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Comment on “A theoretical foundation of ambiguity measurement” [J. Econ. Theory 187 (2020) 105001]

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Abstract

In this paper, we study asymptotic expansions for distorted probabilities under ambiguity, revisiting the framework and analysis of Izhakian (2020b). We argue that the first order terms in these expansions need to be corrected and provide alternatives. We also revisit later results in this paper on the separation of ambiguity and ambiguity attitudes. We argue that a crucial lemma is flawed implying that Izhakian's ambiguity measure $\mho^2$ is not an equivalent way of representing the preferences it is supposed to represent.

JEL classification: D81; D84

Keywords: Ambiguity; Model uncertainty; Probability weighting

1. Introduction

For a long time, the predominant modeling approach in economics for decision-making under uncertainty has been that of an agent who faces a so-called pure risk situation, i.e., an uncertain outcome which is a random draw from a known probability distribution. Starting with Knight (1921), this view has constantly been challenged over the past century, see e.g. Ellsberg (1961), Gilboa and Schmeidler (1989) and Hansen and Sargent (2001). Yet, nevertheless, most empirical

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work in economics and finance still relies on the conceptually simpler reduction of uncertainty to pure risk. This situation is the starting point of an impressive series of papers by Y. Izhakian and his coworkers which can be roughly summarized as follows.

(i) Starting from axiomatic first principles, Izhakian (2017) derives a concrete functional form for preferences under risk and ambiguity.
(ii) Building on that concrete functional form, Izhakian (2020b) provides asymptotic expansions of preferences in terms of quantities which are easy to elicit from observable data. This is similar to the classical Arrow-Pratt approximation from risk theory which can be seen as a justification for measuring risk using the variance. The analogue of the variance in this picture is the proposed ambiguity index $\mho^2$. In particular, just like the variance, $\mho^2$ is intended to be read as a measure of ambiguity that is independent of ambiguity attitudes.
(iii) As a measure of ambiguity, $\mho^2$ has been applied within a short time span to an impressive collection of empirical settings. Izhakian and Yermack (2017) study the effect of ambiguity on the exercise of employee stock options using $\mho^2$ as the measure. Brenner and Izhakian (2018) apply $\mho^2$ to the US equity market and explore the relation between risk, ambiguity, and expected returns. Augustin and Izhakian (2020) study the implication of ambiguity for the pricing of credit default swaps using $\mho^2$. Izhakian (2020a) extends mean-variance preferences to mean-variance-ambiguity preferences, where ambiguity is measured by $\mho^2$. Ben-Rephael and Izhakian (2020) study trading behavior under ambiguity relying on $\mho^2$. Coiculescu et al. (2019) explore the effect of risk and ambiguity on R&D investment using $\mho^2$ as the measure of ambiguity. In addition, $\mho^2$ has been applied to various topics in corporate finance, such as firms' payout policies (Herron and Izhakian, 2020a), mergers and acquisitions (Herron and Izhakian, 2020b), and the corporate capital structure (Izhakian et al., 2020).

In this paper, we take a close look at step (ii) in this development, the connection between preferences and the $\mho^2$ via asymptotic expansions. We begin with a close look at the definition of these preferences, highlighting some issues and smaller errors, that are relatively straightforward to correct. We then argue that there is a flaw in the proof of Izhakian (2020b)'s Theorem 1 which is the key result that establishes the connection between agents' perceived probabilities and the $\mho^2$. This flaw is not a mere technicality. After fixing it, the expansion looks different and is no longer connected to the $\mho^2$ in a way that is obvious to us. Accordingly, the $\mho^2$ also vanishes from later results of the paper such as Theorem 2 or Theorem 5 which connect preferences to the $\mho^2$.

While we offer alternatives to the first main results of Izhakian (2020b), his Theorems 1 and 2, our outlook on repairing some further steps in his analysis is more pessimistic. In particular, his Theorems 5, 6, and 7 on the separation of risk and ambiguity all rely crucially on his Lemma 3. Lemma 3 basically argues that certain covariance terms are always zero, thus simplifying the analysis a lot. We show that Lemma 3 is incorrect, invalidating the claimed separation of risk and ambiguity.[1] As a consequence, also the follow-up theorems, showing that the preference ordering can equivalently be described by $\mho^2$, fail. Given ambiguity, $\mho^2$ presents a total order over prospects. But given only ambiguity, the preference orderings, still depending on risk pref-

[1] Intuitively, when two random variables X and Y have zero covariance, one can write $E[XY] = E[X]E[Y]$ thus achieving a clean separation into two factors associated with each variable. With a non-zero covariance, the expectation of the product can usually not be split in such a way.

erences, are only partial orders. There can be no equivalence between a total order and a partial order (that is not a total order), as is claimed by Izhakian (2020b)'s Theorems 5 and 6.

Let us emphasize that these findings do not invalidate the empirical results of the papers summarized in (iii).[2] While the strong connection between preferences under ambiguity and $\mho^2$ seems lost, $\mho^2$ may well measure ambiguity – just like there are many other measures of risk besides the variance. We do know that it is a non-negative quantity that vanishes only in the absence of ambiguity and that it has had remarkable success in empirical work. However, $\mho^2$ as an ambiguity measure has its limitations. We provide an example in which the ordering according to $\mho^2$ is opposite to the ordering according to the ambiguity-averse preferences it is supposed to represent.

The remainder of this paper is structured as follows. In Section 2, we first introduce the setting, including an explicit link between our setting and the more general framework in Izhakian (2020b).[3] We also point out some minor flaws in the definitions of preferences in Izhakian (2020b). Next, in Section 3 we discuss Izhakian (2020b)'s approximation and provide a counterexample to his Theorem 1. In Section 4 we provide an informal derivation of our alternative asymptotic expansion for perceived probabilities and give a simple numerical example which shows that its accuracy is much higher than that of Izhakian (2020b)'s expansion. Afterwards, we also present rigorous derivations, which can be seen as our analogues of Izhakian (2020b)'s Theorems 1 and 2. In Section 5 we discuss Izhakian (2020b)'s $\mho^2$. We explain the problems with Izhakian (2020b)'s crucial Lemma 3 and argue why $\mho^2$ cannot be used as an equivalent way to represent preferences. We also briefly discuss an alternative way to quantify ambiguity, based on our approximation. Section 6 concludes.

Before we begin, we would like to emphasize that the various problems we identify in Izhakian (2020b) can be understood and verified independently of each other. In particular, our counterexamples to Izhakian's asymptotic expansions hold regardless of whether one resolves the sign errors in the definition of preferences or sticks to the original versions. Similarly, our issues with Lemma 3 and Theorem 5 can be understood regardless of whether one works with the corrected asymptotic expansions or with Izhakian's original version.

2. Setting

Izhakian's model of risk and ambiguity is a two-stage model like, e.g., the smooth ambiguity of Klibanoff et al. (2005). A risky and ambiguous quantity (prospect) X is drawn from a probability distribution which is unknown to the agent. The agent knows however that this probability distribution is in turn drawn from a known probability distribution over a family of probability distributions. In this note, we do not go back to the decision-theoretic basics but start right away with a concrete model which is a special case of Izhakian's.

We consider a random variable Z with values in some set $\mathcal{U}$ and distribution μ as the state of the world. The set $\mathcal{U}$ may be either finite or infinite. Throughout, the operations $E[\cdot]$, $\mathrm{cov}(\cdot,\cdot)$ and $\mathrm{var}(\cdot)$ refer to expected values, covariances, and variances with respect to the distribution of Z. While Z models ambiguity, risk is modeled through the measurable function $P : \mathbb{R} \times \mathcal{U} \to [0, 1]$ which has the property that for any fixed $z \in \mathcal{U}$ the function $P(\cdot, z)$ is a cumulative distribution

[2] Of course, several of these papers do cite some of the flawed theoretical claims from Izhakian (2020b). See, e.g., Section 2.1 of Izhakian et al. (2020) which is basically a summary of Theorem 1 and 2 of Izhakian (2020b) for the case where the reference point lies at $-\infty$.

[3] Since our main interest is in counterexamples, nothing is lost by restricting attention to our somewhat simpler setting.

function. We assume that $P(\cdot, z)$ is smooth in its first argument with (Lebesgue) density function $\phi(\cdot, z)$.[4] We assume that, conditional on $Z = z$, the real-valued, risky and ambiguous quantity X is continuously distributed according to the cumulative distribution function $P(x, z)$ with density $\phi(x, z)$ and support supp(z). Unless the support plays an important role, we define $P(x, z)$ and $\phi(x, z)$ for all $x \in \mathbb{R}$, thus assuming that the density is simply set to zero outside the support of X. We denote by $\bar{P}(x) = E[P(x, Z)]$ the expected cumulative distribution function and note that it is a proper distribution function which coincides with the unconditional distribution of X. The associated density is given by $\bar{\phi}(x) = E[\phi(x, Z)]$. The set of prospects X will be denoted by $\mathcal{X}$.

2.1. *Transformed probabilities*

Next, we introduce Izhakian's preference ordering $\succsim$ over $\mathcal{X}$. An agent who is neutral to the distinction between risk and ambiguity would simply evaluate prospects $X \in \mathcal{X}$ by integrating a suitable utility function $u(x)$ against $\bar{P}(x)$, i.e.,

$$\int_{-\infty}^{\infty} u(x) d\bar{P}(x) = \int_{-\infty}^{\infty} u(x)\bar{\phi}(x) dx,$$

for some utility function u that is assumed to be a continuous, strictly increasing bounded function.[5] We shall denote this class of utility functions by $\mathcal{V}$.

The agent in Izhakian's model calculates instead the integral with respect to a transformed probability distribution. Let Υ be a weighting function that is smooth and strictly increasing. Let Γ be the implied weighting function, defined by $\Gamma(p) = 1 - \Upsilon(1 - p)$. Then we can define two transformed probability distributions, namely, the cumulative distribution function[6]

$$P_{\Upsilon}(x) = \Upsilon^{-1}(E[\Upsilon(P(x, Z))])$$

and the cumulative distribution function

$$P_{\Gamma}(x) = \Gamma^{-1}(E[\Gamma(P(x, Z))]) = 1 - \Upsilon^{-1}(E[\Upsilon(1 - P(x, Z))])$$

with respective densities

$$\phi_{\Upsilon}(x) = \frac{d}{dx} P_{\Upsilon}(x), \text{ and } \phi_{\Gamma}(x) = \frac{d}{dx} P_{\Gamma}(x).$$

The functions $P_{\Upsilon}(x)$ and $P_{\Gamma}(x)$ both coincide with $\bar{P}(x)$ in the ambiguity neutral case, $\Gamma(x) = x$, or in the absence of ambiguity, i.e., Z a.s. constant.

If Υ is concave then by Jensen's inequality,

$$P_{\Upsilon}(x) = \Upsilon^{-1}(E[\Upsilon(P(x, Z))]) \leq \bar{P}(x, Z). \tag{1}$$

This implies that $P_{\Upsilon}(x)$ dominates $\bar{P}(x)$ in the stochastic first-order sense. If Υ is concave, then Γ is convex, implying that $\bar{P}(x)$ dominates $P_{\Gamma}(x)$ in the stochastic first-order sense. For each utility function $u \in \mathcal{V}$ we then have

[4] Throughout this section, we use "smooth" in the sense of "differentiable sufficiently often and otherwise well-behaved." We become more precise in Section 4 when we provide a rigorous justification for our alternative expansion.

[5] Izhakian (2020b) also assumes that there is a reference point k such that $u(k) = 0$. As we will discuss later, we do not need this reference point.

[6] Notice that $\Upsilon^{-1}(z) = 1 - \Gamma^{-1}(1 - z)$.

$$\int_{-\infty}^{\infty} u(x)dP_{\Gamma}(x) \leq \int_{-\infty}^{\infty} u(x)d\bar{P}(x) \leq \int_{-\infty}^{\infty} u(x)dP_{\Upsilon}(x).$$

Thus, if Υ is concave, an economic agent using $P_{\Upsilon}(x)$ to evaluate expected utility reveals ambiguity seeking behavior, while an economic agent using $P_{\Gamma}(x)$ reveals ambiguity aversion.

In the sequel, we shall focus on an ambiguity averse agent and assume, unless explicitly stated otherwise, that Υ is concave. In our setting, this means that the economic agent makes use of the transformed probability distribution $P_{\Gamma}(x)$, with Γ a smooth, strictly increasing, and convex transformation.

2.2. Preference orderings

We can now describe Izhakian's preference ordering over $\mathcal{X}$ in our context. Let $\mathcal{G}$ be the set of relevant Γ functions (smooth, strictly increasing, and convex[7]). Let prospect $X_1 \in \mathcal{X}$ have distribution function $P_1(x, Z)$, with density $\phi_1(x, Z)$, and let prospect $X_2 \in \mathcal{X}$ have distribution function $P_2(x, Z)$, with density $\phi_2(x, Z)$. We shall denote the preference ordering over $\mathcal{X}$ by $\succsim_{(u,\Gamma)}$: We have $X_1 \succsim_{(u,\Gamma)} X_2$ if, for some $(u, \Gamma) \in \mathcal{V} \times \mathcal{G}$,

$$\int_{-\infty}^{\infty} u(x)dP_{1,\Gamma}(x) \geq \int_{-\infty}^{\infty} u(x)dP_{2,\Gamma}(x), \tag{2}$$

where $P_{i,\Gamma}(x) = \Gamma^{-1}(E[\Gamma(P_i(x, Z)])$ with density $\phi_{i,\Gamma}(x)$, $i \in \{1, 2\}$. In terms of densities this is equivalent to

$$\int_{-\infty}^{\infty} u(x)\phi_{1,\Gamma}(x)dx \geq \int_{-\infty}^{\infty} u(x)\phi_{2,\Gamma}(x)dx. \tag{3}$$

In line with Izhakian (2020b)'s Theorem 1, our main interest is then in approximating $P_{\Gamma}(x)$ and its density $\phi_{\Gamma}(x)$.

2.3. Reformulations and reference points

Before we turn to these approximations, we first present some reformulations of the expected utility using $P_{\Gamma}(x)$, and link these reformulations to expressions presented in Izhakian (2020b). We are particularly interested in the dependence on the reference point. Let $\mathbb{P}_{\Gamma}$ be the probability distribution corresponding to P_{Γ}. Thus, $\mathbb{P}_{\Gamma}(X \leq x) = P_{\Gamma}(x)$. We can write

$$\int_{-\infty}^{\infty} u(x)dP_{\Gamma}(x) = \int_{-\infty}^{0} ((1 - \mathbb{P}_{\Gamma}(u(X) \leq x)) - 1)dx + \int_{0}^{\infty} (1 - \mathbb{P}_{\Gamma}(u(X) \leq x))dx. \tag{4}$$

Using

$$1 - P_{\Gamma}(x) = \Upsilon^{-1}(E[\Upsilon(1 - P(x, Z))]), \tag{5}$$

[7] See footnote 4.

we see that (4) is a special case of $V(f)$ as defined in equation (2) of Izhakian (2020b) (where Izhakian's f is replaced by X).

We can also write for some k

$$\int_{-\infty}^{\infty} u(x) d P_{\Gamma}(x) = \int_{-\infty}^{k} u(x) d P_{\Gamma}(x) + \int_{k}^{\infty} u(x) d P_{\Gamma}(x). \tag{6}$$

Using again (5), we can rewrite this equation as

$$\begin{aligned}\int_{-\infty}^{\infty} u(x) d P_{\Gamma}(x) = & \int_{-\infty}^{k} u(x) d\left[1-\Upsilon^{-1}(E[\Upsilon(1-P(x, Z))])\right] \\ & + \int_{k}^{\infty} u(x) d\left[1-\Upsilon^{-1}(E[\Upsilon(1-P(x, Z))])\right] \\ = & -\int_{-\infty}^{k} u(x) d\left[\Upsilon^{-1}(E[\Upsilon(1-P(x, Z))])-1\right] \\ & -\int_{k}^{\infty} u(x) d\left[\Upsilon^{-1}(E[\Upsilon(1-P(x, Z))])\right].\end{aligned}$$

This is a special case of $W(f)$, introduced on page 9 of Izhakian (2020b) as the dual representation of $V(f)$. However, our derivation shows that the plus-sign before his second integral has to be replaced by the minus-sign in our expression, needed to guarantee $V(f) = W(f)$.[8]

Our derivation also reveals that the reference point k does not play the intended role, namely, to distinguish between gains and losses. Indeed, Izhakian (2020b) uses the same transformed probability distribution $P_{\Gamma}(x)$ for values above and below this reference point. This means that the value of k *should* not matter in evaluating the integral in (6) (or its equivalent formulations). However, curiously, and as we discuss below, in Izhakian (2020b), the reference point is assigned an important role, since the utility index $u(x)$ is normalized such that $u(k) = 0$.

One can generalize (6) to give the reference point k its intended role, for example, by using $P_{\Gamma}(x)$ for gains (above the reference point) and using $P_{\Upsilon}(x)$ for losses (below the reference point). Since Υ is concave, this means ambiguity aversion above the reference point and ambiguity seeking behavior below the reference point. Something like this is done, for example, by Barberis and Huang (2008) in the context of prospect theory. In the sequel, we shall not consider this generalization of (6), but focus on the ambiguity averse case without a (spurious) reference point.[9]

8 For example, without correcting the plus-sign, Izhakian (2020b)'s expression does not reduce to expected utility in the absence of ambiguity: If we plug in a deterministic distribution function $P(x)$ with density $\phi(x)$, the dependence on Υ vanishes and we obtain $\int_{-\infty}^{k} u(x)\phi(x)dx - \int_{k}^{\infty} u(x)\phi(x)dx$ rather than $\int_{-\infty}^{\infty} u(x)\phi(x)dx$.

9 Indeed, some later works of Izhakian such as Izhakian et al. (2020) place the reference point at $-\infty$ without placing any constraints on the utility function, possibly recognizing the problem.

3. Izhakian (2020b)'s asymptotic expansion

In this section we first present and discuss Izhakian (2020b)'s approximation as presented in his Theorem 1, reformulated in our context. Next, we present a counterexample invalidating the claims of this theorem. In a nutshell, the term $\text{sgn}(u(x))\bar{\phi}(x)\text{var}(\phi(x, Z))$ plays a crucial role in Izhakian's approximation. We first discuss the factor $\text{sgn}(u(x))$ and then move on to $\bar{\phi}(x)\text{var}(\phi(x, Z))$.

3.1. Izhakian's asymptotic expansion

The main result of Izhakian's Theorem 1 is an asymptotic expansion for $\phi_\Gamma(x)$. Let $\text{sgn}_k(x)$ be equal to 1 if $x > k$ and equal to -1 otherwise. In our setting the asymptotic expansion for $\phi_\Gamma(x)$ presented in Izhakian (2020b) can be translated as follows

$$\phi_\Gamma(x) \approx \bar{\phi}(x) - \text{sgn}_k(x)\mathcal{A}(\bar{P}(x))\bar{\phi}(x)\text{var}(\phi(x, Z)), \tag{7}$$

where

$$\mathcal{A}(x) = \Gamma''(x)/\Gamma'(x) = -\Upsilon''(1-x)/\Upsilon'(1-x)$$

is analogous to the absolute risk aversion of Arrow and Pratt and where "$\approx$" stands for equality up to a higher order error term in the limit of vanishing ambiguity. The sign of $\mathcal{A}(x)$ reveals the attitude towards ambiguity: $\mathcal{A}(x) > 0$ corresponds to ambiguity aversion, $\mathcal{A}(x) < 0$ represents ambiguity seeking behavior, and $\mathcal{A}(x) = 0$ means ambiguity neutrality.

Izhakian's measure of ambiguity, $\mho^2$, is defined as the integral over x of the ambiguity-dependent part in the correction term in (7), i.e.,

$$\mho^2 = \int_{-\infty}^{\infty} \bar{\phi}(x)\text{var}(\phi(x, Z))dx. \tag{8}$$

We shall discuss this measure of ambiguity in Section 5.

The approximation presented in (7) depends on the reference point k. Since Izhakian (2020b) normalizes the increasing utility function u such that $u(k) = 0$, we can rewrite (7) as

$$\phi_\Gamma(x) \approx \bar{\phi}(x) - \text{sgn}(u(x))\mathcal{A}(\bar{P}(x))\bar{\phi}(x)\text{var}(\phi(x, Z)), \tag{9}$$

with $\text{sgn}(u(x)) = 1$ if $u(x) > 0$ and -1 otherwise. This removes the (explicit) dependence of the approximation on k.

It is curious that the factor $\text{sgn}(u(x))$ appears in an approximation to $\phi_\Gamma(x)$ (which depends on Γ but not on u). However, it is easy to see that without a factor like this Izhakian's preference approximation cannot respect ambiguity aversion. Most likely, this was the motivation for including this factor. Ignoring higher order terms, the expected utility using the approximation of the density function becomes

$$\int_{-\infty}^{\infty} u(x)\left[\bar{\phi}(x) - \text{sgn}(u(x))\mathcal{A}(\bar{P}(x))\bar{\phi}(x)\text{var}(\phi(x, Z))\right]dx.$$

In order for this expression to represent ambiguity aversion (corresponding to $\mathcal{A}(x) \geq 0$ for all x) or ambiguity seeking behavior (corresponding to $\mathcal{A}(x) \leq 0$ for all x) for any $\phi(x, Z)$, it is indeed necessary that $u(x)\text{sgn}(u(x)) \geq 0$.

The left hand side density $\phi_\Gamma(x)$ of (9) integrates to one. However, the approximation, i.e., the right hand side of (9), typically does not integrate to one: While the first term in the approximation integrates to one, the second term of the approximation only integrates to zero under special circumstances. For example, if $\operatorname{sgn}(u(x)) < 0$ and $\mathcal{A}(\bar{P}(x)) > 0$, the integral of the second term is strictly positive, but if $\operatorname{sgn}(u(x)) > 0$ and $\mathcal{A}(\bar{P}(x)) > 0$ the integral of the second term becomes strictly negative. Combined with the dependence of the approximation on the sign of $u(x)$ (or on k) these results violate the claim in Izhakian (2020b) that the approximation will be exact, for example, when Γ (or Υ) is linear or quadratic.[10] Indeed, as we will illustrate later, the approximation might be poor if it does not integrate to one. And even if it integrates to one for some given value of k, where the sign of $u(x)$ switches, by changing this value of k the approximation will change, typically no longer integrating to one, and thus impossible to be exact.

As discussed by Izhakian (2020b), the approximation (7) only has a chance to be valid for

$$|\mathcal{A}(\bar{P}(x))| < \frac{1}{\operatorname{var}(\phi(x, Z))},$$

since if this condition is not satisfied, the approximation (9) might become negative, potentially turning ambiguity aversion into ambiguity seeking behavior (or the other way around). However, such a condition is unfortunate, since it requires decreasing ambiguity aversion (or decreasing ambiguity seeking behavior) when the ambiguity increases.

All these considerations raise serious doubts about Theorem 1 of Izhakian (2020b). We next provide an example, invalidating the claims in Theorem 1.[11]

3.2. Counterexample to Theorem 1

We focus on the special case $\Gamma(x) = \exp(x) - 1$, which leads to $\mathcal{A} \equiv 1$ and $\mathcal{A}' \equiv 0$.[12] This choice implies that we can translate the claim of his theorem into our framework as follows: *Suppose that* $\operatorname{var}(\phi(x, Z)) \leq 1$. *Then*

$$\phi_\Gamma(x) = \bar{\phi}(x) - \operatorname{sgn}(u(x))\bar{\phi}(x)\operatorname{var}(\phi(x, Z)) + R_2(\phi(x, Z)). \tag{10}$$

Moreover $R_2(\phi(x, Z)) = o(E[|\phi(x, Z) - \bar{\phi}(x)|^3])$ *as* $|\phi(x, Z) - \bar{\phi}(x)| \to 0$.

Notice that (10) is a statement about deterministic quantities which do not depend on the realization of Z. Thus, $R(\phi(x, Z))$ is not random and the stochastic limit $|\phi(x, Z) - \bar{\phi}(x)| \to 0$ is not entirely straightforward to interpret. Nevertheless, our example is simple enough to fit into this setting without room for doubt. In particular, the example is a valid counterexample no matter whether $\operatorname{sgn}(u(x))$ is -1 or 1.

Our basic idea is to consider a family of random distribution functions $P(\cdot, Z)$ with the property that the support of $P(\cdot, z)$ is $[0, 1]$ for all z. We focus on the case where x is the lower end of the support $x = 0$. We know that $P(0, z) = 0$ for all z and, consequently, $P_\Gamma(0) = \bar{P}(0) = 0$. There is no ambiguity about the cumulative distribution function at the lower end of the support. For the distorted density $\phi_\Gamma(0)$, we find that

[10] According to Izhakian (2020b), the approximation is also exact if the fourth and higher central moments of $\phi(x, Z)$ are equal to zero. These moments can only be zero in the degenerate case, in which case $\operatorname{var}(\phi(x, Z))$ is also zero, making the approximation indeed exact.

[11] In the appendix, we also take a closer look at the proof of Theorem 1, pointing out the place that leads to the error.

[12] The function $\Gamma_\lambda(x) = (\exp(\lambda x) - 1)/\lambda$, is, of course, a special case of a function with constant (positive if $\lambda > 0$ or negative if $\lambda < 0$) absolute risk aversion in the sense of Arrow-Pratt. The limit $\lambda \to 0$ is the ambiguity neutral case $\Gamma_0(x) = x$.

$$\begin{aligned}\phi_\Gamma(0) &= \frac{d}{dx}\log\left(E[\exp(P(x,Z))]\right)\Big|_{x=0} \\ &= \frac{(E[\phi(x,Z)\exp(P(x,Z))]}{E[\exp(P(x,Z))]}\Big|_{x=0} \\ &= \bar{\phi}(0),\end{aligned}$$

since $P(0, Z) = 0$. Thus, in this situation no correction terms are needed as also $\phi_\Gamma(0)$ and $\bar{\phi}(0)$ coincide. However, the leading error term $E[\phi(0, Z)]\text{var}(\phi(0, Z))$ in (10) is not necessarily zero when the density $\phi(0, Z)$ at the lower bound of the support is random. In this situation, since $\phi_\Gamma(0) = \bar{\phi}(0)$ we know that the absolute value of the error term in (10) is given by

$$|R_2(\phi(0,Z))| = \bar{\phi}(0)\text{var}(\phi(0,Z)). \tag{11}$$

We will now make concrete parametric assumptions to show that the claim $R_2(\phi(x, Z)) = o(E[|\phi(x, Z) - \bar{\phi}(x)|^3])$ as $|\phi(x, Z) - \bar{\phi}(x)| \to 0$ is wrong in general.

We assume $P(x, Z) = (1 - \frac{Z}{2})x + \frac{1}{2}Zx^2$ for $x \in [0, 1]$. Z follows the uniform distribution on $[-L, L]$, $L \in [0, 2]$. The corresponding density function is $\phi(x, Z) = 1 - \frac{Z}{2} + Zx$. Moreover, $\bar{\phi}(x) = 1$ and $\text{var}(\phi(x, Z)) = \frac{1}{3}(x - \frac{1}{2})^2L^2$. The example thus models ambiguity around the uniform distribution on $[0, 1]$. To study the limit of vanishing ambiguity of Izhakian's theorem, we can let L go to 0. Izhakian's regularity condition $\text{var}(\phi(x, Z)) \leq 1$ is satisfied for all $x \in [0, 1]$ and all $L \in [0, 2]$. By (11), we know that

$$|R_2(\phi(0,Z))| = \text{var}(\phi(0,Z)) = \frac{1}{12}L^2.$$

Moreover,

$$E[|\phi(0,Z) - \bar{\phi}(0)|^3] = E[|-\frac{Z}{2}|^3] = \frac{1}{32}L^3.$$

We thus have a direct contradiction to the claim that

$$|R_2(\phi(0,Z))| = o(E[|\phi(0,Z) - \bar{\phi}(0)|^3])$$

which boils down to the wrong statement that

$$0 = \lim_{L\to 0}\frac{\frac{1}{12}L^2}{\frac{1}{32}L^3} = \lim_{L\to 0}\frac{8}{3L} = \infty.$$

This counterexample shows that the claimed convergence speed of the remainder term in Izhakian's Theorem 1 is incorrect. Instead, $R_2(\phi(0, Z))$ is of order $O(E[|\phi(0, Z) - \bar{\phi}(0)|^2])$.

4. An alternative asymptotic expansion

Given the findings in the previous section, our main claim is that, in general, the approximation (7) is not more precise than the crude approximation $\phi_\Gamma(x) \approx \bar{\phi}(x)$ while another equally simple approximation, to be introduced in this section, is superior. We first present this alternative asymptotic expansion. Then we illustrate and compare this approximation with the one proposed by Izhakian. We conclude this section by presenting a rigorous derivation of our approximation, to present our analogues to Izhakian's Theorems 1 and 2.

4.1. The alternative asymptotic expansion

To derive the alternative approximation, we simply apply the classical Arrow-Pratt approximation to $P_\Gamma(x)$, noting that for any fixed x this is formally the "certainty equivalent" of $P(x, Z)$ for an agent with "utility function" Γ,

$$P_\Gamma(x) = \Gamma^{-1}(E[\Gamma(P(x, Z))]) \approx \bar{P}(x) + \mathcal{A}(\bar{P}(x))\frac{\text{var}(P(x, Z))}{2} \equiv P_\Gamma^{\approx}(x). \tag{12}$$

Now, formally taking the derivative with respect to x on both sides of this approximation leads to

$$\phi_\Gamma(x) \approx \bar{\phi}(x) + \mathcal{A}(\bar{P}(x))\text{cov}(P(x, Z), \phi(x, Z)) + \mathcal{A}'(\bar{P}(x))\bar{\phi}(x)\frac{\text{var}(P(x, Z))}{2}, \tag{13}$$

since

$$\begin{aligned}\frac{d}{dx}\text{var}(P(x, Z)) &= E[2(P(x, Z) - \bar{P}(x))(\phi(x, Z) - \bar{\phi}(x))]\\ &= 2\,\text{cov}(P(x, Z), \phi(x, Z)).\end{aligned}$$

We see that in this approximation the value of Izhakian (2020b)'s reference point k does not play a role and there are also no restrictions on the magnitude of the ambiguity aversion (if the ambiguity increases). Moreover, since

$$\lim_{x\to\pm\infty} \text{var}(P(x, Z)) = 0,$$

we have

$$\lim_{x\to-\infty} P_\Gamma^{\approx}(x) = \lim_{x\to-\infty} \bar{P}(x) = 0 \text{ and } \lim_{x\to\infty} P_\Gamma^{\approx}(x) = \lim_{x\to\infty} \bar{P}(x) = 1.$$

The approximation has the same end limit values as $\bar{P}(x, Z)$. Thus, like the integral over the density $\phi_\Gamma(x)$, also the integral over the approximation is one. In our previous counterexample with $P(0, Z) = 0$ a.s., we have $\text{cov}(P(0, Z), \phi(0, Z)) = 0$ and $\text{var}(P(0, Z)) = 0$ so the approximation becomes exact, $\phi_\Gamma(0) = \bar{\phi}(0)$. So, we conclude that this approximation does not suffer from the drawbacks of the approximation proposed by Izhakian (2020b).

4.2. Illustration

To illustrate and compare Izhakian's and our approximation in a concrete example, we consider the case $\Gamma(x) = (\exp(\lambda x) - 1)/\lambda$, $\lambda \neq 0$, which leads to $\mathcal{A} \equiv \lambda$ and $\mathcal{A}' \equiv 0$. In this case, Izhakian's approximation (9) becomes

$$\phi_\Gamma(x) \approx \bar{\phi}(x) - \text{sgn}(u(x))\lambda\,\bar{\phi}(x)\text{var}(\phi(x, Z)) \tag{14}$$

while our approximation (13) becomes

$$\phi_\Gamma(x) \approx \bar{\phi}(x) + \lambda\,\text{cov}(P(x, Z), \phi(x, Z)). \tag{15}$$

We take $\lambda = 5$. We assume that Z takes two values, 0 and 1 with equal probability. $P(x, 0)$ is a normal distribution with mean 0 and variance 1 while $P(x, 1)$ is a normal distribution with mean 0 and variance 10. This mixture-of-normals example can be interpreted as a stylized model of ambiguity about the level of volatility in a financial market. Moreover, it is simple enough for computing $\phi_\Gamma(x)$ and its two approximations exactly and to contrast them against $\bar{\phi}(x)$.

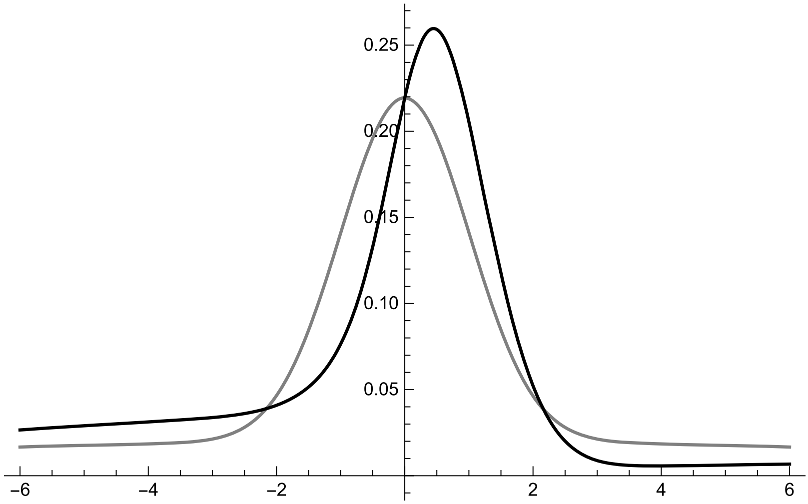

Fig. 1. The black line is the distorted distribution $\phi_\Gamma(x)$, while the grey line is the undistorted mixture distribution $\bar{\phi}(x)$.

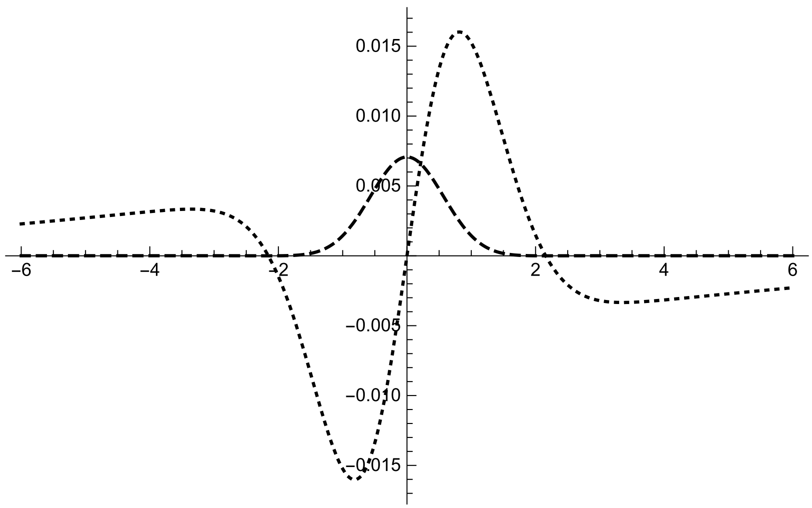

Fig. 2. The dashed line is the term $\bar{\phi}(x)\mathrm{var}(\phi(x, Z))$ in (14), while the dotted line is the term $\mathrm{cov}(P(x, Z), \phi(x, Z))$ in (15).

Fig. 1 shows the difference between the distorted distribution $\phi_\Gamma(x)$ and the undistorted mixture distribution $\bar{\phi}(x)$. We see that this distortion increases the weight of lower tail while moving the mode of the distribution slightly upwards. Fig. 2 shows the main term of the two approximation formulas, Izhakian's $\bar{\phi}(x)\mathrm{var}(\phi(x, Z))$ and our $\mathrm{cov}(P(x, Z), \phi(x, Z))$. The two terms are completely different. While Izhakian's term is symmetric around 0, ours indeed puts additional probability mass on positive values and subtracts it on the negative axis. The latter behavior is in line with the difference we see between $\bar{\phi}$ and ϕ_Γ, considering λ is positive, corresponding to ambiguity averse behavior. Finally, Fig. 3 adds the two approximations to Fig. 1. Our approximation is highly accurate fairly uniformly over all values of x despite the marked distortion. In

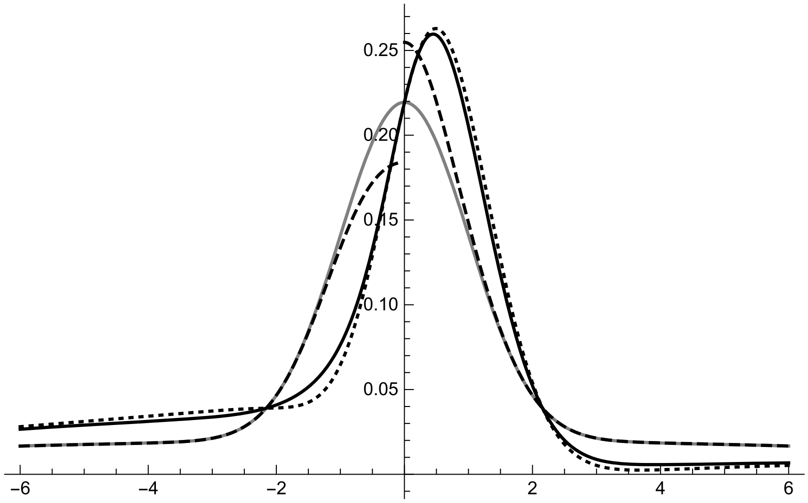

Fig. 3. The solid black line is the distorted distribution $\phi_\Gamma(x)$; the solid grey line is the undistorted mixture distribution $\bar{\phi}(x)$; the dashed black line is the approximation of $\phi_\Gamma(x)$ based on (14); the dotted black line is the approximation of $\phi_\Gamma(x)$ based on (15).

contrast, the approximation based on (14) does not match the shape of ϕ_Γ closely. In the tails, this approximation is much closer to $\bar{\phi}(x)$ than to $\phi_\Gamma(x)$. The discontinuity at 0 comes from the assumption that $u(0) = 0$. This is the only dependence on the utility function in any of the curves: For negative x, we subtract the dashed line from Fig. 2 from the grey line in Fig. 1 while for positive numbers we add it. If we would shift the distributions $P(x, 0)$ and $P(x, 1)$ by a constant along the x-axis, $\phi_\Gamma(x)$, $\bar{\phi}(x)$ and our approximation (15) would be shifted by the same constant while the discontinuity in (14) is fixed in 0, leading to a different curve. Similarly, changing the utility function so that $u(0) \neq 0$ would only shift the discontinuity in the dashed line while leaving everything else unchanged.

Comparing Izhakian's approximation (14) and our approximation (15) under constant absolute risk aversion, an alternative for Izhakian's $\mho^2$ (see (8)) would be

$$\kappa = \int_{-\infty}^{\infty} |\mathrm{cov}(P_\theta(x, Z), \phi_\theta(x, Z))| dx. \tag{16}$$

Without absolute values, the integral would be zero. Therefore, we first take absolute values, before calculating the integral. We shall discuss this alternative measure of ambiguity in Section 5.

4.3. Rigorous derivations

To conclude this section, we provide a rigorous derivation of our alternative expansion. Our results in this subsection can be considered as our analogues to Izhakian's Theorems 1 and 2.

We do not aim at maximal generality here but rather at a simple and transparent result. In particular, we restrict attention to situations where we interpolate between a random variable (or random distribution function) and its expected value using a single parameter $\theta \in [0, 1]$. We thus restrict attention to a particular class of situations with vanishing ambiguity. Besides the simplicity, the main justification for this is that ultimately, we will be interested in the approximation of a particular fixed model. Thus, introducing the limit of vanishing ambiguity is merely a technical step.

Our first intermediate result is essentially the classical Arrow-Pratt approximation for the certainty equivalent of a random outcome. Our one deviation from the text book version of this result is that we rely on the integral form of the remainder term. This will be useful in later steps when we plug in random functions for the random variables and take their derivatives. Then, we need to ensure that derivatives of the remainder term behave nicely. All proofs of this section are in the appendix.

Lemma 4.1. *Let $\Gamma : [0,1] \to \mathbb{R}$ be a strictly increasing, three times continuously differentiable function whose first derivative is bounded away from zero. Consider a random variable Y with expectation $\bar{Y} = E[Y]$. For $\theta \in [0,1]$, define the random variable $Y_\theta = \theta Y + (1-\theta)\bar{Y}$ and the certainty equivalent $Y_\Gamma(\theta) = \Gamma^{-1}(E[\Gamma(Y_\theta)])$. The second-order Taylor expansion of $Y_\Gamma(\theta)$ around $\theta = 0$ is*

$$Y_\Gamma(\theta) = \bar{Y} + \frac{1}{2}\mathcal{A}(\bar{Y})\mathrm{var}(Y_\theta) + \int_0^\theta \frac{1}{2} Y_\Gamma'''(s)(\theta - s)^2 ds, \tag{17}$$

where $\mathcal{A} = \Gamma''/\Gamma'$.

To obtain the asymptotic expansions we need, we now set $Y = P(x, Z)$, replacing the random variable by a random cumulative distribution function. For fixed x, this does not change anything, but, of course, we are ultimately interested in derivatives with respect to x. Thus, the role of $\bar{Y}$ is taken by $\bar{P}(x)$ and Y_θ becomes $P_\theta(x, Z) = \theta P(x, Z) + (1-\theta)\bar{P}(x)$. Therefore, as we increase θ from 0 to 1 we effectively turn risk into ambiguity, interpolating from $\bar{P}(x)$ to $P(x, Z)$. Finally, $\Gamma^{-1}(E[\Gamma(P_\theta(x, Z))])$ is a function of not only θ but also x, and denoted by $P_\Gamma(\theta, x)$. When $\theta = 1$, we have $P_\Gamma(\theta, x) = P_\Gamma(x)$; when $\theta = 0$, $P_\Gamma(\theta, x) = \bar{P}(x)$. Following Lemma 4.1, we thus know that

$$P_\Gamma(\theta, x) = \bar{P}(x) + \frac{1}{2}\mathcal{A}(\bar{P}(x))\mathrm{var}(P_\theta(x, Z)) + \int_0^\theta \frac{1}{2}\frac{\partial^3 P_\Gamma}{\partial \theta^3}(s, x)(\theta - s)^2 ds.$$

The following theorem is our analogue of Izhakian's Theorem 1.

Theorem 4.2. *Let $\phi_\Gamma(\theta, x) = \frac{\partial}{\partial x} P_\Gamma(\theta, x)$ and $\phi_\theta(x, Z) = \frac{\partial}{\partial x} P_\theta(x, Z)$. We assume that $\Gamma : [0,1] \to \mathbb{R}$ is a strictly increasing, four times continuously differentiable function whose first derivative is bounded away from zero. Moreover, we assume that there exists C such that $\phi(x, z) \le C$ for all $x \in \mathbb{R}$ and $z \in \mathcal{U}$. Then we have that*

$$\begin{aligned}\phi_\Gamma(\theta, x) =& \bar{\phi}(x) + \mathcal{A}(\bar{P}(x))\mathrm{cov}(P_\theta(x, Z), \phi_\theta(x, Z)) \\ &+ \frac{1}{2}\mathcal{A}'(\bar{P}(x))\bar{\phi}(x)\mathrm{var}(P_\theta(x, Z)) + \mathcal{R}(\theta),\end{aligned} \tag{18}$$

where $\mathcal{A} = \Gamma''/\Gamma'$ and

$$\mathcal{R}(\theta) = \int_0^\theta \frac{1}{2}\frac{\partial^4 P_\Gamma}{\partial \theta^3 \partial x}(s, x)(\theta - s)^2 ds.$$

In the limit $\theta \to 0$, we have $\mathcal{R}(\theta) = O(\theta^3)$.

Note that

$$\mathrm{var}(P_\theta(x, Z)) = \theta^2 \mathrm{var}(P(x, Z))$$

and

$$\mathrm{cov}(P_\theta(x, Z), \phi_\theta(x, Z)) = \theta^2 \mathrm{cov}(P(x, Z), \phi(x, Z))$$

so that we have two terms of order θ^2. Finally, plugging $\theta = 1$ into (18) and dropping the remainder term, we arrive at our previous approximation (13).

We can use the previous theorem to obtain an asymptotic expansion in terms of preferences. This is our analogue of Izhakian's Theorem 2. In this theorem, we impose conditions on Izhakian's $\mho^2$ (see (8)), on κ (see (16), and on the union of the supports $\mathrm{supp}(z)$, defined as $\mathcal{S} = \cup_{z \in \mathcal{U}} \mathrm{supp}(z)$.

Theorem 4.3. *Suppose that the conditions of Theorem 4.2 are satisfied. In addition, assume $\mho^2 < \infty$ and $\kappa < \infty$ and assume that $\mathcal{S}$ is bounded. Then for $u \in \mathcal{V}$ we have that*

$$\int_{\mathcal{S}} u(x)\phi_\Gamma(\theta, x)dx = \int_{\mathcal{S}} [u(x)\bar{\phi}(x) + \mathcal{A}(\bar{P}(x))\mathrm{cov}(P_\theta(x, Z), \phi_\theta(x, Z)) + \frac{1}{2}\mathcal{A}'(\bar{P}(x))\bar{\phi}(x)\mathrm{var}(P_\theta(x, Z))]dx + \mathcal{R}_u(\theta), \tag{19}$$

where

$$\mathcal{R}_u(\theta) = \int_0^\theta \left[\int_{\mathcal{S}} u(x)\frac{1}{2}\frac{\partial^4 P_\Gamma}{\partial\theta^3\partial x}(s, x)dx \right] (\theta - s)^2 ds.$$

In the limit $\theta \to 0$, we have $\mathcal{R}_u(\theta) = O(\theta^3)$.

5. Measuring ambiguity

In this section, we first discuss Izhakian's measure of ambiguity, $\mho^2$, defined as

$$\mho^2 = \int_{-\infty}^{\infty} \bar{\phi}(x)\mathrm{var}(\phi(x, Z))dx.$$

We write $\mho^2_\Gamma(X)$ whenever we want to emphasize the dependence of $\mho^2$ on the prospect X and the weighting function Γ. We show that this measure is not representing preferences as is claimed in Izhakian (2020b) in his Theorems 5 and 6. Next, we propose and discuss the alternative measure κ, defined by

$$\kappa = \int_{-\infty}^{\infty} |\mathrm{cov}(P_\theta(x, Z), \phi_\theta(x, Z))|dx.$$

5.1. Izhakian's $\mho^2$

The motivation for $\mho^2$ as a measure of ambiguity is its claimed link with the preference ordering as stated in Theorems 5 and 6 of Izhakian (2020b). The claimed link between Izhakian's $\mho^2$ and a preference ordering $\succsim_{(u,\Gamma)}$ over $\mathcal{X}$, represented by some $(u,\Gamma) \in \mathcal{V} \times \mathcal{G}$ (i.e., under ambiguity aversion) is then

$$\mho^2_\Gamma(X_1) \le \mho^2_\Gamma(X_2) \iff X_1 \succsim_{(u,\Gamma)} X_2, \tag{20}$$

assuming $\bar{P}_1(x) = \bar{P}_2(x)$ and thus also $\bar{\phi}_1(x) = \bar{\phi}_2(x)$. One can rewrite the right hand side of (20) more explicitly in terms of u and Γ using (2) or (3). The claimed equivalence (20) is strong and surprising: The preference relation $\succsim_{(u,\Gamma)}$ on the right hand side depends on two non-linear functions, u and Γ, while the summary statistic $\mho^2_\Gamma$ on the left hand side depends only on Γ. In the following, we argue that this is too good to be true.

Izhakian (2020b) arrives at (20) using his Theorem 1 (discussed earlier) and his Lemma 3. In the proof of Theorem 5 (and also of Theorems 6 and 7), Lemma 3 plays a key role in separating the $\mho^2$ from other components of preferences. Translated into our framework, Lemma 3 claims the following:

For $a < b$ in the support of $\bar{\phi}$, denote by $\bar{\phi}_{a,b}$ the restriction of $\bar{\phi}$ to the interval $[a, b]$,

$$\bar{\phi}_{a,b}(x) = \frac{\bar{\phi}(x)1_{\{x\in[a,b]\}}}{\int_a^b \bar{\phi}(x)dx}$$

and define the function $v(x) = \mathrm{var}(\phi(x, Z))$. Then for any other function $h : \mathbb{R} \to \mathbb{R}$

$$\int_a^b h(x)v(x)\bar{\phi}_{a,b}(x)dx = \int_a^b h(x)\bar{\phi}_{a,b}(x)dx \int_a^b v(x)\bar{\phi}_{a,b}(x)dx.$$

This lemma is clearly wrong. In probabilistic terms, the lemma states that when X is distributed with density $\bar{\phi}_{a,b}$ over the interval $[a, b]$ then the random variable $v(X)$ is uncorrelated from any random variable of the form $h(X)$. Effectively, this means that $v(x) = \mathrm{var}(\phi(x, Z))$ must be constant in x which is not supported by the assumptions. To see this, we can pick $h(x) = v(x)$ so that the claim of the lemma becomes

$$\int_a^b v(x)^2\bar{\phi}_{a,b}(x)dx - \left(\int_a^b v(x)\bar{\phi}_{a,b}(x)dx\right)^2 = 0$$

which is the same as $v(X)$ having zero variance when X is distributed according to $\bar{\phi}_{a,b}$ for arbitrary $a < b$. Inspecting Izhakian's proof of the lemma, at least part of the problem seems to be that he uses an incorrect tower property of conditional expectations where the inner conditional expectation is squared, $E[E[U|V]^2] = E[E[U]^2]$. Given the result of Lemma 3, Izhakian (2020b) claims to be able to separate $\mho^2$ from the risk component of the preference ordering, resulting in his Theorems 5 and 6.

Given that Lemma 3 is wrong, the question arises whether there is another way to prove Izhakian (2020b)'s Theorems 5 and 6. We shall now argue why such attempts are doomed to fail, since, contrary to the result of Lemma 3, $\mho^2$ cannot be separated from the risk component of the preference ordering.

The point is that in the claimed equivalence (20) the left hand side does not depend on the utility function u, whereas the right hand side does. Given $\Gamma \in \mathcal{G}$, $\mho^2$ is a total order over $\mathcal{X}$.[13] However, the preference ordering is only a total order over $\mathcal{X}$ given both Γ and u. But since the left hand side of (20) does not depend on u, this equivalence (20) can only be true for a given Γ if the right hand side is true for all $u \in \mathcal{V}$. But with u left free, the preference ordering on the right hand side becomes a partial order and is no longer a total order. Such a right hand side partial order cannot be equivalent to the left hand side total order of (20).

Indeed, for a given $\Gamma \in \mathcal{G}$, and given the left hand side of (20), we should have for all $u \in \mathcal{V}$

$$\int_{-\infty}^{\infty} u(x) dP_{1,\Gamma}(x) \geq \int_{-\infty}^{\infty} u(x) dP_{2,\Gamma}(x). \tag{21}$$

But this means that $P_{1,\Gamma}(x)$ first-order stochastically dominates $P_{2,\Gamma}(x)$, i.e.,

$$\text{for all } x, \;\; P_{1,\Gamma}(x) \leq P_{2,\Gamma}(x). \tag{22}$$

This is a partial order, but not a total order, over $\mathcal{X}$.

If the left hand side of (20) holds, but condition (22) is not satisfied, then there will exist at least one utility function $u \in \mathcal{V}$ with the opposite inequality than given in (21). It is straightforward to construct examples, such that (22) is not satisfied, when comparing some X_1 and X_2, whereas they are always ordered by $\mho^2$. For example, take

$$X_1 \sim P_1(x,z) = \frac{1}{3}(x^3 - x)z + x, \;\; X_2 \sim P_2(x,z) = \frac{1}{2}(x^2 - x)z + x,$$

for $Z \sim U(\{-1,+1\})$, $x \in [0,1]$, and choosing $\Gamma(x) = \exp(x) - 1$. Then clearly X_1 and X_2 are ordered using $\mho^2$, but neither $P_{1,\Gamma}(x)$ nor $P_{2,\Gamma}(x)$ first-order stochastically dominates the other. This can be seen from the fact that the difference $P_{1,\Gamma}(x) - P_{2,\Gamma}(x)$ switches signs at $x = \frac{1}{2}$ as shown in Fig. 4. It follows that the ambiguity measure $\mho^2$ cannot represent the preference orderings as claimed by Izhakian (2020b). We emphasize that this problem is not particular to the $\mho^2$. The counterexample shows that $\succsim_{(u,\Gamma)}$ cannot be represented by any total order that depends on Γ but not on u.

5.2. *Monotonic transformations*

If one still wants to make use of $\mho^2$ as a measure of ambiguity, it is relevant to be aware of the following drawback. In Observation 6 of Izhakian (2020b) it is stated that $\mho^2$ is invariant under monotonic transformations of X. This is clearly true in discrete examples. Suppose that X takes values $x_1, \ldots, x_n$ with random probabilities $p_1(Z), \ldots, p_n(Z)$. Then the discrete version of $\mho^2$, i.e.,

$$\mho^2 = \sum_{i=1}^{n} E[p_i(Z)] \mathrm{var}(p_i(Z)),$$

only depends on the probabilities $p_i(Z)$ and will not change if we compute $\mho^2$ for the distribution of $f(X)$ provided that $f(x_i) \neq f(x_j)$ for all $i \neq j$. In contrast, in the continuous case, $\mho^2$ is not even invariant to scaling X by a positive factor. To see this, suppose, we multiply a continuous

[13] A total (or linear) order is a partial order in which any two elements can be compared.

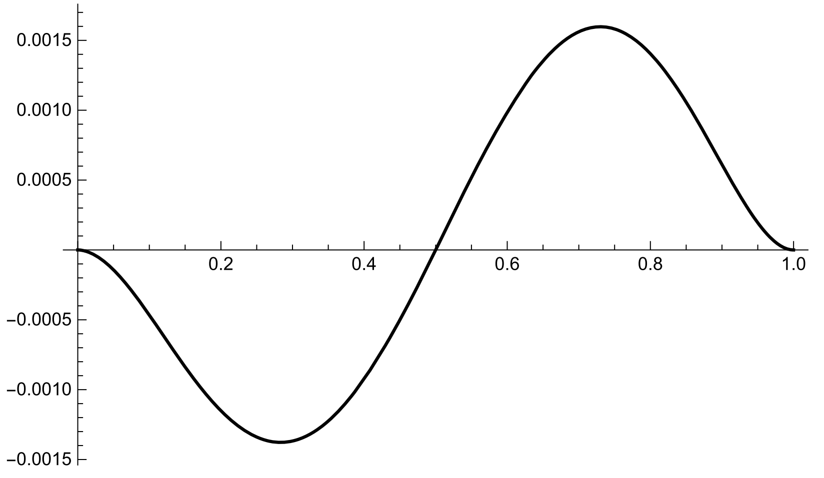

Fig. 4. The black line shows $P_{1,\Gamma}(x) - P_{2,\Gamma}(x)$ as a function of $x \in [0, 1]$.

X with random density $\phi(x, Z)$ by the factor c and write $Y = cX$. The density function of Y is $\frac{1}{c}\phi\left(\frac{y}{c}, Z\right)$, with $\mho^2$ of Y given by

$$\mho^2 = \int_{-\infty}^{\infty} E\left[\frac{1}{c}\phi\left(\frac{y}{c}, Z\right)\right] \operatorname{var}\left(\frac{1}{c}\phi\left(\frac{y}{c}, Z\right)\right) dy$$

$$= \frac{1}{c^2}\int_{-\infty}^{\infty} E[\phi(x, Z)]\operatorname{var}(\phi(x, Z))dx.$$

Thus, after scaling X by a factor c, $\mho^2$ is changed by a factor of c^{-2}.

Therefore, the claim of Observation 6 is limited to discrete distributions. This indicates on the one hand that some caution is needed when transferring intuitions about the $\mho^2$ between the discrete and the continuous case. On the other hand, it hints at the possibility that $\mho^2$ might be a more natural quantity in discrete settings than in continuous ones.

5.3. An alternative ambiguity measure

If we look into our analysis for an alternative to the $\mho^2$, the natural starting point is the term $\operatorname{cov}(P(x, Z), \phi(x, Z))$ from our expansion

$$\phi_\Gamma(x) \approx \bar{\phi}(x) + \lambda \operatorname{cov}(P(x, Z), \phi(x, Z)) \tag{23}$$

in the special case where $\Gamma''(x)/\Gamma'(x) \equiv \lambda$ is constant. If one is mainly interested in measuring the strength of the distortion due to ambiguity without representing preferences, a natural choice of ambiguity measure might be the integral of its absolute value

$$\kappa = \int_{-\infty}^{\infty} |\operatorname{cov}(P(x, Z), \phi(x, Z))| dx.$$

Note however that, just like for the $\mho^2$, the ordering according to this ambiguity measure cannot represent the same ordering as the underlying preferences. Our previous counterexample still applies.

Nevertheless, the κ has some attractive properties. It can be related to the total variation distance d_{tv} (e.g. Tsybakov (2008), p. 83) between the measures μ_Γ and $\bar{\mu}$ associated with the densities ϕ_Γ and $\bar{\phi}$ by rearranging (23),

$$d_{tv}(\mu_\Gamma, \bar{\mu}) = \frac{1}{2} \int_{-\infty}^{\infty} |\phi_\Gamma(x) - \bar{\phi}(x)| dx \approx \frac{|\lambda|\kappa}{2}.$$

In the product of $|\lambda|$ and κ, we thus achieve a separation into a factor that measures the strength of ambiguity attitudes and a factor that measures the degree of ambiguity. Using classical properties of the total variation distance, this implies, e.g., the following universal asymptotic bound on preference distortions,

$$\left| \int_{-\infty}^{\infty} u(x)\phi_\Gamma(x)dx - \int_{-\infty}^{\infty} u(x)\bar{\phi}(x)dx \right| \leq 2d_{tv}(\mu_\Gamma, \bar{\mu}) \lessapprox |\lambda|\kappa$$

for any utility function u that takes values in $[0, 1]$. Note the absolute value on the left hand side: The bound does not say anything about the ordering of the two utilities, only about the size of their absolute difference.

Unlike $\mho^2$, the measure κ is invariant to monotonic transformations. To see this, consider the transformation $Y = f(X)$, where f is strictly monotonic and continuously differentiable with first derivative bounded away from zero. Denoting by $P(x, Z)$ and $\phi(x, Z)$ the random distribution function and density of X, the corresponding quantities for Y are $P_Y(y, Z) = P(f^{-1}(y), Z)$ and $\phi_Y(y, Z) = \frac{1}{f'(f^{-1}(y))}\phi(f^{-1}(y), Z)$. The κ for Y is

$$\kappa = \int_{-\infty}^{+\infty} |\text{cov}(P_Y(x, Z), \phi_Y(x, Z))| dy = \int_{-\infty}^{+\infty} |\text{cov}(P(x, Z), \phi(x, Z))| dx,$$

by the usual substitution rules for integrals. Thus, the κ of Y coincides with the κ of X.

6. Conclusion

In this paper we reject Izhakian (2020b)'s claim that $\mho^2$ is an ambiguity measure that can be used as an equivalent way of representing preferences under risk and ambiguity. Instead, we have to conclude that the search for an ambiguity measure, with the characteristics Izhakian (2020b) is looking for, is still open.

We wish to end this paper by emphasizing that we are not writing this paper lightheartedly. We still consider the research agenda outlined in (i)-(iii) in the introduction a highly important one. Ambiguity research has been mostly restricted to theoretical work for far too long. Finding ways of taking the theory to the data is a timely and important topic. We like to see our paper as a positive contribution to this agenda. It is mostly one particular (but central) technical aspect that worries us, the derivation of the ambiguity measure $\mho^2$.

Data availability

No data was used for the research described in the article.

Appendix A

Comments on the proof of Izhakian's Theorem 1. To understand the problem behind Izhakian's Theorem 1 (Izhakian, 2020b) better, we follow his proof step-by-step. In his proof, he refers to Judd (2003) to do the Taylor expansion of

$$G(x)=\frac{\Upsilon'(1-P(x,Z))}{\Upsilon'(\Upsilon^{-1}(E[\Upsilon(1-P(x,Z))]))}$$

with respect to $\phi(x,Z)$. Judd (2003) states that, given the Taylor expansion of $f(x)$ at $x=a$ as $f(a)+(x-a)f'(a)+R_1$ (where R_1 is the residual term), for the strictly increasing and nonlinear transformation $x=h(y)$, the Taylor expansion of $f(h(y))$ at $y=b=h^{-1}(a)$ is

$$f(h(y))=f(h(b))+(y-b)f'(h(b))h'(b)+R_1=f(a)+(y-b)f'(a)h'(b)+R_1.$$

We apply this result in the context of Theorem 1. The Taylor expansion of $G(x)$ around $x=x_0$ is $G(x)=G(x_0)+(x-x_0)G'(x_0)+R_1$. The nonlinear transformation of x is $y=\phi(x,Z)$, and thus $x=\phi^{-1}(y,Z)$.[14] Then, the Taylor expansion of $H(y)=G(\phi^{-1}(y,Z))$ around $y=y_0=\phi(x_0,Z)$ is

$$H(y)=G(\phi^{-1}(y,Z))=G(x_0)+(\phi(x,Z)-\phi(x_0,Z))G'(x_0)\frac{d\phi^{-1}}{dy}(y_0,Z)+R_1,$$

where

$$\frac{d\phi^{-1}}{dy}(y_0,Z)=\frac{1}{\phi'(x_0,Z)},$$
$$G(x_0)=\frac{\Upsilon'(1-P(x_0,Z))}{\Upsilon'(\Upsilon^{-1}(E[\Upsilon(1-P(x_0,Z))]))},$$

and

$$\begin{aligned}G'(x_0)=&-\frac{\Upsilon''(1-P(x_0,Z))\phi(x_0,Z)}{\Upsilon'(\Upsilon^{-1}(E[\Upsilon(1-P(x_0,Z))]))}\\&+\frac{\Upsilon'(1-P(x_0,Z))\Upsilon''(\Upsilon^{-1}(E[\Upsilon(1-P(x_0,Z))]))E[\Upsilon'(1-P(x_0,Z))\phi(x_0,Z)]}{\Upsilon'(\Upsilon^{-1}(E[\Upsilon(1-P(x_0,Z))]))^3}.\end{aligned}$$

This result is quite different from equation (A.2) in Izhakian (2020b).

Proof of Lemma 4.1. By Taylor's theorem, the second-order approximation of $Y_\Gamma(\theta)$ around $\theta=0$ is

$$Y_\Gamma(\theta)=Y_\Gamma(0)+Y_\Gamma'(0)\theta+\frac{1}{2}Y_\Gamma''(0)\theta^2+R_2(\theta)\tag{24}$$

We work with the integral form of the remainder, i.e. $R_2(\theta)=\int_0^\theta \frac{1}{2}Y_\Gamma'''(s)(\theta-s)^2ds$.

[14] The probability density function ϕ is not strictly increasing in many cases. To satisfy the requirement of Judd (2003), we assume it is strictly increasing and its inverse function is ϕ^{-1}.

The first and second derivative of $Y_\Gamma(\theta)$ with respect to θ are

$$Y'_\Gamma(\theta) = \frac{E[\Gamma'(Y_\theta)(Y-\bar{Y})]}{\Gamma'(\Gamma^{-1}(E[\Gamma(Y_\theta)]))},$$
$$Y''_\Gamma(\theta) = \frac{E[\Gamma''(Y_\theta)(Y-\bar{Y})^2]}{\Gamma'(\Gamma^{-1}(E[\Gamma(Y_\theta)]))} - \frac{\Gamma''(\Gamma^{-1}(E[\Gamma(Y_\theta)]))E[\Gamma'(Y_\theta)(Y-\bar{Y})]^2}{\Gamma'(\Gamma^{-1}(E[\Gamma(Y_\theta)]))^3}.$$

When $\theta = 0$, $Y_\Gamma(0) = \bar{Y}$, $Y'_\Gamma(0) = 0$, and $Y''_\Gamma(0) = \frac{\Gamma''(\bar{Y})}{\Gamma'(\bar{Y})}\mathrm{var}(Y) = \mathcal{A}(\bar{Y})\mathrm{var}(Y)$. Furthermore, $\mathrm{var}(Y_\theta) = \theta^2\mathrm{var}(Y)$. We plug these results into (24), and thus obtain (17) in Lemma 4.1. □

Proof of Theorem 4.2. (18) follows easily by taking the derivative of $P_\Gamma(\theta, x)$ with respect to x. It remains to study the order of the error term

$$\int_0^\theta \frac{1}{2}\frac{\partial^4 P_\Gamma}{\partial\theta^3\partial x}(s,x)(\theta-s)^2 ds.$$

Using the short-hand notation

$$\begin{aligned}
G_1 &= G_1(s,x) = \Gamma'(\Gamma^{-1}(E[\Gamma(P_s(x,Z))]))\\
G_2 &= G_2(s,x) = \Gamma''(\Gamma^{-1}(E[\Gamma(P_s(x,Z))]))\\
G_3 &= G_3(s,x) = \Gamma'''(\Gamma^{-1}(E[\Gamma(P_s(x,Z))]))\\
G_4 &= G_4(s,x) = \Gamma''''(\Gamma^{-1}(E[\Gamma(P_s(x,Z))]))\\
H_1 &= H_1(s,x) = E[\Gamma'(P_s(x,Z))(P(x,Z)-\bar{P}(x))]\\
H_2 &= H_2(s,x) = E[\Gamma''(P_s(x,Z))(P(x,Z)-\bar{P}(x))^2]\\
H_3 &= H_3(s,x) = E[\Gamma'''(P_s(x,Z))(P(x,Z)-\bar{P}(x))^3],
\end{aligned}$$

we have

$$\frac{\partial^3 P_\Gamma}{\partial\theta^3}(s,x) = \frac{H_3}{G_1} - 3\frac{G_2}{G_1^3}H_1H_2 - \left(\frac{G_3}{G_1^4} - 3\frac{G_2^2}{G_1^5}\right)H_1^3,$$

and

$$\begin{aligned}
\frac{\partial^4 P_\Gamma}{\partial\theta^3\partial x}(s,x) =& \frac{1}{G_1}\cdot\frac{\partial H_3}{\partial x} - \frac{H_3}{G_1^2}\cdot\frac{\partial G_1}{\partial x} - 3\frac{H_1H_2}{G_1^3}\cdot\frac{\partial G_2}{\partial x}\\
&- 3\frac{G_2H_2}{G_1^3}\cdot\frac{\partial H_1}{\partial x} - 3\frac{G_2H_1}{G_1^3}\cdot\frac{\partial H_2}{\partial x} + 9\frac{G_2H_1H_2}{G_1^4}\cdot\frac{\partial G_1}{\partial x}\\
&- 3\frac{G_3H_1^2}{G_1^4}\cdot\frac{\partial H_1}{\partial x} + 9\frac{G_2^2H_1^2}{G_1^5}\cdot\frac{\partial H_1}{\partial x} - \frac{H_1^3}{G_1^4}\cdot\frac{\partial G_3}{\partial x}\\
&+ 4\frac{G_3H_1^3}{G_1^5}\cdot\frac{\partial G_1}{\partial x} + 6\frac{G_2H_1^3}{G_1^5}\cdot\frac{\partial G_2}{\partial x} - 15\frac{G_2^2H_1^3}{G_1^6}\cdot\frac{\partial G_1}{\partial x}.
\end{aligned}$$

Moreover, it can be easily verified that

$$\frac{\partial G_1}{\partial x} = \frac{G_2}{G_1} E[\Gamma'(P_s(x,Z)\phi_s(x,Z))]$$
$$\frac{\partial G_2}{\partial x} = \frac{G_3}{G_1} E[\Gamma'(P_s(x,Z)\phi_s(x,Z))]$$
$$\frac{\partial G_3}{\partial x} = \frac{G_4}{G_1} E[\Gamma'(P_s(x,Z)\phi_s(x,Z))]$$
$$\begin{aligned}\frac{\partial H_1}{\partial x} &= E[\Gamma''(P_s(x,Z))(P(x,Z)-\bar{P}(x))\phi_s(x,Z)] \\ &\quad - E[\Gamma'(P_s(x,Z))(\phi(x,Z)-\bar{\phi}(x))]\end{aligned}$$
$$\begin{aligned}\frac{\partial H_2}{\partial x} &= E[\Gamma'''(P_s(x,Z))(P(x,Z)-\bar{P}(x))^2\phi_s(x,Z)] \\ &\quad - 2E[\Gamma''(P_s(x,Z))(P(x,Z)-\bar{P}(x))(\phi(x,Z)-\bar{\phi}(x))]\end{aligned}$$
$$\begin{aligned}\frac{\partial H_3}{\partial x} &= E[\Gamma''''(P_s(x,Z))(P(x,Z)-\bar{P}(x))^3\phi_s(x,Z)] \\ &\quad - 3E[\Gamma'''(P_s(x,Z))(P(x,Z)-\bar{P}(x))^2(\phi(x,Z)-\bar{\phi}(x))].\end{aligned}$$

By assumption Γ and its first four derivatives are bounded. Thus G_1, G_2, G_3, G_4, H_1, H_2, and H_3 are all bounded given $P(x,Z) \in [0,1]$. Furthermore, Γ' and thus $|G_1|$ is bounded away from zero and $\phi(x,Z)$ is also bounded. Thus, $\frac{\partial G_1}{\partial x}$, $\frac{\partial G_2}{\partial x}$, $\frac{\partial G_3}{\partial x}$, $\frac{\partial H_1}{\partial x}$, $\frac{\partial H_2}{\partial x}$, and $\frac{\partial H_3}{\partial x}$ are bounded as well. Overall, $\frac{\partial^4 P_\Gamma}{\partial\theta^3\partial x}(s,x)$ is bounded uniformly in s for every x, so there exist functions $B_1(x)$ and $B_2(x)$ such that

$$B_1(x) \le \frac{\partial^4 P_\Gamma}{\partial\theta^3\partial x}(s,x) \le B_2(x).$$

Therefore, we have

$$\frac{1}{6}B_1(x)\theta^3 \le \int_0^\theta \frac{1}{2}\frac{\partial^4 P_\Gamma}{\partial\theta^3\partial x}(s,x)(\theta-s)^2 ds \le \frac{1}{6}B_2(x)\theta^3.$$

The approximation error term in (18) is of order $O(\theta^3)$ in the limit $\theta \to 0$. □

Proof of Theorem 4.3. We calculate $\int_{\mathcal{S}} u(x)\phi_\Gamma(x)dx$ by substituting the right hand side of (18) for $\phi_\Gamma(x)$. As remainder term, we get

$$\mathcal{R}_u(\theta) = \int_{\mathcal{S}} u(x)\left[\int_0^\theta \frac{1}{2}\frac{\partial^4 P_\Gamma}{\partial\theta^3\partial x}(s,x)(\theta-s)^2 ds\right]dx.$$

This remainder term is equal to

$$\int_{\mathcal{S}} u(x)\phi_\Gamma(x)dx - \int_{\mathcal{S}} u(x)\phi_\Gamma^{\approx}(x)dx,$$

with $\phi_\Gamma^{\approx}(x)$ the right hand side of (18) except $\mathcal{R}(\theta)$. By the assumptions of this theorem, the two integrals in this expression exist. Thus, also the integral $\mathcal{R}_u(\theta)$ exists. Using Fubini's theorem, we can rewrite $\mathcal{R}_u(\theta)$ as

$$\mathcal{R}_u(\theta) = \int_0^\theta \left[\int_{\mathcal{S}} u(x) \frac{1}{2} \frac{\partial^4 P_\Gamma}{\partial \theta^3 \partial x}(s,x) dx \right] (\theta - s)^2 ds.$$

Using the results of the previous theorem, the inner integral is an integral that can be bounded by (measurable) bounded functions, not depending on s, over a bounded interval. Thus, integrating over the bounded set $\mathcal{S}$, we can find constants C_1 and C_2, such that

$$C_1 \leq \int_{\mathcal{S}} u(x) \frac{1}{2} \frac{\partial^4 P_\Gamma}{\partial \theta^3 \partial x}(s,x) dx \leq C_2.$$

But then we find, analogously to the previous theorem,

$$\frac{1}{6} C_1 \theta^3 \leq \mathcal{R}_u(\theta) \leq \frac{1}{6} C_2 \theta^3.$$

This approximation error term is of order $O(\theta^3)$ in the limit $\theta \to 0$. $\square$

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