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Accounting for Stochastic Interest Rates, Stochastic Volatility and a General Dependency Structure in the Valuation of Forward Starting Options

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Abstract

A quantitative analysis on the pricing of forward starting options under stochastic volatility and stochastic interest rates is performed. The main finding is that forward starting options not only depend on future smiles, but also directly on the evolution of the interest rates as well as the dependency structures between the underlying asset, the interest rates and the stochastic volatility: compared to vanilla options, dynamic structures such as forward starting options are much more sensitive to model specifications such as volatility, interest rate and correlation movements. We conclude that it is of crucial importance to take all these factors explicitly into account for a proper valuation and risk management of these securities. The performed analysis is facilitated by deriving closed-form formulas for the valuation of a forward starting options, hereby taking the stochastic volatility, stochastic interest rates as well the dependency structure between all these processes explicitly into account. The valuation framework is derived using a probabilistic approach, enabling a fast and efficient evaluation of the option price by Fourier inverting the forward starting characteristic functions.

Keywords: Forward-starting options, Stochastic Interest Rates, Stochastic Volatility, Correlation Risk, Fourier Inversion.

1 Introduction

Due to the increasing popularity for exotic structures like cliquets and ratchet options, the pricing of forward starting options (which can be seen as natural building blocks for these contracts) recently attracted a lot of attention from both academics and practitioners. Forward starting options belong to the class of path-dependent European-style contracts in the sense that they not only depend on the terminal value of the underlying asset, but also on the asset price at an intermediate point (often dubbed as 'strike determination date'). Typically, a forward starting contract gives the holder a call (or put) option with a strike that is set equal to a fixed proportion of the underlying asset price at this intermediate date. A special form of these options are those on the (future) return of the underlying, which can be seen as a call option on the ratio of the stock price at maturity and the intermediate date. The latter form is often being used by insurance companies to hedge unit-linked guarantees embedded in life insurance products. Additionally, structured products involving forward starting options (like cliquet and ratchet structures) are often tailored for investors seeking for upside potential, while keeping protection against downside movements.

Though forward starting options seem quite simple exotic derivatives, their valuation can be

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demanding, depending on the underlying model. Our pricing takes into account two important factors in the pricing of forward starting options: stochastic volatility and stochastic interest rates, whilst also taking into account the correlation between those processes explicitly. It is hardly necessary to motivate the inclusion of stochastic volatility in a derivative pricing model. Stochastic interest rates are crucial for the pricing of forward starting options because securities with forward starting features often have a long-dated maturities and are therefore much more interest rate sensitive, e.g. see Guo and Hung (2008) or Kijima and Muromachi (2001). The addition of interest rates as a stochastic factor has been the subject of empirical investigations most notably by Bakshi et al. (2000). These authors show that the hedging performance of delta hedging strategies of long-maturity options improves when taking stochastic interest rates into account.

The pricing of forward starting options was first considered by Rubinstein (1991) who provides a closed-form solution for the pricing of forward starting options based on the assumptions of the Black and Scholes (1973) model. Lucić (2003), Hong (2004) and Kruse and Nögel (2005) relax the constant volatility assumption and consider the pricing of forward starting options under Heston (1993) stochastic volatility. The pricing of forward starting options under stochastic volatility with independent stochastic interest rates was considered by Guo and Hung (2008), Ahlip (2008b) and Nunes and Alcaria (2009). The framework employed in this paper distinguishes itself from these models by a closed form pricing formula and an explicit, rather than implicit, incorporation of the correlation between underlying and the term structure of interest rates. The flexibility of stochastic volatility model with (correlated) stochastic rates and the pricing of vanilla call options in such a framework was covered in Ahlip (2008a) and van Haastrecht et al. (2008).

The main goal of this work is performing a quantitative analysis on the pricing of forward starting options under stochastic volatility and stochastic interest rates. In particular we want to investigate the impact of stochastic volatility, stochastic interest rates as well as a realistic dependency structure between all the underlying processes on the valuation of these securities. The analysis is made possible by developing a closed-form solution for the price of a forward starting option in a model in which the instantaneous stochastic volatility is given by the Schöbel and Zhu (1999) model and the interest rates follow Hull and White (1993) dynamics. We explicitly incorporate the correlation between underlying stock and the term structure of interest rates, which is an important empirical characteristic that needs to be taken into account for the pricing and hedging of long-term options, e.g. see Bakshi et al. (2000) or Piterbarg (2005). The setup of the paper is as follows: we discuss the modeling framework and the corresponding forward starting option problem in Section 2 and 3. Using the characteristic function of the log-asset price under the stock price measure (derived in Section 4), we derive in Section 5 the main pricing formulas of the paper. In Section 6 we consider the implementation of these formulas and analyze the valuation and risk management of forward starting option under stochastic volatility, stochastic interest rates and a general correlation structure. Finally, we conclude in Section 7.

2 The modeling framework

Let $(\Omega, \mathcal{F}, \mathcal{Q})$ be a probability space with filtration $\mathcal{F}$ and equivalent (risk-neutral) martingale measure $\mathcal{Q}$, such that the stock price $S(t)$ is governed by the following dynamics:

$$dS(t) = r(t)S(t)dt + \nu(t)S(t)dW_S^{\mathcal{Q}}(t), \qquad S(0) = S_0, \tag{1}$$

$$\nu(t) = \kappa\Big(\psi - \nu(t)\Big)dt + \tau dW_\nu^{\mathcal{Q}}(t), \qquad \nu(0) = \nu_0. \tag{2}$$

Here $\nu(t)$, which follows an Ornstein-Uhlenbeck process, is the (instantaneous) stochastic volatility of the stock $S(t)$. The parameters of the volatility process are the positive constants κ (mean reversion), $\nu(0)$ (short-term mean), ψ (long-term mean) and τ (volatility of the volatility). We assume the interest rates are given by a one-factor Hull and White (1993) model, which dynamics under $\mathcal{Q}$ can be parameterized by

$$r(t) = \beta(t) + x(t), \qquad r(0) = r_0, \tag{3}$$

$$dx(t) = -ax(t)dt + \sigma dW_r^{\mathcal{Q}}(t), \qquad x(0) = 0. \tag{4}$$

Here a (mean reversion) and σ (volatility) are the positive parameters of the model, and where $\beta(t)$ can be used to exactly fit the current term structure of interest rates, e.g. see Pelsser (2000) or Brigo and Mercurio (2006) for further details. The model allows for a general correlation structure between all driving model factors, i.e. the correlation matrix between of the Brownian motions $W_S(t), W_r(t), W_\nu(t)$ is given by

$$\begin{pmatrix} 1 & \rho_{Sr} & \rho_{S\nu} \\ \rho_{Sr} & 1 & \rho_{r\nu} \\ \rho_{S\nu} & \rho_{r\nu} & 1 \end{pmatrix}. \tag{5}$$

Even though the dynamics incorporate stochastic interest rates, stochastic volatility and a general correlation, one can still obtain closed-form formulas for European option prices, which is big advantage in the calibration, see van Haastrecht et al. (2008).

At first sight, one curious property of the model is that the volatility process v(t) affects the sign of the instantaneous correlation between v(t) and ln x(t). Indeed, one can show that

$$\text{Corr}\Big(d\ln x(t), d\nu(t)\Big) = \frac{\rho_{x\nu}\nu(t)\tau}{\sqrt{\nu^2(t)\tau^2}} = \rho_{x\nu}\,\text{sgn}\Big(\nu(t)\Big)dt, \tag{6}$$

This effect is visualized in , where we have plotted a sample path of $x(t)$, $\nu(t)$ and $|v(t)|$.

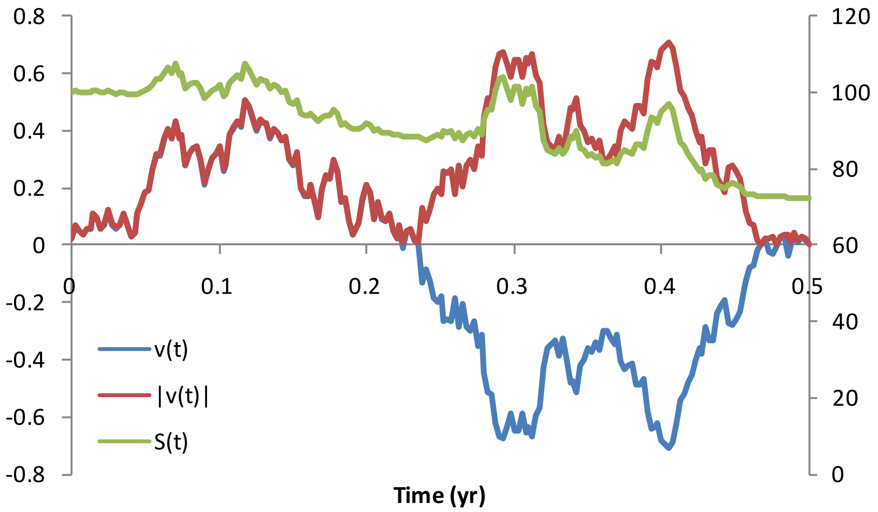

Figure **1**: Sample path of $S(t)$, $\nu(t)$ and $|\nu(t)|$. SZ parameters are $\kappa = \tau = 1$, $nu(0) = \psi =$ 25%, x(0)=100.

Indeed, when $\nu(t)$ is negative and decreasing, the asset price is increasing, contrary to what one would expect from the parameter configuration. The key lies therein that $\nu(t)$ should not be interpreted as the volatility of the underlying asset: it is merely a latent variable which drives the true volatility of the asset, the true volatility being defined as the square root of the variance. Indeed if one applies the Ito-Tanaka theorem to derive the dynamics of $|\nu(t)|$,

$$d|\nu(t)| = \text{sgn}\Big(\nu(t)\Big)d\nu(t) + d\Big(\nu(t)\Big)\langle d\nu(t), d\nu(t)\rangle, \tag{7}$$

leading to an instantaneous correlation of

$$\text{Corr}\Big(d\ln x(t), d\nu(t)\Big) = \frac{\rho_{x\nu}|\nu(t)|}{\sqrt{\nu^2(t)\tau^2}} = \rho_{x\nu}dt, \tag{8}$$

as we would like it to be.

3 Forward starting options

Forward starting options are contracts which not only on their terminal value of the underlying asset, but also on the asset price at an intermediate time between the current time and its expiry time. Kruse and Nögel (2005) consider two types of forward starting options under the Heston (1993) model: European forward starting call options on the underlying asset and on the underlying return. The first structure is prevalent in Employee stock option schemes, while the second category forms a building block for cliquet, ratchet and Unit-Linked insurance options. In both contracts a premium is paid on the purchase date, however the option's life will only start on an intermediate date (in between the purchase and expiry date, dubbed as the strike determination time). Thus, the terminal payoff of these options depends on the underlying asset price at both the maturity and the start date of the underlying option. The next definition formalizes these option types.

Definition 3.1 *The terminal payoff of a European forward starting call option on the underlying asset price S, with a percentage strike of K, strike determination time T_{i-1} and maturity T_i is given by*

$$\Big[S(T_i) - KS(T_{i-1})\Big]^+. \tag{9}$$

The terminal payoff of a European forward starting call option on the return of the underlying asset price S, with an absolute strike of K, determination time T_{i-1} and maturity T_i is given by

$$\left[\frac{S(T_i)}{S(T_{i-1})} - K\right]^+. \tag{10}$$

3.1 The option pricing framework

We can express the price of the forward starting call option price $C_F(T_{i-1}, T_i)$ on the underlying asset, i.e. with terminal payoff (9), in the following expectation under the risk-neutral measure $\mathcal{Q}$

$$C_F(T_{i-1}, T_i) = \mathbb{E}^{\mathcal{Q}}\left[e^{-\int_t^{T_i} r(u)du}\Big(S(T_i) - KS(T_{i-1})\Big)^+\Big|\mathcal{F}_t\right]. \tag{11}$$

Instead of evaluating the expected discounted payoff under the risk-neutral bank account measure, we can also change the underlying probability measure to evaluate this expectation under the stock price probability measure $\mathcal{Q}^S$ (e.g. see Geman et al. (1996)), i.e. with the stock price S as numeraire. Hence, conditional on time t, we can evaluate the price of the forward starting option (11) as

$$\begin{aligned} C_F(T_{i-1}, T_i) &= S(t)\mathbb{E}^{\mathcal{Q}^S}\left[\frac{1}{S(T_i)}\Big(S(T_i) - KS(T_{i-1})\Big)^+\Big|\mathcal{F}_t\right] \\ &= S(t)\mathbb{E}^{\mathcal{Q}^S}\left[\Big(1 - K\frac{S(T_{i-1})}{S(T_i)}\Big)^+\Big|\mathcal{F}_t\right] \\ &= S(t)K\mathbb{E}^{\mathcal{Q}^S}\left[\Big(\frac{1}{K} - \frac{S(T_{i-1})}{S(T_i)}\Big)^+\Big|\mathcal{F}_t\right], \end{aligned} \tag{12}$$

where the last line can be interpreted as put option with strike $\frac{1}{K}$ on the ratio $\frac{S(T_{i-1})}{S(T_i)}$.

In principle it also possible, following the lines of Rubinstein (1991), Guo and Hung (2008) and Ahlip (2008b), to express the forward starting option price as the expected value of a future call option price, i.e.

$$C_F(T_{i-1}, T_i) = S(t)\mathbb{E}^{\mathcal{Q}^S}\left\{\frac{1}{S(T_{i-1})}\mathbb{E}^{\mathcal{Q}^S}\left[\Big(S(T_i) - KS(T_{i-1})\Big)^+\Big|\mathcal{F}_{T_{i-1}}\right]\Big|\mathcal{F}_t\right\}. \tag{13}$$

The above expectation can be evaluated using similar techniques as the evaluation of formula (12), and results in a pricing formula containing two integrals. On the other hand, working out the equivalent expectation (12) results in a pricing formula which only contains one integral. Not only does this make the corresponding implementation more efficient, but even more importantly it has been shown in Andersen and Andreasen (2002) and Lord and Kahl (2007) that the double integral formulation suffers from numerical instabilities whereas the single integral can be implemented in a numerically very stable way. Hence though both approaches are mathematically equivalent, we prefer to work with expectation (12) over the expression in formula (13).

We therefore express the option (12) with log strike $k := \ln \frac{1}{K}$, in terms of the (T-forward) characteristic function $\phi_F(T_{i-1}, T_i, v)$ of the log ratio $\ln \frac{S(T_{i-1})}{S(T_i)}$, i.e.

$$C_F(T_{i-1}, T_i, k) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\Big(e^{(\alpha - iv)k} \psi_F(T_{i-1}, T_i, v) \Big) dv, \tag{14}$$

with

$$\psi_F(T_{i-1}, T_i, v) \quad := \quad \frac{\phi_F\Big(T_{i-1}, T_i, v + (\alpha - 1)i\Big)}{(iv - \alpha)(iv - \alpha + 1)},$$

with $\phi_F(T_{i-1}, T_i, v) := \mathbb{E}^{\mathcal{Q}^S}\Big[\exp\Big(iu \ln \frac{S(T_{i-1})}{S(T_i)}\Big) \big| \mathcal{F}_t\Big]$ and where $\alpha > 1$ has been introduced for Fourier Transform regularization, e.g. see Carr and Madan (1999), Lewis (2001) and Lord and Kahl (2007),

Remark 3.2 *For the pricing of the forward starting option on the underlying asset, it suffices to know the characteristic function $\phi_F(T_{i-1}, T_i, v)$ of $\ln \frac{S(T_{i-1})}{S(T_i)}$ under the stock price probability measure $\mathcal{Q}^S$. For the derivation of this characteristic function, see Section 5.1.*

For the price $C_R(T_{i-1}, T_i)$ of the forward starting call option on the return of the underlying asset, i.e. with terminal payoff (9), the following expectation expectation under the T_i-forward measure holds

$$C_R(T_{i-1}, T_i) = P(t, T_i) \mathbb{E}^{\mathcal{Q}^{T_i}}\left[\Big(\frac{S(T_i)}{S(T_{i-1})} - K \Big)^+ \Big| \mathcal{F}_t \right], \tag{15}$$

i.e. with the corresponding numeraire is now the (pure) discount bond $P(t, T_i)$ maturing at time T_i. One can again write the option (12) with log strike $k := \ln K$, in terms of the (T-forward) characteristic function $\phi(T_{i-1}, T_i, v)$ of the log ratio $\ln \frac{S(T_i)}{S(T_{i-1})}$, i.e.

$$C_R(T_{i-1}, T_i, k) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\Big(e^{-(\alpha + iv)k} \psi(T_{i-1}, T_i, v) \Big) dv, \tag{16}$$

with

$$\psi_R(T_{i-1}, T_i, v) \quad := \quad \frac{\phi_R\Big(T_{i-1}, T_i, v - (\alpha + 1)i\Big)}{(\alpha + iv)(\alpha + 1 + iv)},$$

with $\phi_R(T_{i-1}, T_i, v) := \mathbb{E}^{\mathcal{Q}^{T_i}}\Big[\exp\Big(iu \ln \frac{S(T_i)}{S(T_{i-1})}\Big) \big| \mathcal{F}_t\Big]$ and where $\alpha \in \mathbb{R}^+$ has been introduced for Fourier Transform regularization.

Remark 3.3 *For the pricing of the forward starting option on the return of the underlying asset, it suffices to know the characteristic function $\phi_R(T_{i-1}, T_i, v)$ of $\ln \frac{S(T_i)}{S(T_{i-1})}$ under the T_i-forward probability measure $\mathcal{Q}^{T_i}$. For the derivation of the characteristic function, see Section 5.2.*

The remainder of the paper hence focusses on the derivation of the above characteristic functions.

4 Characteristic function of the log asset price

As a preliminary step towards the general valuation results presented in Section 5, we derive in this section the characteristic function of the log asset price $F(t,T)$ under the stock price measure $\mathcal{Q}^S$ and under the T-forward measure $\mathcal{Q}^T$. To this end, define the T-forward asset price at time t as

$$F(t,T) = \frac{S(t)}{P(t,T)}, \tag{17}$$

where $P(t,T)$ denotes the the price of a (pure) discount bond at time t maturing at time T, hence note that $F(T,T) = S(T)$. Under the risk-neutral measure $\mathcal{Q}$ (where we use the money market bank account as numeraire) the discount bond price follows the process $dP(t,T) = r(t)P(t,T)dt - \sigma B_{\text{HW}}(t,T)P(t,T)dW_r(t)$, where $B_{\text{HW}}(t,T) := \frac{1}{a}\Big(1 - e^{-a(T-t)}\Big)$. Hence, by an application of Ito's lemma, one has the following result for the T-forward stock price process:

$$\begin{aligned} dF(t,T) &= \Big(\rho_{Sr}\nu(t)\sigma B_{\text{HW}}(t,T) + \sigma^2 B_{\text{HW}}^2(t,T)\Big)F(t,T)dt \\ &\quad + \nu(t)F(t,T)dW_S^{\mathcal{Q}}(t) + \sigma B_{\text{HW}}(t,T)F(t,T)dW_r^{\mathcal{Q}}(t). \end{aligned} \tag{18}$$

We will use these dynamics in the following two sections to determine the characteristic function of $\ln F(T)$ under respecitvely the stock price measure and the T-forward measure.

4.1 Characteristic function under the stock price measure $\mathcal{Q}^S$

To determine the dynamics of the forward asset price under the stock price measure, we need to change from the money market account numeraire to the stock price numeraire; we thus need to calculate the corresponding Radon-Nikodým derivative (e.g. see Geman et al. (1996)), which is given by

$$\frac{d\mathcal{Q}^S}{d\mathcal{Q}} = \frac{S(T)B(0)}{S(0)B(T)} = \exp\Big[-\frac{1}{2}\int_0^T \nu^2(u)du + \int_0^T \nu(u)dW_S^{\mathcal{Q}}(u)\Big]. \tag{19}$$

The multi-dimensional version of Girsanov's theorem (e.g. see Brigo and Mercurio (2006)) implies that in our model

$$dW_S^{\mathcal{Q}^S}(t) \mapsto dW_S^{\mathcal{Q}}(t) - \nu(t)dt, \tag{20}$$

$$dW_r^{\mathcal{Q}^S}(t) \mapsto dW_r^{\mathcal{Q}}(t) - \rho_{Sr}\nu(t)dt, \tag{21}$$

$$dW_\nu^{\mathcal{Q}^S}(t) \mapsto dW_\nu^{\mathcal{Q}}(t) - \rho_{S\nu}\nu(t)dt, \tag{22}$$

are $\mathcal{Q}^S$ Brownian motions. Hence under $\mathcal{Q}^S$ we have the following model dynamics for the volatility and interest rate process

$$\begin{aligned} dF(t,T) &= \Big(\nu^2(t) + 2\rho_{Sr}\nu(t)\sigma B_{\text{HW}}(t,T) + \sigma^2 B_{\text{HW}}^2(t,T)\Big)F(t,T)dt \\ &\quad + \nu(t)F(t,T)dW_S^{\mathcal{Q}^S}(t) + \sigma B_{\text{HW}}(t,T)F(t,T)dW_r^{\mathcal{Q}^S}(t) \end{aligned} \tag{23}$$

$$dx(t) = \Big(-ax(t) + \rho_{Sr}\sigma\nu(t)\Big)dt + \sigma dW_r^{\mathcal{Q}^S}(t), \tag{24}$$

$$d\nu(t) = \Big(\kappa\big(\psi - \nu(t)\big) + \rho_{S\nu}\tau\nu(t)\Big)dt + \tau dW_\nu^{\mathcal{Q}^S}(t). \tag{25}$$

We can simplify (23) by switching to logarithmic coordinates and rotating $W_S^{\mathcal{Q}^S}(t)$ and $W_r^{\mathcal{Q}^S}(t)$ to a Brownian motion $W_F^{\mathcal{Q}^S}(t)$. Defining $y(t,T) := \ln\big(F(t,T)\big)$ and an application of Ito's lemma yields the following dynamics:

$$dy(t) = +\frac{1}{2}\nu_F^2(t)dt + \nu_F(t)dW_F^{\mathcal{Q}^S}(t), \tag{26}$$

$$d\nu(t) = \widetilde{\kappa}\Big(\widetilde{\psi} - \nu(t)\Big)dt + \tau dW_\nu^{\mathcal{Q}^S}(t) \tag{27}$$

where $\widetilde{\kappa} := \kappa - \rho_{S\nu}\tau$, $\widetilde{\psi} := \frac{\kappa\psi}{\widetilde{\kappa}}$ and with

$$\nu_F^2(t) := \nu^2(t) + 2\rho_{Sr}\nu(t)\sigma B_{\mathrm{HW}}(t,T) + \sigma^2 B_{\mathrm{HW}}^2(t,T). \tag{28}$$

Note that we now have reduced the system (23) of the three variables $S(t)$, $x(t)$ and $\nu(t)$ under the risk-neutral measure, to the system (26) of two variables $y(t)$ and $\nu(t)$ under the stock price measure. It remains to find the corresponding characteristic function in the reduced system of variables, which is the subject of the now following lemma.

Lemma 4.1 *Under the stock price measure $\mathcal{Q}^S$, the characteristic function of the T-forward asset price $\ln F(T,T) = \ln \frac{S(T)}{P(T,T)} = \ln S(T)$ conditional on the time t filtration $\mathcal{F}_t$ is given by the following closed-form solution:*

$$\begin{aligned} &I\!E^{\mathcal{Q}^S}\Big[\exp\Big(iu\ln F(T,T)\Big)\big|\mathcal{F}_t\Big] \\ &= \exp\Big[A(u,t,T) + B(u,t,T)\ln F(t,T) + C(u,t,T)\nu(t) + \frac{1}{2}D(u,t,T)\nu^2(t)\Big], \end{aligned} \tag{29}$$

where:

$$A(u,t,T) = \frac{1}{2}u(i-u)V(t,T) + \int_t^T \Big[\Big(\widetilde{\kappa}\widetilde{\psi} + \rho_{r\nu}iu\tau\sigma B_{HW}(s,T)\Big)C(s) + \frac{1}{2}\tau^2\Big(C^2(s) + D(s)\Big)\Big]ds, \tag{30}$$

$$B(u,t,T) = iu, \tag{31}$$

$$C(u,t,T) = u(i-u)\frac{\Big(\big(\gamma_3 - \gamma_4 e^{-2\gamma(T-t)}\big) - \big(\gamma_5 e^{-a(T-t)} - \gamma_6 e^{-(2\gamma+a)(T-t)}\big) - \gamma_7 e^{-\gamma(T-t)}\Big)}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}} \tag{32}$$

$$D(u,t,T) = u(i-u)\frac{1-e^{-2\gamma(T-t)}}{\gamma_1 + \gamma_2 e^{-2\gamma(T-t)}}, \tag{33}$$

with:

$$\gamma = \sqrt{(\widetilde{\kappa} - \rho_{S\nu}\tau iu)^2 - \tau^2 u(i-u)}, \qquad \gamma_1 = \gamma + (\widetilde{\kappa} - \rho_{S\nu}\tau iu), \tag{34}$$

$$\gamma_2 = \gamma - (\widetilde{\kappa} - \rho_{S\nu}\tau iu), \qquad \gamma_3 = \frac{\rho_{Sr}\sigma\gamma_1 + \widetilde{\kappa}a\widetilde{\psi} + \rho_{r\nu}\sigma\tau iu}{a\gamma},$$

$$\gamma_4 = \frac{\rho_{Sr}\sigma\gamma_2 - \widetilde{\kappa}a\widetilde{\psi} - \rho_{r\nu}\sigma\tau iu}{a\gamma}, \qquad \gamma_5 = \frac{\rho_{Sr}\sigma\gamma_1 + \rho_{r\nu}\sigma\tau iu}{a(\gamma - a)},$$

$$\gamma_6 = \frac{\rho_{Sr}\sigma\gamma_2 - \rho_{r\nu}\sigma\tau iu}{a(\gamma + a)}, \qquad \gamma_7 = (\gamma_3 - \gamma_4) - (\gamma_5 - \gamma_6),$$

and:

$$V(t,T) := \frac{\sigma^2}{a^2}\Big((T-t) + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}\Big). \quad (35)$$

Proof To determine the characteristic function of $\ln F(T,T)$, we can apply the Feynman-Kac theorem and reduce the problem of finding the characteristic of the forward log-asset price dynamics to solving a partial differential equation. The Feynman-Kac theorem implies that the characteristic function

$$f(t,y,\nu) = \mathbb{E}^{\mathcal{Q}^T}\Big[\exp\big(iuy(T)\big)\big|\mathcal{F}_t\Big], \quad (36)$$

is given by the solution of the following partial differential equation,

$$\begin{aligned} 0 &= f_t + \frac{1}{2}\nu_F^2(t)\Big(f_{yy} + f_y\Big) + \kappa\big(\xi(t) - \nu(t)\big)f_\nu \quad (37)\\ &\quad +\big(\rho_{S\nu}\tau\nu(t) + \rho_{r\nu}\tau\sigma B_{\text{HW}}(t,T)\big)f_{y\nu} + \frac{1}{2}\tau^2 f_{\nu\nu},\\ f(T,y,\nu) &= \exp\Big(iuy(T)\Big), \quad (38) \end{aligned}$$

where the subscripts denote partial derivatives and to ease the notation we dropped the explicit (t,y,ν)-dependence for f. Furthermore we have taken into account that the covariance term $dy(t)d\nu(t)$ is equal to

$$dy(t)d\nu(t) = \big(\nu(t)dW_r^T(t) + \sigma B_{\text{HW}}(t,T)dW_r^T(t)\big)\big(\tau dW_\nu^T(t)\big) = \big(\rho_{S\nu}\tau\nu(t) + \rho_{r\nu}\tau\sigma B_{\text{HW}}(t,T)\big)dt. \quad (39)$$

Some tedious algebra shows that direct substitution of the ansatz

$$f(t,y,\nu) = \exp\Big[A(u,t,T) + B(u,t,T)y(t) + C(u,t,T)\nu(t) + \frac{1}{2}D(u,t,T)\nu^2(t)\Big], \quad (40)$$

solves the partial differential equation (37) and hence proves the theorem. □

4.2 Characteristic function under the T-forward measure $\mathcal{Q}^T$

For the derivation of the characteristic function of $\ln S(T)$ under the T-forward measure we refer the reader to van Haastrecht et al. (2008), in where the following result has been derived.

Lemma 4.2 *Under the T-forward measure $\mathcal{Q}^T$, the characteristic function of the T-forward asset price $\ln F(T,T) = \ln \frac{S(T)}{P(T,T)} = \ln S(T)$ conditional on the time t filtration $\mathcal{F}_t$ is given by the following closed-form solution:*

$$f(t,y,\nu) = \exp\Big[L(u,t,T) + M(u,t,T)y(t) + N(u,t,T)\nu(t) + \frac{1}{2}O(u,t,T)\nu^2(t)\Big], \quad (41)$$

where:

$$
\begin{aligned}
L(u,t,T) &= -\frac{1}{2}u\big(i+u\big)V(t,T) \qquad (42)\\
&\quad + \int_t^T \left[\Big(\kappa\psi + \rho_{r\nu}(iu-1)\tau\sigma B_{HW}(s,T)\Big)N(s) + \frac{1}{2}\tau^2\Big(N^2(s,t,T) + O(s,t,T)\Big)\right]ds,\\
M(u,t,T) &= iu, \qquad (43)\\
N(u,t,T) &= -u\big(i+u\big)\frac{\Big(\big(\delta_3 - \delta_4 e^{-2\delta(T-t)}\big) - \big(\delta_5 e^{-a(T-t)} - \delta_6 e^{-(2\delta+a)(T-t)}\big) - \delta_7 e^{-\delta(T-t)}\Big)}{\delta_1 + \delta_2 e^{-2\delta(T-t)}}, \qquad (44)\\
O(u,t,T) &= -u\big(i+u\big)\frac{1-e^{-2\delta(T-t)}}{\delta_1 + \delta_2 e^{-2\delta(T-t)}}, \qquad (45)
\end{aligned}
$$

with:

$$
\begin{aligned}
\delta &= \sqrt{(\kappa - \rho_{S\nu}\tau iu)^2 + \tau^2 u\big(i+u\big)}, & \delta_1 &= \delta + (\kappa - \rho_{S\nu}\tau iu), \qquad (46)\\
\delta_2 &= \delta - (\kappa - \rho_{S\nu}\tau iu), & \delta_3 &= \frac{\rho_{Sr}\sigma\delta_1 + \kappa a\psi + \rho_{r\nu}\sigma\tau(iu-1)}{a\delta},\\
\delta_4 &= \frac{\rho_{Sr}\sigma\delta_2 - \kappa a\psi - \rho_{r\nu}\sigma\tau(iu-1)}{a\delta}, & \delta_5 &= \frac{\rho_{Sr}\sigma\delta_1 + \rho_{r\nu}\sigma\tau(iu-1)}{a(\delta-a)},\\
\delta_6 &= \frac{\rho_{Sr}\sigma\delta_2 - \rho_{r\nu}\sigma\tau(iu-1)}{a(\delta+a)}, & \delta_7 &= (\delta_3 - \delta_4) - (\delta_5 - \delta_6),
\end{aligned}
$$

and with $V(t,T)$ as in (35).

5 Valuation of forward starting call options

Having done the preliminary work in the previous sections, we are now well prepared to present the general valuation results for the forward starting characteristic functions. The results are provided in the theorems of the following two sections.

5.1 Forward starting characteristic function under the stock price measure

With the help of lemma 4.1, we are now ready to derive the characteristic function of $\ln\frac{S(T_{i-1})}{S(T_i)}$. This characteristic function, provided by the following theorem, can then directly be plugged into the Fourier inversion formula (14) to price the forward starting call option (12) in closed-form.

Theorem 5.1 *Under the stock price measure $\mathcal{Q}^S$, the characteristic function $\phi_F(T_{i-1},T_i,u)$ of $\ln\frac{S(T_{i-1})}{S(T_i)}$ is given by the following closed-form solution:*

$$
\begin{aligned}
\phi_F(T_{i-1},T_i,u) = \;&\cdot\; \exp\left[a_0 + a_1\mu_x + \frac{1}{2}a_1^2\sigma_x^2\Big(1-\rho_{x\nu}^2(t,T_{i-1})\Big)\right]\\
&\cdot\; \frac{\exp\left[a_2\mu_\nu + a_3\mu_\nu^2 + \frac{\left(a_1\sigma_x\rho_{x\nu}^2(t,T_{i-1}) + a_2\sigma_\nu + 2a_3\mu_\nu\sigma_\nu\right)^2}{2(1-2a_3\sigma_\nu^2)}\right]}{\sqrt{1-2a_3\sigma_\nu^2}}, \qquad (47)
\end{aligned}
$$

where:

$$a_0 \;:=\; iu \ln A_{HW}(T_{i-1}, T_i) + A(-u, T_{i-1}, T_i), \quad a_1 := -iuB_{HW}(T_{i-1}, T_i), \tag{48}$$

$$a_2 \;:=\; C(-u, T_{i-1}, T_i) \quad a_3 := \frac{1}{2} D(-u, T_{i-1}, T_i). \tag{49}$$

Proof Recalling the definition (17) for the forward asset price and using lemma 4.1, one can write the following for the characteristic function $\phi_F(T_{i-1}, T_i, u)$ of $\ln \frac{S(T_{i-1})}{S(T_i)}$:

$$\begin{aligned}\phi_F(T_{i-1}, T_i, u) &= \mathbb{E}^{\mathcal{Q}^S}\left[e^{iu \ln \frac{S(T_{i-1})}{S(T_i)}} \Big| \mathcal{F}_t\right] = \mathbb{E}^{\mathcal{Q}^S}\left[e^{iu \ln S(T_{i-1}) - iu \ln S(T_i)} \Big| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathcal{Q}^S}\left[e^{iu \ln P(T_{i-1}, T_i) + iu \ln F(T_{i-1}) - iu \ln F(T_i)} \Big| \mathcal{F}_t\right].\end{aligned}$$

Using the tower law of conditional expectations, i.e. conditioning on the time T_{i-1} filtration $\mathcal{F}_{T_{i-1}}$, we have that

$$\phi_F(T_{i-1}, T_i, u) = \mathbb{E}^{\mathcal{Q}^S}\left[e^{iu \ln P(T_{i-1}, T_i) + iu \ln F(T_{i-1}, T_i)} \mathbb{E}^{\mathcal{Q}^S}\left\{e^{i(-u) \ln F(T_i)} \Big| \mathcal{F}_{T_{i-1}}\right\} \Big| \mathcal{F}_t\right],$$

and note that the inner expectation is the characteristic function of $\ln F(T_{i-1}, T_i)$ evaluated in the point $-u$, i.e. given by lemma 4.1. Hence substituting for this the characteristic function in the above expression, we obtain:

$$\phi_F(T_{i-1}, T_i, u) = \mathbb{E}^{\mathcal{Q}^S}\left[e^{iu \ln P(T_{i-1}, T_i) + A(-u, T_{i-1}, T_i) + C(-uT_{i-1}, T_i)\nu(T_{i-1}) + D(-u, T_{i-1}, T_i)\nu^2(T_{i-1})} \Big| \mathcal{F}_t\right]. \tag{50}$$

In the Gaussian rate model, one has the following expression for the time-T_{i-1} price of a zero-coupon bond $P(T_{i-1}, T_i)$ maturing at time T_i (e.g. see Brigo and Mercurio (2006)):

$$P(T_{i-1}, T_i) = A_{\mathrm{HW}}(T_{i-1}, T_i) e^{-B_{\mathrm{HW}}(T_{i-1}, T_i) x(T_{i-1})}, \tag{51}$$

where

$$A_{\mathrm{HW}}(T_{i-1}, T_i) \;=\; \frac{P^M(t, T_i)}{P^M(t, T_{i-1})} \exp\left[\frac{1}{2}\Big(V(T_{i-1}, T_i) - V(t, T_i) + V(t, T_{i-1})\Big)\right] \tag{52}$$

$$B_{\mathrm{HW}}(T_{i-1}, T_i) \;=\; \frac{1 - e^{-a(T_i - T_{i-1})}}{a} \tag{53}$$

$$V(T_{i-1}, T_i) \;=\; \frac{\sigma^2}{a^2}\left((T_i - T_{i-1}) + \frac{2}{a} e^{-a(T_i - T_{i-1})} - \frac{1}{2a} e^{-2a(T_i - T_{i-1})} - \frac{3}{2a}\right). \tag{54}$$

Hence we can express the characteristic function $\phi_F(T_{i-1}, T_i, u)$ completely in terms of the Gaussian factors $x(T_{i-1})$ and $\nu^2(T_{i-1})$, i.e.

$$\begin{aligned}\phi_F(T_{i-1}, T_i, u) &= \mathbb{E}^{\mathcal{Q}^S}\Big[\exp\Big\{iu \ln A_{\mathrm{HW}}(T_{i-1}, T_i) - B_{\mathrm{HW}}(T_{i-1}, T_i) x(T_{i-1}) + A(-u, T_{i-1}, T_i) \\ &\qquad + C(-u, T_{i-1}, T_i)\nu(T_{i-1}) + D(-u, T_{i-1}, T_i)\nu^2(T_{i-1})\Big\} \Big| \mathcal{F}_t\Big] \\ &=: \mathbb{E}^{\mathcal{Q}^S}\left[\exp\Big\{a_0 + a_1 x(T_{i-1}) + a_2 \nu(T_{i-1}) + a_3 \nu^2(T_{i-1})\Big\} \Big| \mathcal{F}_t\right],\end{aligned} \tag{55}$$

where the last line defines the constants $a_0, \ldots, a_3$. Because the above expression is a Gaussian quadratic form of the variables $x(T_{i-1})$ and $\nu(T_{i-1})$, one can evaluate this expectation completely in terms of the means μ_x, μ_ν, variances σ_x^2, σ_ν^2 and correlation $\rho_{xv}(t, T_{i-1})$ of these two state variables, e.g. see Feuerverger and Wong (2000) or Glasserman (2003). A straightforward evaluation (e.g. by completing the square or by integration the exponential affine function against the bivariate normal distribution) of this Gaussian quadratic expectation results in the characteristic function $\phi_F(T_{i-1}, T_i, u)$ of (47) and hence proves the theorem. $\square$

5.2 Forward starting characteristic function under the T-forward measure

Using lemma 4.2 and similar arguments as in the previous section, we can now also derive the characteristic function of $\ln \frac{S(T_i)}{S(T_{i-1})}$ under the T_i-forward probability measure. This characteristic function can directly be used in the Fourier inversion formula (16) to price the forward starting call option (15) on the return of the underlying asset in closed-form.

Theorem 5.2 *Under the stock price measure $\mathcal{Q}^{T_i}$, the characteristic function $\phi_R(T_{i-1}, T_i, u)$ of $\ln \frac{S(T_{i-1})}{S(T_i)}$ is given by the following closed-form solution:*

$$\phi_R(T_{i-1}, T_i, u) = \cdot \exp\left[b_0 + b_1\mu_x + \frac{1}{2}b_1^2\sigma_x^2\Big(1 - \rho_{x\nu}^2(t, T_{i-1})\Big)\right] \cdot \frac{\exp\left[b_2\mu_\nu + b_3\mu_\nu^2 + \frac{\left(b_1\sigma_x\rho_{x\nu}^2(t,T_{i-1}) + b_2\sigma_\nu + 2b_3\mu_\nu\sigma_\nu\right)^2}{2(1-2b_3\sigma_\nu^2)}\right]}{\sqrt{1 - 2b_3\sigma_\nu^2}}, \tag{56}$$

where:

$$b_0 := -iu \ln A_{HW}(T_{i-1}, T_i) + L(u, T_{i-1}, T_i), \quad b_1 := iuB_{HW}(T_{i-1}, T_i), \tag{57}$$

$$b_2 := N(u, T_{i-1}, T_i) \quad b_3 := \frac{1}{2}O(u, T_{i-1}, T_i). \tag{58}$$

Proof Using analogue arguments as in the proof of theorem 5.1, one can obtain the following expression for the characteristic function ϕ_R of $\ln \frac{S(T_i)}{S(T_{i-1})}$ under the T_i-forward probability measure. Using the tower law of conditional expectations, i.e. conditioning on the time T_{i-1} filtration $\mathcal{F}_{T_{i-1}}$, we have that

$$\phi_R(T_{i-1}, T_i, u) = \mathbb{E}^{\mathcal{Q}^{T_i}}\left[e^{-iu\ln P(T_{i-1},T_i) - iu\ln F(T_{i-1},T_i)}\mathbb{E}^{\mathcal{Q}^{T_i}}\left\{e^{iu\ln F(T_i)}\Big|\mathcal{F}_{T_{i-1}}\right\}\Big|\mathcal{F}_t\right].$$

As the inner expectation is just the characteristic function of $\ln F(T_{i-1}, T_i)$ evaluated in the point u, we can substitute the closed-form expression of lemma 4.1 for this characteristic function in the above expression, i.e.

$$\begin{aligned}\phi_R(T_{i-1}, T_i, u) &= \mathbb{E}^{\mathcal{Q}^{T_i}}\left[e^{-iu\ln P(T_{i-1},T_i) + L(u,T_{i-1},T_i) + M(uT_{i-1},T_i)\nu(T_{i-1}) + O(u,T_{i-1},T_i)\nu^2(T_{i-1})}\Big|\mathcal{F}_t\right] \\ &=: \mathbb{E}^{\mathcal{Q}^{T_i}}\left[\exp\left\{b_0 + b_1x(T_{i-1}) + b_2\nu(T_{i-1}) + b_3\nu^2(T_{i-1})\right\}\Big|\mathcal{F}_t\right]\end{aligned} \tag{59}$$

Note that the only difference with the Gaussian quadratic form (55) are the dynamics of the processes $x(T_{i-1})$ and $\nu(T_{i-1})$, which now instead need to be evaluated under the T_i-forward measure. Hence it can be evaluated in an analogous way as in the proof of theorem 5.1 resulting in the closed-form expression (56) for the characteristic function $\phi_R(T_{i-1}, T_i, u)$ and hence proving the theorem. □

6 Numerical results

To investigate the impact of stochastic volatility and stochastic interest rates on the prices of forward starting options, we will consider the following numerical test cases. As the prices of forward starting options can be calculated in closed-form, a Monte Carlo benchmark against the pricing formulas (14)-(16) forms a standard test case for their implementation. We then explicitly investigate the impact and parameter sensitivities of stochastic interest rates and stochastic volatility on the prices of forward starting options. Finally, we tackle the issue of model risk and compare our framework with the Black and Scholes (1973) and Heston (1993) model, respectively considered in Rubinstein (1991) and Guo and Hung (2008) for the valuation of forward starters.

6.1 Implementation of the option pricing formulas

In this section we consider the practical implementation of the pricing formulas (14) and (16); both the implementation of the inverse Fourier transform, as well as the calculation of the characteristic function underlying this transform, deserve some attention. For the calculation of the inverse Fourier transform we refer the reader to Lord and Kahl (2007), Kilin (2006) and van Haastrecht et al. (2008), where this topic is covered in great detail. Instead we focus on the application specific calculation of the characteristic functions (47) and (56). The calculation of the characteristic functions (47) and (56) is trivial up to the calculation of the constants $A(u, t, T)$ of (30) and $L(u, t, T)$ of (42), which involves the calculation of a numerical integral. Hence we focus on the calculation of $A(u, t, T)$, but a completely analogous reasoning holds for the calculation of $L(u, t, T)$.

It is possible to write a closed-form expression for the remaining integral in (30). As the ordinary differential equation for $D(u, t, T)$ is exactly the same as in the Heston (1993) or Schöbel and Zhu (1999) model, it will involve a complex logarithm and should therefore be evaluated as outlined in Lord and Kahl (2008) in order to avoid any discontinuities. The main problem however lies in the integrals over $C(u, t, T)$ and $C^2(u, t, T)$, which will involve the Gaussian hypergeometric ${}_2F_1(a, b, c; z)$. The most efficient way to evaluate this hypergeometric function (according to Press and Flannery (1992)) is to integrate the defining differential equation. Since all of the terms involved in $D(u, t, T)$ are also required in $C(u, t, T)$, numerical integration of the second part of (30) seems to be the most efficient method for evaluating $A(u, t, T)$. Note that we hereby conveniently avoid any issues regarding complex discontinuities altogether. It remains to have a closer look at the implementation of the numerical integral of $A(u, t, T)$ and $L(u, t, T)$.

We compute the prices for short and long term forward starting option for a range of strikes and where we use fixed-point Gaussian-Legendre quadrature to compute the numerical integral in (30) and (42). Hereby we vary the number of quadrature points to determine how many points are needed in the test cases to obtain a certain accuracy. The numerical results together with the corresponding Monte Carlo estimates (using 10^6 sample paths) can be found in Table **1** and **2** below.

strike level	CF(4)	CF(8)	CF(16)	CF(1024)	MC (±95% interval)
50%	65.31	65.26	65.26	65.26	65.30 (±0.31)
75%	53.94	53.85	53.85	53.85	53.89 (±0.29)
100%	44.97	44.85	44.85	44.85	44.90 (±0.27)
125%	37.80	37.65	37.65	37.65	37.71 (±0.25)
150%	32.00	31.82	31.82	31.82	31.89 (±0.24)

Table **1**: Closed-form solution prices (CF(N)) using N quadrature points for $A(u, T_1, T_2)$ in (30) and Monte Carlo prices (MC) of the forward starting call option (12) for $t = 0$, $T_1 = 5$, $T_2 = 15$ and $P(t, T_1) = P(t, T_2) = 1.0$ and model parameters $\kappa = 1.00$, $\nu(0) = \psi = 0.20$, $a = 0.02$, $\sigma = 0.01$, $\tau = 0.50$, $\rho_{S\nu} = -0.70$, $\rho_{Sr} = 0.30$ and $\rho_{r\nu} = 0.15$.

strike level	CF(1)	CF(2)	CF(4)	CF(1024)	MC (±95% interval)
50%	50.23	50.24	50.24	50.24	50.27 (±0.05)
75%	26.77	26.79	26.79	26.79	26.80(±0.04)
100%	8.56	8.39	8.39	8.39	8.39 (±0.03)
125%	2.07	2.04	2.04	2.04	2.05 (±0.02)
150%	0.69	0.69	0.69	0.69	0.69 (±0.01)

Table **2**: Closed-form solution prices (CF(N)) using N quadrature points for $L(u, T_1, T_2)$ in (42) and Monte Carlo prices (MC) of the forward starting return call option (15) for $t = 0$, $T_1 = 1$, $T_2 = 2$ and $P(t, T_1) = P(t, T_2) = 1.0$ and model parameters $\kappa = 0.30$, $\nu(0) = \psi = 0.15$, $\tau = 0.20$, $a = 0.05$, $\sigma = 0.01$, $\rho_{S\nu} = -0.40$, $\rho_{Sr} = 0.20$ and $\rho_{r\nu} = 0.10$.

From the tables we see that the characteristic functions (47) and (56) underlying the option price formulas can be calculated very accurately, using only a small number of quadrature points; the prices of short term options (Table **1**) and long term options (Table **2**) can be calculated within a base points accuracy by using respectively just two and eight quadrature points for the calculation of the integral in $A(u, t, T)$ and $L(u, t, T)$. Note hereby that the corresponding Monte Carlo confidence interval is also larger in test case of Table **2**, due to the longer dated maturity. Combining the efficient calculation of characteristic functions (47) and (56) with the efficient Fourier inversion techniques, we can all in all conclude the pricing of forward starting options can be done fast, highly accurate and in closed-form using the latter methods.

6.2 Impact of stochastic interest rates and stochastic volatility

In this section we will cover the impact of stochastic volatility and (correlated) stochastic interest rates on the prices of forward starting options. That is, we investigate qualitative aspects of our extended framework in comparison to deterministic (or independent) interest rates and volatility assumptions. Rubinstein (1991) considered the pricing of a vanilla forward starting option in the Black and Scholes (1973) framework; as both interest rates and volatilities are deterministic in this model, the prices of a forward starting options are (up to deterministic discounting effects) equal for all forward starting dates. The constant volatility assumption has been relaxed by Lucić (2003), Hong (2004) and Kruse and Nögel (2005), who consider the pricing of forward starting options under Heston (1993) stochastic volatility. The impact of stochastic volatility can seen from the top graphs of Figure 2.

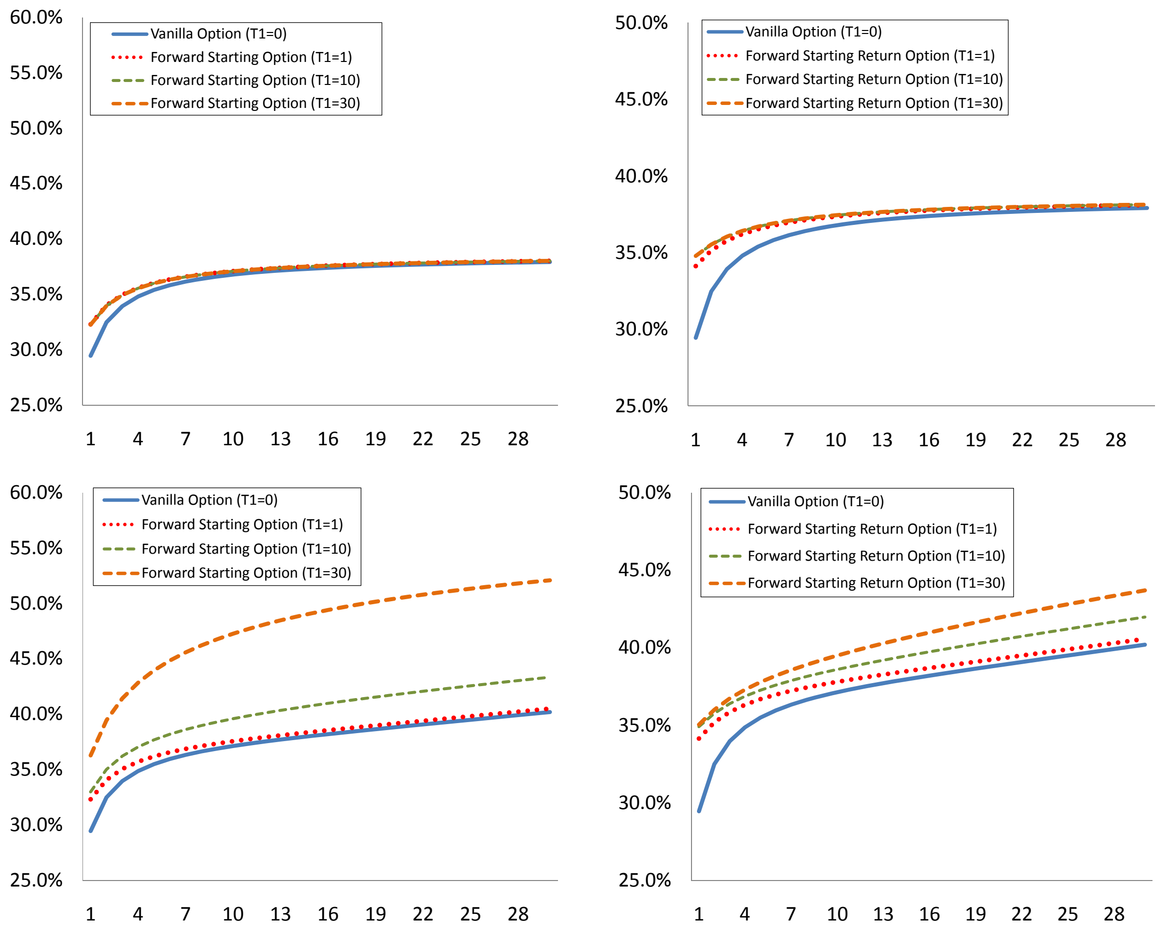

Figure **2**: The figures plot, for different option maturities, the impact of stochastic interest rates on the forward implied volatility structure of an underlying call (left pictures) and return call option (right pictures). Parameters are $\kappa = 1.0$, $\nu(t) = \psi = 0.20$, $\tau = 0.5$, $\rho_{S\nu} = -0.70$, $\rho_{Sr} = \rho_{r\nu} = 0$ and $P(t,s) = \exp\big(-0.04(s-t)\big)$ for all $s > t$. The top figures plot the volatility structure for deterministic interest rates, whilst the bottom figures plot the volatility structure for stochastic interest rates case with parameters $a = 0.02$ and $\sigma = 0.01$.

Compared to constant volatility, the addition of stochastic volatility increases the future uncertainty about the underlying option price which is hence reflected in higher implied volatilities for longer forward starting dates. Intuitively this effect is rather appealing as this coincides with market prices for forward starting structures where the writer of such an option wants to be compensated for the extra (future) volatility risk he is exposed to. Furthermore it is interesting to note from the figures that these effects are more apparent where the underlying option has a short maturity, which effect may be explained by the mean reverting property of stochastic volatility that is less severe for a short term option hence increasing the future volatility risk. Finally note from the top two graphs of Figure 2 that with deterministic rates the long-term uncertainty approaches a limit (or a stationary state) as the forward starting date or the underlying option maturity increase. For example the implied volatilities for forward starting options with a forward date of ten and thirty years are exactly equal, which is counterintuitive as the term structure of implied volatilities remains increasing for long-dated options and in general does not flatten out nor approaches a limit, for instance see the implied volatility quotes in long-maturity equity markets (readily available from MarkIT or Bloomberg) or the over-the-counter FX quotes in Piterbarg (2005) or Andreasen (Global Derivatives Conference 2006).

The inconsistency in the way the market and the latter models look at long-dated implied volatility structures, more likely suggests that these models lack an extra factor in their pricing frameworks; this conjecture is supported by Guo and Hung (2008) and Kijima and Muromachi (2001), who claim that stochastic interest rates are crucial for the pricing of forward starting options as these securities are often much more interest rate sensitive due to their long term nature. In fact if we look at the bottom graphs of Figure **2**, where we add stochastic interest rates to the framework with stochastic volatility, we see that the implied volatilities increase for longer forward starting and maturity dates. These model effects also correspond with a general feature of the interest rate market: the market's view on the uncertainty of long-maturity bonds is often much higher than that of shorter bond, reflecting the increasing impact of stochastic interest rates for long dated structures. In this sense stochastic interest rates do seem to incorporate the larger uncertainty the writers of the forward starting options are exposed to.

The addition of stochastic interest rates as independent factor for the pricing of forward starting options has been investigated in Guo and Hung (2008) and Nunes and Alcaria (2009). Though one step in the right direction, the independency assumption is certainly not supported by empirical analysis (e.g. see Baur (2009)) nor do the exotic option markets (such as hybrid equity-interest rate options) price these derivatives in this way, e.g. see Andreasen (Global Derivatives Conference 2007) or Antonov et al. (2008); from Figure **3** and **4** of Appendix C, we see that correlated stochastic interest rates can have a big impact on the prices of forward starting options. From Figure **3** we can see that for a positive rate-asset correlation coefficient the prices of forward starting options increase and vice versa for a negative correlation coefficient. In particular note from Figure **3** that, though the correlation coefficient between the interest rates and the stock also affects the implied volatility structure of the current time vanilla options, the effects on the prices of forward starting options are much more pronounced. Forward starting options are thus not only more interest rate and volatility sensitive, but are also much more exposed to correlation risks. This is not surprising as a joint movement in both the interest rates as the asset price not only affects the future discounting, but more importantly also the (joint) asset price distribution. All in all, we can conclude that because forward starting options are very sensitive to future interest rate movements, volatility smiles as well as their dependency structure with the underlying asset, it is very important to take all these stochastic quantities into account for a proper pricing and risk management of these derivatives.

7 Conclusion

We performed a quantitative analysis on the valuation of forward starting options, where we explicitly accounted for stochastic volatility, stochastic interest rates as well as a general dependency structure between all underlying processes. The analysis was made possible by the development of closed-form formulas involving the pricing of the two main forward starting structures, currently present in the literature and the financial markets. Using a probabilistic approach, we derived closed-form expressions for the characteristic functions of the assets underlying the forward starting options. We then demonstrated how forward starting options can be priced efficiently and in closed-form by Fourier inverting these forward starting characteristic functions. An additional advantage of this technique is that our modeling framework can include jumps as a trivial extension, since we already work in the Fourier option pricing domain.

Our results are of great practical importance as the derivative markets for long-dated dynamic securities such as forward starting options have grown very rapidly over the last decade; compared to vanilla options, these structures directly depend on future volatility smiles, the term structure of interest rates as well as their dependency structure with the underlying asset. Moreover, as these contracts often incorporate long-dated maturities, we found that it is of crucial importance to take stochastic interest rates, volatility and a general correlation structure into account for a proper valuation and hedging of these securities: not doing so leads to serious mispricings, not to mention the potential hedge errors. Compared to other models, the analysis performed in our framework stands out by modeling both the stochastic volatility and interest rates, as well as taking a general correlation structure between all underlying drivers explicitly into account.

Besides investigating the behaviour of these dynamic derivatives, our formulas can also be used to directly price or hedge financial contracts. For instance unit-linked guarantees embedded in life insurance products, being sold in large amounts by insurance companies, can be priced in closed-form relying on our formulas. The same applies for cliquet options, which are heavily traded in over-the-counter markets, and CEO/employee stock option plans. Furthermore, there is a big intercourse between forward starting options considered here and over-the-counter exotic structures such as ratchet options and pension contracts, as these form the natural building blocks and hedge instruments for such contracts. Finally, as all the above-mentioned products explicitly depend on the future volatility smiles, the term structure of interest rates as well as their dependency structure with the underlying asset, we judge that a proper valuation framework should account for all these characteristics.

A Calculation of the moments for the rate processes under different measures

Stock price measure

For the computation of the characteristic functions from theorems 5.1 one needs the first two moments of $x(T_{i-1})$ and $\nu(T_{i-1})$ (conditional on the time t filtration $\mathcal{F}_t$) under the stock price measure $\mathcal{Q}^S$. For completeness, we will therefore explicitly provide the analytical expressions for these moments: integrating the dynamics (23) and using Fubini's theorem, results (after some algebra) in the following explicit solutions:

$$\nu(T_{i-1}) = \widetilde{\psi} + \Big(\nu(t) - \widetilde{\psi}\Big)e^{-\widetilde{\kappa}(T_{i-1}-t)} + \tau \int_t^{T_{i-1}} e^{-\widetilde{\kappa}(T_{i-1}-u)} dW_\nu^{\mathcal{Q}^S}(u),$$

$$x(T_{i-1}) = \rho_{Sr}\sigma\Big(\frac{\widetilde{\psi}}{a}\big[1 - e^{-a(T_{i-1}-t)}\big] + \frac{\nu(t) - \widetilde{\psi}}{a - \widetilde{\kappa}}\big[e^{-\widetilde{\kappa}(T_{i-1}-t)} - e^{-a(T_{i-1}-t)}\big]\Big)$$
$$+ \frac{\rho_{Sr}\sigma\tau}{(a - \widetilde{\kappa})} \int_t^{T_{i-1}} \big[e^{-\widetilde{\kappa}(T_{i-1}-u)} - e^{-a(T_{i-1}-u)}\big] dW_\nu^{\mathcal{Q}^S}(u) + \sigma \int_t^{T_{i-1}} e^{-a(T_{i-1}-u)} dW_r^{\mathcal{Q}^S}(u).$$

Using Ito's isometry, one therefore has that the pair $\Big(\nu(T), x(T)\Big)$, under the stock price measure and conditional on $\mathcal{F}_t$, follow a bivariate normal distribution with means μ_ν,μ_x, variances σ_ν^2, σ_x^2 and correlation $\rho_{x\nu}(t, T_{i-1})$ respectively given by

$$\mu_\nu = \widetilde{\psi} + \Big(\nu(t) - \widetilde{\psi}\Big)e^{-\widetilde{\kappa}(T_{i-1}-t)} \tag{60}$$

$$\sigma_\nu^2 = \frac{\tau^2}{2\widetilde{\kappa}}\Big(1 - e^{-2\widetilde{\kappa}(T_{i-1}-t)}\Big), \tag{61}$$

$$\mu_x = \rho_{Sr}\sigma\Big(\frac{\widetilde{\psi}}{a}\big[1 - e^{-a(T_{i-1}-t)}\big] + \frac{\nu(0) - \widetilde{\psi}}{(a - \widetilde{\kappa})}\big[e^{-\widetilde{\kappa}(T_{i-1}-t)} - e^{-a(T_{i-1}-t)}\big]\Big), \tag{62}$$

$$\sigma_x^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2, \tag{63}$$

$$\rho_{x\nu}(t, T_{i-1}) = \frac{\rho_{r\nu}\sigma\tau}{\sigma_x\sigma_\nu(a + \widetilde{\kappa})}\Bigg[1 - e^{-(a+\widetilde{\kappa})(T_{i-1}-t)}\Bigg], \tag{64}$$

where:

$$\sigma_1 = \sigma\sqrt{\frac{1 - e^{-2a(T_{i-1}-t)}}{2a}},$$

$$\sigma_2 = \frac{\rho_{Sr}\sigma\tau}{a - \widetilde{\kappa}}\sqrt{\frac{1}{2\widetilde{\kappa}} + \frac{1}{2a} - \frac{2}{(\widetilde{\kappa} + a)} - \frac{e^{-2\widetilde{\kappa}(T_{i-1}-t)}}{2\widetilde{\kappa}} - \frac{e^{-2a(T_{i-1}-t)}}{2a} + \frac{2e^{-(\widetilde{\kappa}+a)(T_{i-1}-t)}}{(\widetilde{\kappa} + a)}},$$

$$\rho_{12} = \rho_{r\nu}\frac{\sigma^2\rho_{Sr}\tau}{\sigma_1\sigma_2(a - \widetilde{\kappa})}\Bigg[\frac{1 - e^{-(a+\widetilde{\kappa})(T_{i-1}-t)}}{(a + \widetilde{\kappa})} - \frac{1 - e^{-2a(T_{i-1}-t)}}{2a}\Bigg].$$

T-forward measure

For computation of the characteristic functions from theorem 5.2, the first moments of $x(T_{i-1})$ and $\nu(T_{i-1})$ under the T_i-forward measure $\mathcal{Q}^{T_i}$ are needed; one can obtain the following explicit solutions for $x(T_{i-1})$ and $\nu(T_{i-1})$ by direct integration of the corresponding T_i-forward dynamics, i.e.

$$\begin{aligned} x(T_{i-1}) &= x(t)e^{-a(T_{i-1}-t)} - M^{T_i}(t,T_{i-1}) + \sigma \int_t^{T_{i-1}} e^{-a(T_{i-1}-u)} dW_r^{T_i}(u), \\ \nu(T_{i-1}) &= \nu(t)e^{-\kappa(T_{i-1}-t)} + \int_t^{T_{i-1}} \kappa\xi(u)e^{-\kappa(T_{i-1}-u)}du + \int_t^{T_{i-1}} \tau e^{-\kappa(T_{i-1}-u)} dW_\nu^T(u), \end{aligned}$$

where

$$\xi(u) := \psi - \frac{\rho_{r\nu}\sigma\tau}{a\kappa}\left(1 - e^{a(T_i-u)}\right), \tag{65}$$

$$M^{T_i}(t,T_{i-1}) := \frac{\sigma^2}{a^2}\left(1 - e^{-a(T_{i-1}-t)}\right) - \frac{\sigma^2}{2a^2}\left(e^{-a(T_i-T_{i-1})} - e^{-a(T_i+T_{i-1}-2t)}\right). \tag{66}$$

Hence, from Ito's isometry, we immediately have that the pair $\left(\nu(T_{i-1}), x(T_{i-1})\right)$, under the T_i-forward measure and conditional on $\mathcal{F}_t$, follows a bivariate normal distribution, respectively with means μ_ν, μ_x, variances $\sigma_n u^2$, σ_x^2 and correlation $\rho_{x\nu}(t,T_{i-1})$ given by

$$\begin{aligned} \mu_\nu &= \nu(t)e^{-\kappa(T_{i-1}-t)} + \left(\psi - \frac{\rho_{r\nu}\sigma\tau}{a\kappa}\right)\left(1 - e^{-\kappa(T_{i-1}-t)}\right) \\ &\quad - \frac{\rho_{r\nu}\sigma\tau}{a(\kappa+a)}\left(e^{-a(T_i-t)-\kappa(T_{i-1}-t)} - e^{-a(T_i-T_{i-1})}\right), \end{aligned} \tag{67}$$

$$\sigma_\nu^2 = \frac{\tau^2}{2\kappa}\left(1 - e^{-2\kappa(T_{i-1}-t)}\right), \tag{68}$$

$$\mu_x = x(t)e^{-a(T_{i-1}-t)} - M^{T_i}(t,T_{i-1}), \tag{69}$$

$$\sigma_x^2 = \frac{\sigma^2}{2a}\left(1 - e^{-2a(T_{i-1}-t)}\right), \tag{70}$$

$$\rho_{x\nu}(t,T_{i-1}) = \frac{\rho_{r\nu}\sigma\tau}{\sigma_x\sigma_\nu(a+\kappa)}\left[1 - e^{-(a+\kappa)(T_{i-1}-t)}\right]. \tag{71}$$

B Impact of the rate-asset correlation coefficient on the forward starting options

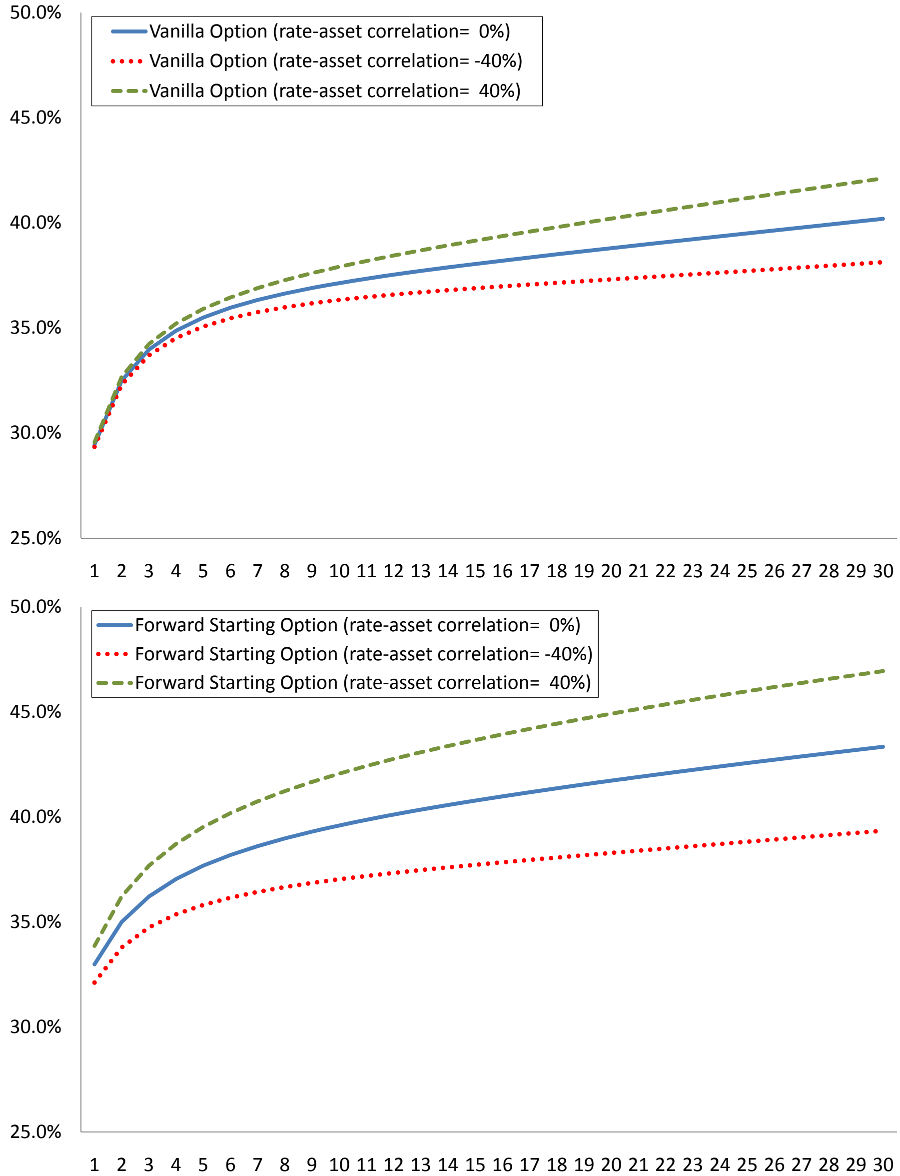

Figure **3**: Impact of the rate-asset correlation ρ_{Sr} on the (forward) implied volatility structure. for different underlying call option maturities. Parameters are $\kappa = 1.0$, $\nu(t) = \psi = 0.20$, $\tau = 0.5$, $a = 0.02$, $\sigma = 0.01$, $\rho_{S\nu} = -0.70$, $\rho_{r\nu} = 0$ and $P(t,s) = \exp\bigl(-0.04(s-t)\bigr)$ for all $s > t$. The top figure shows the impact of this correlation on the volatilities of the current time (vanilla) options, whereas the bottom figure plots these volatility structures for forward starting call options with strike determination date $T_1 = 10$ year.

C Impact of the rate-volatility correlation coefficient on the forward starting options

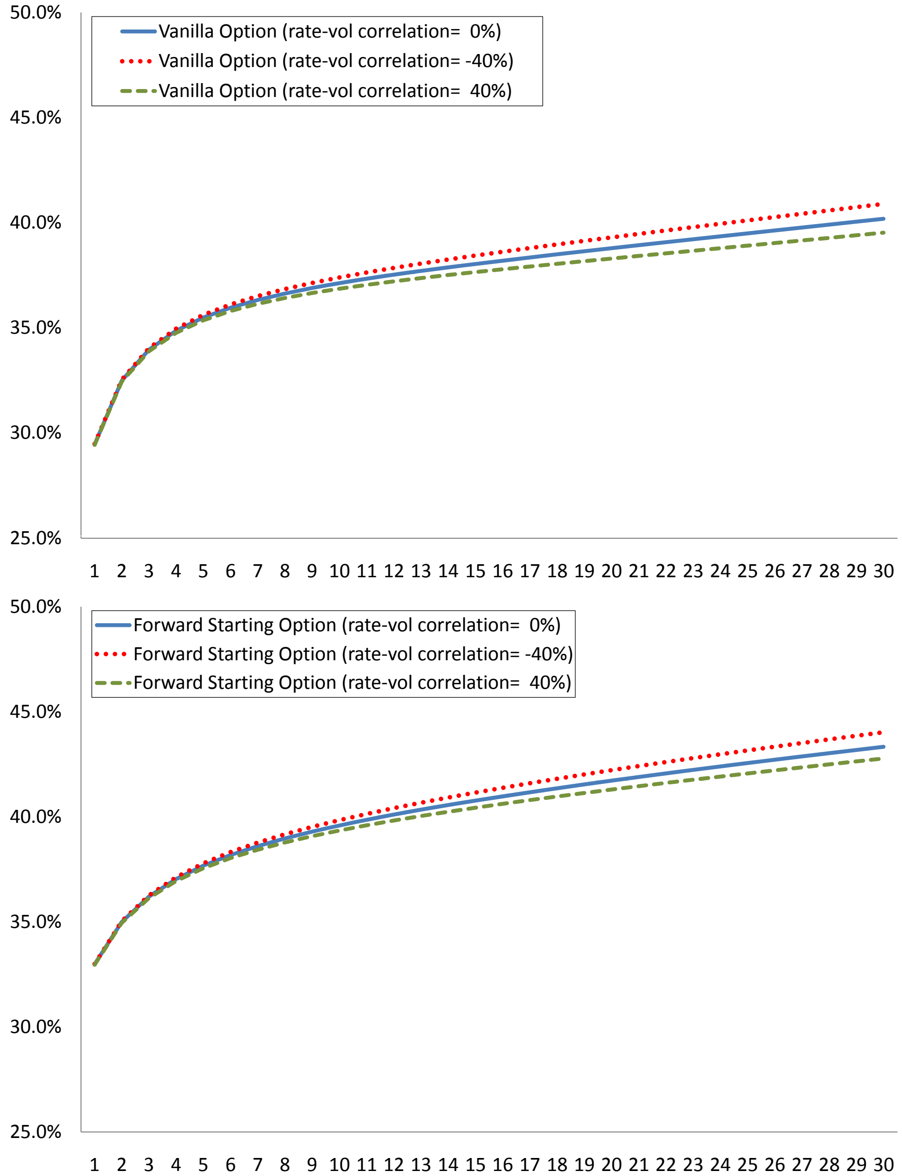

Figure **4**: Impact of the rate-volatility correlation $\rho_{r\nu}$ on the (forward) implied volatility structure. for different underlying call option maturities. Parameters are $\kappa = 1.0$, $\nu(t) = \psi = 0.20$, $\tau = 0.5$, $a = 0.02$, $\sigma = 0.01$, $\rho_{S\nu} = -0.70$, $\rho_{Sr} = 0$ and $P(t,s) = \exp\big(-0.04(s-t)\big)$ for all $s > t$. The top figure graphs the impact of this correlation on the volatilities of the current time (vanilla) options, whereas the bottom figure plots these volatility structures for forward starting call options with strike determination date $T_1 = 10$ year.

References

R. Ahlip. Foreign exchange options under stochastic volatility and stochastic interest rates. *International Journal of Theoretical and Applied Finance*, 11(3):277–294, 2008a.

R. Ahlip. Forward start options under stochastic volatility and stochastic interest rates. *International Journal of Theoretical and Applied Finance*, 2008b.

L. Andersen and J. Andreasen. Volatile volatilities. *Risk*, 15(2):163–168, 2002.

J. Andreasen. Closed form pricing of fx options under stochastic rates and volatility. Global Derivatives Conference 2006.

J. Andreasen. A lego approach to hybrid modeling: The baest. Global Derivatives Conference 2007.

A. Antonov, M. Arneguy, and N. Audet. Markovian projection to a displaced volatility heston model. http://ssrn.com/abstract=1106223, 2008.

S. Bakshi, C. Cao, and Z. Chen. Pricing and hedging long-term options. *Journal of Econometrics*, 94:277–318, 2000.

D.G. Baur. Stock-bond co-movements and cross-country linkages. http://ssrn.com/abstract=978249, 2009.

F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3), 1973.

D. Brigo and F. Mercurio. *Interest rate models - theory and practice*. Springer Finance, 2006.

P. Carr and D.B. Madan. Option valuation using the fast fourier transform. *Journal of Computational Finance*, 2:61–73, 1999.

A. Feuerverger and A.C.M. Wong. Computation of value-at-risk for nonlinear portfolios. *Journal of Risk*, 3(1), 2000.

H. Geman, E. Karoui, and J.C. Rochet. Changes of numeraire, changes of probability measures and pricing of options. *Journal of Applied Probability*, 32:443–548, 1996.

P. Glasserman. *Monte Carlo Methods in Financial Engineering*. Springer Verlag, 2003.

J. Guo and M. Hung. A generalization of rubinstein's "pay now, choose later". *The Journal of Futures Markets*, 28(5):488–515, 2008.

S.L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

G. Hong. Forward smile and derivatives pricing, 2004. http://www-cfr.jbs.cam.ac.uk/archive/PRESENTATIONS/seminars/2004/hong.pdf.

J. Hull and A. White. One factor interest rate models and the valuation of interest rate derivative securities. *Journal of Financial and Quantitative Analysis*, 28(2), 1993.

F. Kilin. Accelerating the calibration of stochastic volatility models. http://mpra.ub.uni-muenchen.de/2975/1/MPRA_paper_2975.pdf, 2006.

S. Kruse and U. Nögel. On the pricing of forward starting options in hestons model on stochastic volatility. *Finance and Stochastics*, 9(2):233–250, 2005.

A. Lewis. A simple option formula for general jump-diffusion and other exponential lévy processes. http://ssrn.com/abstract=282110, 2001.

R. Lord and C. Kahl. Optimal fourier inversion in semi-analytical option pricing. *Journal of Computational Finance*, 10(4), 2007.

R. Lord and C. Kahl. Complex logarithms in Heston-like models. http://ssrn.com/abstract=1105998, 2008.

V. Lucić. Forward-start options in stochastic volatility models. *Wilmott Magazine*, 2003.

M. Kijima and Muromachi. Pricing equity swaps in a stochastic interest rate economy. *Journal of Derivatives*, 2001.

J.P.V. Nunes and T.R.V. Alcaria. Two extensions to forward start options valuation. http://ssrn.com/abstract=1329370, 2009.

A. Pelsser. *Efficient Methods For Valuing Interest Rate Derivatives*. Springer Finance, 2000.

V. Piterbarg. A multi-currency model with fx volatility skew. http://ssrn.com/abstract=685084, 2005.

W.H. Press and B. P. Flannery. *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, second edition, 1992.

M. Rubinstein. Pay now, choose later. *Risk*, 4(13), 1991.

R. Schöbel and J. Zhu. Stochastic volatility with an ornstein uhlenbeck process: An extension. *European Finance Review*, 4:23–46, 1999.

A. van Haastrecht, R. Lord, A. Pelsser, and D. Schrager. Pricing long-maturity equity and fx derivatives with stochastic interest rates and stochastic volatility. http://ssrn.com/abstract_id=1125590, 2008.