Network for Studies on Pensions, Aging and Retirement

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Risk-Neutral Valuation of Real Estate Derivatives

Discussion Paper 10/2009 - 048

October, 2009

OFRC WORKING PAPER SERIES

RISK-NEUTRAL VALUATION OF REAL ESTATE DERIVATIVES

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Technical Paper No. 2009-02
October 2009

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ABSTRACT

We propose a novel and intuitive risk-neutral valuation model for real estate derivatives. We first model the underlying efficient market price of real estate and then construct the observed index value with an adaptation of the price update rule by Blundell and Ward (1987). The resulting index behavior can easily be analyzed and closed-form pricing solutions are derived for forwards, swaps and European put and call options. We demonstrate the application of the model by valuing a put option on a house price index. Autocorrelation in the index returns appears to have a large impact on the option value. We also study the effect of an over- or undervalued real estate market. The observed effects are significant and as expected.

Keywords: Real estate derivatives; Option pricing; Incomplete markets; Price discovery; Autoregressive models; Seasonality, Stochastic volatility.
JEL Classification: C51/D52/G13

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1 Introduction

Recently, the interest in real estate derivatives has surged. This interest has for instance been fueled by the introduction of real estate futures on the Chicago Mercantile Exchange (CME) in 2006. These futures give investors the opportunity to directly manage house price risk. Currently, trading is possible using 20 regional indices and two composite indices. See the recent papers by Bertus, Hollans, and Swidler (2008) and Shiller (2008) for more information. Geltner and Fisher (2007) and Fabozzi, Shiller, and Tunaru (2009) also provide a good overview of other real estate derivatives markets, such as swap trading on the U.K. Investment Property Database index (IPD) or the U.S. NCREIF Property Index (NPI).

Currently, the most mature property derivatives market is the U.K. IPD derivatives market. At the end of 2008 some GBP 19.3 billion of swaps referenced IPD indices. In the beginning of 2009, trading in IPD derivatives has decreased significantly, however, mostly because less deals between banks were executed with Lehman Brothers exiting the market and several other banks cutting back on new business activities. The U.S. CME futures market does not yet have much liquidity, with only occasional trades. Property derivatives markets in France and Germany are also still very small.

In this paper, we develop a novel and intuitive risk-neutral valuation model for real estate derivatives. Our main goal is to value derivatives which are coupled to private real estate indices with a significant degree of autocorrelation. It is well known from the real estate literature (see for example Geltner, MacGregor, and Schwann (2003) for an overview) that autocorrelation can occur in appraisal-based indices because appraisers slowly update past prices with new market information. Transaction-based indices can also exhibit a positive autocorrelation because private real estate markets are less informationally efficient than public securities markets. As a result, the price discovery and information aggregation functions of the private real estate market are less effective. This can cause noisy prices and inertia in asset values (and returns).[1]

A significantly positive autocorrelation implies a (partial) predictability of future returns and opportunities for arbitrageurs. It is not possible, however, to trade the assets which constitute a private real estate index in a liquid market and at low costs. In practice, the index is thus not a tradable asset and arbitrage possibilities are very limited. This can also cause significant problems for suppliers of real estate derivatives since they cannot easily trade the underlying assets and (delta) hedge their positions. Nevertheless, derivatives markets for forward and swap contracts are emerging in recent years.

Geltner and Fisher (2007) note, however, that a 2006 survey of (potential) market participants identified a lack of confidence in how real estate derivatives should be priced. They also note that this concern is understandable, since the underlying asset cannot be traded in a frictionless market. This makes it impossible to use classic pricing formulas for derivatives (such as the relationship between spot and forward prices), since these formulas only apply under strict no-arbitrage assumptions. Geltner and Fischer (2007) argue, however, that the valuation of real estate derivatives is still possible using equilibrium pricing rules, provided that the dynamic behavior of the underlying real estate index is properly taken into account. In this paper we take the next step by proposing a quantitative risk-neutral valuation model which can be used for actual pricing purposes.

A small body of related research exists in the equity option literature. Lo and Wang (1995) study the effect of predictability of asset returns in a continuous-time model. They propose an adjustment of the Black-Scholes (1973) pricing formula for stock options to account for the effect of predictability. Jokivuolle (1998) develops a discrete-time model to derive an analytical pricing formula for options on a stock index which exhibits positive correlation due to infrequent trading of the underlying stocks. He assumes that the unobservable true liquidation value of the index follows a random walk process. The observed (autocorrelated) index is then modeled as the weighted average of current and past returns. More recently, Liao and Chen (2006) derived a closed-form formula for a European option on an asset with returns following a first-order moving average process.

The real estate literature also contains a few pioneering papers on risk-neutral valuation. Early examples of risk-neutral valuation techniques are given by Kau et al. (1990), Buetow and Albert (1998) and Buttimer and Kau (1997). The last paper is especially important, because it describes how a risk-neutral valuation model can be used to value derivatives which are related to commercial real estate indices.

[1] See Shiller (2008, p. 4) and the references given in that paper.

Throughout their paper, these authors assume that the real estate index follows a random walk process with drift. By construction, such a process leads to uncorrelated index returns. Buttimer and Kau (1997) note, however, that their model can also be used in case of autocorrelated indices provided that a proper transformation can be found to switch (back and forth) between the autocorrelated index (which is observed) and the uncorrelated variable (which is explicitly modeled).

A different approach is followed by Shiller and Weiss (1999) in their paper on home equity insurance. They first fit the observed real estate returns with a simple autoregressive (AR) model with one lag. Using this model, the conditional returns and volatilities can be determined analytically. The assumption is then made that options on the house price index can be valued using an adaptation of the familiar Black and Scholes (1973) equation. This adaptation consists of replacing the expected risk-free return with the expected real-world return and the implied volatility with the estimated value from the AR model. One aspect of this approach is adopted by us, namely modeling the real estate returns with an AR model. We view the second, heuristic, step (in which the real-world return is directly used as an input for a risk-neutral valuation formula) as problematic, however.

Our approach circumvents this problem by following the approach of Jokivuolle (1998). We thus explicitly model the (underlying) 'efficient market' value of the real estate index and then construct the observed index. We argue that an adaptation of the price update rule proposed by Blundell and Ward (1987) serves this purpose well. The first modification of this update rule is straightforward and consists of adding multiple lag terms. This leads to an AR model which can be estimated using standard econometric techniques. A second modification is more fundamental from a valuation perspective and consists of using the accrued value of past observations. Empirical results confirm that the volatility and autocorrelation of a (transaction-based) Dutch house price index can be replicated quite well (on an annual basis) with our model. As a second example, we consider monthly data for the U.S. 10 city S&P/Case-Shiller house price index. Our analysis shows that modeling seasonality and stochastic volatility is important for such monthly data.

The remainder of this paper is organized as follows. In Section 2 we introduce our theoretical framework and analyze the properties of the real-world and risk-neutral process for real estate indices with autocorrelation. We also explain how the real estate model can be coupled to a stochastic interest rate model. Section 3 contains closed-form pricing formulas for forwards, swaps and European options. In Section 4 we estimate the real estate model using historical information for house price indices in the Netherlands and the U.S. Section 5 discusses the valuation of a European put option on a house price index using Monte Carlo simulation. We also assess the quality of the derived closed-form option pricing formula. Section 6 concludes.

2 Theoretical framework

2.1 Notational conventions

Our risk-neutral model consists of a discrete-time model for the observed real estate index in combination with continuous-time models for the efficient market process of real estate and for interest rates. The current point in time is denoted as $t = 0$. Time is measured with respect to the period between two price updates of the real estate index. Unless stated otherwise (and without loss of generality), we assume that the time step between two price updates is equal to one year. Hence, $t = 1$ corresponds to one year ahead, $t = 2$ to two years ahead, etc. In the continuous-time models, the non-integer points in time are also sampled. To avoid confusion, we therefore denote the continuous-time variable with τ in the remainder of this paper.

2.2 Real-world process

2.2.1 Price update model

We model the real-world process of a real estate index with an adaptation of the price update rule proposed by Blundell and Ward (1987).[2] They suggest that the new price is a weighted sum of the current market price and the last period's price. More precisely, they propose the following price update rule:[3]

$$a(t) = Ky(t) + (1-K)a(t-1), \tag{2.1}$$

where $a(t)$ is the current price, $a(t-1)$ is the previous price, $y(t)$ is the 'true' market price and K is a constant, $0 \leq K \leq 1$. The parameter K is commonly referred to as the 'confidence' parameter. If K is close to 1, recent market information is weighted heavily; if K is small, the emphasis is more on past price information. The simple price update rule in Eq. (2.1) is frequently used to model appraisal smoothing in real estate indices. See Geltner, MacGregor, and Schwann (2003) for an extensive overview of research in this area. We show in this paper that this price update rule can also be used to describe the dynamic behavior of autocorrelated transaction-based indices.

The model in Eq. (2.1) is equivalent to an exponentially weighted moving average (EWMA) model, see Hull (2009, p. 479–480). By substituting the expression for $a(t-1)$ in $a(t)$, the expression for $a(t-2)$ in $a(t-1)$, etcetera, we find that

$$a(t) = K\sum_{i=1}^{m}(1-K)^{i-1}y(t-i+1) + (1-K)^m a(t-m), \tag{2.2}$$

where $1 \leq m \leq t$. This equation shows that the current value $a(t)$ partly consists of a basket of previous y-terms, where the weight of these terms decreases at an exponential speed (as controlled by the K parameter). By setting m equal to t we also see that the weight of the index value at time 0, $a(0)$ is equal to $(1-K)^t$ at time t.

It is important to note that the price update rule in Eq. (2.1) does not account properly for the time value of money because the previous value $a(t-1)$ is not accrued. From a valuation perspective this leads to a systematic underperformance of the real estate index. To correct for this effect, we adapt the price update rule in Eq. (2.1) and accrue the past index value with the expected (annual) return π:

$$a(t) = Ky(t) + (1-K)(1+\pi)a(t-1). \tag{2.3}$$

Using accrued prices is common practice when appraisers set new prices for real estate objects. In this case, the reference price level is often formed by previous transactions for similar objects with a correction for the price increase (or decrease) of the real estate market up to the current point in time. The expected return is not modelled in detail at this point to keep the analysis as simple as possible. In practice the expected return may depend (positively) on the amount of risk associated with the real estate investment. In the risk-neutral model the expected return is coupled directly to the level of the interest rate, as we explain in detail in Section 2.3.

Substitution of $a(t-1)$ in $a(t)$, $a(t-2)$ in $a(t-1)$, etcetera, again yields the EWMA form of Eq. (2.3):

$$a(t) = K\sum_{i=1}^{m}(1-K)^{i-1}y^*(t-i+1) + (1-K)^m a^*(t-m), \tag{2.4}$$

where $1 \leq m \leq t$ and

$$\begin{aligned} a^*(t-m) &\equiv a(t-m)(1+\pi)^m, \\ y^*(t-i+1) &\equiv y(t-i+1)(1+\pi)^{i-1}. \end{aligned} \tag{2.5}$$

Equation (2.4) is thus equivalent to Eq. (2.2) if we accrue past values.

[2] A much earlier application of this update rule can be found in Brown (1959).

[3] Equation (2.1) assumes that the price update rule is constant over time. Generalizations with time-varying parameters can be found in Brown and Matysiak (2002).

We can easily determine the evolution of annual returns based on Eq. (2.3):

$$r^a(t) = K\frac{y(t-1)}{a(t-1)}r^y(t) + (1-K)\frac{a(t-2)}{a(t-1)}(1+\pi)r^a(t-1), \tag{2.6}$$

where $r^a(t) \equiv a(t)/a(t-1) - 1$ is the index return and $r^y(t) \equiv y(t)/y(t-1) - 1$ is the unobserved return, both using annual compounding. A much simpler expression is derived when the index series are expressed in logarithms, see Geltner, MacGregor, and Schwann (2003). In this case, continuously compounded returns can be expressed as log differences:

$$r_c^a(t) = K^* r_c^y(t) + (1-K^*)r_c^a(t-1), \tag{2.7}$$

and thus

$$r_c^y(t) = \frac{1}{K^*}r_c^a(t) - \frac{1-K^*}{K^*}r_c^a(t-1), \tag{2.8}$$

where $r_c^a(t)$ is the index return and $r_c^y(t)$ is the unobserved market return, both using continuous compounding. The parameter K^* has a similar interpretation as the confidence parameter K. This parameter determines what fraction of the index return is explained by the unobserved market return (the remaining fraction is explained by the past index return). Note that the effect of accrual disappears when we take log differences (that is, when we assume that past values accrue with the same return π).

2.2.2 The efficient market process

We now assume that the underlying market returns follow a random walk process with drift:

$$r_c^y(t) = \pi + \epsilon(t), \tag{2.9}$$

where $\epsilon(t)$ is a normally-distributed, serially-uncorrelated noise term with zero mean and variance σ_ϵ^2. Note that the drift parameter π is assumed to be constant here to keep the analysis as simple as possible. However, in successive periods of appreciation and depreciation of the price levels this assumption is not always valid. A more appropriate specification would then be to allow π to change over time. An example of such a model is a local linear trend model, which is for example used in Francke (2009).

The confidence parameter K^* can be calculated from the first-order autoregressive (AR) process that we obtain by substituting Eq. (2.9) in Eq. (2.7):

$$r_c^a(t) = K^*\pi + (1-K^*)r_c^a(t-1) + K^*\epsilon(t). \tag{2.10}$$

K^* is thus equal to 1 minus the first-order autocorrelation of the index returns. In practice, we can also neglect the difference between K in Eq. (2.6) and K^* in Eq. (2.7) because (on average) $y(t-1) \approx a(t-1)$ and $\frac{a(t-2)}{a(t-1)} \approx \frac{1}{1+\pi}$. Under these simplifying assumptions, the functional form of Eqs. (2.6) and (2.7) becomes the same. Annually and continuously compounded returns also have almost the same first-order autocorrelation. It thus follows that $K \approx K^*$.

A word of caution is appropriate at this point. The assumption that the underlying market returns follow a random walk with drift is probably too strong for the private real estate market since these markets are less informationally efficient than public securities markets, see also Geltner, MacGregor, and Schwann (2003). We should therefore be careful not to directly equate the underlying random walk process with the true market process. A better interpretation would be to state that the observed index returns can be modeled using an underlying efficient market process in combination with the price update rule in Eq. (2.1). In the remainder of this paper, we therefore refer to $y(t)$ as the efficient market price at time t. We will show in Section 2.4 that assuming an underlying efficient market process makes it possible to easily analyze the properties of the constructed real estate index with autocorrelation. This facilitates the derivation of several pricing formulas in Section 3.

2.2.3 Price update rules with multiple lags

The price update rule in Eq. (2.1) can easily be extended with multiple lag terms. For the general case of p lags (with $p \geq 1$) we have:

$$a(t) = Ky(t) + \sum_{i=1}^{p} \omega_i a(t-i), \text{ where } K + \sum_{i=1}^{p} \omega_i = 1. \tag{2.11}$$

The generalization of Eq. (2.10) then becomes:

$$r_c^a(t) = K^*\pi + \sum_{i=1}^{p} \omega_i^* r_c^a(t-i) + K^*\epsilon(t), \text{ where } K^* + \sum_{i=1}^{p} \omega_i^* = 1. \tag{2.12}$$

Equation (2.12) is an AR model of order p. To estimate the expected return π, the weights ω_i^* and the variance σ_ϵ^2 of an AR(p) model different approaches can be followed, see Lütkepohl (2006) or Steehouwer (2005, p. 43–129) for detailed overviews. Note that the restriction that the sum of the weights should be equal to one does not complicate the estimation of the model since both K^* and π are free parameters.

A simple approach is to estimate the model parameters with an ordinary least squares (OLS) regression method. This method basically minimizes the one-step-ahead prediction errors. An alternative approach is to choose the weights in such a way that the autocovariance function of the AR process is exactly equal to the autocovariance function of the observed real estate index. This correspondence can be achieved by using the Yule-Walker equations, see Steehouwer (2005, p. 46).

To decide which model is most appropriate several order selection criteria have been proposed in the literature, see for example Steehouwer (2005, p. 82–84). These selection criteria typically choose the model order in such a way that the prediction error is minimized while putting a penalty on the number of parameters estimated. The estimation of the real estate model is discussed in detail in Section 4.

2.2.4 Seasonality

Seasonality in real estate returns can become important when modeling quarterly or monthly returns. Let us assume that we have already modeled the seasonally adjusted index $a(t)$ using Eq. (2.1) or (2.11). We can then add a seasonal component $g(t)$ to obtain the index value with seasonality, $\tilde{a}(t)$:

$$ln(\tilde{a}(t)) = ln(a(t)) + g(t), \tag{2.13}$$

or, equivalently:

$$\tilde{a}(t) = a(t)e^{g(t)}. \tag{2.14}$$

Different approaches can be used to estimate the $g(t)$ function. A natural assumption is to assume that seasonality does not have a net effect on an annual basis. For simplicity, one can also assume that the seasonal pattern is constant over time. Given these assumptions one could then use so-called dummy variables in the OLS regression. These dummy variables are equal to one for the respective periods. For example, a January dummy is equal to one for all January (log) returns and zero for all other months; a February return is equal to one for all February returns and zero otherwise, etcetera. The $g(t)$ function is then easily constructed using the estimated weights of the dummy variables. Another (even simpler) method consists of detrending the log index and then fitting a (shifted) sinus function with a period of one year to the data. The first approach (i.e., a regression on monthly dummies) is used in Section 4.3.

2.3 Risk-neutral process

2.3.1 Introduction

In a risk-neutral world all individuals are indifferent to risk and expect to earn on all securities a return equal to the (instantaneous) risk-free rate. Assuming that the world is risk neutral greatly facilitates the valuation of options: the option payoffs can simply be discounted along the path of the short rate for each scenario. It is also important to note that risk-neutral valuation gives the correct price of an option in *all* worlds (also the risk-averse real world), not just in the risk-neutral world.

2.3.2 Process for interest rates

We model the evolution of the short interest rate in this paper with the familiar one-factor Hull-White (HW) model, see Hull (2009, p. 688–689). Within the large family of interest rate models, the HW model is a typical example of a 'no-arbitrage' model. Such a model produces interest-rate scenarios which are consistent with the current term structure. This no-arbitrage feature is extremely important for option pricing applications, since a small error in the underlying bond prices can cause large errors in the price of interest-rate options, see Hull (2009, p. 686).

Technically speaking, the one-factor HW model assumes that the risk-neutral process for the nominal short rate r_N is as follows:

$$dr_N(\tau) = \kappa \left(\frac{\theta(\tau)}{\kappa} - r_N(\tau) \right) d\tau + \sigma_1 dZ_1(\tau). \tag{2.15}$$

We denote time in this equation with the symbol τ to indicate that we now use a continuous-time model. This model assumes that the short interest rate fluctuates around the mean reversion level $\theta(\tau)/\kappa$. The parameter κ controls the amount of mean-reversion. The θ-function is deterministic and chosen in such a way that the model satisfies the no-arbitrage constraint. The one-factor HW model is in fact an extension of the Vasicek (1977) model in the sense that the mean reversion level is time-dependent instead of constant. σ_1 controls the volatility of the Wiener process dZ_1.

2.3.3 Price update model

We now derive the risk-neutral process for real estate indices with autocorrelation. The risk-neutral process for the evolution of the index value can be derived analogously to Eq. (2.3):

$$\begin{aligned} a(t) &= Ky(t) + (1-K)\pi(t)a(t-1), \\ \text{where } \pi(t) &\equiv \exp\left(\int_{t-1}^{t} r_N(\tau)d\tau \right) \exp(-q). \end{aligned} \tag{2.16}$$

The expected return π is thus a time-dependent function in a risk-neutral world and depends on the level of the (short) interest rate and the direct return. More precisely, the term $\exp\left(\int_{t-1}^{t} r_N(\tau)d\tau\right)$ is the risk-free return on a bank account between time $t-1$ and t. The term $\exp(-q)$ is a correction for the direct return q associated with real estate investments. By setting q equal to zero a total return index is modeled.

Substitution of $a(t-1)$ in $a(t)$, $a(t-2)$ in $a(t-1)$, etcetera, again yields the following EWMA form of Eq. (2.16):

$$a(t) = K \sum_{i=1}^{m} (1-K)^{i-1} y^*(t-i+1) + (1-K)^m a^*(t-m), \tag{2.17}$$

where $1 \leq m \leq t$ and

$$\begin{aligned} a^*(t-m) &\equiv a(t-m)\exp\left(\int_{t-m}^{t} r_N(\tau)d\tau \right) \exp(-qm), \\ y^*(t-i+1) &\equiv y(t-i+1)\exp\left(\int_{t-i+1}^{t} r_N(\tau)d\tau \right) \exp(-q(i-1)). \end{aligned} \tag{2.18}$$

Equation (2.16) can be extended for the general price update model with *p* lag terms:

$$a(t) = Ky(t) + \sum_{i=1}^{p} \omega_i a^*(t-i) \quad \text{where} \quad K + \sum_{i=1}^{p} \omega_i = 1. \tag{2.19}$$

The EWMA form of Eq. (2.19) can also be derived. Let us assume that $T > t \geq 0$. We can now substitute $a^*(T-1)$ in $a(T)$, $a^*(T-2)$ in $a^*(T-1)$, etcetera, until an expression is obtained with only the terms $y^*(t+1), \cdots, y^*(T)$ and $a^*(t), \cdots, a^*(t-p+1)$:

$$a(T) = \sum_{i=1}^{T-t} c_i y^*(t+i) + \sum_{i=1}^{p} d_i a^*(t-i+1), \tag{2.20}$$

where $c_{T-t} = K$. Explicit expressions for the c_i and d_i coefficients of this equation can be determined using a software package which is able to perform symbolic algebra calculations.[4]

2.3.4 The efficient market process

We also need to specify the risk-neutral process for the underlying efficient market price. Analogously to Eq. (2.9), we use a random walk process with drift (geometric Brownian motion):

$$dy(\tau) = (r_N(\tau) - q)y(\tau)d\tau + \sigma_2 y(\tau)dZ_2, \tag{2.21}$$

where the volatility σ_2 is constant and dZ_2 follows a Wiener process. By means of Ito's lemma it can be shown that $\ln y(\tau)$ is governed by the following process, see Hull (2009, p. 270–271):

$$d\ln y(\tau) = \left(r_N(\tau) - q - \sigma_2^2/2\right) d\tau + \sigma_2 dZ_2. \tag{2.22}$$

For numerical reasons Eq. (2.22) is commonly used in practice instead of Eq. (2.21). Note that these equations are equivalent to the Black-Scholes (1973) price process for a dividend paying stock in case of stochastic interest rates.

2.3.5 Model extensions: Real interest rates, inflation, stochastic volatility

It is also possible to model real interest rates and inflation in a consistent way. Brigo and Mercurio (2006, p. 646–647) for example develop a consistent risk-neutral model for nominal and real interest rates as well as the CPI index. To keep the analysis as simple as possible, we do not discuss such an extended model in this paper. Including inflation may be very important for practical applications, however, since real estate cash flows (like rental income or maintenance costs) are often inflation-linked.

Another extension consists of modeling stochastic volatility. This is especially important when considering high-frequency data, like monthly or quarterly returns. An example is given in Section 4.3 for monthly U.S. house price data. A quite general stochastic volatility model is the constant elasticity of variance (CEV) model:

$$\begin{aligned} dy(\tau) &= (r_N(\tau) - q)y(\tau)d\tau + \sqrt{V(\tau)}y(\tau)dZ_2, \\ dV(\tau) &= \lambda(\gamma - V(\tau))dt + \sigma_3 V(\tau)^\beta dZ_3. \end{aligned} \tag{2.23}$$

This model is a natural extension of the geometric Brownian motion in Eq. (2.21) and has for example been studied by Jones (2003). The λ parameter controls the speed of mean reversion of the variance $V(\tau)$. The γ parameter denotes the mean reversion level and σ_3 controls the volatility of the variance process. The initial variance should, of course, also be specified as a boundary condition. The elasticity parameter β must satisfy $0.5 \leq \beta \leq 1.0$ to retain the uniqueness of option prices. Both limiting cases are in fact well-known stochastic volatility models. For $\beta = 1/2$ we have the model of Heston (1993) and for $\beta = 1$ we have the continuous-time GARCH model as in Nelson (1990). Maximum likelihood estimation of the CEV model parameters, based on option prices, is discussed in detail in an excellent paper by Aït-Sahalia and Kimmel (2007).

[4] The following *Mathematica* program for example writes $a(10)$ in the form of Eq. (2.20) for a price update model with two lags:
a[t_] := Ky[t]+ w1 a[t-1] + (1 - K - w1)a[t-2]   (* price update equation with two lags *)
a[0] = a0   (* index value at time 0 *)
a[-1] = am1   (* index value at time -1 *)
Simplify[a[10]]   (* evaluate a(10) and simplify the expression *)

2.4 Martingale properties

2.4.1 The efficient market process

If there are no arbitrage opportunities, the expected price of a traded security has to increase in the same way as a bank account in a risk-neutral world, see Hull (2009, p. 630). To verify this no-arbitrage restriction, we consider the realization of the efficient market price $y(t)$ and the nominal bank account $B_N(t) \equiv B(0) \exp\left(\int_0^t r_N(\tau)d\tau\right)$ up to time t and determine the expected value of the ratio $y(T)/B_N(T)$ for $T > t \geq 0$. Let us first consider the situation where all direct returns are reinvested in the index (i.e., we have a total return index). This situation can also be modeled by setting the direct return q equal to zero. We then have that (see Appendix A.1 for the proof):

$$E_Q\left[\frac{y(T)}{B_N(T)}|F_t\right] = \frac{y(t)}{B_N(t)}, \tag{2.24}$$

where $E_Q[y(T)/B_N(T)\,|F_t]$ means that the expected value of $y(T))/B_N(T)$ in a risk-neutral world and conditional on the filtration up to time *t* is considered. The expected value of $y(T)/B_N(T)$ is thus constant for $T > t \geq 0$. That is, this ratio is a zero-drift (martingale) process. A total return index thus satisfies the martingale requirement for traded securities if its dynamics is governed by Eq. (2.22). This martingale property is <u>not</u> satisfied by a price index if $q > 0$. In this case:

$$E_Q\left[\frac{y(T)}{B_N(T)}|F_t\right] = \frac{y(t)}{B_N(t)} \exp\left(-q(T-t)\right), \tag{2.25}$$

see again Appendix A.1. A price index is thus not a tradable asset if direct returns are paid out.

2.4.2 The real estate index process

We now consider the realization of the real estate index $a(t)$ up to time t and determine the expected value of the ratio $a(T)/B_N(T)$ for $T > t \geq 0$. In Appendix A we also prove that

$$E_Q\left[\frac{a(T)}{B_N(T)}|F_t\right] = \frac{\exp\left(-q(T-t)\right)}{B_N(t)}\left[y(t)(1-\alpha_{K,T}(t)) + a(t)\alpha_{K,T}(t)\right], \tag{2.26}$$

where $\alpha_{K,T}(t) \equiv (1-K)^{T-t}$. $a(T)/B_N(T)$ is thus a martingale if $a(t) = y(t)$ and $q = 0$. Since $a(t) = y(t)$ holds in general only when $K = 1$, Eq. (2.16) does <u>not</u> represent (the risk-neutral process of) a tradable asset when $K < 1$. Arbitrage opportunities would thus exist in case of a complete market when trading an autocorrelated real estate index. The reverse argument also holds: the index value may well be different from the efficient market price, but active trading in the index is not possible in this case: otherwise arbitrageurs would quickly force the index value toward the efficient market price.

Another important observation is that the future development of a total return real estate index with autocorrelation (i.e., $q = 0$ and $K < 1$) is unbiased if the index is in equilibrium at time *t* (i.e., when $a(t) = y(t)$). With 'unbiased' we here mean that $E_Q[a(T)/B_N(T)] = a(t)/B_N(t)$ for $T > t$. Stated otherwise, if the real estate index starts from an equilibrium situation, the expected return is in line with the return on a risk-free bank account. As a consequence, the pricing formulas for linear instruments (forwards and swaps) all collapse to the classic no-arbitrage formulas if a total return real estate index is in equilibrium at the valuation date. This will be proved more formally in the next section.

A generalization of Eq. (2.26) also exists for the price update model with more than one lag term, as specified in Eq. (2.19). Using the same procedure as in Appendix A.2, the counterpart of Eq. (2.26) follows:

$$\begin{aligned} E_Q\left[\frac{a(T)}{B_N(T)}|F_t\right] &= \frac{\exp\left(-q(T-t)\right)}{B_N(t)}\left[y(t)\sum_{i=1}^{T-t} c_i + \sum_{i=1}^{p} d_i \hat{a}(t-i+1)\right], \\ \text{where } \hat{a}(t-i+1) &\equiv a(t-i+1)\exp\left(\int_{t-i+1}^{t} r_N(\tau)d\tau\right)\exp\left(-q(i-1)\right). \end{aligned} \tag{2.27}$$

Incorporating seasonality is also straightforward (see Eq. (2.14)):

$$E_Q\left[\frac{\tilde{a}(T)}{B_N(T)}|F_t\right] = \exp(g(T))E_Q\left[\frac{a(T)}{B_N(T)}|F_t\right], \tag{2.28}$$

where $\tilde{a}(t)$ is the index value with seasonality.

2.5 Conclusions

We have developed a simple and intuitive risk-neutral model for autocorrelated real estate indices. This model can be coupled to existing risk-neutral models for interest rates, inflation, stochastic volatility, etcetera. By studying the martingale properties of the real estate index we find that the no-arbitrage restriction is only satisfied under very specific conditions (i.e., for total return indices without autocorrelation). In general, arbitrage possibilities thus exist. These cannot be exploited easily, however, since the underlying index cannot be traded actively. We will use the derived results in the next section to derive pricing formulas for various real estate derivatives.

3 Pricing formulas

In this section, we derive pricing formulas for derivatives which are linked to autocorrelated real estate indices. To keep the analysis as transparent as possible, we first present results for the simple price update model with one lag term and then for a model with multiple lags.

3.1 Forwards

We can easily determine the price of a forward contract on a real estate index. Let us assume that the forward contract expires at time $T > t$ and that the agreed-upon delivery price is $F_T(t)$. For the owner of the forward contract, the payoff at time *T* is then equal to the difference between $F_T(t)$ and the index $a(T)$. If we denote the price of this contract at time *t* as $f(t)$, we have that:

$$f(t) = B_N(t)E_Q\left[\frac{F_T(t) - a(T)}{B_N(T)}|F_t\right]. \tag{3.1}$$

Using Eq. (2.26) we then arrive at the following result:

$$f(t) = D(t,T)F_T(t) - \exp\left(-q(T-t)\right)\left[y(t)(1-\alpha_{K,T}(t)) + a(t)\alpha_{K,T}(t)\right], \tag{3.2}$$

where $D(t,T)$ denotes the price at time t of a zero-coupon bond which matures at time $T > t$. The scaling factor $\alpha_{K,T}(t)$ is equal to $(1-K)^{T-t}$. The price of this forward contract is equal to zero if

$$F_T(t) = \frac{\exp\left(-q(T-t)\right)}{D(t,T)}\left[y(t)(1-\alpha_{K,T}(t)) + a(t)\alpha_{K,T}(t)\right] \quad \text{if } f(t) = 0. \tag{3.3}$$

When the index value $a(t)$ is also equal to the efficient market price $y(t)$ we obtain the classic relationship between the (spot) value of the index and the forward price:

$$F_T(t) = \frac{\exp\left(-q(T-t)\right)}{D(t,T)}a(t) \;\; \text{if } y(t) = a(t) \;\; \text{and } f(t) = 0. \tag{3.4}$$

The above analysis is easily extended to more general price update models with seasonality and multiple lag terms. By substituting Eq. (2.28) in Eq. (3.1) we arrive at:

$$\tilde{F}_T(t) = \frac{\exp\left(g(T)\right)\exp\left(-q(T-t)\right)}{D(t,T)}\left[y(t)\sum_{i=1}^{T-t} c_i + \sum_{i=1}^{p} d_i\hat{a}(t-i+1)\right], \tag{3.5}$$

with $\tilde{F}_T$ the forward price including the seasonal component.

Geltner and Fisher (2007) note that the forward market can signal that the real estate market is over- or undervalued. Equation (3.3) makes this price discovery function of the forward market explicit: using an estimate for the confidence parameter *K*, together with the actual forward price $F(T)$ and the index value $a(t)$, the underlying efficient market price $y(t)$ can be derived. The accuracy of the extracted efficient market price is of course strongly depending on the degree of liquidity and density in the forward market. A reliable price reporting system is also crucial. Geltner and Fisher (2007) mention that the U.K. IPD swap market appears to be performing the price discovery function well since IPD swap prices have fallen dramatically in 2006, even as the IPD index itself has continued to climb. The IPD swap market has thus correctly signaled overvaluation in the U.K. property market.

If there is a liquid forward market it also becomes possible to (delta) hedge movements of the underlying efficient market price. For example, if we calculate the sensitivity $\partial F_T(t)/\partial y(t)$ using Eq. (3.3) we find that:

$$\partial F_T(t)/\partial y(t) = \frac{\exp\left(-q(T-t)\right)}{D(t,T)}[1-\alpha_{K,T}(t)+K\alpha_{K,T}(t)], \tag{3.6}$$

where we have used Eq. (2.16) to determine $\partial a(t)/\partial y(t)$. Changes of the efficient market price $y(t)$ are thus directly reflected in changes of the forward price $F_T(t)$. This is an important result because we implicitly assumed (see Section 2.3.4) that continuous trading in the underlying efficient market index is possible. This, obviously, cannot be achieved by trading in the primary real estate market (due to a limited liquidity and high trading costs). Using forward contracts it however becomes possible to replicate the efficient market process in good approximation, provided this secondary market is sufficiently liquid. This also provides a method to replicate the cash flows of more complicated real estate derivatives (like options) using delta hedging. A liquid forward market would thus serve as the foundation of the risk-neutral valuation method developed in this paper.

We should also note that in case of stochastic interest rates forward and futures prices are not equal. This is caused by the daily settlement procedure for futures contracts. Assume for instance that the real estate index is strongly positively correlated with interest rates. When the real estate index increases, the gain of a long futures contract is invested with a high probability at an above average interest rate. The opposite holds when the real estate index drops and the resulting loss probably needs to be financed at a below average interest rate. It thus follows that in case of a positive correlation between the real estate index and interest rates a long futures contract will be more attractive than a long forward contract. Other factors may also cause significant differences between forward and futures contracts (like taxes and transaction costs).

3.2 Swaps

We assume that the swap contract starts at time $T_0 \geq t$ and ends at time $T_n > T_0$. To fix the notation: the owner of a receiver swap receives the price return of the real estate index in each period and pays the floating rate. The floating payments are based on the index values at the beginning of each period. The floating rate can, for example, be the LIBOR spot rate. We also assume (without loss of generality) that the cash flows are swapped annually. Results for a total return index can be obtained by setting *q* equal to zero in the equations below.

We first determine the value of the swaplet which is active during the time interval $[T_{k-1}, T_k]$, where $1 \leq k \leq n$. If we denote the price of this swaplet as $\Pi_k(t)$ we can use the following result by Björk and Clapham (2002):[5]

$$\Pi_k(t) = LB_N(t)E_Q\left[\frac{a(T_k)}{B_N(T_k)} - \frac{a(T_{k-1})}{B_N(T_{k-1})}|F_t\right]. \tag{3.7}$$

where *L* is a scaling parameter which can be used to set the notional amount of the swap to the right

[5]We do not assume (as is the case in Björk and Clapham (2002)) that direct returns are reinvested. This situation is, however, easily obtained by setting q equal to zero in the derived equations.

amount.[6] The total value of the swap, $\Pi(t)$, is thus equal to:

$$\Pi(t) = \sum_{k=1}^{n} \Pi_k(t) = LB_N(t)E_Q\left[\frac{a(T_n)}{B_N(T_n)} - \frac{a(T_0)}{B_N(T_0)}\,|F_t\right]. \tag{3.8}$$

Substituting Eq. (2.26) and rearranging terms we find that

$$\begin{aligned}\Pi(t) &= L\exp\left(-q(T_N-t)\right)\left[y(t)(1-\alpha_{K,T_N}(t)) + a(t)\alpha_{K,T_N}(t)\right]\\ &- L\exp\left(-q(T_0-t)\right)\left[y(t)(1-\alpha_{K,T_0}(t)) + a(t)\alpha_{K,T_0}(t)\right].\end{aligned} \tag{3.9}$$

When $a(t) = y(t)$ and $q = 0$ the value of the swap contract is equal to zero. The same holds if $K = 1$ and $q = 0$. This is in line with the result obtained by Björk and Clapham (2002), who prove that the value of a total return real estate swap is exactly equal to zero if the real estate index follows a random walk process with drift.[7]

Results for the general model with multiple lags and seasonality can easily be derived by substituting Eq. (2.28) in Eq. (3.8). If the swap market is sufficiently liquid it also becomes possible to (delta) hedge movements of the underlying efficient market price. For example, if we calculate the sensitivity $\partial\Pi_T(t)/\partial y(t)$ using Eqs. (3.9) and (2.16) we find that:

$$\begin{aligned}\partial\Pi(t)/\partial y(t) &= L\exp\left(-q(T_N-t)\right)\left[1-\alpha_{K,T_N}(t) + K\alpha_{K,T_N}(t)\right]\\ &- L\exp\left(-q(T_0-t)\right)\left[1-\alpha_{K,T_0}(t) + K\alpha_{K,T_0}(t)\right].\end{aligned} \tag{3.10}$$

A liquid swap market can thus also be used to delta hedge more complicated derivatives. If there is no access to either a liquid forward or swap market the risk-neutral valuation approach cannot be applied. One should then resort to methods developed for pricing in incomplete markets. A good example of this approach is given in the recent paper by Syz and Vanini (2009). They study the effect of market frictions (like transaction costs, transaction time, and short sale constraints) to explain why property returns are swapped against a rate that can deviate significantly from LIBOR.

3.3 European options

We value the option at time *t*. The option expires at time $T > t \geq 0$ and cannot be exercised before that date. If $K = 1$, an exact pricing formula exists. This formula is a modification of the familiar Black-Scholes (1973) equation. The crucial modification is an adjustment of the implied volatility parameter to account for the effect of stochastic interest rates, see Brigo and Mercurio (2006, p. 888). For clarity, the adjusted implied volatility will be denoted below as σ_2^*.

If $K < 1$, a simple, approximating pricing formula can be derived. Equation (2.17) shows that the index value at time T is equal to a weighted sum of $T - t$ lognormal distributions. We thus have an Asian basket option. To value this option, we first calculate the first moment M_1 and the second moment M_2 of the exact probability distribution at time T, see Hull (2009, p. 578–579):

$$\begin{aligned}M_1 &= M_{1,0} + \sum_{i=1}^{T-t} M_{1,i},\\ M_{1,0} &= \exp\left((r_{N,T}(t) - q)(T-t)\right)a(t)(1-K)^{(T-t)},\\ M_{1,i} &= \exp\left((r_{N,T}(t) - q)(T-t)\right)y(t)K(1-K)^{i-1},\end{aligned} \tag{3.11}$$

and

$$M_2 = \sum_{i=0}^{T-t} M_{1,i}^2 \exp\left(\sigma_2^{*2} i\right) + \sum_{i<j} M_{1,i}M_{1,j}\exp\left(\sigma_2^{*2} i\right). \tag{3.12}$$

[6]The notional is thus not a fixed amount but is adjusted periodically by the appreciation and depreciation of the index, see the description in Buttimer and Kau (1997, p. 22).

[7]Patel and Pereira (1996) have recently extended this result by considering the effect of counterparty default risk. They find that the swap price is no longer equal to zero in this case because a compensation for the additional risk is required. Syz and Vanini (2009) study the effect of market frictions (like transaction costs, transaction time, and short sale constraints) on the real estate swap market.

Using these moments, we can fit the exact distribution with an approximating lognormal distribution. This approach was first proposed by Levy (1992). The forward price $F_T(t)$ and the implied volatility σ can then be approximated using the following equation, see also Hull (2009, p. 565):

$$F_T(t) = M_1; \quad \sigma = \sqrt{\frac{1}{T-t}\ln\left(\frac{M_2}{M_1}\right)}. \tag{3.13}$$

Closed-form pricing formulas for European put and call options are then given by the familiar Black-Scholes (1973) equation:

$$\begin{aligned} p(t) &= \exp\left(r_{N,T}(t)(T-t)\right)\left[KN(-d_2) - F_T(t)N(-d_1)\right], \\ c(t) &= \exp\left(r_{N,T}(t)(T-t)\right)\left[F_T(t)N(d_1) - KN(d_2)\right], \end{aligned} \tag{3.14}$$

where

$$d_1 = \frac{\ln(F_T(t)/K) + (r_{N,T}(t) + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}. \tag{3.15}$$

$p(t)$ here denotes the price of a put option and $c(t)$ the price of a call option.

Equations (3.12)-(3.15) are also valid for models with multiple lag terms and seasonality. Equation (3.11) should be generalized, however, in this case. The proper form is:

$$\begin{aligned} M_1 &= M_{1,0} + \sum_{i=1}^{T-t} M_{1,i}, \\ M_{1,0} &= \exp(g(T))\exp\left((r_{N,T}(t) - q)(T-t)\right)\sum_{i=1}^{p} d_i \hat{a}(t-i+1), \\ M_{1,i} &= \exp(g(T))\exp\left((r_{N,T}(t) - q)(T-t)\right) y(t) c_i. \end{aligned} \tag{3.16}$$

The accuracy of this option pricing formula is tested in Section 5.3. It is important to note that alternative pricing techniques for Asian options are available in the literature. Lord (2006) provides an overview of the current state-of-the-art in this field. A detailed discussion of these advanced methods is outside the scope of this paper. We certainly recommend them, however, when option pricing with a very high accuracy is required. We also note that an accurate valuation of American options is possible using the least squares Monte Carlo method by Longstaff and Schwartz (2001).

3.4 Conclusions

Our real estate valuation model can be analyzed theoretically and closed-form pricing formulas have been derived. The formulas for forwards and swaps are exact. These formulas collapse to the classic no-arbitrage results if the real estate index follows an efficient market process. If this is not the case, due to autocorrelation in the index returns, deviations from the no-arbitrage price occur if the index level is not equal to the efficient market price.

Given actual market prices for forwards or swaps, the derived pricing formulas can thus be used to estimate the difference between the current index level and the efficient market price. This facilitates the price discovery process: information about market over- or undervaluation can be extracted from the derivatives markets for forwards or swaps. This, in turn, can also make the primary real estate market more efficient because the price update process becomes more effective. We also demonstrated that forward and swap contracts can be used to replicate the underlying efficient market process. Given a liquid forward or swap market the developed risk-neutral valuation model is thus well-founded and more complicated derivatives can be hedged and priced.

For European real estate options a simple, approximate closed-form solution is derived. Since these options are essentially Asian (basket) options, existing algorithms for Asian stock options can be applied to further improve the accuracy. Accurate valuation of American real estate options is also possible using the Monte Carlo method proposed by Longstaff and Schwartz (2001).

The accuracy of the proposed valuation model is of course strongly depending on the ability of the valuation model to properly capture the dynamic behavior of the real estate index. Model selection and estimation issues are therefore discussed in detail in the next section.

4 Estimation of the model

4.1 Introduction

If liquid real estate option markets would exist, our real estate model could be calibrated (at least partly) to market data. For example, for equity markets the standard approach is to fit the model parameters as good as possible to prices of equity options with different maturities and strike levels. The market-implied volatilities (and sometimes also the market-implied correlations) are thus the key input for the estimation of the model. This approach is currently not feasible for real estate, however, due to a lack of trading in real estate options. We thus have to resort to an estimation of the model using historical index data. In Section 4.2 we first consider annual data for the Dutch residential market. In Section 4.3 we consider monthly data for the U.S. residential market.

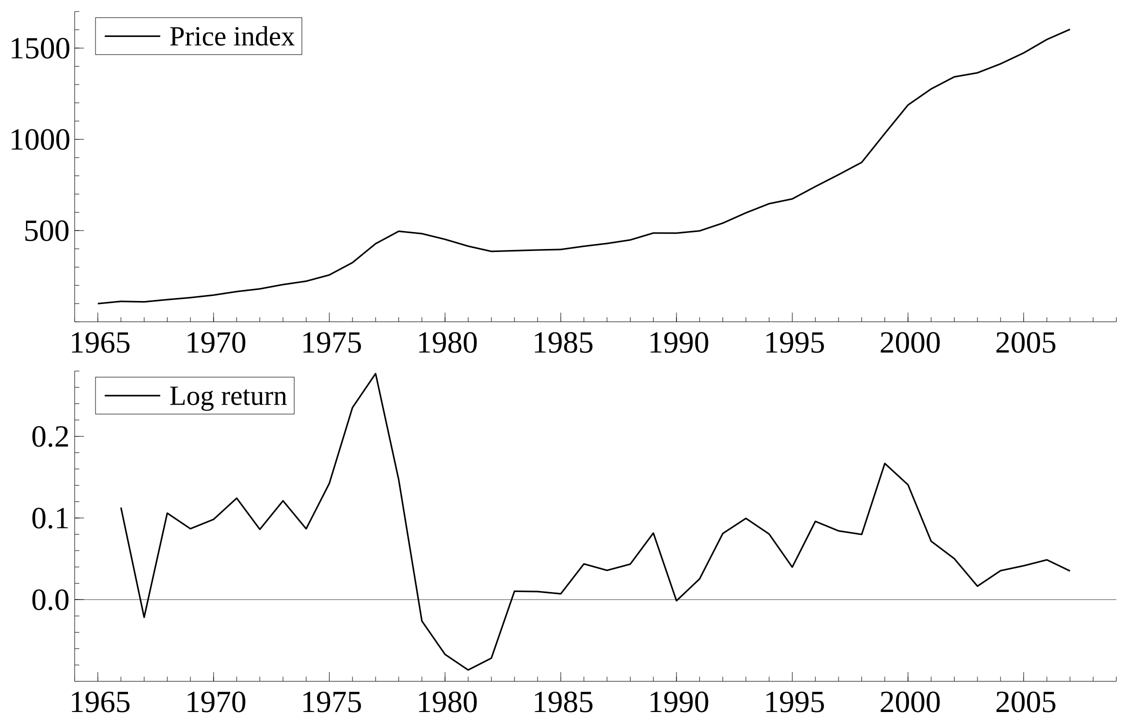
4.2 Dutch house price index

4.2.1 Real estate model

As a first example, we consider a transaction-based index of Dutch house prices. We use annual returns for the period 1966-2007. The data from 1985-2007 are from the Dutch Association of Realtors and Property Consultants (NVM). The data from 1966-1984 are average weighted sales prices of existing homes from Statistics Netherlands (CBS), except for 1975 which is an extrapolation of the NVM figures. For more information about this index, see De Vries et al. (2007) and Francke (2009).

Figure 4.1 shows the development of the (log) price index and the corresponding annual (log) returns. We can see that house prices increased until 1977, when they experienced a sharp fall. From the mid-1980s until the early 2000s, house prices exhibited a sharp increase. The most recent period, the late 2000s, which coincides with the global financial crisis, witnesses a much slower increase of house prices.

Figure 4.1: The CBS / NVM index of Dutch house prices.



We now compare the quality of price update models with up to 3 lags. The parameters of Eq. (2.12) are estimated with ordinary least squares (OLS) regression using annual log returns for the period 1969-2007. By applying the augmented Dickey-Fuller test, the null hypothesis of a unit root is rejected at the 5% level for these returns, i.e. the process is stationary. The main results are summarized in Table 4.1. More detailed information about the OLS regression results can be found in Appendix B.

Table 4.1: Characteristics of the estimated price update models.

	1 lag	*2 lags*	*3 lags*
K^*	0.27	0.37	0.39
w_1^*	0.73	0.99	0.97
w_2^*	N.A.	-0.35	-0.29
w_3^*	N.A.	N.A.	-0.07
π	0.061	0.061	0.062
$K^*\pi$	0.016	0.022	0.024
σ_ϵ	0.188	0.126	0.116
$K^*\sigma_\epsilon$	0.050	0.046	0.046
Durbin-Watson	1.39	1.87	1.86
Jarque-Bera (JB)	4.18	0.50	0.73
P-value JB	0.12	0.78	0.69

Because we assume in our valuation model that the efficient market returns follow a random-walk process with drift, it is important to check if the residuals indeed have a serial correlation close to zero. In this case, the Durbin-Watson test statistic should be close to 2. Table 4.1 shows that this is indeed the case for the models with 2 or 3 lags. The residuals are also (in good approximation) normally distributed for these models: the Jarque-Bera test statistic is close to zero for the models with 2 and 3 lags.

There exist several order selection criteria to select the best AR model. These criteria typically choose the model order in such a way that the prediction error is minimized while putting a penalty on the number of parameters estimated. The prediction error is here measured using the maximum likelihood residual variance. This variance is not corrected for the number of parameters estimated. The number of parameters is equal to the order of the AR model plus an additional parameter for the strength of the random walk process. We use the final prediction error criterion (FPE), the Akaike information criterion (AIC), the Schwarz criterion (SC), the Hannan and Quinn criterion (HQ)[8] and the criterion for autoregressive transfer functions (CAT). The order for which the value of the criterion is minimized is seen as the model which is closest to the true model and is therefore the 'optimal' order. Each of the criteria assumes that the models are estimated including a constant term.

Table 4.2 shows that the model with 2 lags is unanimously selected. This is not surprising since the residual error ($K^*\sigma_\epsilon$) does not further decrease for the more complex model with 3 lags. An AR model with 2 lags thus appears to be the best candidate for the situation at hand.

Table 4.2: Selecting the optimal order of the estimated price update models.

	1 lag	*2 lags*	*3 lags*	*selected order*
1. FPE	0.00268	0.00242	0.00253	2
2. AIC	-5.92	-6.03	-5.98	2
3. SC	-5.84	-5.90	-5.81	2
4. HQ	-5.88	-5.96	-5.89	2
5. CAT	-383	-424	-404	2

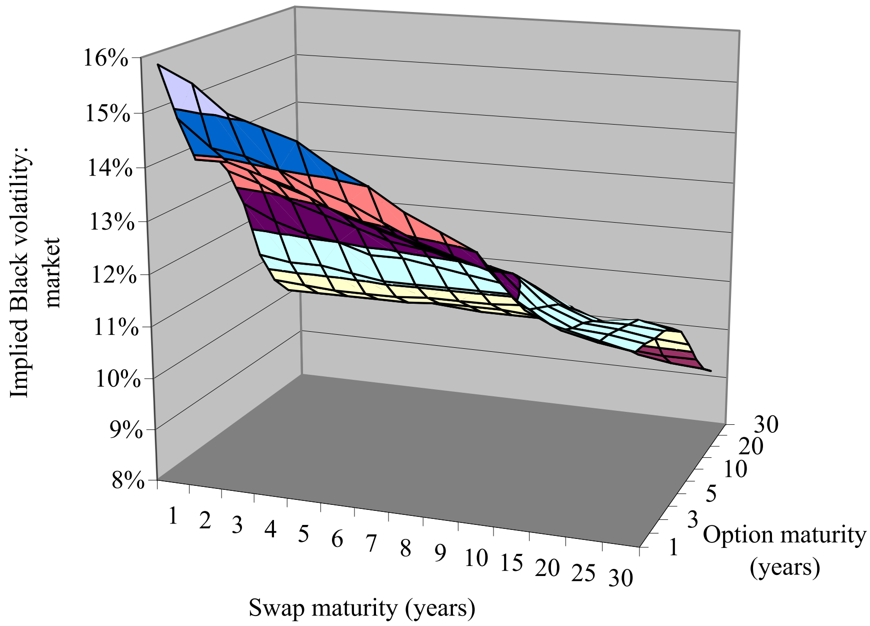
4.2.2 Interest rate model

The valuation date is December 31, 2007. We use the Euro zero-coupon swap curve as published by Bloomberg as the reference nominal interest rate curve. We estimate the stochastic interest rate

[8] The constant c that is used in the HQ criterion is set equal to 1.1.

model (a one-factor Hull-White model) using market prices of forward-at-the-money options on Euro swap contracts (data also from Bloomberg). The two parameters of the one-factor Hull-White model (the mean-reversion parameter κ and the volatility σ_1) are estimated using a large set of swaptions, with option and swap maturities ranging from 1 to 30 years. Swaption prices are typically quoted in terms of implied (Black) volatilities. Figure 4.2 gives a graphical overview of these volatilities. We use

Figure 4.2: Market-implied volatilities for a large set of swaptions with different option and swap maturities as of December 31, 2007.



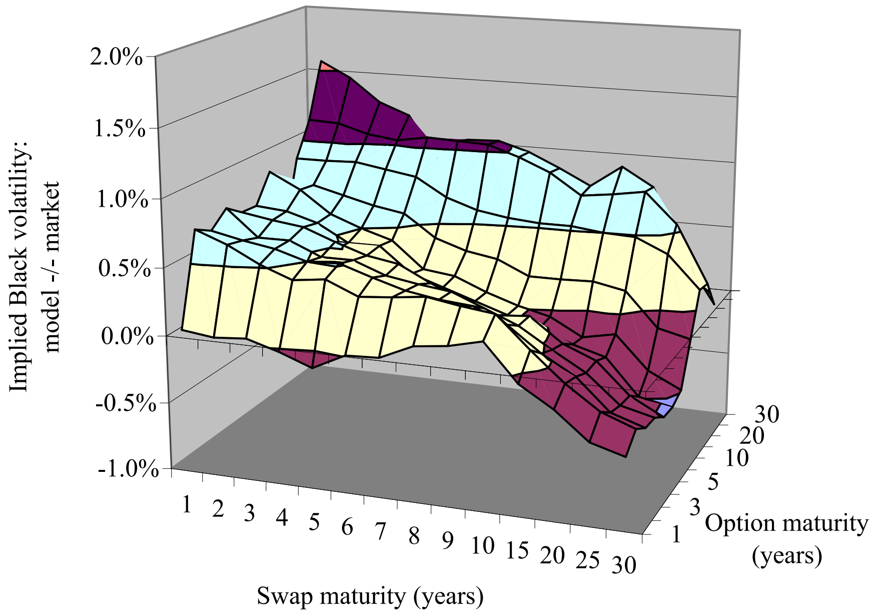
the Levenberg-Marquardt least-squares algorithm to find the optimal model parameters. The Hull-White parameters with the best fit are a mean reversion κ of ≈ 0.024 and a volatility σ_1 of $\approx$ 0.0068. A comparison between the model and market prices is shown in Figure 4.3. In this figure we show the difference between the model and market implied volatility for the entire set of swaptions. The average error is equal to 0.4 % point; the maximum error is 1.6 % point (for a 30-year swaption on a 1 year swap). The fit is reasonably good for the most liquid swaptions (with short to medium term option and swap maturities). For less liquid options (for example with very long option maturities) the deviations can become larger. We also used a more elaborate two-factor Hull-White model. This does not improve the results significantly, however, so we continue with the simple one-factor model in the remainder of this paper.

The correlation between the Wiener processes for the short interest rate (see Eq. (2.15)) and the efficient market process (see Eq. (2.22)) is estimated using historical data. More precisely, we consider the correlation between historical changes in the short interest rate and the derived efficient market returns for the period 1969–2007. This correlation is equal to -0.03.

4.2.3 Scenario characteristics

The total expected return on owner-occupied housing is the expected house price appreciation plus a convenience yield, see De Jong, Driessen, and van Hemert (2007). A convenience yield represents the (non-monetary) benefits from the housing services. When we assume that the convenience yield is a constant fraction of the house value we can model this aspect by setting the direct return q equal to the convenience yield. De Jong, Driessen, and van Hemert (2007) refer to the convenience yield as an imputed rent and give an estimate of 0.67% per year for the U.S. housing market. The same percentage is used in this section for the Dutch housing market.

Figure 4.3: Quality of fit of the estimated Hull-White model. We here show the difference between the model-implied volatility and the market-implied volatility for a large set of swaptions with different option and swap maturities.



It is also important to specify the ratio of the initial index price level $a(0)$ and the efficient market price $y(0)$. If $a(0) > y(0)$, the house market is overvalued; if $a(0) < y(0)$, the house market is undervalued. The question whether the Dutch housing market is overvalued or not is currently a hotly debated topic. Francke, Vujic, and Vos (2009) investigated this issue using different models. Unfortunately, all models estimate the overvaluation of the Dutch market differently, ranging between approximately 0% and 12% overvaluation in 2007 (our valuation date). Because the precise amount of overvaluation in 2007 thus cannot be determined very accurately, we first take a neutral stance by assuming that the house market is in equilibrium in 2007. The effect of over- or undervaluation is then studied in a sensitivity analysis in Section 5.2.

The scenario characteristics of the nominal short interest rate and the autocorrelated price returns for the price update model with two lags are shown in Figure 4.4. Recall that the model with two lags has been selected as the best candidate in Section 4.2.1. The real estate returns are generated using the price update rule in Eq. (2.19) and the model parameters in Table 4.1. To illustrate the effect of the stochastic interest rates, we show real estate returns for deterministic interest rates (see Panel B) and stochastic interest rates (see Panel C). In the simulation with deterministic interest rates the volatility parameter σ_1 of the Hull-White model is set equal to zero. Comparing the results in Panel B and Panel C we see that the selected high interest rate scenario leads to relatively high real estate returns in Panel C. This effect is typical for a risk-neutral model since the expected return is equal to the short interest rate, see Eq. (2.22). When interest rates become stochastic (as in Panel C of Figure 4.4), the volatility of the real estate returns increases slightly ($\approx$ 0.3 % point). The autocorrelations also increase slightly in this case. Recall that the expected return is governed by the evolution of the short interest rate in a risk-neutral world. Because the short rate has a high autocorrelation (see Panel A of Figure 4.4) the autocorrelation of the real estate returns also increases. These effects are also visible in Table 4.3, where we compare the statistical properties of the scenarios in Figure 4.4 with the historical returns in Figure 4.1. Note that the model with deterministic interest rates fits the historical statistics very well. For the complete model with stochastic interest rates the volatility and autocorrelations increase slightly.

Figure 4.4: Scenario characteristics of the short interest rate and the real estate returns.

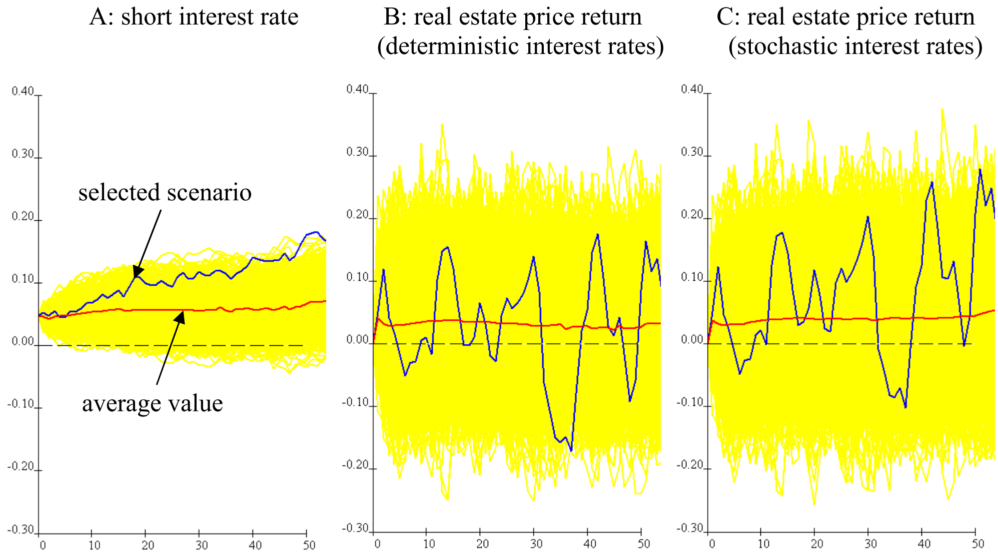


Table 4.3: Comparing the statistical properties of the estimated price update model with the historical data.

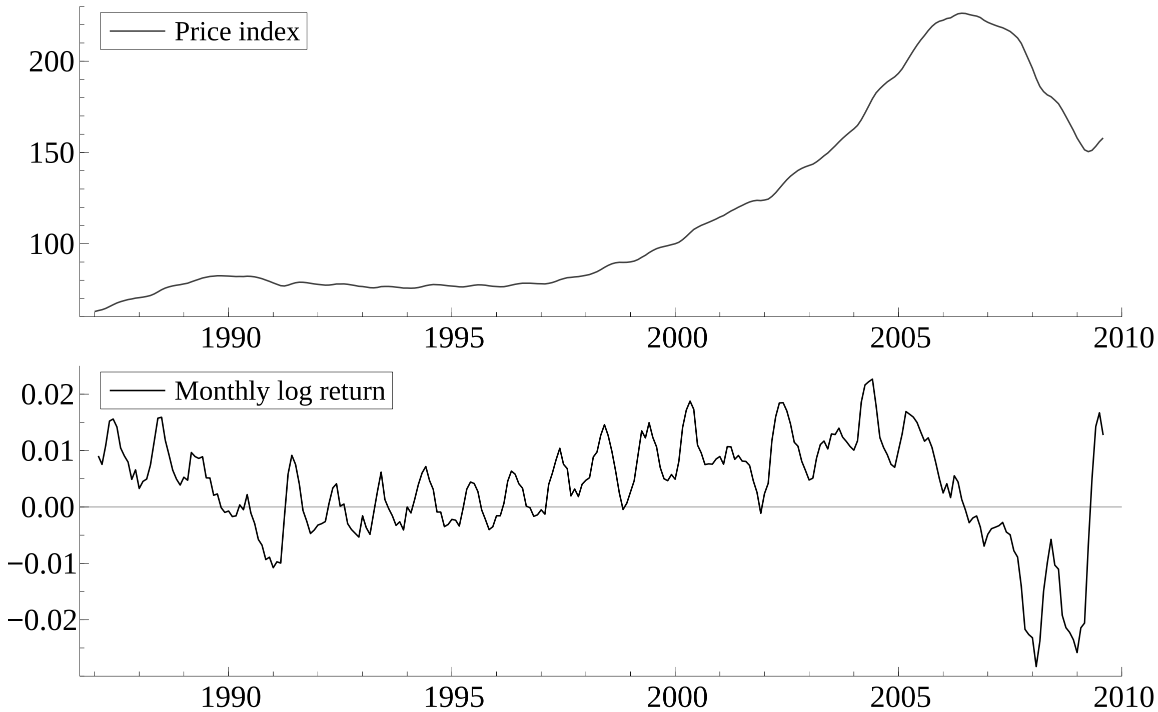
	original series	*scenario model deterministic interest rates*	*scenario model stochastic interest rates*
Stdev. a(t)	0.073	0.072	0.075
Cor. $\{a(t), a(t-1)\}$	0.67	0.73	0.75
Cor. $\{a(t), a(t-2)\}$	0.34	0.37	0.43
Cor. $\{a(t), a(t-3)\}$	0.06	0.11	0.19

4.3 Example 2: U.S. house price index

As a second example, we consider the S&P/Case-Shiller Home Price Indices.[9] These indices measure the residential housing market in metropolitan regions across the United States. All indices are constructed using the repeat sales pricing technique. This methodology collects data on single-family home re-sales, and captures re-sold sale prices to form sale pairs. The S&P/Case-Shiller Home Price Indices are calculated monthly and published with a two month lag. The index point for each reporting month is based on sales pairs found for that month and the preceding two months. This index family consists of 20 regional indices and two composite indices as aggregates of the regions.

We here consider the 10 city composite index, which tracks the house price in the original 10 S&P/Case-Shiller indices. We use the monthly index levels from January 1987 through August 2009 (as published on October 29, 2009). An overview of the index development over this period is given in Figure 4.5. Notice the sharp increase (more than 250%) of the house price in the period 1997 through 2006, followed

Figure 4.5: Development of the 10 city composite S&P/Case-Shiller index between January 1987 and August 2009.



by the sharp decline in prices in the last years. The last few months, prices are slowly recovering. Looking more closely, we can see that the log returns exhibit an oscillatory pattern with a period of approximately one year. This is due to a seasonal effect. In addition, the volatility of the log returns appears to be non-stationary over time. For example, the volatility in the quiet upward trending market (1995-2005) is much lower than during the recent market crash.

We model seasonality and stochastic volatility as follows. First, we estimate an autoregressive (AR) model with seasonal dummies using OLS (see the description in Section 2.2.4). We allow for at most 14 lags, i.e. a lookback period of at most 14 months.[10] We estimate the optimal model using the automatic model selection option in PCGive.[11] We then remove all AR coefficients which are not significant and

[9] More information about the Case-Shiller indices, including historical data and information about the index construction, can be found on http://www.homeprice.standardandpoors.com.

[10] Results do not improve significantly when we use more than 14 lags.

[11] See Doornik and Hendry (2007).

extend this model with a GARCH(1,1) stochastic volatility model, see Bollerslev (1986). This model can be written in the following form:

$$(\sigma(t))^2 = \alpha_0 + \alpha_1 (r_c^a(t-1))^2 + \beta_1 (\sigma(t-1))^2, \quad (4.1)$$

where $\sigma(t)$ is the volatility of the monthly log return $r_c^a(t)$. The AR coefficients, the coefficients of the seasonal dummies and the GARCH(1,1) coefficients (α_0, α_1, and β_1) are then determined by maximum likelihood estimation in PCGive. The characteristics of the estimated model are summarized in Table 4.4.

Table 4.4: Characteristics of the estimated Case-Shiller model, including seasonality and stochastic volatility. The AR model consists of 14 lags, of which only the 5 significant coefficients are included. Three seasonal dummies (for the months March, April and July) are also included. The α_0, α_1 and β_1 parameters are the coefficients of the GARCH(1,1) stochastic volatility model.

Model with 14 lags	*Coefficients*	*Standard Error*	*t Stat*	*P-value*
$r_c^a(t-1)$	1.162	0.040	32.9	0.000
$r_c^a(t-3)$	-0.346	0.055	-6.99	0.000
$r_c^a(t-5)$	0.112	0.034	3.84	0.000
$r_c^a(t-11)$	0.116	0.030	4.27	0.000
$r_c^a(t-14)$	-0.083	0.029	-3.03	0.003
$r_c^a(March)$	0.00163	0.00042	2.72	0.007
$r_c^a(April)$	0.00149	0.00048	2.75	0.006
$r_c^a(July)$	-0.00143	0.00047	-2.89	0.004
α_0	9.04e-7	4.99e-7	1.31	0.192
α_1	0.233	0.087	2.66	0.008
β_1	0.582	0.154	2.85	0.005

The residuals of the regression have a serial correlation close to zero: the Durbin-Watson test statistic is equal to 2.1. The residuals are also (in good approximation) normally distributed: the Jarque-Bera test statistic is equal to 1.5 (with a P-value of 0.46).

The results are shown in more detail in Figure 4.6. The top left figure shows how well the model fits the historical data period. The top right figure displays the (scaled) residuals. The evolution of conditional volatility is displayed in the bottom left figure. Notice the increasing volatility in the most recent period. The histogram in the bottom right figure shows the histogram of the residuals. Notice that these are (approximately) normally distributed.

Figure 4.7 displays the autocorrelation function (ACF) of the squared residuals. Autocorrelations are mostly positive when only the AR model (with seasonal dummies) is used. These positive autocorrelations, which signal the presence of stochastic volatility, are suppressed when using a GARCH(1,1) model. The GARCH(1,1) model thus successfully describes the stochastic volatility component.

5 Application

5.1 Introduction

The developed valuation model for real estate derivatives is explored further in this section. As an example, we consider a house owner who buys a (hypothetical) at-the-money European put option with a maturity of 10 years on his/her house. The underlying index is the Dutch house price index that we introduced in Section 4.2. We use the price update model with two lags and the parameter settings in Table 4.1. The direct return q is again set equal to 0.67%. In Section 5.2 we present an accurate valuation of the put option using Monte Carlo valuation. Section 5.3 tests the accuracy of the approximating closed-form pricing formula that we derived in Section 3.3.

It is important to note that a property derivative market does currently not exist in the Netherlands. The example given in this section is thus only provided to illustrate the developed valuation framework. We

Figure 4.6: Estimation results for the 10 city composite S&P/Case-Shiller index.

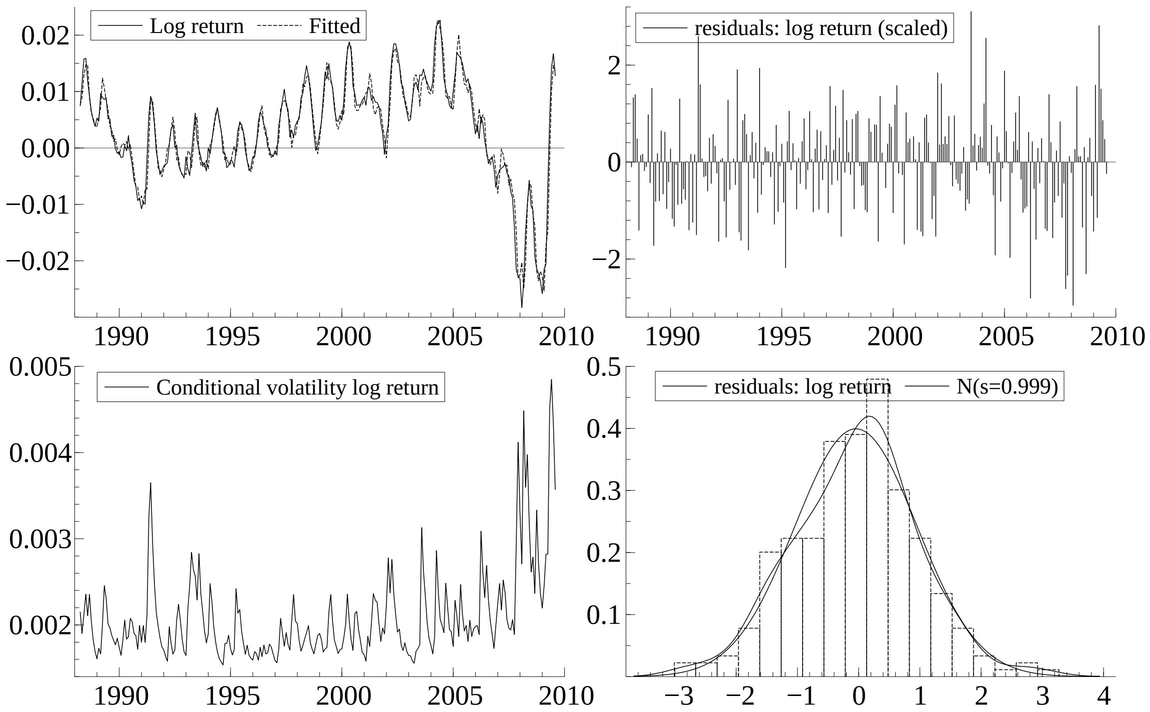
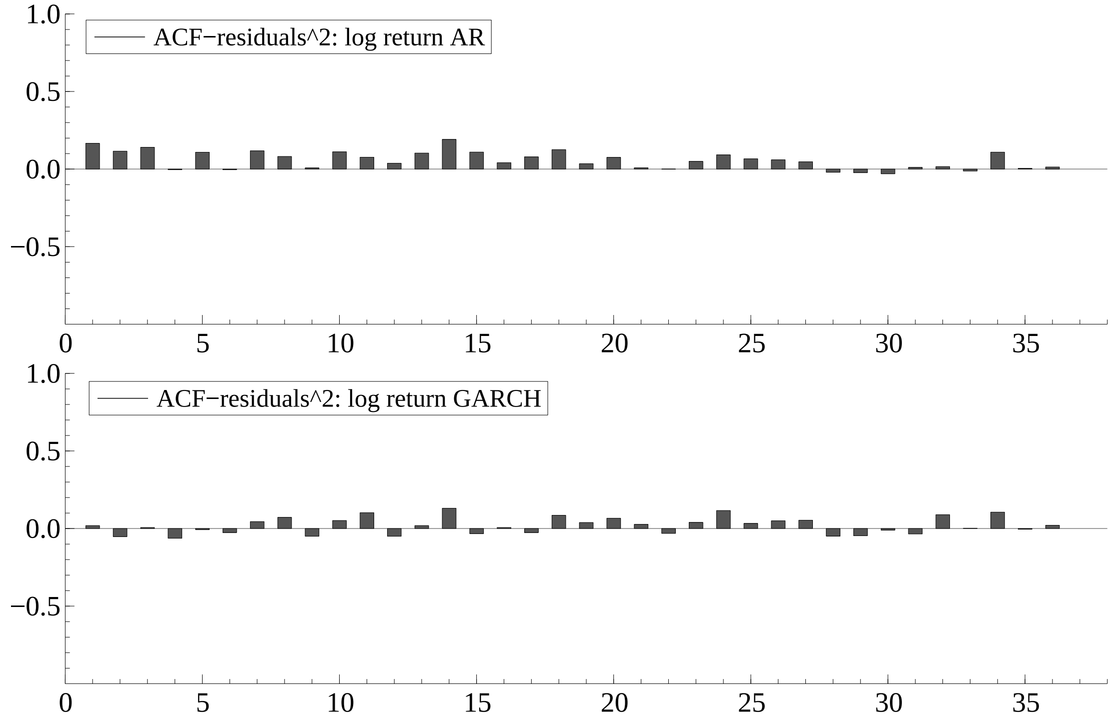


Figure 4.7: Autocorrelation function (ACF) of the squared residuals for an AR model without stochastic volatility (top) and a GARCH(1,1) stochastic volatility model (bottom). Note that the positive autocorrelations, which signal the presence of stochastic volatility, are suppressed when using a GARCH(1,1) model.



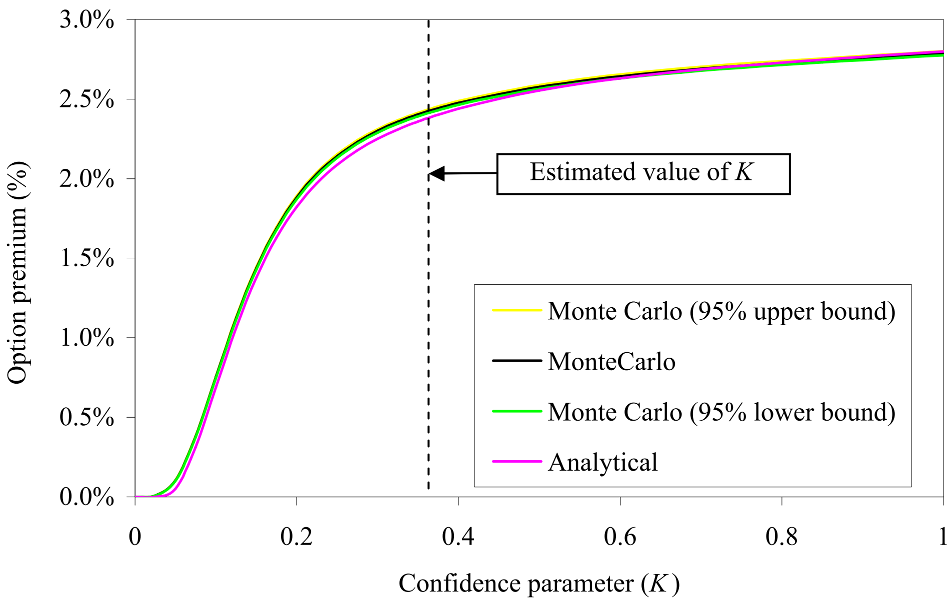
should also note that our valuation model assumes that continuous trading in the underlying efficient market price is possible. We explained before (see Section 3.1 and 3.2) that it is possible to replicate continuous trading in the underlying index using forward or swap contracts. Once a liquid forward or swap market has emerged in the Netherlands, the applied valuation framework can thus be applied to value (arbitrary complex) property derivatives. In practice, it is of course also important to distinguish between global and more local real estate risk. Derivatives trading typically focuses on the main (metropolitan) areas, which can make it more difficult to hedge house price risk in local areas using derivatives.

5.2 Monte Carlo results

We value this option using Monte Carlo simulation by generating 1,000,000 scenarios and discounting the option payoffs back to time zero along the path of the short interest rate.[12] For the estimated risk-neutral model the option price is 2.43% ($\pm$ 0.01%) of the notional amount. To determine the effect of autocorrelation in the index returns on the option value we also generated results with alternative model parameters. For these alternative models, the confidence parameter K is varied between zero and one. The other weights (w_1 and w_2) are proportionally scaled up or down in order to keep the sum of all weights equal to one.

Figure 5.1 shows the results. Recall that returns are highly correlated if $K = 0$; if $K = 1$, returns are

Figure 5.1: Price of a 10-year at-the-money put option on a house price index as a function of the confidence parameter K.



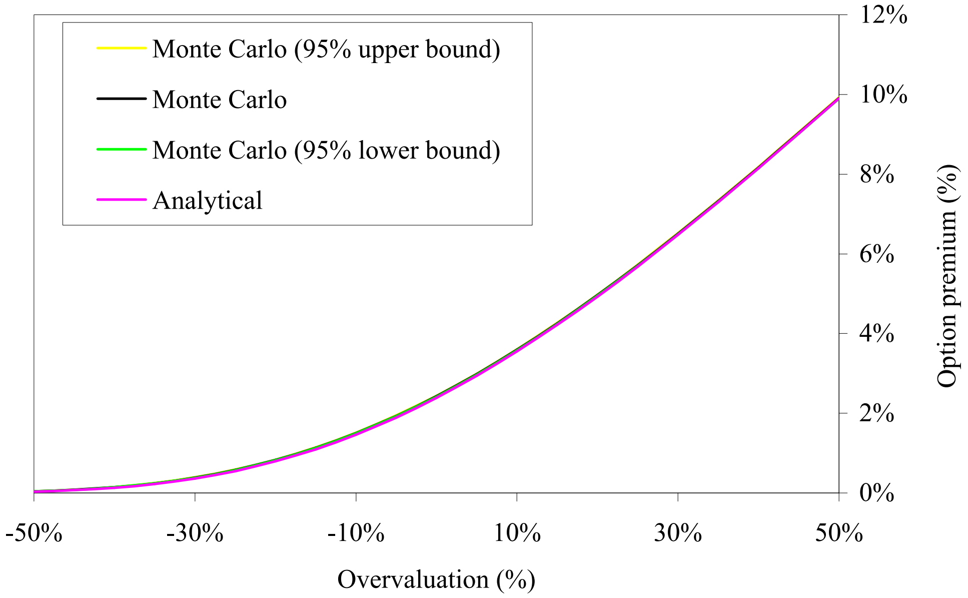
almost completely uncorrelated. Also keep in mind that the estimated value of K is equal to 0.37, as indicated in the graph. The option premiums are expressed as percentages of the notional amount. The upper and lower bound Monte Carlo lines indicate the 95% confidence interval ($\pm$ 1.96 σ) around the Monte Carlo prices. Figure 5.1 clearly shows that the option premiums decrease when the autocorrelation of the returns increases (i.e. K decreases). This is due to the smaller (cumulative) volatility of the autocorrelated real estate returns.

We can model overvaluation and undervaluation of the real estate market by setting the current efficient market level $y(0)$ lower respectively higher than the current index level $a(0)$, see Eq. (2.19). The results are shown in Figure 5.2. In this figure, the confidence parameter is set equal to the default value (0.37). Overvaluation is measured as $(a(0) - y(0))/y(0)$. The option premiums increase, to be expected, when

[12] We use antithetic sampling to reduce the standard error of the Monte Carlo estimate.

the initial index level is higher than the efficient market price (and vice versa). Information about the degree of under- or overvaluation of the real estate market may be obtained by using information in the forward or swap market (see Geltner and Fisher (2007) and Section 3 of this paper) or by using information in the public real estate market, see Geltner, MacGregor, and Schwann (2003).

Figure 5.2: Effect of over- or undervaluation on the option price ($K = 0.37$).



5.3 Quality of the closed-form solution

The quality of the approximate analytical pricing formula that we derived in Section 3.3 is also investigated in Figure 5.1.[13] The analytical pricing formula is very accurate for high values of the confidence parameter K. In this case, the terminal probability distribution of the index value is determined to a large extent by only a few lognormal distributions. The terminal distribution function can be fitted well with a single lognormal distribution in this case, so the approximation error is small. When K decreases, the terminal probability distribution of the index value becomes the sum of a series of different lognormal functions. As a result, the fit with one lognormal function deteriorates. However, the quality of the approximation remains quite satisfactory. Figure 5.2 also demonstrates that the agreement between the Monte Carlo price and the analytical price is very good in case of under- or overvaluation. For alternative approximation methods, the interested reader is referred to the overview in Lord (2006).

6 Conclusions

We propose a new and intuitive risk-neutral valuation model for real estate derivatives which are linked to autocorrelated indices. Following Jokivuolle (1998), we first model the (unobserved) underlying market fluctuations using a simple random walk process with drift. We then reconstruct the observed index using an adaptation of the price update rule by Blundell and Ward (1987).

The first modification of the update rule by Blundell and Ward (1987) is straightforward and consists of adding multiple lag terms. This leads to an autoregressive (AR) model which can be estimated using standard econometric techniques. A second modification is more fundamental from a valuation perspective and consists of using the accrued value of past observations. We show, using real (annual

[13] We use Eq. (3.16) because the applied price update model has multiple lags.

and monthly) data, that this model is able to reproduce the dynamic behavior of a transaction-based house price index with autocorrelation. For high-frequency data (like monthly house prices) aspects like seasonality and stochastic volatility can become important. It is possible to model such phenomena as well within the developed framework.

The resulting model has also been analyzed analytically and closed-form pricing solutions have been derived for forwards, swaps and European put and call options. The developed model can be applied once a liquid forward or swap market has been established. In this case it becomes possible to (approximately) replicate the underlying efficient market process. The risk-neutral assumption of continuous trading in the underlying asset is then satisfied and arbitrarily complex derivatives can be hedged and priced. Given actual market prices for forwards or swaps, the derived pricing formulas can also be used to estimate the difference between the current index level and the efficient market price. This facilitates the price discovery process: information about market over- or undervaluation can be extracted from the derivatives markets for forwards or swaps. This, in turn, can also make the primary real estate market more efficient because the price update process is facilitated.

As an example, we value a (hypothetical) European put option on a house price index. We first generate benchmark (Monte Carlo) results and then test our (approximate) closed-form pricing formula. This example highlights the strong effect of autocorrelation in the underlying index on the option price. As is well known in the real estate literature, a high degree of autocorrelation reduces the (annual) volatility. This causes lower option prices, since the time value of the option decreases in this case. Using the proposed model, the effect of over- or undervaluation of the real estate market is also studied. The observed effects are significant and as expected.

Our technique is quite general and can be applied for different purposes. First, it can be used to price existing derivatives in real estate markets, see the examples in Buttimer and Kau (1997), Bertus, Hollans, and Swidler (2008) and Geltner and Fisher (2007). Our technique can also be used for the valuation of so-called hybrid forms of sales, see Kramer (2008). In this case a housing corporation sells a house with a discount to the tenant. In addition, there is a profit and loss sharing mechanism when the house is sold in the future. By determining the present value of the future profits and losses, the corporation can determine if the initial discount (given to the home buyer) is reasonable. This information can also be used when the corporation reports on a pure market-value basis and includes the present value of future profits and losses on the balance sheet.

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A Proofs

A.1 Proof of Eq. (2.25)

Using Eq. (2.22) we find that

$$E_Q\left[\frac{y(T)}{B_N(T)}|F_t\right] = E_Q\left[\frac{y(t)\exp\left(\int_t^T r_N(\tau)d\tau - \sigma_2^2(T-t)/2 - q(T-t) + \sigma_2 Z_2(T-t)\right)}{B_N(t)\exp\left(\int_t^T r_N(\tau)d\tau\right)}|F_t\right]. \quad \text{(A.1)}$$

The right-hand side of this expression can be simplified to

$$\frac{y(t)}{B_N(t)}\exp\left(-q(T-t)\right)\exp\left(-\sigma_2^2(T-t)/2\right)E_Q\left[\exp\left(\sigma_2 Z_2(T-t)\right)|F_t\right]. \quad \text{(A.2)}$$

Since we also have that

$$E_Q\left[\exp\left(\sigma_2 Z_2(T-t)\right)|F_t\right] = \exp\left(\sigma_2^2(T-t)/2\right), \quad \text{(A.3)}$$

we arrive at Eq. (2.25).

A.2 Proof of Eq. (2.26)

The proper starting point for the analysis is Eq. (2.17), since this equation enables us to write $a(T)$ as a basket of previous (accrued) efficient market prices and the (accrued) index value at time *t*. This becomes clear when we set *t* equal to *T* and *m* equal to $T-t$ in Eq. (2.17):

$$a(T) = K\sum_{i=1}^{T-t}(1-K)^{i-1}y^*(T-i+1) + (1-K)^{T-t}a^*(t). \quad \text{(A.4)}$$

Let us first determine whether the accrued prices $y^*(T-i+1)$, at time *T* and conditional on the filtration up to time *t*, are martingales for $1 \le i \le T-t$ if $q=0$. This is indeed the case, since

$$\begin{aligned} E_Q\left[\frac{y^*(T-i+1)}{B_N(T)}|F_t\right] &= E_Q\left[\frac{y(T-i+1)\exp\left(\int_{T-i+1}^T r_N(\tau)d\tau\right)}{B_N(T-i+1)\exp(\int_{T-i+1}^T r_N(\tau)d\tau)}\exp(-q(i-1))|F_t\right] \\ &= E_Q\left[\frac{y(T-i+1)}{B_N(T-i+1)}\exp(-q(i-1))|F_t\right]. \end{aligned} \quad \text{(A.5)}$$

Using Eq. (2.25) we also find that

$$E_Q\left[\frac{y^*(T-i+1)}{B_N(T)}|F_t\right] = \frac{y(t)}{B_N(t)}\exp\left(-q(T-t)\right). \quad \text{(A.6)}$$

The following result then easily follows:

$$E_Q\left[\frac{a(T)}{B_N(T)}|F_t\right] = \frac{\exp\left(-q(T-t)\right)}{B_N(t)}\left(y(t)K\sum_{i=1}^{T-t}(1-K)^{i-1} + a(t)(1-K)^{T-t}\right). \quad \text{(A.7)}$$

Since $K\sum_{i=1}^{T-t}(1-K)^{i-1} = 1-(1-K)^{T-t}$, Eq. (2.26) is obtained.

B OLS regression results

The results of the OLS regression for annual returns of the Dutch house price index in Section 4.2.1 are reported in Table B.1 and B.2 for price update models with up to 3 lags.

Table B.1: OLS regression results for price update models with up to 3 lags.

Number of lags	1	2	3
R Square	0.542	0.607	0.609
Adjusted R Square	0.529	0.585	0.576
Standard Error	0.050	0.047	0.048
Observations (1969-2007)	39	39	39

Table B.2: OLS coefficients and statistics.

Model with 1 lag	*Coefficients*	*Standard Error*	*t Stat*	*P-value*
Intercept	0.016	0.011	1.463	0.152
r_c^a*(t-1)*	0.735	0.111	6.611	0.000
Model with 2 lags	*Coefficients*	*Standard Error*	*t Stat*	*P-value*
Intercept	0.022	0.011	2.091	0.044
r_c^a*(t-1)*	0.987	0.147	6.723	0.000
r_c^a*(t-2)*	-0.352	0.144	-2.441	0.020
Model with 3 lags	*Coefficients*	*Standard Error*	*t Stat*	*P-value*
Intercept	0.024	0.012	2.110	0.042
r_c^a*(t-1)*	0.968	0.153	6.327	0.000
r_c^a*(t-2)*	-0.291	0.190	-1.532	0.135
r_c^a*(t-3)*	-0.072	0.143	-0.504	0.617

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