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Optimal Investment Policy for Pension Funds with Transaction Costs

The Finite Horizon Case

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Optimal Investment Policy for Pension Funds with Transaction Costs: the Finite Horizon Case

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Abstract

In this article, we investigate a CRRA investor as a representative of pension plan participants, who has a finite investment horizon and is subject to the proportional transaction costs. He attempts to maximize his utility by trading between stock and money market account. A set of ordinary differential equations are derived first for analytical solution. We then alternatively propose the binomial tree method in order to numerially solve the dynamic maximization problem. Result shows that optimal investment policy is horizon-dependent: the no-transaction region slightly broadens over time and only in the very last moment the investor dramatically reduces transaction. When allowing for intermediate consumption, the investor finds it optimal to keep a larger fraction of wealth in the money market account and reduce trading frequency. In particular, any shock in transaction cost rate, risk, risk premium, relative risk aversion coefficient and time discounting parameter affects shorter-horizon investor more distinctly for both cases.

Key words: pension funds, optimal investment policy, lifecycle investment, proportional transaction costs, finite horizon, no-transaction region, binomial tree method.

Contents

1 Introduction

Traditional economic models on optimal investment and consumption policy have been extended on many directions ever since Merton (1971,1973) published his pioneer article. In these benchmark models investors can trade asset continuously at any time without incurring any kind of costs. However, in the capital market, an asset is also featured by its liquidity in addition to the commonly used risk and return. Trading is most of the time costly.

Working with models that incorporate liquidity considerations, in whatever shape, requires a thorough re-examination of mainstream theory of financial markets. It is particularly relevant to pension funds when considering the impact of reduced liquidity and feedback effects in the markets. Pension funds allocate the wealth of their funds to many different classes of investments on behalf of their plan participants. The central question is: how to incorporate illiquidity into the current valuation and optimal investment strategy.

There are various ways to model illiquidity in an optimal investment problem. We decide to use transaction cost as a proxy for illiquidity due to the following two advantages: first it allows for mathematical flexibility and tractability; secondly it enables references and comparisons with a large body of previous literature. Constantinides (1979,1986) has shown that the optimal transaction policy is to maintain the ratio of the dollar amount invested in the riskless asset to that in the risky asset within a certain range, represented by the buy boundary and sell boundary. Consequently three regions are identified depending on the portfolio ratio: the no-transaction region, the buy region and the sell region

We formulate the continuous-time dynamic maximization problem and derive the optimal conditions. In particular, we investigate a CRRA investor as a representative of pension plan participants, who has a finite horizon and is subject to proportional transaction costs when trading stock as well as money market account. Mathematically these conditions are boiled down to a set of ordinary differential equations. In contrast to the infinite horizon model in which stationary solutions can be obtained, the value function and the corresponding two boundaries strongly depend on horizon, hence it is extremely time-consuming to analytically solve this system. Liu and Loewinstein (2002) solved the deterministic finite horizon problem by making use of the exponentially distributed horizon but their picture of the sell and buy boundary is incomplete.

Alternatively we consider a numerical method in this paper, namely the binomial tree method, to search for the two moving boundaries at each period. This method is first proposed by Gennotte and Jung (1992). As its name implies, the binomial tree is a diagram representing possible paths that are followed by the stock price up to the terminal time. It is a commonly used approach for option pricing. We divide the investment horizon into a large number of small time intervals. Therefore both time and state are discretized, which enables a numerical approximation. Working backwards from the terminal time, all the boundaries could be located. The advantage of numerical method is obvious: it allows for more flexible specifications of capital market. On the other hand the stability of numerical solution is in general not guaranteed, especially for high dimension problems.

Liu and Loewenstein (2002) claimed that proportional transaction costs together with a finite horizon would imply a time-varying, largly buy-and-hold trading strategy. Appying binomial tree method we successfully find out the overall pattern of the moving boundaries and analyze the impact of transaction cost rate, risk, risk premium and relative risk aversion on the no-transaction region. However, we find no supporting evidence on the second statement that it is consistent with the life-cycle investment advice.

Another contribution of this paper is that we examine the optimal investment policy when allowing intermediate consumption, which has not been done yet. One might expect that transaction costs have two opposite impacts on consumption. The income effect depresses current consumption while the substitution effect shifts consumption to the earlier stage. Both consumption and investment decisions have to be made simutaneously in real life: how much to consume and whether to adjust portfolio composition. Again the finite horizon closed-form solution is difficult to obtain so that we find a way to realize it by using binomial tree method. A new dimension is thus introduced to the tree representing the consumption policy.

The remainder of the paper is organized as follows. Section 2 reviews the literature on the optimal investment policy concerning illiquidity and transaction costs. Section 3 describes the continuous time model in which the investor has finite horizon. Four cases are considered: no-transaction costs case, proportional transaction costs case, life-cycle investment with and without transaction costs. Section 4 gives the mathematical formulation of dynamic programming and binomial tree method that we apply to find numerical solution. Section 5 presents the key results of the binomial tree method. Section 6 discusses possible extensions and summarizes the paper.

2 Literature review

2.1 Asset pricing, optimal investment policy considering illiquidity

The presence of illiquidity in the capital market complicates the problem of asset pricing and optimal investment policy. Traditional models assume that agent can trade asset continuously at any time without occuring costs, as in Merton (1971,1973), who showed that optimal portfolio consists stock and bond at a constant ratio, and in addition, optimal consumption is affine in total wealth. However, existence of human wealth, housing, proprietorships certainly casts doubts on the assumption. Moreover, stocks, bonds, and financial derivatives cannot be traded without any costs. Obviously asset allocation without taking account of illiquidity is suboptimal and consequently asset pricing models need to be adjusted accordingly. Since late 60's, research on asset illiquidity and its implication has emerged intensively in the finance literature and gradually gained more popularity.

Among all, two approaches to model illiquid asset, namely transaction costs and lock-up period, attract most attention. Transaction costs apply while trading takes place, including fixed transaction cost and proportional transaction cost. The pioneer work of Constantinides (1979, 1986) proved that optimal consumption policy and investment policy deviate from no transaction costs case when considering proportional transaction costs. The optimal investment policy is characterized by two reflecting barriers, such that investor stops trading asset in the no-transaction region. Since investors accomodate large transaction costs by drastically reducing trading volume and frequency, transaction costs only have a second order effect on equilibrium price. Constantinides' study and the subsequent works, such as Dumas and Luciano (1991), Liu and Loewenstein (2002), Liu (2004) have become the building blocks of optimal investment policy considering transaction costs which our model is based on. We will review relevant literacture in more details in the next section.

Amihud and Mendelson (1986) established a theoretical model to investigate return-spread relationship. By introducing the expected spread-adjusted return, they derived clientele effect and spread-return relationship in asset pricing with bid-spread ask. Clientele effect means that assets with higher spread are allocated in equilibrium to portfolios with longer expected holding period, while spread-return relationship addresses that in equilibrium the observed market gross return is an increasing and concave piecewise linear function of the relative spread. The implication of their model is that investors with longer expected holding periods will earn higher returns net of transaction costs in market equilibrium.

To further address the asset pricing issue with liquidity risk, Acharya and Pedersen (2005) derived a liquidity-adjusted CAPM. Different from the aforementioned literacture, liquidity-adjusted CAPM states that the required execess return is the sum of expected liquidity costs and four types of risk-premium. Thus additional risks caused by illiquidity are also priced. For the sake of simplicity, we will disregard liquidity risk in our model. Only expected level of illiquidity, which we models as transaction costs, is already sufficient to induce significant deviation from no transaction costs case.

Using transaction cost as a proxy for illiquidity allows for endogenous holding period. Alternatively, illiquid asset is non-tradable during the lock-up period. The latter definition may give rise to one obvious disadvantage: since lock-up period is predetermined, investor cannot rebalance their portfolio by adjustment. Following the definition by Longstaff (1999), the illiquid asset can only be traded at a certain date. Both optimal portfolio choice and asset evaluation become strikingly different from fully liquid case. The less patient agent tilts his portfolio towards the liquid asset, to prepare for supporting an optimal consumption path, for the illiquid asset cannot generate cash flow during the lock-up period. The liquid asset becomes more valuable while the illiquid asset becomes less valuable when introducing illiquidity in general. Similarly, Schwartz and Tebaldi (2004) assumed that illiquid asset cannot be traded, but generates a liquid dividend. Schwartz and Tebaldi (2004) found that the optimal asset allocation and consumption critically depend on the endowment and characteristics of the illiquid asset.

There are also other variations to model illiquidity. Being unable to initiate or unwind a portfolio position instantly is often observed in financial market. Longstaff (2001) investigated optimal portfolio selection when investors are restricted to trading strategies that are of bounded variation. The investor exposed to this kind of risk tends to take less risk, and large price discount is possible. The contribution of this article is that it focuses on the endogenous effects of illiquidity on trading strategies and security values.

2.2 Optimal investment policy on the presence of transaction costs

As discussed above, there are various ways to model illiquidity in an optimal investment problem. We decided to use transaction cost as a proxy for illiquidity due to the following two advantages: first it allows for mathematical flexibility and tractability; secondly it enables references and comparisons with a large body of previous literacture. We can show that the optimal investment policy is horizon-dependent and largely buy and hold near the terminal time. All the papers reviewed in this section differ on two dimensions: whether it allows for intermediate consumption, and whether it assumes infinite horizon or finite horizon.

Following Constantinides (1979, 1986), finance literature have mainly focused on the infinite horizon maximization problem. In this group of models, it is commonly assumed the investor has a CRRA utility preference (on final wealth or consumption stream) which he maximizes by trading risky and riskless assets subject to transaction costs. Davis and Norman (1990) considered a infinite horizon maximization problem with intermediate consumption. Davis and Norman (1990) showed that the proportion of wealth held in stock should be kept between the upper and lower barrier. Dumas and Luciano (1991) studied maximizing expected terminal utility of wealth without intermidiate consumption as the horizon gets very large. They obtained a stationary portfolio rule in the limit. The portfolio policy is in the form of two control barriers between which portfolio proportions fluctuate. Compared to Constantinides (1986) who only provided approximate solutions, Davis and Norman (1990), Dumas and Luciano (1991) derived closed-form solutions. However, the asymptotic analysis presented in these articles and other infinite horizon type models all suffer from contradicting the horizon dependence of optimal investment. When combining transaction costs and finite horizon, results of optimal investment policy can show substantial difference.

According to the limited quantity of literacture, optimal investment policy in a finite horizon problem has been treated analytically and numerically. Our model is first inspired by Liu and Loewenstein (2002), who examined the optimal investment problem for an individual attempting to maximize his expected CRRA utility without intermediate consumption at an uncertain time, which is assumed to be the n-th jump time of an independent Poisson process (thus the horizon is exponentially distributed). It can be shown that the optimal transaction policy is to maintain the ratio of the dollar amount invested in the riskless asset to that in the risky asset within a certain range, represented by the buy boundary and sell boundary. Moreover, the trading boundaries for the exponentially distributed horizon converge to those of the deterministic finite horizon. Liu and Loewenstein (2002) concluded that the presence of transaction costs together with a finite horizon would yield consistent results with life-cycle investment advice which suggests younger investors allocate more wealth to stocks than older investors and all investors follow a largely buy and hold strategy,

Gennotte and Jung (1992), also examined the effect of proportional transaction costs on optimal portfolio selection for an agent maximizing his expected utility of terminal wealth. As the boundaries characterizing optimal investment policy cannot be derived analytically, Gennotte and Jung (1992) developed an efficient and tractable numerical algorithm to obtain the boundaries, namely, binomial approximation. Backward induction is applied to derive the boundaries. This approach has an obvious advantage: it allows for flexibility regarding asset return dynamics. Depending on the parameter values, the multi-period binomial distribution converges to one of several common distributions governing evolution of asset returns. Davis, Panas and Zariphopoulou (1993) took advantage of convergence of discretization schemes based on the binomial approximation of the stock price in order to solve the european option pricing problem with transaction costs, for computing the price involves solving two stochastic optimal control problems. Similarly, Balduzzi and Lynch (1999) discreted both time and state to numerically compute the optimal investment policy for an investor with finite horizon.

Chellathurai and Draviam (2007) considered a finite horizon portfolio selection problem without intermediate consumption when fixed and/or proportional transaction costs are presented. Chellathurai and Draviam (2007) derived a set of time-dependent differential equations which coincide with those of Davis and Norman (1990). A monotone upwind finite-difference scheme is developed to discretize the differential equations. Chellathurai and Draviam (2007) concluded that in the presence of proportional transaction costs, trading is infinitesimal when it takes place, while in the presence of fixed transaction costs, the random trade is instead lump-sum.

Several extensions were made to the simple case which only concerns two assets, proportional transaction costs and constant investment opportunity set. Liu (2004) showed optimal policy with multiple assets involves a constant amount investment in each stock, and the optimal consumption is an affine function of total wealth, much similar to the results of Merton. When both proportional and fixed transaction costs are presented, the optimal policy for any stock is characterized by four numbers, the buy boundary, sell bounday, buy target and sell target. Optimal policy is to transact immediately to the buy target or sell target when reaching the corresponding boundary. Moreover, Liu (2004) showed that correlation between assets does not alter the general rule. Jang, Koo, Liu and Loewenstein (2007) found that in contrast to the standard literature, transaction costs can have a first order effect, and investor responds to changes in either regime by adjusting consumption and investment policies if the market conditions change over time. These models only consider infinite horizon problem, but they make known that to investigate how multiple assets, fixed transaction costs and regime-switching affect optimal investment (and consumption) policy would be both interesting and worthwhile.

3 Model

3.1 No transaction costs case

Assume in the economy pension funds can trade two assets continuously. The first asset is a money market account growing at a constant rate r, while the second asset is a risky security (the stock). The pension fund takes these prices as given and chooses quantities without any transaction cost. Further assume that the securities pay no dividend, and taxes on capital gains are zero. [1] Uncertainty in the model is generated by a standard Brownian motion w. We write down the two equations governing the dynamics of the money market account and the stock:

$$dB_t = rB_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dw_t$$

Denote the admissible trading strategies by (D, I). The processes D and I are the cumulative amount of sales and purchases of the stock. The two processes satisfy $D(0) = I(0) = 0$ and both are nondecreasing, right continuous adapted. The evolution of the amount invested in the money market account and the stock can be expressed as:[2]

$$db_t = rb_t dt - dI_t + dD_t$$

$$ds_t = \mu s_t dt + \sigma s_t dw_t + dI_t - dD_t$$

Pension funds all face a risk-return trade-off of providing a safe pension at low cost. The decision-making in a multiple member and multiple objective pension plan depends on the pension fund governance, the financial position of the fund and risk attitudes. Typically solvency positions, indexation quality and asset-liability risk are considered as primary objectives.

It is difficult to build a full-fledged objective function for a pension fund with heterogeneous members and a sponsor. For tractable quantitative derivation and insightful analytical solutions to optimal investments of a pension fund, the preferences are usually simplified to CRRA utility function of final wealth, that is, $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$ for $\gamma > 0, \gamma \neq 1$. [3]

On behalf of the plan participants, the pension fund chooses optimal investment strategies D and I so as to maximize the final wealth at a deterministic time T.

[1] An extension to the case of dividend payment or taxes on capital gains would be of some interests.

[2] s and b are traditional notations for the fraction of wealth invested in stocks and money market account, but in this article we use them to denote the amount invested in stocks and money market account

[3] The emphasis on final wealth or funding ratio ignores intermediate payout.

Define the value function at time t as:

$$V(b,s,t;T) = \underset{(D,I)}{\text{Max}} E[\frac{(b_T + s_T)^{1-\gamma}}{1-\gamma}]$$

Assumption 1: the parameter values satisfy:

$$0 < \frac{\mu - r}{\gamma\sigma^2} < 1$$

It guarantees that b and s would be chosen to be strictly positive.

The pension fund's problem can therefore be written as:

$$V(b,s,t;T) = \underset{(s_t:t>0)}{\text{Max}} E[\frac{(b_T + s_T)^{1-\gamma}}{1-\gamma}] \tag{1}$$

subject to:

$$d(b_t + s_t) = r(b_t + s_t)dt + (\mu - r)s_t dt + \sigma s_t dw_t \tag{2}$$

The wealth of the pension fund $W \equiv s + b$, with dynamics given by:

$$dW_t = (rb_t + \mu s_t)dt + \sigma s_t dw_t \tag{3}$$

To solve the dynamic optimization problem, we first derive the Hamilton-Jacobi-Bellman(HJB) partial differential equation. Start with the Bellman equation:

$$V(b,s,t;T) = \underset{s}{\text{Max}} E[V(b', s', t + \Delta t; T)] \tag{4}$$

where s' and b' denote the amount invested in money market account and stock at time $t + \Delta t$. Hence:

$$0 = \underset{s}{\text{Max}} E[V(b', s', t + \Delta t; T) - V(b,s,t;T)]$$

Dividing by Δt and let it go to 0, the Bellman equation becomes:

$$0 = \underset{s}{\text{Max}} \frac{1}{dt} E[dV] \tag{5}$$

Ito's lemma states:

$$dV = \frac{dV}{dt}dt + \frac{dV}{dW}dW + \frac{1}{2}\frac{d^2V}{dW^2}(dW)^2$$

Rewrite this by substituting the SDE (stochastic differential equation) for dW:

$$dV = (\frac{dV}{dt} + (rb + \mu s)\frac{dV}{dW} + \frac{1}{2}\sigma^2 s^2 \frac{d^2V}{dW^2})dt + \sigma s \frac{dV}{dW} dw$$

Applying it to the Bellman equation, we get the HJB equation:

$$V_t + V_W(rb + \mu s) + \frac{1}{2}V_{WW}\sigma^2 s^2 = 0 \tag{6}$$

with the terminal condition:

$$V(s, b, T; T) = \frac{{W_T}^{1-\gamma}}{1-\gamma} \tag{7}$$

To get rid of b, use the condition $W = b + s$. Equation (57) thus becomes:

$$V_t + V_W(rW + (\mu - r)s) + \frac{1}{2}V_{WW}\sigma^2 s^2 = 0 \tag{8}$$

Observing the homogeneity of the objective function, the restriction and the terminal condition, we conjecture that value function V must be linear to $\frac{W^{1-\gamma}}{1-\gamma}$. Rewrite $V(s, b, t; T)$ as $H(t; T) \cdot \frac{W^{1-\gamma}}{1-\gamma}$ to replace the value functions in the HJB equation:

$$H'\frac{W^{1-\gamma}}{1-\gamma} + HW^{-\gamma}(rW + (\mu - r)s) - \frac{\gamma}{2}HW^{-\gamma-1}\sigma^2 s^2 = 0$$

First order condition on s gives the optimal amount invested in stock:

$$s^* = \frac{\mu - r}{\gamma\sigma^2}W \tag{9}$$

It can be easily shown that optimal investment policy is characterized by one constant, that is, the ratio of the two asset values:

$$\theta^* = \frac{s^*}{b^*} = \frac{\frac{\mu-r}{\gamma\sigma^2}W}{(1 - \frac{\mu-r}{\gamma\sigma^2})W} = \frac{\mu - r}{\gamma\sigma^2 - \mu + r} \tag{10}$$

The result is also given by Merton(1971,1973) that without transaction costs optimal policy involves investing a constant fraction of wealth in the stock, independent of the investor's horizon. As long as this ratio is positive, the pension fund always holds the stock in its portfolio.

Now we will proceed to find the solution to the HJB equation.[4] Substituting s by its optimal value $\frac{\mu-r}{\gamma\sigma^2}W$ in the HJB equation and rearranging terms gives:

$$\frac{H'}{1-\gamma} + Hr + H\frac{(\mu - r)^2}{\gamma\sigma^2} = 0 \tag{11}$$

We also obtain the differential equation of function H:

$$-\frac{H'}{H} \equiv r(1-\gamma) - \delta \tag{12}$$

[4]The derivation of value function is necessary for comparison purpose. Allowing for intermediate consumption in section 3.3, we will prove that instead of a constant H, the coefficient function becomes horizon-dependent.

where

$$\delta = -\frac{1}{2}(1-\gamma)\frac{(\mu - r)^2}{\gamma\sigma^2} \tag{13}$$

Together with the terminal condition:

$$H(T;T) = \frac{V(s,b,T;T)}{W^{1-\gamma}/(1-\gamma)} = 1 \tag{14}$$

it implies that the horizon-dependent solution to the investment problem is:

$$V(s,b,t;T) = e^{(r(1-\gamma)-\delta)(T-t)}\frac{W^{1-\gamma}}{1-\gamma}$$

This is the maximized lifetime expected utility at time t under optimal investment policy.

3.2 Proportional transaction costs

In this section we examine the optimal investment policy when proportional transaction costs are presented. As shown by Constantinides(1979, 1986) and Taksar, Klass and Assaf(1983), among all, an investment policy is simple in a sense that it is characterized by two reflecting barriers, the buy boundary $\overline{\lambda}$ and the sell boundary $\underline{\lambda}$, with $\underline{\lambda} < \overline{\lambda}$. The investor stops transacting as far as the the portfolio ratio $\frac{b_t}{s_t}$ falls in the no-transaction region $[\underline{\lambda}, \overline{\lambda}]$, while he immediately transactes to the closest boundary when the ratio falls outside. In line with the proportional nature of transaction costs, the optimal trading size in continuous time model is always infinitely small so as to keep the portfolio ratio in the interval of no-transaction region.

To capture the idea that purchasing stock and bond both involves transaction costs, the proportional transaction cost rate k is defined as the amount of one asset the investor can buy by selling one unit amount of the other, with $0 \leq k < 1$, as in Dumas and Luciano (1991) where they introduced the term 'conversion ratio'. This definition reflects the two-way property of transaction costs.[5]

Now we can restate the pension fund's problem.

The finite horizon value function:

$$V(b,s,t;T) = \underset{(b_t,s_t:t>0)}{\text{Max}} E[\frac{(b_T+s_T)^{1-\gamma}}{1-\gamma}] \tag{15}$$

[5]There are several ways to model proportional transaction costs. For instance, to capture the idea that the pension fund can buy the stock at the ask price $S_t^A = S_t$, while sell it at the bid price $S_t^B = (1-k)S_t$, proportional transaction cost rate is defined as the bid-spread ask rate k. According to this definition, transaction costs only take place when selling the stock and deplete the money market account.

subject to:

$$db_t = rb_t dt - dI_t + (1-k)dD_t$$

$$ds_t = \mu s_t dt + \sigma s_t dw_t + (1-k)dI_t - dD_t$$

Assumption 2: the value function $V(b,s,t;T)$ is once continuously differentiable in b and twice continuously differentiable in s.

At each point in time, the three regions are identified depending on the portfolio ratio: the no-transaction region, the buy region, and the sell region, as in Davis and Norman(1990), Liu and Loewenstein(2002). At the no-transaction region, the value function must satisfy the modified HJB equation:

$$V_t + V_b rb + V_s \mu s + \frac{1}{2} V_{ss} \sigma^2 s^2 = 0 \qquad \underline{\lambda} \leq \frac{b}{s} \leq \overline{\lambda} \tag{16}$$

The HJB is obtained by applying Ito's Lemma to the Bellman equation (5):

$$dV = (\frac{dV}{dt} + rb\frac{dV}{db} + \mu s\frac{dV}{ds} + \frac{1}{2}\sigma^2 s^2 \frac{d^2V}{ds^2})dt + \sigma s \frac{dV}{ds} dw$$

For $\frac{b}{s} > \overline{\lambda}$, in the buy region, the marginal rate of substitution of the money market account for the stock must equal $\frac{1}{1-k}$:[6]

$$V_b = (1-k)V_s \qquad \frac{b}{s} > \overline{\lambda} \tag{17}$$

For $\frac{b}{s} < \underline{\lambda}$, the marginal rate of substitution of the money market account for the stock must equal $1-k$:[7]

$$(1-k)V_b = V_s \qquad \frac{b}{s} < \underline{\lambda} \tag{18}$$

In addition, the terminal condition must be satisfied

$$V(s,b,T;T) = \frac{(b_T + s_T)^{1-\gamma}}{1-\gamma} \tag{19}$$

[6]When in the buy region, sell dI amount of b to purchase $(1-k)dI$ amount of s. The value matching condition is satisfied such that there is no jump in the value of the problem.

$$V(b,s) = V(b - dI, s + (1-k)dI)$$

Expanding the right hand side, it follows that marginal cost of decreasing investment in the money market account equals $1-k$ times marginal benefit of increasing investment in stock.

[7]Proof is similar by using the quality:

$$V(b,s) = V(b + (1-k)dD, s - dD)$$

The terminal condition reflects the fact that at the time the representative individual retires (the horizon point), the pension fund has to turn all the investment into cash and pay out as pension benefits.

It is difficult to solve this system of partial differential equations(PDEs). We introduce a new value function in order to get an equivalent system of ordinary differential equations(ODEs). The value function $V(s,b,t;T)$ is homogeneous of degree $1-\gamma$ for all positive numbers in (s,b), as shown by Fleming and Soner (1993). Define $h = \frac{b}{s}$, for a new value function $g : (-\infty, +\infty) \times [0,T] \to \Re$,[8] homogeneity gives: [9]

$$V(b,s,t;T) = s^{1-\gamma} g(h,t;T) \tag{20}$$

The no-transaction region, buy region and sell region thus can be characterized by two horizon-dependent boundaries $\overline{\lambda}(t;T)$ and $\underline{\lambda}(t,T)$. By applying the chain rule, we find the new value function and its derivatives with respect to h and t:

$$V = s^{1-\gamma} g$$

$$V_t = s^{1-\gamma} g_t$$

$$V_b = s^{-\gamma} g_h$$

$$V_s = (1-\gamma)s^{-\gamma} g - bs^{-\gamma-1} g_h$$

$$V_{ss} = \gamma(\gamma-1)s^{-\gamma-1} g + 2\gamma b s^{-\gamma-2} g_h + b^2 s^{-\gamma-3} g_{hh}$$

Substituting the new value function and its derivatives into the modified HJB equation and the two equations of marginal substitution on the buy region and sell region, one obtains a system of ODEs.

On the no-transaction region:

$$\frac{1}{2} g_{hh}\sigma^2 h^2 + g_h(\gamma\sigma^2 - (\mu - r))h + g(1-\gamma)(\mu - \gamma\sigma^2/2) + g_t = 0 \qquad \underline{\lambda}(t;T) \le h \le \overline{\lambda}(t,T) \tag{21}$$

On the buy region:

$$(\frac{1}{1-k} + h) g_h(h,t;T) = (1-\gamma) g(h,t;T) \qquad h > \overline{\lambda}(t,T) \tag{22}$$

[8] Solvency would require $g : (k-1,\infty) \times [0,T] \to \Re$

[9] $V(b,s,t;T) = s^{1-\gamma} g(h,1,t;T)$, from now on we compress $g(h,1,t;T)$ to $g(h,t;T)$ for the sake of simplicity

On the sell region:

$$(1-k+h)g_h(h,t;T) = (1-\gamma)g(h,t;T) \qquad h < \underline{\lambda}(t,T) \tag{23}$$

In addition, same as in the previous problem, the terminal condition must be satisfied:

$$g(h,T;T) = \frac{(h_T+1)^{1-\gamma}}{1-\gamma} \tag{24}$$

In the infinite horizon models, the search for the limite is identical to the search for the stationary solution, for which i) the value function $V(b,s,t;T)$ is independent of time t; ii) the buy boundary $\overline{\lambda}$ and the sell boundary $\underline{\lambda}$ are independent of time, in addition to being independent of s and b, as in Dumas and Luciano(1991).

In contrast, the finite horizon model highlights horizon dependence of the value function $g(h,t;T)$ and the corresponding two boundaries $\overline{\lambda}(t;T)$, $\underline{\lambda}(t,T)$. Since the two free boundaries are moving over time, it is difficult to solve the ODEs system depicted above. In section 4 we use numerical approximations to compute the optimal investment policy when the pension fund is assumed to have a finite investment horizon.

3.3 Lifecycle investment without transaction costs

From this section on we consider the optimal investment policy as well as the consumption-savings decision from a lifecycle perspective. The lifecycle model allows for flexible intermediate consumption. Pension funds, especially collective DC plans should take optimal consumption choice into account while making strategic decisions on behalf of their participants. As documented by earlier literature, proportional transaction costs reduce the consumption rate, though some debated on whether the effect is weak or not.

Following Merton (1971), assume there is a single perishable consumption good as numeraire. The pension plan participants derive utility from intertermporal consumption c of this good and the terminal wealth at time T. Pension plan participants are impatient. Their time preference can be summerized by a discount rate ρ. The consumption is made through the money market account. The participant has a CRRA utility function over consumption and terminal wealth, as defined in the previous section. We ignore labor income in this context.

Assumption 3: The participant makes intermediate consumption decision on the admissible consumption space $\mathbb{C}$, which satisfies $\int_0^t |c_s|ds < \infty, \forall t \in [0,T]$.

Assumption 4: Consumption is made through the money market account.

The pension fund's problem becomes:

$$J(c,b,s,t;T) = \underset{(c_t,b_t,s_t:t>0)}{\text{Max}} E[\int_0^T e^{-\rho t}\frac{{c_t}^{1-\gamma}}{1-\gamma}dt + e^{-\rho T}\frac{(b_T+s_T)^{1-\gamma}}{1-\gamma}] \tag{25}$$

subject to:

$$db_t = rb_t dt - c_t dt - dI_t + dD_t$$

$$ds_t = \mu s_t dt + \sigma s_t dw_t + dI_t - dD_t$$

The constraints above are equivalent to:

$$dW_t = (rb_t + \mu s_t - c_t)dt + \sigma s_t dw_t \tag{26}$$

The value function should also satisfy the terminal condition:

$$J(c,b,s,T;T) = \frac{(b_T+s_T)^{1-\gamma}}{1-\gamma} \tag{27}$$

The first term of the value function J represents discounted utility from consumption flows, while the second term captures the idea that terminal wealth gives utility to the participant as well, for he can finance his retirement consumption by using the benefit payments from time T onwards. To solve optimal consumption and investment policy, the technique of stochastic dynamic optimization is used again.

Start with the Bellman equation:

$$J(c,b,s,t;T) = \underset{c,s}{\text{Max}}\left\{\frac{c^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho}E[J(c',b',s',t+\Delta t;T)]\right\} \tag{28}$$

The actual utility over the time interval of length Δt is $\frac{c^{1-\gamma}}{1-\gamma}\Delta t$, and the discounting over such time interval is expressed by $\frac{1}{1+\rho\Delta t}$. Therefore the Bellman equation becomes:

$$J(c,b,s,t;T) = \underset{c,s}{\text{Max}}\left\{\frac{c^{1-\gamma}}{1-\gamma}\Delta t + \frac{1}{1+\rho\Delta t}E[J(c',b',s',t+\Delta t;T)]\right\} \tag{29}$$

Multiplying both LHS and RHS by a factor of $1+\rho\Delta t$ and rearranging the terms, we get:

$$\rho J\Delta t = \underset{c,s}{\text{Max}}\left\{\frac{c^{1-\gamma}}{1-\gamma}\Delta t(1+\rho\Delta t) + E[\Delta J]\right\}$$

Dividing by Δt and let it go to 0, the Bellman equation becomes:

$$\rho J = \underset{s}{\text{Max}}\left\{\frac{c^{1-\gamma}}{1-\gamma} + \frac{1}{dt}E[dJ]\right\} \tag{30}$$

Ito's lemma states:

$$dJ = (\frac{dJ}{dt} + (rb + \mu s - c)\frac{dJ}{dW} + \frac{1}{2}\sigma^2 s^2 \frac{d^2 J}{dW^2})dt + \sigma s \frac{dJ}{dW} dw$$

Applying it to the Bellman equation, we get the corresponding HJB equation:

$$\frac{c^{1-\gamma}}{1-\gamma} + J_t + J_W(rb + \mu s - c) + \frac{1}{2} J_{WW}\sigma^2 s^2 - \rho J = 0 \tag{31}$$

We derive optimal consumption policy from the HJB equation. First order condition with respect to consumption on the HJB equation yields:

$$0 = \frac{\partial}{\partial c}\frac{c^{1-\gamma}}{1-\gamma} - J_W$$

which gives the optimal consumption:

$$c^* = (J_W)^{-\frac{1}{\gamma}} \tag{32}$$

Substituting optimal consumption into HJB equation yields:

$$\frac{c^{*1-\gamma}}{1-\gamma} + J_t + J_W(rb + \mu s - c^*) + \frac{1}{2} J_{WW}\sigma^2 s^2 - \rho J = 0$$

To eliminate b from the equation, use the condition $W = b + s$:

$$\frac{c^{*1-\gamma}}{1-\gamma} + J_t + J_W(rW + (\mu - r)s - c^*) + \frac{1}{2} J_{WW}\sigma^2 s^2 - \rho J = 0$$

We conjecture that the value function J must be linear to $\frac{W^{1-\gamma}}{1-\gamma}$. It takes the form $J(c, b, s, t; T) = M(t; T) \cdot \frac{W^{1-\gamma}}{1-\gamma}$ for a horizon-dependent function $M(t; T) > 0$, $\forall t \in [0, T]$.

Replacing c^* by $(J_W)^{-\frac{1}{\gamma}} = M^{-\frac{1}{\gamma}} W$, J_t by $M' \frac{W^{1-\gamma}}{1-\gamma}$ and J by $M \frac{W^{1-\gamma}}{1-\gamma}$ in the HJB equation, it follows that:

$$\frac{M^{-\frac{1-\gamma}{\gamma}}}{1-\gamma} W^{1-\gamma} + M' \frac{W^{1-\gamma}}{1-\gamma} + MW^{-\gamma}(rW + (\mu - r)s - M^{-\frac{1}{\gamma}} W) - \frac{\gamma}{2} MW^{-\gamma-1}\sigma^2 s^2 - \rho M \frac{W^{1-\gamma}}{1-\gamma} = 0$$

First order condition on s gives the optimal amount invested in stock:

$$s^* = \frac{\mu - r}{\gamma\sigma^2} W \tag{33}$$

We have same result as in the previous section that without transaction costs optimal investment policy involves investing a constant fraction of wealth in the stock, independent of the investor's horizon. As long as $\mu > r$, the pension fund always holds the stock in its portfolio. Allowing for intermediate consumption does not change optimal investment policy. The ratio of the amount invested in stock and money market account is:

$$\pi^* = \frac{s^*}{b^*} = \frac{\frac{\mu - r}{\gamma\sigma^2}W}{(1 - \frac{\mu - r}{\gamma\sigma^2})W} = \frac{\mu - r}{\gamma\sigma^2 - \mu + r} \tag{34}$$

Now replacing s by the optimal value $s^* = \frac{\mu - r}{\gamma\sigma^2}W$ in the HJB equation and rearrange, we find the ODE of M on t:

$$M^{-\frac{1-\gamma}{\gamma}}\frac{\gamma}{1-\gamma} + \frac{M'}{1-\gamma} + Mr + M\frac{(\mu - r)^2}{2\gamma\sigma^2} - M\frac{\rho}{1-\gamma} = 0$$

Formalizing it to:

$$\frac{dM}{dt} = -\gamma M^{-\frac{1-\gamma}{\gamma}} - ((1-\gamma)r + \frac{(1-\gamma)(\mu - r)^2}{2\gamma\sigma^2} - \rho)M \tag{35}$$

with the terminal condition:

$$M(T, T) = 1 \tag{36}$$

We can derive M at each time t numerically by discretization $M_t = M_{t-1} + \Delta M_t$ and work backward from the terminal time. Optimal consumption contains a horizon-dependent fraction of wealth, which is independent of wealth at hand:[10]

$$c_t^* = M(t, T)^{-\frac{1}{\gamma}} W_t \tag{37}$$

3.4 Lifecycle investment with transaction costs

Considering the proportional transaction costs described in section 3.2, one may not feel too surprised to see that optimal investment policy is characterized by a no-transaction region, a buy region and a sell region. But the new message brought by a lifecycle model is that if the introduction of transaction costs also has an impact on optimal consumption policy, then pension fund participants should take this into account when making consumption-savings decisions. In the simple lifecycle model of constantinides (1986)[11], consumption policy is set to be a constant fraction of riskless asset. We release this assumption in the following analysis.

[10]It can be easily shown that in the infinite horizon case, optimal consumption is a constant proportion of wealth:

$$c_t^* = \frac{1}{\gamma}(\rho - (1-\gamma)r - \frac{(1-\gamma)(\mu - r)^2}{2\gamma\sigma^2})W_t$$

as given by Merton (1973)

[11]Labor income is safely ignore again.

As for whether transaction costs affect consumption positively or negatively, we still do not reach any concrete conclusion up till now. What we can say is that transaction costs have two opposite effects on consumption. The income effect depresses consumption since transaction costs deplete the capital gains and hence wealth at hand. On the other hand, the substitution effect shifts consumption to the earlier stage as current consumption becomes less costly than future consumption in terms of transaction costs.

Based on section 3.2 and 3.3, we are now able to build a model to quantify the impact of transaction costs on optimal investment and consumption policy.

The pension fund's problem is:

$$J(c,b,s,t;T) = \underset{(c_t,b_t,s_t:t>0)}{\text{Max}} E[\int_0^T e^{-\rho t}\frac{{c_t}^{1-\gamma}}{1-\gamma}dt + e^{-\rho T}\frac{(b_T+s_T)^{1-\gamma}}{1-\gamma}] \tag{38}$$

subject to:

$$db_t = rb_t dt - c_t dt - dI_t + (1-k)dD_t$$
$$ds_t = \mu s_t dt + \sigma s_t dw_t + (1-k)dI_t - dD_t$$

The value function should also satisfy the terminal condition, that all the stock holding must be transformed to cash at time T:

$$J(c,b,s,T;T) = \frac{(b_T+s_T)^{1-\gamma}}{1-\gamma} \tag{39}$$

The two boundary equations in the sell region and the buy region are the same as in section 3.2. In the no-transaction region, the corresponding HJB equation is similar to that in section 3.3. To obtain the HJB, apply Ito's lemma:

$$dJ = (\frac{dJ}{dt} + (rb-c)\frac{dJ}{db} + \mu s\frac{dJ}{ds} + \frac{1}{2}\sigma^2 s^2\frac{d^2J}{ds^2})dt + \sigma s\frac{dJ}{ds}dw$$

to the Bellman equation (30). We have:

$$\frac{c^{1-\gamma}}{1-\gamma} + J_t + J_b(rb-c) + J_s\mu s + \frac{1}{2}J_{ss}\sigma^2 s^2 - \rho J = 0 \qquad \underline{\lambda} \le \frac{b}{s} \le \overline{\lambda} \tag{40}$$

$$J_b = (1-k)J_s \qquad \frac{b}{s} > \overline{\lambda} \tag{41}$$

$$(1-k)J_b = J_s \qquad \frac{b}{s} < \underline{\lambda} \tag{42}$$

Substituting optimal consumption into HJB equation $c^* = (J_b)^{-\frac{1}{\gamma}}$ yields:

$$\frac{({J_b}^{-\frac{1}{\gamma}})^{1-\gamma}}{1-\gamma} + J_t + J_b(rb - J_b^{-\frac{1}{\gamma}}) + J_s\mu s + \frac{1}{2}J_{ss}\sigma^2 s^2 - \rho J = 0 \qquad \underline{\lambda} \le \frac{b}{s} \le \overline{\lambda}$$

The value function $J(c, s, b, t; T)$ is homogeneous of degree $1 - \gamma$ for all positive numbers in (s, b). Define $h = \frac{b}{s}$, for a new value function $f : (-\infty, +\infty) \times [0, T] \rightarrow \Re$ so that $J(c, b, s, t; T) = s^{1-\gamma} f(c, h, t; T)$ The no-transaction region, buy region and sell region thus can be characterized by two horizon-dependent boundaries $\overline{\lambda}(t; T)$ and $\underline{\lambda}(t, T)$. We derive the new value function and its derivatives as follows:

$$J = s^{1-\gamma} f$$

$$J_t = s^{1-\gamma} f_t$$

$$J_b = s^{-\gamma} f_h$$

$$J_s = (1 - \gamma)s^{-\gamma} f - bs^{-\gamma-1} f_h$$

$$J_{ss} = \gamma(\gamma - 1)s^{-\gamma-1} f + 2\gamma bs^{-\gamma-2} f_h + b^2 s^{-\gamma-3} f_{hh}$$

Substituting the new value function and its derivatives with respect to h into the modified HJB equation and the two boundary equations on the buy region and the sell region, one obtains a system of ODEs.

On the no-transaction region:

$$\frac{\gamma}{1-\gamma} {f_h}^{1-\frac{1}{\gamma}} + \frac{1}{2} f_{hh} \sigma^2 h^2 + f_h(\gamma\sigma^2 - (\mu - r))h + f((1 - \gamma)(\mu - \gamma\sigma^2/2) - \rho) + f_t = 0$$

$$\underline{\lambda}(t; T) \leq h \leq \overline{\lambda}(t, T) \tag{43}$$

On the buy region:

$$(\frac{1}{1-k} + h) f_h(h, t; T) = (1 - \gamma) f(h, t; T) \qquad h > \overline{\lambda}(t, T) \tag{44}$$

On the sell region:

$$(1 - k + h) f_h(h, t; T) = (1 - \gamma) f(h, t; T) \qquad h < \underline{\lambda}(t, T) \tag{45}$$

In addition, the following terminal condition must be satisfied:

$$f(h, T; T) = \frac{(h_T + 1)^{1-\gamma}}{1 - \gamma} \tag{46}$$

This set of ODEs are similar to that derived in section 3.2 except for the first term in HJB equation due to intermediate consumption. The two moving boundaries which depend on the horizon largely complicate comuptation of closed-form solution, thus we look upon numerical approximations to find a solution. We present the algorithm to obtain these boundaries in section 4.

4 Numerical approximation

To our knowledge, no closed form solution exists for the finite horizon maximization problem on presence of transaction costs, as the two free boudarie move over time. It is therefore tempting to use numerical methods, such as finite difference method and binomial tree method, as in Chellathurai and Draviam (2007), Gennotte and Jung (1994). These methods are widely used for option pricing when the holder has early exercise decisions to make prior to maturity. In this chapter, we present a algorithm to derive the two boundaries between which portfolio ratio is allowed to fluctuate. Dynamic programming technique is applied to restate the problems described in section 3.2 and 3.4, and then we develop a multi-period binomial tree model to approximate the underlying stock return dynamics, in our case, a log normal distribution.[12]

4.1 Dynamic programming formulation

We first of all rewrite the continuous time model to a discrete time model. At date $t = 0, 1, ...T - 1$, pension fund can rebalance the investment. For the sake of convenience, we still keep using the same notations as in the previous section.

The problem of the pension fund is:

$$V(b, s, t; T) = \underset{(b_t, s_t : t > 0)}{\text{Max}} E[\frac{(b_T + s_T)^{1-\gamma}}{1 - \gamma}] \tag{47}$$

The proportional transaction costs take place both when selling and purchasing the stock, thus we distinguish the following two cases:

The wealth dynamics of money market account and stock in discrete time when purchasing the stock:

$$b_{t+1} = (b_t - u_t)r$$
$$s_{t+1} = (s_t + (1 - k)u_t)z_{t+1}$$

The wealth dynamics of money market account and stock in discrete time when selling the stock:

$$b_{t+1} = (b_t + (1 - k)u_t)r$$
$$s_{t+1} = (s_t - u_t)z_{t+1}$$

where u_t is positive in both cases.

Next we format the problem by the dynamic programming algorithm:

$$V(b_t, s_t, t; T) = \underset{u_t}{\text{Max}} \ E_t[V(b_{t+1}, s_{t+1}, t + 1; T)] \tag{48}$$

[12] We will show later that by choosing appropriate values for the parameters, the multi-period binomial distribution could converge to one of many common-seen distributions for the stock return.

with the boudary condition:

$$V(b_T, s_T, T; T) = \frac{(b_T + s_T)^{1-\gamma}}{1-\gamma} \tag{49}$$

Recalling that the optimal investment policy is characterized by two numbers, which divide the investment space to three regions, the sell region, the buy region and the no-transaction region. In the discrete time setting, the same rule applies. The buy boundary and sell bounday are both functions of time (horizon), which we denote by $\overline{\lambda}_t$ and $\underline{\lambda}_t$ respectively. Furthermore, denote the money market account to stock ratio at time t as h_t, and accordingly a new value function $g(h_t, t; T)$ so that $V(b_t, s_t, t, T) = s^{1-\gamma} g(h_t, t; T)$, with the boundary condition $g(h_T, T; T) = \frac{(h_T+1)^{1-\gamma}}{1-\gamma}$, noticing that V is homogeneous of degree $1-\gamma$ on b_t, s_t.

Preliminary result shows:

$$V(b_t, s_t, t; T) = \begin{cases} g(\overline{\lambda}_t, t; T) s_t^{1-\gamma} & \text{if} \quad h_t > \overline{\lambda}_t \\ E_t[V(b_t r, s_t z_{t+1}, t+1; T)] & \text{if} \quad \underline{\lambda}_t \leq h_t \leq \overline{\lambda}_t \\ g(\underline{\lambda}_t, t; T) s_t^{1-\gamma} & \text{if} \quad h_t < \underline{\lambda}_t \end{cases} \tag{50}$$

On the no-transaction region, i.e. $\underline{\lambda}_t < h_t < \overline{\lambda}_t$, no trade is made and the amount invested in money market account and stock evolves according to the return dynamics.

On the buy region, i.e. $h_t > \overline{\lambda}_t$, pension fund chooses optimal control u_t to rebalance the portfolio. Stock holding increases to $s_t + (1-k)u_t$, while the amount invested in money market account decreases to $b_t - u_t$. Noticing that $\overline{\lambda}_t$ maximizes the value function $g(h_t, t; T)$. Pension fund always finds it optimal to rebalance the portfolio in order to reach the ratio $\overline{\lambda}_t$ at time t. The relationship of $g(h_t, t; T)$ and $g(\overline{\lambda}_t, t; T)$ can be found.

On the sell region, i.e. $h_t < \underline{\lambda}_t$, selling u_t amount of stock incurs transaction costs ku_t, the amount invested in money market account only increases to $b_t + (1-k)u_t$. It is always optimal to transact to the sell boundary. The relationship of $g(h_t, t; T)$ and $g(\underline{\lambda}_t, t; T)$ is also found.

$g(h_t, t; T)$ is completely determined by h_t. At the terminal date, $g(h_T, T; T)$ represents indirect utility attached to an unit investment in stock. The derivation of the moving boundaries reduces to the identification of function g for all t and h_t. In the next section, we use backward induction to compute these values for which three equations characterize the indirect value function g.

Proposition 1: The rules governing the indirect utility function at buy boundary, no-transaction region and sell boundary can be expressed as:

$$g(h_t,t;T)=\begin{cases} (\frac{h_t(1-k)+1}{\overline{\lambda}_t(1-k)+1})^{1-\gamma}g(\overline{\lambda}_t,t;T) & \text{if} \quad h_t>\overline{\lambda}_t \\ E_t[(\frac{h_t}{h_{t+1}}r)^{1-\gamma}g(h_{t+1},t+1;T)] & \text{if} \quad \underline{\lambda}_t\le h_t\le\overline{\lambda}_t \\ (\frac{h_t+1-k}{\underline{\lambda}_t+1-k})^{1-\gamma}g(\underline{\lambda}_t,t;T) & \text{if} \quad h_t<\underline{\lambda}_t \end{cases} \tag{51}$$

Proof: See Appendix 1.

4.2 The binomial tree method

Binomial tree is a diagram representing possible paths that might be followd by the stock price up to the terminal date. Binomial tree approach is a popular technique for pricing options, first proposed by Cox, Ross and Rubinstein (1979). By assuming that the stock price follows a multistep binomial tree, we can treat each node of the tree separately and work back from the terminal date. A large number of steps consisted in the binomial tree certainly improves accuracy of the approximation, but also requires considerable computational effort. Figlewski and Gao (1999) proposed adaptive mesh model to graft a high resolution tree on to a low resolution tree. Moreover, trinomial trees can be used as an alternative to binomial trees.

To illustrate the binomial tree model, we first divide the investment horizon into a large number of small time intervals. The stock price can either go up or go down at each time point. The stock earns a return of u for an up movement and a return of d for a down movement. The probability of an up movement is denoted by p, while that of down movement by $1-p$. The riskless money market account always earns a return of r. To exclude arbitrage, assume $d<r<u$.[13]

The precision of the results increases with the number of initial values. Starting from any time t, conditionally on the value h_t, h_{t+1} follows a binomial distribution:

$$h_{t+1}=\begin{cases} \frac{r}{u}h_t & \text{with probability} \quad p \\ \frac{r}{d}h_t & \text{with probability} \quad 1-p \end{cases}$$

We refer to the $i-th$ node at time t as the (t,i) node, where $0\le t\le T$ and $1\le i\le t+N$, where N is related to the precision of the search and increases linearly on time. Introduce new notations $h_{t,i}$ and $g(h_t,i)$ to represent the portfolio ratio and indirect value function at the (t,i) node. Following the algorithms of Gennotte and Jung (1994), we start from deriving the no-transaction region one period before terminal date, since directly starting from the terminal date and searching for the boundaries one period before terminal date without any restriction dramatically increase computational effort. The boundaries at period $T-1$ are given by:

[13] We implicitely assume d,u,p are all constant. In a more generalized form, these parameters can be time-dependent

$$\overline{\lambda}_{T-1} = \frac{uM - d}{r(1-M)}, \text{ where } M = \left(\frac{p(u - r(1+k))}{(1-p)(r(1+k)-d)}\right)^{-\frac{1}{\gamma}} \tag{52}$$

$$\underline{\lambda}_{T-1} = \frac{u\widetilde{M} - d}{r(1-\widetilde{M})}, \text{ where } \widetilde{M} = \left(\frac{p(u - r(1-k))}{(1-p)(r(1-k)-d)}\right)^{-\frac{1}{\gamma}} \tag{53}$$

To find $g(h_t, i)$ at all nodes, we compute $g(h_{T-1}, i)$ by using the boundary condition:

$$g(h_{T-1}, i) = \frac{(h_{T-1,i} + 1)^{1-\gamma}}{1-\gamma} \tag{54}$$

The grid points starting from the smallest value increase with i:

$$h(T-1, i) = \underline{\lambda}_{T-1}\left(\frac{u}{d}\right)^{\frac{i-1}{L}}$$

where the positive integer L sets the precision of the search. With larger L, the boundaries become smoother. On the other hand, we have to make sure that following either an up or down movement the points at a earlier period lead to ratios belonging to the next period. Therefore the number of points decreases with the speed proportional to L as we move backwards. [14] Moreover, using binomial distribution, the portfolio ratio on all nodes can be computed straightforward. For example, the ith node at time $T-2$ is given by:

$$h(T-2, i) = \underline{\lambda}_{T-1}\left(\frac{u}{r}\right)\left(\frac{u}{d}\right)^{\frac{i-1}{L}}$$

Optimal investment policy is derived backwards from period $T-1$ on to the initial time. Noticing that the number of nodes on the binomial tree decreases as we work backwards, it might be the case that there is not enough point left at some period. In this situation, we modify the set of initial values on $T-1$ and repeat the same procedure which we will demonstrate in detail. By selecting a set of initial points around the no-transaction region, we can always obtain the boundaries.

Then we derive the value function two periods prior to the terminal time $g(h_{T-2}, i)$. Under the assumption that no transaction takes place at time $T-2$, we use the second expression from equation (51) to compute each $g(h_{T-2}, i)$:

$$g(h_{T-2}, i) = E_t[(\frac{h_{T-2,i}}{h_{T-1,j}} r)^{1-\gamma} g(h_{T-1}, j)] \qquad \text{where } j = i, i+L$$

An up movement from node $(T-2, i)$ arrives at the node $(T-1, i)$ and a down movement at the node $(T-1, i+L)$. The expectation is a weighted average with weight p and $1-p$:

[14]We verified that $L = 100$ is sufficiently high to yield a smooth curve

$$h_{T-1,i} = \frac{r}{u} h_{T-2,i}, \quad h_{T-1,i+L} = \frac{r}{d} h_{T-2,i}$$

Equation (51) therefore reduces to:

$$g(h_{T-2}, i) = pu^{1-\gamma} g(h_{T-1}, i) + (1-p) d^{1-\gamma} g(h_{T-1}, i+L) \tag{55}$$

Repeat this procedure for each period and for all nodes at time $T-2$ so that we can attach values under no-transaction assumption to these nodes on the binomial tree.

Now we turn to the procedure to find buy boundary and sell bounday two periods priror to the terminal time $T-1$. If the optimal investment policy is to transact (either purchase or sell stock), trade from one node to its neighbors must yield positive profit. (We have computed the value function at each node.)

We start from the first node, i.e. $i = 1$, which is characterized by the smallest possible portfolio ratio. If we select appropriate set of initial values, the sell boundary should locate in the interior of the set. Pension fund has incentive to sell stock on this extreme scenario in which stock performs very well. The value function $g(h_{T-2}, 1)$ is known as we have computed before. On the sell region, i.e. $h_{T-2,1} < \overline{\lambda}_t$, the value function resulting from selling an additional amount of stock must increase. We compute this value function by using the third expression in equation (51), with h_t replaced by $h_{T-2,1}$, $\underline{\lambda}_t$ by $h_{T-2,2}$, and $g(\underline{\lambda}_t, t; T)$ by $g(h_{T-2}, 2)$:

$$g(h_{T-2}, 1) = (\frac{h_{T-2,1} + 1 - k}{h_{T-2,2} + 1 - k})^{1-\gamma} g(h_{T-2}, 2) \tag{56}$$

The coefficient $(\frac{h_{T-2,1}+1-k}{h_{T-2,2}+1-k})^{1-\gamma}$ reflects the cost of rebalancing the portfolio.

Compare the value function resulting from selling an additional unit of stock (equation (56)) with the value function derived from no-transaction (equation (55)). If the former is larger, repeat this procedure for increasing values of $h_{T-2,i}$ to $h_{T-2,i+1}$, until additional stock sale cannot increases the value function. The level of $h_{T-2,i}$ for which this first occurs is the sell boundary at time $T-2$: $\underline{\lambda}_T - 2$.

The buy boundary can be derived in a similar fashion for which we start from the largest possible portfolio ratio and compute the value functions resulting form an additional amount of stock purchase. Now we know both boundaries $\overline{\lambda}_{T-2}$ and $\underline{\lambda}_{T-2}$ at time $T-2$. Using the following criteria, replace the value function at all nodes $(T-2, i)$ by the largest possible value:

$$g(h_{T-2}, i) = \begin{cases} (\frac{h_{T-2,i}(1-k)+1}{\overline{\lambda}_{T-2}(1-k)+1})^{1-\gamma} g(\overline{\lambda}_{T-2}) & \text{if} \quad h_{T-2,i} > \overline{\lambda}_{T-2} \\ \text{does not change} & \text{if} \quad \underline{\lambda}_{T-2} \leq h_{T-2,i} \leq \overline{\lambda}_{T-2} \\ (\frac{h_{T-2,i}+1-k}{\underline{\lambda}_{T-2}+1-k})^{1-\gamma} g(\underline{\lambda}_{T-2}) & \text{if} \quad h_{T-2,i} < \underline{\lambda}_{T-2} \end{cases} \tag{57}$$

Note that $g(\overline{\lambda}_{T-2})$ and $g(\underline{\lambda}_{T-2})$ represent the value function derived at the buy and sell boundaries. We use these new values to derive $g(h_{T-3}, i)$ under no-transaction assumption. Repeat the above sequential procedure for each period from $T-3$ to 0 so that the moving boundaries are identified.

Now we wish to choose the parameters r, u, d, p in a way such that the multiplicative binomial probability distribution converges to the lognormal distribution in the limit.[15]

Recall the stock earns a return of u for an up movement with probability p and a return of d for a down movement with probability $1-p$. If every year consists of n period, we consider dividing up the original longer time period (a year) into n shorter periods Δt (a month, a day or even less). When n goes to infinity, the multiplicative binomial probability distribution leads to lognormal distribution. It is much easier to perform the calculation with the natural logarithm of the rate of return, which is $log(u)$ with probability p and $log(d)$ with probability $1-p$.

Let the stock price at the beginning of this year and next year be S_0 and S_1. Consequently the annual log stock return is $log\left(\frac{S_1}{S_0}\right)$. Let j denote the number of upward movements occuring during each year. Hence we can write down the expectation and variance of log annual stock return:

$$E\left(log\left(\frac{S_1}{S_0}\right)\right) = log\left(\frac{u}{d}\right)E(j) + nlog(d)$$
$$Var\left(log\left(\frac{S_1}{S_0}\right)\right) = \left(log\left(\frac{u}{d}\right)\right)^2 + Var(j)$$

where $E(j) = np$ and $Var(j) = np(1-p)$ since each of the upward movements has probability p.

Recall the lognormal case:

$$dS_t = \mu S_t dt + \sigma S_t dw_t$$

or equivalently

$$dlogS_t = \mu dt + \sigma dw_t$$

The drift μ and volatility σ^2 reflect the empirical values of expectations and variance of log annual stock return. To find appropriate values for the parameters

[15]Cox, Ross and Robinstein (1979) showed that both a continuous process (lognormal distribution) and a jump process can be derived from the multiplicative binomial process by simply choosing appropriate u and d. In the case of jump process, the stock price usually moves in a smooth deterministic way, but occasionally experiences sudden discontinuous shocks. In this article, we only explore the lognormal case.

u, d, p, we want the mean and variance of the continuously compounded return to coincide with that of the actual stock prices as n goes to infinity. Combining all of this, we have:

$$E\left(log\left(\frac{S_1}{S_0}\right)\right) = n\left(plog\left(\frac{u}{d}\right) + log(d)\right) = \mu$$

$$Var\left(log\left(\frac{S_1}{S_0}\right)\right) = np(1-p)\left(log\left(\frac{u}{d}\right)\right)^2 = \sigma^2$$

Moreover, we have $d = \frac{1}{u}$. Thus, two unknowns with two equations, a little algebra gives the only correct values for the parameters:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad p = \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}) \tag{58}$$

The parameters are chosen in such a way that the limiting expectations and variances coincide. However, expectation and variance only describe certain aspects of the probability distribution. We need to verify that the multiplicative binomial distribution converges to the lognormal distribution by examining the remaining important aspects, especially, the skewness of the distribution and other higher order moments. Cox, Ross and Rubinstein (1979) have verified that the multipicative binomial distribution indeed includes the lognormal distribution as a limiting case, since the higher order properties of the distribution, such as skewness become less and less important relative to its standard deviation as n goes to infinity.

For the money market account, we also need to decide the riskless return for the same time interval Δt. Unlike in section 3, we will use the symbol $\widehat{r}$ to refer to the empirical annual interest rate. The continuously compounded rate of return must coincide with that of the actual riskless return, then:

$$r = e^{\widehat{r}\Delta t} \tag{59}$$

There is one potential problem when using binomial process to approximate lognormal distribution, that is, for cetain range of values of the parameters, trade may never occur. If $\frac{r}{1-k} \geq u$, the marginal investment in the stock yields u, which at most equals the return on money market account $\frac{r}{1-k}$ in the up state, and it is strictly inferior in the down state. Hence optimal investment policy at all portfolio ratio is not to purchase stock. If $d \geq r(1-k)$, the marginal investment in the stock yields d, which at least equals the return on money market account $r(1-k)$ in the down state, and it is strictly superior in the up state. Hence optimal investment at all portfolio ratio is not to sell stock.

To make sure that boundaries at each time exist (or possibly exist), in addition to the non-arbitrage assumption which requires $d < r < u$, we impose an extra condition:

$$\frac{d}{1-k} < r < u(1-k)$$

Note that from equation (58) as the number of periods n goes to infinity, all u, d, r tend to 1, while p tends to $\frac{1}{2}$. For sufficiently large n, the boundaries do not exist for all values of proportional transaction cost rate k. We use a moderate number of periods which balances accuracy of results and validity of the method.

4.3 Apply binomial tree method to the lifecycle model

Allowing for intermediate consumption brings a new dimension to the search for optimal investment policy. We skip the large body of similar derivation to those formulated in section 4.1 and 4.2, and only write down the expressions of independent interests in this section. We start by deriving the set of equations which govern the evolution of indirect utility function. Same notations are used here as in section 3. Homogeneity guarantees $J(c_t, b_t, s_t, t; T) = s^{1-\gamma} f(c_t, h_t, t; T)$ with the boundary condition $g(h_T, T, T) = \frac{(h_T+1)^{1-\gamma}}{1-\gamma}$, since at the terminal time all the wealth will be consumed.

The dynamic programming algorithm is written as:

$$J(c_t, b_t, s_t, t; T) = \underset{c_t, b_t, s_t}{\text{Max}} \frac{{c_t}^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} E_t[J(c_{t+1}, b_{t+1}, s_{t+1}, t+1; T)] \tag{60}$$

We define the no-transaction region with respect to investment, but not consumption. Even without rebalancing the portfolio composition, both h_t and h_{t+1} change due to intermediate consumption. Therefore, $h_{t+1} = \frac{b_t - c_t}{s_t} z_{t+1}$, which can be further simplified by using binomial tree method. It boils down to a weighted average, since z_{t+1} is either u or d depending on the state.

On the buy region, i.e. $h_t > \overline{\lambda_t}$. Similar to section 4.1, the relationship of $f(y_t, h_t, t; T)$ and $f(\overline{y_t}, \overline{\lambda_t}, t; T)$ can be found. Notice here we have:

$$h_{t+1} = \frac{b_t - u_t - c_t}{s_t + (1-k)u_t} \cdot \frac{r}{z_{t+1}}$$

The nominator of the first term on RHS includes three variables, the money market account b_t, sales of the money market account u_t and consumption c_t. Consumption only depletes the money market account and thus cannot exceed the money market account.

On the sell region, i.e. $h_t < \underline{\lambda_t}$, using the same logic, the relationship of $f(y_t, h_t, t; T)$ and $f(\underline{y_t}, \underline{\lambda_t}, t; T)$ can be found.

Proposition 2: When allowing for intermediate consumption, the rules governing the indirect utility function at buy boundary, no-transaction region and sell boundary can be expressed as:

$$f(c_t, h_t, t; T) = \begin{cases} \left(\frac{\overline{\lambda_t}(h_t(1-k)+1) - y_t(\overline{\lambda_t}(1-k)+1)}{\overline{\lambda_t}(\overline{\lambda_t}(1-k)+1) - \overline{y_t}(\overline{\lambda_t}(1-k)+1)}\right)^{1-\gamma} \left(f(\overline{y_t}, \overline{\lambda_t}, t; T) - \frac{\overline{y_t}^{1-\gamma}}{1-\gamma}\right) + \frac{y_t^{1-\gamma}}{1-\gamma} & \text{if } h_t > \overline{\lambda}_t \\ \frac{(\frac{c_t}{s_t})^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} E_t[(\frac{h_t}{h_{t+1}} r)^{1-\gamma} f(c_{t+1}, h_{t+1}, t+1; T)] & \text{if } \underline{\lambda}_t \leq h_t \leq \overline{\lambda}_t \\ \left(\frac{\underline{\lambda_t}(h_t+1-k) - y_t(\underline{\lambda_t}+1-k)}{\underline{\lambda_t}(\underline{\lambda_t}+1-k) - \underline{y_t}(\underline{\lambda_t}+1-k)}\right)^{1-\gamma} \left(f(\underline{y_t}, \underline{\lambda_t}, t; T) - \frac{\underline{y_t}^{1-\gamma}}{1-\gamma}\right) + \frac{y_t^{1-\gamma}}{1-\gamma} & \text{if } h_t < \underline{\lambda}_t \end{cases} \quad (61)$$

Proof: See Appendix 2.

Now we turn to investigate the application of binomial tree method again. Since intermediate consumption is allowed, now two decisions have to be made at each node of the binomial tree: how much to consume and whether to adjust portfolio composition. A new dimension is thus introduced to the tree representing the consumption policy. The investment and consumption decisions are made simutaneously in reality. We replicate this procedure as follows:

Starting at any node (t, i), immediate consumption has three effects that it realizes instant utility, reduces future consumption and brings down the portfolio ratio $h_{t,i}$. The third effect is implemented by moving along the i dimension for a given time t. Given such trade-off, optimal consumption policy balances these three opposite effects on the value function. It's neither too high, so that the portfolio ratio remains in a appropriate range, nor too low, so that impatient participants get satisfaction from intermediate consumption. Optimal consumption is time-dependent: a higher proportion of wealth is consumed as getting closer to the terminal time. And finally it reaches the highest level, 100 percent, at the terminal time. Backward induction enables computation of optimal consumption alongside the binomial tree.

A few technical details need to be mentioned here briefly. The set of consumption policies in our model cannot cover all possibilities. The reason is that all the control variables are discretized, with some artificial bounds. The values lying beyond the bounds are not accessible. Take the node with smallest portfolio ratio at time i for example, no consumption is available since by assumption it represents the lowest holding in money market account and consumption would further reduce such holding.

The rest of the search is same to that of the no-consumption case. To derive the sell and buy boundary, we start from the smallest and highest portfolio ratio, respectively. A boundary is obtained when marginal purchase or sale cannot improve the indirect value function. All the moving boundaries can be identified by backward induction.

5 Results and further analysis on optimal investment policy

In the Section 4 we have demonstrated the binomial tree method that yields a good approximation of the sell and buy boundaries along time. In this section we examine the pattern of the boundaries' movements over time and provide further analysis of the optimal policies for the pension fund on behalf of its participants who only have finite investment horizon.

We choose $T = 5$, and each year consists of 20 periods. The binomial tree hence has 100 steps, which is believed to generate reasonable results. Furthermore, we observe that increasing number of periods per year does not change the results. Since the boundaries converge to certain point as we move away from the terminal time to a remote period, we set the time horizon T not too large. The properties of the moving boundaries are summerized bellow:

5.1 Overall pattern of the moving boundaries

The no-transaction region broadens as the investor approaches the terminal time. Figure 1 reports the sell (upper) and buy (lower) boundaries of the portfolio ratio for parameter values $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$.[16] when proportional transaction costs apply to both stock and money market account. Since the expected return earned over the remaining time decreases as one moves torwards the terminal time, the incremental return gained from adjusting the portfolio declines. Especially near the terminal date, incremental return becomes minimal. This is why investor tends to transact less and less as time passes by and eventually avoids trading. Such behavior is more distinct near the terminal time that the boundairies start to explode without bounds.

The sell boundary shifts upwards alongside time monotonically, from 0.2918 at period 1 to 0.4603 at the last period, consistent with the evidence documented in the literature. On the other hand, the movement of the buy boundary is non-monotonic, opposed to results given by Liu and Loewenstein (2004), Gennotte and Jung (1994). It first slightly goes up from 0.2887 at period 1 to 0.3043 at the end of the fourth year, since then it fluctuates between 0.3041 to 0.3045 for about half a year, and quickly decreases to 0.2001 for the remaining periods of time.

Combining these facts, no-transaction region slightly shifts upwards to the stock for most of the time and eventually gets wide enough to restraint from trading, indicating that the optimal investment policy for the pension fund is to move out of money market account and into the stock gradually and slowly over a very long horizon then

[16] All the parameter values are on a annual base

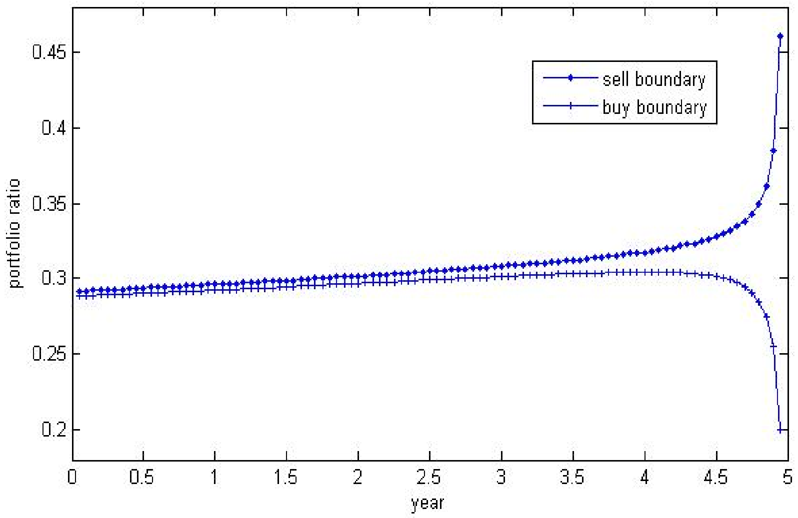

Figure 1: The no-transaction region broadens as number of remaining periods to terminal time decreases. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$

reduce trading dramatically near terminal time. Interestingly, the no-transaction region narrows and converges to a constant width as the number of remaining periods increases, consistent with the infinite horizon model. The speed of the convergence is rather fast that it happens within a few years, as found by Gennotte and Jung (1994).

Therefore, our model does not support the view that presence of transaction costs together with a finite horizon would yield consistent results with life-cycle investment advice. Life-cycle model predicts that youth invests more in risky assets than the elderly and the investment gradually shifts to riskless asset in the lifespan, however our model suggests that only in the very last moment investor dramatically reduces transaction. Indeed no-transaction region broadens over time, but the timing when it significantly appears to a large extent characterizes the optimal investment policy. In our model, it occurs nearly at the end of the investment horizon.

5.2 Changes in transaction costs

The sell and buy boundaries are sensitive to the transaction costs. With higher transaction costs, the width of the no-transaction region increases at all time. The rationale is that trading becomes infrequent if each incurs higher costs. Transaction costs also shift the no-transaction region towards the money market account because the demand for stock is decreasing in the transation cost rate, as reported by Constantinides (1986). However, these effects are not uniform over time. The width of no-transaction region is more sensitive to transaction costs when getting closer to the terminal time, indicating that pension fund should adjust optimal investment policy

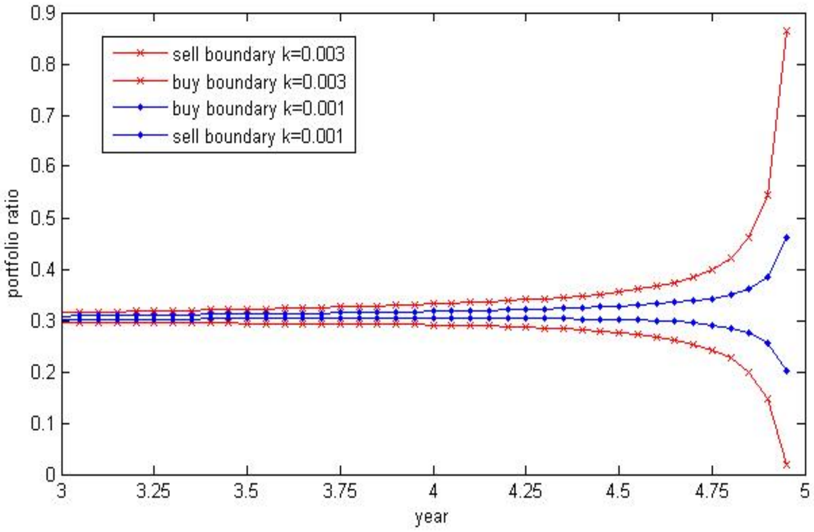

Figure 2: The no-transaction region widens at all time as proportional transaction costs increase. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$ and 0.003

more for shorter-horizon (within a few years) participants when transaction costs change. In section 5.1 we mentioned that near the terminal time the boundairies start to explode without bounds, thus one can expect a heavier effect of transaction costs accordingly.

Graphs below will only show the pattern of the boundaries from the third year on, since at earlier time it has no particular interests as the no-transaction region narrows down and converges to a certain width.

Figure 2 compares the no-transaction region with different transaction costs, $k = 0.001$ and $k = 0.003$. For $k = 0.003$, The sell boundary and buy boundary at the last period are 0.8652 and 0.0178 respectively, significantly different from those derived with $k = 0.001$. In contrast, boundaries converge as the number of remaining periods increases, so they do not exhibit too much difference at an early period. The convergence slows down when transaction costs increase.[17] In addition, the buy boundary is decreasing in transaction cost rate faster than the rate at which the sell boundary is increasing in transaction cost rate, especially near terminal time. (The sell boundary only doubles while the buy boundary decreases to one order of magnitude smaller.)

[17] The speed of convergence increases when σ increases, or γ increases. See figure 3 and figure 5.

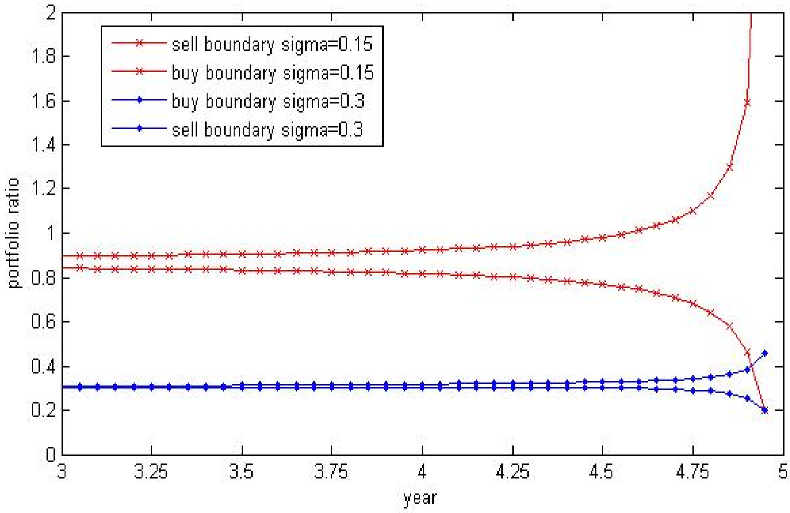

Figure 3: The no-transaction region narrows and moves downwards as volatility of the stock increases. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$ and 0.15, $\gamma = 3$, $k = 0.001$

5.3 Changes in risk and risk premium

Changes in risk and risk premium of stock affect the optimal investment policy, which is not a surprising result for anyone with a little knowlegde of optimal investment theory. A risk-averse agent requires a positive risk premium for the risks involved in holding the stock.

In our model, volatility of the stock σ represents the level of risk. Figure 3 demonstrates the effect of volatility on no-transaction region. When σ increases from 0.15 to 0.3, both boundaries move downwards and the width of no-transaction region narrows down. This implies that the risk-averse investor is more in favor of the money market account when stock becomes more risky and thus he tends to invest less in the stock on average. In addition, he trades more frequently in order to keep the portfolio ratio more stable when stock becomes more risky. This opposes the result of Liu and Lowenstein (2004), who found that no-transaction region widens when σ increases.

Once again, we observe a time-dependent effect of risk: in particular, the boundaries become extremely sensitive to σ upon the termial time, which coincides with the result of Liu and Lowenstein (2004), that the boundaries decrease at a significantly higher rate for a shorter-horizon investor as the risk increases. The optimal investment policy of pension fund should react accordingly for participants with different horizon. Especially for short-horizon participants, optimal investment policy is sensitive to risk.

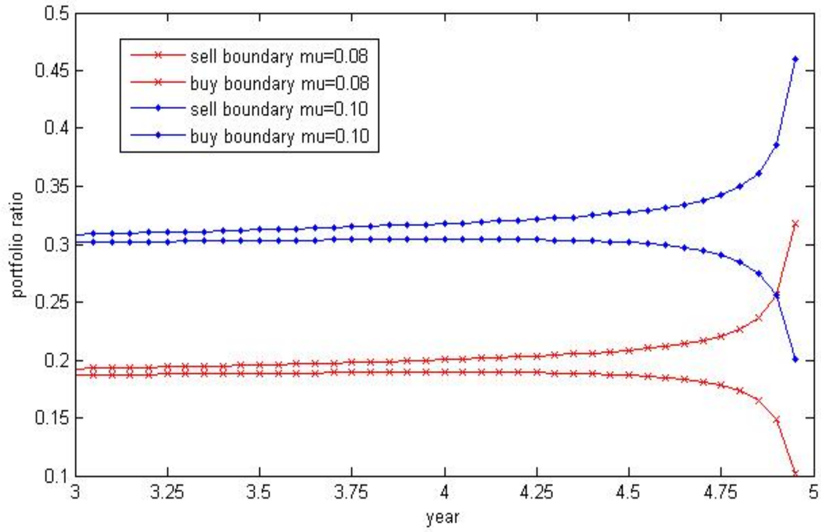

Figure 4: The no-transaction region moves upwards as the expected return of the stock increases. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$ and 0.08, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$

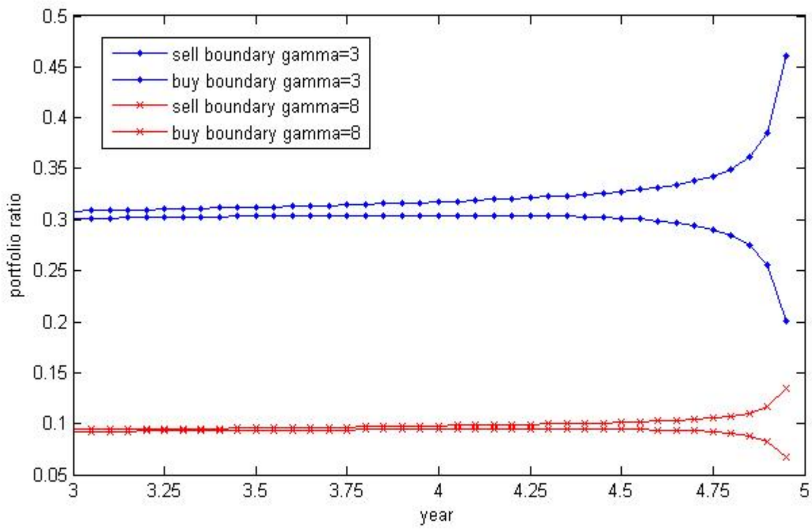

Figure 5: The no-transaction region narrows and moves downwards as the relative risk aversion coefficient increases. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$ and 8, $k = 0.001$

Now we turn to examine the change in risk premium. Throughout the paper, we keep the interest rate a constant $r_f = 0.04$, so risk premium only depends on the expected return of the stock. Figure 3 demonstrates the effect of expected return of the stock on no-transaction region. When expected return of the stock increases from 0.08 to 0.1,[18] stock become more attractive, all else being equal, so the investor would move out of the money market account and move into the stock. The no-transaction region consequently shifts upwards. The width of no-transaction region is left unchanged. However the horizon-dependence of the effect on no-transaction region is not significant. The buy and sell boundaries with risk premium of 0.04 are parallel to those with risk premium of 0.06.

5.4 Changes in relative risk aversion coefficient

A more risk-averse investor tends to invest more in the money market account to avoid undesirable risk of holding stock: the fluctuation of the portfolio value. Figure 5 shows the effect of increasing relative risk aversion coefficient γ on the no-transaction region. When γ increases from 3 to 8, both boundaries move downwards and the width of no-transaction region narrows down, indicating that the investor with a higher risk aversion coefficient holds less stock in his portfolio, and trades more in order to keep a more stable portfolio ratio. The effect is also horizon-dependent. It appears to be more distinct for shorter-horizon investors. Pension fund thus should take into account of the horizon when considering optimal investment policy for different risk-averse participants.

Moreover, an increase in σ and γ has similar effect on no-transaction region. Comparing Figure 5 and Figure 3, we could see the similarity.

5.5 Analysis of the moving boundaries when allowing for intermediate consumption

Same as before, the no-transaction region broadens when getting closer to the terminal time. Figure 6 reports the sell and buy boundaries for the parameter values $r_f = 0.04$, $T = 0.10$, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$, $\rho = 0$ Consumption does not change the property that the total expected return decreases as one moves towards the terminal time, so that the incremental return gained from adjusting the portfolio declines. Therefore, investor tends to trade less and less when allowing for intermediate consumption.

The sell boundary shifts upwards monotonically as time goes by, and the pattern is particularly obvious from the third year onwards. On the other hand, the behavior of the buy boundary is quite irregular. It slightly goes up for three years, and then

[18] This is equivalent to that the risk premium increases from 0.04 to 0.06, given a interest rate of 0.04

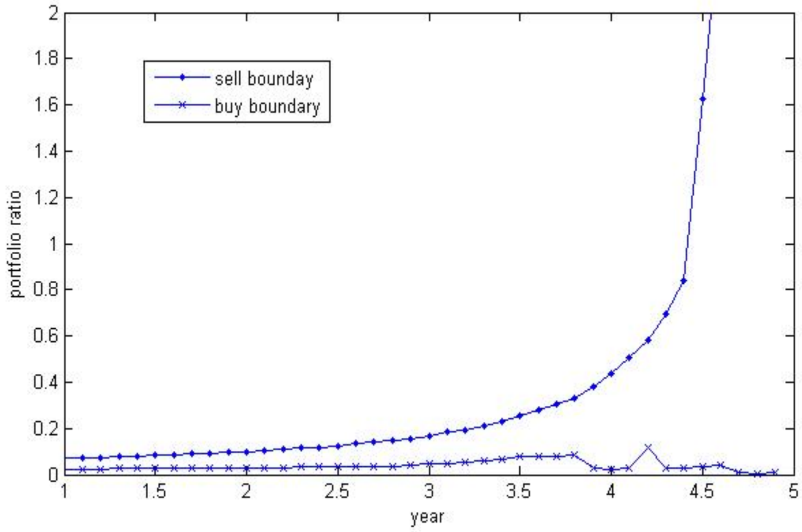

Figure 6: The no-transaction region broadens as number of remaining periods to terminal time decreases when allowing for intermediate consumption. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$, $\rho = 0$

flutuates intensively around this level.[19]Therefore, the widening of the no-transaction region is solely determined by the sell region.

Compared to the no-consumption case, several new features draw our attention. First, the no-transaction region moves downwards. Since consumption only depletes money market account (cash at hand), and optimal investment policy is determined in such a way that maximizes the expected utility of the lifetime's consumption stream, the investor therefore needs to maintain substaintial amount of wealth in the money market account, in order to support a desired level of consumption. Stock has to be sold before realizing for consumption. If cash at hand is not enough and transaction cost rate is high, the investor will find himself refrained by the liquidity constarint, which reduces the lifetime utility. The stock to money market account ratio is systematically lower than that of the no-consumption case for this reason.

Secondly, the buy boundary stays at a very low level for the lifetime. It is always below the buy boundary of the no-consumption case, though flutuates intensively. This is due to the fact that investor favors money market account more, so that he only buys stock when the portfolio ratio reaches a very low level.

Thirdly, the sell boundary starts to shift upwards much earlier when intermediate consumption is allowed. The buy boundary remains at a very low level at all time, thus the no-transaction region widens much earlier too. Compared to the no-consumption case, the width of the no-transaction region is 0.1230 at the beginning

[19]Relative to the low level the buy boundary remains at, the flutuation is rather intensive.

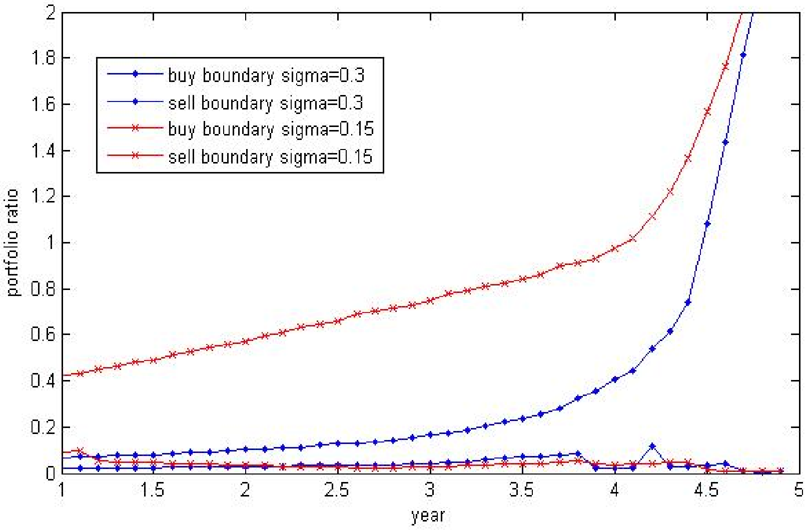

Figure 7: The no-transaction region shifts downwards and narrows down as the stock volatility increases when allowing for intermediate consumption. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$ and 0.15, $\gamma = 3$, $k = 0.001$, $\rho = 0$

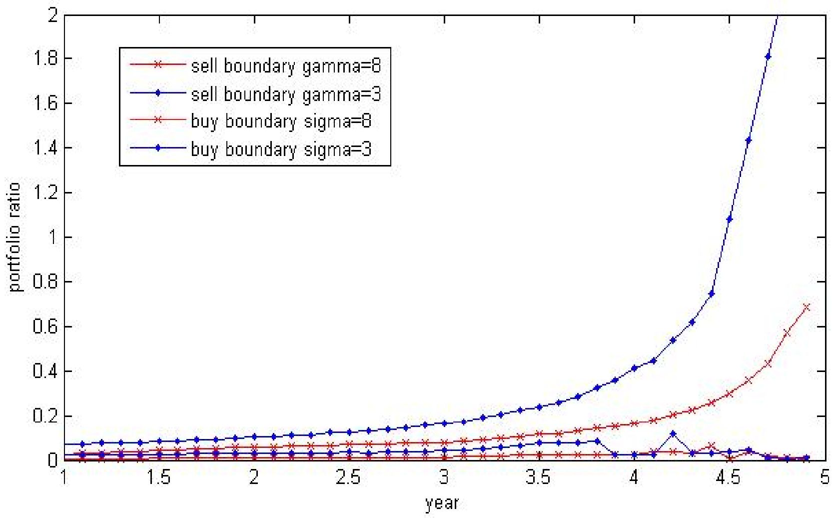

Figure 8: The no-transaction region shifts downwards and narrows down as the relative risk aversion coefficient increases when allowing for intermediate consumption. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$ and 8, $k = 0.001$, $\rho = 0$

of the third year, almost expanded by a factor of 2 (the width is only 0.0069 for the former). The intuition is straightforward: adjusting portfolio has two effects: it increases the return over the remaining time, and it incurs proportional transaction costs. With intermediate consumption, the incremental return gained from adjusting the portfolio declines even faster as time goes by, because consumption reduces the total wealth. We also find that the optimal proportion of wealth (and money market account) consumed by the investor is mostly an increasing function of time, which even strengthens the second effect. Hence, the investor finds it optimal to avoid trading at a much earlier stage.

Once more, increasing risk and relative risk aversion coefficient have similar effects on the width and location of the no-transaction region: it shifts downwards and narrows down, which we will not further discuss. Figure 7 and Figure 8 demonstrate these effects. The effect is more obvious if we plot it in terms of portfolio weight instead of portfolio ratio. See Appendix 3 for an example.

However, the effects of proportional transaction cost rate and expected stock returns show impressive difference. Figure 9 and Figure 10 demonstrate the effects of increasing transaction cost rate from 0.001 to 0.02, and increasing expected stock return from 0.10 to 0.12. In a finite-period model, we see from the graph that the effects of both increasing transaction cost and expected stock return become much smaller when allowing for intermediate consumption, almost neglegible before the fourth year. The buy boundary is not sensitive to transaction cost rate and expected stock return, while the sell boundaries only display significant difference near the terminal time.

The width of the no-transaction region is particularly sensitive to the time discounting parameter. Figure 11 shows the effect of increasing the time discounting parameter from 0 to 0.2. A high ρ implies a impatient preference, with which the investor tends to consume a large proportion of his wealth. Adjusting the portfolio thus only yields relatively small incremental gain. Consequently the sell boundary starts to shift upwards much earlier.

Combining all these evidences, we conclude that optimal investment of the pension fund becomes much different when considering intermediate consumption of the participants. The portfolio ratio systematically goes down for all parameter values. Pension fund also finds it optimal to avoid trading at a much ealier stage. Moreover, optimal investment policy is sensitive to stock volatility, relative risk aversion coefficient and time discounting parameter, and is not sensitive to proportional transaction cost rate, expected stock return, especially at an early stage.

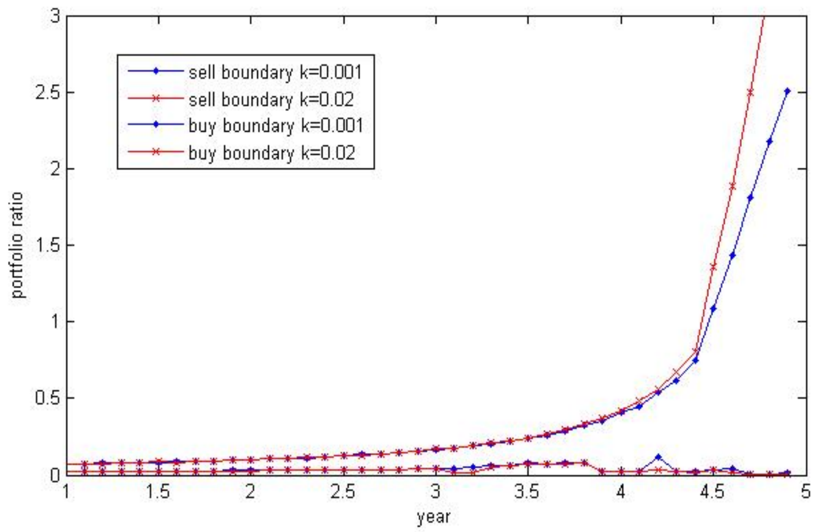

Figure 9: The no-transaction region slightly broadens as the proportional transaction cost rate increases when allowing for intermediate consumption. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$ and 0.02, $\rho = 0$

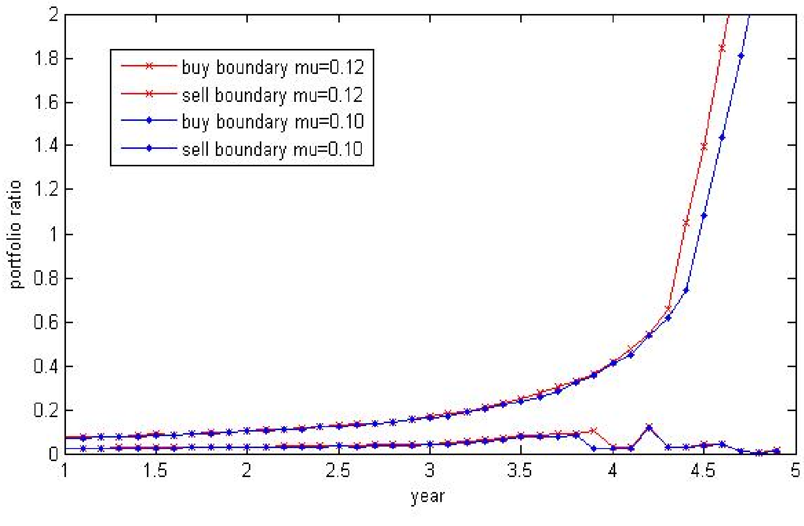

Figure 10: The no-transaction region slightly shifts upwards as the expected stock return increases when allowing for intermediate consumption. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$ and 0.12, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$, $\rho = 0$

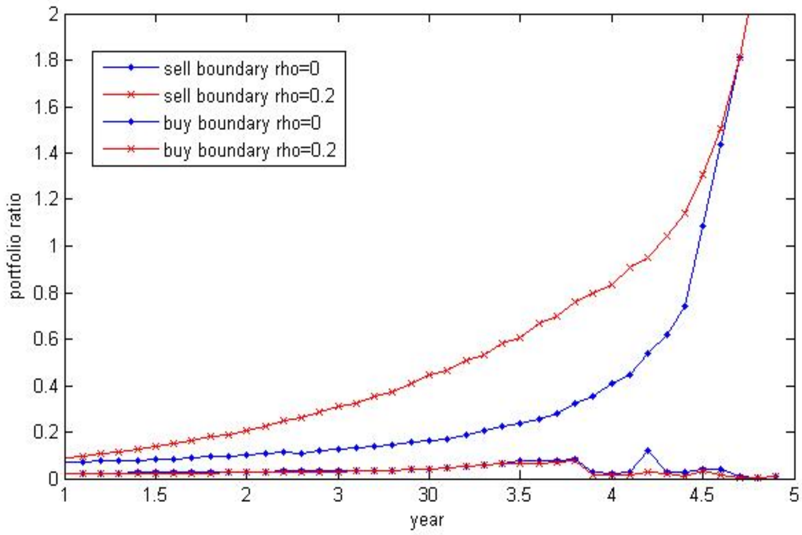

Figure 11: The no-transaction region slightly broadens as the time discounting parameter increases when allowing for intermediate consumption. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$, $k = 0.001$, $\rho = 0$, and 0.2

6 Discussion and conclusion

In this article we study the optimal investment policy for pension funds when regarding transaction costs. We have shown that in contrast to the infinite horizon case, the optimal investment policy, reprensented by the sell (upper) and buy (lower) boundary, is strongly horizon-dependent for an investor who attempts to maximize his utility on a finite horizon. Pension fund, on behalf of its participants who only have finite investment horizon, should refrain from trading when the portfolio ratio falls in the no-transaction region, which is by large time varying.

Four specific cases are examined respectively: no-transaction cost case, proportional transaction costs case, life-cycle investment with and without transaction costs. According to the set of ordinary differential equations direved for the buy region, no-transaction region and sell region, we do confirmed that optimal investment policy is horizon-dependent. However, since analytical solving this dynamic control problem is extremly time consuming, as pointed out by Gennotte and Jung (1992), we alternatively adopt a numerical method, namely binomial tree method, to search for the moving boundaries. Working backwards from the second last period, the searching procedure is simple and efficient: both boundaries can always be properly located at each time point.

As for the no-consumption case, the no-transaction region broadens and slightly shifts upwards as the investor approaches the terminal time. Eventually the boundaries explode without bounds. The sell boundary shifts upwards alongside time monotonically, while the movement of the buy boundary is non-monotonic. Combining the behavior of sell and buy boundary, it suggests that the optimal investment policy for pension fund is to move out of the money market account and into the stock gradually over a long time then reduce trading dramatically near the terminal time. Consistent with the stationary solution of the infinite horizon model, the no-transaction region narrows and converges to a certain width as the number of remaining periods increases.

Life-cycle model suggests that youth invests more in risky assets than the elderly, however our model shows that only in the very last moment investor dramatically reduces transaction. Therefore our model does not support the conclusion of Liu and Loewenstein (2002) which states that presence of transaction costs together with a finite horizon would give consistent results with life-cyle investment advice.

Moreover we have checked how changes in parameter values might affect the optimal investment policy. First, with higher transaction costs, the width of the no-transaction region increases and the demand for stock is decreasing. Secondly, when the volatility of the stock increases, both boundaries move downwards and the width of the no-transaction region narrows down, implying that the investor favors money market account more and trade more frequently. Thirdly, the no-transaction region shift upwards when expected return of the stock increases, meaning that the investor moves out of the money market account and into the stock. Fourth, an increase in relative risk aversion coefficient has similar effect on no-transaction region with an increase in volatility. Except for the increase in expected return, all these effects appear to be horizon-dependent, more distinct for shorter-horizon investors. Pension fund thus should take into consideration of the horizon when making optimal investment policy for its participants.

Following the no-consumption case, optimal investment policy when allowing for intermediate consumption can also be found in a similar fashion. Once again the sell boundary shifts upwards monotonically, while the behavior of the buy boundary is irregular. Compared to the no-consumption case, several new features draw our attention. First, the no-transaction region moves downwards. The reason is that investor needs to keep a large fraction of wealth in the money market account in order to sustain the optimal consumption. Secondly, the buy boundary stays at a very low level at all time, and the investor only buys stock when the portfolio ratio drops to a very low level. Thirdly, the no-transaction region widens much earlier, because incremental return gained from adjusting the portfolio declines as wealth gets consumed overtime.

Furthermore, optimal investment policy is only sensitive to stock volatility, relative risk aversion parameter and time discounting parameter. We conclude that considering the intermediate consumption of the participants, optimal portfolio ratio for pension funds goes down and trading becomes less frequent at a much earlier stage.

There are several directions in which the model presented in this article can be refined. First, to sketch a more general picture of optimal investment policy in a finite horizon, extension to multiple risky assets can be done. Magill and Constantinides (1976) pointed out that n assets imply 3^n possible transaction region. Even the stationary solution is extremely difficult to obtain. In particular, pension fund participants often have large specific investments in human capital and housing, so the optimal investment policy should also take these factors into account. Our method can be directly applied to the case with deterministic labor income. Nevertheless incoporating risky human capital and housing largely complicates the problem. It is a special case of the multiple risky assets model.

Secondly, it is still not clear how the investor adjusts consumption optimally when he is subject to proportional transaction costs. As we mentioned before, income effect and substitution effect work simutaneously at opposite directions. In this article we emphasis on optimal investment policy rather than consumption policy, though we have confirmed in the numerical results that consumption indeed increases over time. Futher research could focus on precisely investigating the causal relationship of transaction costs and consumption rate.

Thirdly, Some methodological concerns of the numerical method remain open to discussion. Instead of money market account to stock ratio, using the portfolio weight as the state variable might markedly improve numerical stability. The former can be extremely large, while the latter always falls between 0 and 1. In the no-consumption case, our result is inaccurate for large Merton values, which could be possiblly avoided by converting portfolio ratio to portfolio weight at the dynamic programming formulation stage. Additional to this, we still need to elaborate more on the behavior of the buy boundary in the life-cycle model. Last but not the least, the finite-difference method seems to be a tempting alternative to the binomial tree method, especially when considering intermediate consumption.

7 References

Acharya, V. V., and L. H. Pedersen, 2005, Asset Pricing with Liquidity Risk, Journal of Financial Economics, 77, 2, 375-410.

Amihud, Y., and H. Mendelson, 1986, Asset Pricing and the Bid-ask Spread, Journal of Financial Economics, 17, 2, 223-249.

Balduzzi, P., and A. W. Lynch, 1999, Transaction Ccosts and Predictability: Some Utility Cost Calculations, Journal of Financial Economics, 52, 1, 47-78.

Browne, S., M. A. Milevsky, and T. S. Salisbury, 2003, Asset Allocation and the Liquidity Premium for Illiquid Annuities, Journal of Risk and Insurance, 70, 3, 509526.

Cocco, J. F., F. J. Gomes, and P. J. Maenhout, 2005, Consumption and Portfolio Choice over the Life Cycle, The Review of Financial Studies, 18, 2, 491-533.

Constantinides, G. M., 1979, Multiperiod Consumption and Investment Behavior with Convex Transaction Costs, Management Science, 25, 1127-1137.

Constantinides, G. M., 1986, Capital Market Equilibrium with Transaction Costs, Journal of Political Economy, 94, 842-862.

Chellathurai, T., and T. Draviam, 2005, Dynamic Portfolio Selection with Nonlinear Transaction Costs, Mathematical, Physical and Engineering Sciences, 461, 3183-3212.

Cox, J., S. Ross, and M. Rubinstein, 1979, Options Pricing: A Simplified Approach, Journal of FinianiciaEl coniomlics, 7, 3, 229-263.

Davis, M. H. A., and A. R. Norman, 1990, Portfolio Selection with Transaction Costs, Mathematics of Operations Research, 15, 676-713.

Davis, M. H. A., V. G. Panas, and T. Zariphopoulou, 1993, European Option Pricing with Transaction Costs, SIAM Journal of Control and Optimization, 31, 470-493.

Dixit, A. K., and R. S. Pindyck, 1994, Investment under Uncertainty, Princeton University Press, Princeton, New Jersey.

Dumas, B., and E. Luciano, 1991, An Exact Solution to a Dynamic Portfolio Choice Problem Under Transaction Costs, Journal of Finance, 46, 577-595.

Gennotte, G., and A. Jung 1994, Investment Strategies Under Transaction Costs: The Finite Horizon Case, Management Science, 40, 385-404.

John, C. H., 2005, Options, Futures, and Other Derivatives, Prentice Hall, ISBN: 0131499084.

Jong, F. D., P. Schotman, and B. Weker, 2008, Strategic Asset Allocation, Netspar Panel Paper, No. 8.

Liu, H., and M. Loewenstein, 2002, Optimal Portfolio Selection with Transaction Costs and Finite Horizons, The Review of Financial Studies, 15, 805-835.

Liu, H., 2004, Optimal Consumption and Investment with Transaction Costs and Multiple Risky Assets, The Journal of Finance, 59, 1, 289-338.

Liu, H., B. G. Jang, H. K. Koo, and M. Loewenstein, 2007, Liquidity Premia and Transactions Costs, The Journal of Finance, 62, 5, 2329-2366.

Longstaff, F. A., 2001, Optimal Portfolio Choice and the Valuation of Illiquid Securities, Review of Financial Studies, 14, 407-431.

Longstaff, F. A., 2009, Portfolio Claustrophobia: Asset Pricing in Markets with Illiquid Assets, The American Economic Review, 99, 4, 1119-1144.

Magill, M. J. P., and G. M. Constantinides, 1976, Portfolio Selection with Transactions Costs, Journal of Economic Theory, 13, 2, 245-263.

Merton, R. C., 1969, Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case, Review of Economics and Statistics, 51, 247-257.

Merton, R. C., 1971, Optimum Consumption and Portfolio Rules in a Continuous Time Model, Journal of Economic Theory, 3, 373-413.

Merton, R. C., 1973, An Intertemporal Capital Asset Pricing Model, Econometrica, 41, 867-887.

Shreve, S. E., and H. M. Soner, 1994, Optimal Investment and Consumption with Transaction Costs, Annals of Applied Probability, 4, 609-692.

Taksar, M., M. Klass, and D. Assaf, 1988, Diffusion Model for Optimal Portfolio Selection in Presence of Brokerage Fees, Mathematics of Operation research, 13, 2, 277-294.

Tebaldi, C., and E. S. Schwartz, 2006, Illiquid Assets and Optimal Portfolio Choice, NBER Working Paper Series, No. w12633.

Zakamouline, V. I., 2002, A Unified Approach to Portfolio Optimization with Linear Transaction Costs, Mathematical Methods of Operations Research, 62, 2, 319-343.

A Appendix 1

On the no-transaction region, i.e. $\underline{\lambda}_t < h_t < \overline{\lambda}_t$, the amount invested in money market account and stock evolve according to the return dynamics:

$$V(b_t, s_t, t; T) = E_t[V(b_t r, s_t z_{t+1}, t+1; T)]$$

Transformation of the value function V to g both on LHS and RHS implies:

$$g(h_t, t; T) = E_t[s_t^{\gamma-1}(s_t z_{t+1})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{62}$$

For a given h_t, h_{t+1} is a random variable, determined by:

$$h_{t+1} = h_t \frac{r}{z_{t+1}}$$

so that we can write z_{t+1} as a function of the portfolio ratio $z_{t+1} = \frac{h_t}{h_{t+1}} r$. Replacing z_{t+1} in equation (62) by the portfolio ratio, we have:

$$g(h_t, t; T) = E_t[(\frac{h_t}{h_{t+1}} r)^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{63}$$

On the buy region, i.e. $h_t > \overline{\lambda}_t$:

$$V(b_t, s_t, t; T) = g(\overline{\lambda}_t, t; T) s^{1-\gamma}$$

where $\overline{\lambda}_t$ maximizes $g(h_t, t; T)$.

$$V(b_t, s_t, t; T) = \underset{u_t}{\text{Max}}\ E_t[V((b_t - u_t)r, (s_t + (1-k)u_t)z_{t+1}, t+1; T)]$$

Transformation of the value function V to g both on LHS and RHS implies:

$$g(h_t, t; T) = \underset{u_t}{\text{Max}}\ E_t[s_t^{\gamma-1}((s_t + (1-k)u_t)z_{t+1})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{64}$$

For the same reason, h_{t+1} is determined by:

$$h_{t+1} = \frac{b_t - u_t}{s_t + (1-k)u_t} \cdot \frac{r}{z_{t+1}}$$

Hence replacing $(s_t + (1-k)u_t)z_{t+1}$ in equation (64) by $\frac{(b_t - u_t)r}{h_{t+1}}$ gives the following expression:

$$g(h_t, t; T) = \underset{u_t}{\text{Max}}\ E_t[s_t^{\gamma-1}(\frac{(b_t - u_t)r}{h_{t+1}})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{65}$$

$$= \underset{u_t}{\text{Max}}\ E_t[(\frac{r}{h_{t+1}})^{1-\gamma}(h_t - \frac{u_t}{s_t})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{66}$$

Noticing that $\overline{\lambda}_t$ maximizes the value function $g(h_t, t; T)$, on the buy region, pension fund always finds it optimal to rebalance the portfolio in order to reach the ratio

$\overline{\lambda}_t$ at time t. The optimal control at time t is therefore determined uniquely by h_t and $\overline{\lambda}_t$ and can be expressed as a function of the state variables.

From $h_t = \frac{b_t}{s_t}$ and $\overline{\lambda}_t = \frac{b_t - u_t}{s_t + (1-k)u_t}$, we get $u_t = \frac{(h_t - \overline{\lambda}_t)s_t}{(1-k)\overline{\lambda}_t + 1}$. Plugging into equation (66), it can be easily shown:

$$g(h_t, t; T) = E_t[(\frac{r}{h_{t+1}})^{1-\gamma}(\frac{\overline{\lambda}_t(h_t(1-k)+1)}{\overline{\lambda}_t(1-k)+1})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{67}$$

$$g(\overline{\lambda}_t, t; T) = E_t[(\frac{r}{h_{t+1}})^{1-\gamma}(\frac{\overline{\lambda}_t(\overline{\lambda}_t(1-k)+1)}{\overline{\lambda}_t(1-k)+1})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{68}$$

The second equation is obtained from the first equation, with h_t replaced by $\overline{\lambda}_t$.

Combining the above two equations, the relationship of $g(h_t, t; T)$ and $g(\overline{\lambda}_t, t; T)$ is found:

$$g(h_t, t; T) = (\frac{h_t(1-k)+1}{\overline{\lambda}_t(1-k)+1})^{1-\gamma} g(\overline{\lambda}_t, t; T) \qquad \text{for} \quad h_t > \overline{\lambda}_t \tag{69}$$

On the sell region, i.e. $h_t < \underline{\lambda}_t$:

$$V(b_t, s_t, t; T) = g(\underline{\lambda}_t, t; T)s^{1-\gamma}$$

where $\underline{\lambda}_t$ maximizes $g(h_t, t; T)$.

$$V(b_t, s_t, t; T) = \underset{u_t}{\text{Max}} \; E_t[V((b_t + (1-k)u_t)r, (s_t - u_t)z_{t+1}, t+1; T)]$$

Transformation of the value function V to g on both sides gives:

$$g(h_t, t; T) = \underset{u_t}{\text{Max}} \; E_t[s_t^{\gamma-1}((s_t - u_t)z_{t+1})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{70}$$

$$= \underset{u_t}{\text{Max}} \; E_t[s_t^{\gamma-1}(\frac{(b_t + (1-k)u_t)r}{h_{t+1}})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{71}$$

$$= E_t[(\frac{r}{h_{t+1}})^{1-\gamma}(\frac{\underline{\lambda}_t(h_t + 1 - k)}{\underline{\lambda}_t + 1 - k})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{72}$$

The first equality holds by the use of homogeneity of the value function V. The second equality holds because $h_{t+1} = \frac{b_t + (1-k)u_t}{s_t - u_t} \cdot \frac{r}{z_{t+1}}$ while the third equality holds because $h_t = \frac{b_t}{s_t}$ and $\underline{\lambda}_t = \frac{b_t + (1-k)u_t}{s_t - u_t}$.

Replacing h_t by $\underline{\lambda}_t$, we have:

$$g(\underline{\lambda}_t, t; T) = E_t[(\frac{r}{h_{t+1}})^{1-\gamma}(\frac{\underline{\lambda}_t(\underline{\lambda}_t + 1 - k)}{\underline{\lambda}_t + 1 - k})^{1-\gamma} g(h_{t+1}, t+1; T)] \tag{73}$$

The relationship of $g(h_t, t; T)$ and $g(\underline{\lambda}_t, t; T)$ is also found:

$$g(h_t, t; T) = (\frac{h_t + 1 - k}{\underline{\lambda}_t + 1 - k})^{1-\gamma} g(\underline{\lambda}_t, t; T) \qquad \text{for} \quad h_t < \underline{\lambda}_t \tag{74}$$

Proposition 1 is proved.

B Appendix 2

Similar to section 4.2, on the no-transaction region we have:

$$f(c_t, h_t, t; T) = \frac{(\frac{c_t}{s_t})^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} E_t[(\frac{h_t}{h_{t+1}} r)^{1-\gamma} f(c_{t+1}, h_{t+1}, t+1; T)] \tag{75}$$

$h_{t+1} = \frac{b_t - c_t}{s_t} z_{t+1}$, which can be further simplified by using binomial tree method. It boils down to a weighted average, since z_{t+1} is either u or d depending on the state.

On the buy region, i.e. $h_t > \overline{\lambda_t}$:

$$f(c_t, h_t, t; T) = \underset{c_t, h_t}{\text{Max}} \frac{(\frac{c_t}{s_t})^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} E_t[{s_t}^{\gamma-1}((s_t + (1-k)u_t) z_{t+1})^{1-\gamma} f(c_{t+1}, h_{t+1}, t+1; T)] \tag{76}$$

where:

$$h_{t+1} = \frac{b_t - u_t - c_t}{s_t + (1-k)u_t} \cdot \frac{r}{z_{t+1}}$$

Replacing $(s_t + (1-k)u_t) z_{t+1}$ by $\frac{(b_t - u_t - c_t) r}{h_{t+1}}$, the following equation can be obtained:

$$f(c_t, h_t, t; T) = \underset{c_t, h_t}{\text{Max}} \frac{(\frac{c_t}{s_t})^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} E_t[(\frac{r}{h_{t+1}})^{1-\gamma} (h_t - \frac{u_t + c_t}{s_t})^{1-\gamma} f(c_{t+1}, h_{t+1}, t+1; T)] \tag{77}$$

To eliminate u_t, we review the relationship of h_t and $\overline{\lambda_t}$:

$$h_t = \frac{b_t}{s_t}, \quad \overline{\lambda_t} = \frac{b_t - u_t}{s_t + (1-k)u_t}$$

Substituting $u_t = \frac{(ht - \overline{\lambda_t}) s_t}{(1-k)\overline{\lambda_t} + 1}$ into the equation, and writting $\frac{c_t}{s_t}$ as y_t we get:

$$f(y_t, h_t, t; T) = \frac{{y_t}^{1-\gamma}}{1-\gamma} + \frac{1}{1+\rho} E_t[(\frac{r}{h_{t+1}})^{1-\gamma} (\frac{\overline{\lambda_t}(h_t(1-k)+1)}{\overline{\lambda_t}(1-k)+1} - y_t)^{1-\gamma} f(c_{t+1}, h_{t+1}, t+1; T)] \tag{78}$$

A little algebra would yield the relationship of $f(y_t, h_t, t; T)$ and $f(\overline{y_t}, \overline{\lambda_t}, t; T)$:

$$f(y_t, h_t, t; T) = \left(\frac{\overline{\lambda_t}(h_t(1-k)+1) - y_t(\overline{\lambda_t}(1-k)+1)}{\overline{\lambda_t}(\overline{\lambda_t}(1-k)+1) - \overline{y_t}(\overline{\lambda_t}(1-k)+1)} \right)^{1-\gamma} \left(f(\overline{y_t}, \overline{\lambda_t}, t; T) - \frac{\overline{y_t}^{1-\gamma}}{1-\gamma} \right) + \frac{y_t^{1-\gamma}}{1-\gamma} \tag{79}$$

On the sell region, i.e. $h_t < \underline{\lambda_t}$, using the same logic, the relationship of $f(y_t, h_t, t; T)$ and $f(\underline{y_t}, \underline{\lambda_t}, t; T)$ can be found:

$$f(y_t, h_t, t; T) = \left(\frac{\underline{\lambda_t}(h_t+1-k) - y_t(\underline{\lambda_t}+1-k)}{\underline{\lambda_t}(\underline{\lambda_t}+1-k) - \underline{y_t}(\underline{\lambda_t}+1-k)} \right)^{1-\gamma} \left(f(\underline{y_t}, \underline{\lambda_t}, t; T) - \frac{\underline{y_t}^{1-\gamma}}{1-\gamma} \right) + \frac{y_t^{1-\gamma}}{1-\gamma} \tag{80}$$

Proposition 2 is proved.

C Appendix 3

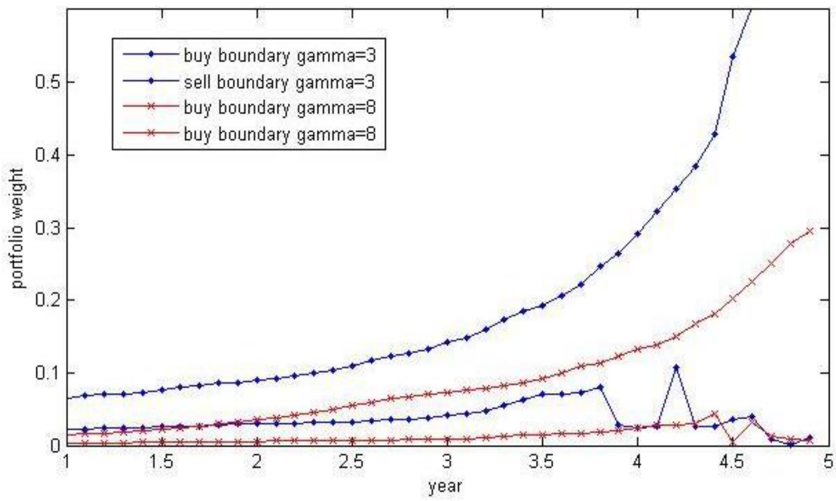

Figure 12: The no-transaction region shifts downwards and narrows down as the relative risk aversion coefficient increases when allowing for intermediate consumption. The assumed parameter values are $r_f = 0.04$, $\mu = 0.10$, $\sigma = 0.30$, $\gamma = 3$ and 8, $k = 0.001$, $\rho = 0$

Increasing relative risk aversion coefficient shifts the no-transaction region downwards and narrows down its width. The pattern is more obvious when using portfolio weight, which always lies between 0 and 1. Figure 12 reports this effect.