta E_t C_{t+1}^{1-\gamma} \frac{\rho_t}{\rho_{t+1}} Z_\alpha \sigma = 0. \tag{24}$$

This implies that the optimal portfolio rule is

$$\omega_{SF}^{\ *} = \frac{\beta E_t \dfrac{\rho_t}{\rho_{t+1}} \left(\dfrac{C_{t+1}}{C_t}\right)^{1-\gamma} Z_\alpha}{\sigma} + \frac{\lambda + Z_\alpha}{\sigma}.$$

---

[2] For some certain values of α, for example when α goes to the positive infinity, the constraint can also be non-binding. We will first analyze the normal case when the constraint is binding, and then discuss the reasonable range of α in section 4.5.

According to the Euler Equation, when $i = f$, the following equation holds,

$$E_t(\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}) = \frac{1}{R_f} = e^{-r_f},$$

therefore, the optimal portfolio allocation to risky asset is

$$\omega_{SF}^{\ *} = \frac{\beta^{\frac{1}{\gamma}} \frac{\rho_t}{\rho_{t+1}} \exp(r_f \frac{1-\gamma}{\gamma}) Z_\alpha}{\sigma} + \frac{\lambda + Z_\alpha}{\sigma}. \quad (25)$$

From expression (25), it is only obvious that the optimal portfolio consists of a larger proportion of risky assets when the Sharpe ratio λ of the risky asset increases. Here Z_α is the α-th quantile of a standard normal distribution. Apart from the subjective time discount factor β and risk aversion level γ, we have noticed that the optimal weight to risky asset under the safety first constraint is also largely dependent on the ratio between the rhos in the subsequent years. Before presenting a further analysis, first understanding the intuition of the rho-ratio is worthwhile.

According to the expressions of ρ_{t+1} and ρ_t in (14), they are constants derived from the log-linearization method (see Appendix B for the details), which only depend on the unconditional expectation of the consumption-wage ratio for period $t+1$ and t, respectively. Therefore, they are endogenous. More importantly, the values have different signs, leading to a negative ρ_t / ρ_{t+1}. The reason is as follows: in the first period, the agents' consumption is subject to their wage income, thus $c_t - y_t$ should not be positive and ρ_t should be negative. In the second period, individuals can consume up to their whole wealth M_{t+1} (m_{t+1} is its logarithm), which including their actual consumption (since ρ_{t+1} is *planned* in the first period, it is very well possible to deviate from the actual value in the second period) and savings. Hence, it gives a positive value to ρ_{t+1}. Given the setup of our two period model (constant investment opportunity set, no stochastic wage, and myopic investors), and the intuition of the rho-ratio, we expect it to be a constant over time[3]. In section 4, an empirical analysis for the rho-ratio will provide more comprehensive interpretation.

After analyzing the rho-ratio, we can conclude at this stage that the optimal portfolio weight of financial wealth to risky asset is a constant throughout the individuals' lifespan. Regarding to the one-to-one relationship between the optimal portfolio choice and other endogenous parameters, for instance, γ and α, it is pretty unclear analytically because of the possibility that some parameters are endogenously determined by others. For instance, since labor income is assumed to be riskless, only one risk exists in the market - the equity risk, which determines the uncertainty in the next period's consumption. Therefore, the concern on the tail risk of the planned consumption-wage ratio α may be endogenously related to the constant relative risk aversion γ. Besides, the presence of the rho-ratio makes it even harder to explore the causality effect of these parameters on the optimal portfolio choice. We will leave these discussions to the next section, where the study of the linkage between the optimal weight and other parameters will be conducted numerically.

One important point worthy strengthening is that there should be a reasonable range for the level of the critical value α, since it is a necessity for the safety first constraint (13) (or equivalently, the log-linearized safety first constraint (16), we will refer to (16) in the following discussion) to be binding. By definition, α should have a positive value. It cannot, however, go beyond a certain level, which is the boundary condition for the constraint (16) to bind. In other word, when α is given a sufficiently large value, the left hand side of the expression (16) is not necessary to be lower than or equal to the right hand side. As a result, the safety first constraint does not play a role in the optimization, which makes the case of a safety first investor converges to that of a standard investor. In section 4.5, we will investigate the possible choices of α.

When comparing the optimal portfolio allocation between (19) and (25), i.e., without and with the safety-first constraint, the role of the safety-first constraint in shifting the investors' choice towards risky asset is not obvious. We derive the difference of the fraction in (26) and it is notable that the numerator is not necessarily negative. The sign

[3] Of course the rhos could be time vaying because the consumption-wage ratio or the wealth-wage ratio may change over time. But we expect the ratio to be a constant.

of the numerator is again determined by the magnitude of the rho-ratio, once the values of other parameters are set.

$$\Delta \omega^* = \omega_{SF}{}^* - \omega^* = \frac{\gamma \beta^{\frac{1}{\gamma}} \frac{\rho_t}{\rho_{t+1}} \exp(r_f \frac{1-\gamma}{\gamma}) Z_\alpha + (\gamma - 1)\lambda + \gamma Z_\alpha}{\gamma \sigma} \tag{26}$$

Next, we move to the investigation of the optimal consumption path for the safety first investor. Given the complexity of the expression for the optimal weight, we would rather analyze it qualitatively based on equation (21), instead of presenting an exact expression. All in all, during employment, the consumption-wage ratio is also a constant since the optimal portfolio composition remains unchanged. It is the same as the outcome from the standard investor's problem. The reason is straightforward: according to the financial market setup, the investment opportunity set does not show a time-varying feature. Intuitively, the consumption-wage ratio in the safety first model should have a smaller value than that from the model in section 3.2.2, since people are more cautious about their relative consumption in the second period.

Up to now, all the analyses are still based on the employment period. When retirement is taken into account, the consumption-wage ratio will change, because of people's awareness in the last period of the working life. At age 64, individuals adjust their consumption to prepare for the decrease of income in the next period, when retirement starts. Given expression (21), when the growth rate decreases to zero during the retirement period, or even it becomes negative in the last working period, the consumption-wage ratio in the previous period starts to increase. It implies that people tend to consume more and invest a smaller proportion of their income at age 64, if they want to ensure that they will have a certain level of consumption-wage ratio in the next period.

A comparison between the result of the retirement period with that in subsection 3.2.2 yields a conclusion that, when the riskiness of the planned consumption (relative to income) is concerned, agents with safety first preference would rather consume more than before, and become reluctant to invest in the risky asset.

4. Simulation

In this section, we assign the parameters values that are consistent with the literature and then simulate the outcome for the optimal portfolio choice and consumption path. The table below summarizes the value and description for each parameter.

Table 1 Summary of Parameter Value

The table contains a summary of the value for each parameter that is going to use in the calibration. See text for the motivation of the choice of the value. For the rest, the value is chosen consistent with the literature.

Preference Parameters			
α	0.05	Critical level on consumption-wage ratio	
β	0.98	Time discount factor	
γ	5	Relative risk aversion	
c	1.4	Reference point of consumption-wage ratio	Working Period
	1.3		Retirement
Financial Market Parameters			
r_f	0.01	Risk free rate	
μ	0.05	Expected return of risky asset	
σ	0.2	Volatility of risky asset	
Other Parameters			
η	0.6	Replacement rate	
T	25	Age that agent starts working	
R	65	Retirement age	
T	85	Life expectance	

One important remark is that the value of the reference point should be chosen very carefully because it is crucial in the safety first constraint. In order to cope with it, we refer to the empirical data to determine the value for c. A detailed description of the computational procedure will be demonstrated in section 4.1. According to its definition in this paper, we finalize the value of c to be 1.4 before retirement and 1.3 after retirement. The reason why the value is lower in retirement is straightforward: agents receive a less income and become less aggressive regarding their wealth-wage ratio being above a certain level. Note that each number here is in real-terms, since we do not consider inflation risk.

The rest of this section is structured as follows. We start to figure out the empirical value for the rho-ratio, which is of great importance in determining the value for the optimal weight under the safety first constraint. Then in section 4.2, we simulate the performance of the financial market for each period and calculate the optimal portfolio choice. Section 4.3 discusses the optimal consumption path over life time and section 4.4 illustrates the planned consumption. Section 4.5 contains a sensitivity analysis for some preference parameters of interest.

4.1 Empirical Value of the Rho-Ratio

In order to get some empirical interpretations of the rho-ratio, we collect data on consumption, wealth, and income data from U.S. Department of Commerce, Bureau of Economic Analysis[4]. All the data are on an annual basis, from 1950 to 2009, and in a chained (2005) dollar[5] term. More specifically, we use *per capita personal consumption expenditure* as a proxy for consumption, which includes the consumption expenditure on non-durable goods and services. As to the measure of personal income, *per capita disposable personal income* is selected. Lastly, *per capita gross domestic product* is also collected to be a fair proxy for wealth. It is also useful to point out that personal wealth in our model might be lower than the gross domestic product because there is no social welfare. Roughly speaking, the range of the consumption-income ratio is from 0.8 to 0.9, and that of wealth-income ratio is from 1.3 to 1.4 (that's why we choose 1.3 for a critical point in working life and 1.4 for that in retirement).

According to the formulas of the rhos in section 3.2.3, we first calculate the value of ρ_t and ρ_{t+1} for each year, and then take the ratio between them in two subsequent years. The results are visualized in Figure 2. Both the values and the ratio of rho tend to be more stable in the most recent years. As we suspect the data at the beginning of the period is

[4] http://www.bea.gov/national/nipaweb

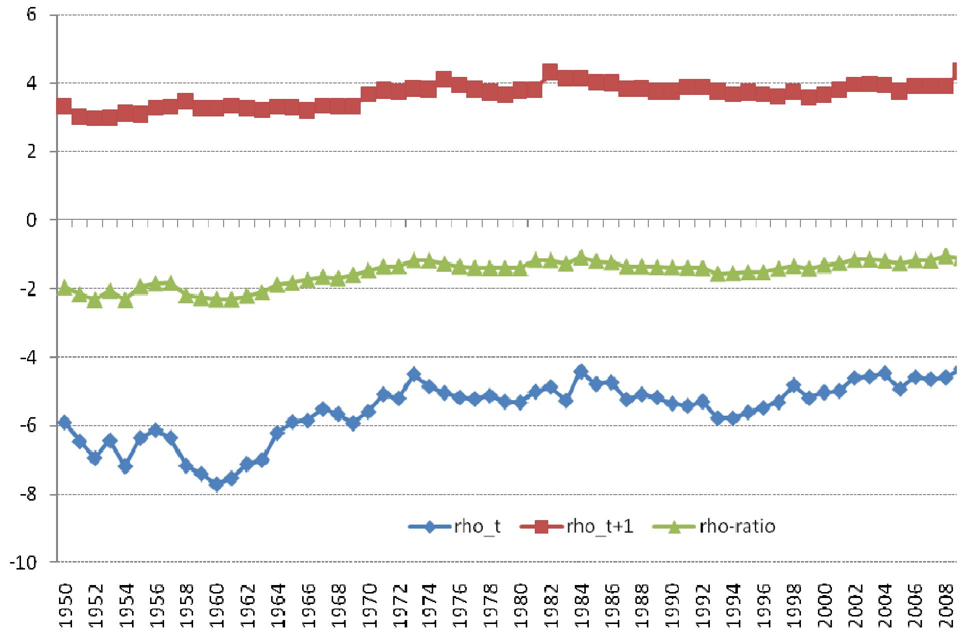
[5] A method of adjusting real dollar amounts for inflation over time, so as to allow comparison of figures from different years. Chained dollars generally reflect dollar figures computed with 2005 as the base year.

somehow influenced by the statistical method "Chained dollars", so we pay more attention to the recent periods.

The rho for the wealth-income ratio stays around 4, which is consistent with the common economic intuition that the wealth-income ratio itself is very stable. On the other hand, the rho for the consumption-income ratio is more volatile, but the fluctuation magnitude is not large. The persistence of the wealth-income ratio and consumption-income ratio also guarantees the extent of accuracy of our log-linearization methodology. As to the ratio between ρ_t and ρ_{t+1}, it fluctuates around -1, which demonstrates a very stable feature.

Figure 2 Time Evolution of rho_t, rho_t+1 and rho-ratio

This figure demonstrates the value of rho_t, rho_t+1 and the corresponding rho-ratio for the period from 1950 to 2009. Data is collected from U.S. Department of Commerce, Bureau of Economic Analysis. See text for detailed description. The value is calculated based on the formulas given in section 3.2.3.



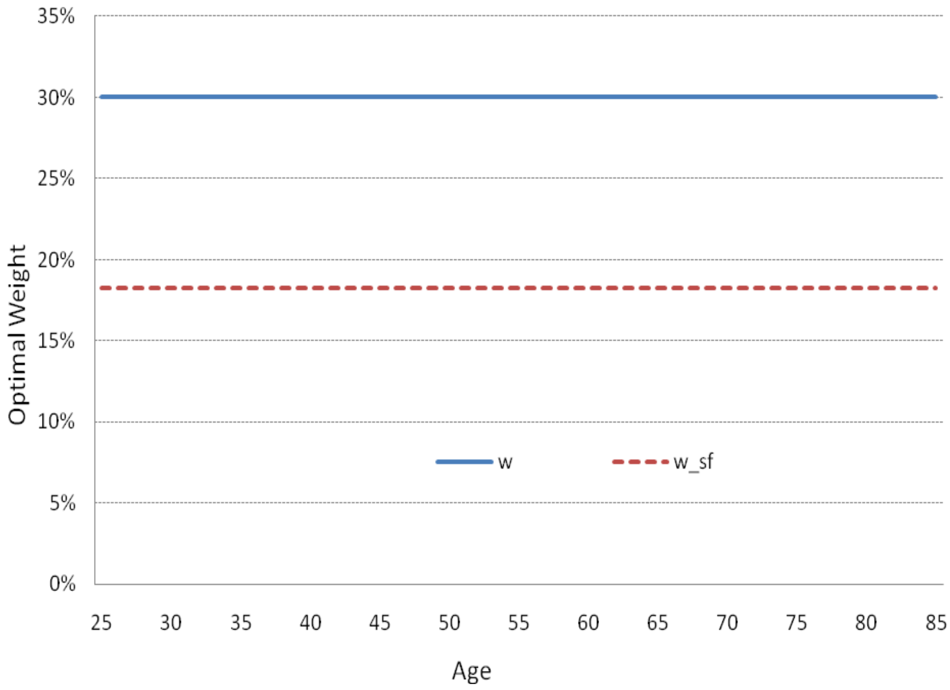
Nevertheless, before we use the value to simulate the results, it is worthwhile to mention the inaccuracy of those proxies. For example, gross domestic product might not be a good indicator of the personal wealth since there is no central planner involved in our model. In addition, according to our model assumption, the rho for the next period is only a plan, but not realized yet, while our empirical data provides the actual information, which may lead to a deviation as well. Thus, we decide to take the value -0.85 for the rho-ratio. However, because of value of the rho-ratio is virtually unkown, our choice of the rho-ratio is really important to determine the optimal weight. One caveat is that the optimal weight is very sensitive to the choice of the rho-ratio.

4.2 Optimal Portfolio Choice

Based on the parameter set described in Table 1 and the choice of the rho-ratio, we calculate the optimal portfolio choice for the normal investor and safety first investor. The result is shown in Figure 3.

Figure 3 Optimal Portfolio Choice

This figure contains a comparison between the normal investor and the safety first investor in terms of their optimal portfolio allocation. The critical level of safety first investor is 0.05.



According to Figure 3, the optimal fraction of the financial wealth to be allocated to the risky asset is 30% for non-safety first investors, but 18.92% for safety first investors. It is rather consistent with our intuition: after considering the risk of the consumption-wage ratio dropping below a certain level, the safety first investors are more risk averse than their counterparty, which result in a less proportion of risky holding in their portfolio. Note that the life cycle pattern of the fraction is a flat line for both of them since the investment opportunity set is not time-varying, and we assume no wage risk.

4.3 Optimal Consumption

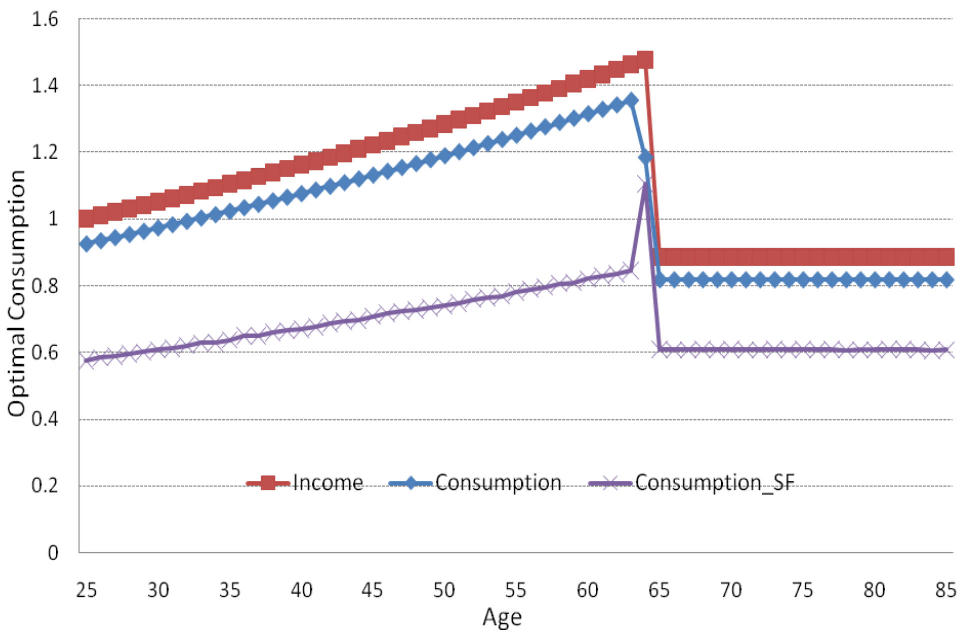
As discussed in section 3.2.3, we know that both ρ_t and ρ_{t+1} are endogenous parameters. After the value of the rho-ratio is obtained, the values of ρ_t and ρ_{t+1} can be figured out by iteration (Viceira, 2001), because the expressions of optimal consumption in (20) and (21) contain the rhos, while the expressions for ρ_{t+1} and ρ_t in (14) are functions of the log consumption-wage ratio, which provides a nonlinear mapping of ρ_{t+1} and ρ_t on themselves. We first give an initial value for ρ_t , and the value for ρ_{t+1} is calculated automatically given the value of the rho-ratio is known. Then based on the parameter set we compute the optimal consumptions and then get a new value for the rhos. The iteration continues until the absolute difference in two subsequent iterations is less than 0.01. Thus, the convergence is achieved and the value for the optimal consumption is obtained.

A comparison between the optimal consumption for safety first investors and their counterparty is illustrated in Figure 4. As predicted, the flat consumption-wage ratio leads to an increase in the consumption as the wage income rises until one period before the retirement. Note that the consumptions for both investors are below the income all the time, ensuring a borrowing constraint of the individuals to be held. A notable distance between the consumption of the safety first investors and the standard investors indicate the importance of the safety first constraint in the individuals' decision making: when people care about their consumption (wealth) in the next period, they tend to be much more conservative in their current consumption.

The most striking changes occur in the last employment period. At age 64, people take into account their decreased income in the next period – the starting of retirement – and make adjustment accordingly. However, two types of investors behave differently: an ordinary investor consumes less and saves more financial wealth to invest in the financial market, while a safety first investor inversely consumes more and has less financial market participation. The totally opposite decisions are attributed to the additional concern about the tail risk of the planned consumption-wage (wealth-wage) ratio. Investors, who do not care about the consumption (wealth) in the next period, may save more financial wealth and participate in the financial market so as to hedge the decrease in the income. On the other hand, agents with the safety first preference in mind may consume more and are reluctant to have the financial market exposure. Because the composition of the portfolio is not time varying, the absolute amount that is allocated to the risky asset is smaller if they invest less into financial market. As a result, when confronted with a reduction in income, they would rather consume more than investing. Nevertheless, even in the last period of working life, safety first agents still consume less than their counterparty, implying the effect of the safety first constraint.

Figure 4 Optimal Consumption Path

This figure gives a comparison of the optimal consumption between safety first investors and the normal investors. In addition, the age-dependent wage is also included as a reference. All the results are obtained from the calibration as described in the text.



In the retirement, both types of agents keep their consumption proportional to their (pension) income constantly. The gap between them is reduced, indicating that safety first agents consume relatively more. The rationale is the same as before: they want to have more consumption instead of financial investment. Given the fixed portfolio allocation, less investment means less equity exposure, and agents can control as much as possible the uncertainty in their consumption (wealth) in the next period.

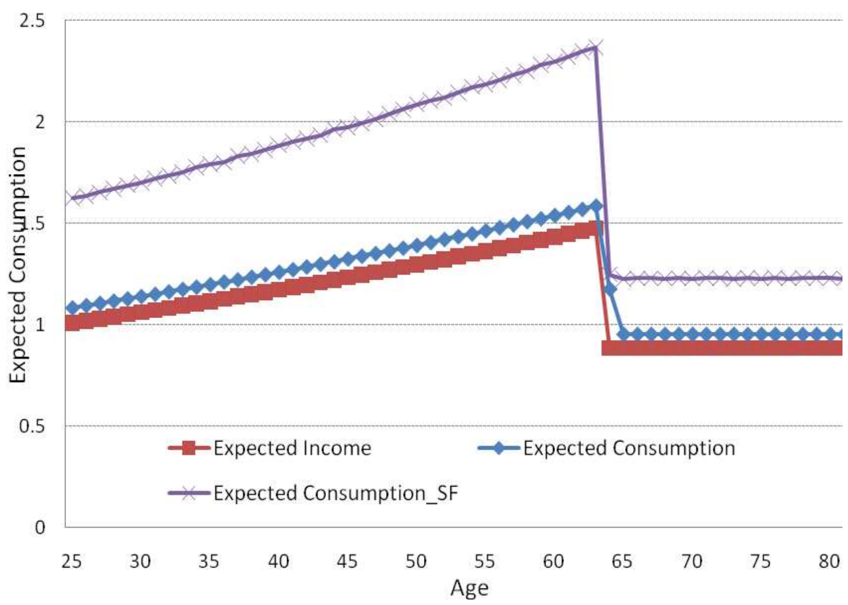
However, it is important to point out that, in reality, people start planning for their retirement much earlier than just one year ahead. So the seemingly striking spikes in Figure 4 are expected to be more spread.

4.4 Planned Consumption

Given agents' choices of consumption and portfolio allocation in the current period, let's look at the agents' plan of their total wealth in the next period. Figure 5 unveils the outcome.

Figure 5 Expected Consumption

This figure gives a comparison of the planned consumption between safety first investors and the normal investors. In addition, the expected wage is also included as a reference. All the results are obtained from the calibration as described in the text.

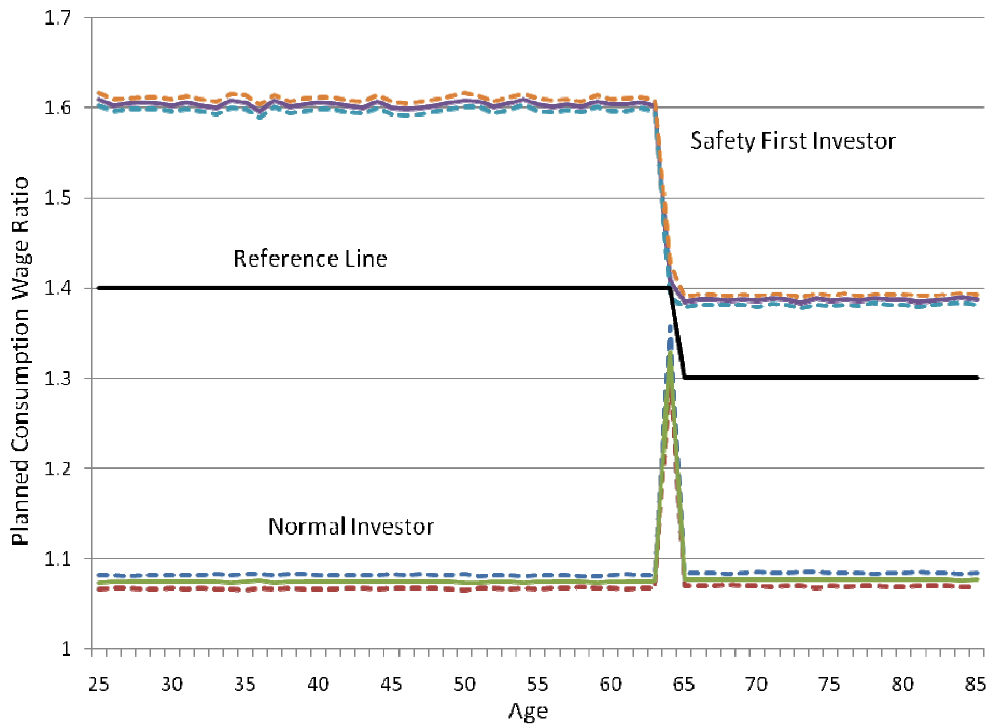


Similar to the optimal consumption in the decision period, the expected consumption is also following the increase of the expected income for the planning period. The expected consumption of the safety first investor is much higher than that of the standard investors: it stays above 1.5 in employment period and reduces to be slightly above 1.3 in retirement, while the normal investor only has a small amount of surplus in their consumption relative to the income all the time.

At age 64, when the retirement is approaching, the expected consumption also drops markedly, both for the safety first and the standard investors, but the magnitude of reduction is more substantial for the former. The drops are driven by two reasons. Firstly, the expected wage is reduced when the retirement starts. Even if the consumption-wage ratio remains unchanged, the value of the expected consumption will drop. Secondly, for the safety first investors, the expected consumption must be lower since they decide to consume a larger proportion of their income in their decision period and have less investment. For the ordinary investors, who have less consumption and more financial market participation in the last working period, the planned consumption is amplified enormously. However, the increase is not sufficient to compensate the decrease in the income. But nevertheless, it drops less severe than that of the safety first investor.

Figure 6 Planned Consumption-Wage Ratio

This figure gives a comparison of the planned consumption-wage ratio between safety first investors and the normal investors. In addition, the reference line for the safety first investor is also included as a benchmark. All the results are obtained from the calibration as described in the text.



In order to further analyze the consumption-wage ratio and test whether our safety first target is satisfied, we include the comparison of the consumption-wage ratio between the two types of investors. As shown in Figure 6, the three lines in the top and in the bottom are from the safety first investors and the standard investors, respectively. The solid line in the middle is the expected consumption and the dotted lines are the corresponding 5% and 95% quantile. The three lines are not clearly distinguished because the standard deviation of the planned consumption-wage ratio is quite small. For the safety first investor, the medium standard deviation over the life span is only 0.26%, while that for the normal investor is 0.43%. Although hard to observe graphically, the planned consumption-wage ratio for the safety investor is indeed much lower than their counterparty.

The reference line is also illustrated as a benchmark for comparison. Being above the reference line for all the age group and less risk inherited, the planned consumption-wage ratio for the safety first investor is exactly demonstrating a safety first feature, which satisfies the demand of controlling the probability of reaching disaster required by the safety first investor.

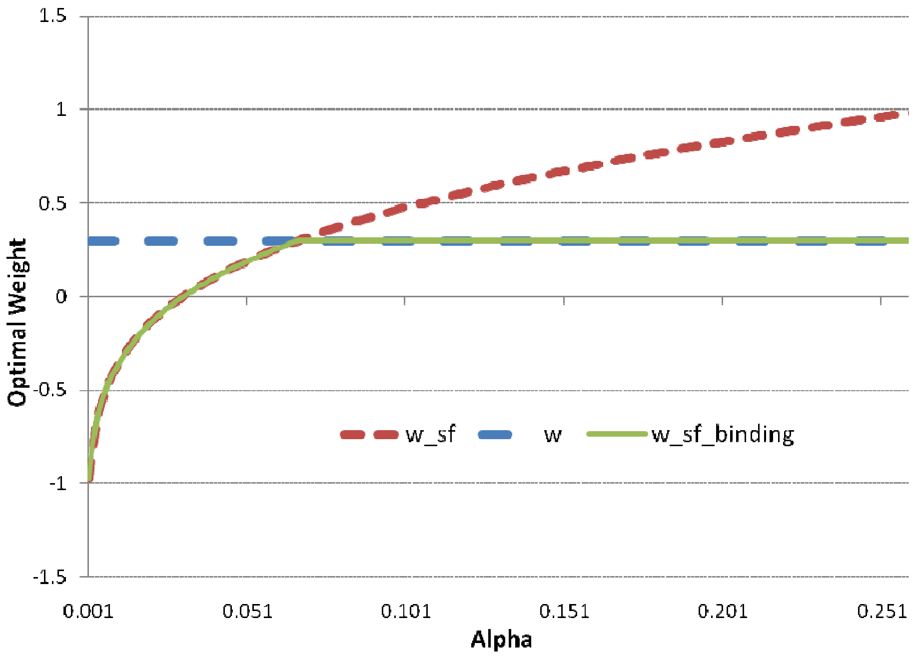
The drop and the spike in the ratio are the results from the opposite change in the consumption behavior in the decision period, which is also discussed in section 4.3. They are also consistent with our observation in Figure 5, where the expected consumption for the safety first investor decreases more notably than the non-safety first counterparty.

4.5 Sensitivity Analysis

In this subsection, given the importance of α, we will first check whether or not our choice of α is reasonable. As discussed in section 3.2.3, the influence of α on the optimal weight ends when α is too large, since the safety first constraint will not bind and the optimization becomes the same as that for a standard investor. Based on the set of the parameter value in Table 1, and the formula of the optimal weight for a safety first investor in (25), we vary the value of α and calculate the corresponding optimal portfolio allocation to the risky asset. Figure 7 describe the relationship between α and the optimal portfolio choice. The portfolio weight for a standard investor is also include, which is a straight line at 0.3. The concave and dotted line for the safety first investor demonstrates that when α increases, the optimal weight to the risky asset should also increase, which is consistent with the intuition that, the agents relax their critical level indicates a high risk tolerance. The concavity indicates the marginal effect of the choice of α on the portfolio allocation is more substantial when α is already low. It is also quite intuitive because the high persistence of the consumption-wage ratio implies that the distribution of the consumption-wage ratio is highly centralized around the mean, and has a less widely spread tail. Hence a small change in the quantile has a relatively large impact.

Figure 7 Influence of Alpha on the Optimal Weight

Figure 7 gives a graphical demonstration of the effect of the value for alpha on the optimal weight to the risky asset. The tow dotted lines are from the optimal portfolio weight for a safety first investor and a standard investor, respectively. When alpha increases to a certain level, the safety first constraint is not binding and there is no difference between the portfolio choice between the two types investors, which is reflected by the solid line.



Nevertheless, when α is increased to approximately 0.07, the two dotted lines come to a cross, stating a boundary value of α. If α continues growing, the constraint will not bind and there is no difference between the safety first investor and the standard investor. Thus the optimal weight for a safety first investor must diminish to that of a standard investor. Putting all together, the solid line in Figure 7 shows the optimal weight of the safety first investor when the safety first constraint is binding, no matter what is the value of α .

Thus, the necessity of keeping the safety first constraint binding leads to a non-linear relationship between α and the optimal portfolio choice. As illustrated in Figure 7, our choice of the value for α , which is 0.05, falls in a reasonable range. Based on that, we will present a sensitivity analysis of the optimal portfolio allocation and consumption, when the values of the preference parameters, namely α and β , are changed. Note that the sensitivity analysis of γ will not be conducted, given the importance of α in this thesis and the potential endogeneity problem between α and γ . The input value and the outcome are summarized in Table 2.

Table 2 Summarized Outcomes of Sensitivity Analyses

This table contains the sensitivity of the optimal consumption and optimal weight towards the changes of the parameter value. The life time is divided into three parts, pre-retirement: the period from age 25 to age 63, age 64, and post retirement.

		α			β		
		0.04	0.05	0.06	0.97	0.98	0.99
	Optimal Weight	9.79%	18.92%	25.48%	16.85%	18.92%	19.67%
Pre-retirement	Consumption-Wage	-0.5316	-0.5481	-0.5611	-0.5451	-0.5481	-0.5513
	Expected Consumption-Wage	0.4581	0.4724	0.4835	0.4697	0.4724	0.4751
Age 64	Consumption-Wage	-0.2808	-0.2898	-0.2910	-0.2883	-0.2898	-0.2901
	Expected Consumption-Wage	0.3353	0.3426	0.3468	0.3411	0.3426	0.3439
Post-Retirement	Consumption-Wage	-0.3657	-0.3762	-0.3834	-0.3743	-0.3762	-0.3774
	Expected Consumption-Wage	0.3178	0.3270	0.3333	0.3253	0.3270	0.3281

When we decrease the critical level of the consumption-wage ratio α from 0.05 to 0.04, the optimal weight reduces to 9.79%. It implies that more concern about the tail risk of consumption-wage ratio will result in a more cautious investment. On the contrary, the allocation to the risky asset increases to 25.48% when the probability constraint is released to 0.06. Furthermore, if the time discount factor β is increased from 0.98 to 0.99, the optimal portfolio will consist of 19.67% risky asset, a slight increase from 18.92%. It is due to the fact that, when β increases, people care less about when to consume. As a result, they can afford more investment risk through adjusting their consumption.

Nevertheless, the safety first investors are still more risk averse since the weight is still lower than 30%, which is the portfolio composition of the normal investor. A comparison between the outcomes from the two parameters yields that α has a larger effect on the optimal weight, pointing out the importance of the safety first constraint.

As to the consumption-wage ratio in the decision period, an increase in α leads to a decrease in the optimal consumption because the investor's constraint on the planned consumption-wage ratio is less severe, more financial risk exposure is acceptable. When β is increased to 0.99, meaning consumption in different times does not differ quite much, current consumption is reduced, as expected, in order to absorb more equity shocks. Due to a smaller consumption in the decision period and a larger allocation to the risky asset, the planned consumption-wage ratio over lifespan is larger when α increases. A similar change is also observed when we increase the value for β. All in all, the overall life cycle pattern for the consumption-wage ratio remains the same no matter the parameter value increases or decreases. Therefore, the sensitivity analysis concludes that the result is quite robust to the value of α and β.

Note that we only focus on the sensitivity of the choice of some preference parameters, namely α and β, in this section. Of course other parameters, for example the rho-ratio and the reference point c, are also crucial to the optimal solution: the value of the rho-ratio plays an important role in determining the optimal portfolio weight, while the reference point has a large impact on the optimal consumption path. Indeed, given the expression (25), the optimal asset allocation is very sensitive to the value of the rho-ratio. But nevertheless, due to the fact that they are virtually unknown (at least to me), we just use the empirical data to calibrate the model.

5. Conclusion

Since it was proposed by Roy (1952), the idea of safety first has seen an increasing popularity. Especially after the burst of the financial crisis in 2008, it becomes more interesting to investigate. Besides, safety first also plays an important function in the retirement of an aging population, because it aims at reducing the uncertainty of the future prospects.

This thesis solves a life cycle model of consumption and portfolio choice when a safety first constraint on the future intermediate consumption is incorporated. In the absence of a closed-form solution for the optimization problem, we impose an assumption that the investors are myopic and we use an approximation method based on the log-linearization in Campbell and Viceira (1999) and Viceira (2001). The model is proposed as a preliminary investigation on the effect of the safety first on the agents' decision making over the lifespan.

With realistic parameter values, the model predicts that the incorporation of the safety first idea into the individual's decision making has an impact on the optimal asset allocation and optimal consumption. When having concerns on the tail risk of relative consumption to income, the safety first agents become more risk averse and are less motivated to allocate their financial wealth to the risky asset than the standard agents. The optimal portfolio composition remains constant over time, because the investment opportunity set is not time varying. Thus, on average, safety first investors have less equity participation, which provides an explanation for the limited stock participation.

In addition, safety first concerns also discourage the agents' current consumption. They are reluctant to consume at the current period so as to accumulate enough wealth for ensuring a certain level of consumption in the next period. When retirement is approaching, the agent starts preparing at age 64, because the assumption on the planning horizon is only one year ahead (Of course, agents are expected to start preparation much earlier in reality). Two types of investors demonstrate completely different consumption behavior. Instead of investing, the safety first investors would rather consume more and have a smaller exposure to the financial market, while the standard investors consume less and save more financial wealth to invest, which is used to hedge the loss of income.

The strikingly opposite decision is actually driven by the safety first concern of the investors: the portfolio composition is fixed over time, thus as less financial wealth is invested in the financial market, the absolute amount that is allocated into the risky asset is less, which results in smaller risk exposure.

Consequently, the planned consumption-wage ratio for the safety first investor is indeed demonstrating a safety first feature: with the mean, 5%, and 95% quantile being above the reference line for all the age group, and a standard deviation of 0.26%, it satisfies the demand of controlling the probability of reaching disaster. Moreover, it is also robust to the changes in the preference parameters α and β. On the contrary, the planned consumption-wage ratio for the standard investor is much worse, with a higher standard deviation and a much lower value – always below the reference line.

Nevertheless, there is always a trade-off between the sophistication and computational simplicity of a model. The availability of the analytical solution brings along some limitations to the model. First of all, given our assumption of the myopic investors, our solution is not an optimum throughout the life cycle, but only a static and periodical optimization. We expect that more rationale people will have their decisions much closer to the one that is derived from a dynamic optimization, which should be superior to our case. Secondly, we exclude any background risk. Neither wage risk nor longevity risk is included in our analysis. Moreover, we assume a simple setting of the financial market, which does not incorporate the mean reversion of the equity premium or the stochastic interest rate. A possible extension could include either additional risk that has been mentioned. Last but not least, our result is based on our specification of the safety first constraint and it is sensitive to the choice of some parameters, for example, the rho-ratio. A different specification will lead to a different optimal portfolio choice and consumption. For instance, an alternative of the specification is to impose the safety first constraint on the accumulated wealth upon retirement: individuals should at least afford to purchase an annuity, which guarantees their income throughout the retirement.

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Appendix

Appendix A. Derivation of Hamilton-Jacobi-Bellman Equation

To apply the stochastic dynamic programming, the value function is defined as:

$$J(W,Y,t) \equiv \max E_t\left[\int_t^T e^{-\beta s} U(C_t) ds\right].$$

The control variables are the consumption C_t and the portfolio allocation parameter ω_t, while the state variables are the current value of total wealth W_t and the wage Y_t. Then we derive the Hamilton-Jacobi-Bellman (HJB) differential equation to solve the dynamic optimization. First start with the bellman equation

$$J(W,Y,t) = \max E_t[J(W,Y,t+\Delta t)],$$

which leads to

$$\max E_t[J(W,Y,t+\Delta t) - J(W,Y,t)] = 0.$$

Divide Δt and let it go to zero, we get

$$\max \frac{1}{dt} E_t[dJ(W,Y,t)] = 0.$$

According to Ito's lemma,

$$dJ(W,Y,t) = J_C dt + J_W dW + J_t dt + J_Y dY + 0.5 J_{WW} d(W)^2.$$

Applying it to the bellman equation, the optimality conditions of $J(W,Y,t)$ is then formulated in following HJB equation, which gives the equation (3) in the text.

$$0 = \max\left\{e^{-\beta t}U(C) + J_W\left[[(1-\omega)r_f + \omega\mu]W - C + Y\right] + J_t + J_Y gY + 0.5\omega^2\sigma^2 W^2 J_{WW}\right\} \quad \text{(A1)}$$

Time subscripts are dropped for simplicity, and the remaining subscripts denote partial derivatives. Based on the HJB equation, we can further compute the optimum for the control variable C_t and ω_t .We take the first order condition (FOC) on the control variables of the HJB equation (A1), which gives

$$e^{-\beta t}U_C(C_t^*) - J_W = 0$$
$$J_W(\mu - r_f) + \omega^* \sigma^2 W J_{WW} = 0.$$

Rearranging the FOC, we have that

$$(C_t^*)^{-\gamma} = e^{\beta t} J_W$$
$$\omega^* W = -\frac{J_W}{J_{WW}} \frac{\mu - r_f}{\sigma^2}. \qquad \text{(A2)}$$

which is included in the text as (4). Thus, substituting the optimal value for the control variables in (A2) into the HJB condition (A1) yields:

$$0 = \frac{\gamma}{1-\gamma} e^{-\beta t} (e^{\beta t} J_W)^{\frac{\gamma-1}{\gamma}} + J_W (rW + Y) + J_t + J_Y gY - 0.5 \frac{J_W^2}{J_{WW}} \frac{(\mu - r_f)^2}{\sigma^2} \qquad \text{(A3)}$$

Observe the existence of the J function in (A3), the form of the J function should be obtained in order to work out the expression for the optimality. Motivated by the closed-form solution for the constant wage used in Merton (1971), we assume that the J function takes the following form, which is common in the literature (for example, Huang, Milevsky, and Wang, 2008).

$$J(W, Y, t) = \frac{h(t)[W + k(t)Y]^{1-\gamma}}{1-\gamma}. \qquad \text{(A4)}$$

After guessing the form of the J function, we substitute (A4) into (A3). The HJB condition then becomes a function of h and k. The prime symbol denotes the derivative with respect to time.

$$0 = \frac{\gamma}{1-\gamma} (e^{\beta t})^{-\frac{1}{\gamma}} h^{-\frac{1}{\gamma}} (W + kY) + (rW + Y) + \frac{h'}{h} \frac{(W + kY)}{1-\gamma} + Yk' + 0.5 \frac{1}{\gamma} (W + kY) \frac{(\mu - r)^2}{\sigma^2} + kgY.$$

Define the income-wealth ratio as $x \equiv \frac{Y}{W}$, then the HJB equation becomes

$$0 = \frac{\gamma}{1-\gamma} (e^{\beta t})^{-\frac{1}{\gamma}} h^{-\frac{1}{\gamma}} (1 + kx) + (r + gx) + \frac{h'}{h} \frac{(1 + kx)}{1-\gamma} + xk' + 0.5 \frac{1}{\gamma} (1 + kx) \frac{(\mu - r_f)^2}{\sigma^2} + kgx.$$

This can be viewed as a linear equation for x and for the solution to exist, we must have that the coefficients vanish for all possible value of x, which leads to the following two Ordinary Differential Equations (ODE) for h and k:

$$\frac{\gamma}{1-\gamma} (e^{\beta t})^{-\frac{1}{\gamma}} h^{-\frac{1}{\gamma}} + r_f + \frac{h'}{h} \frac{1}{1-\gamma} + 0.5 \frac{1}{\gamma} \frac{(\mu - r_f)^2}{\sigma^2} = 0,$$
$$-r_f k + 1 + k' + kg = 0.$$

Using the zero terminal condition, the closed-form solution of the above ODE can be obtained for the pre-retirement period:

$$h(t;T) = e^{-\beta t}\left(\frac{e^{A(T-t)}-1}{A}\right)^{\gamma},\ k(t;R) = \frac{1}{g-r_f}(e^{(g-r_f)(R-t)}-1),$$

where

$$A = -\frac{\beta}{\gamma} + \frac{1-\gamma}{\gamma}\left(r_f + 0.5\frac{(\mu - r_f)^2}{\gamma\sigma^2}\right).$$

Thus, after the forms of $h(t)$ and $k(t)$ are derived, the exact expression for the J function is given for the employment period based on our earlier discussion in (A4),

$$J(W,Y,t) = e^{-\beta t}\left(\frac{e^{A(T-t)}-1}{A}\right)^{\gamma}\frac{[W_t - \frac{1}{g-r_f}(e^{(g-r\ )(R-t)}-1)Y_t]^{1-\gamma}}{1-\gamma},\ \text{when}\ t < R,$$

and that for post-retirement period is

$$J(W,Y,t) = e^{-\beta t}\left(\frac{e^{A(T-t)}-1}{A}\right)^{\gamma}\frac{W_t^{1-\gamma}}{1-\gamma},\ \text{when}\ R \le t < T.$$

Appendix B. Log-Linearization Approximation Method

B.1. Derivation of the Log-Linear Intertemporal Budget Constraint

Rewrite the budget constraint (10) as

$$(\frac{C_{t+1}}{Y_{t+1}}-1)\frac{Y_{t+1}}{Y_t}=(1-\frac{C_t}{Y_t})R_{t+1}^p \tag{B1}$$

and in logs, we have

$$\log[\exp(c_{t+1}-y_{t+1})-1]+g=\log[1-\exp(c_t-y_t)]+r_{t+1}^p \tag{B2}$$

where the lowercase letters denote the variables in logs, for example, we define $c_{t+1}=\log(C_{t+!})$ and $g=\log(1+G)$.

It is noticeable that the nonlinear function of the log consumption-labor ratio $\log[\exp(c_{t+1}-y_{t+1})-1]$ appears on both side of the equation (B2). Following Campbell (1993), it can be linearized by applying the first-order Taylor expansion around the mean value of $(c_{t+1}-y_{t+1})$, i.e. $E(c_{t+1}-y_{t+1})$. The resulting approximation is

$$\log[\exp(c_{t+1}-y_{t+1})-1]\approx\log\{\exp[E(c_{t+1}-y_{t+1})]-1\}$$
$$+\frac{\exp[E(c_{t+1}-y_{t+1})]}{\exp[E(c_{t+1}-y_{t+1})]-1}[(c_{t+1}-y_{t+1})-E(c_{t+1}-y_{t+1})]$$

Similarly, we can get a linear approximation for $\log[1-\exp(c_t-y_t)]$. Rearranging the equation using the approximation term, the following approximation must hold

$$k+g+\rho_{t+1}(c_{t+1}-y_{t+1})-\rho_t(c_t-y_t)-r_{t+1}^p\approx 0$$

where

$$\rho_{t+1}=\frac{\exp[E(c_{t+1}-y_{t+1})]}{\exp[E(c_{t+1}-y_{t+1})]-1},$$

$$\rho_t=\frac{\exp[E(c_t-y_t)]}{\exp[E(c_t-y_t)]-1},$$

and

$$k=-\rho_{t+1}\log\rho_{t+1}+(\rho_{t+1}-1)\log(\rho_{t+1}-1)+(1-\rho_t)\log(1-\rho_t)+\rho_t\log(-\rho_t)$$

This gives the equation (14) in the text.

B.2. Derivation of the Log-Linear Safety-First Constraint

Given the intertemporal budget constraint in the form of (B1), the constraint (13) is equivalent to

$$\Pr_t[(1-\frac{C_t}{Y_t})R_{t+1}^p \frac{Y_t}{Y_{t+1}} + 1 < c] \le \alpha$$

$$\Leftrightarrow \Pr_t[R_{t+1}^p < \frac{(c-1)(1+G)}{(1-\frac{C_t}{Y_t})}] \le \alpha \qquad \text{(B3)}$$

Denote the α-fractile of the distribution of R_{t+1}^p to be $q_\alpha(R_{t+1}^p)$, the constraint (B3) can be reformulated as:

$$q_\alpha(R_{t+1}^p) \le \frac{(c-1)(1+G)}{(1-\frac{C_t}{Y_t})} \qquad \text{(B4)}$$

Note that the distribution of the portfolio return R_{t+1}^p is lognormal. In order to obtain the quantile of a lognormal distribution, the first step is to derive the mean and variance of the log portfolio return r_{t+1}^p. Applying Campbell-Viceira's log-linearize technique, the mean and the variance of the log portfolio return r_{t+1}^p can be approximated:

$$E_t r_{t+1}^p \approx r_f + \omega(E_t r_{t+1}^e - r_f) + 0.5\omega(1-\omega)Var_t(r_{t+1}^e)$$
$$= r_f + \omega(\mu - r_f) + 0.5\omega(1-\omega)\sigma^2$$

$$Var_t(r_{t+1}^p) \approx \omega^2 Var_t(r_{t+1}^e) = \omega^2\sigma^2$$

Taking the exponential of a α-quantile from a normal distribution gives the expression of the α-quantile from a lognormal distribution. Let Z_α denote the α-quantile of a standard normal distribution, the constraint (B4) is equivalent to

$$q_\alpha(R_{t+1}^p) = \exp\left[r_f + \omega(\mu - r_f) + 0.5\omega(1-\omega)\sigma^2 + Z_\alpha\omega\sigma\right] \le \frac{(c-1)(1+G)}{(1-\frac{C_t}{Y_t})} \qquad \text{(B5)}$$

Since a logarithm function is monotonic, taking the logarithm on both sides of (B5) we have expression (16) in the text.